Mathematics Learning Centre

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Sigma notation

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1 Sigma Notation

1.1 Understanding Sigma Notation

The symbol Σ (capital sigma) is often used as shorthand notation to indicate the sum of a number of similar terms. Sigma notation is used extensively in statistics.

For example, suppose we weigh five children. We will denote their weights by x_1, x_2, x_3, x_4 and x_5.

The sum of their weights $x_1 + x_2 + x_3 + x_4 + x_5$ is written more compactly as $\sum_{j=1}^{5} x_j$.

The symbol Σ means 'add up'. Underneath Σ we see $j = 1$ and on top of it 5. This means that j is replaced by whole numbers starting at the bottom number, 1, until the top number,5, is reached.

Thus

$$\sum_{j=2}^{5} x_j = x_2 + x_3 + x_4 + x_5,$$

and

$$\sum_{j=2}^{4} x_j = x_2 + x_3 + x_4.$$

So the notation $\sum_{j=1}^{n} x_j$ tells us:

a. to add the scores x_j,

b. where to start: x_1,

c. where to stop: x_n (where n is some number).

Now take the weights of the children to be $x_1 = 10$kg, $x_2 = 12$kg, $x_3 = 14$kg, $x_4 = 8$kg and $x_5 = 11$kg. Then the total weight (in kilograms) is

$$\begin{aligned} \sum_{i=1}^{5} x_i &= x_1 + x_2 + x_3 + x_4 + x_5 \\ &= 10 + 12 + 14 + 8 + 11 \\ &= 55. \end{aligned}$$

Notice that we have used i instead of j in the formula above. The j is what we call a dummy variable - any letter can be used, ie,

$$\sum_{j=1}^{n} x_j = \sum_{i=1}^{n} x_i.$$

Now let us find $\sum_{i=1}^{4} 2x_i$ where $x_1 = 2$, $x_2 = 3$, $x_3 = -2$ and $x_4 = 1$.

Again, starting with $i = 1$ we replace the expression $2x_i$ with its value and add up the terms until $i = 4$ is reached. So,

$$\begin{aligned}\sum_{i=1}^{4} 2x_i &= 2x_1 + 2x_2 + 2x_3 + 2x_4 \\ &= 2(2) + 2(3) + 2(-2) + 2(1) \\ &= 4 + 6 - 4 + 2 \\ &= 8.\end{aligned}$$

Similarly, let us find $\sum_{k=1}^{3}(x_k - 4)$ where $x_1 = 7$, $x_2 = 4$, $x_3 = 1$.

Here,

$$\begin{aligned}\sum_{k=1}^{3}(x_k - 4) &= (x_1 - 4) + (x_2 - 4) + (x_3 - 4) \\ &= (7 - 4) + (4 - 4) + (1 - 4) \\ &= 3 + 0 + (-3) \\ &= 0.\end{aligned}$$

Notice that this is different from $\sum_{k=1}^{3} x_k - 4$ where $x_1 = 7$, $x_2 = 4$, $x_3 = 1$.

In this case, we have,

$$\begin{aligned}\sum_{k=1}^{3} x_k - 4 &= x_1 + x_2 + x_3 - 4 \\ &= 7 + 4 + 1 - 4 \\ &= 8.\end{aligned}$$

We use brackets to indicate what should be included in the sum. In the previous example, there were no brackets, so the '4' was not included in the sum.

Example: Write out in full: $\sum_{k=1}^{5} x^k$.

Solution: $x^1 + x^2 + x^3 + x^4 + x^5$.

We also use sigma notation in the following way:

$$\sum_{j=1}^{4} j^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30.$$

This is the same principle: replace j in the expression (this time j^2) by whole numbers starting with 1 and ending with 4 , and add.

1.1.1 Exercises

1. Evaluate $\sum_{i=1}^{4} x_i$ where $x_1 = 5$, $x_2 = 2$, $x_3 = 3$, $x_4 = 8$.

2. Evaluate $\sum_{k=1}^{n} 5x_k$ where $x_1 = 10$, $x_2 = 14$, $x_3 = -2$, and $n = 3$.

3. Find $\mu = \frac{1}{5}\sum_{j=1}^{5} x_j$ where the $x_1 = 10$kg, $x_2 = 12$kg, $x_3 = 14$kg, $x_4 = 8$kg and $x_5 = 11$kg are the weights of 5 children. (μ is the mean weight of the children.)

4. Find the value of $\sum_{i=1}^{3}(x_i - \mu)^2$ where $x_1 = 105$, $x_2 = 100$, $x_3 = 95$, and $\mu = 100$.

1.2 Rules of summation

We will prove three rules of summation. These rules will allow us to evaluate formulae containing sigma notation more easily and allow us to derive equivalent formulae.

Rule 1: If c is a constant, then

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i.$$

To see why Rule 1 is true, let's start with the left hand side of this equation,

$$\begin{aligned}\sum_{i=1}^{n} cx_i &= cx_1 + cx_2 + cx_3 + \cdots + cx_n \\ &= c(x_1 + x_2 + x_3 + \cdots + x_n) \\ &= c\sum_{i=1}^{n} x_i\end{aligned}$$

as required.

Suppose that $\sum_{i=1}^{5} x_i = 55$ as in a previous example. Then $\sum_{i=1}^{5} 3x_i = 3\sum_{i=1}^{5} x_i = 3 \times 55 = 165$.

Rule 2: If c is a constant, then

$$\sum_{i=1}^{n} c = nc.$$

This rule looks a bit strange as there is no 'x_i'. The left hand side of this formula means 'sum c, n times'. That is,

$$\begin{aligned}\sum_{i=1}^{n} c &= \overbrace{c + c + \cdots + c}^{n} \\ &= n \times c \\ &= nc.\end{aligned}$$

For example, $\sum_{i=1}^{5} 2 = 5 \times 2 = 10$.

Rule 3:

$$\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i.$$

To prove this rule, let's start with the left hand side,

$$\begin{aligned}
\sum_{i=1}^{n}(x_i + y_i) &= (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) + \cdots + (x_n + y_n) \\
&= (x_1 + x_2 + x_3 + \cdots + x_n) + (y_1 + y_2 + y_3 + \cdots + y_n) \\
&= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i.
\end{aligned}$$

For example, if $\sum_{i=1}^{7} x_i = 21$ and $\sum_{i=1}^{7} y_i = 35$ then $\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{7} x_i + \sum_{i=1}^{7} y_i = 21 + 35 = 56$.

1.2.1 Exercises

In the following exercises you may assume that $\sum_{i=1}^{5} x_i = 37$, $\sum_{i=1}^{5} y_i = 12$, $\sum_{i=1}^{5} x_i^2 = 303$, $\sum_{i=1}^{5} y_i^2 = 50$ and $\sum_{i=1}^{5} x_i y_i = 105$.

Evaluate the following expressions:

1. $\sum_{i=1}^{5} 2y_i$

2. $\sum_{i=1}^{5} x_i - 1$

3. $\sum_{i=1}^{5}(x_i - 1)$

4. $(\sum_{i=1}^{5} x_i)^2$

5. $\sum_{i=1}^{5}(2x_i + y_i)$

6. $\sum_{i=1}^{5}(2x_i + 3y_i)$

7. $\sum_{i=1}^{5}(2x_i - 5y_i + 3)$

8. $\sum_{i=1}^{5}(x_i - 2y_i)^2$

1.3 Using Sigma Notation in Statistics

Here are some examples of how sigma notation is used in statistics:

The formula for a mean of a group of N scores, is

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

A measure of how spread out the scores are, called the variance, has the following formula:

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2.$$

For example, the number of customers having lunch at a certain restaurant on 7 weekdays were $x_1 = 92$, $x_2 = 84$, $x_3 = 70$, $x_4 = 76$, $x_5 = 66$, $x_6 = 80$, $x_7 = 71$.

The mean is

$$\begin{aligned}
\mu &= \frac{1}{N}\sum_{i=1}^{N} x_i \\
&= \frac{1}{7}(92 + 84 + 70 + 76 + 66 + 80 + 71) \\
&= \frac{539}{7} \\
&= 77.
\end{aligned}$$

Note: There are 7 scores so $N = 7$.

The variance is

$$\begin{aligned}
\sigma^2 &= \frac{1}{7}\sum_{i=1}^{7}(x_i - 77)^2 \\
&= \frac{1}{7}[(15)^2 + (7)^2 + (-7)^2 + (-1)^2 + (-11)^2 + (3)^2 + (-6)^2] \\
&= \frac{1}{7}[225 + 49 + 49 + 1 + 121 + 9 + 36] \\
&= \frac{1}{7}[490] \\
&= 70.
\end{aligned}$$

An alternative formula for variance is

$$\sigma^2 = \frac{1}{N}(\sum_{i=1}^{N} x_i^2 - N\mu^2)$$

For the above example we get:

$$\begin{aligned}
\sigma^2 &= \frac{1}{N}[x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 - N\mu^2] \\
&= \frac{1}{7}[92^2 + 84^2 + 70^2 + 76^2 + 66^2 + 80^2 + 71^2 - 7(77)^2] \\
&= \frac{1}{7}[8464 + 7056 + 4900 + 5776 + 4356 + 6400 + 5041 - 7(5929)] \\
&= \frac{1}{7}[490] \\
&= 70
\end{aligned}$$

as before.

We can use the rules to show that two formulae for variance are equivalent, since

$$\sum_{i=1}^{N}(x_i - \mu)^2 = \sum_{i=1}^{N} x_i^2 - N\mu^2.$$

$$\begin{aligned}
\sum_{i=1}^{N}(x_i - \mu)^2 &= \sum_{i=1}^{N}(x_i^2 - 2\mu x_i + \mu^2) \\
&= \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} 2\mu x_i + \sum_{i=1}^{N} \mu^2 \\
&= \sum_{i=1}^{N} x_i^2 - 2\mu \sum_{i=1}^{N} x_i + N\mu^2 && \text{since } \mu \text{ is a constant} \\
&= \sum_{i=1}^{N} x_i^2 - 2\mu \times N\mu + N\mu^2 && \text{since } \textstyle\sum_{i=1}^{N} x_i = N\mu \\
&= \sum_{i=1}^{N} x_i^2 - N\mu^2
\end{aligned}$$

1.3.1 Exercises

1. Find the variance of the weights of the five children (in Exercise 1.1.1 number 3), using each of the above formulae for σ^2.

2. During a 5 week period, a salespersons weekly income (in dollars) was $x_1 = 400$, $x_2 = 250$, $x_3 = 175$, $x_4 = 300$, $x_5 = 375$.

Calculate $\mu = \frac{1}{5}\sum_{i=1}^{5} x_i$ and $\sigma^2 = \frac{1}{5}(\sum_{i=1}^{5} x_i^2 - 5\mu^2)$.

3. An insurance company is concerned about the length of time required to process claims. The length of time, measured in days, taken to process 7 claims produced the data $x_1 = 23$, $x_2 = 20$, $x_3 = 22$, $x_4 = 25$, $x_5 = 24$, $x_6 = 23$, $x_7 = 21$.

Evaluate the mean μ and variance σ^2 for these data.

1.4 Answers to Exercises

Answers to Exercise 1.1.1

1. 18 **2.** 110 **3.** 11kg **4.** 50

Answers to Exercise 1.2.1

1. $\sum_{i=1}^{5} 2y_i = 2\sum_{i=1}^{5} y_i = 24$

2. $\sum_{i=1}^{5} x_i - 1 = 37 - 1 = 36$

3. $\sum_{i=1}^{5}(x_i - 1) = \sum_{i=1}^{5} x_i - \sum_{i=1}^{5} 1 = 37 - 5(1) = 32$

4. $(\sum_{i=1}^{5} x_i)^2 = (37)^2 = 1369$ Note this is different from $\sum_{i=1}^{5} x_i^2 = 303.$

5. $\sum_{i=1}^{5}(2x_i + y_i) = 2\sum_{i=1}^{5} x_i + \sum_{i=1}^{5} y_i = 2(37) + 12 = 86$

6. $\sum_{i=1}^{5}(2x_i + 3y_i) = 2\sum_{i=1}^{5} x_i + 3\sum_{i=1}^{5} y_i = 2(37) + 3(12) = 110$

7. $\sum_{i=1}^{5}(2x_i - 5y_i + 3) = 2\sum_{i=1}^{5} x_i - 5\sum_{i=1}^{5} y_i + \sum_{i=1}^{5} 3 = 2(37) - 5(12) + 5(3) = 29$

8. $\sum_{i=1}^{5}(x_i - 2y_i)^2 = \sum_{i=1}^{5}(x_i^2 - 4x_iy_i + 4y_i^2) = \sum_{i=1}^{5} x_i^2 - 4\sum_{i=1}^{5} x_iy_i + 4\sum_{i=1}^{5} y_i^2 = 303 - 420 + 200 = 83$

Answers to Exercise 1.3.1

1. $\sigma^2 = 4$

2. $\mu = 300$, $\sigma^2 = 6750$

3. $\mu = 22.57$ to two decimal places, $\sigma^2 = 2.53$
taking the mean as 22.57 and using the formula $\sigma^2 = \frac{1}{N}\Sigma_{i=1}^{N}(x_i - \mu)^2$.
If, however the formula $\sigma^2 = \frac{1}{N}[\Sigma_{i=1}^{N} x_i^2 - N\mu^2]$ is used, then the answer $\sigma^2 = 2.60$ is obtained. This discrepancy is due to round off error and can be avoided by using $\mu = 22.571429$ in the above formula.