Stochastic Processes

University of Tehran

Instructor: Dr. Ali Olfat — Fall 2022

Homework 8

Due : 1401/10/19

Problem 1

Let $X(t)$ be a WSS process with zero mean and autocorrelation function $R_X(\tau) = \Lambda(\tau)$. Find the KL expansion of $X(t)$ for $t \in [0, T]$, where $T \leqslant 1$ is known.

Problem 2

Let $x(n)$ be a sequence of i.i.d. random variables with $Pr\{x(n) = 1\} = Pr\{x(n) = -1\} = 0.5$. Define the process $y(n) = 0.8y(n-1) + x(n)$.

(a) Find the PSD of $y(n)$.

(b) Prove that $y(n)$ is a Markov process.

(c) Let $z(n) = x(n-1) + x(n)$. Find the pdf of $z(n)$.

(d) Find the mean, the autocorrelation function, and the PSD of $z(n)$.

Problem 3

Consider the difference equation $y(n) = 0.3y(n-1) + x(n)$, where $x(n)$ is a stationary white noise with $R_X(m) = \delta(m)$.

(a) If the equation is valid for $-\infty < n < \infty$, find PSD and autocorrelation function of $y(n)$.

(b) If the differential equation is valid for $n \geq 0$ and $y(n) = 0$ for $n < 0$, find the autocorrelation function of $y(n)$.

Problem 4

Consider a Gaussian random process with the power spectral density as shown in Figure 1. Give a representation of $X(t)$ using linear operations on independent zero-mean unit-variance Gaussian random variables, in accordance to the Nyquist sampling theorem. Identify explicitly any function you use, and also identify how many R.V.s (samples) you use per unit of time. Try to minimize the number of $\mathcal{N}(0,1)$ random variables. (This is a way of simulating a random process with specified spectral density.)

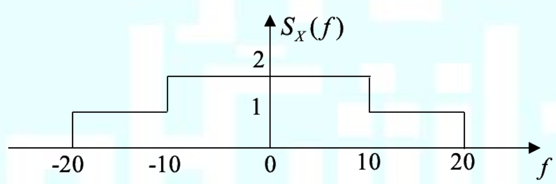

Figure 1

Problem 5

Let $v(n)$ be a WSS white process with unit variance. The process $x(n)$ is related to $v(n)$ as:

$$\sum_{k=1}^{+\infty}(k+1)^2 3^{-k} x(n-k) = v(n)$$

(a) Prove that $x(n)$ is an $ARMA(N, M)$ process and find N and M.

(b) Does $x(n)$ have an $MA(\infty)$ model? If yes, find it. If no, why?

Problem 6

Let $x(t)$ be a zero-mean wide-sense stationary stochastic process with spectrum

$$S_X(f) = \begin{cases} 1 + \cos(20\pi f), & |f| \leqslant \frac{1}{20} \\ 0, & |f| > \frac{1}{20} \end{cases}$$

Consider the Continuous to Discrete (C/D) converter with sampling period of T depicted in the Figure 2, i.e., $y(n) = x(nT)$ for all $n \in \mathbb{Z}$.

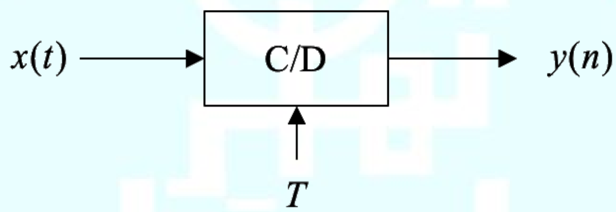

Figure 2

(a) Assume that the sampling period is $T = 10sec$. Determine $C_Y(n, m)$ and $S_Y(f)$. Is it possible to reconstruct $x(t)$ from $y(n)$ in mean-square sense? Show how the reconstruction can be achieved.

(b) Assume that the sampling period is $T = 20sec$. Determine $C_Y(n, m)$ and $S_Y(f)$. What is the best linear estimate of $x(t)$ based on a finite number of $y(n)$'s. Can such a linear function of $y(n)$'s reconstruct $x(t)$ in mean-square sense?

Problem 7

Let $x(n)$ be a discrete stationary random process with the PSD:

$$S_X(f) = \frac{4}{5 - 4\cos(2\pi f)}$$

(a) Find the innovation process of $x(n)$.

(b) Has $x(n)$ an AR model? If yes, find it.

(c) Has $x(n)$ an MA model? If yes, find it.

(d) Find LMMSE estimation of $x(n)$ given $x(n-1)$ and its MSE.

(e) Find LMMSE estimation of $x(n)$ given $\{x(n-1), x(n-2)\}$ and and its MSE.

(f) Find LMMSE estimation of $x(n)$ given $\{x(n-1), x(n+1)\}$ and and its MSE. compare the estimator of part f and e.