Stochastic Processes

University of Tehran

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Homework 5

Due : 1401/9/8

Problem 1

Let $X(t)$ be a Markov stochastic process and assume $t_1 \leq t_2 \leq \cdots \leq t_n$. Prove that:

$$f_{X_1}\Big(x_1; t_1 \mid X(t_2) = x_2, \ldots, X(t_n) = x_n\Big) = f_{X_1}\Big(x_1; t_1 \mid X(t_2) = x_2\Big)$$

Problem 2

Let $X(t)$ be Poisson process with uniform density λ. The process $X(t)$ jumps at Poisson points $t_i \geq 0, i = 1, 2, \ldots$. Find the pdf of the random variable $T_n = t_{i+n} - t_i$ for $n \geq 1$.

Problem 3

Let $W(t) = \int_0^t X(\alpha) d\alpha$ be a Wiener process where $X(t)$ is a zero-mean stationary white Gaussian process with $R_X(\tau) = N_0 \delta(\tau)$. Suppose we want to estimate $W(2)$ given $W(1)$ by MMSE criterion. Find the estimator and its mean square error.

Problem 4

Consider the Wiener process $W(t)$ of problem 3, and define $Y(t) = W^2(t)$.

(a) Find the pdf $f_Y(y;t)$.

(b) Is $Y(t)$ an independent increment process? Explain.

Problem 5

The stochastic process $X(t)$ is defined as

$$X(t) = A\cos(2\pi Ft + \theta)$$

where A is a constant, F is a uniform random variable on $[2, 4]$, and θ is a uniform random variable on $[0, 2\pi]$ and independent of F. Find the mean and the autocorrelation function of $X(t)$. Is $X(t)$ wide sense stationary?

Problem 6

Let $X(t)$ be a zero-mean stationary (WSS) Gaussian process with autocorrelation function $R_X(\tau)$. Define the stochastic process $Y(t) = Ae^{jX(t)}$, where A is a Poisson random variable with parameter a and independent of $X(t)$. Find the mean and the autocorrelation function of $Y(t)$. Is $Y(t)$ wide sense stationary?

Problem 7

Let $X(t)$ be a zero-mean stationary (WSS) Gaussian process with autocorrelation $R_X(\tau) = sinc^2(\tau)$. Suppose that $X_1 = X(0), X_2 = X(\frac{1}{2})$ and $X_3 = X(1)$.

(a) Determine the value of $Y = E\{X_3 \mid X_2\}$.

(b) Find the value of $Pr\{|X_1 + 3Y| > 1\}$.

(c) Define

$$\begin{cases} Z_1(t) = X_1\cos(t) + X_3\sin(t) \\ Z_2(t) = X_1\sin(t) + X_3\cos(t) \end{cases}$$

Find the autocorrelation and cross-correlation functions of $Z_1(t)$ and $Z_2(t)$. Are $Z_1(t)$ and $Z_2(t)$ jointly wide sense stationary? Are they individually wide sense stationary?

(d) Find the pdf of $Z_3(t) = X_3 t + X_2$.

(e) Find the variance of $\frac{1}{n}\sum_{k=1}^{n} X\left(\frac{k}{2}\right)$.

Problem 8

Let $U(n), n \in \mathbb{Z}$ be an i.i.d. sequence of Gaussian random variables, with zero mean and unit variance. Let $X(t)$ denote the continuous-time random process obtained by linearly interpolating between the U's, i.e. $X(t) = U(t)$ for any $t = n \in \mathbb{Z}$, and $X(t)$ is affine on each interval of the form $[n, n+1]$ for $n \in \mathbb{Z}$.

(a) Find and sketch the first order marginal density $f_X(x;t)$.

(b) Is the random process $X(t)$ wide sense stationary? Justify your answer.