الف) -1

$\langle f_1, f_2 \rangle_{-\frac{\pi}{2}, \frac{\pi}{2}} = 0 \longrightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos 2x\,dx \longrightarrow x = u \rightarrow dx = du$

$\cos 2x\,dx = dv \rightarrow v = \frac{1}{2}\sin 2x$

$\longrightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos 2x\,dx = \frac{x}{2}\sin 2x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2x = 0 + \frac{1}{4}\cos 2x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$ متعامدند

ب)

$\langle f_1, f_2 \rangle_{-\frac{\pi}{2}, \frac{\pi}{2}} = 0 \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sinh x \sin x\,dx \longrightarrow \sinh x = u \rightarrow \cosh x\,dx = du$

$\sin x\,dx = dv \rightarrow -\cos x = v$

$\longrightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sinh x \sin x\,dx = -\sinh x \cos x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cosh x \cos x\,dx = 0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cosh x \cos x\,dx$

$\cosh x = u \rightarrow \sinh x\,dx = du$

$\cos x\,dx = dv \rightarrow v = \sin x$

$\longrightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cosh x \cos x\,dx = \sin x \cosh x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sinh x \sin x\,dx$

$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sinh x \sin x = \sin x \cosh x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \longrightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sinh x \sin x\,dx = \frac{\sin x \cosh x}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{\cosh(\frac{\pi}{2}) + \cosh(-\frac{\pi}{2})}{2}$ متعامد نیست

الف) -2

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi}{\ell}x + b_n \sin\frac{n\pi}{\ell}x)$

$[f(x)]^2 = \frac{a_0^2}{4} + \left[(a_1\cos\frac{n\pi}{\ell}x + b_n\sin\frac{n\pi}{\ell}x) + (a_2\cos\frac{n\pi}{\ell}x + b_2\sin\frac{n\pi}{\ell}x) + \dots\right]^2$

انتگرال هایی که باید بسیم را به صورت پارامتری می‌نویسیم که شلوغ نشود.

$\frac{1}{\ell}\int_{-\ell}^{\ell} (a_n\cos\frac{n\pi}{\ell}x + b_n\sin\frac{n\pi}{\ell}x)^2 = \frac{1}{\ell}\left(\int_{-\ell}^{\ell} a_n^2\cos^2\frac{n\pi}{\ell}x + \int_{-\ell}^{\ell} b_n^2\sin^2\frac{n\pi}{\ell}x + \int_{-\ell}^{\ell} a_n b_n \sin 2\frac{n\pi}{\ell}x\right)$

$\int_{-\ell}^{\ell} a_n^2\cos^2\frac{n\pi}{\ell} = \frac{a_n^2}{2}\int_{-\ell}^{\ell} 1 + \cos\frac{2n\pi}{\ell} = \ell a_n^2 \qquad \int_{-\ell}^{\ell} b_n^2 \sin^2\frac{n\pi}{\ell} = \frac{b_n^2}{2}\int_{-\ell}^{\ell} 1 - \sin 2\frac{n\pi}{\ell}x$

$= \ell b_n^2 \qquad \int_{-\ell}^{\ell} a_n b_n \sin 2\frac{n\pi}{\ell}x = 0 \longrightarrow \frac{1}{\ell}\int_{-\ell}^{\ell} (a_n\cos\frac{n\pi}{\ell}x + b_n\sin\frac{n\pi}{\ell}x)^2 = a_n^2 + b_n^2$

$\frac{1}{\ell}\int_{-\ell}^{\ell} (a_n\cos\frac{n\pi}{\ell}x + b_n\sin\frac{n\pi}{\ell}x)(a_m\cos\frac{m\pi}{\ell}x + b_m\sin\frac{m\pi}{\ell}x)$

$= \frac{1}{\ell}\left(\int_{-\ell}^{\ell} a_n a_m \cos\frac{n\pi}{\ell}x\cos\frac{m\pi}{\ell}x + \int_{-\ell}^{\ell} b_n b_m \sin\frac{n\pi}{\ell}x\sin\frac{m\pi}{\ell}x + \int_{-\ell}^{\ell} a_n b_m \cos\frac{n\pi}{\ell}x \sin\frac{m\pi}{\ell}x\right.$

$$+\int_{-l}^{l} a_m b_n \cos\frac{m\pi}{l}x \sin\frac{n\pi}{l}x) \xrightarrow[\text{[illegible]}]{\text{[illegible]}} \int_{-l}^{l} \sin nu \sin vdv = \int \cos nu \cos nu du = 0 \quad \text{if } m \neq n$$

که در این سوال $m \neq n$ می باشد.

$$\int_{-l}^{l} a_n b_m \cos\frac{n\pi}{l}x \sin\frac{n\pi}{l}x\, dx = \int_{-l}^{l} a_m b_n \cos\frac{m\pi}{l}x \sin\frac{n\pi}{l}x = 0 \quad \text{if } m \neq n$$

که در بعضی جاها $m=n$ است.

$$\longrightarrow \frac{1}{l}\int_{-l}^{l}[f(x)]^2dx = \frac{1}{l}\int_{-l}^{l} \frac{a_0^2}{4} + \left[(a_1\cos\frac{\pi}{l}x + b_1\sin\frac{\pi}{l}x) + (a_2\cos\frac{2\pi}{l}x + b_2\sin\frac{2\pi}{l}x) + \cdots\right]^2$$

$$\longrightarrow \frac{1}{l}\int_{-l}^{l}[f(x)]^2dx = \frac{1}{l}\left(\frac{2la_0^2}{4} + \sum_{n=1}^{\infty} la_n^2 + lb_n^2\right) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

ب)

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x)$$

$$g = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n\cos\frac{n\pi}{l}x + d_n\sin\frac{n\pi}{l}x)$$

$$f.g = \frac{a_0c_0}{4} + \frac{a_0}{2}\sum_{n=1}^{\infty}(c_n\cos\frac{n\pi}{l}x + d_n\sin\frac{n\pi}{l}x) + \frac{c_0}{2}\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x)$$

$$+\sum_{n=1}^{\infty} a_nc_n\cos^2\frac{n\pi}{l}x + b_nd_n\sin^2\frac{n\pi}{l}x + a_nd_n\sin\frac{n\pi}{l}x\cos\frac{n\pi}{l}x + b_nc_n\sin\frac{n\pi}{l}x\cos\frac{n\pi}{l}x)$$

$$\rightarrow \frac{1}{l}\int_{-l}^{l} f.g = \frac{1}{l}\int_{-l}^{l}\frac{a_0c_0}{4} + \frac{a_0}{2}\frac{1}{l}\sum\int_{-l}^{l} c_n\cos\frac{n\pi}{l}x + d_n\sin\frac{n\pi}{l}x\,dx + \frac{c_0}{2}\frac{1}{l}\sum\int_{-l}^{l} a_n\cos\frac{n\pi}{l}x$$

$$+\sin\frac{n\pi}{l}x\,dx + \frac{1}{l}\sum\int_{-l}^{l} a_nc_n\cos^2\frac{n\pi}{l}x + b_nd_n\sin^2\frac{n\pi}{l}x + a_nd_n\sin\frac{n\pi}{l}x\cos\frac{n\pi}{l}x + b_nc_n\sin\frac{n\pi}{l}x\cos\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\frac{2a_0c_0l}{4} + 0 + 0 + \frac{1}{l}\sum_{n=1}^{\infty} la_nc_n + lb_nd_n = \frac{a_0c_0}{2} + \sum_{n=1}^{\infty} a_nc_n + b_nd_n$$

3- الف)

$$y(x) = \sinh x \quad -\pi \leq x < \pi \qquad f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sinh x\,dx = \frac{1}{2\pi}\cosh x\Big|_{-\pi}^{\pi} = 0 \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sinh x\cos nx\,dx$$

$$U = \cos nx \rightarrow du = -n\sin nx \qquad \longrightarrow \overset{0}{\cosh x\cos nx\Big|_{-\pi}^{\pi}} + n\int_{-\pi}^{\pi}\sin nx\cosh x\,dx \qquad U = \sin nx \rightarrow du = n\cos nx$$

$$\sinh x\,dx = dv \rightarrow \cosh x = v \qquad \cosh x\,dx = dv \rightarrow v = \sinh x$$

$$\int_{-\pi}^{\pi} \sin nx \cosh x\,dx = \sin nx \sinh x \Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} \sinh x \cos nx$$

$$\to \frac{1}{\pi}\int_{-\pi}^{\pi} \sinh x \cos nx\,dx = -\frac{n^2}{\pi}\int_{-\pi}^{\pi} \sinh x \cos nx\,dx \to \int_{-\pi}^{\pi} \sinh x \cos nx\,dx = 0 \to a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sinh x \sin nx\,dx \qquad \int_{-\pi}^{\pi} \sinh x \sin nx\,dx \longrightarrow \begin{matrix} u = \sin nx \to du = n\cos nx \\ \sinh x\,dx = dv \to v = \cosh x \end{matrix}$$

$$\int \sinh x \sin nx\,dx = \sin nx \cosh x - n\int_{-\pi}^{\pi} \cosh x \cos nx\,dx \quad \begin{matrix} u = -n\cos nx \to du = +n^2 \sin nx \\ \cosh x\,dx = dv \to v = \sinh x \end{matrix}$$

$$\to -n\int \cosh x \cos nx\,dx = -n \sinh x \cos nx - \int n^2 \sin nx \sinh x$$

$$\longrightarrow \int \sinh x \cos nx\,dx = \frac{\cosh x \sin nx - n\sinh x \cos nx}{n^2+1}$$

$$\to b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sinh x \sin nx\,dx = \frac{e^{-\pi}\left((e^{2\pi}+1)\sin n\pi + (1-e^{2\pi})\, n\cos(n\pi)\right)}{\pi(n^2+1)}$$

$$= \frac{(1-e^{2\pi})\, n\cos(n\pi)}{\pi(n^2+1)} \to \begin{cases} n = \text{زوج} : \dfrac{n(1-e^{2\pi})}{\pi(n^2+1)} \\ n = \text{فرد} : \dfrac{(e^{2\pi}-1)n}{\pi(n^2+1)} \end{cases}$$

$$\to y(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n(1-e^{2\pi})}{\pi(n^2+1)} \sin nx$$

ب) توجه : پاسخ بخش ب و ج سوال 3 را در پایان پاسخ ها نوشتم

4- الف)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} (x-\pi)^2\,dx = \frac{2\pi^3}{3} \times \frac{1}{2\pi} = \frac{\pi^3}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (x-\pi)^2 \cos nx\,dx \qquad \begin{matrix} (x-\pi)^2 = u \to 2(x-\pi)dx = du \\ \cos nx\,dx = dv \to v = \frac{1}{n}\sin nx \end{matrix}$$

$$\longrightarrow \frac{(x-\pi)^2 \sin nx}{n} - \int_0^{2\pi} \frac{2(x-\pi)\sin nx}{n}\,dx \qquad \begin{matrix} u = x-\pi \to du = dx \\ \sin nx\,dx = dv \to v = -\frac{\cos nx}{n} \end{matrix}$$

$$= \frac{2(x-\pi)\cos nx}{n^2} - \int_0^{2\pi} \frac{2\cos nx}{n^2}\,dx \xrightarrow{u=nx} -\frac{2}{n^3}\int \cos u\,du = -\frac{2\sin nx}{n^3}$$

$$a_n=\frac{1}{\pi}\left(\frac{(x-\pi)^2\sin nx}{n}+\frac{2(x-\pi)\cos nx}{n^2}-\frac{2\sin nx}{n^3}\right)\Bigg|_0^{2\pi}=\frac{4\pi}{n^2}\times\frac{1}{\pi}=\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}(x-\pi)^2\sin nx=0$$ مشابه قبلی با جزء به جزء

ج)

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\left(1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots\right)=I$$

$$f(x)=(x-\pi)^2=\frac{\pi^2}{3}+4\left(\cos x+\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x+\frac{1}{16}\cos 4x+\cdots\right)$$

$$f(\pi)=0=\frac{\pi^2}{3}+4\underbrace{\left(-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\frac{1}{25}+\cdots\right)}_{-I}$$

$$\frac{\pi^2}{3}-4I=0\rightarrow I=\frac{\pi^2}{12}\rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$$

5-

$$f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)+\frac{2(-1)^{n+1}}{n^2}\sin(nx)$$

$$\int_{-\pi}^{\pi}f(x)(\sin^3x+2\cos^2\tfrac{x}{2})=\underbrace{\int_{-\pi}^{\pi}f(x)\sin^3x\,dx}_{I}+\underbrace{\int_{-\pi}^{\pi}f(x)2\cos^2\tfrac{x}{2}}_{II}\qquad \sin^3x=\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$$

$$I=\int_{-\pi}^{\pi}\frac{3}{4}f(x)\sin x\,dx-\int_{-\pi}^{\pi}\frac{1}{4}f(x)\sin 3x\,dx=\frac{3}{4}\pi b_1-\frac{1}{4}\pi b_3$$

$$II=\int_{-\pi}^{\pi}f(x)(1+\cos x)dx=\int_{-\pi}^{\pi}f(x)dx+\int_{-\pi}^{\pi}f(x)\cos x\,dx=2\pi a_0+\pi a_1$$

$$\int_{-\pi}^{\pi}f(x)(\sin^3x+2\cos^2\tfrac{x}{2})dx=\frac{3}{4}\pi b_1-\frac{1}{4}\pi b_3+2\pi a_0+\pi a_1=\frac{3\pi}{4}\times 2-\frac{\pi}{4}\times\frac{2}{9}$$

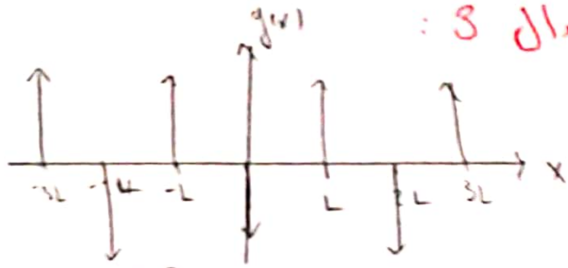
$$+\frac{2\pi^3}{3}-\pi=\frac{2\pi^3}{3}+\frac{3\pi}{2}-\pi-\frac{\pi}{18}=\frac{2\pi^3}{3}+\frac{4\pi}{9}$$

ادامه سوال 3 :

ب)

$$g(x)=\sum_{k=-\infty}^{\infty}(-1)^{k+1}\delta(x-kL)$$

رسم :



$$\rightarrow T=2L$$

می‌دانیم $\int_b^c \delta(x-a)f(x)dx=\begin{cases} f(a) & b<a<c \\ 0 & a<b \text{ or } c<a \end{cases}$

$$a_0=\frac{1}{2L}\int_{-L}^{L}\sum_{k=-\infty}^{\infty}(-1)^{k+1}\delta(x-kL)\,dx$$

دم : طبق قضیه فوبینی یا مجاز به تعویض مکان انتگرال، سیگما هستیم

$$\rightarrow a_0=\frac{1}{2L}\sum_{k=-\infty}^{\infty}(-1)^{k+1}\int_{-L}^{L}\delta(x-kL)dx=\frac{1}{2L}\int_{-L}^{L}(\delta(x+L)-\delta(x)+\delta(x-L))dx$$

$$=\frac{1}{2L}\left(\int_{-L}^{L}\delta(x+L)dx-\int_{-L}^{L}\delta(x)dx+\int_{-L}^{L}\delta(x-L)dx\right)=0$$

$$a_n=\frac{1}{L}\sum_{k=-\infty}^{\infty}(-1)^{k+1}\int_{-L}^{L}\delta(x-kL)\cos\frac{n\pi x}{L}dx=\frac{1}{L}\int_{-L}^{L}(\delta(x+L)-\delta(x)+\delta(x-L))\cos\frac{n\pi x}{L}$$

$$=\frac{1}{L}\left(\int_{-L}^{L}\delta(x+L)\cos\frac{n\pi x}{L}dx-\int_{-L}^{L}\delta(x)\cos\frac{n\pi x}{L}dx\int_{-L}^{L}\delta(x-L)\cos\frac{n\pi x}{L}dx\right)=\frac{1}{L}(\cos n\pi-1)$$

$$=\frac{\cos n\pi-1}{L}\rightarrow\begin{cases}0 & n\in E\\ -\frac{2}{L} & n\in O\end{cases}$$

$$b_n=\frac{1}{L}\sum_{k=-\infty}^{\infty}(-1)^{k+1}\int_{-L}^{L}\delta(x-kL)\sin\frac{n\pi x}{L}=\frac{1}{L}\int_{-L}^{L}(\delta(x+L)-\delta(x)+\delta(x-L))\sin\frac{n\pi x}{L}dx$$

$$=\frac{1}{L}\left(\int_{-L}^{L}\delta(x+L)\sin\frac{n\pi x}{L}dx-\int_{-L}^{L}\delta(x)\sin\frac{n\pi x}{L}dx+\int_{-L}^{L}\delta(x-L)\sin\frac{n\pi x}{L}dx\right)=\frac{1}{L}(\sin(n\pi))=0$$

$$g(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\longrightarrow a_0=b_n=0\quad a_n=\begin{cases}0 & n\in E\\ -\frac{2}{L} & n\in O\end{cases}$$

$$\Longrightarrow g(x)=\sum_{\text{فرد } n}-\frac{2}{L}\cos\frac{n\pi x}{L}\rightarrow g(x)=-\frac{2}{L}\left(\cos\frac{\pi x}{L}+\cos\frac{3\pi x}{L}+\cos\frac{5\pi x}{L}+\cdots\right)$$

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12)

$h(x) = (x+\cos^2 x)^2 + x - x\cos 2x \xrightarrow[\cos^2 x = \frac{1+\cos 2x}{2}]{} h(x) = x^2 + 2x + \frac{3}{8} + \cos 2x + \frac{\cos 4x}{8}$

$a_0 = \frac{1}{\pi}\int_0^\pi h(x)\,dx = \frac{1}{\pi}\int_0^\pi \left(x^2 + 2x + 3/8 + \cos 2x + \frac{\cos 4x}{8}\right)dx = \frac{\pi^2}{3} + \pi + \frac{3}{8}$

دوره تناوب $T = \pi$ داریم پس سری فوریه باید به صورت : $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi}{T}nx + b_n \sin\frac{2\pi}{T}nx$ باشد

$a_n = \frac{2}{\pi}\int_0^\pi h(x)\cos 2nx\,dx = \frac{2}{\pi}\Big(\int_0^\pi x^2\cos 2nx\,dx + \underbrace{\int_0^\pi 2x\cos 2nx\,dx}_{0} + \underbrace{\int_0^\pi \frac{3}{8}\cos 2nx\,dx}_{0} + \underbrace{\int_0^\pi \cos 2x\cos 2nx}_{0}$

$+ \underbrace{\int_0^\pi \frac{\cos 4x}{8}\cos 2nx\,dx}_{0}\Big) = \frac{1}{n^2} + 0 + 0 + 0 + 0 \longrightarrow a_n = \frac{1}{n^2}$

$b_n = \frac{2}{\pi}\int_0^\pi h(x)\sin 2nx\,dx = \frac{2}{\pi}\Big(\int_0^\pi x^2\sin 2nx\,dx + \int_0^\pi 2x\sin 2nx\,dx + \underbrace{\int_0^\pi \frac{3}{8}\sin 2nx}_{0}$

$+ \underbrace{\int_0^\pi \cos 2x\sin 2nx\,dx}_{0} + \underbrace{\int_0^\pi \frac{\cos 4x}{8}\sin 2nx\,dx}\Big) = \frac{-\pi}{n} - \frac{2}{n} + 0 + 0 + 0$

$b_n = \frac{-\pi - 2}{n}$

$h(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos 2nx + b_n\sin 2nx$

$\longrightarrow h(x) = \frac{\pi^2}{3} + \pi + \frac{3}{8} + \sum_{n=1}^{\infty} \frac{\cos 2nx}{n^2} - \frac{(\pi+2)}{n}\sin 2nx$