Evidence lower bound

Given a probabilistic model $p(x, z)$ over observed random variable X and a latent random variable Z, the **evidence lower bound (ELBO)** is a lower bound for the evidence. Recall, the **evidence** for a probabilistic model is the marginal likelihood of the observed data:

$$p(x) = \int p(x, z)\, dz$$

Intuitively, if our model is correct, then the observed data should have a high probability under the model. Hence this quantity is "evidence" for the model.

If we introduce a new distribution $q(z)$ over the latent random variable Z, we can use Jensen's inequality to derive a lower bound to the evidence. Recall, Jensen's Inequality states

$$E[f(X)] \leq f(E[X])$$

where f is a convex function. Then, we derive the ELBO as a lower bound to the evidence as follows:

$$\begin{aligned}
\log p(x) &= \log \int p(x, z)\, dz \\
&= \log \int p(x, z) \frac{q(z \mid \phi)}{q(z)}\, dz \\
&= \log \left(E_{Z \sim q} \left[\frac{p(x, Z)}{q(Z)} \right] \right) \\
&\geq E_{Z \sim q} \left(\log \frac{p(x, Z)}{q(Z)} \right) && \text{Jensen's Inequality} \\
&= E_{Z \sim q} [\log p(x, Z)] - E_{Z \sim q} [\log q(Z)] \\
&= E_{Z \sim q} [\log p(x, Z)] - H(q)
\end{aligned}$$

where $H(q)$ is the entropy of q. This final quantity is the ELBO:

$$ELBO(q) := E_{Z \sim q} [\log p(x, Z)] - H(q)$$

(Definition 1).

Definition 1 *Given two distributions $p(x)$ and $q(x)$ over the same support, the **evidence lower bound** of $p(x)$ using $q(x)$ is given by:*

$$ELBO_p(q) := E_{X \sim q} [\log p(X)] - H(q)$$

where $H(q)$ is the entropy of q.

Relationship to KL-divergence

The difference between the evidence and the ELBO is precisely the KL-divergence between $p(z \mid x)$ and $q(z)$:

$$\begin{aligned} KL(q(z) \parallel p(z \mid x)) &= E_{Z \sim q}\left[\log \frac{q(Z)}{p(Z \mid x)}\right] \\ &= E_{Z \sim q}\left[\log q(Z)\right] - E_{Z \sim q}\left[\log p(Z \mid x)\right] \\ &= E_{Z \sim q}\left[\log q(Z)\right] - E_{Z \sim q}\left[\log \frac{p(Z, x)}{p(x)}\right] \\ &= E_{Z \sim q}\left[\log q(Z)\right] - E_{Z \sim q}\left[\log p(Z, x)\right] + E_{Z \sim q}\left[\log p(x)\right] \\ &= \log p(x) - \left(E_{Z \sim q}\left[\log p(x, Z)\right] - E_{Z \sim q}\left[\log q(Z)\right]\right) \\ &= \log p(x) - \text{ELBO} \end{aligned}$$

This insight forms the basis of the variational inference algorithm.

