Stochastic Processes

University of Tehran

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Homework 2

Due : 1401/8/2

Problem 1

X and Y are jointly Gaussian random variables with $\mathcal{N}(\eta_x, \eta_y, \sigma_x^2, \sigma_y^2, \rho_{xy})$, i.e.

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}} \exp\left(\frac{-1}{2(1-\rho_{xy}^2)}\left[\frac{(x-\eta_x)^2}{\sigma_x^2} - \frac{2\rho_{xy}(x-\eta_x)(y-\eta_y)}{\sigma_x\sigma_y} + \frac{(y-\eta_y)^2}{\sigma_y^2}\right]\right)$$

(a) Find the conditional pdfs $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

(b) Compute the conditional mean, $\mathbb{E}_{X|Y}\{X|Y=y\}$, and the conditional variance $\sigma_{X|Y}^2 = \mathbb{E}_{X|Y}\{X^2|Y=y\} - \left(\mathbb{E}_{X|Y}\{X|Y=y\}\right)^2$.

(c) Prove that the random variables $\begin{cases} Z = aX + bY \\ W = cX + dY \end{cases}$ are also jointly normal.

Problem 2

Let X and Y be two random variables with the following joint density.

$$f_{XY}(x,y) = \begin{cases} xe^{-x(y+1)} & x>0, y>0 \\ 0 & \text{Otherwirse} \end{cases}$$

(a) Compute $\operatorname{cov}(X,Y)$.

(b) Compute $\mathbb{E}_{X|Y}(X|Y=y)$.

(c) Compute $\mathbb{E}_{Y|X}(Y|X=x)$.

(d) Compute $\mathbb{E}_{Y|X}(X^2Y|X=x)$.

Problem 3

Let $\{X_1, X_2, \ldots, X_n, \ldots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables and suppose N is an integer random variable independent of X_i for $i = 1, 2, \ldots$. Let $Z = \sum_{k=1}^{N} X_K$.

(a) Find the characteristic function of Z in terms of characteristic function of X_i.

(b) Find the mean and the variance of Z.

(c) Repeat parts (a) and (b) for the special case where X_is are Normal RVs as $\mathcal{N}(\eta, \sigma^2)$, and N is a Geometric RV with parameter p.

Problem 4

X is a Gaussian random variables with mean η and variance σ^2. Compute the mean of $Z = sin(aX)$. where a is a known constant.

Problem 5

Suppose that X is a binomial random variable with parameters n and p, i.e. $X \sim Binomial(n, p)$. Find $\mathbb{E}\left\{\frac{1}{X+1}\right\}$.

Problem 6

Let $\{X_1, X_2, \ldots, X_n, \ldots\}$ be a sequence of discrete independent and identically distributed (i.i.d) random variables with the following pmf,

$$P\{X_i = k\} = -\frac{(1-p)^k}{k\log(p)};\ k \geq 1,\ 0 < p < 1$$

(a) Find the probability generating function (PGF) of X_i.

(b) Find the probability mass function (pmf) of $Y = \sum_{k=1}^{N} X_k$, where N is a Poisson random variable independent of X_i for $i = 1, 2, \ldots$, with parameter λ.

Problem 7

Suppose that the random variables $\{X_1, X_2, \ldots, X_n\}$ are i.i.d., each with the pdf of $f_X(x)$ and the cdf of $F_X(x)$. Find the pdf of the followings in terms of $f_X(x)$ and $F_X(x)$.

(a) Find the pdf of $Y_1 = \max\{X_1, X_2, \ldots, X_n\}$ in terms of $f_X(x)$ and $F_X(x)$.

(b) Find the pdf of $Y_2 = \min\{X_1, X_2, \ldots, X_n\}$ in terms of $f_X(x)$ and $F_X(x)$.

(c) Find the joint pdf of Y_1 and Y_2 defined in part a and b in terms of $f_X(x)$ and $F_X(x)$.