Iterative Candidate Removal Plurality Voting

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Abstract

Plurality voting is the simplest voting procedure where the candidate that receives the most votes wins. In this paper we look at the convergence properties of plurality voting in iterative voting games. Additionally, we introduce a new voting procedure called Iterative Candidate Removal (ICR) Plurality Voting that was an attempt to arrive at the same outcome as plurality voting but with fewer steps required for convergence. We simulated elections under different voter behavior models and saw that there was no distinct advantage to ICR over plurality and in fact ICR converges to plurality for large number of voters and candidates.

Index Terms

Voting, Iterative Voting, Plurality Voting, Iterative Candidate Removal Plurality Voting, ICR Voting

I. Introduction

Voting is a well studied area of Game Theory and Social Choice Theory. Voting is the process of selecting the preferred choice among the group of voters who each have their preferences. The most obvious usage of voting are political elections, where each voter internally ranks each political candidate on who they want to win the election then the voters submit their vote. This type of voting can be interpreted as a standard form incomplete information game because the voters submit their vote simultaneously and without knowledge of the preferences of other voters (assuming voters vote at the same time). The main type of voting which is of interest for this paper is iterative voting, where the winner of the election is known after all votes have been submitted and each voter is allowed to change his vote an infinite number of times. The voting process is complete when no voter wants to change their vote, which is the notion of equilibrium or convergence which we will define later.

Both types of voting are susceptible to strategic voting, where voters decide not to vote for their number one preferred candidate. For example, in the United States the Republican and Democrat parties have the largest support bases out of all political parties and so candidates from these parties win the majority of elections. A voter that prefers an Independent candidate may choose to vote for either a Republican or Democrat candidate instead since the Independent candidate has no practical chance of winning. While strategic voting is not a focus of this paper, many of the notions described later were initially derived to analyze strategic games.

Plurality voting is the most basic voting game where the candidate with most votes is the winner. In this paper we look at iterative plurality voting and we compare it to a new voting method called Iterative Candidate Removal Plurality Voting (ICR voting). ICR voting is the same as iterative plurality voting except the least popular candidate is removed after each voter has the chance to change their vote. In Section II we define the notations necessary to analyze voting procedures. In Section III, we note that plurality has *finite direct reply property (FDRP)* which guarantees convergence in iterative games. In Section IV we explain plurality and ICR in more depth. Lastly, in Section V, we compare the voting methods by analyzing how fast each converges and how often they arrive at the same results by simulations under different parameters and voting behaviors[1].

II. Definitions

We adopt a lot of the notations, terminologies, and definitions used in [1]. Let N be the set of all voters and let M be the set of all candidates. We say that each voter $n \in N$ has a *preference profile* L_n which is a ranking of all $m \in M$ and $L = \{L_1, L_2, \cdots, L_N\}$. Let $rank_n(m)$ be the rank of candidate m for voter n and $rank_n(m_1) > rank_n(m_2)$ if voter n prefers candidate m_1 to candidate m_2. For the purpose of this paper, assume the ranking is strict, i.e. $\forall n \in N, \forall m_1, m_2 \in L_n, m_1 \neq m_2 \implies rank_n(m_1) \neq rank_n(m_2)$. In other words, no voter sees any two candidates as equals, the voter always has a strict preference. A voter's preference profile is only known to himself, it is not public information.

[1]Repository of simulation code: https://github.com/SoroushTAMU/ECEN756FinalProject

Let $A = \{a_1, a_2, \cdots, a_{|N|}\}$ be the votes submitted by each voter, $A \in M^{|N|}$. Let a_{-n} be the votes submitted by everyone except voter n. We define $a_{n,1}$ to be a better reply than $a_{n,2}$ if going from $a_{n,2}$ to $a_{n,1}$ more likely results in a more preferred election outcome, holding a_{-n} the same. Additionally, we say $a_{n,i}$ is a best reply for voter n if $a_{n,i}$ is a better or equal reply to $a_{n,j}, \forall j \in M$, holding a_{-n} the same. A is a *Pure Nash Equilibrium (PNE)* if $\forall$ voter $n \in N, a_n$ is a best reply to a_{-n}.

III. Equilibrium Properties

In this section we discuss convergence to equilibrium in iterative games. Iterative games, unlike sequential games, allow voters to change their vote an unlimited amount of times. At each iteration, one voter is chosen and is asked if they would like to change their vote. The voter is chosen either randomly or in order, and how that voter is chosen impacts convergence.

We define a voting function $f : A \mapsto M$ as being the voting procedure which takes each vote and produces a winning candidate. Let a game be defined as the tuple of voting procedure, preference profiles, and voting profiles $G = (f, L, A)$. To analyze convergence, G induces a *local improvement graph* $H_G(f, L)$ which is a directed graph where the vertices are all possible voting profiles $A \in M^{|N|}$ and directed edges going from vertices with single step deviations a to b if b is a better reply to a for some voter. It easily follows that any vertex a that does not have any outgoing edges is a PNE since no other one step deviation voting profile is beneficial to any voter.

We can now analyze convergence of voting procedures by looking at the induced graphs. In iterative games, at each iteration one voter is chosen to be asked if they want to change their vote. If they have a better reply, the state of the game G goes from $a_{n,i}$ to $a_{n,j}$, with a_{-n} remaining constant. This continues until no voter has any incentive to change their vote, i.e. PNE has been reached.

Definition 1 (Finite Improvement Property)**.** *A game G has the finite improvement property (FIP) if the corresponding graph H_G has no cycles [1].*

If a graph H_G has no cycles then no matter the initial state, it will eventually reach a state that does not have any outgoing edges which is a PNE. Hence, a voting procedure f that has the FIP is guaranteed to always converge to a PNE. This is a strong property that, however, is not guaranteed to be met by any voting procedure except for the dictatorship voting procedure [1]. A slightly weaker but still useful condition is the finite direct reply property (FDRP). A direct reply is a change of vote that immediately changes the outcome of the election. This gives us,

Definition 2 (Finite Direct Reply Property)**.** *A game G has the finite direct reply property (FDRP) if G converges for any initial state, any order the voters are chosen to change their vote, and any direct reply paths [2].*

FDRP is an important properties for two main reasons: 1. It converges regardless of how you choose voters to change their vote and 2. G converges for any initial state, meaning it doesn't matter if the voters were initially voting completely different than their preferences. Hence, if the voters were initially strategically voting, i.e. not voting true to their preferences, we will still arrive at an equilibrium eventually.

IV. Plurality and ICR

Plurality is the simplest voting procedure where the candidate with most number of votes is the winner. However, plurality voting does not satisfy the Majority Criterion, which states that the winner of an election should obtain at least 50% of the total number of votes. This has led to the creation of the plurality with elimination or instant runoff voting procedures which conducts plurality until a candidate has over 50% of the votes.

Iterative plurality is of interest because it is FDRP.

Theorem 1. *f^{PL} is FDRP. Moreover, any path of direct replies will converge after at most m^2n^2 steps, where $m = |M|$, $n = |N|$ [2].*

One of the motivations of this project was to see if we could improve on the number of steps until convergence. As we will see in Section V, the number of steps until convergence is actually very loose upper bound in practice. Nonetheless, we introduce a variation of plurality voting which we thought could improve convergence speed. The Iterative Candidate Removal (ICR) plurality voting procedure is the same as iterative plurality except after each voter has the chance to change his vote the least popular candidate is removed from the election. While elimination voting methods already exist, to our knowledge this is the first time this specific method has been implemented.

Theorem 2. *f^{ICR} will converge after at most $n(m-1)$ steps, where $m = |M|$, $n = |N|$.*

Proof. Let a round be one pass over all voters asking them if they want to change their vote. Since ICR eliminates a candidate after every round, there will be at most $m-1$ rounds. Each of those rounds will contain at most n steps. Thus, f^{ICR} is guaranteed to converge in at most $n(m-1)$ steps. □

Indeed, while upper bound number of steps for convergence for ICR is smaller than the upper bound number of steps plurality is guaranteed to converge, this does not necessarily mean ICR converges faster in practice. Nor does it mean that ICR will reach same outcome as plurality. This is what we want to discover via simulations in the Section V.

V. Simulations and Results

We simulate voting with different number of voters, different number of candidates, and different underlying voting behaviors. We compare plurality vs ICR convergence under two different voting behavior models: impartial culture (IC) and the behavior model described in [3] which wasn't named so we call it Amorim behavior. There are $|M|!$ possible strict orderings of the candidates (i.e. possible preference profiles) and under IC behavior each has uniform probability of being chosen [4]. The *Condorcet Criterion* states that in an election the candidate that wins the most head-to-heads versus all other candidates should be the winner. [5] shows that IC behavior is the *worst-case behavior* that maximizes the probability of violating the Condorcet Criterion and is also an unrealistic behavior model. Nonetheless, it is widely used in simulations because of its computational simplicity.

The Amorim behavior model [3] is based off the work of [6] and associates each candidate with a Gaussian random variable with different means but same variances. Each voter samples from each Gaussian distribution and then the sampled values are sorted. The preference profile is the resulting order of candidates after sorting. More formally,

Algorithm 1 Amorim Behavior Preference Generation

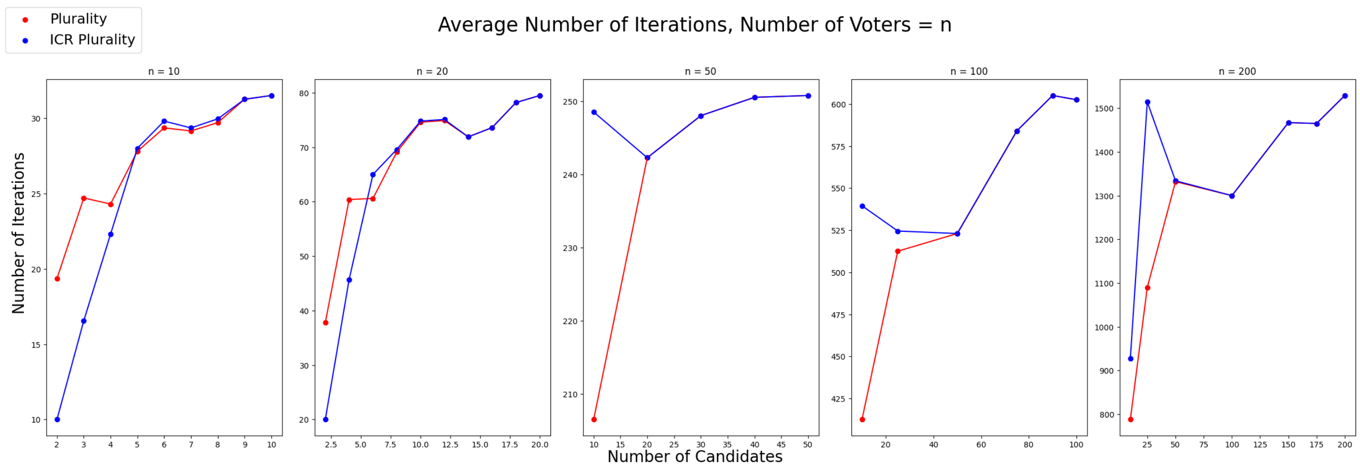
$\forall m \in M$, define $N_m(\mu_m, \sigma)$
for $n \in N$ **do**
  $array \leftarrow []$
  for $m = 1, 2, \cdots, |M|$ **do**
    Sample $s \sim N_m(\mu_m, \sigma)$, insert s into *array*
  end for
  $L_n = ArgumentSort(array)$
end for

A. Simulation Parameters

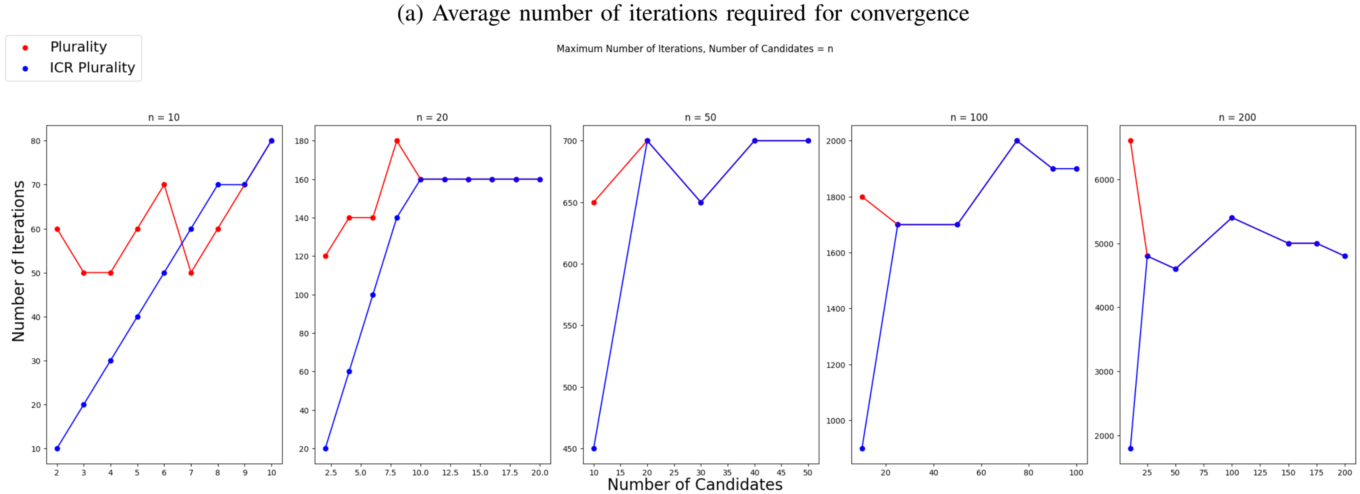
We look at up to 200 voters and 100 candidates for each simulation. For each combination of parameters we conduct 200 simulations. Each simulation conducts iterative plurality and ICR with same initial votes and same preference profiles for all voters. Since iterative plurality is *FDRP* we can simply ask voters in order to change their vote. Ties are broken by Lexicographical ordering, i.e. if candidate 1 has same number of votes as candidate 2 then if we are declaring a winner candidate 1 is the winner and if we are declaring a loser (for removing a candidate) candidate 2 is the loser. For Amorim behavior preference generation, we space the means out for each candidate uniformly on $[0, 1]$ and set $\sigma = 0.5$, as was done in [3].

B. Impartial Culture (IC) Behavior Modeling

Here we analyze results when preference profiles are generated by the IC assumption. Figure 1 compares iterative plurality and ICR number of iterations until convergence. First thing to note is that neither required the theoretical upper bound number of step to converge. Next, for relatively few number of candidates compared to number of voters, ICR converges slower in expectation but faster in worst case scenario. However, past some point, they perform equivalently. This is a common theme among all the results because ICR will eventually start removing candidates that no one was voting for so ICR becomes equivalent to plurality past some point.



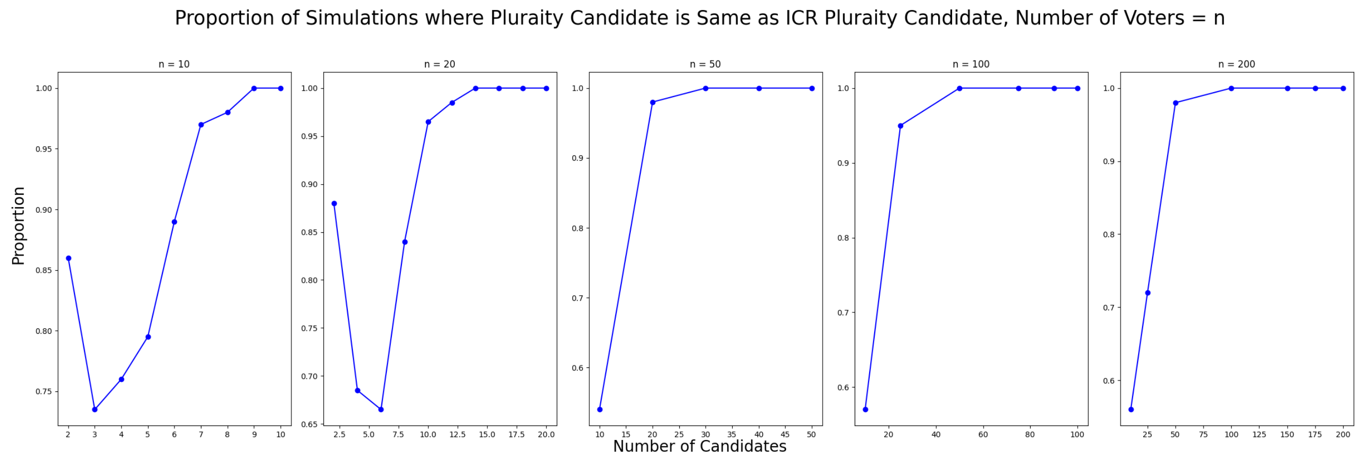
(a) Average number of iterations required for convergence



(b) Maximum number of iterations required for convergence

Fig. 1. Comparison of number iterations required for convergence for plurality and ICR with IC behavior modeling.

Figure 2 below shows that when number of candidates is relatively small compared to number of voters, ICR's outcome is different than the plurality outcome in at most 50% of simulations. However, as number of voters and number of candidates increase, the outcome of ICR becomes more aligned with the plurality outcomes.



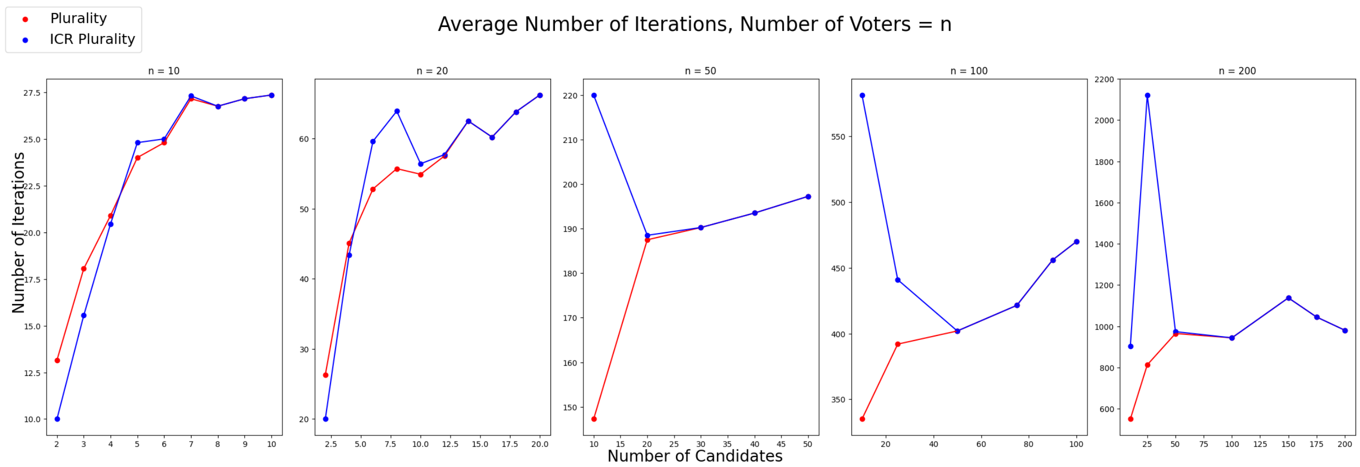
(a) Proportions of simulations where plurality and ICR yield same outcome

Fig. 2. Comparison of outcomes of voting procedures with IC behavior modeling.

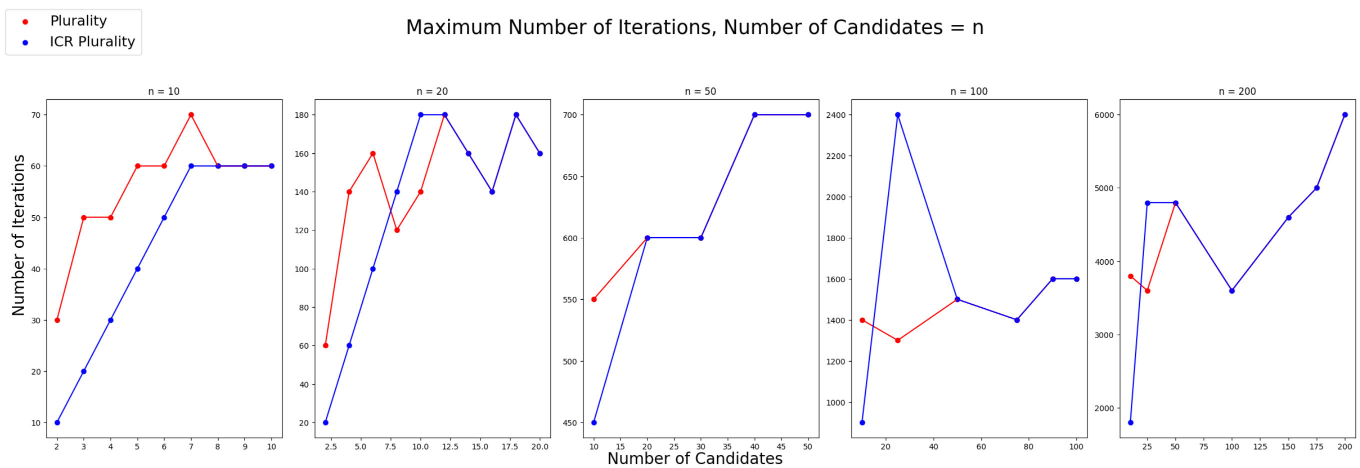
C. Amorim Behavior Modeling

Now we do the same analysis when preference profiles are generated by the Amorim assumption. Figure 3 below shows that under Amorim behavior modeling ICR tends to perform equally as good or slightly better than plurality when number of candidates is low. When number of candidates and number of voters increase, ICR either performs equally as good as plurality or worse.

Figure 4 shows the proportion of simulations where the results of plurality and ICR were the same. The results are extremely similar to the results of Figure 2, which shows that modeling behavior is not the main determining factor of whether or not they two voting procedures produce equivalent outcomes.

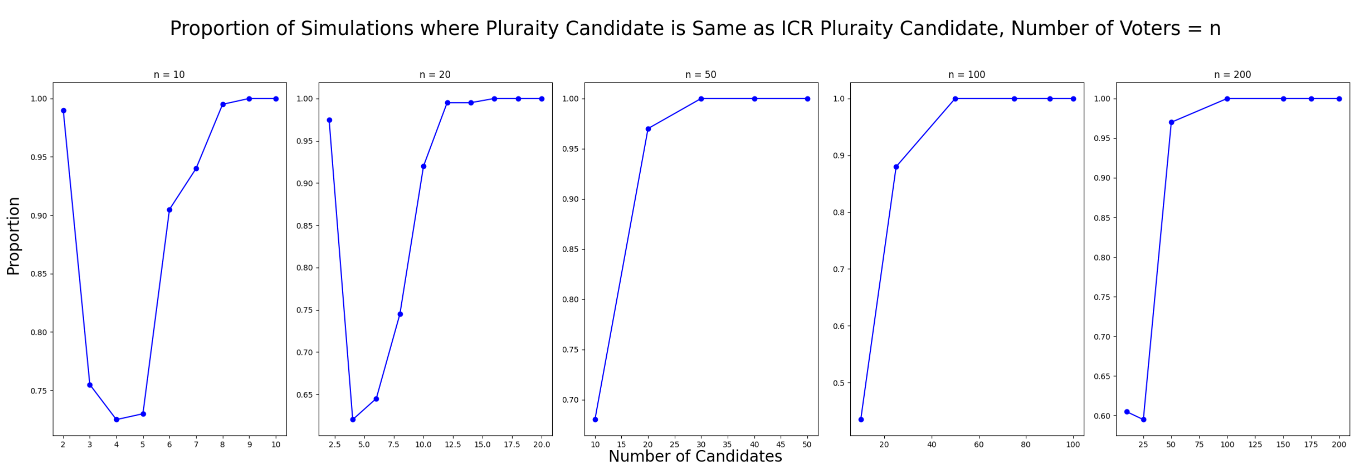


(a) Average number of iterations required for convergence



(b) Maximum number of iterations required for convergence

Fig. 3. Comparison of number iterations required for convergence for plurality and ICR with Amorim behavior modeling.



(a) Proportions of simulations where plurality and ICR yield same outcome

Fig. 4. Comparison of outcomes of voting procedures with Amorim behavior modeling.

VI. Conclusion

ICR voting was introduced to see if we could achieve same outcomes as iterative plurality in fewer steps. Our results showed that under IC and Amorim voting behaviors, number of steps to convergence and election outcomes had significant variance up until a ratio of candidates to voters after which the results became identical. For future work, it would be interesting to analyze iterative elimination algorithm of other voting procedures such as Borda counting to see if results are similar regardless of voting procedure. Veto voting is also FDRP, so perhaps the results of running an iterative elimination veto voting procedure would yield similar results to what we have shown.

References

[1] R. Meir, "Strategic voting," *Synthesis lectures on artificial intelligence and machine learning*, vol. 13, no. 1, pp. 1–167, 2018.
[2] R. Meir, M. Polukarov, J. S. Rosenschein, and N. R. Jennings, "Iterative voting and acyclic games," *Artificial Intelligence*, vol. 252, pp. 100–122, 2017.
[3] G. Barros Correa de Amorim, A. P. Cabral Seixas Costa, D. Costa Morais, and L. Chaves Rego, "Random-subset voting," *Journal of Artificial Societies and Social Simulation*, vol. 21, no. 1, p. 10, 2018. [Online]. Available: http://jasss.soc.surrey.ac.uk/21/1/10.html
[4] M. B. Garman and M. I. Kamien, "The paradox of voting: Probability calculations," *Behavioral Science*, vol. 13, no. 4, pp. 306–316, 1968.
[5] I. Tsetlin, M. Regenwetter, and B. Grofman, "The impartial culture maximizes the probability of majority cycles," *Social Choice and Welfare*, vol. 21, no. 3, pp. 387–398, 2003.
[6] J. R. Chamberlin and F. Featherston, "Selecting a voting system," *The Journal of Politics*, vol. 48, no. 2, pp. 347–369, 1986.