تاریخ تحویل: 99/9/29 — ریاضیات مهندسی — کلاس دکتر ... — دانشجو: ... — شماره دانشجویی: 810198472

1-

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \\ u(0,t)=1 \quad \lim_{x\to -\infty} u(x,t)=0 \\ u(x,0)=e^x \quad u_t(x,0)=0 \end{cases} \xrightarrow{\mathcal{L}} \frac{d^2 U}{dx^2} = s^2U(x,s) - su(x,0) - u_t(x,0)$$

$$\to \frac{d^2 U}{dx^2} = s^2U - se^x$$

$$\to U'' - s^2 U = -se^x \qquad U_h(x,s) = Ae^{sx} + Be^{-sx}$$

$$U_p(x,s) = A_0 e^x \xrightarrow[\text{در معادله}]{\text{جایگذاری}} A_0 e^x - s^2 A_0 e^x = -se^x \to A_0(1-s^2) = -s$$

$$\to A_0 = \frac{s}{s^2-1} \qquad U(x,s) = Ae^{sx} + Be^{-sx} + \frac{s}{s^2-1}e^x$$

$$u(0,t)=1 \xrightarrow{\mathcal{L}} U(0,s) = \frac{1}{s} \ (1) \qquad \lim_{x\to -\infty} u(x,t) = 0 \xrightarrow{\mathcal{L}} \lim_{x\to -\infty} U(x,s) = 0 \ (2)$$

$$\xrightarrow{\text{اعمال شرط}} (2) \to B=0 \qquad U(0,s) = \frac{1}{s} \to A + \frac{s}{s^2-1} = \frac{1}{s}$$

$$\to A = \frac{-1}{s(s^2-1)} = \frac{-1}{s(s^2-1)} \to U(x,s) = \frac{-1}{s(s^2-1)}e^{sx} + \frac{s}{s^2-1}e^x$$

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{-e^{sx}}{s(s^2-1)} + \frac{s}{s^2-1}e^x\right] = \mathcal{L}^{-1}\left[\frac{-e^{sx}}{s(s^2-1)}\right] + \mathcal{L}^{-1}\left[\frac{s}{s^2-1}e^x\right]$$

$$\frac{-1}{s(s^2-1)} = \frac{A_0}{s} + \frac{A_1}{s-1} + \frac{A_2}{s+1} \qquad (s-1)(s+1)A_0 + s(s+1)A_1 + s(s-1)A_2 = -1$$

$$\to -A_0 = -1 \to A_0 = +1 \quad 2A_1 = -1 \to A_1 = -\frac{1}{2} \quad 2A_2 = -1 \to A_2 = -\frac{1}{2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2-1)}\right] = \mathcal{L}^{-1}\left[+\frac{1}{s}\right] + \mathcal{L}^{-1}\left[-\frac{\frac{1}{2}}{s-1}\right] + \mathcal{L}^{-1}\left[-\frac{\frac{1}{2}}{s+1}\right] = +u(t) - \frac{1}{2}e^t - \frac{1}{2}e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{e^{sx}}{s(s^2-1)}\right] = \left[+H(t+x) - \frac{1}{2}e^{t+x} - \frac{1}{2}e^{-t+x}\right] H(t+x)$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2-1}e^x\right] = \cosh t\, e^x \qquad u(x,t) = \left[H(t+x) - \frac{1}{2}e^{t+x} - \frac{1}{2}e^{-t+x}\right] H(t+x)$$

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$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} - \sin \pi x \sin t \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u_t(x,0) = 0 \end{cases} \xrightarrow{\mathcal{L}} \frac{d^2 U}{dx^2} = s^2 U(x,s) - sU(x,0) - U_t(x,0) - \frac{\sin \pi x}{1+s^2}$$

$$\rightarrow \frac{d^2 U}{dx^2} = s^2 U - \frac{\sin \pi x}{1+s^2} \rightarrow U'' - s^2 U = -\frac{\sin \pi x}{1+s^2}$$

$$U(x,s) = U_h(x,s) + U_p(x,s) \Rightarrow \quad U_h(x,s): \ U'' - s^2 U = 0 \rightarrow r^2 - s^2 = 0 \rightarrow r = \pm s$$

$$\rightarrow U_h(x,s) = Ae^{sx} + Be^{-sx} \qquad U_p(x,s): \ U'' - s^2 U = \frac{-1}{1+s^2} \sin \pi x$$

حدس: $U_p = c \sin \pi x \xrightarrow{\text{جایگذاری}} -\pi^2 c \sin \pi x - s^2 c \sin \pi x = \dfrac{-1}{1+s^2} \sin \pi x$

$$\rightarrow c(\pi^2 + s^2) = +\frac{1}{1+s^2} \rightarrow c = \frac{1}{s^2+1} \times \frac{1}{\pi^2 + s^2}$$

$$\rightarrow U(x,s) = Ae^{sx} + Be^{-sx} + \frac{1}{(s^2+1)(s^2+\pi^2)} \sin \pi x \qquad \begin{aligned} U(0,s) &= \mathcal{L}[u(0,t)] = 0 \\ U(1,s) &= \mathcal{L}[u(1,t)] = 0 \end{aligned}$$

$$\rightarrow A = B = 0 \rightarrow U(x,s) = \frac{1}{(s^2+1)(s^2+\pi^2)} \sin \pi x \qquad u(x,t) = \mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+\pi^2)}\right] \sin \pi x$$

$$\mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(t-x) g(x)\, dx \qquad F(s) = \frac{1}{s^2+1} \xrightarrow{\mathcal{L}^{-1}} f(t) = \sin t$$

$$G(s) = \frac{1}{(s^2+\pi^2)} \xrightarrow{\mathcal{L}^{-1}} g(t) = \frac{1}{\pi} \sin \pi t \rightarrow \mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+\pi^2)}\right]$$

$$= \frac{1}{\pi} \int_0^t \sin(t-x) \sin \pi x\, dx \overset{\text{حل}}{=} \frac{\pi \sin t - \sin \pi t}{\pi^3 - \pi}$$

$$\rightarrow u(x,t) = \sin \pi x \left[\frac{\pi \sin t - \sin \pi t}{\pi^3 - \pi}\right]$$

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3- به نظرم $u(l,0)$ اشتباه تایپی است.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = 3\sin\left(\dfrac{2\pi x}{l}\right) \end{cases}$$

$$\xrightarrow{\mathcal{L}} \frac{d^2 U}{dx^2} = sU(x,s) - u(x,0)$$

$$\rightarrow U'' - sU = -3\sin\frac{2\pi x}{l}$$

$$U(x,s) = U_h(x,s) + U_p(x,s)$$

جواب عمومی: $U'' - sU = 0 \rightarrow U_h(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$

جواب خصوصی: $U'' - sU = -3\sin\frac{2\pi x}{l}$ $\quad U = A\sin\frac{2\pi x}{l}$ فرض می‌کنیم

$$\rightarrow U' = A\frac{2\pi}{l}\cos\frac{2\pi x}{l} \rightarrow U'' = -A\frac{4\pi^2}{l^2}\sin\frac{2\pi x}{l} \xrightarrow{\text{جایگذاری}} -A\frac{4\pi^2}{l^2}\sin\frac{2\pi x}{l} - sA\sin\frac{2\pi x}{l} = -3\sin\frac{2\pi x}{l}$$

$$\rightarrow -A\frac{4\pi^2}{l^2} - sA = -3 \rightarrow A\left(s + \frac{4\pi^2}{l^2}\right) = 3 \rightarrow A = \frac{3}{s + \frac{4\pi^2}{l^2}}$$

$$\rightarrow U(x,s) = U_h(x,s) + U_p(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + \frac{3}{s + \frac{4\pi^2}{l^2}}\sin\frac{2\pi x}{l}$$

$$u(0,t) = 0 \xrightarrow{\mathcal{L}} U(0,s) = 0 \qquad u(l,t) = 0 \xrightarrow{\mathcal{L}} U(l,s) = 0$$

$$\rightarrow U(0,s) = 0 \rightarrow A + B = 0 \rightarrow A = -B \quad U(l,s) = 0 \rightarrow Ae^{\sqrt{s}l} + Be^{-\sqrt{s}l} = 0 \quad B(e^{-\sqrt{s}l} - e^{\sqrt{s}l}) = 0 \rightarrow B = 0$$

$$\rightarrow A = B = 0 \rightarrow U(x,s) = \frac{3}{s + \frac{4\pi^2}{l^2}}\sin\frac{2\pi x}{l} \qquad u(x,t) = \mathcal{L}^{-1}[U(x,s)]$$

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{3}{s + \frac{4\pi^2}{l^2}}\right]\sin\frac{2\pi x}{l} \qquad \frac{3}{s + \frac{4\pi^2}{l^2}} = \frac{\frac{2\pi}{l}}{s + \frac{4\pi^2}{l^2}} \times \frac{3l}{2\pi}$$

$$\rightarrow \mathcal{L}^{-1}\left[\frac{3}{s + \frac{4\pi^2}{l^2}}\right] = \frac{3l}{2\pi}\sin\frac{2\pi}{l}t$$

$$\rightarrow u(x,t) = \frac{3l}{2\pi}\sin\frac{2\pi}{l}t\ \sin\frac{2\pi}{l}x$$

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$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2}\dfrac{\partial u}{\partial t} = 0 \\ \lim\limits_{x \to \infty} u(x,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

از تبدیل فوریه نسبت به x (فوریه مکانی) استفاده می‌کنیم. $t \xrightarrow{\mathcal{F}} \omega$, $x \xrightarrow{\mathcal{F}_x} k_x$

$$\mathcal{F}_x\left[\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial u}{\partial t}\right] = 0$$

$$\rightarrow (jk_x)^2 U(k_x,t) - \frac{1}{c^2}\frac{dU}{dt} = 0 \qquad -k_x^2 U - \frac{1}{c^2}\frac{dU}{dt} = 0 \xrightarrow{\times -1}$$

$$U'_t + c^2 k_x^2 U = 0 \rightarrow U(k_x,t) = D(k_x)\, e^{-c^2 k_x^2 t}$$

فرم بسته‌ی پاسخ ما شرط مرزی مربوط به کراندار بودن در $x=\infty, x=-\infty$ را ارضا می‌کند. پس می‌توانیم جلو رویم.

$$u(x,0) = f(x) \xrightarrow{\mathcal{F}} U(k_x,0) = F(k_x) \qquad U(k_x,0) = D = F(k_x)$$

$$\rightarrow U(k_x,t) = F(k_x)\, e^{-c^2 t k_x^2} \rightarrow u(x,t) = \mathcal{F}_x^{-1}\left[F(k_x)\, e^{-c^2 t k_x^2}\right]$$

داریم: $e^{-bx^2} \xrightarrow{\mathcal{F}_x} \sqrt{\frac{\pi}{b}}\, e^{\frac{-k_x^2}{4b}} \rightarrow \mathcal{F}_x^{-1}\, e^{-\frac{k_x^2}{4b}} = \dfrac{e^{-bx^2}}{\sqrt{\frac{\pi}{b}}}$

$$\frac{1}{4b} = +c^2 t \rightarrow b = \frac{1}{4c^2 t} \rightarrow \mathcal{F}_k^{-1}\left[e^{-c^2 t k_x^2}\right] = \frac{e^{\frac{-x^2}{4c^2 t}}}{\sqrt{4\pi c^2 t}}$$

$$\rightarrow \mathcal{F}^{-1}\left[F(k_x)\, G(k_x)\right] = \int_{-\infty}^{\infty} f(x)\, g(\xi - x)\, dx$$

$$\rightarrow u(x,t) = \int_{-\infty}^{\infty} f(\xi - x)\, \frac{e^{\frac{-x^2}{4c^2 t}}}{\sqrt{4\pi c^2 t}}\, dx = \int_{-\infty}^{\infty} \frac{1}{2c\sqrt{\pi t}}\, f(\xi - x)\, e^{\frac{-x^2}{4c^2 t}}\, dx$$

$$\text{or} \quad u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2c\sqrt{\pi t}}\, f(x)\, e^{\frac{-(\xi - x)^2}{4c^2 t}}\, dx$$

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