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A Brief History of Hamiltonian Graphs

R. J. Wilson
Faculty of Mathematics
The Open University
Milton Keynes, England
and
Department of Mathematics
The Colorado College
Colorado Springs, Colorado, USA

Dedicated to the memory of Gabriel Dirac

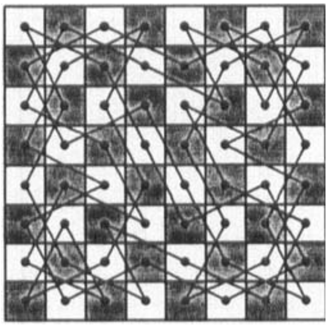
A graph is hamiltonian if it contains a closed cycle passing through every vertex. In this paper we outline the history of hamiltonian graphs from the early studies on the knight's tour problem to Gabriel Dirac's important paper of 1952.

1 The knight's tour problem

The earliest known example of a hamiltonian-type problem is that of the knight's tour problem on a chessboard: *Can a knight visit all sixty-four squares of a chessboard just once and return to its starting position?*

Chessboard problems of this kind have a long history (see [17]), and solutions are known to exist as far back as the 14th century. A solution to the knight's tour problem is given in Figure 1.

Methods of solution for the knight's tour problem were given at the end of the 17th century by De Montmort and De Moivre. The first *systematic* approach to the problem, however, seems to have been due to Euler, in a paper [7] written for the Academy of the Royal Society of Berlin in 1759. Euler had been interested in chess problems for some time and corresponded with both Goldbach and Bertrand on the subject. In this paper Euler described various methods for solving the problem and discussed the analogous problem for chessboards of different shapes and sizes. In particular, he showed that no solution is possible for chessboards with an odd number of squares.



50	11	24	63	14	37	26	35
23	62	51	12	25	34	15	38
10	49	64	21	40	13	36	27
61	22	9	52	33	28	39	16
48	7	60	1	20	41	54	29
59	4	45	8	53	32	17	42
6	47	2	57	44	19	30	55
3	58	5	46	31	56	43	18

Figure 1.

The next significant advance came in 1771, when Vandermonde wrote a paper [20] on *problèmes des situation* for the Académie Royale des Sciences in Paris. Among the subjects discussed in this paper was a systematic general method for the knight's tour problem, in which the square in row a, column b was represented by the symbol $\genfrac{}{}{0pt}{}{a}{b}$, and a knight's move corresponded to a pair of symbols ($\genfrac{}{}{0pt}{}{a}{b}$ to $\genfrac{}{}{0pt}{}{a\pm1}{b\pm2}$ or $\genfrac{}{}{0pt}{}{a\pm2}{b\pm1}$). Vandermonde referred to Euler's 1759 paper, remarking that whereas "that great geometer presupposes that one has a chessboard to hand, I have reduced the problem to simple arithmetic." An extract from Vandermonde's paper appears in [3].

In the 19th century the problem continued to attract much attention, and was discussed at length in Rouse Ball's book on recreational mathematics [17] and in a three-volume work by Jaenisch [9]. Particular interest centred around finding solutions with some extra property, such as symmetry. An interesting example of a special solution (due to Jaenisch) is that given in Figure 1, where the successive knight's-moves give rise to a magic square whose rows and columns all sum to 260.

2 The work of Kirkman

The Reverend Thomas Penyngton Kirkman (1806–95) was rector of the parish of Croft-with-Southworth in Lancashire for about fifty years. His parish duties were not onerous, and he spent much of his spare time working on mathematics. He was a fine mathematician whose significant contributions to both group theory and the theory of block designs have been much underrated. A detailed appraisal of his work is given in [2].

Among his mathematical interests was the study of polyhedra (or *polyedra*, as he preferred to call them). In particular, he was fascinated by the following question: *For which polyhedra does there exist a closed cycle passing through every vertex?*

In [10] he gave a sufficient condition for the existence of such a cycle, but this proved to be incorrect. He also showed that any polyhedron with even-sided faces but an odd number of vertices has no such cycle. We can obtain such a polyhedron if we "cut in two the cell of a bee," giving the diagram in Figure 2. Nowadays we observe that since such a diagram is a bipartite graph of odd order, and since any cycle must visit the two partite sets of vertices alternately, no cycle through every vertex can exist. Kirkman's argument was not as simple as this.

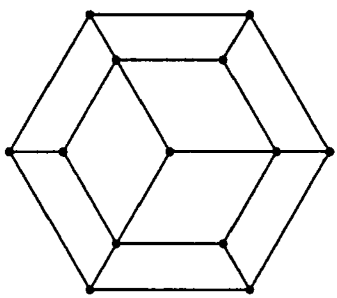


Figure 2.

In 1880, Tait [18] conjectured that every *trivalent* polyhedron has such a cycle. This conjecture intrigued Kirkman who remarked [11] that it "mocks alike at doubt and proof." If it were true one could deduce that every such polyhedron is 3-edge-colourable, and hence obtain a simple proof of the four colour theorem. (At the time Tait made his conjecture, Kempe's 'proof' of the four colour theorem was still regarded as correct.) However, in 1946 Tutte [19] produced an example of a trivalent polyhedron which contains no hamiltonian cycle (see Figure 3).

3 Hamilton's icosian game

Kirkman was not the only mathematician interested in cycles on polyhedra. William Rowan Hamilton's involvement arose from a suggestion from his friend and biographer John T. Graves that the subject might be relevant to his work on quaternions and non-commutative algebra.

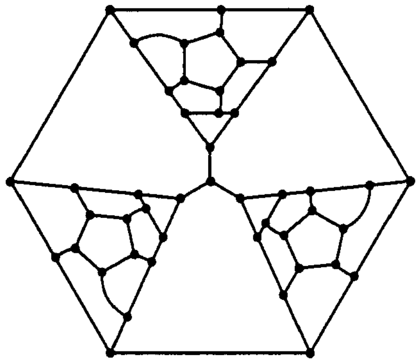


Figure 3.

Hamilton's interest soon led to the invention of the *icosian calculus* [8], a system consisting of the symbols ι, κ and λ and the relations

$$\iota^2 = \kappa^3 = \lambda^5 = 1 \quad \text{and} \quad \lambda = \iota\kappa.$$

It follows from these relations that if $\mu = \iota\kappa^2 = \lambda\kappa$, then

$$\mu^5 = 1, \quad \lambda\mu^2\lambda = \mu\lambda\mu, \quad \text{and} \quad (\lambda^3\mu^3\lambda\mu\lambda\mu)^2 = 1.$$

By interpreting λ as 'turn left' and μ as 'turn right', Hamilton used this last relation to obtain a closed circuit of faces on an icosahedron. This leads, by duality, to a cycle passing through all the vertices of a dodecahedron (see Figure 4).

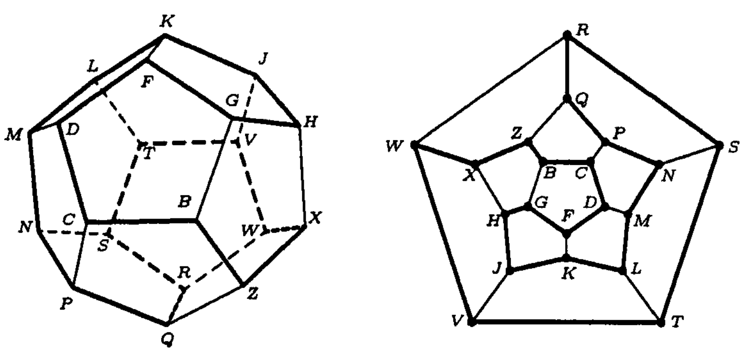


Figure 4.

Hamilton became increasingly intrigued by these ideas, and invented the *icosian game*, a solid or flat dodecahedron with holes at the vertices and pegs to be inserted into the holes according to certain instructions (see [3]). The object was to find, by experimentation, the number of hamiltonian cycles starting with a given set of vertices. The game was marketed under the title *A voyage round the world*, with the vertices B, C, D, ..., Z standing for *Brussels*, *Canton*, *Delhi*, ..., *Zanzibar*.

Further studies into Hamilton's icosian game were made by Hermary (see [15]) and A. Kowalewski ([13], [14]). Hermary showed that any hamiltonian cycle can be obtained as the boundary of the flat figure obtained by unfolding the dodecahedron (see Figure 5). Using this representation he showed that the number of possible hamiltonian cycles with a given number k of starting vertices is given by the following table:

k	1	2	3	4	5	...
cycles	30	20	10	6 or 4	4 or 2	...

Kowalewski related the icosian game to the so-called *Buntordnungsproblem*. In these papers he was the first to present the Petersen graph as the graph whose vertices are pairs of numbers from the set $\{1,2,3,4,5\}$ with vertices joined by an edge whenever the corresponding pairs are disjoint (see Figure 6).

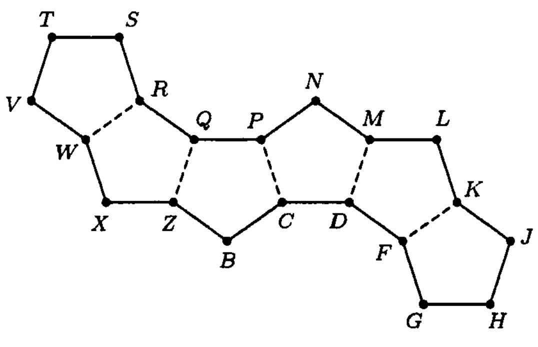


Figure 5.

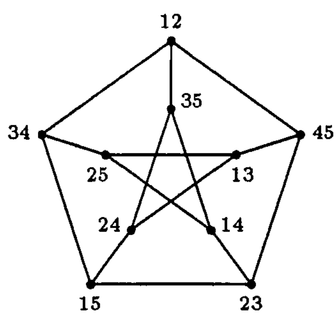


Figure 6.

4 Lucas' Récréations Mathématiques

In Volume 2 of his four-volume work on recreational mathematics [15], Lucas described a number of problems which are related to hamiltonian cycles in graphs. The most well-known of these (attributed by Berge [1] to Kirkman) is as follows:

Les Rondes Enfantines: A number of children dance around in a circle. How can we arrange the dances so that each child dances next to every other child just once?

To solve this problem he noted first that the total number of children must be odd ($2k + 1$, say), and that the arrangement of children in each dance corresponds to a hamiltonian cycle in K_{2k+1}. A solution to the problem is obtained by finding k edge-disjoint hamiltonian cycles in K_{2k+1}. Lucas found these cycles by placing $2k$ of the vertices symmetrically in a ring, and the remaining vertex on a diameter (see Figure 7). The first hamiltonian cycle is the zig-zag pattern shown. The remaining cycles are then obtained by fixing the ring of vertices and rotating the zig-zag pattern. For $k = 5$, this yields the hamiltonian cycles

$a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ a$	(shown)
$a\ c\ e\ b\ g\ d\ i\ f\ k\ h\ j\ a$	(one step)
$a\ e\ g\ c\ i\ b\ k\ d\ j\ f\ h\ a$	(two steps)
$a\ g\ i\ e\ k\ c\ j\ b\ h\ d\ f\ a$	(three steps)
$a\ i\ k\ g\ j\ e\ h\ c\ f\ b\ d\ a$	(four steps)

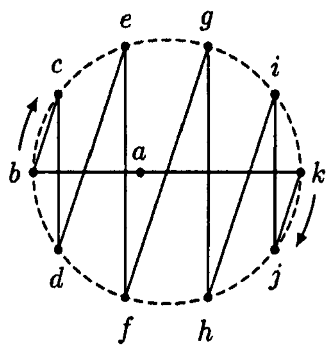


Figure 7.

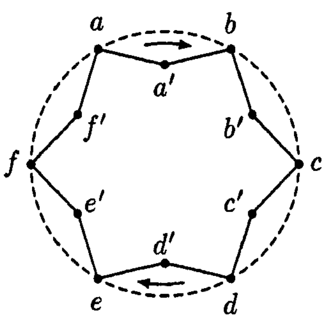


Figure 8.

Another problem of a similar type is the following:

Les Rondes Alternées: A number of girls and boys dance around in a circle with boys and girls alternating. How can we arrange the dances so that each girl dances next to each boy just once?

We illustrate the solution when there are six boys (a, b, c, d, e, f) and six girls (a', b', c', d', e', f'). Lucas first drew the star-shaped diagram shown in

Figure 8. To obtain the required cycles he rotated the star inside the circle, giving the solution

$$\begin{array}{ll} a\ a'\ b\ b'\ c\ c'\ d\ d'\ e\ e'\ f\ f'\ a & \text{(shown)} \\ a\ e'\ b\ f'\ c\ a'\ d\ b'\ e\ c'\ f\ d'\ a & \text{(two steps)} \\ a\ c'\ b\ d'\ c\ e'\ d\ f'\ e\ a'\ f\ b'\ a & \text{(four steps)} \end{array}$$

The solution for a general *even* number of girls and boys is similar. The analogous question for an *odd* number of girls and boys was answered by Dirac in 1972 [6].

5 From 1890 to 1940

In the following fifty years there were a number of results of greater or less significance. We mention four of them briefly:

(i) In 1891, Brunel wrote a short note [4] in which he considered the graph with 64 vertices, corresponding to the squares of a chessboard, and 168 edges, corresponding to the possible knight's-moves. He then related a solution of the knight's tour problem to the determination of the 168 edge-currents in a related electrical network.

(ii) In 1934, Rédei [16] proved the important result that every tournament has a directed path passing through every vertex. Surprisingly, the natural extension of this result, that every strongly-connected tournament has a hamiltonian cycle, was not proved until 1957.

(iii) The whole subject of traversability was given a new lease of life in 1936 with the appearance of Dénes König's classic text *Theorie der endlichen und unendlichen Graphen* [12]. Eulerian and hamiltonian graphs were discussed in detail, and the term "hamiltonian" was established for all time. Gabriel Dirac was a great admirer of König's book.

(iv) In 1939, Vászonyi [21] showed the existence of two-way infinite hamiltonian paths in the n-dimensional square lattice L_n, for each $n \geq 1$; the case $n = 2$ is shown in Figure 9. He also found a sequence of knight's moves covering each vertex of L_n just once.

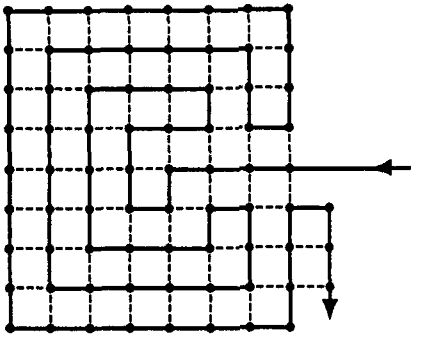


Figure 9.

6 Dirac's 1952 paper

We conclude this brief survey by outlining Dirac's fundamental paper of 1952 [5]. He had just completed his doctoral thesis for King's College, London, and the paper contains many of the important results in this thesis. It is divided into two parts, with part A devoted to properties of cycles of graphs, and part B applying these results to the colouring of graphs. Many results which are now in the 'folk-lore' of the subject made their first appearance in this paper.

We now state the main results, with brief comments on each:

Theorem 1. *If a graph G has no cut-vertex, and has a path of length l, then G contains a cycle of length greater than $\sqrt{2l}$.*

Dirac's proof of this theorem used Menger's theorem. He remarked that the bound $\sqrt{2l}$ can be improved to $2\sqrt{l}$, but that this is much harder to prove.

Theorem 2. *If G is a finite graph with* $\deg(v) \geq d$ *for each vertex v, then G contains a cycle of length at least $d + 1$.*

A similar result holds for infinite graphs with no cut-vertex. Dirac used Theorem 2, and the fact that every vertex-critical graph has minimum degree at least $k - 1$, to deduce that every k-chromatic graph contains a cycle of length at least k, for $k \geq 3$.

The most well-known result in the paper is the following, now called *Dirac's theorem*:

Theorem 3. *If G is connected, with* $\deg(v) \geq d$ *for each vertex v, and if G has at most $2d$ vertices, then G is hamiltonian.*

Using Theorem 3, Dirac proved that, for $k \geq 3$, a k-chromatic graph with at most $2k - 2$ vertices has a k-chromatic hamiltonian subgraph, and that a critical k-chromatic graph with at most $2k-1$ vertices is hamiltonian.

Another consequence of Theorem 3 is that, if $\chi(G) = k$, then G contains K_k or a cycle of length at least $k + 2$. Moreover, if $0 \leq r \leq k - 1$ and $\chi(G) = k$, then G contains K_{k-r} or a cycle of length at least $k + r + 2$.

Theorem 4. *If G is 2-connected with* $\deg(v) \geq d$ *for each vertex v, and if G has at least $2d$ vertices, then G contains a cycle of length at least $2d$.*

The proof of this also used Menger's theorem. Finally, using Theorem 1 and a counting argument, Dirac proved the following result:

Theorem 5. *Let G be a 2-connected graph with n vertices. Then the length of a longest cycle tends to ∞ as $n \to \infty$, provided that every vertex-degree remains bounded.*

Since 1952 much progress has been made in the study of hamiltonian graphs. Details of some of this work can be found elsewhere in this volume.

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