Stochastic Processes

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Homework 4

Due : 1401/8/24

Problem 1

Let $\{Z_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. random variables where each Z_k is Gaussian with zero mean and unit variance. Define:

$$X_k = 0.5X_{k-1} + Z_k,\ k = 1, 2, \ldots$$

and assume X_0 to be independent of the Z_ks.

(a) Find a distribution for X_0 to make every X_k to have the same distribution.

(b) Find $E\{X_{k+n}X_k\}$ as a function of n.

Problem 2

Consider the probability space (Ω, F, P) with $\Omega = [0, 1]$ and $P\{(a, b]\} = b - a$. Determine in what sense the following random sequences converge, and what are their limits.

(a) $X_n(\omega) = e^{-n\omega},\ n \geq 0.$

(b) $X_n(\omega) = sin\left(\omega + \frac{1}{n}\right),\ n \geq 1.$

(c) $X_n(\omega) = cos^n(\omega),\ n \geq 0.$

Problem 3

The members of the sequence of jointly independent random variables X_n have pdfs of the form:

$$f_{X_n}(x) = \left(1 - \frac{1}{n}\right)\frac{1}{\sigma\sqrt{2\pi}} exp\left[\frac{-1}{2\sigma^2}\left(x - \frac{n-1}{n}\sigma\right)^2\right] + \frac{1}{n}\sigma exp(-\sigma x)u(x)$$

Determine whether or not the random sequence $\{X_n\}$ converges in:

(a) the mean square sense,

(b) probability,

(c) distribution.

Problem 4

Let $\{X_n\}$ be a sequence of random variables that converges in probability to a random variable X, i.e.

$$X_n \xrightarrow{p} X$$

Assume that the pdfs $f_{X_n}(x)$ of X_ns are such that for some $N > 0$, $f_{X_n}(x) = 0$, for $|x| > x_0$ and for all $n > N$. Show that X_n also converges to X in mean square sense.

Problem 5

Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables, each with uniform distribution on the interval $[0, 1]$. Define the sequence $\{Y_n\}_{n=1}^{\infty}$ as:

$$Y_n = n\big(1 - max(X_1, X_2, \ldots, X_n)\big), \quad \text{for} \quad n = 1, 2, \ldots$$

Let Y be an exponential random variable with parameter $\lambda = 1$, i.e. pdf of $f_Y(y) = e^{-y}u(y)$. Prove that the sequence $\{Y_n\}$ converges to Y in distribution, i.e.

$$Y_n \xrightarrow{dist} Y.$$

Problem 6

Suppose $\{W_k\}_{k=1}^{\infty}$ are independent Gaussian random variables with mean zero and variance $\sigma^2 > 0$. Define the sequence $\{X_k\}_{k=1}^{\infty}$ recursively by $X_0 = 0$ and $X_k = \frac{1}{2}(X_{k-1} + W_k)$. Determine in what senses, m.s., p. and d., the sequence $\{X_k\}_{k=1}^{\infty}$ converges.

Problem 7

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of continuous random variables each with the CDF

$$F_{X_n}(x) = \frac{e^{nx}}{1 + e^{nx}}, \quad x \in \mathbb{R}$$

(a) Find $\lim_{n \to +\infty} F_{X_n}(x)$. Can the result be considered as a CDF?

(b) Does $\{X_n\}_{n=1}^{\infty}$ converge in distribution? If yes, find the random variable that it converges to. If no, explain your reasoning.