1	تاریخ تحویل: 01 / 10 / 12	نام درس: فرآیندهای تصادفی تکلیف سری دهم	سروش مس‌فروش مشهد شماره دانشجویی: 810198472

مسئله 3.

$X(t) = A\cos(2\pi Y t + \Theta)$, $A>0$ known constant, $Y \sim f_Y(y)$

$\Theta \sim f_\Theta(\theta) = 1/2\pi \quad 0\le\theta\le 2\pi$, $S_X(f) = ?$

درگام اول باید همبستگی حساب کنیم.

$$R_X(t+\tau,t) = E\{X(t+\tau)X^*(t)\} \longrightarrow E\{X(t+\tau)X(t)\}$$

$$= E\{A\cos(2\pi Y(t+\tau)+\theta)\,A\cos(2\pi Y t+\theta)\} = A^2 E\{\cos(2\pi Y(t+\tau)+\theta)\cos(2\pi Y t+\theta)\} = A^2 E\{\tfrac{1}{2}[\cos(4\pi Y t + 2\pi Y\tau + 2\theta) + \cos(2\pi Y\tau)]\}$$

$$= \frac{A^2}{2} E\{\cos(4\pi Y t + 2\pi Y\tau + 2\theta) + \cos(2\pi Y\tau)\}$$

حال باید از عبارت فوق میانگین زمانی بگیریم.

$$\overline{R_X(t+\tau,t)} = \langle R_X(t+\tau,t)\rangle$$

$$= \frac{A^2}{2} E\{\underbrace{\overset{0}{\langle \cos(4\pi Y t + 2\pi Y\tau + 2\theta)\rangle}}_{\text{تابع برحسب t است.}} + \langle\cos(2\pi Y\tau)\rangle\}$$

$$= \frac{A^2}{2} E\{\cos(2\pi Y\tau)\} , \qquad S_X(f) = \mathcal{F}(\overline{R_X(t+\tau,t)})$$ طبق Wiener-Khinchin:

$$\longrightarrow S_X(f) = \frac{A^2}{2} E\{\mathcal{F}[\cos(2\pi Y\tau)]\} = \frac{A^2}{4} E\{\delta(f-Y) + \delta(f+Y)\}$$

$$E\{\delta(f-Y)\} = \int_{-\infty}^{\infty} \delta(f-y) f_Y(y)\,dy = f_Y(f), \quad E\{\delta(f+Y)\} = \int_{-\infty}^{\infty} \delta(f+y) f_Y(y)\,dy$$

$$= f_Y(-f), \longrightarrow S_X(f) = \frac{A^2}{4}[f_Y(f) + f_Y(-f)]$$

مسئله 4.

a) $$S_X(f) = \frac{4(\pi^2 f^2 + 1)}{(4\pi^2 f^2 + 1)(4\pi^2 f^2 + 9)} \qquad s = j2\pi f$$

$$\longrightarrow S_X(s) = \frac{4 - s^2}{(1-s^2)(9-s^2)} \longrightarrow S_X(s) = \frac{(2+s)(2-s)}{(1+s)(1-s)(3+s)(3-s)}$$

$$X(t) \longrightarrow \underset{\text{Whitening}}{\boxed{\Gamma(s)}} \longrightarrow \underset{\text{Innovation}}{\boxed{L(s) = \frac{1}{\Gamma(s)}}} \longrightarrow X(t)$$ دیاگرام متناظر را در نظر بگیرید:

$$L(s) = \frac{(2+s)}{(1+s)(3+s)} , \qquad S_X(s) = L(s)L^*(-s) ,$$

$$L(s) = \frac{0.5}{1+s} + \frac{0.5}{3+s} \xrightarrow{\mathcal{L}^{-1}} l(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

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b.1) $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ $\quad R_Y(\tau) = e^{-|\tau|}$

$S_Y(f) = \mathcal{F}[e^{-|t|}] = \frac{2}{1+4\pi^2 f^2}$, $S_Y(f) = |H(f)|^2 S_X(f)$

$\longrightarrow |H(f)|^2 = \frac{S_Y(f)}{S_X(f)} = \dfrac{\frac{2}{1+4\pi^2 f^2}}{\frac{4\pi^2 f^2+4}{(4\pi^2 f^2+1)(4\pi^2 f^2+9)}} \longrightarrow |H(f)|^2 = \frac{9+4\pi^2 f^2}{2(1+4\pi^2 f^2)}$

$\longrightarrow |H(f)|^2 = \frac{(3+j2\pi f)(3-j2\pi f)}{2(1+j2\pi f)(1-j2\pi f)}$ $\xrightarrow{\text{یک نمونه علّی}}$ $H(f) = \frac{3+j2\pi f}{\sqrt{2}(1+j2\pi f)}$

$\longrightarrow H(f) = \frac{3}{\sqrt{2}(1+j2\pi f)} + \frac{j2\pi f}{\sqrt{2}(1+j2\pi f)}$, $h(t) = \mathcal{F}^{-1}[H(f)]$

$\longrightarrow h(t) = 3\sqrt{2}e^{-2t}u(t) + \mathcal{F}^{-1}\{\frac{j2\pi f}{\sqrt{2}(1+j2\pi f)}\}$, می‌دانیم: $x'(t) \xleftrightarrow{\mathcal{F}} j2\pi f X(f)$

$\longrightarrow \mathcal{F}^{-1}[\frac{j2\pi f}{\sqrt{2}(1+j2\pi f)}] = \frac{\partial}{\partial t}\mathcal{F}^{-1}[\frac{1}{\sqrt{2}(1+j2\pi f)}] = \frac{\partial}{\partial t}[\sqrt{2}e^{-2t}u(t)] = -2\sqrt{2}e^{-t}u(t) + \sqrt{2}\delta(t)$

$\longrightarrow h(t) = 3\sqrt{2}e^{-2t}u(t) - 2\sqrt{2}e^{-2t}u(t) + \sqrt{2}\delta(t) \longrightarrow h(t) = \sqrt{2}e^{-2t}u(t) + \sqrt{2}\delta(t)$

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a.1) سوال 6.

$X \sim U(-2,3)$, $f_X(x) = \frac{1}{5} \quad -2<x<3$

$X(t) = e^{-X}\cos(2\pi X t)$, $\quad R_X(t+\tau,t) = E\{X(t+\tau)X(t)^*\} = E\{X(t+\tau)X(t)\}$

$= E\{e^{-X}\cos(2\pi X(t+\tau))e^{-X}\cos(2\pi X t)\} = \frac{1}{2}E\{e^{-2X}(\cos(4\pi Xt+2\pi X\tau)+\cos(2\pi X\tau))\}$

$\longrightarrow R_X(t+\tau,t) = \frac{1}{2}E\{e^{-2X}(\cos(4\pi Xt+2\pi X\tau)+\cos(2\pi X\tau))\}$

$\langle R_X(t+\tau,t)\rangle = \frac{1}{2}E\{\langle e^{-2X}(\cos(4\pi Xt+2\pi X\tau)+\cos(2\pi X\tau))\rangle\}$

$= \frac{1}{2}E\{\langle e^{-2X}\cos(2\pi X\tau)\rangle\} = \frac{1}{2}E\{e^{-2X}\cos(2\pi X\tau)\}$

$\overline{R_X(t+\tau,t)} = \frac{1}{2}E\{e^{-2X}\cos(2\pi X\tau)\} = \frac{1}{10}\int_{-2}^{3} e^{-2x}\cos(2\pi x\tau)\,dx$

Wiener-Khinchin: $S_X(f) = \mathcal{F}[\overline{R_X(\tau)}] = \mathcal{F}[R_X(\tau)] = \frac{1}{10}\int_{-2}^{3} e^{-2x}\mathcal{F}[\cos(2\pi x\tau)]\,dx$

$= \frac{1}{20}\int_{-2}^{3} e^{-2x}[\delta(f-x)+\delta(f+x)]\,dx = \frac{1}{20}\int_{-2}^{3} e^{-2x}\delta(f-x)\,dx + \frac{1}{20}\int_{-2}^{3} e^{-2x}\delta(f+x)\,dx$

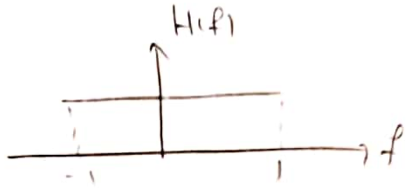
انتگرال های فوق با توجه به بازه‌های مختلف f پاسخ های متفاوتی می‌دهند. لذا خواهیم داشت:

if $2<f<3 \to S_X(f)=\frac{1}{20}e^{-2f}$ if $-2<f<2 \to S_X(f)=\frac{1}{20}e^{-2f}+\frac{1}{20}e^{2f}$. if $-3<f<-2 \to S_X(f)=\frac{1}{20}e^{2f}$

پس در نهایت خواهیم داشت:

$$S_X(f)=\begin{cases}\frac{1}{20}e^{-2f} & 2<f<3\\ \frac{1}{20}e^{-2f}+\frac{1}{20}e^{2f} & -2<f<2\\ \frac{1}{20}e^{2f} & -3<f<-2\end{cases}$$

b.) $X(t)\longrightarrow\boxed{LPF}\longrightarrow Y(t)$



می‌دانیم: $S_Y(f)=S_X(f)\,|H(f)|^2 \to S_Y(f)=\frac{1}{20}(e^{-2f}+e^{2f})=\frac{1}{10}\cosh(2f)\quad |f|<1$

$$P_Y=\int_{-\infty}^{\infty}S_Y(f)\,df=\int_{-1}^{1}\frac{1}{10}\cosh(2f)\,df=\frac{1}{20}\sinh(2f)\Big|_{-1}^{1}$$

$$\Rightarrow P_Y=\frac{\sinh(2)-\sinh(-2)}{20}=\frac{\sinh(2)}{10}\approx 0.36268604078$$

مسئله 7.

$X(t)\to m_X=0,\quad R_X(\tau)=e^{-|\tau|}$

$$Y(t)=\int_0^2 X(t-s)\,ds=\int_{-\infty}^{\infty}\Pi(s-1)\,X(t-s)\,ds\ ,\quad Y(t)=\Pi(t-1)*X(t)$$

$X(t)\longrightarrow\boxed{h(t)}\longrightarrow Y(t)$, $h(t)=\Pi(t-1)$, $S_Y(f)=|H(f)|^2S_X(f)$

$$S_X(f)=\mathcal{F}[R_X(\tau)]=\frac{2}{1+4\pi^2f^2}\ ,\quad H(f)=\mathcal{F}[\Pi(t-1)]=e^{-j2\pi f}\mathcal{F}[\Pi(t)]$$

$$\to H(f)=e^{-j2\pi f}\operatorname{sinc}(f)\ ,\quad |H(f)|^2=\operatorname{sinc}^2(f)\ ,\quad S_Y(f)=\frac{2\operatorname{sinc}^2(f)}{1+4\pi^2f^2}$$

پس پاسخ نهایی مسئله به صورت زیر است:

$$S_Y(f)=\frac{2\operatorname{sinc}^2(f)}{1+4\pi^2f^2}$$

مسئله 2.

a.) $X(t)\to m_X=0$, Stationary and Gaussian Random process

$S_X(f)=\Lambda(f)\ ,\quad X'(t)=\frac{\partial}{\partial t}X(t)\qquad \underline{X}=\begin{bmatrix}X(t)\\X'(t)\\X(t-1)\end{bmatrix}\qquad m_{\underline{X}}=E\{\underline{X}\}$

$$=\begin{bmatrix}E\{X(t)\}\\E\{X'(t)\}\\E\{X(t-1)\}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\qquad \overline{R_X(\tau)=R_X(\tau)}\ ,\quad S_X(f)=\mathcal{F}[R_X(\tau)]=\Lambda(f)$$

می‌دانیم: $\mathcal{F}[\operatorname{sinc}^2(t)]=\Lambda(f)\ ,\longrightarrow R_X(\tau)=\operatorname{sinc}^2(\tau)$

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$X'(t): X(t) \longrightarrow \boxed{h(t)} \longrightarrow X'(t) \qquad h(t) = \delta'(t) \,,\; S_{X'}(f) = |H(f)|^2 S_X(f)$

$\rightarrow S_{X'}(f) = 4\pi^2 f^2 \Lambda(f) \qquad H(f) = j2\pi f$ در ادامه به ماتریس کوواریانس نیاز داریم:

$C_X = [C_{ij}]_{n\times n} \qquad C_{ij} = Cov(x_i, x_j) = E\{x_i x_j^*\} - m_{xi} m_{xj}$

$$\rightarrow C_X = \begin{bmatrix} E\{x_1^2\} & E\{x_1 x_2^*\} & E\{x_1 x_3^*\} \\ E\{x_2 x_1^*\} & E\{x_2^2\} & E\{x_2 x_3^*\} \\ E\{x_3 x_1^*\} & E\{x_3 x_2^*\} & E\{x_3^2\} \end{bmatrix}$$

می‌دانیم $E\{x_i^2\} = R_{x_i}(0)$

$E\{x_1^2\} = R_X(0) = \text{Sinc}^2(0) = 1$

تعریف: $X_1 = X(t),\; X_2 = X'(t),\; X_3 = X(t-1)$

$E\{x_2^2\} = \int_{-\infty}^{\infty} S_{X'}(f)\,df = \int_{-1}^{1} 4\pi^2 f^2 (1-|f|)\,df = \frac{2\pi^2}{3}$

$E\{x_3^2\} = E\{(X(t-1))^2\} = R_X(0) = 1 \,,\quad E\{x_i x_j^*\} = 0 \quad i \neq j$

$E\{x_1 x_2^*\} = \int_{-\infty}^{\infty} S_{XX'}(f)\,df = \int_{-\infty}^{\infty} S_X(f) H^*(f)\,df = \int_{-1}^{1} \underbrace{\Lambda(f)}_{\text{زوج}} \underbrace{(-j2\pi f)}_{\text{فرد}}\,df = 0$

$E\{x_1 x_3^*\} = E\{X(t-1)X(t)\} = R_X(t-t+1) = R_X(1) = 0$

$E\{x_2 x_3^*\} = E\{X'(t) X^*(t-1)\} = R_{X'X}(1) = 0$

$E\{x_i x_j^*\} = 0 \quad i \neq j$ با توجه به قضیه بالا * طبق اثبات قبل:

$$C_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2\pi^2}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

می‌دانیم: $f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}} |C_X|^{1/2}} \exp\left\{-\frac{1}{2}[x - m_X]^T C_X^{-1} [x - m_X]\right\}$

$$\rightarrow f_X(x) = \frac{1}{2\pi^{3/2}\sqrt{\frac{2\pi^2}{3}}} \exp\left\{-\frac{1}{2}\left([X(t), X'(t), X(t-1)] \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} X(t) \\ X'(t) \\ X(t-1) \end{bmatrix}\right)\right\}$$

معکوس C_X: $\det(C_X) = \frac{2\pi^2}{3} \qquad C_X^T = C_X \,,\quad Adj(C_X) = \begin{bmatrix} \frac{2\pi^2}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2\pi^2}{3} \end{bmatrix}$

$$\rightarrow C_X^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2\pi^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow f_X(x) = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{\frac{2\pi^2}{3}}} \exp\left\{-\frac{1}{2}\left[X^2(t) + \frac{3}{2\pi^2}X'^2(t) + X^2(t-1)\right]\right\}$$

b.) $X(t) \longrightarrow \boxed{h(t) = \delta'(t)} \longrightarrow X'(t)$ در قسمت a انجام شد.

$S_X(f) = \Lambda(f) \qquad H(f) = j2\pi f \,,\quad S_{X'}(f) = S_X(f)|H(f)|^2 = 4\pi^2 f^2 \Lambda(f)$

$S_{XX'}(f) = S_X(f) H^*(f) = -j2\pi f \Lambda(f)$

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c) $x'(t+\tau),\ x'(t)$ independent r.v.s $\to Cov(x'(t+\tau), x'(t)) = 0$, $m_{x'(t+\tau)} = m_{x'(t)} = 0$

$\to E\{x'(t'+\tau), x'(t)\} = 0 \to R_{x'}(\tau) = 0$, $R_{x'}(\tau) = R_x(\tau) * h(\tau) * h^*(-\tau)$

$h(t) = \delta'(t)$, $R_{x'}(\tau) = R_x(\tau) * \delta'(\tau) * \delta'(-\tau) = \frac{\partial^2}{\partial\tau^2} R_x(\tau)$, $R_x(\tau) = \text{sinc}^2(\tau)$

$\longrightarrow \frac{\partial^2}{\partial\tau^2}\text{sinc}^2(\tau) = 0$ If we solve the differential equation, the corresponding τs shall be found.

مسئله ۱

a) $x(t) = \sum_{k=-\infty}^{\infty} A_k P(t-kT)$ $A_k \in \{-1, 1\}$, $P(A_k=1) = P(A_k=-1) = 1/2$

$P(t)$, Energy signal pulse. $R_x(t+\tau, t) = E\{x(t+\tau)x^*(t)\}$

$$= E\left\{\sum_{k=-\infty}^{\infty} A_k P(t+\tau-kT) \sum_{l=-\infty}^{\infty} A_l^* P(t-lT)\right\} = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} E\{A_k A_l^*\} p(t+\tau-kT)p(t-lT)$$

$$R_A[k,l] = E\{A_k A_l^*\} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases} \to R_x(t+\tau, t) = \sum_{k=-\infty}^{\infty} p(t+\tau-kT)p(t-kT)$$

استفاده از این روابط کار را بسیار سخت می‌کند و راه به جایی نمی‌برد. از کانولوشن استفاده می‌کنیم.

$x(t) = \sum_{k=-\infty}^{\infty} A_k \delta(t-kT) * p(t)$ حال انگار طوری شده است که $\sum_{k=-\infty}^{\infty} A_k\delta(t-kT)$ از فیلتر $p(t)$ عبور داده شده است. لذا دیاگرام زیر را در نظر بگیرید.

$$\sum_{k=-\infty}^{\infty} A_k \delta(t-kT) \longrightarrow \boxed{p(t)} \longrightarrow x(t)$$

We know: $S_x(f) = |P(f)|^2 S_\alpha(f)$, $\alpha(t) \triangleq \sum_{k=-\infty}^{\infty} A_k \delta(t-kT)$

$$R_\alpha(t+\tau, t) = E\{\alpha(t+\tau)\alpha^*(t)\} = E\left\{\sum_{k=-\infty}^{\infty} A_k \delta(t+\tau-kT) \sum_{l=-\infty}^{\infty} A_l^* \delta(t-kl)\right\}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} E\{A_k A_l^*\}\delta(t+\tau-kT)\delta(t-kl), \quad \underbrace{E\{A_k A_l^*\}}_{R_A[k-l]} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

$k-l=i$

$$\to R_\alpha(t+\tau, t) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_A(i)\,\delta((l-k)T+\tau)\,\delta(t-lT)$$

$$\to R_\alpha(t+\tau, t) = \sum_{i=-\infty}^{\infty} R_A(i)\delta(\tau-iT) \sum_{l=-\infty}^{\infty}\delta(t-lT) \qquad \overline{R_\alpha(t+\tau, t)} = \langle R_\alpha(t+\tau, t)\rangle_T$$

$$\left\langle \sum_{l=-\infty}^{\infty}\delta(t-lT)\right\rangle = \frac{1}{T}\int_T \sum_{l=-\infty}^{\infty}\delta(t-lT)\,dt = 1/T.$$

$$\to R_\alpha(t+\tau, t) = \delta(\tau)\cdot 1/T = \frac{\delta(\tau)}{T}, \quad S_\alpha(f) = \mathcal{F}\left(\frac{\delta(\tau)}{T}\right) = \frac{1}{T}$$

$$S_x(f) = |P(f)|^2 S_\alpha(f) \to S_x(f) = \frac{|P(f)|^2}{T}$$

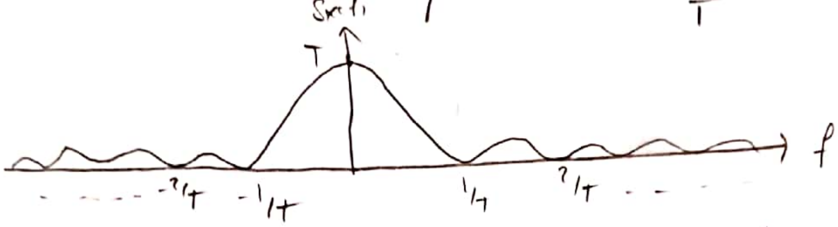
b.) $S_X(f) = \dfrac{|P(f)|^2}{T}$, $P(t) = \Pi\left(\dfrac{t-0.5T}{T}\right)$ $\quad \Pi\left(\dfrac{t-0.5T}{T}\right) = \begin{cases} 1 & 0<t<T \\ 0 & o.w \end{cases}$

$F\{P(t)\} = \int_{-\infty}^{\infty} p(t)\, e^{-j2\pi f t}\, dt = \int_0^T e^{-j2\pi f t}\, dt = \dfrac{-1}{j2\pi f} e^{-j2\pi f t}\Big|_0^T = \dfrac{1-e^{-j2\pi f T}}{j2\pi f}$

$= \dfrac{e^{-j\pi f T}}{\pi f}\left(\dfrac{e^{j\pi f T}-e^{-j\pi f T}}{2j}\right) = e^{-j\pi f T}\dfrac{\sin(\pi f T)}{\pi f}$ $\quad \operatorname{sinc}(x) \triangleq \dfrac{\sin(\pi x)}{\pi x}$

$\rightarrow P(f) = e^{-j\pi f T}\, T \operatorname{sinc}(fT)$, $\quad S_X(f) = \dfrac{|P(f)|^2}{T}$, $\quad \dfrac{T^2 \operatorname{sinc}^2(fT)}{T}$

$\rightarrow S_X(f) = T \operatorname{sinc}^2(fT)$



همچنین شکل با MATLAB رسم شده است.

مسأله 5.

a.) $X_1(t), X_2(t) \longrightarrow R_{X_1Y_2}(t_1,t_2)$

$X_1(t) \longrightarrow \boxed{h_1(t)} \longrightarrow Y_1(t) \qquad X_2(t) \longrightarrow \boxed{h_2(t)} \longrightarrow Y_2(t)$

نا ایستا: $\begin{cases} R_{XY}(t_1,t_2) = R_X(t_1,t_2) * h^*(t_2) \\ R_{YX}(t_1,t_2) = R_X(t_1,t_2) * h(t_1) \\ R_Y(t_1,t_2) = R_X(t_1,t_2) * h(t_1) * h^*(t_2) \end{cases}$ $\quad \xrightarrow{\langle\,\rangle} \quad \begin{cases} \overline{R_{XY}(\tau)} = \overline{R_X(\tau)} * h^*(-\tau) \\ \overline{R_{YX}(\tau)} = \overline{R_X(\tau)} * h(\tau) \\ \overline{R_Y(\tau)} = \overline{R_X(\tau)} * h(\tau) * h(-\tau) \end{cases}$

$R_{X_1Y_2}(t_1,t_2) = E\{X_1(t_1)Y_2^*(t_2)\}$, $Y_2(t_2) = \int_{-\infty}^{\infty} h_2(\alpha)\, X_2(t-\alpha)\, d\alpha$

$\rightarrow R_{X_1Y_2}(t_1,t_2) = E\left\{X(t_1)\left(\int_{-\infty}^{\infty} h_2(\alpha)\, X_2(t_2-\alpha)\, d\alpha\right)^*\right\} = E\left\{X(t_1)\int_{-\infty}^{\infty} h_2^*(\alpha)\, X_2^*(t_2-\alpha)\, d\alpha\right\}$

$= E\left\{\int_{-\infty}^{\infty} X_1(t_1)\, h_2^*(\alpha)\, X_2^*(t-\alpha)\, d\alpha\right\} = \int_{-\infty}^{\infty} \underbrace{E\{X_1(t_1)\, X_2^*(t_2-\alpha)\}}_{R_{X_1X_2}(t_1,t_2-\alpha)}\, h_2^*(\alpha)\, d\alpha$

$\rightarrow R_{X_1Y_2}(t_1,t_2) = \int_{-\infty}^{\infty} R_{X_1X_2}(t_1,t_2-\alpha)\, h_2^*(\alpha)\, d\alpha = R_{X_1X_2}(t_1,t_2) * h_2^*(t_2)$

, $R_{X_1Y_2}(t_1,t_2) = R_{X_1X_2}(t_1,t_2) * h^*(t_2)$, $\quad$ Wiener-Khinchin: $S_X(f) = F[\overline{R_X(\tau)}]$

$\overline{R_{X_1Y_2}(\tau)} = \langle R_{X_1Y_2}(t+\tau,t)\rangle = \overline{R_{X_1Y_2}(\tau)} * h^*(-\tau)$, $\quad S_{X_1Y_2}(f) = S_{X_1X_2}(f)\, H^*(f)$

b) $R_{Y_1Y_2}(t_1,t_2) = E\{Y_1(t_1)Y_2^*(t_2)\} = E\{\int_{-\infty}^{\infty} h_1(\alpha)X_1(t_1-\alpha)d\alpha \,(\int_{-\infty}^{\infty} h_2(\beta)X_2(t-\beta))^*\}$

$= E\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2^*(\beta)X_1(t-\alpha)X_2^*(t-\beta)\,d\alpha\, d\beta\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2^*(\beta)E\{X_1(t-\alpha)X_2^*(t-\beta)\}d\alpha\, d\beta$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2^*(\beta)\, R_{X_1X_2}(t_1-\alpha, t_2-\beta)\,d\alpha\, d\beta = \int_{-\infty}^{\infty}(R_{X_1X_2}(t_1-\alpha,t_2) * h_2^*(t_2))h_1(\alpha)d\alpha$

$= R_{X_1X_2}(t_1,t_2) * h_2^*(t_2) * h_1(t_1) \rightarrow R_{Y_1Y_2}(t_1,t_2) = R_{X_1X_2}(t_1,t_2) * h_1(t_1) * h_2^*(t_2)$

داریم: $\overline{R_{Y_1Y_2}(\tau)} = \overline{R_{X_1X_2}(\tau)} * h_1(\tau) * h_2^*(-\tau)$, Wiener-Khinchin: $S_X(f) = \mathcal{F}\overline{(R_X(\tau))}$

$\rightarrow S_{Y_1Y_2}(f) = S_{X_1X_2}(f)H_1(f)H_2^*(f) \rightarrow S_{Y_1Y_2}(f) = S_{X_1X_2}(f)H_1(f)H_2^*(f)$

c) $X(t_1)$ $X(t_2)$ jointly stationary $\rightarrow \begin{cases} R_{X_1X_2}(t_1,t_2) = R_{X_1X_2}(t_1-t_2) \\ X(t_1) \rightarrow \text{Stationary} \\ X(t_2) \rightarrow \text{Stationary} \end{cases}$

ابتدا ثابت می‌کنیم اگر یک فرآیند WSS را به یک سیستم اعمال کنیم، خروجی نیز WSS است.

$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$ $\quad X(t)$ is WSS : $\begin{cases} m_X(t) = m_X \\ R_X(t_1,t_2) = R_X(t_1-t_2) \end{cases}$

$m_Y(t) = m_X(t) * h(t) = \int_{-\infty}^{\infty} h(\alpha)m_X(t-\alpha)d\alpha = \int_{-\infty}^{\infty} h(\alpha)m_X\,d\alpha = m_X H(0) = m_Y$ مستقل از t

$R_Y(t_1,t_2) = E\{Y(t_1)Y^*(t_2)\} = E\{\int_{-\infty}^{\infty} h(\alpha)X(t_1-\alpha)d\alpha\,(\int_{-\infty}^{\infty} h(\beta)X(t_2-\beta)d\beta)^*\}$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h^*(\beta)E\{X(t_1-\alpha)X(t_2-\beta)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h^*(\beta)R_X(t_1-\alpha, t_2-\beta)$

$= R_X(t_1,t_2) * h(t_1) * h^*(t_2) = R_X(\tau) * h(\tau) * h^*(-\tau)$, $Y(t)$ is WSS

$R_Y(t_1,t_2) = R_{XY}(t_1,t_2) * h^*(t_2) = R_X(t_1,t_2) * h^*(t_2) * h(t_1)$ از قبل داریم:

$\rightarrow R_Y(t_1,t_2) = \int_{-\infty}^{\infty} h(\alpha)R_{XY}(t_1-\alpha,t_2)d\alpha = \int_{-\infty}^{\infty} h(\alpha)R_{XY}(t_1-\alpha-t_2)d\alpha \xrightarrow[t_2=t]{t_1=t+\tau}$

$R_Y(t+\tau,t) = \int_{-\infty}^{\infty} h(\alpha)R_{XY}(\tau-\alpha) \longrightarrow R_Y(\tau) = h(\tau) * R_{XY}(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau)$

پس $Y_1(t), Y_2(t)$ هردو WSS هستند. حال باید رابطه $R_{Y_1Y_2}(t_1,t_2) = R_{Y_1Y_2}(t_1-t_2)$ را بررسی نماییم.

$R_{Y_1Y_2}(t+\tau,t) = E\{Y_1(t+\tau)Y_2^*(t)\} = E\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2^*(\beta)X_1(t+\tau-\alpha)X_2^*(t-\beta)\,d\alpha\, d\beta\}$

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$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)\, h_2^*(\beta)\, E\{x_1(t+\tau-\alpha)\, x_2(t-\beta)\}\, d\alpha\, d\beta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)\, h_2^*(\beta)\, R_{x_1x_2}(t+\tau-\alpha, t-\beta)\, d\alpha\, d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)\, h_2^*(\beta)\, R_{x_1x_2}(\tau+\beta-\alpha)\, d\alpha\, d\beta = R_{x_1x_2}(\tau) * h_1(\tau) * h_2^*(-\tau)$$

پس نتیجه می‌گیریم که: $R_{y_1y_2}(t+\tau, t) = R_{y_1y_2}(\tau) \rightarrow y_1(t), y_2(t)$ are jointly WSS.

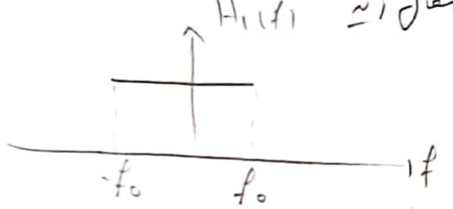
d.) $\forall f \quad S_{y_1y_2}(f) = 0 \quad h_1(t)$ & $h_2(t)$ are unknown.

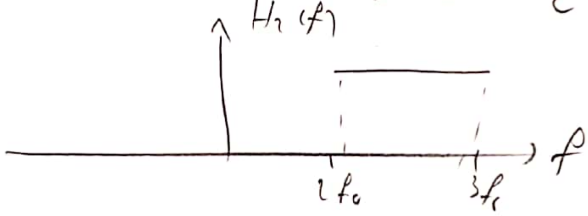
$$S_{y_1y_2}(f) = S_{x_1x_2}(f)\, H_1(f)\, H_2^*(f)$$

درستی b به دست آمد.

$$\rightarrow S_{x_1x_2}(f)\, H_1(f)\, H_2^*(f) = 0 \quad \forall f \qquad S_{x_1x_2}(f) \neq 0 \rightarrow H_1(f)\, H_2^*(f) = 0$$

وقتی چنین اتفاقی رخ می‌دهد یعنی فیلترها در حوزه فرکانس هم پوشانی ندارند. مثلاً اشکال زیر





پس به تعبیری فیلترهای $h_1(t)$, $h_2(t)$ دو فیلتر Orthogonal هستند

Question 1 Power Spectral Density plot

Let us imagine the following process.

$$X(t) = \sum_{k=-\infty}^{\infty} A_k p(t - kT), \quad p(t) = \Pi\left(\frac{t - 0.5T}{T}\right)$$

The code to plot is as follows.

```
clc
clear all
f=-10:0.01:10;
T = 1;% Can be changed as you wish.
SX = T.*sinc(f.*T).*sinc(f.*T);
plot(f,SX,'r')
grid on
title('Power Spectral Density of PAM for p(t) = Pi(t-0.5T/T)','Interpreter','latex')
xlabel('Frequency(Hertz)','Interpreter','latex');
ylabel('PSD','Interpreter','latex');

```

Figure 1: Code to plot PSD

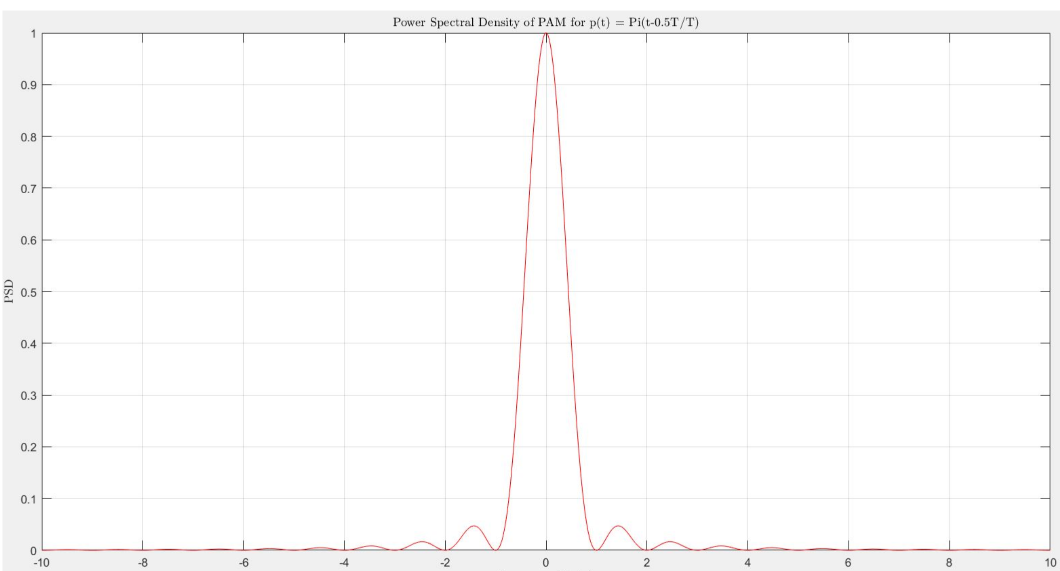


Figure 2: Power Spectral Density of PAM for p(t) $=\Pi\left(\frac{t-0.5T}{T}\right)$