Applications of Mathematics in Philosophy: Four Case Studies

Hannes Leitgeb

University of Bristol

February 2007

Introduction

Many philosophers still deny that mathematical methods can play a substantive role in philosophy.

Introduction

Many philosophers still deny that mathematical methods can play a substantive role in philosophy.

Paradigmatic case: Kant in the *Critique of Pure Reason*.

According to Kant's *Transcendental Doctrine of Method*, philosophy cannot be developed according to the *definitions-axioms-proofs* scheme.

This is because mathematics deals with pure intuitions, whereas philosophy deals with pure concepts (or so Kant says).

Introduction

Many philosophers still deny that mathematical methods can play a substantive role in philosophy.

Paradigmatic case: Kant in the *Critique of Pure Reason*.

According to Kant's *Transcendental Doctrine of Method*, philosophy cannot be developed according to the *definitions-axioms-proofs* scheme.

This is because mathematics deals with pure intuitions, whereas philosophy deals with pure concepts (or so Kant says). However,

- in the meantime, mathematics has developed into a theory of abstract structures in general
- the progress in logic shows that the "space of concepts" has itself an intricate mathematical structure
- if Platonists such as Gödel are right, there is a rational intuition of concepts!?

Claims:

When philosophical theories get sufficiently complex, they are in need of mathematics.

Claims:

When philosophical theories get sufficiently complex, they are in need of mathematics.

The necessary bridge between philosophy and mathematics is often supplied by logic broadly understood.

Claims:

When philosophical theories get sufficiently complex, they are in need of mathematics.

The necessary bridge between philosophy and mathematics is often supplied by logic broadly understood.

Plan of the talk: *Examples*

- Similarity, Properties, and Hypergraphs
- Nonmonotonic Logic and Dynamical Systems
- Belief Revision for Conditionals and Arrow's Theorem
- Semantic Paradoxes and Probability

Claims:

When philosophical theories get sufficiently complex, they are in need of mathematics.

The necessary bridge between philosophy and mathematics is often supplied by logic broadly understood.

Plan of the talk: *Examples*

- Similarity, Properties, and Hypergraphs
- Nonmonotonic Logic and Dynamical Systems
- Belief Revision for Conditionals and Arrow's Theorem
- Semantic Paradoxes and Probability
- (If time permits: Meaning Similarity and Compositionality)

Similarity, Properties, and Hypergraphs

G. Frege, A.N. Whitehead & B. Russell: definition by logical abstraction; abstracting equivalence classes from equivalence relations

Similarity, Properties, and Hypergraphs

G. Frege, A.N. Whitehead & B. Russell: definition by logical abstraction; abstracting equivalence classes from equivalence relations

B. Russell, R. Carnap: generalized abstraction;
abstracting "similarity classes" from similarity relations

Similarity, Properties, and Hypergraphs

G. Frege, A.N. Whitehead & B. Russell: definition by logical abstraction;
abstracting equivalence classes from equivalence relations

B. Russell, R. Carnap: generalized abstraction;
abstracting "similarity classes" from similarity relations
$\hookrightarrow$ abstraction for the *empirical* domain (where transitivity often fails)

Similarity, Properties, and Hypergraphs

G. Frege, A.N. Whitehead & B. Russell: definition by logical abstraction; abstracting equivalence classes from equivalence relations

B. Russell, R. Carnap: generalized abstraction;
abstracting "similarity classes" from similarity relations
$\hookrightarrow$ abstraction for the *empirical* domain (where transitivity often fails)

• $\langle S, \sim \rangle$ is a similarity structure on S :iff
$\sim \subseteq S \times S$ is reflexive and symmetric.

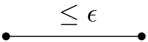
Similarity, Properties, and Hypergraphs

G. Frege, A.N. Whitehead & B. Russell: definition by logical abstraction; abstracting equivalence classes from equivalence relations

B. Russell, R. Carnap: generalized abstraction;
abstracting "similarity classes" from similarity relations
$\hookrightarrow$ abstraction for the *empirical* domain (where transitivity often fails)

• $\langle S, \sim \rangle$ is a similarity structure on S :iff
$\sim \subseteq S \times S$ is reflexive and symmetric.

E.g.: *metric* similarity (for colours,...)



Similarity, Properties, and Hypergraphs

G. Frege, A.N. Whitehead & B. Russell: definition by logical abstraction; abstracting equivalence classes from equivalence relations

B. Russell, R. Carnap: generalized abstraction;
abstracting "similarity classes" from similarity relations
$\hookrightarrow$ abstraction for the *empirical* domain (where transitivity often fails)

• $\langle S, \sim \rangle$ is a similarity structure on S :iff
$\sim \subseteq S \times S$ is reflexive and symmetric.

E.g.: *metric* similarity (for colours,...)

• $\langle S, P \rangle$ is a property structure on S :iff
$P \subseteq \wp(S)$, $\varnothing \notin P$, and
for every $x \in S$ there is an $X \in P$, s.t. $x \in X$.

Similarity, Properties, and Hypergraphs

G. Frege, A.N. Whitehead & B. Russell: definition by logical abstraction; abstracting equivalence classes from equivalence relations

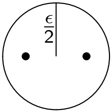
B. Russell, R. Carnap: generalized abstraction;
abstracting "similarity classes" from similarity relations
$\hookrightarrow$ abstraction for the *empirical* domain (where transitivity often fails)

• $\langle S, \sim \rangle$ is a similarity structure on S :iff
$\sim \subseteq S \times S$ is reflexive and symmetric.

E.g.: *metric* similarity (for colours,...)

• $\langle S, P \rangle$ is a property structure on S :iff
$P \subseteq \wp(S)$, $\varnothing \notin P$, and
for every $x \in S$ there is an $X \in P$, s.t. $x \in X$.

E.g.: (colour) spheres



Every property structure on S determines a similarity structure on S:
(cf. Leibniz: "Peter is similar to Paul" reduces to "Peter is A now and Paul is A now")

Every property structure on S determines a similarity structure on S:
(cf. Leibniz: "Peter is similar to Paul" reduces to "Peter is A now and Paul is A now")

Definition (Determined similarity structure)

$\langle S, \sim_P \rangle$ is determined by $\langle S, P \rangle$:iff
for all $x, y \in S$: $x \sim_P y$ iff there is an $X \in P$, such that $x, y \in X$.

Every property structure on S determines a similarity structure on S: (cf. Leibniz: "Peter is similar to Paul" reduces to "Peter is A now and Paul is A now")

Definition (Determined similarity structure)

$\langle S, \sim_P \rangle$ is determined by $\langle S, P \rangle$:iff
for all $x, y \in S$: $x \sim_P y$ iff there is an $X \in P$, such that $x, y \in X$.

Every similarity structure on S determines a property structure on S:

$X \subseteq S$ is a *clique* of $\langle S, \sim \rangle$:iff for all $x, y \in X$: $x \sim y$.

Every property structure on S determines a similarity structure on S:
(cf. Leibniz: "Peter is similar to Paul" reduces to "Peter is A now and Paul is A now")

Definition (Determined similarity structure)

$\langle S, \sim_P \rangle$ is determined by $\langle S, P \rangle$:iff
for all $x, y \in S$: $x \sim_P y$ iff there is an $X \in P$, such that $x, y \in X$.

Every similarity structure on S determines a property structure on S:

$X \subseteq S$ is a *clique* of $\langle S, \sim \rangle$:iff for all $x, y \in X$: $x \sim y$.

Definition (Determined property structure)

$\langle S, P^{\sim} \rangle$ is determined by $\langle S, \sim \rangle$:iff
$P^{\sim} = \{X \subseteq S \,|\, X \text{ is a maximal clique of } \langle S, \sim \rangle\}$.

Every property structure on S determines a similarity structure on S: (cf. Leibniz: "Peter is similar to Paul" reduces to "Peter is A now and Paul is A now")

Definition (Determined similarity structure)

$\langle S, \sim_P \rangle$ is determined by $\langle S, P \rangle$:iff
for all $x, y \in S$: $x \sim_P y$ iff there is an $X \in P$, such that $x, y \in X$.

Every similarity structure on S determines a property structure on S:

$X \subseteq S$ is a *clique* of $\langle S, \sim \rangle$:iff for all $x, y \in X$: $x \sim y$.

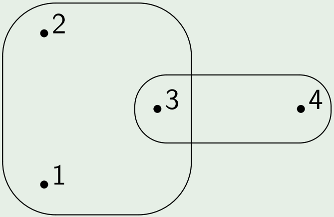
Definition (Determined property structure)

$\langle S, P^{\sim} \rangle$ is determined by $\langle S, \sim \rangle$:iff
$P^{\sim} = \{X \subseteq S \,|\, X \text{ is a maximal clique of } \langle S, \sim \rangle\}$.

Special case: equivalence classes $\rightleftharpoons$ equivalence relations ✓

Example

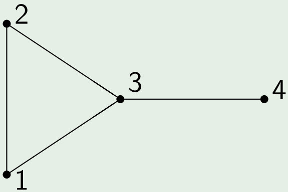
(Faithful, full) $S_1 = \{1, 2, 3, 4\}$, $P_1 = \{\{1, 2, 3\}, \{3, 4\}\}$:



Given: $\langle S_1, P_1 \rangle$

Example

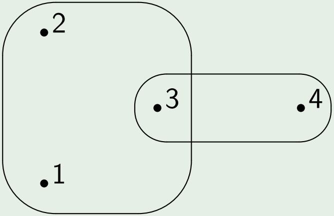
(Faithful, full) $S_1 = \{1, 2, 3, 4\}$, $P_1 = \{\{1, 2, 3\}, \{3, 4\}\}$:



Determined: $\langle S_1, \sim_{P_1} \rangle$

Example

(Faithful, full) $S_1 = \{1, 2, 3, 4\}$, $P_1 = \{\{1, 2, 3\}, \{3, 4\}\}$:

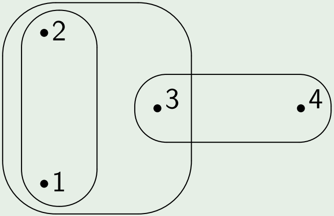


Determined: $\langle S_1, P^{\sim_{P_1}} \rangle = \langle S_1, P_1 \rangle$ ✓

But this method of abstraction does not work in *all* cases...
(as N. Goodman and others observed)

Example

(Faithful, not full) $S_2 = \{1, 2, 3, 4\}$, $P_2 = \{\{1, 2\}, \{1, 2, 3\}, \{3, 4\}\}$:

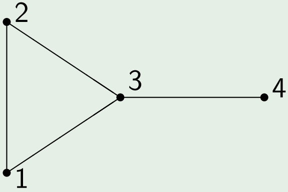


Given: $\langle S_2, P_2 \rangle$

But this method of abstraction does not work in *all* cases...
(as N. Goodman and others observed)

Example

(Faithful, not full) $S_2 = \{1, 2, 3, 4\}$, $P_2 = \{\{1, 2\}, \{1, 2, 3\}, \{3, 4\}\}$:

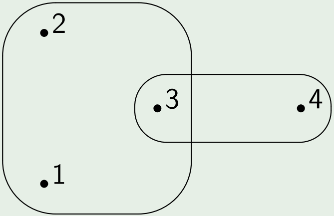


Determined: $\langle S_2, \sim_{P_2} \rangle$

But this method of abstraction does not work in *all* cases...
(as N. Goodman and others observed)

Example

(Faithful, not full) $S_2 = \{1, 2, 3, 4\}$, $P_2 = \{\{1, 2\}, \{1, 2, 3\}, \{3, 4\}\}$:

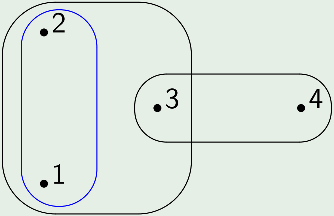


Determined: $\langle S_2, P^{\sim_{P_2}} \rangle$

But this method of abstraction does not work in *all* cases...
(as N. Goodman and others observed)

Example

(Faithful, not full) $S_2 = \{1, 2, 3, 4\}$, $P_2 = \{\{1, 2\}, \{1, 2, 3\}, \{3, 4\}\}$:

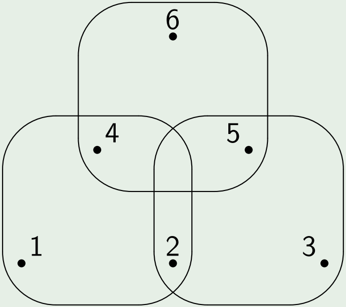


$\langle S_2, P^{\sim_{P_2}} \rangle \neq \langle S_2, P_2 \rangle$

Example

(Full, not faithful)

$S_3 = \{1, 2, 3, 4, 5, 6\}$, $P_3 = \{\{1, 2, 4\}, \{2, 3, 5\}, \{4, 5, 6\}\}$:

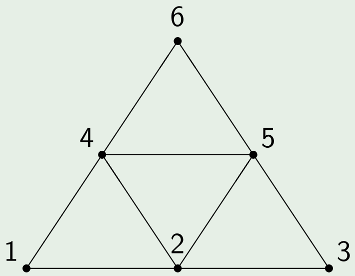


Given: $\langle S_3, P_3 \rangle$

Example

(Full, not faithful)

$S_3 = \{1,2,3,4,5,6\}$, $P_3 = \{\{1,2,4\},\{2,3,5\},\{4,5,6\}\}$:

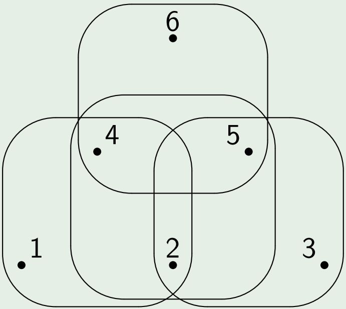


Determined: $\langle S_3, \sim_{P_3} \rangle$

Example

(Full, not faithful)

$S_3 = \{1, 2, 3, 4, 5, 6\}$, $P_3 = \{\{1, 2, 4\}, \{2, 3, 5\}, \{4, 5, 6\}\}$:

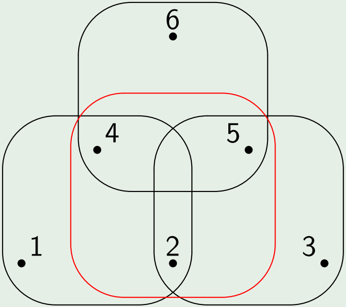


Determined: $\langle S_3, P^{\sim_{P_3}} \rangle$

Example

(Full, not faithful)

$S_3 = \{1,2,3,4,5,6\}$, $P_3 = \{\{1,2,4\},\{2,3,5\},\{4,5,6\}\}$:

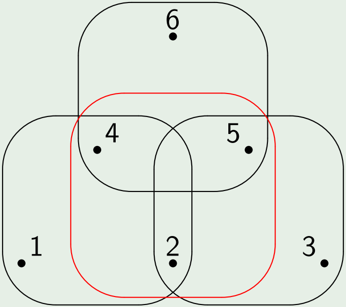


$\langle S_3, P^{\sim_{P_3}} \rangle \neq \langle S_3, P_3 \rangle$

Example

(Full, not faithful)

$S_3 = \{1, 2, 3, 4, 5, 6\}$, $P_3 = \{\{1, 2, 4\}, \{2, 3, 5\}, \{4, 5, 6\}\}$:



$\langle S_3, P^{\sim_{P_3}} \rangle \neq \langle S_3, P_3 \rangle$

QUESTION: If similarity is determined by properties, under which conditions can the latter be reconstructed from the former?

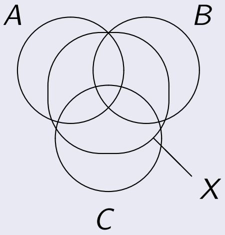
$\hookrightarrow$ *Hypergraph theory!*

$\hookrightarrow$ *Hypergraph theory!*

Theorem (Gilmore; cf. Berge 1989)

Let $\langle S, \sim_P \rangle$ be determined by $\langle S, P \rangle$, with S finite:

1. *$\langle S, \sim_P \rangle$ is faithful with respect to $\langle S, P \rangle$ iff*
   (a) for all $A, B, C \in P$ there is an $X \in P$, such that
   $$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq X$$

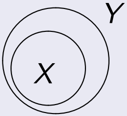


$\hookrightarrow$ *Hypergraph theory!*

Theorem (Gilmore; cf. Berge 1989)

Let $\langle S, \sim_P \rangle$ be determined by $\langle S, P \rangle$, with S finite:

1. *$\langle S, \sim_P \rangle$ is faithful with respect to $\langle S, P \rangle$ iff*
   (a) for all $A, B, C \in P$ there is an $X \in P$, such that
   $$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq X$$
2. *If $\langle S, \sim_P \rangle$ is full with respect to $\langle S, P \rangle$, then*
   (b) there are no $X, Y \in P$, such that $X \subsetneq Y$.



$\hookrightarrow$ *Hypergraph theory!*

Theorem (Gilmore; cf. Berge 1989)

Let $\langle S, \sim_P \rangle$ be determined by $\langle S, P \rangle$, with S finite:

1. *$\langle S, \sim_P \rangle$ is faithful with respect to $\langle S, P \rangle$ iff*
   (a) for all $A, B, C \in P$ there is an $X \in P$, such that
   $$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq X$$
2. *If $\langle S, \sim_P \rangle$ is full with respect to $\langle S, P \rangle$, then*
   (b) there are no $X, Y \in P$, such that $X \subsetneqq Y$.
3. *$\langle S, \sim_P \rangle$ is faithful & full with respect to $\langle S, P \rangle$ iff (a)+(b).*

$\hookrightarrow$ *Hypergraph theory!*

Theorem (Gilmore; cf. Berge 1989)

Let $\langle S, \sim_P \rangle$ be determined by $\langle S, P \rangle$, with S finite:

1. *$\langle S, \sim_P \rangle$ is faithful with respect to $\langle S, P \rangle$ iff*
   (a) for all $A, B, C \in P$ there is an $X \in P$, such that
   $$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq X$$
2. *If $\langle S, \sim_P \rangle$ is full with respect to $\langle S, P \rangle$, then*
   (b) there are no $X, Y \in P$, such that $X \subsetneq Y$.
3. *$\langle S, \sim_P \rangle$ is faithful & full with respect to $\langle S, P \rangle$ iff (a)+(b).*

Proof: By induction over $|S|$.

$\hookrightarrow$ *Hypergraph theory!*

Theorem (Gilmore; cf. Berge 1989)

Let $\langle S, \sim_P \rangle$ be determined by $\langle S, P \rangle$, with S finite:

1. *$\langle S, \sim_P \rangle$ is faithful with respect to $\langle S, P \rangle$ iff*
   (a) for all $A, B, C \in P$ there is an $X \in P$, such that
   $$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq X$$
2. *If $\langle S, \sim_P \rangle$ is full with respect to $\langle S, P \rangle$, then*
   (b) there are no $X, Y \in P$, such that $X \subsetneqq Y$.
3. *$\langle S, \sim_P \rangle$ is faithful & full with respect to $\langle S, P \rangle$ iff (a)+(b).*

Proof: By induction over $|S|$.

Russell, *Our Knowledge of the External World*: ✓ (Helly's Theorem)
Carnap, *The Logical Structure of the World*: $\not\checkmark$

$\hookrightarrow$ *Hypergraph theory!*

Theorem (Gilmore; cf. Berge 1989)

Let $\langle S, \sim_P \rangle$ be determined by $\langle S, P \rangle$, with S finite:

1. *$\langle S, \sim_P \rangle$ is faithful with respect to $\langle S, P \rangle$ iff*
   (a) for all $A, B, C \in P$ there is an $X \in P$, such that
   $$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq X$$
2. *If $\langle S, \sim_P \rangle$ is full with respect to $\langle S, P \rangle$, then*
   (b) there are no $X, Y \in P$, such that $X \subsetneqq Y$.
3. *$\langle S, \sim_P \rangle$ is faithful & full with respect to $\langle S, P \rangle$ iff (a)+(b).*

Proof: By induction over $|S|$.

Russell, *Our Knowledge of the External World*: ✓ (Helly's Theorem)
Carnap, *The Logical Structure of the World*: $\not\checkmark$
If determination starts with similarity, then "reconstruction" works.

$\hookrightarrow$ *Hypergraph theory!*

Theorem (Gilmore; cf. Berge 1989)

Let $\langle S, \sim_P \rangle$ be determined by $\langle S, P \rangle$, with S finite:

1. *$\langle S, \sim_P \rangle$ is faithful with respect to $\langle S, P \rangle$ iff (a) for all $A, B, C \in P$ there is an $X \in P$, such that $(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq X$*
2. *If $\langle S, \sim_P \rangle$ is full with respect to $\langle S, P \rangle$, then (b) there are no $X, Y \in P$, such that $X \subsetneq Y$.*
3. *$\langle S, \sim_P \rangle$ is faithful & full with respect to $\langle S, P \rangle$ iff (a)+(b).*

Proof: By induction over $|S|$.

Russell, *Our Knowledge of the External World*: ✓ (Helly's Theorem)
Carnap, *The Logical Structure of the World*: $\not\checkmark$
If determination starts with similarity, then "reconstruction" works.
If $n = |S|$ and $|P| > \binom{n}{\lfloor n/2 \rfloor}$ then $\langle S, \sim_P \rangle$ is not full. (Sperner's Theorem)

$\hookrightarrow$ *Hypergraph theory!*

Theorem (Gilmore; cf. Berge 1989)

Let $\langle S, \sim_P \rangle$ be determined by $\langle S, P \rangle$, with S finite:

1. *$\langle S, \sim_P \rangle$ is faithful with respect to $\langle S, P \rangle$ iff*
   (a) for all $A, B, C \in P$ there is an $X \in P$, such that
   $$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq X$$
2. *If $\langle S, \sim_P \rangle$ is full with respect to $\langle S, P \rangle$, then*
   (b) there are no $X, Y \in P$, such that $X \subsetneqq Y$.
3. *$\langle S, \sim_P \rangle$ is faithful & full with respect to $\langle S, P \rangle$ iff (a)+(b).*

Proof: By induction over $|S|$.

Russell, *Our Knowledge of the External World*: ✓ (Helly's Theorem)
Carnap, *The Logical Structure of the World*: $\not\checkmark$
If determination starts with similarity, then "reconstruction" works.
If $n = |S|$ and $|P| > \binom{n}{\lfloor n/2 \rfloor}$ then $\langle S, \sim_P \rangle$ is not full. (Sperner's Theorem)
(See Leitgeb 2007, JPL. In work: Monograph on a new *Logischer Aufbau*.)

Nonmonotonic Logic and Dynamical Systems

Here is a problem in the philosophy of mind:

Nonmonotonic Logic and Dynamical Systems

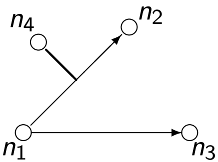
Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.

Nonmonotonic Logic and Dynamical Systems

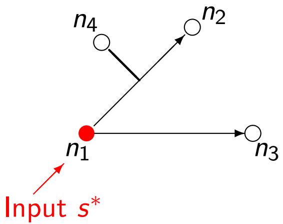
Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.



Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.

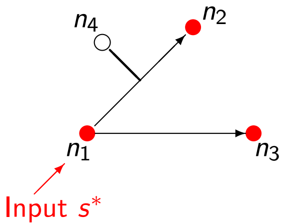


$F^1_{s^*}(s_0)$

Nonmonotonic Logic and Dynamical Systems

Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.

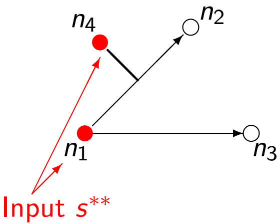


$F^2_{s^*}(s_0)$

Nonmonotonic Logic and Dynamical Systems

Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.

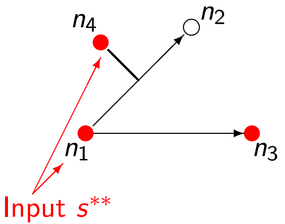


$F^1_{s^{**}}(s_0)$

Nonmonotonic Logic and Dynamical Systems

Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.



$F^2_{s^{**}}(s_0)$

Nonmonotonic Logic and Dynamical Systems

Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.
- On the other hand, these brains seem to have beliefs, draw inferences, and so forth, the contents of which can be expressed by *sentences*:

Nonmonotonic Logic and Dynamical Systems

Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.
- On the other hand, these brains seem to have beliefs, draw inferences, and so forth, the contents of which can be expressed by *sentences*:

  x believes that $\neg\varphi$
  x infers that $\varphi \vee \psi$ from φ

Nonmonotonic Logic and Dynamical Systems

Here is a problem in the philosophy of mind:

- On the one hand, (human, animal, robot) brains seem to be physical systems that can be described in terms of differential or difference equations, i.e., as *dynamical systems*.

- On the other hand, these brains seem to have beliefs, draw inferences, and so forth, the contents of which can be expressed by *sentences*:

  x believes that $\neg\varphi$
  x infers that $\varphi \vee \psi$ from φ

We take these to be distinct but *compatible* perspectives on the same phenomenon.

So we have to associate *system states* with *propositions*:
system states carry information that can be expressed linguistically!

What might a theory of such *interpreted dynamical systems* look like?

Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

1. S is a non-empty set (the set of states)

Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

1. S is a non-empty set (the set of states)
2. $ns : S \rightarrow S$ (the internal next-state function)

Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

1. S is a non-empty set (the set of states)
2. $ns : S \to S$ (the internal next-state function)
3. $\leqslant \subseteq S \times S$ is a partial order (the information ordering) on S,

Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

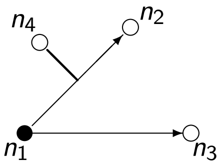
1. S is a non-empty set (the set of states)
2. $ns : S \rightarrow S$ (the internal next-state function)
3. $\leqslant \subseteq S \times S$ is a partial order (the information ordering) on S, s.t. for all s, $s' \in S$ there is a supremum $sup(s, s') \in S$ with respect to $\leqslant$.

Definition

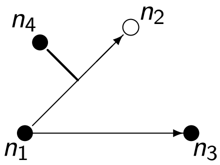
A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

1. S is a non-empty set (the set of states)
2. $ns : S \to S$ (the internal next-state function)
3. $\leqslant \subseteq S \times S$ is a partial order (the information ordering) on S, s.t. for all s, $s' \in S$ there is a supremum $sup(s, s') \in S$ with respect to $\leqslant$.

E.g.: $S = \{s | s : N \to \{0, 1\}\}$ (with $N = \{n_1, n_2, n_3, n_4\}$)



$\leqslant$



Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

1. S is a non-empty set (the set of states)
2. $ns : S \to S$ (the internal next-state function)
3. $\leqslant \subseteq S \times S$ is a partial order (the information ordering) on S, s.t. for all s, $s' \in S$ there is a supremum $sup(s, s') \in S$ with respect to $\leqslant$.

The internal dynamics of such systems is given by the iteration $ns (= ns^1), ns^2, ns^3, \ldots$ of the mapping ns.

Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

1. S is a non-empty set (the set of states)
2. $ns : S \to S$ (the internal next-state function)
3. $\leqslant \subseteq S \times S$ is a partial order (the information ordering) on S, s.t. for all s, $s' \in S$ there is a supremum $sup(s, s') \in S$ with respect to $\leqslant$.

The internal dynamics of such systems is given by the iteration $ns(= ns^1), ns^2, ns^3, \ldots$ of the mapping ns.

Now we add an input which is regarded to activate a fixed state $s^* \in S$:

Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

1. S is a non-empty set (the set of states)
2. $ns : S \to S$ (the internal next-state function)
3. $\leqslant \subseteq S \times S$ is a partial order (the information ordering) on S, s.t. for all s, $s' \in S$ there is a supremum $sup(s, s') \in S$ with respect to $\leqslant$.

The internal dynamics of such systems is given by the iteration $ns(= ns^1), ns^2, ns^3, \ldots$ of the mapping ns.

Now we add an input which is regarded to activate a fixed state $s^* \in S$: the "next" state of the system is given by the superposition of s^* with the next internal state $ns(s)$, i.e., we actually iterate

$$F_{s^*}(s) := sup(s^*, ns(s))$$

Definition

A triple $\mathcal{S} = \langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system :iff

1. S is a non-empty set (the set of states)
2. $ns : S \to S$ (the internal next-state function)
3. $\leqslant \subseteq S \times S$ is a partial order (the information ordering) on S, s.t. for all s, $s' \in S$ there is a supremum $sup(s, s') \in S$ with respect to $\leqslant$.

The internal dynamics of such systems is given by the iteration $ns(= ns^1), ns^2, ns^3, \ldots$ of the mapping ns.

Now we add an input which is regarded to activate a fixed state $s^* \in S$: the "next" state of the system is given by the superposition of s^* with the next internal state $ns(s)$, i.e., we actually iterate

$$F_{s^*}(s) := sup(s^*, ns(s))$$

Fixed points s_{stab} of F_{s^*} are the "answers" which the system gives to s^*.

Finally, we assign formulas to states of these systems.

Finally, we assign formulas to states of these systems.

Let $\mathcal{L}$ be a propositional language:

Finally, we assign formulas to states of these systems.

Let $\mathcal{L}$ be a propositional language:

Definition

A quadruple $\mathcal{S}_{\mathfrak{I}} = \langle S, ns, \leqslant, \mathfrak{I} \rangle$ is an interpreted ordered system :iff

Finally, we assign formulas to states of these systems.

Let $\mathcal{L}$ be a propositional language:

Definition

A quadruple $\mathcal{S}_{\mathfrak{I}} = \langle S, ns, \leqslant, \mathfrak{I} \rangle$ is an interpreted ordered system :iff

1. $\langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system

Finally, we assign formulas to states of these systems.

Let $\mathcal{L}$ be a propositional language:

Definition

A quadruple $\mathcal{S}_{\mathfrak{I}} = \langle S, ns, \leqslant, \mathfrak{I} \rangle$ is an interpreted ordered system :iff

1. $\langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system
2. $\mathfrak{I} : \mathcal{L} \to S$ (the interpretation mapping) has "nice" properties, such as

Finally, we assign formulas to states of these systems.

Let $\mathcal{L}$ be a propositional language:

Definition

A quadruple $\mathcal{S}_{\mathfrak{I}} = \langle S, ns, \leqslant, \mathfrak{I} \rangle$ is an interpreted ordered system :iff

1. $\langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system
2. $\mathfrak{I} : \mathcal{L} \to S$ (the interpretation mapping) has "nice" properties, such as
    - for all $\varphi, \psi \in \mathcal{L}$: $\mathfrak{I}(\varphi \wedge \psi) = sup(\mathfrak{I}(\varphi), \mathfrak{I}(\psi))$

Finally, we assign formulas to states of these systems.

Let $\mathcal{L}$ be a propositional language:

Definition

A quadruple $\mathcal{S}_{\mathfrak{I}} = \langle S, ns, \leqslant, \mathfrak{I} \rangle$ is an interpreted ordered system :iff

1. $\langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system
2. $\mathfrak{I} : \mathcal{L} \to S$ (the interpretation mapping) has "nice" properties, such as
    - for all $\varphi, \psi \in \mathcal{L}$: $\mathfrak{I}(\varphi \wedge \psi) = sup(\mathfrak{I}(\varphi), \mathfrak{I}(\psi))$
    - for every $\varphi \in \mathcal{L}$: there is a unique $\mathfrak{I}(\varphi)$-stable state

Finally, we assign formulas to states of these systems.

Let $\mathcal{L}$ be a propositional language:

Definition

A quadruple $\mathcal{S}_{\mathfrak{I}} = \langle S, ns, \leqslant, \mathfrak{I} \rangle$ is an interpreted ordered system :iff

1. $\langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system
2. $\mathfrak{I} : \mathcal{L} \to S$ (the interpretation mapping) has "nice" properties, such as
    - for all $\varphi, \psi \in \mathcal{L}$: $\mathfrak{I}(\varphi \wedge \psi) = sup(\mathfrak{I}(\varphi), \mathfrak{I}(\psi))$
    - for every $\varphi \in \mathcal{L}$: there is a unique $\mathfrak{I}(\varphi)$-stable state

      $\vdots$

Definition

1. $\mathcal{S}_{\mathfrak{I}} \vDash \varphi \Rightarrow \psi$:iff
   if s_{stab} is the unique $\mathfrak{I}(\varphi)$-stable state, then $\mathfrak{I}(\psi) \leqslant s_{stab}$

Finally, we assign formulas to states of these systems.

Let $\mathcal{L}$ be a propositional language:

Definition

A quadruple $\mathcal{S}_{\mathfrak{I}} = \langle S, ns, \leqslant, \mathfrak{I} \rangle$ is an interpreted ordered system :iff

1. $\langle S, ns, \leqslant \rangle$ is an ordered discrete dynamical system
2. $\mathfrak{I} : \mathcal{L} \rightarrow S$ (the interpretation mapping) has "nice" properties, such as
    - for all $\varphi, \psi \in \mathcal{L}$: $\mathfrak{I}(\varphi \wedge \psi) = sup(\mathfrak{I}(\varphi), \mathfrak{I}(\psi))$
    - for every $\varphi \in \mathcal{L}$: there is a unique $\mathfrak{I}(\varphi)$-stable state

      $\vdots$

Definition

1. $\mathcal{S}_{\mathfrak{I}} \vDash \varphi \Rightarrow \psi$:iff
   if s_{stab} is the unique $\mathfrak{I}(\varphi)$-stable state, then $\mathfrak{I}(\psi) \leqslant s_{stab}$
2. $\mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}}) = \{\varphi \Rightarrow \psi \,|\, \mathcal{S}_{\mathfrak{I}} \vDash \varphi \Rightarrow \psi\}$
   (the conditional theory corresponding to $\mathcal{S}_{\mathfrak{I}}$).

Theorem (Representation)

- *For all $\mathcal{S}_{\mathfrak{I}}$: $\mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$ is closed under the rules of the system C(umulativity) of nonmonotonic logic (cf. KLM 1990).*
- *Furthermore, for all consistent $\mathcal{TH}$ that are closed under these rules there is an interpreted system $\mathcal{S}_{\mathfrak{I}}$, such that $\mathcal{TH} = \mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$.*

Theorem (Representation)

- *For all $\mathcal{S}_{\mathfrak{I}}$: $\mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$ is closed under the rules of the system C(umulativity) of nonmonotonic logic (cf. KLM 1990).*
- *Furthermore, for all consistent $\mathcal{TH}$ that are closed under these rules there is an interpreted system $\mathcal{S}_{\mathfrak{I}}$, such that $\mathcal{TH} = \mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$.*

Proof: (i) induction over formulas; (ii) construction of systems.

Theorem (Representation)

- *For all $\mathcal{S}_{\mathfrak{J}}$: $\mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{J}})$ is closed under the rules of the system C(umulativity) of nonmonotonic logic (cf. KLM 1990).*
- *Furthermore, for all consistent $\mathcal{TH}$ that are closed under these rules there is an interpreted system $\mathcal{S}_{\mathfrak{J}}$, such that $\mathcal{TH} = \mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{J}})$.*

Proof: (i) induction over formulas; (ii) construction of systems.

There is a corresponding representation theorem for *hierarchical* interpreted systems and the nonmonotonic system C+L(oop).

Theorem (Representation)

- *For all $\mathcal{S}_{\mathfrak{I}}$: $\mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$ is closed under the rules of the system C(umulativity) of nonmonotonic logic (cf. KLM 1990).*
- *Furthermore, for all consistent $\mathcal{TH}$ that are closed under these rules there is an interpreted system $\mathcal{S}_{\mathfrak{I}}$, such that $\mathcal{TH} = \mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$.*

Proof: (i) induction over formulas; (ii) construction of systems.

There is a corresponding representation theorem for *hierarchical* interpreted systems and the nonmonotonic system C+L(oop).

$$\frac{\varphi_0 \Rightarrow \varphi_1,\ \varphi_1 \Rightarrow \varphi_2, \ldots, \varphi_{j-1} \Rightarrow \varphi_j, \varphi_j \Rightarrow \varphi_0}{\varphi_0 \Rightarrow \varphi_j}\ \text{(Loop)}$$

Theorem (Representation)

- *For all $\mathcal{S}_{\mathfrak{I}}$: $\mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$ is closed under the rules of the system C(umulativity) of nonmonotonic logic (cf. KLM 1990).*
- *Furthermore, for all consistent $\mathcal{TH}$ that are closed under these rules there is an interpreted system $\mathcal{S}_{\mathfrak{I}}$, such that $\mathcal{TH} = \mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$.*

Proof: (i) induction over formulas; (ii) construction of systems.

There is a corresponding representation theorem for *hierarchical* interpreted systems and the nonmonotonic system C+L(oop).

Application: Layered neural networks are particularly nice examples of hierarchical systems. Distributed representation can be computational.

Theorem (Representation)

- *For all $\mathcal{S}_{\mathfrak{I}}$: $\mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$ is closed under the rules of the system C(umulativity) of nonmonotonic logic (cf. KLM 1990).*
- *Furthermore, for all consistent $\mathcal{TH}$ that are closed under these rules there is an interpreted system $\mathcal{S}_{\mathfrak{I}}$, such that $\mathcal{TH} = \mathcal{TH}_{\Rightarrow}(\mathcal{S}_{\mathfrak{I}})$.*

Proof: (i) induction over formulas; (ii) construction of systems.

There is a corresponding representation theorem for *hierarchical* interpreted systems and the nonmonotonic system C+L(oop).

Application: Layered neural networks are particularly nice examples of hierarchical systems. Distributed representation can be computational.

(Leitgeb 2001, *Artificial Intelligence*.
Leitgeb 2004, *Inference in the Low Level*, Kluwer-Springer.
Leitgeb 2005, *Synthese* & French edition of *Scientific American*.)

Belief Revision for Conditionals and Arrow's Theorem

Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

Belief Revision for Conditionals and Arrow's Theorem

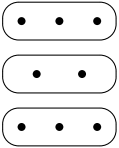
Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,

Belief Revision for Conditionals and Arrow's Theorem

Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

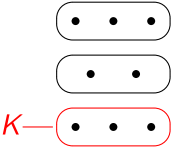
Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,



Belief Revision for Conditionals and Arrow's Theorem

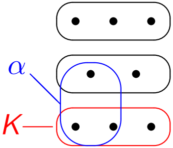
Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,



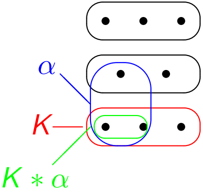
Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,



Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

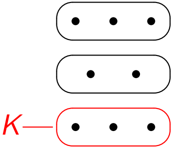
Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,



Belief Revision for Conditionals and Arrow's Theorem

Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

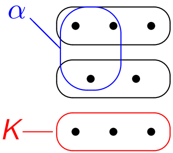
Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,



Belief Revision for Conditionals and Arrow's Theorem

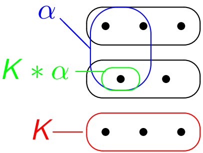
Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,



Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

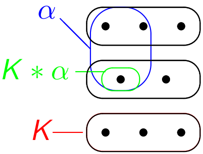
Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,



Belief Revision for Conditionals and Arrow's Theorem

Belief revision theorists (cf. AGM 1985) are interested in how to revise one's belief set K rationally in the face of new evidence α, i.e.:
For which β is it the case that $\beta \in K * \alpha$?

Grove 1988: representation theorem for all operators $*$ which satisfy the AGM axioms – any $*$ can be put in one-to-one correspondence to a ranked model of truth value assignments for a propositional language $\mathcal{L}$, i.e.,



Now let us add a new conditional sign $\Rightarrow$ to our language $\mathcal{L}$ and postulate as a new axiom:

Ramsey test $\alpha \Rightarrow \beta \in K$ iff $\beta \in K * \alpha$

Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"

Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"

•

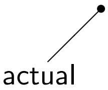
Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"



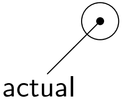
Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"



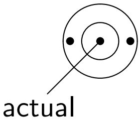
Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"



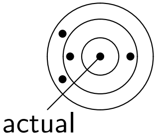
Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"



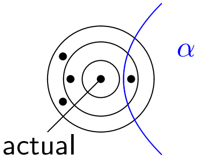
Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"



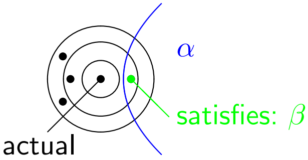
Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"



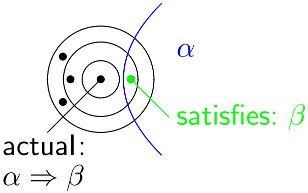
Theorem (Gärdenfors)

The axioms of $*$ are inconsistent with the Ramsey test for conditionals (given weak non-triviality assumptions).

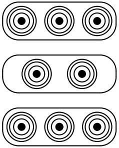
Here is a new take on this result:

The logic of subjunctive conditionals was proved valid (by D. Lewis) with respect to a semantics that is similar to the semantics for $*$:

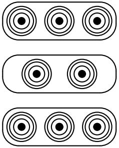
"If $\underbrace{\text{you left Max Bell Room 159 now}}_{\alpha}$, then $\underbrace{\text{I would be sad}}_{\beta}$"



So for a language with Lewis-type conditionals, ranked models for $*$ rather look like...

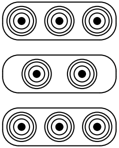


So for a language with Lewis-type conditionals, ranked models for $*$ rather look like...



...and Gärdenfors' Impossibility Theorem expresses the fact that the "small" ranking in the least layer cannot correspond to the "big" ranking as suggested by the Ramsey test.

So for a language with Lewis-type conditionals, ranked models for $*$ rather look like...



...and Gärdenfors' Impossibility Theorem expresses the fact that the "small" ranking in the least layer cannot correspond to the "big" ranking as suggested by the Ramsey test.

But this sounds familiar:

Arrow's theorem (1951): There is no function that extends any given set of individual rankings $\leqslant_i$ of alternatives (for fixed individuals $i \in \{1, \ldots, n\}$) to a social ranking $\leqslant$, such that certain axioms are satisfied.

E.g., Pareto:

(P) If $x \leqslant_i y$ for all i, then $x \leqslant y$.

Once can show: the assumptions in Arrow's theorem can be formulated on the basis of $*$ and $\Rightarrow$, where Arrow's individuals and alternatives are "coded" by $\mathcal{L}$-worlds.

Once can show: the assumptions in Arrow's theorem can be formulated on the basis of $*$ and $\Rightarrow$, where Arrow's individuals and alternatives are "coded" by $\mathcal{L}$-worlds.

It follows that there is a belief revision counterpart of Arrow's theorem:

Theorem

K∗1–K∗8, IIA, P, ND are jointly inconsistent (given background assumptions on belief sets and on the number of possible worlds involved).

Proof: Translation of Arrow's proof.

Once can show: the assumptions in Arrow's theorem can be formulated on the basis of $*$ and $\Rightarrow$, where Arrow's individuals and alternatives are "coded" by $\mathcal{L}$-worlds.

It follows that there is a belief revision counterpart of Arrow's theorem:

Theorem

K∗1–K∗8, IIA, P, ND are jointly inconsistent (given background assumptions on belief sets and on the number of possible worlds involved).

Proof: Translation of Arrow's proof.

The Pareto condition P is just the left-to-right direction of the Ramsey test for conditionals:

(P) For all $\mathcal{L}_{\Rightarrow}$-consistent belief sets K, for all $\alpha, \beta \in \mathcal{L}$:
if $\alpha \Rightarrow \beta \in K$, then $\beta \in K * \alpha$.

Once can show: the assumptions in Arrow's theorem can be formulated on the basis of $*$ and $\Rightarrow$, where Arrow's individuals and alternatives are "coded" by $\mathcal{L}$-worlds.

It follows that there is a belief revision counterpart of Arrow's theorem:

Theorem

$K*1$–$K*8$, IIA, P, ND are jointly inconsistent (given background assumptions on belief sets and on the number of possible worlds involved).

Proof: Translation of Arrow's proof.

The Pareto condition P is just the left-to-right direction of the Ramsey test for conditionals:

(P) For all $\mathcal{L}_{\Rightarrow}$-consistent belief sets K, for all $\alpha, \beta \in \mathcal{L}$:
if $\alpha \Rightarrow \beta \in K$, then $\beta \in K * \alpha$.

(Leitgeb 2005, unpublished manuscript.
Leitgeb & Segerberg 2007, *Synthese KRA*.
Leitgeb 2007, "Beliefs in Conditionals vs. Conditional Beliefs", *Topoi*.)

Semantic Paradoxes and Probability

In view of Tarski's result, we know that we cannot have a type-free truth predicate and at the same time *accept* all T-biconditionals.

Semantic Paradoxes and Probability

In view of Tarski's result, we know that we cannot have a type-free truth predicate and at the same time *accept* all T-biconditionals.

Indeed, there is no subjective probability measure P that assigns to each formula of the form

$$Tr(`\alpha') \leftrightarrow \alpha$$

probability 1 (with $\alpha \in \mathcal{L}$ for $\mathcal{L}$ semantically closed).

Semantic Paradoxes and Probability

In view of Tarski's result, we know that we cannot have a type-free truth predicate and at the same time *accept* all T-biconditionals.

Indeed, there is no subjective probability measure P that assigns to each formula of the form

$$Tr(`\alpha') \leftrightarrow \alpha$$

probability 1 (with $\alpha \in \mathcal{L}$ for $\mathcal{L}$ semantically closed).

New approach: Instead of believing that $Tr(`\alpha')$ and $\alpha$ are *equivalent*, we might rather assign the same *degree of belief* to $Tr(`\alpha')$ and α!

Semantic Paradoxes and Probability

In view of Tarski's result, we know that we cannot have a type-free truth predicate and at the same time *accept* all T-biconditionals.

Indeed, there is no subjective probability measure P that assigns to each formula of the form

$$Tr(`\alpha') \leftrightarrow \alpha$$

probability 1 (with $\alpha \in \mathcal{L}$ for $\mathcal{L}$ semantically closed).

New approach: Instead of believing that $Tr(`\alpha')$ and $\alpha$ are *equivalent*, we might rather assign the same *degree of belief* to $Tr(`\alpha')$ and α!

Formalized: let $\mathcal{L}$ be the first-order language of arithmetic extended by Tr.

QUESTION: Is there a function $P : \mathcal{L} \rightarrow [0, 1]$, such that

- P satisfies the analogues of standard probability axioms.
- P satisfies:

$$P(Tr(`\alpha')) = P(\alpha)$$

- P assigns 1 to all commutation axioms for Tr.

Well, almost!

Well, almost!

Theorem

1. *There is no σ-additive measure P that satisfies all the above.*

Well, almost!

Theorem

1. *There is no σ-additive measure P that satisfies all the above.*
2. *There is a finitely additive measure P that satisfies all the conditions above except for σ-additivity.*

Well, almost!

Theorem

1. *There is no σ-additive measure P that satisfies all the above.*
2. *There is a finitely additive measure P that satisfies all the conditions above except for σ-additivity.*

Proof: 1. McGee's fixed-point formula. 2. Hahn-Banach.

Well, almost!

Theorem

1. *There is no σ-additive measure P that satisfies all the above.*
2. *There is a finitely additive measure P that satisfies all the conditions above except for σ-additivity.*

Proof: 1. McGee's fixed-point formula. 2. Hahn-Banach.

What about a "Liar" sentence λ with $P(\lambda \leftrightarrow \neg Tr(`\lambda'))=1$?

$$\begin{aligned} P(\lambda) &= P(\neg Tr(`\lambda')) \\ &= 1 - P(Tr(`\lambda')) \\ &= 1 - P(\lambda) \end{aligned}$$

Hence, $P(\lambda) = P(\neg\lambda) = \frac{1}{2}$.

Well, almost!

Theorem

1. *There is no σ-additive measure P that satisfies all the above.*
2. *There is a finitely additive measure P that satisfies all the conditions above except for σ-additivity.*

Proof: 1. McGee's fixed-point formula. 2. Hahn-Banach.

What about a "Liar" sentence λ with $P(\lambda \leftrightarrow \neg Tr(`\lambda')) = 1$?

$$\begin{aligned} P(\lambda) &= P(\neg Tr(`\lambda')) \\ &= 1 - P(Tr(`\lambda')) \\ &= 1 - P(\lambda) \end{aligned}$$

Hence, $P(\lambda) = P(\neg\lambda) = \frac{1}{2}$.

(Leitgeb 2005, unpublished manuscript.)

Similar methods can be used to investigate *type-free probability*.

Conclusions

We found:

- Hypergraph theory can be used to determine under which conditions properties can be reconstructed from similarity.
- Dynamical systems theory can be used to justify systems of nonmonotonic logic. Both together throw new light on the symbolic computationalism vs. connectionism debate.
- Social choice theory can be used to improve our understanding of limitative results on belief revision with conditionals.
- Functional analysis can be used to support a doxastic account of type-free truth.

Even in philosophy, *calculemus*!

Meaning Similarity and Compositionality

Goodman (1972): Linguistic expressions are hardly ever synonymous but much more often merely *similar* in meaning.

Meaning Similarity and Compositionality

Goodman (1972): Linguistic expressions are hardly ever synonymous but much more often merely *similar* in meaning.

Churchland vs. Fodor&Lepore on Meaning Similarity vs. Compositionality

Meaning Similarity and Compositionality

Goodman (1972): Linguistic expressions are hardly ever synonymous but much more often merely *similar* in meaning.

Churchland vs. Fodor&Lepore on Meaning Similarity vs. Compositionality

QUESTION: Which general postulates does a semantic resemblance relation $\sim$ on a language $\mathcal{L}$ satisfy?

SIM $\sim\subseteq \mathcal{L} \times \mathcal{L}$ is reflexive and symmetric.

SIM $\sim \subseteq \mathcal{L} \times \mathcal{L}$ is reflexive and symmetric.

CONN $\sim$ is connected.

SIM $\sim \subseteq \mathcal{L} \times \mathcal{L}$ is reflexive and symmetric.

CONN $\sim$ is connected.

So: for all $\alpha, \beta \in \mathcal{L}$ there is a sequence

$$\alpha = \gamma_1 \sim \gamma_2 \sim \ldots \sim \gamma_n = \beta$$

SIM $\sim \subseteq \mathcal{L} \times \mathcal{L}$ is reflexive and symmetric.

CONN $\sim$ is connected.

NON-TRIV No tautology stands in the $\sim$-relation to any contradiction.

SIM $\sim \subseteq \mathcal{L} \times \mathcal{L}$ is reflexive and symmetric.

CONN $\sim$ is connected.

NON-TRIV No tautology stands in the $\sim$-relation to any contradiction.

COM$\neg$ For all $\alpha, \beta \in \mathcal{L}$: if $\alpha \sim \beta$ then $\neg\alpha \sim \neg\beta$.

COM$\wedge$ For all $\alpha, \beta, \gamma, \delta \in \mathcal{L}$: if $\alpha \sim \beta$ and $\gamma \sim \delta$ then $\alpha \wedge \gamma \sim \beta \wedge \delta$.

COM$\vee$ For all $\alpha, \beta, \gamma, \delta \in \mathcal{L}$: if $\alpha \sim \beta$ and $\gamma \sim \delta$ then $\alpha \vee \gamma \sim \beta \vee \delta$.

SIM $\sim \subseteq \mathcal{L} \times \mathcal{L}$ is reflexive and symmetric.

CONN $\sim$ is connected.

NON-TRIV No tautology stands in the $\sim$-relation to any contradiction.

COM$\neg$ For all $\alpha, \beta \in \mathcal{L}$: if $\alpha \sim \beta$ then $\neg\alpha \sim \neg\beta$.

COM$\wedge$ For all $\alpha, \beta, \gamma, \delta \in \mathcal{L}$: if $\alpha \sim \beta$ and $\gamma \sim \delta$ then $\alpha \wedge \gamma \sim \beta \wedge \delta$.

COM$\vee$ For all $\alpha, \beta, \gamma, \delta \in \mathcal{L}$: if $\alpha \sim \beta$ and $\gamma \sim \delta$ then $\alpha \vee \gamma \sim \beta \vee \delta$.

Theorem

No relation satisfies SIM, CONN, NON-TRIV, COM$\neg$, COM$\wedge$, COM$\vee$.

Proof: "Shrink" $\sim$-chains up to logical equivalence.

SIM $\sim \subseteq \mathcal{L} \times \mathcal{L}$ is reflexive and symmetric.

CONN $\sim$ is connected.

NON-TRIV No tautology stands in the $\sim$-relation to any contradiction.

COM$\neg$ For all $\alpha, \beta \in \mathcal{L}$: if $\alpha \sim \beta$ then $\neg\alpha \sim \neg\beta$.

COM$\wedge$ For all $\alpha, \beta, \gamma, \delta \in \mathcal{L}$: if $\alpha \sim \beta$ and $\gamma \sim \delta$ then $\alpha \wedge \gamma \sim \beta \wedge \delta$.

COM$\vee$ For all $\alpha, \beta, \gamma, \delta \in \mathcal{L}$: if $\alpha \sim \beta$ and $\gamma \sim \delta$ then $\alpha \vee \gamma \sim \beta \vee \delta$.

Theorem

No relation satisfies SIM, CONN, NON-TRIV, COM$\neg$, COM$\wedge$, COM$\vee$.

Proof: "Shrink" $\sim$-chains up to logical equivalence.

Conclusion: Either meaning similarity has to go or compositionality has to be "softened".

(Leitgeb 2006, unpublished draft.)