Exploration versus exploitation in reinforcement learning: a stochastic control approach

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Abstract

We consider reinforcement learning (RL) in continuous time and study the problem of achieving the best trade-off between exploration of a black box environment and exploitation of current knowledge. We propose an entropy-regularized reward function involving the differential entropy of the distributions of actions, and motivate and devise an exploratory formulation for the feature dynamics that captures repetitive learning under exploration. The resulting optimization problem is a resurrection of the classical relaxed stochastic control. We carry out a complete analysis of the problem in the linear-quadratic (LQ) case and deduce that the optimal control distribution for balancing exploitation and exploration is Gaussian. This in turn interprets and justifies the widely adopted Gaussian exploration in RL, beyond its simplicity for sampling. Moreover, the exploitation and exploration are reflected respectively by the mean and variance of the Gaussian distribution. We also find that a more random environment contains more learning opportunities in the sense that less exploration is needed, other things being equal. As the weight of exploration decays to zero, we prove the convergence of the solution to the entropy-regularized LQ problem to that of the classical LQ problem. Finally, we characterize the cost of exploration, which is shown to be proportional to the entropy regularization weight and inversely proportional to the discount rate in the LQ case.

Key words. Reinforcement learning, exploration, exploitation, entropy regularization, stochastic control, linear–quadratic, Gaussian.

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1 Introduction

Reinforcement learning (RL) is currently one of the most active and fast developing subareas in machine learning. It has been successfully applied to solve large scale real world, complex decision-making problems in recent years, including playing perfect-information board games such as Go (AlphaGo/AlphaGo Zero, Silver et al. (2016), Silver et al. (2017)), achieving human-level performance on video games (Mnih et al. (2015)), and autonomous driving (Levine et al. (2016), Mirowski et al. (2016)). An RL agent does not pre-specify a structural model or a family of models, but instead gradually learns the best (or near-best) strategies based on trial and error, through interactions with the random (black box) environment and incorporation of the responses of these interactions in order to improve the overall performance. This is a "kill two birds with one stone" case: the agent's actions (controls) serve both as a means to explore (learn) and a way to exploit (optimize). Since exploration is inherently costly in terms of both resources and time, a natural and crucial question in RL is to address the dichotomy between exploration of uncharted territory and exploitation of existing knowledge. Such question exists in both the stateless RL settings represented by the multi-armed bandit problem, and the more general multi-state RL settings with delayed reinforcement (Sutton and Barto (2018), Kaelbling et al. (1996)). More specifically, the agent must balance between greedily exploiting what has been learned so far to choose actions that yield near-term higher rewards, and continually exploring the environment to acquire more information to potentially achieve long-term benefits. Extensive studies have been conducted to find optimal strategies for trading off exploitation and exploration.¹

However, most of the contributions to balancing exploitation and exploration do not include exploration formally as a part of the optimization objective; the attention has mainly focused on solving the classical optimization problem maximizing the accumulated rewards, while exploration is typically treated separately as an ad-hoc chosen exogenous component, rather than being endogenously derived as a part of the solution to the overall RL problem. The recently proposed discrete-time entropy-regularized (also termed as "entropy-augmented" or "softmax") RL formulation, on the other hand, explicitly incorporates exploration into the optimization objective, with a trade-off weight imposed on the entropy of the exploration strategy (Ziebart et al. (2008), Nachum et al. (2017a), Fox et al. (2015)). An exploratory distribution with a greater entropy signifies a higher level of exploration, and is hence more favorable on the exploration front. The extreme case of a Dirac measure having minimal entropy implies no exploration, reducing to the case of classical optimization with complete knowledge about the underlying model. Recent works have de-

¹For the multi-armed bandit problem, well-known strategies include Gittins-index approach (Gittins (1974)), Thompson sampling (Thompson (1933)), and upper confidence bound algorithm (Auer et al. (2002)), with sound theoretical optimality established (Russo and Van Roy (2013, 2014)). For general RL problems, various efficient exploration methods have been proposed that have been proved to induce low sample complexity (see, e.g., Brafman and Tennenholtz (2002), Strehl and Littman (2008), Strehl et al. (2009)) among other advantages.

voted to the designing of various algorithms to solve the entropy regulated RL problem, where numerical experiments demonstrate remarkable robustness and multi-modal policy learning (Haarnoja et al. (2017), Haarnoja et al. (2018)).

In this paper, we study the trade-off between exploration and exploitation for RL in a continuous time setting with both continuous control (action) and state (feature) spaces.² Such continuous-time formulation is especially appealing if the agent can interact with the environment at ultra-high frequency aided by the modern computing resources.³ Meanwhile, an in-depth and comprehensive study of the RL problem also becomes possible which leads to elegant and insightful results, once cast in continuous time, thanks in no small measure to the tools of stochastic calculus and differential equations.

Our first main contribution is to propose an entropy-regularized reward function involving the differential entropy for probability distributions over the continuous action space, and motivate and devise an "exploratory formulation" for the state dynamics that captures repetitive learning under exploration in the continuous-time limit. Existing theoretical works on exploration mainly concentrate on the analysis at the algorithmic level, e.g., proving convergence of the proposed exploration algorithms to the solutions of the classical optimization problems (see, e.g., Singh et al. (2000), Jaakkola et al. (1994)), yet they rarely look into the learning algorithms' impact on changing significantly the underlying dynamics. Indeed, exploration not only substantially enriches the space of control strategies (from that of Dirac measures to that of all possible probability distributions) but also, as a result, enormously expands the reachable space of states. This, in turn, sets out to change the underlying state transitions and system dynamics.

We show that our exploratory formulation can account for the effects of learning in both the rewards received and the state transitions observed from the interactions with the environment. It, thus, unearths the important characteristics of learning at a more refined and in-depth level, beyond the theoretical analysis of mere learning algorithms. Intriguingly, this formulation of the state dynamics coincides with the relaxed control framework in classical control theory (El Karoui et al. (1987), Kurtz and Stockbridge (1998, 2001)), which was motivated by entirely different reasons. Relaxed controls were introduced to mainly deal with the theoretical question of whether an optimal control exists. The approach is essentially a randomization device to convexify the universe of control strategies. To our best knowledge, our paper is the first to bring back the fundamental ideas and formulation of the relaxed control, guided by a practical motivation: exploration and learning.

We then carry out a complete analysis of the continuous-time entropyregularized RL problem, assuming that the original dynamic system is linear in both control and state and the original reward function is quadratic in the

²The terms "feature" and "action" are typically used in the RL literature, whose counterparts in the control literature are "state" and "control" respectively. Since this paper uses the control approach to study RL, we will interchangeably use these terms whenever there is no confusion.

 $^{^3\}mathrm{A}$ notable example is high frequency stock trading.

two. This type of linear—quadratic (LQ) problems has occupied the center stage for research in classical control theory for its elegant solutions and its ability to approximate more general nonlinear problems. One of the most important, conceptual contributions of this paper is to show that the optimal control distribution for balancing exploitation and exploration is *Gaussian*. As is well known, a pure exploitation optimal distribution is Dirac, and a pure exploration optimal distribution is uniform. Our results reveal that Gaussian is the right choice if one seeks a balance between those two extremes. Moreover, we find that the mean of this optimal distribution is a feedback of the current state *independent* of the intended exploration level, whereas the variance is a linear function of the entropy regulating weight (also called the "temperature parameter") *irrespective* of the current state. This result highlights a *separation* between exploitation and exploration: the former is reflected in the mean and the latter in the variance of the final optimal actions.

There is yet another intriguing result. All other things being equal, the more volatile the original dynamic system is, the smaller the variance of the optimal Gaussian distribution is. Conceptually, this implies that an environment reacting more aggressively to an action in fact contains more learning opportunities and hence is less costly for learning.

Another contribution of the paper is to establish the connection between the solvability of the exploratory LQ problem and that of the classical LQ problem. We prove that as the exploration weight in the former decays to zero, the optimal Gaussian control distribution and its value function converge respectively to the optimal Dirac measure and the value function of the latter, a desirable result for practical learning purposes.

Finally, we observe that, beyond the LQ problems, the Gaussian distribution remains, formally, optimal for a much larger class of control problems, namely, problems with drift and volatility linear in control, and reward functions linear or quadratic in control. Such family of problems can be seen as the local-linear-quadratic approximation to more general stochastic control problems whose state dynamics are linearized in the control variables and the reward functions are approximated by quadratic functions in controls locally (Todorov and Li (2005), Li and Todorov (2007)). Note that although such iterative LQ approximation generally has different parameters at different local state-action pairs, our result on the optimality of Gaussian distribution under the exploratory LQ framework still holds at any local point, and therefore justifies, from a stochastic control perspective, why Gaussian distribution is commonly used in practice for exploration (Haarnoja et al. (2017), Haarnoja et al. (2018), Nachum et al. (2017b)), beyond its simplicity for sampling.

The paper is organized as follows. In section 2, we motivate and propose the relaxed stochastic control problem involving an exploratory state dynamics and an entropy-regularized reward function. We then present the Hamilton-Jacobi-Bellman (HJB) equation and the optimal control distribution for general entropy-regularized stochastic control problems in section 3. In section 4, we study the special LQ problem in both the state-independent and state-dependent reward cases, corresponding respectively to the multi-armed bandit

problem and general RL problem in discrete time. We discuss the connections between the exploratory LQ problem and the classical LQ problem in section 5, establish the equivalence of the solvability of the two and the convergence result for vanishing exploration, and finally characterize the cost of engaging exploration. Finally, section 6 concludes. Some technical proofs are relegated to appendices.

2 An Entropy-Regularized Stochastic Control Problem

We introduce an entropy-regularized stochastic control problem and provide its motivation in the context of RL.

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\geq 0})$ in which we define an $\{\mathcal{F}_t\}_{t\geq 0}$ -adapted Brownian motion $W=\{W_t, t\geq 0\}$. An "action space" U is given, representing the constraint on an agent's decisions ("controls" or "actions"). An admissible control $u=\{u_t, t\geq 0\}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$ -adapted measurable process taking value in U. Denote by \mathcal{U} the set of all admissible controls.

The classical stochastic control problem is to control the state (or "feature") dynamics

$$dx_t^u = b(x_t^u, u_t)dt + \sigma(x_t^u, u_t)dW_t, \ t > 0; \quad x_0^u = x$$
 (1)

so as to achieve the maximum expected total discounted reward represented by the value function

$$w\left(x\right):=\sup_{\mathcal{U}}\mathbb{E}\left[\left.\int_{0}^{\infty}e^{-\rho t}r\left(x_{t}^{u},u_{t}\right)dt\right|x_{0}^{u}=x\right],\tag{2}$$

where r is the reward function and $\rho > 0$ is the discount rate.

In the classical setting, where the model is fully known (namely when the functions b, σ and r are fully specified) and the dynamic programming is applicable, the optimal control can be derived and represented as a deterministic mapping from the current state to the action space $U, u_t^* = \mathbf{u}^*(x_t^*)$ where \mathbf{u}^* is the optimal feedback policy (or "law"). This feedback policy is derived at t = 0 and will be carried out through $[0, \infty)$.

In contrast, under the RL setting when the underlying model is not known and therefore dynamic learning is needed, the agent employs exploration to interact with and learn the unknown environment through trial and error. This exploration can be modelled by a distribution of controls $\pi = \{\pi_t(u), t \geq 0\}$ over the control space U from which each "trial" is sampled. We can therefore extend the notion of controls to distributions⁴. The agent executes a control for N rounds over the same time horizon, while at each round, a classical control is sampled from the distribution π . The reward of such a policy becomes

⁴A classical control $u = \{u_t, t \geq 0\}$ can be regarded as a Dirac distribution (or "measure") $\pi = \{\pi_t(u), t \geq 0\}$ where $\pi_t(\cdot) = \delta_{u_t}(\cdot)$. In a similar fashion, a feedback policy $u_t = \mathbf{u}(x_t^u)$ can be embedded as a Dirac measure $\pi_t(\cdot) = \delta_{\mathbf{u}(x_t^u)}(\cdot)$, parameterized by the current state x_t^u .

accurate enough when N is large. This procedure, known as *policy evaluation*, is considered as a fundamental element of most RL algorithms in practice (see, e.g., Sutton and Barto (2018)). Hence, for evaluating such a policy distribution in our continuous-time setting it is necessary to consider the limiting situation as $N \uparrow \infty$.

In order to quickly capture the essential idea, let us first examine the special case when the reward depends only on the control, namely, $r(x_t^u, u_t) = r(u_t)$. One then considers N identical independent copies of the control problem in the following way: at round i, i = 1, 2, ..., N, a control u^i is sampled under the (possibly random) control distribution π , and executed for its corresponding copy of the control problem (1)–(2). Then, at each fixed time t, from the law of large numbers and under certain mild technical conditions, it follows that the average reward over $[t, t + \Delta t]$, with Δt small enough, should satisfy

$$\frac{1}{N} \sum_{i=1}^{N} e^{-\rho t} r(u_t^i) \Delta t \xrightarrow{\text{a.s.}} \mathbb{E} \left[e^{-\rho t} \int_U r(u) \pi_t(u) du \Delta t \right], \quad \text{as} \quad N \to \infty.$$

For a general reward $r(x_t^u, u_t)$ which depends on the state, we first need to describe how exploration might alter the state dynamics (1), by defining appropriately its "exploratory" version. For this, we look at the effect of repetitive learning under a given control distribution π for N rounds. Let W_t^i , $i=1,2,\ldots,N$, be N independent sample paths of the Brownian motion W_t , and x_t^i , $i=1,2,\ldots,N$, be the copies of the state process respectively under the controls u^i , $i=1,2,\ldots,N$, each sampled from π . Then the increments of these state process copies are, for $i=1,2,\ldots,N$,

$$\Delta x_t^i \equiv x_{t+\Delta t}^i - x_t^i \approx b(x_t^i, u_t^i) \Delta t + \sigma(x_t^i, u_t^i) \left(W_{t+\Delta t}^i - W_t^i\right), \quad t \ge 0. \tag{3}$$

Then, each such process x^i , $i=1,2,\ldots,N$, can be viewed as an independent sample from the exploratory state dynamics X^{π} . The superscript π of X^{π} indicates that each x^i is generated according to the classical dynamics (3), with the corresponding u^i sampled independently under the policy π .

It then follows from (3) and the law of large numbers that, as $N \to \infty$,

$$\frac{1}{N} \sum_{i=1}^N \Delta x_t^i \; \approx \; \frac{1}{N} \sum_{i=1}^N b(x_t^i, u_t^i) \Delta t + \frac{1}{N} \sum_{i=1}^N \sigma(x_t^i, u_t^i) \left(W_{t+\Delta t}^i - W_t^i\right)$$

$$\stackrel{\text{a.s.}}{\longrightarrow} \quad \mathbb{E}\left[\int_{U}b(X_{t}^{\pi},u)\pi_{t}(u)du\Delta t\right] + \mathbb{E}\left[\int_{U}\sigma(X_{t}^{\pi},u)\pi_{t}(u)du\right]\mathbb{E}\left[W_{t+\Delta t} - W_{t}\right]$$

$$= \mathbb{E}\left[\int_{U} b(X_{t}^{\pi}, u) \pi_{t}(u) du \Delta t\right]. \tag{4}$$

In the above, we have implicitly applied the (reasonable) assumption that both π_t and X_t^{π} are independent of the increments of the Brownian motion sample paths, which are identically distributed over $[t, t + \Delta t]$.

Similarly, as $N \to \infty$,

$$\frac{1}{N} \sum_{i=1}^{N} \left(\Delta x_t^i \right)^2 \approx \frac{1}{N} \sum_{i=1}^{N} \sigma^2(x_t^i, u_t^i) \Delta t \xrightarrow{\text{a.s.}} \mathbb{E} \left[\int_U \sigma^2(X_t^{\pi}, u) \pi_t(u) du \Delta t \right]. \tag{5}$$

As we see, not only Δx_t^i but also $(\Delta x_t^i)^2$ are affected by repetitive learning under the given policy π .

Finally, as the individual state x_t^i is an independent sample from X_t^{π} , we have that Δx_t^i and $(\Delta x_t^i)^2$, $i=1,2,\ldots,N$, are the independent samples from ΔX_t^{π} and $(\Delta X_t^{\pi})^2$, respectively. As a result, the law of large numbers gives that as $N \to \infty$,

$$\frac{1}{N} \sum_{i=1}^{N} \Delta x_t^i \xrightarrow{\text{a.s.}} \mathbb{E}\left[\Delta X_t^{\pi}\right] \quad \text{and} \quad \frac{1}{N} \sum_{i=1}^{N} (\Delta x_t^i)^2 \xrightarrow{\text{a.s.}} \mathbb{E}\left[(\Delta X_t^{\pi})^2\right].$$

This interpretation, together with (4) and (5), motivates us to propose the exploratory version of the state dynamics

$$dX_t^{\pi} = \tilde{b}(X_t^{\pi}, \pi_t)dt + \tilde{\sigma}(X_t^{\pi}, \pi_t)dW_t, \ t > 0; \quad X_0^{\pi} = x,$$
 (6)

where the coefficients $\tilde{b}(\cdot,\cdot)$ and $\tilde{\sigma}(\cdot,\cdot)$ are defined as

$$\tilde{b}(X_t^{\pi}, \pi_t) := \int_U b(X_t^{\pi}, u) \,\pi_t(u) du,\tag{7}$$

and

$$\tilde{\sigma}(X_t^{\pi}, \pi_t) := \sqrt{\int_U \sigma^2(X_t^{\pi}, u) \, \pi_t(u) du}. \tag{8}$$

We will call (6) the exploratory formulation of the controlled state dynamics, and $\tilde{b}(\cdot,\cdot)$ and $\tilde{\sigma}(\cdot,\cdot)$ in (7) and (8), respectively, the exploratory drift and the exploratory volatility.⁵

In a similar fashion,

$$\frac{1}{N} \sum_{i=1}^{N} e^{-\rho t} r(x_t^i, u_t^i) \Delta t \xrightarrow{\text{a.s.}} \mathbb{E} \left[e^{-\rho t} \int_U r(X_t^{\pi}, u) \pi_t(u) du \Delta t \right], \quad \text{as} \quad N \to \infty.$$
(10)

$$\mathbb{L}[f](x,u) := \frac{1}{2}\sigma^2(x,u)f''(x) + b(x,u)f'(x), \ x \in \mathbb{D}, \ u \in U.$$

In the classical relaxed control framework, the controlled dynamics are replaced by the sixtuple $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, X^{\pi}, \pi)$, such that $X_0^{\pi} = x$ and

$$f(X_t^{\pi}) - f(x) - \int_0^t \int_U \mathbb{L}[f](X_t^{\pi}, u)\pi_t(u)duds, \quad t \ge 0, \quad \text{is a } \mathbb{P} - \text{martingale.}$$
 (9)

It is easy to verify that our proposed exploratory formulation (6) agrees with the above martingale formulation. However, even though the mathematical formulations are equivalent, the motivations of the two are entirely different. Relaxed control was introduced to mainly deal with the existence of optimal controls, whereas the exploratory formulation here is motivated by learning and exploration in RL.

⁵The exploratory formulation (6), inspired by repetitive learning, is consistent with the notion of relaxed control in the control literature (see, e.g., El Karoui et al. (1987); Kurtz and Stockbridge (1998, 2001); Fleming and Nisio (1984)). Indeed, let $f: \mathbb{D} \to \mathbb{R}$ be a bounded and twice continuously differentiable function, and define the infinitesimal generator associated to the classical controlled process (1) as

Hence the reward function r in (2) needs to be modified to its exploratory version

$$\tilde{r}\left(X_t^{\pi}, \pi_t\right) := \int_U r\left(X_t^{\pi}, u\right) \pi_t(u) du. \tag{11}$$

If, on the other hand, the model is fully known, exploration would not be needed and the control distributions would all degenerate into the Dirac measures, and we are in the realm of the classical stochastic control. Thus, in the RL context, we need to add a "regularization term" to account for model uncertainty and to encourage exploration. We use Shanon's differential entropy to measure the degree of exploration:

$$\mathcal{H}(\pi_t) := -\int_U \pi_t(u) \ln \pi_t(u) du,$$

where π_t is a control distribution.

We therefore introduce the following entropy-regularized value function

$$V(x) := \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\int_U r(X_t^u, \pi_t) \pi_t(u) du - \lambda \int_U \pi_t(u) \ln \pi_t(u) du\right) dt \middle| X_0^{\pi} = x\right]$$
(12)

where $\lambda > 0$ is an exogenous exploration weight parameter capturing the tradeoff between exploitation (the original reward function) and exploration (the entropy), and $\mathcal{A}(x)$ is the set of the admissible control distributions (which may in general depend on x).

It remains to specify $\mathcal{A}(x)$. Denote by $\mathcal{B}(U)$ the Borel algebra on U and by $\mathcal{P}(U)$ the set of probability measures on U that are absolutely continuous with respect to the Lebesgue measure. The admissible set $\mathcal{A}(x)$ contains all measure-valued processes $\pi = \{\pi_t, t \geq 0\}$ satisfying:

- i) for each $t \geq 0$, $\pi_t \in \mathcal{P}(U)$ a.s.;
- ii) for each $A \in \mathcal{B}(U)$, $\{\pi_t(A), t \geq 0\}$ is \mathcal{F}_t -progressively measurable;
- iii) the stochastic differential equation (SDE) (6) has a unique solution $X^{\pi} = \{X_t^{\pi}, t \geq 0\}$ if π is applied;
 - iv) the expectation on the right hand side of (12) is finite.

3 HJB Equation and Optimal Control Distributions

We present the general procedure for solving the optimization problem (12). To this end, applying the classical Bellman's principle of optimality, we have

$$V(x) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[\left.\int_{0}^{s} e^{-\rho t} \left(\tilde{r}\left(X_{t}^{\pi}, \pi_{t}\right) + \lambda \mathcal{H}\left(\pi_{t}\right)\right) dt + e^{-\rho s} V\left(X_{s}^{\pi}\right)\right| X_{0}^{\pi} = x\right], \ s > 0.$$

Proceeding with standard arguments, we deduce that V satisfies the following Hamilton-Jacobi-Bellmam (HJB) equation

$$\rho V(x) = \max_{\pi \in \mathcal{P}(U)} \left(\tilde{r}(x,\pi) - \lambda \int_{U} \pi(u) \ln \pi(u) du + \frac{1}{2} \tilde{\sigma}^{2}(x,\pi) V''(x) \right)$$

$$+\tilde{b}(x,\pi)V'(x)$$
, $x \in \mathbb{D}$, (13)

or

$$\rho V(x) = \max_{\pi \in \mathcal{P}(U)} \int_{U} \left(r(x, u) - \lambda \ln \pi(u) + \frac{1}{2} \sigma^{2}(x, u) V''(x) + b(x, u) V'(x) \right) \pi(u) du,$$
(14)

where, with a slight abuse of notation, we have denoted its generic solution by V. Note that $\pi \in \mathcal{P}(U)$ if and only if

$$\int_{U} \pi(u)du = 1 \quad \text{and} \quad \pi(u) \ge 0 \text{ a.e. on } U.$$
(15)

Solving the (constrained) maximization problem on the right hand side of (14) then yields the "feedback-type" optimizer

$$\pi^*(u;x) = \frac{\exp\left(\frac{1}{\lambda}\left(r(x,u) + \frac{1}{2}\sigma^2(x,u)V''(x) + b(x,u)V'(x)\right)\right)}{\int_U \exp\left(\frac{1}{\lambda}\left(r(x,u) + \frac{1}{2}\sigma^2(x,u)V''(x) + b(x,u)V'(x)\right)\right)du}.$$
 (16)

This leads to an optimal measure-valued process

$$\pi_t^* = \pi^*(u; X_t^*) = \frac{\exp\left(\frac{1}{\lambda}\left(r(X_t^*, u) + \frac{1}{2}\sigma^2(X_t^*, u)V''(X_t^*) + b(X_t^*, u)V'(X_t^*)\right)\right)}{\int_U \exp\left[\frac{1}{\lambda}\left(r(X_t^*, u) + \frac{1}{2}\sigma^2(X_t^*, u)V''(X_t^*) + b(X_t^*, u)V'(X_t^*)\right)\right]du},$$
(17)

where $\{X_t^*, t \geq 0\}$, solves (6) when the feedback control distribution $\pi_t^*(\cdot; X_t^*)$ is applied and assuming that $\pi_t^* \in \mathcal{A}(x)$.

The formula (17) elicits qualitative understanding about an optimal exploration. We further investigate this in the next section.

4 The Linear–Quadratic Case

We focus on the family of entropy-regularized problems with linear state dynamics and quadratic rewards, in which 6

$$b(x, u) = Ax + Bu$$
, and $\sigma(x, u) = Cx + Du$, $x, u \in \mathbb{R}$ (18)

where A, B, C, D are given constants with $D \neq 0$, and

$$r(x,u) = -\left(\frac{M}{2}x^2 + Rxu + \frac{N}{2}u^2 + Px + Qu\right), \ x, u \in \mathbb{R}$$
 (19)

where $M \geq 0, N > 0, R, P, Q \in \mathbb{R}$.

In the classical control literature, this type of linear–quadratic (LQ) control problems is one of the most important in that it admits elegant and simple solutions and, furthermore, more complex, nonlinear problems can be approximated

 $^{^6}$ We assume both the state and the control are scalar-valued for notational simplicity. There is no essential difficulty with these being vector-valued.

by LQ problems. As is standard with LQ control, we assume that the control is unconstrained, namely, $U = \mathbb{R}$.

Fix an initial state $x \in \mathbb{R}$. For each measured-valued control $\pi \in \mathcal{A}(x)$, we define its mean and variance processes μ_t , σ_t^2 , $t \geq 0$, namely,

$$\mu_t := \int_{\mathbb{R}} u \pi_t(u) du$$
 and $\sigma_t^2 := \int_{\mathbb{R}} u^2 \pi_t(u) du - \mu_t^2$.

Then (6) can be rewritten as

$$dX_t^{\pi} = (AX_t^{\pi} + B\mu_t) dt + \sqrt{C^2(X_t^{\pi})^2 + 2CDX_t^{\pi}\mu_t + D^2(\mu_t^2 + \sigma_t^2)} dW_t$$
$$= (AX_t^{\pi} + B\mu_t) dt + \sqrt{(CX_t^{\pi} + D\mu_t)^2 + D^2\sigma_t^2} dW_t, \ t > 0; \quad X_0^{\pi} = x.$$
(20)

Further, define

$$L(X_t^{\pi}, \pi_t) := \int_{\mathbb{R}} r(X_t^{\pi}, u) \pi_t(u) du - \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du.$$

We will be working with the following assumption for the entropy-regularized LQ problem.

Assumption 1 The discount rate satisfies $\rho \gg 2A + C^2$.

Here, $a\gg b$ means a-b is sufficiently larger than 0. This assumption requires a sufficiently large discount rate, or (implicitly) a sufficiently short planning horizon. Such an assumption is standard in an infinite horizon problem with running rewards.

We are now ready to define the admissible control set $\mathcal{A}(x)$ for the entropy-regularized LQ problem as follows: $\pi \in \mathcal{A}(x)$, if

- for each $t \geq 0$, $\pi_t \in \mathcal{P}(\mathbb{R})$ a.s.;
- for each $A \in \mathcal{B}(\mathbb{R})$, $\{\pi_t(A), t \geq 0\}$ is \mathcal{F}_t -progressively measurable;
- for each $t \ge 0$, $\mathbb{E}\left[\int_0^t \left(\mu_s^2 + \sigma_s^2\right) ds\right] < \infty$;
- there exists $\delta \in (0, \rho (2A + C^2))$, such that

$$\liminf_{T \to \infty} e^{-\delta T} \int_0^T \mathbb{E}\left[\mu_t^2 + \sigma_t^2\right] dt = 0;$$

• with X^{π} solving (20), $\mathbb{E}\left[\int_0^{\infty} e^{-\rho t} |L(X_t^{\pi}, \pi_t)| dt \mid X_0^{\pi} = x\right] < \infty$.

Under the above conditions, it is immediate that for any $\pi \in \mathcal{A}(x)$, both the drift and volatility terms of (20) satisfy a global Lipschitz condition in the state variable; hence the SDE (20) admits a unique strong solution X^{π} .

Next, we solve the entropy-regularized stochastic LQ problem

$$V(x) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\int_{\mathbb{R}} r(X_t^{\pi}, u) \pi_t(u) du - \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du\right) dt\right]$$
(21)

with r as in (19) and X^{π} as in (20).

In the following two subsections, we respectively derive explicit solutions for the case of state-independent reward and of general state-dependent reward, respectively.

4.1 The case of state-independent reward

We start with the technically less challenging case when $r(x,u) = -\left(\frac{N}{2}u^2 + Qu\right)$, namely, the reward is state independent. In this case, the system dynamics become irrelevant. However, the problem is still interesting in its own right as it corresponds to the state-independent RL problem, which is known as the continuous-armed bandit problem in the continuous-time setting (see, e.g., Mandelbaum (1987); Kaspi and Mandelbaum (1998)).

Following the derivation in the previous section, the optimal control distribution in feedback form, (16), reduces to

$$\pi^* (u; x) = \frac{\exp\left(\frac{1}{\lambda} \left(\left(-\frac{N}{2}u^2 + Qu \right) + \frac{1}{2}(Cx + Du)^2 v''(x) + (Ax + Bu)v'(x) \right) \right)}{\int_{\mathbb{R}} \exp\left(\frac{1}{\lambda} \left(\left(-\frac{N}{2}u^2 + Qu \right) + \frac{1}{2}(Cx + Du)^2 v''(x) + (Ax + Bu)v'(x) \right) \right) du}$$

$$= \frac{\exp\left(-\left(u - \frac{CDxv''(x) + Bv'(x) - Q}{N - D^2v''(x)} \right)^2 / \frac{2\lambda}{N - D^2v''(x)} \right)}{\int_{\mathbb{R}} \exp\left(-\left(u - \frac{CDxv''(x) + Bv'(x) - Q}{N - D^2v''(x)} \right)^2 / \frac{2\lambda}{N - D^2v''(x)} \right) du}. \tag{22}$$

So the optimal control distribution appears to be *Gaussian*. More specifically, at any present state x, the agent should embark exploration according to the Gaussian distribution with mean and variance given, respectively, by $\frac{CDxv''(x)+Bv'(x)-Q}{N-D^2v''(x)}$ and $\frac{\lambda}{N-D^2v''(x)}$. We note that it is required in the expression (22) that $N-D^2v''(x)>0$, $x\in\mathbb{R}$, a condition that will be justified and discussed later on.

Remark 2 If we examine the derivation of (22) more closely, we easily see that the optimality of Gaussian distribution still holds so long as the state dynamics is linear in control and the reward is quadratic in control whereas their dependence on state can be generally nonlinear.

Substituting (22) back to (13), the HJB equation becomes, after straightforward calculations,

$$\rho v(x) = \frac{\left(CDxv''(x) + Bv'(x) - Q\right)^2}{2(N - D^2v''(x))} + \frac{\lambda}{2} \left(\ln\left(\frac{2\pi e\lambda}{N - D^2v''(x)}\right) - 1\right) + \frac{1}{2}C^2x^2v''(x) + Axv'(x).$$
(23)

In general, this nonlinear equation has multiple smooth solutions, even among quadratic polynomials that satisfy $N - D^2v''(x) > 0$. One such solution is a constant, given by

$$v(x) \equiv v = \frac{Q^2}{2\rho N} + \frac{\lambda}{2\rho} \left(\ln \frac{2\pi e\lambda}{N} - 1 \right), \tag{24}$$

with the corresponding optimal control distribution (22) being

$$\pi^* (u; x) = \frac{e^{-\left(u + \frac{Q}{N}\right)^2 / \frac{2\lambda}{N}}}{\int_{\mathbb{R}} e^{-\left(u + \frac{Q}{N}\right)^2 / \frac{2\lambda}{N}} du}.$$

Note that the classical LQ problem with the state-independent reward function $r(x,u) = -\left(\frac{N}{2}u^2 + Qu\right)$ clearly has the optimal control $u^* = -\frac{Q}{N}$, which is the mean of the optimal Gaussian control distribution π^* . The following result establishes that this constant v is indeed the value function V.

Henceforth, we denote, for notational convenience, by $\mathcal{N}(\cdot|\mu,\sigma^2)$ the distribution function of a Gaussian random variable with mean μ and variance σ^2 .

Theorem 3 If $r(x,u) = -(\frac{N}{2}u^2 + Qu)$, then the value function in (21) is given by

$$V(x) = \frac{Q^2}{2\rho N} + \frac{\lambda}{2\rho} \left(\ln \frac{2\pi e\lambda}{N} - 1 \right), \ x \in \mathbb{R},$$

and the optimal control distribution is Gaussian: $\pi_t^*(u) = \mathcal{N}\left(u \mid -\frac{Q}{N}, \frac{\lambda}{N}\right)$, $t \geq 0$. The associated optimal state process solves the SDE

$$dX_t^* = \left(AX_t^* - \frac{BQ}{N}\right)dt + \sqrt{\left(CX_t^* - \frac{DQ}{N}\right)^2 + \frac{\lambda D^2}{N}}dW_t, \ X_0^* = x, \quad (25)$$

which can be explicitly expressed as follows:

i) If
$$C \neq 0$$
, then

$$X_t^* = F(W_t, Y_t), \quad t \ge 0,$$
 (26)

where the function

$$F(z,y) = \sqrt{\frac{\lambda}{N}} \left| \frac{D}{C} \right| \sinh \left(|C|z + \sinh^{(-1)} \left(\sqrt{\frac{N}{\lambda}} \left| \frac{C}{D} \right| \left(y - \frac{DQ}{CN} \right) \right) \right) + \frac{DQ}{CN},$$

and the process Y_t , $t \ge 0$, is the unique pathwise solution to the random ordinary differential equation

$$\frac{dY_t}{dt} = \frac{AF(W_t, Y_t) - \frac{BQ}{N} - \frac{1}{2} \left(\left(CF(W_t, Y_t) - \frac{DQ}{N} \right)^2 + \frac{\lambda D^2}{N} \right)}{\frac{\partial}{\partial y} F(z, y) \big|_{z=W_t, y=Y_t}}, Y_0 = x.$$

ii) If
$$C = 0$$
, $A \neq 0$, then

$$X_t^* = xe^{At} - \frac{BQ}{AN}(1 - e^{At}) + \frac{|D|}{N}\sqrt{Q^2 + \lambda N} \int_0^t e^{A(t-s)}dW_s, \quad t \ge 0.$$
 (27)

Proof. See Appendix A.

The above solution suggests that when the reward is independent of the state, so is the optimal control distribution $\mathcal{N}(\cdot|-\frac{Q}{N},\frac{\lambda}{N})$. This is intuitive as objective (12) does not explicitly distinguish between states.⁷

A remarkable feature of the derived optimal distribution $\mathcal{N}(\cdot \mid -\frac{Q}{N}, \frac{\lambda}{N})$ is that the mean coincides with the optimal control of the original non-exploratory LQ problem, whereas the variance is determined by the parameter λ . In the context of continuous-armed bandit problem, the mean is concentrated on the current incumbent of the best arm and the variance is determined by the parameter. The more weight put on the level of exploration, the more spread out the exploration around the current best arm. This type of exploration/exploitation strategies is clearly intuitive, and in turn, it gives a guidance on how to actually choose the parameter in practice: it is nothing else than the variance of the exploration the agent wishes to engage.

However, we shall see in the next section that when the reward depends on local state, the HJB equation does not admit a constant solution. Consequently, the optimal control distribution must be of a feedback form, depending on where the state is at any given time.

4.2 The case of state-dependent reward

We now consider the case of a general state-dependent reward, i.e.,

$$r(x,u) = -\left(\frac{M}{2}x^2 + Rxu + \frac{N}{2}u^2 + Px + Qu\right), \quad x, u \in \mathbb{R}.$$

We start with the following lemma that will be used for the verification arguments.

Lemma 4 For each $\pi \in \mathcal{A}(x)$, $x \in \mathbb{R}$, the unique solution X_t^{π} , $t \geq 0$, to the SDE (20) satisfies

$$\lim_{T \to \infty} \inf e^{-\rho T} \mathbb{E}\left[\left(X_T^{\pi} \right)^2 \right] = 0.$$
(30)

$$\sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[-\int_0^\infty e^{-\rho t} \left(\int_U \pi_t(u) \ln \pi_t(u) du\right) dt \,\middle|\, X_0^\pi = x\right]. \tag{28}$$

This problem becomes relevant when $\lambda \uparrow \infty$ in the entropy-regularized objective (21), corresponding to the extreme case when the least informative (or the highest entropy) distribution is favored for pure exploration without considering exploitation (i.e., without maximizing any rewards). To solve problem (28), we can pointwisely maximize the integrand there, leading to the state-independent optimization problem

$$\sup_{\pi \in \mathcal{P}(U)} \left(-\int_{U} \pi(u) \ln \pi(u) du \right). \tag{29}$$

If the action space U is a finite interval, say [a,b] on \mathbb{R} , then it is straightforward that the optimal control distribution π^* is, for all $t \geq 0$, the uniform distribution on the interval [a,b]. This is in accordance with the traditional static setting where uniform distribution achieves maximum entropy on any finite interval (see, e.g., Shannon (2001)).

 $^{^{7}}$ Similar observation can be made for the (state-independent) pure entropy maximization formulation, where the goal is to solve

Proof. See Appendix B.

Following an analogous argument as in (22), we deduce that

$$\pi^*(u;x) := \mathcal{N}\left(u \,\middle|\, \frac{CDxv''(x) + Bv'(x) - Rx - Q}{N - D^2v''(x)} \,,\, \frac{\lambda}{N - D^2v''(x)}\right).$$

Then, denoting by $\mu^*(x)$ and $(\sigma^*(x))^2$ the mean and variance of π^* given above, we have

$$\begin{split} \rho v(x) &= \int_{\mathbb{R}} -\left(\frac{M}{2}x^2 + Rxu + \frac{N}{2}u^2 + Px + Qu\right)\mathcal{N}\left(u \ \middle| \mu^*(x), (\sigma^*(x))^2\right) du \\ &+ \lambda \ln\left(\sqrt{2\pi e}\sigma^*(x)\right) + v'(x) \int_{\mathbb{R}} (Ax + Bu)\mathcal{N}\left(u \ \middle| \mu^*(x), (\sigma^*(x))^2\right) du \\ &+ \frac{1}{2}v''(x) \int_{\mathbb{R}} (Cx + Du)^2 \mathcal{N}\left(u \ \middle| \mu^*(x), (\sigma^*(x))^2\right) du \\ &= -\frac{M}{2}x^2 - \frac{N}{2}\left(\left(\frac{CDxv''(x) + Bv'(x) - Rx - Q}{N - D^2v''(x)}\right)^2 + \frac{\lambda}{N - D^2v''(x)}\right) \\ &- (Rx + Q)\frac{CDxv''(x) + Bv'(x) - Rx - Q}{N - D^2v''(x)} - Px + \lambda \ln\sqrt{\frac{2\pi e\lambda}{N - D^2v''(x)}} \\ &+ Axv'(x) + Bv'(x)\frac{CDxv''(x) + Bv'(x) - Rx - Q}{N - D^2v''(x)} + \frac{1}{2}C^2x^2v''(x) \\ &+ \frac{1}{2}D^2\left(\left(\frac{CDxv''(x) + Bv'(x) - Rx - Q}{N - D^2v''(x)}\right)^2 + \frac{\lambda}{N - D^2v''(x)}\right)v''(x) \\ &+ CDxv''(x)\frac{CDxv''(x) + Bv'(x) - Rx - Q}{N - D^2v''(x)}. \end{split}$$

Reorganizing, we arrive at the HJB equation

$$\rho v(x) = \frac{\left(CDxv''(x) + Bv'(x) - Rx - Q\right)^2}{2(N - D^2v''(x))} + \frac{\lambda}{2} \left(\ln\left(\frac{2\pi e\lambda}{N - D^2v''(x)}\right) - 1\right) + \frac{1}{2}(C^2v''(x) - M)x^2 + (Av'(x) - P)x.$$
(31)

Under Assumption 1 and the additional condition $R^2 < MN$, one smooth solu-

tion to the HJB equation (31) is given by $v(x) = \frac{1}{2}k_2x^2 + k_1x + k_0$ where⁸

$$k_2 := \ \ \tfrac{1}{2} \tfrac{(\rho - (2A + C^2))N + 2(B + CD)R - D^2M}{(B + CD)^2 + (\rho - (2A + C^2))D^2}$$

$$-\frac{1}{2}\frac{\sqrt{((\rho-(2A+C^2))N+2(B+CD)R-D^2M)^2-4((B+CD)^2+(\rho-(2A+C^2))D^2)(R^2-MN)}}{(B+CD)^2+(\rho-(2A+C^2))D^2};$$
(35)

$$k_1 := \frac{P(N - k_2 D^2) - QR}{k_2 B(B + CD) + (A - \rho)(N - k_2 D^2) - BR},$$
(36)

and

$$k_0 := \frac{(k_1 B - Q)^2}{2\rho(N - k_2 D^2)} + \frac{\lambda}{2\rho} \left(\ln \left(\frac{2\pi e \lambda}{N - k_2 D^2} \right) - 1 \right). \tag{37}$$

For the particular solution given by $v(x) = \frac{1}{2}k_2x^2 + k_1x + k_0$, we can verify that $k_2 < 0$, due to Assumption 1 and $R^2 < MN$ (which holds automatically if R = 0, M > 0 and N > 0). Hence v is concave, a property that is essential in proving that it is actually the value function. On the other hand, $N - D^2v''(x) = N - k_2D^2 > 0$, ensuring that k_0 is well defined.

Theorem 5 Suppose $D \neq 0$, $R^2 < MN$ and the reward function is given by

$$r(x,u) = -\left(\frac{M}{2}x^2 + Rxu + \frac{N}{2}u^2 + Px + Qu\right).$$

Furthermore, suppose that Assumption 1 holds. Then, the value function is given by

$$V(x) = \frac{1}{2}k_2x^2 + k_1x + k_0, \quad x \in \mathbb{R},$$
(38)

where k_2 , k_1 and k_0 are as in (35), (36) and (37), respectively. Moreover, the optimal control distribution is Gaussian, given in a feedback form by

$$\pi_t^*(u) = \mathcal{N}\left(u \ \left| \frac{(k_2(B+CD)-R)X_t^* + k_1B - Q}{N - k_2D^2} \right., \ \frac{\lambda}{N - k_2D^2} \right.\right), \ t \ge 0.$$
(39)

$$\rho a_2 = \frac{(a_2(B+CD)-R)^2}{N-a_2D^2} + a_2(2A+C^2) - M,$$
(32)

$$\rho a_1 = \frac{(a_1 B - Q)(a_2 (B + CD) - R)}{N - a_2 D^2} + a_1 A - P,$$
(33)

$$\rho a_0 = \frac{(a_1 B - Q)^2}{2(N - a_2 D^2)} + \frac{\lambda}{2} \left(\ln \left(\frac{2\pi e \lambda}{N - a_2 D^2} \right) - 1 \right). \tag{34}$$

This system has in general multiple solutions, leading to multiple quadratic solutions to the HJB equation (31).

The values of k_2 , k_1 and k_0 can be found by substituting $v(x) = \frac{1}{2}k_2x^2 + k_1x + k_0$ into (31) and solving the resulting equation in k_2 , k_1 and k_0 . On the other hand, there are multiple solutions to (31). Indeed, applying an generic quadratic function ansatz $v(x) = \frac{1}{2}a_2x^2 + a_1x + a_0$, $x \in \mathbb{R}$ in (31) yields a system of algebraic equations

Finally, the optimal state process solves the SDE

$$dX_{t}^{*} = \left[\left(A + \frac{B(k_{2}(B+CD)-R)}{N-k_{2}D^{2}} \right) X_{t}^{*} + \frac{B(k_{1}B-Q)}{N-k_{2}D^{2}} \right] dt + \sqrt{\left[\left(C + \frac{D(k_{2}(B+CD)-R)}{N-k_{2}D^{2}} \right) X_{t}^{*} + \frac{D(k_{1}B-Q)}{N-k_{2}D^{2}} \right]^{2} + \frac{\lambda D^{2}}{N-k_{2}D^{2}} dW_{t},$$

$$(40)$$

 $X_0^* = x$, which can be expressed as follows:

i) if
$$C + \frac{D(k_2(B+CD)-R)}{N-k_2D^2} \neq 0$$
, then
$$X_t^* = F(W_t, Y_t), \quad t \geq 0,$$
(41)

where the function

$$F(z,y) = \frac{\sqrt{\tilde{D}}}{|\tilde{C}_1|} \sinh \left(|\tilde{C}_1|z + \sinh^{(-1)} \left(\frac{|\tilde{C}_1|}{\sqrt{\tilde{D}}} \left(y + \frac{\tilde{C}_2}{\tilde{C}_1} \right) \right) \right) - \frac{\tilde{C}_2}{\tilde{C}_1},$$

and the process Y_t , $t \geq 0$, is the unique pathwise solution to the random ordinary differential equation

$$\frac{dY_t}{dt} = \frac{\tilde{A}F(W_t, Y_t) + \tilde{B} - \frac{1}{2}\left(\left(\tilde{C}_1 F(W_t, Y_t) + \tilde{C}_2\right)^2 + \tilde{D}^2\right)}{\frac{\partial}{\partial y} F(z, y)\big|_{z=W_t, y=Y_t}}, \ Y_0 = x,$$

with
$$\tilde{A}:=A+\frac{B(k_2(B+CD)-R)}{N-k_2D^2},\ \tilde{B}:=\frac{B(k_1B-Q)}{N-k_2D^2},\ \tilde{C_1}:=C+\frac{D(k_2(B+CD)-R)}{N-k_2D^2}$$

$$\tilde{C}_2 := \frac{D(k_1B - Q)}{N - k_2D^2}$$
 and $\tilde{D} := \frac{\lambda D^2}{N - k_2D^2}$.

ii) If
$$C + \frac{D(k_2(B+CD)-R)}{N-k_2D^2} = 0$$
, $\tilde{A} \neq 0$, then
$$X_t^* = xe^{\tilde{A}t} - \frac{\tilde{B}}{\tilde{A}}(1 - e^{\tilde{A}t}) + \sqrt{\tilde{C}_1^2 + \tilde{D}} \int_0^t e^{\tilde{A}(t-s)} dW_s, \quad t \geq 0.$$
(42)

Proof. See Appendix C. \blacksquare

For the state-dependent reward case, we see that actions over U ($U = \mathbb{R}$) also depend on the current state X_t^* , which are selected according to a state-dependent Gaussian distribution (39). Compared with the state-independent case studied earlier, the variance of this distribution is smaller: $\frac{\lambda}{N-k_2D^2} < \frac{\lambda}{N}$ because $N-k_2D^2 > N$ (recall that $k_2 < 0$). Moreover, the greater D the smaller the variance, indicating less exploration is required. Recall that D is the coefficient of the control in the diffusion term of the state dynamics. So the need for exploration is reduced if an action has a greater impact on volatility

of the system dynamics. This hints that a more volatile environment renders more learning opportunities.

On the other hand, the mean of the Gaussian distribution does not explicitly depend on λ . The implication is that the agent should concentrate on the most promising region in the action space while randomly selecting actions to interact with the unknown environment. It is intriguing that the entropy-regularized RL formulation separates the exploitation from exploration, through respectively the mean and variance of the resulting optimal Gaussian distribution.

Finally, as noted earlier (see Remark 2), the optimality of the Gaussian distribution is still valid for problems with dynamics

$$dx_t = (A(x_t) + B(x_t)u_t) dt + (C(x_t) + D(x_t)u_t) dW_t,$$

and reward function in the form $r(x, u) = r_2(x)u^2 + r_1(x)u + r_0(x)$, where the functions A, B, C, D, r_2, r_1 and r_0 are possibly nonlinear.

5 The Cost and Effect of Exploration

Motivated by the necessity of exploration in face of the typically unknown environment in a RL setting, we have formulated and analyzed a new class of stochastic control problems that combine entropy-regularized criteria and relaxed controls. We have also derived explicit solutions and presented verification results for the representative LQ problems. A natural question is to quantify the cost and effect of the exploration. This can be done by comparing our results to the ones on the classical stochastic LQ problems which have neither entropy regularization nor control relaxation.

We carry out this comparative analysis next.

5.1 The classical LQ problem

We first recall the classical stochastic LQ control problem in an infinite horizon with discounted reward. Let $\{W_t, t \geq 0\}$ be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ that satisfies the usual conditions. The controlled state process $x = \{x_t, t \geq 0\}$ solves

$$dx_t = (Ax_t + Bu_t) dt + (Cx_t + Du_t) dW_t, \quad t \ge 0, \quad x_0 = x, \tag{43}$$

with given constants A, B, C and D, and the process $u = \{u_t, t \ge 0\}$ being a (standard, non-relaxed) control.

The value function is defined as in (2),

$$w(x) := \sup_{\mathcal{U}} \mathbb{E} \left[\int_0^\infty e^{-\rho t} r(x_t, u_t) dt \middle| x_0 = x \right],$$

for $x \in \mathbb{R}$, where the reward function $r(\cdot, \cdot)$ is given by (19).

The associated HJB equation is

$$\rho w(x) = \max_{u \in \mathbb{R}} \left(r(x, u) + \frac{1}{2} (Cx + Du)^2 w''(x) + (Ax + Bu) w'(x) \right)$$

$$= \max_{u \in \mathbb{R}} \left(-\frac{1}{2} \left(N - D^2 w''(x) \right) u^2 + \left(CDxw''(x) + Bw'(x) - Rx - Q \right) u \right)$$

$$+ \frac{1}{2} (C^2 w''(x) - M) x^2 + (Aw'(x) - P) x$$

$$= \frac{(CDxw''(x) + Bw'(x) - Rx - Q)^2}{2(N - D^2 w''(x))} + \frac{1}{2} (C^2 w''(x) - M) x^2 + (Aw'(x) - P) x,$$
(44)

with the maximizer being

$$u^*(x) = \frac{CDxw''(x) + Bw'(x) - Rx - Q}{N - D^2w''(x)}, \ x \in \mathbb{R}.$$

The optimality of u^* follows easily if $N - D^2w''(x) > 0$ holds.

To rigorously demonstrate that a quadratic function, say $w(x) = \frac{1}{2}\hat{a}_2x^2 + \hat{a}_1x + \hat{a}_0$, $x \in \mathbb{R}$, with $N - \hat{a}_2D^2 > 0$, which solves the HJB equation (44), is indeed the value function and, more generally, to establish a verification theorem for the above feedback optimal policy, extra conditions on the model parameters and restrictions on the admissible strategies are needed, similar to the entropy-regularized LQ problem case. In the next section, we establish solvability equivalence between the standard LQ problem and the entropy-regularized LQ problem.

5.2 Equivalence of solvability of classical and exploratory models

Given a reward function $r(\cdot, \cdot)$ and a classical controlled process (1), the relaxed formulation (6) under the entropy-regularized objective typically makes the problem more technically challenging to solve, compared to the corresponding classical stochastic optimization problem.

In this section, we demonstrate that there is equivalence between the solvability of the exploratory stochastic LQ problem and that of its classical counterpart, in the sense that the value function and optimal control of one problem lead immediately to those of the other. Such equivalence enables us to readily establish the convergence result as the exploration weight λ decays to zero. Furthermore, it makes it possible to quantify the exploration cost to be defined in the sequel.

Theorem 6 The following two statements (a) and (b) are equivalent.

(a) The function $V(x) = \frac{1}{2}\alpha_2 x^2 + \alpha_1 x + \alpha_0 + \frac{\lambda}{2\rho} \left(\ln \left(\frac{2\pi e\lambda}{N - \alpha_2 D^2} \right) - 1 \right)$, $x \in \mathbb{R}$, with $\alpha_2 < 0$, solves the HJB equation (31) and the corresponding optimal Gaussian distribution is given by

$$\pi_t^* = \mathcal{N}\left(u \left| \frac{(\alpha_2(B+CD)-R)X_t^* + \alpha_1B - Q}{N - \alpha_2D^2} \right., \left. \frac{\lambda}{N - \alpha_2D^2} \right), \right.$$

with $\{X_t^*, t \geq 0\}$ being the corresponding optimal controlled state process solving (20).

(b) The function $w(x) = \frac{1}{2}\alpha_2 x^2 + \alpha_1 x + \alpha_0$, $x \in \mathbb{R}$, with $\alpha_2 < 0$, solves the HJB equation (44), and the corresponding optimal control is given by

$$u_t^* = \frac{(\alpha_2(B + CD) - R)x_t^* + \alpha_1 B - Q}{N - \alpha_2 D^2},$$

with $\{x_t^*, t \geq 0\}$ being the corresponding classical controlled state process solving (43).

Proof. See Appendix D.

The above equivalence between statement (a) and (b), together with the verification arguments for the entropy-regularized and classical stochastic LQ problems, yield that if one problem is solvable, so is the other, and conversely, if one is not solvable, neither is the other.

5.3 Cost of exploration

We define the exploration cost for a general RL problem to be the difference between the discounted accumulated rewards following the respective optimal strategies under the classical objective (2) and the exploratory objective (12). Note that the equivalence between solvability established in the previous subsection is important for this definition since the cost can be well defined only if both the classical and the exploratory problems are solvable.

Specifically, let the classical maximization problem (2) with the state dynamics (1) have the value function $w(\cdot)$ and an optimal strategy $\{u_t^*, t \geq 0\}$, and the corresponding exploratory problem have the value function $V(\cdot)$ and an optimal control distribution $\{\pi_t^*, t \geq 0\}$. Then, we define the exploration cost as

$$\mathcal{C}^{u^*,\pi^*}(x) := w(x) - \left(V(x) + \lambda \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\int_U \pi_t^*(u) \ln \pi_t^*(u) du\right) dt \middle| X_0 = x\right]\right),\tag{45}$$

for $x \in \mathbb{D}$.

The term in the parenthesis represents the total discounted rewards incurred by π^* after taking out the contribution of the entropy term to the value function $V(\cdot)$ for the exploratory problem. The exploration cost hence measures the best

outcome due to the explicit inclusion of exploratory strategies in the entropy-regularized objective, relative to the benchmark $w(\cdot)$ that is the best possible objective value should the model be fully known.

We next compute the exploration cost for the LQ case. This cost is surprisingly simple: it depends only on two "agent-specific" parameters: the exploration weight parameter λ and the discounting parameter ρ .

Indeed, if statement (a) (or (b), equivalently) holds, then both the augmented and classical stochastic LQ problems are solvable, leading to the exploration cost given by, for $x \in \mathbb{R}$,

$$C^{u^*,\pi^*}(x) = w(x) - \left(V(x) - \frac{\lambda}{2\rho} \ln\left(\frac{2\pi e\lambda}{N - \alpha_2 D^2}\right)\right) = \frac{\lambda}{2\rho},$$

where we have used the fact that the differential entropy of Gaussian distribution $\mathcal{N}(\cdot | \mu, \sigma^2)$ is $\ln(\sigma \sqrt{2\pi e})$, as well as $V(\cdot)$, π^* in (a), and $w(\cdot)$ in (b).

So, the exploration cost for stochastic LQ problems can be completely predetermined by the learning agent through choosing her individual λ and ρ , since it does not rely on the specific (unknown) linear state dynamics, nor the quadratic reward structure.

Moreover, the exploration cost depends on λ and ρ in a rather intuitive way: it increases as λ increases, due to more emphasis placed on exploration, or as ρ decreases, indicating an effectively longer horizon for exploration.⁹

5.4 Vanishing exploration

Another desirable result is when the weight on the exploration weight λ goes to zero, the entropy-regularized LQ problem converges to its classical counterpart. The following result makes this precise.

Proposition 7 Assume that statement (a) (or equivalently, (b)) holds. Then, for each $t \ge 0$,

$$\lim_{\lambda \to 0} \pi_t^* = \delta_{u_t^*} \quad weekly. \tag{46}$$

Moreover, the associated value functions converge pointwisely, i.e., for each $x \in \mathbb{R}$, $\lim_{\lambda \to 0} |V(x) - w(x)| = 0$.

Proof. See Appendix E.

⁹The connection between a discounting parameter and an effective length of time horizon is well known in the discrete-time discounted reward formulation $\mathbb{E}[\sum_{t\geq 0} \gamma^t R_t]$ for classical Markov Decision Processes (see, e.g., Derman (1970)). This infinite horizon discounted problem can be viewed as an undiscounted, finite horizon problem with a random termination time T that is geometrically distributed with parameter $1-\gamma$. Hence, an effectively longer horizon with mean $\frac{1}{1-\gamma}$ is applied to the optimization problem as γ increases. Since a smaller ρ in the continuous-time objective (2) or (12) corresponds to a larger γ in the discrete-time objective, we can see the similar effect of a decreasing ρ on the effective horizon of continuous-time problems.

6 Conclusions

The main contributions of this paper are *conceptual*: casting the RL problem in a continuous-time setting and with the aid of stochastic control and stochastic calculus, we interpret and explain why Gaussian distribution is best for exploration in RL. This finding is independent of the specific parameters of the underlying dynamics and reward function structure, so long as the dependence on actions is linear in the former and quadratic in the latter. The same can be said about other important results, such as the separation between exploration and exploitation in the mean and variance of the resulting Gaussian distribution, and the cost of exploration. The explicit form of the optimal Gaussian does indeed depend on the model specifications which are unknown in the RL context. In implementing RL algorithms based on our results, we will need to turn the problem into an MDP problem by discretizing the time, and then learn the parameters of the optimal Gaussian distribution following standard RL procedure (e.g. the so-called Q-learning). For that our result is again useful: it suggests that we only need to learn among the class of simpler Gaussian policies, i.e., $\pi = \mathcal{N}(\cdot | \theta_1 x + \theta_2, \phi)$ (cf. (39)), rather than generic (nonlinear) parametrized Gaussian policy $\pi_{\theta,\phi} = \mathcal{N}(\cdot | \theta(x), \phi(x))$. We expect that this simpler function form can considerably increase the learning speed.

Appendix A: Proof of Theorem 3

Proof. Following the HJB equation (13), we deduce that for any $\pi \in \mathcal{A}(x)$, $x \in \mathbb{R}$,

$$\rho v \ge \int_{\mathbb{R}} -\left(\frac{N}{2}u^2 + Qu\right)\pi_t(u)du - \lambda \int_{\mathbb{R}} \pi_t(u)\ln \pi_t(u)du. \tag{47}$$

Indeed, one can directly verify the above inequality by recalling the constrained problem (15) for the maximization of the right hand side of (47). In turn, examining the first and second variation of the associated unconstrained problem

$$\mathcal{L}(\pi_t) = \int_{\mathbb{R}} -\left(\frac{N}{2}u^2 + Qu\right)\pi_t(u)du - \lambda \int_{\mathbb{R}} \pi_t(u)\ln \pi_t(u)du + \beta \left(\int_{\mathbb{R}} \pi_t(u)du - 1\right),$$

for a Lagrange multiplier $\beta \in \mathbb{R}$, give the maximizer

$$\pi_t^* \equiv \pi^* = \mathcal{N}\left(u \mid -\frac{Q}{N}, \frac{\lambda}{N}\right), \quad t \ge 0,$$

at which the equality in (47) is also attained. Note that, since $(\pi_t)_{t\geq 0} \in \mathcal{A}(x)$, $\mathcal{L}(\pi_t)$ is well defined. The first and second order differentiation with respect to ε under the integral in $\mathcal{L}(\pi_t + \varepsilon h)$ are justified by the dominated convergence theorem, if $h \in \mathcal{P}(\mathbb{R})$ is such that the first and second variations are integrable. In turn, for any $\pi \in \mathcal{A}(x)$ and $T \geq 0$,

$$e^{-\rho T}v = v - \int_0^T e^{-\rho t} \rho v dt$$

$$\leq v + \mathbb{E}\left[\int_0^T e^{-\rho t} \left(\int_{\mathbb{R}} \left(\frac{N}{2}u^2 + Qu\right) \pi_t(u) du + \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du\right) dt\right].$$

Since $\pi \in \mathcal{A}(x)$, the dominated convergence theorem yields, as $T \to \infty$, that

$$v \ge \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\int_{\mathbb{R}} -\left(\frac{N}{2}u^2 + Qu\right) \pi_t(u) du - \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du\right) dt\right]$$

and, thus, $v \geq V(x)$, for $x \in \mathbb{R}$. On the other hand,

$$\rho v = \int_{\mathbb{R}} -\left(\frac{N}{2}u^2 + Qu\right)\pi_t^*(u)du - \lambda \int_{\mathbb{R}} \pi_t^*(u)\ln \pi_t^*(u)du,$$

and it is also easily verified that $\pi^* \in \mathcal{A}(x)$. Replacing the inequalities by equalities in the above argument and sending T to infinity, we conclude that

$$V(x) \equiv v = \frac{Q^2}{2\rho N} + \frac{\lambda}{2\rho} \left(\ln \frac{2\pi e\lambda}{N} - 1 \right),$$

for $x \in \mathbb{R}$.

The augmented state dynamics (20) under π^* take the form

$$dX_t^* = \left(AX_t^* - \frac{BQ}{N}\right)dt + \sqrt{\left(CX_t^* - \frac{DQ}{N}\right)^2 + \frac{\lambda D^2}{N}}dW_t,\tag{48}$$

after the substitution of $\mu_t^* = -\frac{Q}{N}$ and $(\sigma_t^*)^2 = \frac{\lambda}{N}$, $t \ge 0$, in (20). If $C \ne 0$, equation (48) can be solved using the Doss-Saussman transforma-

If $C \neq 0$, equation (48) can be solved using the Doss-Saussman transformation (see, for example, Karatzas and Shreve (1991), pp 295-297). Specifically, we deduce that (48) has a unique solution, given by

$$X_t^*(\omega) = F(W_t(\omega), Y_t(\omega)), \quad t \ge 0,$$

for all $\omega \in \Omega$, with $F : \mathbb{R}^2 \to \mathbb{R}$ solving the ODE

$$\frac{\partial F}{\partial z} = \sqrt{\left(CF - \frac{DQ}{N}\right)^2 + \frac{\lambda D^2}{N}}, \quad F(0, y) = y. \tag{49}$$

The process Y_t , $t \geq 0$ is the unique pathwise solution to the random ODE

$$\frac{d}{dt}Y_t(\omega) = G(W_t(\omega), Y_t(\omega)), \quad Y_0(\omega) = x, \tag{50}$$

with the function

$$G(z,y) := \frac{AF(z,y) - \frac{BQ}{N} - \frac{1}{2} \left(\left(CF(z,y) - \frac{DQ}{N} \right)^2 + \frac{\lambda D^2}{N} \right)}{\frac{\partial}{\partial y} F(z,y)}.$$

It is then easy to verify that both equations (49) and (50) have a unique solution. Solving (49), we obtain that

$$F(z,y) = \sqrt{\frac{\lambda}{N}} \left| \frac{D}{C} \right| \sinh \left(|C|z + \sinh^{(-1)} \left(\sqrt{\frac{N}{\lambda}} \left| \frac{C}{D} \right| \left(y - \frac{DQ}{CN} \right) \right) \right) + \frac{DQ}{CN},$$

and, in turn, the form of the function G(z, y).

If C = 0, (48) becomes

$$dX_t^* = \left(AX_t^* - \frac{BQ}{N}\right) dt + \frac{|D|}{N}\sqrt{Q^2 + \lambda N} dW_t,$$

whose unique solution is given in (27) when $A \neq 0$.

Appendix B: Proof of Lemma 4

Proof. Since $\pi \in \mathcal{A}(x)$, the SDE (20) has a unique solution X_t^{π} defined for all $t \geq 0$. Applying Itô's formula, we obtain

$$d(X_t^{\pi})^2 = \left((2A + C^2) (X_t^{\pi})^2 + 2(B + CD)\mu_t X_t^{\pi} + D^2 \sigma_t^2 \right) dt$$
$$+2X_t^{\pi} \sqrt{(CX_t^{\pi} + D\mu_t)^2 + D^2 \sigma_t^2} dW_t.$$

Now, define the stopping times τ_n^{π} , $n \geq 1$,

$$\tau_n^{\pi} := \inf \left\{ t \ge 0 : \int_0^t \left(X_s^{\pi} \right)^2 \left(\left(C X_s^{\pi} + D \mu_s \right)^2 + D^2 \sigma_s^2 \right) \, ds \ge n \right\}.$$

Then, it follows that for fixed T > 0,

$$\mathbb{E}\left[\left(X_{T\wedge\tau_{n}^{\pi}}^{\pi}\right)^{2}\right]=x^{2}+\mathbb{E}\left[\int_{0}^{T\wedge\tau_{n}^{\pi}}\left(\left(2A+C^{2}\right)\left(X_{t}^{\pi}\right)^{2}+2(B+CD)\mu_{t}X_{t}^{\pi}+D^{2}\sigma_{t}^{2}\right)\,dt\right].$$

By the standard estimate $\mathbb{E}\left[\sup_{0\leq t\leq T}(X_t^\pi)^2\right]\leq K(1+x^2)e^{KT}$, for some constant K>0 depending only on the Lipschitz constant and the time T, we can apply the dominated convergence theorem and Fubini's theorem, yielding, as $n\to\infty$,

$$\mathbb{E}\left[\left(X_T^{\pi}\right)^2\right] = x^2 + \int_0^T (2A + C^2) \mathbb{E}\left[\left(X_t^{\pi}\right)^2\right] + 2(B + CD) \mathbb{E}\left[\mu_t X^{\pi_t}\right] + D^2 \mathbb{E}\left[\sigma_t^2\right] dt.$$

Using the inequality $2|\mu_t X_t^{\pi}| \leq \varepsilon^2 |X_t^{\pi}|^2 + \frac{1}{\varepsilon^2} |\mu_t|^2$, we deduce that

$$\mathbb{E}\left[\left(X_T^{\pi}\right)^2\right] \leq x^2 + \int_0^T (2A + C^2 + \varepsilon^2 |B + CD|) \mathbb{E}\left[\left(X_t^{\pi}\right)^2\right] dt$$

$$+ \int_0^T \left(\frac{|B + CD|}{\varepsilon^2} + D^2 \right) \mathbb{E} \left[\mu_t^2 + \sigma_t^2 \right] dt.$$

Under Assumption 1, we can choose ε such that $\hat{\rho} := 2A + C^2 + \varepsilon^2 |B + CD| < \rho - \delta$. Gronwall's inequality then implies

$$\mathbb{E}\left[\left(X_T^{\pi}\right)^2\right] \leq \left(x^2 + C_1 \int_0^T \mathbb{E}\left[\mu_t^2 + \sigma_t^2\right] dt\right) e^{\hat{\rho}T},$$

where $C_1 := \frac{|B+CD|}{\varepsilon^2} + D^2$ is a constant. It follows that

$$\liminf_{T \to \infty} e^{-\rho T} \mathbb{E}\left[\left(X_T^\pi \right)^2 \right] \leq \liminf_{T \to \infty} e^{-(\rho - \hat{\rho})T} \int_0^T \mathbb{E}\left[\mu_t^2 + \sigma_t^2 \right] \, dt = 0,$$

since $\pi \in \mathcal{A}(x)$, for $x \in \mathbb{R}$.

Appendix C: Proof of Theorem 5

Proof. Following the HJB equation (13), we deduce that for each $(\pi_t)_{t\geq 0} \in \mathcal{A}(x)$, $x \in \mathbb{R}$,

$$\rho v(x) \ge \tilde{r}(x, \pi_t) - \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du + \frac{1}{2} v''(x) \tilde{\sigma}^2(x, \pi_t)$$
$$+ v'(x) \tilde{b}(x, \pi_t), \tag{51}$$

with \tilde{r} , \tilde{b} and $\tilde{\sigma}$ given by (11), (7) and (8). Define the stopping times $\tau_n^{\pi} := \{t \geq 0 : \int_0^t \left(e^{-\rho t}v'(X_t^{\pi})\tilde{\sigma}(X_t^{\pi},\pi_t)\right)^2 dt \geq n\}$, for $n \geq 1$. Then, applying Itô's formula to $e^{-\rho t}v(X_{t\wedge\tau_n^{\pi}}^{\pi})$, $0 \leq t \leq T$, we obtain

$$e^{-\rho(T\wedge\tau_n^{\pi})}v(X_{T\wedge\tau_n^{\pi}}^{\pi}) = v(x) + \int_0^{T\wedge\tau_n^{\pi}} e^{-\rho t} \left(-\rho v(X_t^{\pi}) + \frac{1}{2}v''(X_t^{\pi})\tilde{\sigma}^2(X_t^{\pi}, \pi_t)\right) + v'(X_t^{\pi})\tilde{b}(X_t^{\pi}, \pi_t) dt + \int_0^{T\wedge\tau_n^{\pi}} e^{-\rho t}v'(X_t^{\pi})\tilde{\sigma}(X_t^{\pi}, \pi_t) dW_t.$$

Taking expectations and using that the stochastic integral is a true martingale yield

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_n^{\pi})}v(X_{T\wedge\tau_n^{\pi}}^{\pi})\right] =$$

$$v(x) + \mathbb{E}\left[\int_0^{T\wedge\tau_n^{\pi}} e^{-\rho t} \left(-\rho v(X_t^{\pi}) + \frac{1}{2}v''(X_t^{\pi})\tilde{\sigma}^2(X_t^{\pi}, \pi_t) + v'(X_t^{\pi})\tilde{b}(X_t^{\pi}, \pi_t)\right) dt\right]$$

$$\leq v(x) - \mathbb{E}\left[\int_0^{T\wedge\tau_n^{\pi}} e^{-\rho t} \left(\tilde{r}(X_t^{\pi}, \pi_t) - \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du\right) dt\right],$$

where we have also used the inequality (51). From classical results, we have that $\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t^{\pi}|^2\right]\leq K(1+x^2)e^{KT}$, for some constant K>0 depending only on the Lipschitz constant for (20) and T>0. Letting $n\to\infty$ then yields

$$\mathbb{E}\left[e^{-\rho T}v(X_T^{\pi})\right] \leq v(x) - \mathbb{E}\left[\int_0^T e^{-\rho t} \left(\tilde{r}(X_t^{\pi}, \pi_t) - \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du\right) dt\right],$$

where we have used the dominated convergence theorem and that $\pi \in \mathcal{A}(x)$. Furthermore, Lemma 4 gives that $\liminf_{T\to\infty} e^{-\rho T} \mathbb{E}\left[(X_T^{\pi})^2\right] = 0$. Together with $k_2 < 0$, this implies that $\limsup_{T\to\infty} \mathbb{E}\left[e^{-\rho T}v(X_T^{\pi})\right] \geq 0$. Applying the dominated convergence theorem once more yields

$$v(x) \ge \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\tilde{r}(X_t^{\pi}, \pi_t) - \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du\right) dt\right],$$

for each $x \in \mathbb{R}$ and $\pi \in \mathcal{A}(x)$. Hence, $v(x) \geq V(x)$, for all $x \in \mathbb{R}$. On the other hand, we deduce that the right hand side of (51) is maximized at

$$\pi^*(u;x) = \mathcal{N}\left(u \left| \begin{array}{c} CDxv''(x) + Bv'(x) - Rx - Q \\ N - D^2v''(x) \end{array} \right., \ \frac{\lambda}{N - D^2v''(x)}\right).$$

Therefore,

$$\mathbb{E}\left[e^{-\rho T}v(X_T^*)\right] = v(x) - \mathbb{E}\left[\int_0^T e^{-\rho t}\left(\tilde{r}(X_t^*, \pi_t^*) - \lambda \int_{\mathbb{R}} \pi_t^*(u) \ln \pi_t^*(u) du\right) dt\right].$$

Using that $k_2 < 0$ implies $\liminf_{T \to \infty} \mathbb{E}\left[e^{-\rho T}v(X_T^*)\right] \leq 0$, and applying the dominated convergence theorem yields

$$v(x) \le \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(\tilde{r}(X_t^*, \pi_t^*) - \lambda \int_{\mathbb{R}} \pi_t^*(u) \ln \pi_t^*(u) du\right) dt\right],$$

for any $x \in \mathbb{R}$, provided that $\pi^* \in \mathcal{A}(x)$. This proves that $v(\cdot)$ is indeed the value function.

It remains to show that $\pi^* \in \mathcal{A}(x)$. First, we verify that there exists $\delta \in (0, \rho - (2A + C^2))$, such that

$$\liminf_{T \to \infty} e^{-\delta T} \int_0^T \mathbb{E}\left[(\mu_t^*)^2 + (\sigma_t^*)^2 \right] dt = 0.$$

By the form of the π^* as in (39), it suffices to show that

$$\liminf_{T \to \infty} e^{-\delta T} \mathbb{E}\left[(X_T^*)^2 \right] = 0,$$
(52)

with X^* solving the SDE (40). Itô's formula yields, for any $T \geq 0$,

$$(X_T^*)^2 = x^2 + \int_0^T \left(2\left(\tilde{A}X_t^* + \tilde{B}\right)X_t^* + (\tilde{C}_1X_t^* + \tilde{C}_2)^2 + D^2 \right) dt$$

$$+ \int_{0}^{T} 2X_{t}^{*} \sqrt{\left(\tilde{C}_{1}X_{t}^{*} + \tilde{C}_{2}\right)^{2} + \tilde{D}^{2}} dW_{t}. \tag{53}$$

Following similar arguments as in Lemma 8 in Appendix, we can show that the leading terms of $\mathbb{E}[(X_T^*)^2]$ are, as $T \to \infty$, $e^{(2\tilde{A} + \tilde{C}_1^2)T}$ and $e^{\tilde{A}T}$. On the other hand, the case $2\tilde{A} + \tilde{C_1}^2 \leq \tilde{A}$ implies $\tilde{A} \leq 0$ and, hence, (52) easily follows for any $\delta > 0$. Therefore, it suffices to study the dominant term $e^{(2\tilde{A} + \tilde{C_1}^2)T}$. We have

$$2\tilde{A} + \tilde{C}_1^2 - \delta = 2A + \frac{2B(k_2(B+CD)-R)}{N-k_2D^2} + \left(C + \frac{D(k_2(B+CD)-R)}{N-k_2D^2}\right)^2 - \delta$$

$$= 2A + C^2 - \delta + \frac{2(B+CD)(k_2(B+CD)-R)}{N-k_2D^2} + \frac{D^2(k_2(B+CD)-R)^2}{(N-k_2D^2)^2}$$

$$= 2A + C^2 - \delta + \frac{k_2(2N-k_2D^2)(B+CD)^2}{N-k_2D^2} - \frac{2RN(B+CD)-D^2R^2}{N-k_2D^2}. (54)$$

Notice that the first fraction is nonpositive due to $k_2 < 0$, while the second fraction is bounded for any $k_2 < 0$. Therefore, under Assumption 1 on the range of ρ , we can find $\delta \in (0, \rho - (2A + C^2))$ that is large enough so that $2\tilde{A} + \tilde{C_1}^2 - \delta < 0$. This proves the desired result (52). Next, we establish the admissibility constraint $\mathbb{E}\left[\int_0^\infty e^{-\rho t} |L(X_t^*, \pi_t^*)| dt\right] < 0$

 ∞ . Using the definition of L and the form of r(x, u) yield

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \left| L(X_t^*, \pi_t^*) \right| dt \right]$$

$$= \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left| \int_{\mathbb{R}} r(X_t^*, u) \pi_t^*(u) du - \lambda \int_{\mathbb{R}} \pi_t(u) \ln \pi_t(u) du \right| dt \right]$$

$$= \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left| \int_{\mathbb{R}} -\left(\frac{M}{2} \left(X_t^*\right)^2 + RX_t^* u + \frac{N}{2} u^2 + PX_t^* + Qu\right) \pi_t^*(u) du + \frac{\lambda}{2} \ln \left(2\pi e(\sigma_t^*)^2\right) \right| dt \right].$$

Noticing that according to (39), the mean of π^* is linear in X^* while the variance is constant. It is hence sufficient to prove that $\mathbb{E}\left[\int_0^\infty e^{-\rho t}(X_t^*)^2 dt\right] < \infty$, which follows easily since, as shown in (54), $\rho > 2\tilde{A} + \tilde{C_1}^2$ under Assumption 1. The remaining admissibility conditions for π^* are easily checked. Finally, the unique strong solution (41) is derived from the Doss-Saussman transformation as in the proof of Theorem 3. ■

Appendix D: Proof of Theorem 6

Proof. We only show that (a) \Rightarrow (b), as the other direction follows with similar arguments. First, a comparison between the two HJB equations (31) and (44) yields that if $V(\cdot)$ in (a) solves the former, then $w(\cdot)$ in (b) solves the latter, and vice versa.

The optimality of the classical control u_t^* in (b) follows from the first-order condition as well as that $\alpha_2 < 0$ by the assumption in (a). It remains to show that the candidate optimal control policy $\{u_t^*, t \geq 0\}$ is indeed admissible. We first compute $\mathbb{E}[(X_T^*)^2]$ and $\mathbb{E}[(x_T^*)^2]$.

To this end, recall the exploratory state dynamics of X^* from (20) under the Gaussian policy π^* in (a)

$$\begin{split} dX_t^* &= \left(AX_t^* + B\frac{(\alpha_2(B+CD)-R)X_t^* + \alpha_1B - Q}{N-\alpha_2D^2}\right) \ dt \\ &+ \sqrt{\left(CX_t^* + D\frac{(\alpha_2(B+CD)-R)X_t^* + \alpha_1B - Q}{N-\alpha_2D^2}\right)^2 + \frac{\lambda D^2}{N-\alpha_2D^2}} \ dW_t \\ &= (A_1X_t^* + A_2) \ dt + \sqrt{\left(B_1X_t^* + B_2\right)^2 + C_1} \ dW_t, \\ \text{where } A_1 &:= A + \frac{B(\alpha_2(B+CD)-R)}{N-\alpha_2D^2}, \ A_2 &:= \frac{B(\alpha_1B-Q)}{N-\alpha_2D^2}, \ B_1 &:= C + \frac{D(\alpha_2(B+CD)-R)}{N-\alpha_2D^2}, \\ B_2 &:= \frac{D(\alpha_1B-Q)}{N-\alpha_2D^2} \ \text{and} \ C_1 &:= \frac{\lambda D^2}{N-\alpha_2D^2}. \end{split}$$

Itô's formula gives

$$d(X_t^*)^2 = ((2A_1 + B_1^2)(X_t^*)^2 + 2(A_2 + B_1B_2)X_t^* + B_2^2 + C_1) dt$$
$$+2X_t^* \sqrt{(B_1X_t^* + B_2)^2 + C_1} dW_t.$$

Following similar arguments as in the proof of Lemma 4, we have for any $T \geq 0$,

$$\mathbb{E}\left[(X_T^*)^2\right] = x^2 + \int_0^T \left((2A_1 + B_1^2)\mathbb{E}\left[(X_t^*)^2\right] + 2(A_2 + B_1B_2)\mathbb{E}\left[X_t^*\right]\right) dt + (B_2^2 + C_1)T.$$
(55)

On the other hand, a similar argument yields that for the state process x^* under the candidate optimal control u^* in (b) and $T \ge 0$,

$$\mathbb{E}\left[(x_T^*)^2\right] = x^2 + \int_0^T \left((2A_1 + B_1^2)\mathbb{E}\left[(x_t^*)^2\right] + 2(A_2 + B_1B_2)\mathbb{E}\left[x_t^*\right]\right) dt + B_2^2T.$$
(56)

By assumption of (a), the optimal control distribution π^* in (a) is admissible, and hence $\lim\inf_{T\to\infty}e^{-\delta T}\mathbb{E}[(X_T^*)^2]=0$, for some $\delta\in(0,\rho-(2A+C^2))$. Then, from Lemma 8 in Appendix F, the comparison between $\mathbb{E}\left[(X_T^*)^2\right]$ and $\mathbb{E}\left[(x_T^*)^2\right]$, given in (55) and (56), yields that $\liminf_{T\to\infty}e^{-\delta T}\mathbb{E}[(x_T^*)^2]=0$, as desired. To verify the admissible condition $\mathbb{E}\left[\int_0^\infty e^{-\rho t}|r(x_t^*,u_t^*)|\,dt\right]<\infty$, it suffices

To verify the admissible condition $\mathbb{E}\left[\int_0^\infty e^{-\rho t} |r(x_t^*, u_t^*)| dt\right] < \infty$, it suffices to show that $\mathbb{E}\left[\int_0^\infty e^{-\rho t} (x_t^*)^2 dt\right] < \infty$, due to the quadratic structure of the reward (19) and the linearity of the control u_t^* in the state x_t^* , as shown in (b).

The assumption that π^* in the statement (a) is admissible leads to that $\mathbb{E}\left[\int_0^\infty e^{-\rho t}|L(X_t^*,\pi_t^*)|dt\right]<\infty$, which, based on the form of π_t^* in (a), further implies that $\mathbb{E}\left[\int_0^\infty e^{-\rho t}(X_t^*)^2dt\right]<\infty$.

According to Lemma 8 in Appendix F, the comparison between (55) and (56) gives $\mathbb{E}\left[\int_0^\infty e^{-\rho t}(x_t^*)^2 dt\right] < \infty$. The other admissibility conditions are clearly satisfied by u^* in (b). We hence conclude that $\{u_t^*, t \geq 0\}$ is the optimal control. The proof is complete.

Appendix E: Proof of Proposition 7

Proof. We first show that, for each fixed $t \geq 0$, as $\lambda \to 0$ the optimal Gaussian control distribution $\pi_t^* = \pi^*(\cdot; X_t^*)$ in statement (a) converges weakly to the Dirac measure $\delta_{u_t^*}$, with u_t^* given in statement (b).

For t = 0, we have $X_0^* = x_0^* = x \in \mathbb{R}$, and the convergence follows easily. Given a fixed t > 0, we show that, for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{\lambda \to 0} \mathbb{E}\left[\int_{\mathbb{R}} f(u)\pi^*(u; X_t^*) du\right] = \mathbb{E}\left[\int_{\mathbb{R}} f(u)\delta_{u_t^*}(du)\right]. \tag{57}$$

To this end, let μ_t^* and $(\sigma_t^*)^2$ be the mean and variance of $\pi_t^* = \pi^*(u; X_t^*)$, respectively. Then,

$$\mathbb{E}\left[\int_{\mathbb{R}} f(u)\pi^{*}(u; X_{t}^{*})du\right] = \mathbb{E}\left[\int_{\mathbb{R}} f(u)\frac{1}{\sqrt{2\pi}\sigma_{t}^{*}}e^{-\frac{(u-\mu_{t}^{*})^{2}}{2(\sigma_{t}^{*})^{2}}}du\right]$$

$$= \mathbb{E}\left[\int_{\mathbb{R}} f(\sigma_{t}^{*}u + \mu_{t}^{*})\frac{1}{\sqrt{2\pi}}e^{-\frac{u^{2}}{2}}du\right] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathbb{E}\left[f(\sigma_{t}^{*}u + \mu_{t}^{*})\right]e^{-\frac{u^{2}}{2}}du,$$

where we have applied Fubini's theorem, considering that f is bounded.

Based on the explicit expressions for μ_t^* and $(\sigma_t^*)^2$ from statement (a), we have that, for each $u \in \mathbb{R}$ and $t \geq 0$, the random variable $\mu_t^* + \sigma_t^* u$ satisfies

$$\mu_t^* + \sigma_t^* u = \frac{(\alpha_2(B + CD) - R)X_t^* + \alpha_1 B - Q}{N - \alpha_2 D^2} + \frac{\lambda u}{N - \alpha_2 D^2}$$
$$= \frac{(\alpha_2(B + CD) - R)}{N - \alpha_2 D^2} X_t^* + \frac{\alpha_1 B - Q + \lambda u}{N - \alpha_2 D^2}.$$

Thus, it is a Gaussian random variable, since for each fixed t > 0, X_t^* is a Gaussian random variable under \mathbb{P} , with mean and variance computed as N(t) and $M(t) - N(t)^2$ from Lemma 8 in Appendix C. It then follows that the mean and variance of $\mu_t^* + \sigma_t^* u$ are given, respectively, by

$$\alpha_t(u) := \frac{(\alpha_2(B+CD)-R)N(t) + \alpha_1B - Q + \lambda u}{N - \alpha_2D^2}$$

and

$$\beta_t(u) = \left(\frac{(\alpha_2(B+CD)-R)}{N-\alpha_2D^2}\right)^2 (M(t)-N(t)^2).$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{E}\left[f(\sigma_t^* u + \mu_t^*)\right] e^{-\frac{u^2}{2}} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\beta_t(u)}} f(y) e^{-\frac{(y-\alpha_t(u))^2}{2\beta_t(u)}} dy \right) e^{-\frac{u^2}{2}} du. \tag{58}$$

On the other hand, since for each fixed t > 0, x_t^* is a Gaussian random variable with mean N(t) and variance $\hat{M}(t) - N(t)^2$ (see Lemma 8 in Appendix F), we conclude that u_t^* is also Gaussian with mean and variance given, respectively, by

$$\hat{\alpha}_t = \frac{(\alpha_2(B+CD)-R)N(t) + \alpha_1B - Q}{N - \alpha_2D^2}$$

and

$$\hat{\beta}_t = \left(\frac{(\alpha_2(B+CD)-R)}{N-\alpha_2D^2}\right)^2 (\hat{M}(t)-N(t)^2).$$

In turn, the right hand side of (57) becomes

$$\mathbb{E}\left[\int_{\mathbb{R}} f(u)\delta_{u_t^*}(du)\right] = \mathbb{E}\left[f(u_t^*)\right] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\hat{\beta}_t}} f(y)e^{-\frac{(y-\hat{\alpha}_t)^2}{2\hat{\beta}_t}} dy.$$
 (59)

Notice that, as $\lambda \to 0$, $\alpha_t(u) \to \hat{\alpha}_t$ and $\beta_t(u) \to \hat{\beta}_t$, for all $u \in \mathbb{R}$. Indeed, the former is obvious while the latter follows from the fact that $M(t) \to \hat{M}(t)$, as $\lambda \to 0$, as shown in the proof of Lemma 8 in Appendix F.

Sending $\lambda \to 0$ in (58) and using the dominated convergence theorem we obtain

$$\lim_{\lambda \to 0} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\beta_t(u)}} f(y) e^{-\frac{(y-\alpha_t(u))^2}{2\beta_t(u)}} dy \right) e^{-\frac{u^2}{2}} du$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\hat{\beta}_t}} f(y) e^{-\frac{(y-\hat{\alpha}_t)^2}{2\hat{\beta}_t}} dy = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\hat{\beta}_t}} f(y) e^{-\frac{(y-\hat{\alpha}_t)^2}{2\hat{\beta}_t}} dy,$$

and the weak convergence result (57) follows.

The pointwise convergence follows easily using the form of $V(\cdot)$ and $w(\cdot)$ in statements (a) and (b), respectively.

Appendix F: Lemma 8

Lemma 8 For any T > 0, let

$$\mathbb{E}\big[(X_T^*)^2\big] = x^2 + \int_0^T (2A_1 + B_1^2) \mathbb{E}\big[(X_t^*)^2\big] + 2(A_2 + B_1 B_2) \mathbb{E}\big[X_t^*\big] \ dt + (B_2^2 + C_1) T$$

and

$$\mathbb{E}[(x_T^*)^2] = x^2 + \int_0^T (2A_1 + B_1^2) \mathbb{E}[(x_t^*)^2] + 2(A_2 + B_1 B_2) \mathbb{E}[x_t^*] dt + B_2^2 T,$$
as in (55) and (56), respectively. Then, $\lim \inf_{T \to \infty} e^{-\delta T} \mathbb{E}[(X_T^*)^2] = 0$ if and

only if $\lim_{T\to\infty} e^{-\delta T} \mathbb{E}\left[\left(x_T^*\right)^2\right] = 0$, for any $\delta \in (0, \rho - (2A + C^2))$. Moreover, $\mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(X_t^*\right)^2 dt\right] < \infty \text{ if and only if } \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(x_t^*\right)^2 dt\right] < \infty.$

Proof. We start by solving for $\mathbb{E}[(X_T^*)^2]$ and $\mathbb{E}[(x_T^*)^2]$ explicitly, for any T > 0. Recall that X^* satisfies the dynamics

$$X_T^* = x + \int_0^T (A_1 X_t^* + A_2) dt + \int_0^T \sqrt{(B_1 X_t^* + B_2)^2 + C_1} dW_t,$$

for T > 0, with $X_0^* = x$.

Define the stopping time τ_n , $n \geq 1$ as $\tau_n := \{t > 0 : \int_0^t (B_1 X_s^* + B_2)^2 + C_1 ds \geq n\}$. Then, it follows that the stopped stochastic integral is a martingale and hence

$$\mathbb{E}\left[X_{T\wedge\tau_n}^*\right] = x + \mathbb{E}\left[\int_0^{T\wedge\tau_n} (A_1 X_t^* + A_2) \ dt\right].$$

The standard estimate that $\mathbb{E}\left[\sup_{0 \leq t \leq T} (X_t^*)^2\right] \leq K(1+x^2)e^{KT}$, for some constant K > 0 depending only on the Lipschitz constant and the time T allows us to apply Fubini's theorem and dominated convergence theorem, yielding

$$\mathbb{E}[X_T^*] = x + \int_0^T (A_1 \mathbb{E}[X_t^*] + A_2) \ dt.$$

Denote $N(t) := \mathbb{E}[X_t^*]$, for $t \geq 0$. Then, we obtain the ODE

$$\frac{dN(t)}{dt} = A_1N(t) + A_2, \quad N(0) = x,$$

whose solution is $N(t) = \left(x + \frac{A_2}{A_1}\right) e^{A_1 t} - \frac{A_2}{A_1}$, if $A_1 \neq 0$, and $N(t) = x + A_2 t$, otherwise. In turn, the function $M(t) := \mathbb{E}\left[\left(X_t^*\right)^2\right]$, $t \geq 0$, solves the ODE

$$\frac{dM(t)}{dt} = (2A_1 + B_1^2)M(t) + 2(A_2 + B_1B_2)N(t) + B_2^2 + C_1, \quad M(0) = x^2.$$

Similarly, $\mathbb{E}\big[x_t^*\big]=N(t),\,t\geq 0$ and if $\hat{M}(t):=\mathbb{E}\big[(x_t^*)^2\big],\,t\geq 0,$ then

$$\frac{d\hat{M}(t)}{dt} = (2A_1 + B_1^2)\hat{M}(t) + 2(A_2 + B_1B_2)N(t) + B_2^2, \quad \hat{M}(0) = x^2.$$

We next find explicit solutions to the above ODEs corresponding to various conditions on the parameters.

(a) If $A_1 = B_1^2 = 0$, then direct computation gives $N(t) = x + A_2 t$, and

$$M(t) = x^{2} + A_{2}(x + A_{2}t)t + (B_{2}^{2} + C_{1})t,$$

$$\hat{M}(t) = x^2 + A_2(x + A_2t)t + B_2^2t.$$

The conclusion of the Lemma follows.

(b) If $A_1 = 0$ and $B_1^2 \neq 0$, we have $N(t) = x + A_2 t$, and

$$M(t) = \left(x^2 + \frac{2(A_2 + B_1 B_2) \left(A_2 + B_1^2 (x + B_2^2 + C_1)\right)}{B_1^4}\right) e^{B_1^2 t}$$
$$-\frac{2(A_2 + B_1 B_2) \left(A_2 + B_1^2 (x + B_2^2 + C_1)\right)}{B_1^4},$$
$$\hat{M}(t) = \left(x^2 + \frac{2(A_2 + B_1 B_2) \left(A_2 + B_1^2 (x + B_2^2)\right)}{B_1^4}\right) e^{B_1^2 t}$$
$$-\frac{2(A_2 + B_1 B_2) \left(A_2 + B_1^2 (x + B_2^2)\right)}{B_1^4}.$$

The conclusion of the Lemma also follows.

(c) If $A_1 \neq 0$ and $A_1 + B_1^2 = 0$, then $N(t) = \left(x + \frac{A_2}{A_1}\right) e^{A_1 t} - \frac{A_2}{A_1}$. Further computation yields

$$\begin{split} M(t) &= \left(x^2 + \frac{A_1(B_2^2 + C_1) - 2A_2(A_2 + B_1B_2)}{A_1^2}\right) e^{A_1t} \\ &+ \frac{2(A_2 + B_1B_2)(A_1x + A_2)}{A_1} t e^{A_1t} - \frac{A_1(B_2^2 + C_1) - 2A_2(A_2 + B_1B_2)}{A_1^2}, \\ \hat{M}(t) &= \left(x^2 + \frac{A_1B_2^2 - 2A_2(A_2 + B_1B_2)}{A_1^2}\right) e^{A_1t} \\ &+ \frac{2(A_2 + B_1B_2)(A_1x + A_2)}{A_1} t e^{A_1t} - \frac{A_1B_2^2 - 2A_2(A_2 + B_1B_2)}{A_1^2}. \end{split}$$

It is also straightforward to see that the conclusion of the Lemma follows. (d) If
$$A_1 \neq 0$$
 and $2A_1 + B_1^2 = 0$, we have $N(t) = \left(x + \frac{A_2}{A_1}\right)e^{A_1t} - \frac{A_2}{A_1}$, and

$$M(t) = \frac{2(A_2 + B_1 B_2)(A_1 x + A_2)}{A_1^2} e^{A_1 t} + \frac{A_1(B_2^2 + C_1) - 2A_2(A_2 + B_1 B_2)}{A_1^2} t + x^2 - \frac{2(A_2 + B_1 B_2)(A_1 x + A_2)}{A_1^2},$$

$$\hat{M}(t) = \frac{2(A_2 + B_1 B_2)(A_1 x + A_2)}{A_1^2} e^{A_1 t} + \frac{A_1 B_2^2 - 2A_2(A_2 + B_1 B_2)}{A_1^2} t + x^2 - \frac{2(A_2 + B_1 B_2)(A_1 x + A_2)}{A_1^2},$$

and we easily conclude.

(e) If
$$A_1 \neq 0$$
, $A_1 + B_1^2 \neq 0$ and $2A_1 + B_1^2 \neq 0$, then we arrive at $N(t) = \left(x + \frac{A_2}{A_1}\right)e^{A_1t} - \frac{A_2}{A_1}$, and

$$M(t) =$$

$$\left(x^2 + \frac{2(A_2 + B_1B_2)(A_1x + A_2)}{A_1(A_1 + B_1^2)} + \frac{A_1(B_2^2 + C_1) - 2A_2(A_2 + B_1B_2)}{A_1(2A_1 + B_1^2)}\right)e^{(2A_1 + B_1^2)t}$$
$$- \frac{2(A_2 + B_1B_2)(A_1x + A_2)}{A_1(A_1 + B_1^2)}e^{A_1t} - \frac{A_1(B_2^2 + C_1) - 2A_2(A_2 + B_1B_2)}{A_1(2A_1 + B_1^2)},$$

$$\hat{M}(t) = \left(x^2 + \frac{2(A_2 + B_1 B_2)(A_1 x + A_2)}{A_1(A_1 + B_1^2)} + \frac{A_1 B_2^2 - 2A_2(A_2 + B_1 B_2)}{A_1(2A_1 + B_1^2)}\right) e^{(2A_1 + B_1^2)t}$$
$$- \frac{2(A_2 + B_1 B_2)(A_1 x + A_2)}{A_1(A_1 + B_1^2)} e^{A_1 t} - \frac{A_1 B_2^2 - 2A_2(A_2 + B_1 B_2)}{A_1(2A_1 + B_1^2)}.$$

The conclusion then easily follows and the proof of the Lemma is complete.

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