Chapter 2
Tensor Algebra

This chapter contains an introduction to tensor algebra. After defining covectors and dual bases, the space of covariant two-tensor is introduced. Then, the results derived for this space are extended to the general space of the (r, s)-tensors.

2.1 Linear Forms and Dual Vector Space

Definition 2.1. Let E be a vector space on $\Re$. The map

$$\boldsymbol{\omega} : E \to \Re \tag{2.1}$$

is said to be a ***linear form***, a 1-**form**, or a ***covector*** on E if

$$\boldsymbol{\omega}(a\mathbf{x} + b\mathbf{y}) = a\boldsymbol{\omega}(\mathbf{x}) + b\boldsymbol{\omega}(\mathbf{y}), \tag{2.2}$$

$\forall a, b \in \Re$ and $\forall \mathbf{x}, \mathbf{y} \in E$.[1]

The set E^* of all linear forms on E becomes a vector space on $\Re$ when we define the sum of two linear forms $\boldsymbol{\omega}, \boldsymbol{\sigma} \in E^*$ and the product of the scalar $a \in \Re$ with the linear form $\boldsymbol{\omega}$ in the following way

$$(\boldsymbol{\omega} + \boldsymbol{\sigma})(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x}) + \boldsymbol{\sigma}(\mathbf{x}), \quad (a\boldsymbol{\omega})(\mathbf{x}) = a\,\boldsymbol{\omega}(\mathbf{x}),\ \forall \mathbf{x} \in E. \tag{2.3}$$

Theorem 2.1. *Let E_n be a vector space with finite dimension n. Then, E^* has the same dimension n. Moreover, if $(\mathbf{e}_i)$ is a basis of E_n, then the n covectors such that*

$$\boldsymbol{\theta}^i(\mathbf{e}_j) = \delta^i_j \text{ [2]} \tag{2.4}$$

---

[1] For the contents of Chaps. 2–9, see [7, 8, 10, 11, 13, 14].

[2] Here δ^i_j is the Kronecker symbol

$$\delta^i_j = \begin{cases} 0, i \neq j, \\ 1, i = j. \end{cases}$$

A. Romano, *Classical Mechanics with Mathematica*®, Modeling and Simulation in Science, Engineering and Technology, DOI 10.1007/978-0-8176-8352-8_2,

define a basis of E^*.

Proof. First, we remark that, owing to (2.4), the linear forms $\boldsymbol{\theta}^i$ are defined over the whole space E_n since, $\forall \mathbf{x} = x^i \mathbf{e}_i \in E_n$, we have that

$$\boldsymbol{\theta}^i(\mathbf{x}) = \boldsymbol{\theta}^i(x^j \mathbf{e}_j) = x^j \boldsymbol{\theta}^i(\mathbf{e}_j) = x^i. \tag{2.5}$$

To show that E_n^* is n-dimensional, it is sufficient to verify that any element of E_n^* can be written as a unique linear combination of the covectors $(\boldsymbol{\theta}^i)$. Owing to the linearity of any $\boldsymbol{\omega} \in E_n^*$, we have that

$$\boldsymbol{\omega}(\mathbf{x}) = x^i \boldsymbol{\omega}(\mathbf{e}_i) = x^i \omega_i, \tag{2.6}$$

where we have introduced the notation

$$\omega_i = \boldsymbol{\omega}(\mathbf{e}_i). \tag{2.7}$$

On the other hand, (2.5) allows us to write (2.6) in the following form:

$$\boldsymbol{\omega}(\mathbf{x}) = \omega_i \boldsymbol{\theta}^i(\mathbf{x}), \tag{2.8}$$

from which, owing to the arbitrariness of $\mathbf{x} \in E_n$, it follows that

$$\boldsymbol{\omega} = \omega_i \boldsymbol{\theta}^i. \tag{2.9}$$

To prove the theorem, it remains to verify that the quantities ω_i in (2.9) are uniquely determined. If another representation $\boldsymbol{\omega} = \omega_i' \boldsymbol{\theta}^i$ existed, then we would have $\mathbf{0} = (\omega_i - \omega_i')\boldsymbol{\theta}^i$, i.e.,

$$(\omega_i - \omega_i')\boldsymbol{\theta}^i(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in E_n.$$

Finally, from (2.4) it follows that $\omega_i' = \omega_i$. □

Remark 2.1. Note that (2.6) gives the *value* of the linear map $\boldsymbol{\omega}$ when it is applied to the vector $\mathbf{x}$, whereas (2.9) supplies the linear map $\boldsymbol{\omega}$ as a linear combination of the n linear maps $\boldsymbol{\theta}^i$.

By (2.4), the dual basis $\boldsymbol{\theta}^i$ of E_n^* is associated with the basis $(\mathbf{e}_i)$ of E_n. Consequently, to a basis change $(\mathbf{e}_i) \to (\mathbf{e}_i')$ in E_n expressed by (1.15) corresponds a basis change $(\boldsymbol{\theta}^i) \to (\boldsymbol{\theta}'^i)$ in E_n^*. To determine this basis change, since it is

$$\boldsymbol{\theta}'^i(\mathbf{x}) = x'^i, \quad \boldsymbol{\theta}^i(\mathbf{x}) = x^i, \tag{2.10}$$

and (1.16) holds, we have that

$$\boldsymbol{\theta}'^i(\mathbf{x}) = (A^{-1})^i_j x^j = (A^{-1})^i_j \boldsymbol{\theta}^j(\mathbf{x}).$$

In view of the arbitrariness of the vector $\mathbf{x} \in E_n$, from the preceding relation we obtain the desired transformation formulae of the dual bases:

$$\boldsymbol{\theta}'^i = (A^{-1})^i_j \boldsymbol{\theta}^j, \quad \boldsymbol{\theta}^i = A^i_j \boldsymbol{\theta}'^j. \tag{2.11}$$

The transformation formulae of the components ω^i of the linear form $\boldsymbol{\omega}$ corresponding to the basis change (2.11) are obtained recalling that, since $\boldsymbol{\omega}$ is a vector of E_n^*, its components are transformed with the inverse matrix of the dual-basis change. Therefore, it is

$$\omega_i' = A_i^j \omega_j, \quad \omega_i = (A^{-1})_i^j \omega_j'. \tag{2.12}$$

Remark 2.2. Bearing in mind the foregoing results, we can state that the components of a covector relative to a dual basis are transformed according to the *covariance law* (i.e., as the bases of the vector space E_n). By contrast, the dual bases are transformed according to the *contravariance law* (i.e., as the components of a vector $\mathbf{x} \in E_n$).

Remark 2.3. Since the vector spaces E_n and E_n^* have the same dimension, it is possible to build an isomorphism between them. In fact, choosing a basis $(\mathbf{e}_i)$ of E_n and a basis $(\boldsymbol{\theta}^i)$ of E_n^*, an isomorphism between E_n and E_n^* is obtained by associating the covector $\boldsymbol{\omega} = \sum_i x^i \boldsymbol{\theta}^i$ with the vector $\mathbf{x} = x^i \mathbf{e}_i \in E_n$. However, owing to the different transformation character of the components of a vector and a covector, the preceding isomorphism *depends on the choice of the bases* $(\mathbf{e}_i)$ and $(\boldsymbol{\theta}^i)$. Later we will show that, when E_n is a Euclidean vector space, it is possible to define an isomorphism between E_n and E_n^* that does not depend on the aforementioned choice, i.e., it is *intrinsic*.

2.2 Biduality

We have already proved that E_n^* is itself a vector space. Consequently, it is possible to consider its dual vector space E_n^{**} containing all the linear maps $G : E_n^* \to \Re$. Moreover, to any basis $(\boldsymbol{\theta}^i) \in E_n^*$ we can associate the dual basis $(\mathbf{f}_i) \in E_n^{**}$ defined by the conditions [see (2.4)]

$$\mathbf{f}_i(\boldsymbol{\theta}^j) = \delta_i^j, \tag{2.13}$$

so that any $\mathbf{F} \in E_n^{**}$ admits a unique representation in this basis:

$$\mathbf{F} = F^i \mathbf{f}_i. \tag{2.14}$$

At this point we can consider the idea of generating "ad libitum" vector spaces by the duality definition. But this cannot happen since E_n and E_n^{**} are isomorphic, i.e., they can be identified. In fact, let us consider the linear map

$$\mathbf{x} \in E_n \to \mathbf{F}_{\mathbf{x}} \in E_n^{**}$$

such that

$$\mathbf{F}_{\mathbf{x}}(\boldsymbol{\omega}) = \boldsymbol{\omega}(\mathbf{x}), \quad \forall \boldsymbol{\omega} \in E_n^*. \tag{2.15}$$

To verify that (2.15) is an isomorphism, denote by $(\mathbf{e}_i)$ a basis of E_n, $(\boldsymbol{\theta}_i)$ its dual basis in E_n^*, and $\mathbf{f}_i$ the dual basis of $(\boldsymbol{\theta}_i)$ in E_n^{**}. In these bases, (2.15) assumes the form

$$F^i \omega_i = x^i \omega_i, \tag{2.16}$$

which, owing to the arbitrariness of $\boldsymbol{\omega}$, implies that

$$F^i = x^i, \quad i = 1, \ldots, n. \tag{2.17}$$

The correspondence (2.17) refers to the bases $(\mathbf{e}_i)$, $(\boldsymbol{\theta}_i)$, and $(\mathbf{f}_i)$. To prove that the isomorphism (2.17) does not depend on the basis, it is sufficient to note that the basis change (1.15) in E_n determines the basis change (2.11) in E_n^*. But $(\mathbf{f}_i)$ is the dual basis of $(\boldsymbol{\theta}_i)$, so that it is transformed according to the formulae

$$\mathbf{f}'_i = A_i^j \mathbf{f}_j.$$

Consequently, the components x^i of $\mathbf{x} \in E_n$ and the components F^i of $\mathbf{F} \in E_n^{**}$ are transformed in the same way under a basis change.

We conclude by remarking that the foregoing considerations allow us to look at a vector as a linear map on E_n^*. In other words, we can write (2.15) in the following way:

$$\mathbf{x}(\boldsymbol{\omega}) = \boldsymbol{\omega}(\mathbf{x}). \tag{2.18}$$

2.3 Covariant 2-Tensors

Definition 2.2. A bilinear map

$$\mathbf{T} : E_n \times E_n \to \Re$$

is called a ***covariant*** 2-***tensor*** or a $(0,2)$-***tensor***.

With the following standard definitions of addition of two covariant 2-tensors and multiplication of a real number a by a covariant 2-tensor

$$(\mathbf{T}_1 + \mathbf{T}_2)(\mathbf{x}, \mathbf{y}) = \mathbf{T}_1(\mathbf{x}, \mathbf{y}) + \mathbf{T}_2(\mathbf{x}, \mathbf{y}),$$
$$(a\mathbf{T})(\mathbf{x}, \mathbf{y}) = a\mathbf{T}(\mathbf{x}, \mathbf{y}),$$

the set $T_2(E_n)$ of all covariant 2-tensors on E_n becomes a vector space.

Definition 2.3. The ***tensor product*** $\boldsymbol{\omega} \otimes \boldsymbol{\sigma}$ of $\boldsymbol{\omega}, \boldsymbol{\sigma} \in E_n^*$ is a covariant 2-tensor such that

$$\boldsymbol{\omega} \otimes \boldsymbol{\sigma}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\omega}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in E_n. \tag{2.19}$$

Theorem 2.2. *Let* $(\mathbf{e}_i)$ *be a basis of* E_n *and let* $(\boldsymbol{\theta}^i)$ *be the dual basis in* E_n^*. *Then* $(\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j)$ *is a basis of* $T_2(E_n)$, *which is a* n^2*-dimensional vector space.*

Proof. Since $\mathbf{T} \in T_2(E_n)$ is bilinear, we have that

$$\mathbf{T}(\mathbf{x}, \mathbf{y}) = \mathbf{T}(x^i \mathbf{e}_i, y^j \mathbf{e}_j) = x^i y^j \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j).$$

Introducing the ***components*** of the covariant 2-tensor $\mathbf{T}$ in the basis $(\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j)$

$$T_{ij} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j), \tag{2.20}$$

the foregoing relation becomes

$$\mathbf{T}(\mathbf{x}, \mathbf{y}) = T_{ij} x^i y^j. \tag{2.21}$$

On the other hand, in view of (2.19) and (2.5), it also holds that

$$\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j (\mathbf{x}, \mathbf{y}) = x^i y^j, \tag{2.22}$$

and (2.21) assumes the form

$$\mathbf{T}(\mathbf{x}, \mathbf{y}) = T_{ij} \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j (\mathbf{x}, \mathbf{y}).$$

Since this identity holds for any $\mathbf{x}, \mathbf{y} \in E_n$, we conclude that the set $(\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j)$ generates the whole vector space $T_2(E_n)$:

$$\mathbf{T} = T_{ij} \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j. \tag{2.23}$$

Then, the covariant 2-tensors $(\boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j)$ form a basis of $T_2(E_n)$ if they are linearly independent. To prove this statement, it is sufficient to note that from the arbitrary linear combination

$$a_{ij} \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j = 0$$

we obtain

$$a_{ij} \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j (\mathbf{e}_h, \mathbf{e}_k) = a_{ij} \delta_h^i \delta_k^j = 0,$$

so that $a_{hk} = 0$ for any choice of the indices h and k. $\square$

The basis change (1.15) in E_n determines the basis change (2.11) in E_n^* and a basis change

$$\boldsymbol{\theta}'^i \otimes \boldsymbol{\theta}'^j = (A^{-1})_h^i (A^{-1})_k^j \boldsymbol{\theta}^h \otimes \boldsymbol{\theta}^k, \quad \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j = A_h^i A_k^j \boldsymbol{\theta}'^h \otimes \boldsymbol{\theta}'^k \tag{2.24}$$

in $T_2(E_n)$. On the other hand, it also holds that

$$\mathbf{T} = T_{ij} \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j = T'_{ij} \boldsymbol{\theta}'^i \otimes \boldsymbol{\theta}'^j, \tag{2.25}$$

and taking into account (2.24) we obtain the following transformation formulae of the components of a covariant 2-tensor $\mathbf{T} \in T_2(E_n)$ under a basis change (2.24):

$$T'_{ij} = A_i^h A_j^k T_{hk}. \tag{2.26}$$

Remark 2.4. If $\mathbb{T} = (T_{ij})$ is a matrix whose elements are the components of $\mathbf{T}$, and $\mathbb{A} = (A^i_j)$ is a matrix of the basis change (1.15), then the matrix form of (2.26) is

$$\mathbb{T}' = \mathbb{A}^T \mathbb{T} \mathbb{A}. \tag{2.27}$$

Definition 2.4. A ***contravariant*** 2-***tensor*** or a $(2, 0)$-***tensor*** is a bilinear map

$$\mathbf{T} = E_n^* \times E_n^* \to \Re. \tag{2.28}$$

It is evident that the set $T^2(E_n)$ of all contravariant 2-tensors becomes a vector space by the introduction of the standard operations of addition of two contravariant 2-tensors and the product of a 2-tensor by a real number.

Definition 2.5. The ***tensor product*** of two vectors $\mathbf{x}, \mathbf{y} \in E_n$ is the contravariant 2-tensor

$$\mathbf{x} \otimes \mathbf{y}(\boldsymbol{\omega}, \boldsymbol{\sigma}) = \boldsymbol{\omega}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{y}), \quad \forall \boldsymbol{\omega}, \boldsymbol{\sigma} \in E_n^*. \tag{2.29}$$

Theorem 2.3. *Let* $(\mathbf{e}_i)$ *be a basis of the vector space* E_n. *Then* $(\mathbf{e}_i \otimes \mathbf{e}_j)$ *is a basis of* $T^2(E_n)$, *which is an* n^2*-dimensional vector space.*

Proof. $\forall \boldsymbol{\omega}, \boldsymbol{\sigma} \in E_n^*$,

$$\mathbf{T}(\boldsymbol{\omega}, \boldsymbol{\sigma}) = \mathbf{T}(\omega_i \boldsymbol{\theta}^i, \sigma_j \boldsymbol{\theta}^j) = \omega_i \sigma_j \mathbf{T}(\boldsymbol{\theta}^i, \boldsymbol{\theta}^j).$$

By introducing the ***components*** of the contravariant 2-tensor $\mathbf{T}$ relative to the basis $(\mathbf{e}_i \otimes \mathbf{e}_j)$

$$T^{ij} = \mathbf{T}(\boldsymbol{\theta}^i, \boldsymbol{\theta}^j), \tag{2.30}$$

we can write

$$\mathbf{T}(\boldsymbol{\omega}, \boldsymbol{\sigma}) = T^{ij} \omega_i \sigma_j. \tag{2.31}$$

Since

$$\mathbf{e}_i \otimes \mathbf{e}_j(\boldsymbol{\omega}, \boldsymbol{\sigma}) = \omega_i \sigma_j, \tag{2.32}$$

and $\boldsymbol{\omega}, \boldsymbol{\sigma}$ in (2.31) are arbitrary, we obtain the result

$$\mathbf{T} = T^{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \tag{2.33}$$

which shows that $(\mathbf{e}_i \otimes \mathbf{e}_j)$ generates the whole vector space $T^2(E_n)$. Further, it is a basis of $T^2(E_n)$ since, in view of (2.32), any linear combination

$$a^{ij} \mathbf{e}_i \otimes \mathbf{e}_j = 0$$

implies that $a^{ij} = 0$ for all the indices $i, j = 1, \ldots, n$. $\square$

The basis change (1.15) in E_n determines the basis change

$$\mathbf{e}'_i \otimes \mathbf{e}'_j = A_i^h A_j^k \mathbf{e}_h \otimes \mathbf{e}_k \tag{2.34}$$

in $T^2(E_n)$. The contravariant 2-tensor $\mathbf{T}$ can be represented in both bases by

$$\mathbf{T} = T'^{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j = T^{ij}\mathbf{e}_h \otimes \mathbf{e}_k, \tag{2.35}$$

so that, taking into account (2.34), we derive the following transformation formulae for the components of $\mathbf{T}$:

$$T'^{ij} = (A^{-1})^i_h (A^{-1})^j_k T^{hk}. \tag{2.36}$$

Exercise 2.1. Verify that the matrix form of (2.36) is [see (2.27)]

$$\mathbb{T}' = (\mathbb{A}^{-1})\mathbb{T}(\mathbb{A}^{-1})^T. \tag{2.37}$$

Definition 2.6. A ***mixed*** 2-***tensor*** or a $(1,1)$-***tensor*** is a bilinear map

$$\mathbf{T} : E_n^* \times E_n \to \Re. \tag{2.38}$$

Once again, the set $T_1^1(E_n)$ of all mixed 2-tensors becomes a vector space by the introduction of the standard operations of addition of two mixed 2-tensors and the product of a mixed 2-tensor by a real number.

Definition 2.7. The ***tensor product*** of a vector $\mathbf{x}$ and a covector $\boldsymbol{\sigma} \in E_n^*$ is the mixed 2-tensor

$$\mathbf{x} \otimes \boldsymbol{\omega}(\boldsymbol{\sigma}, \mathbf{y}) = \mathbf{x}(\boldsymbol{\sigma})\boldsymbol{\omega}(\mathbf{y}) = \boldsymbol{\sigma}(\mathbf{x})\boldsymbol{\omega}(\mathbf{y}), \quad \forall \boldsymbol{\sigma}, \mathbf{y} \in E_n. \tag{2.39}$$

Theorem 2.4. *Let* $(\mathbf{e}_i)$ *be a basis of the vector space* E_n *and let* $(\boldsymbol{\theta}^i)$ *be the dual basis in* E_n^**. Then* $(\mathbf{e}_i \otimes \boldsymbol{\theta}^j)$ *is a basis of* $T_1^1(E_n)$*, which is an* n^2*-dimensional vector space.*

In view of this theorem we can write

$$\mathbf{T} = T^i_j \mathbf{e}_i \otimes \boldsymbol{\theta}^j, \tag{2.40}$$

where the components of the mixed 2-tensor are

$$T^i_j = \mathbf{T}(\boldsymbol{\theta}^i, \mathbf{e}_j). \tag{2.41}$$

Moreover, in the basis change $(\mathbf{e}_i \otimes \boldsymbol{\theta}^j) \to (\mathbf{e}'_i \otimes \boldsymbol{\theta}'^j)$ given by

$$\mathbf{e}_i \otimes \boldsymbol{\theta}^j = A_i^h (A^{-1})^j_k \mathbf{e}_h \otimes \boldsymbol{\theta}^k, \tag{2.42}$$

the components of $\mathbf{T}$ are transformed according to the formulae

$$T'^i_j = (A^{-1})^i_h A^k_j T^h_k. \tag{2.43}$$

Exercise 2.2. Verify that the matrix form of (2.43) is [see (2.27)]

$$\mathbb{T}' = (\mathbb{A}^{-1})\mathbb{T}\mathbb{A}. \tag{2.44}$$

2.4 (r, s)-Tensors

Definition 2.8. An (r, s)-tensor is a multilinear map

$$\mathbf{T} : E_n^{*r} \times E_n^s \to \Re. \tag{2.45}$$

It is quite clear how to transform the set $T_s^r(E_n)$ of all (r, s)-tensors (2.45) in a real vector space.

Definition 2.9. Let $\mathbf{T} \in T_s^r(E_n)$ be an (r, s)-tensor and let $\mathbf{L} \in T_q^p(E_n)$ be a (p, q)-tensor. Then the ***tensor product*** of these two tensors is the $(r + p, s + q)$-tensor $\mathbf{T} \otimes \mathbf{L} \in T_{s+q}^{r+p}(E_n)$ given by

$$\begin{aligned}
&\mathbf{T} \otimes \mathbf{L}(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{r+p}, \mathbf{x}_1, \ldots, \mathbf{x}_{s+q}) \\
&\quad = \mathbf{T}(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_r, \mathbf{x}_1, \ldots, \mathbf{x}_s)\, \mathbf{L}(\boldsymbol{\sigma}_{r+1}, \ldots, \boldsymbol{\sigma}_{r+p}, \mathbf{x}_{r+1}, \ldots, \mathbf{x}_{s+q}).
\end{aligned} \tag{2.46}$$

By imposing that the associative property holds, the tensor product can be extended to any number of factors. Then we introduce the following definition of the tensor product of vectors and covectors:

$$\begin{aligned}
&\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_r \otimes \boldsymbol{\omega}_1 \otimes \cdots \otimes \boldsymbol{\omega}_s(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_r, \mathbf{u}_1, \ldots, \mathbf{u}_s) = \\
&\quad \mathbf{x}_1(\boldsymbol{\sigma}_1) \cdots \mathbf{x}_r(\boldsymbol{\sigma}_r) \boldsymbol{\omega}_1(\mathbf{u}_1) \cdots \boldsymbol{\omega}_s(\mathbf{u}_s).
\end{aligned} \tag{2.47}$$

With the procedure revealed in the previous section it is possible to prove the following theorem.

Theorem 2.5. *The dimension of the vector space $T_s^r(E_n)$ of the $(r + s)$-tensors on E_n is n^{r+s}, and*

$$\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_r} \otimes \boldsymbol{\theta}^{j_1} \otimes \cdots \otimes \boldsymbol{\theta}^{j_s} \tag{2.48}$$

is a basis of it.

In (2.48), $(\mathbf{e}_i)$ is a basis of E_n and $(\boldsymbol{\theta}^j)$ the dual basis in E_n^*. Instead of Eqs. (2.40)–(2.43), we now obtain

$$\mathbf{T} = T_{j_1 \cdots j_s}^{i_1 \cdots i_r} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_r} \otimes \boldsymbol{\theta}^{j_1} \otimes \cdots \otimes \boldsymbol{\theta}^{j_s}, \tag{2.49}$$

$$T_{i^1,\ldots,i_n}^{j_1,\ldots,j_s} = \mathbf{T}(\boldsymbol{\theta}^{j_1}, \ldots, \boldsymbol{\theta}^{j_s}, \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}), \tag{2.50}$$

$$\mathbf{e}'_{i_1} \otimes \cdots \otimes \mathbf{e}'_{i_r} \otimes \boldsymbol{\theta}'^{j_1} \otimes \cdots \otimes \boldsymbol{\theta}'^{j_s}$$
$$= A^{h_1}_{i_1} \cdots A^{h_r}_{i_r} (A^{-1})^{j_1}_{k_1} \cdots (A^{-1})^{j_s}_{k_s} \mathbf{e}_{h_1} \otimes \cdots \otimes \mathbf{e}_{h_r} \otimes \boldsymbol{\theta}^{k_1} \otimes \cdots \otimes \boldsymbol{\theta}^{k_s}, \quad (2.51)$$

$$T'^{i_1 \cdots i_r}_{j_1 \cdots j_s} = (A^{-1})^{i_1}_{h_1} \cdots (A^{-1})^{i_r}_{h_r} A^{k_1}_{j_1} \cdots A^{k_s}_{j_s} T^{h_1 \cdots h_r}_{k_1 \cdots k_s}. \quad (2.52)$$

2.5 Tensor Algebra

In the previous sections we defined the addition of tensors belonging to the *same* tensor space. On the other hand, the tensor product of two tensors that might belong to different tensor spaces defines a new tensor that belongs to *another* tensor space. In conclusion, the tensor product is not an internal operation. However, it is possible to introduce a suitable set that, equipped with the aforementioned operations, becomes an algebra.

Let us consider the *infinite* direct sum (Sect. 1.4)

$$TE_n = \oplus_{r,s \in \mathbb{N}} T^r_s E_n \quad (2.53)$$

whose elements are *finite* sequences $\{a, \mathbf{x}, \boldsymbol{\omega}, \mathbf{T}, \mathbf{L}, \mathbf{K}, \ldots\}$, where $a \in \Re$, $\mathbf{x} \in E_n$, $\boldsymbol{\omega} \in E^*_n$, $\mathbf{T} \in T^2_0(E_n)$, $\mathbf{K} \in T^0_2(E_n)$, $\mathbf{L} \in T^1_1(E_n)$, etc. With the introduction of this set, the multiplication by a scalar, the addition, and the tensor product become internal operations and the set TE_n, equipped with them, is called a ***tensor algebra***.

2.6 Contraction and Contracted Multiplication

In the tensor algebra TE_n we can introduce two other internal operations: the contraction and the contracted product.

Theorem 2.6. *Denote by* $(\mathbf{e}_i)$ *a basis of the vector space* E_n *and by* $(\boldsymbol{\theta}^i)$ *the dual basis in* E^*_n*. Then, for any pair of integers* $1 \leq h, k \leq n$*, the linear map*

$$C_{h,k} : \mathbf{T} = T^{i_1 \cdots h \cdots i_r}_{j_1 \cdots k \cdots j_s} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_r} \otimes \boldsymbol{\theta}^{j_1} \otimes \cdots \otimes \boldsymbol{\theta}^{j_s} \in T^r_s(E_n)$$
$$\to C_{h,k}(\mathbf{T}) = T^{i_1 \cdots h \cdots i_r}_{j_1 \cdots h \cdots j_s} \mathbf{e}_{i_1} \otimes \cdots \mathbf{e}_{i_{h-1}} \otimes \mathbf{e}_{i_{h+1}} \otimes \cdots \otimes \mathbf{e}_{i_r} \otimes$$
$$\boldsymbol{\theta}^{j_1} \otimes \cdots \otimes \boldsymbol{\theta}^{i_{k-1}} \otimes \boldsymbol{\theta}^{i_{k+1}} \otimes \cdots \otimes \boldsymbol{\theta}^{j_s} \in T^{r-1}_{s-1}(E_n), \quad (2.54)$$

which is called a ***contraction****, to any tensor* $\mathbf{T} \in T^r_s(E_n)$ *associates a tensor* $C_{h,k}(\mathbf{T}) \in T^{r-1}_{s-1}(E_n)$*, which is obtained by equating the contravariant index* h *and the covariant index* k *and summing over these indices.*

Proof. To simplify the notations, we prove the theorem for a $(2,1)$-tensor $\mathbf{T} = T^{ij}_h \mathbf{e}_i \otimes \mathbf{e}_j \otimes \boldsymbol{\theta}^h$. For this tensor $C_{1,1}(\mathbf{T}) = T^{hj}_h \mathbf{e}_j$, and it will be sufficient to prove that, under a basis change, the quantities T_h^{hj} are transformed as vector components. Since in a basis change we have that

$$T_h^{\prime ij} = (A^{-1})_l^i (A^{-1})_m^j A_h^p T_p^{lm},$$

it also holds that

$$T_h^{\prime hj} = (A^{-1})_l^h (A^{-1})_m^j A_h^p T_p^{lm} = (A^{-1})_m^j T_h^{hm},$$

and the theorem is proved. □

The preceding theorem makes it possible to give the following definition.

Definition 2.10. A ***contracted multiplication*** is a map

$$(\mathbf{T},\mathbf{L}) \in T_s^r(E_n) \times T_q^p(E_n) \to C_{h,k}(\mathbf{T}\otimes\mathbf{L}) \in T_{s+q-1}^{r+p-1}(E_n). \tag{2.55}$$

For instance, if $\mathbf{T} = T_h^{ij}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \boldsymbol{\theta}^h$ and $\mathbf{L} = L_m^l \mathbf{e}_l \otimes \boldsymbol{\theta}^m$, then $C_{i,m}(\mathbf{T}\otimes\mathbf{L}) = T^{hj} L_h^l \mathbf{e}_j \otimes \mathbf{e}_l$.

Theorem 2.7. *The n^{r+s} quantities*

$$T_{j_1\cdots j_s}^{i_1\cdots i_r} \tag{2.56}$$

are the components of an $(r+s)$-tensor $\mathbf{T}$ if and only if any contracted multiplication of these quantities by the components of a (p,q)-tensor $\mathbf{L}$, $p \le s$, $q \le r$ generates an $(r-q, s-p)$-tensor.

Proof. The definition of contracted multiplication implies that the condition is necessary. For the sake of simplicity, we prove that the condition is sufficient considering the quantities T_h^{ij} and a $(0,1)$-tensor $\mathbf{L}$. In other words, we suppose that the quantities $T_h^{ij} L_i$ are transformed as the components of a $(1,1)$-tensor. Then we have that

$$T_h^{\prime ij} L_i' = (A^{-1})_l^j A_h^m T_m^{kl} L_k.$$

Since $L_k = (A^{-1})_k^r L_r'$, the theorem is proved. □

Remark 2.5. Let $\mathbf{T} : \mathbf{x} \in E_n \to \mathbf{y} \in E_n$ be a linear map and denote by $(\mathbf{e}_i)$ a basis of E_n and by (T_j^i) the matrix of $\mathbf{T}$ relative to the basis $(\mathbf{e}_i)$. In terms of components, the map $\mathbf{T}$ becomes

$$y^i = T_j^i x^j, \tag{2.57}$$

where y^i are the components of the vector $\mathbf{y} = \mathbf{T}(\mathbf{x})$ in the basis $(\mathbf{e}_i)$. Owing to the previous theorem, the quantities (T_j^i) are the components of a $(1,1)$-tensor relative to the basis $(\mathbf{e}_i \otimes \boldsymbol{\theta}^j)$. It is easy to verify that any $(1,1)$-tensor $\mathbf{T}$ determines a linear map $\mathbf{T} : E_n \to E_n$.

2.7 Exercises

1. Determine all the linear maps that can be associated with an (r, s)-tensor, where $r + s \leq 3$.
2. Let $(\mathbf{e}_i)$ be a basis of a vector space E_n, and let $(\boldsymbol{\theta}^i)$ be the dual basis. Verify that a tensor that has the components (δ^i_j) in the basis $(\mathbf{e}_i \otimes \boldsymbol{\theta}^j)$ has the same components in any other basis.
3. Let $(\mathbf{e}_i)$ be a basis of a two-dimensional vector space E_2, and denote by $(\boldsymbol{\theta}^i)$ the dual basis. Determine the value of the covariant 2-tensor $\boldsymbol{\theta}^1 \otimes \boldsymbol{\theta}^2 - \boldsymbol{\theta}^2 \otimes \boldsymbol{\theta}^1$ when it is applied to the pairs of vectors $(\mathbf{e}_1, \mathbf{e}_2)$, $(\mathbf{e}_2, \mathbf{e}_1)$, and $(\mathbf{x}, \mathbf{y})$ and recognize geometric meaning of each result.
4. The components of a $(1, 1)$-tensor $\mathbf{T}$ of the vector space E_3 relative to a basis $(\mathbf{e}_i \otimes \theta^j)$ are given by the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Determine the vector corresponding to $\mathbf{x} = (1, 0, 1)$ by the linear endomorphism determined by $\mathbf{T}$.

5. In the basis $(\mathbf{e}_i)$ of the vector space E_3, two vectors $\mathbf{x}$ and $\mathbf{y}$ have the components $(1, 0, 1)$ and $(2, 1, 0)$, respectively. Determine the components of $\mathbf{x} \otimes \mathbf{y}$ relative to the basis $(\mathbf{e}'_i \otimes \mathbf{e}'_j)$, where

$$\begin{aligned} \mathbf{e}'_1 &= \mathbf{e}_1 + \mathbf{e}_3, \\ \mathbf{e}'_2 &= 2\mathbf{e}_1 - \mathbf{e}_2, \\ \mathbf{e}'_3 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3. \end{aligned}$$

6. Given the $(0, 2)$-tensor $T_{ij} \boldsymbol{\theta}^i \otimes \boldsymbol{\theta}^j$ of $T_2(E_2)$, where

$$(T_{ij}) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

determine if there exists a new basis in which its components become

$$(T'_{ij}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

7. For which $(1, 1)$-tensor (T^i_j) of $T^1_1(E_3)$ does the linear map F defined by the matrix (T^i_j) satisfy the condition $F(\mathbf{x}) = a\mathbf{x}$, $\forall \mathbf{x} \in E_3$ and $\forall a \in \Re$?
8. Given the $(1, 1)$-tensor $T^i_j \mathbf{e}_i \otimes \theta^j$ of $T_2(E_2)$, verify that $T^1_1 + T^2_2$ and $\det(T^i_j)$ are invariant with respect to a change of basis.

9. Prove that if the components of a $(0, 2)$-tensor $\mathbf{T}$ satisfy either of the conditions

$$T_{ij} = T_{ji}, \quad T_{ij} = -T_{ji}$$

in a given basis, then they satisfy the same conditions in any other basis.

10. Given $(0, 2)$-tensors that in the basis $(\mathbf{e}_i)$, $i = 1, 2, 3$, have the components

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 1 \end{pmatrix},$$

$$\mathbf{T}_2 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix},$$

determine the covector $\boldsymbol{\omega}_{\mathbf{u}}$ depending on $\mathbf{u}$ such that

$$\boldsymbol{\omega}_{\mathbf{u}}(\mathbf{v}) = \mathbf{T}_1(\mathbf{u}, \mathbf{v}),$$

$$\boldsymbol{\omega}_{\mathbf{u}}(\mathbf{v}) = \mathbf{T}_2(\mathbf{u}, \mathbf{v})$$

$\forall \mathbf{v}$. Further, find the vectors $\mathbf{u}$ such that

$$\mathbf{T}_1(\mathbf{u}, \mathbf{v}) = 0, \quad \mathbf{T}_2(\mathbf{u}, \mathbf{v}) = 0, \quad \forall \mathbf{v}.$$

http://www.springer.com/978-0-8176-8351-1

Classical Mechanics with Mathematica®
Romano, A.
2012, XIV, 506 p. 127 illus., Hardcover
ISBN: 978-0-8176-8351-1
A product of Birkhäuser Basel