تاریخ تحویل: 99/8/26 | آمار و احتمال مهندسی — تمرین سری پنجم | مدرس: ... — شماره دانشجویی: 810198472

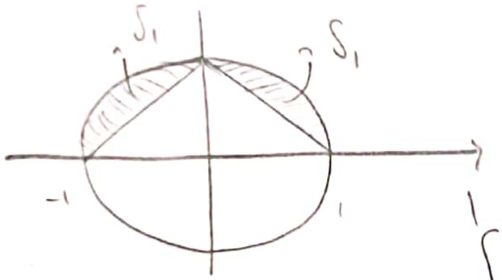
2 - الف)

$$f_{XY}(x,y)=\alpha \rightarrow \int_{-1}^{1}\int_{0}^{1-x^2} \alpha \, dy\,dx = 1 \rightarrow \alpha \int_{-1}^{1}(1-x^2)\,dx = 1$$

$$\rightarrow \frac{4}{3}\alpha = 1 \rightarrow \alpha = \frac{3}{4} \qquad f_{XY}(x,y)=\frac{3}{4}$$ (درداخل ناحیه مدنظر)

$$f_X(x)=\int_{0}^{1-x^2}\frac{3}{4}\,dy = \frac{3}{4}(1-x^2) \rightarrow f_X(x)=\begin{cases}\frac{3}{4}(1-x^2) & -1\le x\le 1\\ 0 & o.w\end{cases}$$

$$f_Y(y)=\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\frac{3}{4}\,d = \frac{3}{2}\sqrt{1-y} \rightarrow f_Y(y)=\begin{cases}\frac{3}{2}\sqrt{1-y} & 0\le y\le 1\\ 0 & o.w\end{cases}$$



ب) با رسم شکل خواهیم داشت:

به دلیل تقارن یک سمت را حساب و ضرب در 2 می کنیم

$$\int_{0}^{1}\int_{1-x}^{1-x^2}\frac{3}{4}\,dy\,dx = \frac{3}{4}\int_{0}^{1}(x-x^2)\,dx = \frac{3}{4}\times\frac{1}{6}=\frac{1}{8}$$

$$Ans = \iint_{S_1} + \iint_{S_2} = 2\times\frac{1}{8}=\frac{1}{4}$$

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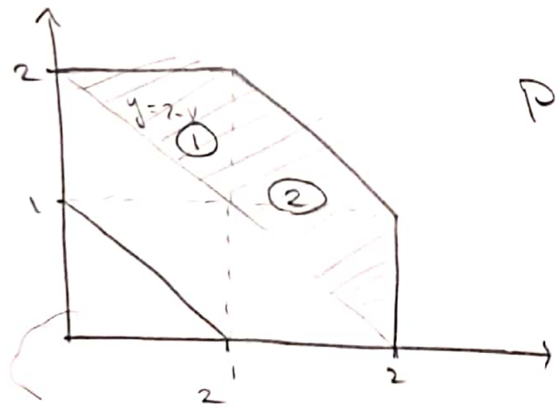
4 - الف)



$$P_r\{X+Y\ge 2\} = \int_{0}^{1}\int_{2-x}^{2}\frac{x}{3}\,dy\,dx + \int_{1}^{2}\int_{2-x}^{3-x}\frac{x}{3}\,dy\,dx$$

$$\frac{x^3}{9}\Big|_0^1 + \frac{x^2}{6}\Big|_1^2 = \frac{1}{2}+\frac{1}{9}=\frac{11}{18}$$

ب)

$$f_X(x)=\begin{cases}\int_{1-x}^{2}\frac{x}{3}\,dy = \frac{x}{3}(x+1) & 0\le x\le 1\\ \int_{0}^{-x+3} x/3\,dy = x/3\,(3-x) & 1\le x\le 2\\ 0 & o.w\end{cases}$$

$0 \le y \le 1 \rightarrow f_Y(y) = \int_{y-1}^{2} \frac{x}{3}\,dx = \frac{4-(y-1)^2}{6}$ $\quad 1 \le y \le 2 \rightarrow f_Y(y) = \int_{3-y}^{} \frac{x}{3}\,dx = \frac{(y-3)^2}{6}$

$$\rightarrow f_Y(y) = \begin{cases} \frac{4-(y-1)^2}{6} & 0 \le y \le 1 \\ \frac{(y-3)^2}{6} & 1 \le y \le 2 \\ 0 & o.w \end{cases}$$

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۶ ـ الف)

$$P_{XY}[i,j] = \begin{cases} \frac{c}{2^{\min(i,j)}} & i,j \ge 0,\ |i-j| \le 1 \\ 0 & o.w \end{cases}$$

زوج های مرتب برقرار را در نظر می‌گیریم

$(i,j) \rightarrow (0,0) \quad (0,1) \quad (1,0) \quad (1,1) \quad (1,2) \quad (2,1)$

$\qquad\qquad\ \ c \qquad\ \ c \qquad\ \ c \qquad \frac{c}{2} \qquad \frac{c}{2} \qquad \frac{c}{2}$

$$\rightarrow \sum_{all\,j}\sum_{all\,i} P_{XY}[i,j] = 1 \rightarrow c+c+c+\frac{c}{2}+\frac{c}{2}+\frac{c}{2}+\dots = 0$$

$$\rightarrow 3c + \frac{3c}{2} + \dots + = 0 \rightarrow \frac{3c}{1-1/2} = 1 \rightarrow 6c = 1 \rightarrow c = \frac{1}{6}$$

با توجه به شرط $|i-j| \le 1$ عملا $P_Y[k]$ و $P_X[k]$ باهم برابرند، یکی را حل کنیم انگار دیگری را نیز داریم.

$$P_X[k] = \sum_{all\,k} P_{XY}[k,j]$$

$$\text{if } k=0 \rightarrow (0,1)+(0,0) = 2c = \frac{1}{3} \qquad \begin{cases} k=1 \rightarrow (1,0)+(1,1)+(1,2) = c+\frac{c}{2}+\frac{c}{2} \\ k=2 \rightarrow (2,1)+(2,2)+(2,3) = \frac{c}{2}+\frac{c}{4}+\frac{c}{4} \\ \vdots \end{cases}$$

$$\text{for } k \ge 1 \rightarrow \frac{c}{2^{k-1}} + \frac{2c}{2^k} = \frac{c}{2^{k-2}}$$

مدل سازی عمل بالا به نتیجه زیر را خواهد داد

$$\rightarrow c = \frac{1}{6} \rightarrow \frac{1}{3 \times 2^{k-1}} = \frac{2}{3}\,2^{-k} \rightarrow P_X[k] = P_Y[k] = \begin{cases} \frac{1}{3} & k=0 \\ \frac{2}{3}\,2^{-k} & k \ge 1 \\ 0 & o.w \end{cases}$$

ب) $Pr\{X=Y\} \to Pr\{X=i, Y=j\} \to i=j$

$(0,0)+(1,1)+(2,2)+\cdots = c+\frac{c}{2}+\frac{c}{4}+\cdots \to \frac{c}{1-\frac{1}{2}}=2c=\frac{1}{3}$

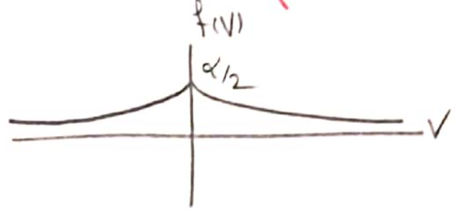
$Pr\{1\le X^2+Y^2\le 5\} \to Pr\{X=i, Y=j\} \quad 1\le i^2+j^2\le 5$

$\to (0,1)+(1,0)+(1,1)+(2,1)+(1,2) = c+c+\frac{c}{2}+\frac{c}{2}+\frac{c}{2}=\frac{7c}{2}=\frac{7}{12}$

8- الف)

$Y=X+V \qquad k=1 \to Pr\{Y\le 1, X=1\} = Pr\{1+V\le 1\}$

$=\frac{1}{2}Pr\{V\le 0\} = \frac{1}{2}\int_{-\infty}^{0}\frac{\alpha}{2}e^{-\alpha|v|}dv = \frac{\alpha}{4}\int_{-\infty}^{0}e^{\alpha v}dv$



$=\frac{1}{4}e^{\alpha v}\Big|_{-\infty}^{0}=\frac{1}{4}$

$k=-1 \to Pr\{Y\le -1, X=-1\} = \frac{1}{2}Pr\{V-1\le -1\} = Pr\{V\le 0\}=\frac{1}{4}$

$Pr\{X=1, Y\le -1\}+Pr\{X=1, Y\le 1\}=\frac{1}{2}$

ب) مشابه آن را می‌گیریم.

10- بسط تیلور $M_X(s)$ را می‌نویسیم --

$M_X(s)=M_X(0)+M'_X(0)s+\frac{M''_X(0)s^2}{2!}+\cdots$

$M_X(s)=1-\frac{s}{2}+\frac{s^2}{2\times 2!}-\frac{s^3}{2\times 3!}+\frac{s^4}{2\times 4!}\cdots=\frac{1}{2}+\frac{1}{2}-\frac{s}{2}+\frac{s^2}{2\times 2!}-\frac{s^3}{2\times 3!}+\frac{s^4}{2\times 4!}$

$=\frac{1}{2}+\frac{1}{2}\underbrace{\left(1-s+\frac{s^2}{2!}-\frac{s^3}{3!}+\cdots\right)}_{e^{-s}}=\frac{1}{2}+\frac{e^{-s}}{2}\xrightarrow{\mathcal{L}^{-1}}\mathcal{L}^{-1}\left(\frac{1}{2}\right)+\mathcal{L}^{-1}\left(\frac{e^{-s}}{2}\right)$

$=\frac{1}{2}\delta[k]+\frac{1}{2}\delta[k+1]\xrightarrow{\int}F_X[k]=\frac{1}{2}U[k]+\frac{1}{2}U[k+1]$

12- الف)

$M_{XY}(s_1,s_2)=\frac{2}{(s_1-1)(s_1+s_2-2)} \qquad m_{kr}=\frac{\partial^{k+r}}{\partial s_1^k\partial s_2^r}M_{XY}(s_1,s_2)$

$k=1, r=0 \to \eta_X=\frac{-2(2s_1+s_2-3)}{(s_1-1)^2(s_1+s_2-2)^2}\Big|_{s_1=s_2=0}=\frac{3}{2} \qquad k=2, r=0 \quad E\{X^2\}=\frac{4(3s_1^2+(3s_2-9)s_1+s_2^2-5s_2+7)}{(s_1-1)^3(s_1+s_2-2)^3}$

$\Big|_{s_1=s_2=0} \to E\{X^2\} = 7/2 \to \sigma_X^2 = \frac{7}{2} - \frac{9}{4} = \frac{5}{4} \to \sigma_X = \frac{\sqrt{5}}{2}$

$\begin{matrix} r=1 \\ k=0 \end{matrix} \to \eta_y = \frac{-2}{(S_1-1)(S_2+S_1-2)^2}\Big|_{S_1=S_2=0} = \frac{1}{2} \qquad \begin{matrix} r=2 \\ k=0 \end{matrix} \to E\{y^2\} = \frac{4}{(S_1-1)(S_2+S_1-2)^3} = \frac{1}{2}$

$\sigma_y^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \to \sigma_y = \frac{1}{2} \qquad \sigma_{xy} = E\{xy\} - \eta_x \eta_y$

$E\{xy\} = \frac{\partial^2}{\partial s_1 \partial s_2} M_{xy}(s_1, s_2) = \frac{2(3S_1+S_2-4)}{(S_1-1)^2(S_1+S_2-2)^3} = \frac{-8}{-8} = 1$

$\to \sigma_{xy} = 1 - \frac{3}{4} = \frac{1}{4} \qquad r_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{\frac{1}{4}}{\frac{\sqrt{5}}{4}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

ب)

$M_{xy}(S_1, S_2) = \frac{2}{(S_1-1)(S_1+S_2-2)} \qquad Z = X - Y \to X = Z + Y$

$M_Z(S) = E\{e^{SX - SY}\} = M_{XY}(S, -S) = \frac{2}{(S-1)(-2)} = \frac{1}{-S+1}$

اگر Y و Z مستقل باشند، می‌دانیم $X = Z + Y$، باید $M_X(S) = M_Z(S) M_Y(S)$ باشد. این شرط را بررسی می‌کنیم

$M_X(S) = M_{XY}(S, 0) = \frac{2}{(S-1)(S-2)}$ توزیع حاشیه‌ای

$M_Z(S) = \frac{1}{-S+1} \qquad M_Y(S) = M_{XY}(0, S) = \frac{2}{2-S}$

$\to M_Y(S) M_Z(S) = \frac{1}{(1-S)} \cdot \frac{2}{2-S} = \frac{(-1)\times(-1)\times 2}{(S-1)(S-2)} = \frac{2}{(S-1)(S-2)} = M_X(S)$

Y و Z مستقل‌اند.

8-ب) در اینجا باید از استقلال بین X و V استفاده کنیم.

$Y = X + V$

$\text{if } x=-1 \to f_X(x) = \frac{\delta(x+1)}{2} \quad (P\{X=1\} \cdot \delta(x-x_i)) \qquad f_Y(y) = f_V(y+1) = \frac{1}{2}\alpha e^{-\alpha|y+1|}$

$\to f_{XY}(x,y) = \frac{1}{4}\alpha e^{-\alpha|y+1|}\delta(x+1)$ ① $\quad \text{if } x=1 \to f_X(x) = \frac{\delta(x-1)}{2} \quad f_Y(y) = f_V(y-1) = \frac{1}{2}\alpha e^{-\alpha|y-1|}$

$\to f_{XY}(x,y) = \frac{1}{4}\alpha e^{-\alpha|y-1|}\delta(x-1)$ ② $\quad \to f_{XY}(x,y) = ① + ② = \frac{1}{4}\alpha e^{-\alpha|y+1|}\delta(x+1) + \frac{1}{4}\alpha e^{-\alpha|y-1|}\delta(x-1)$