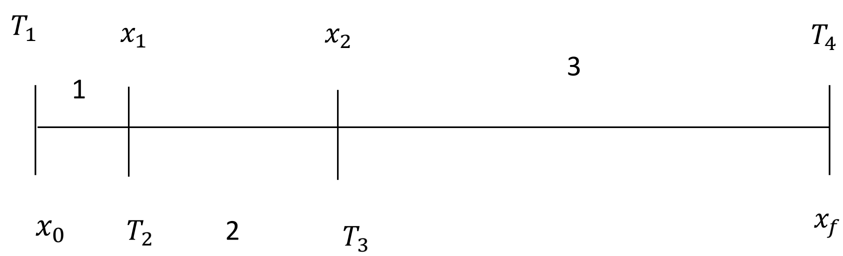
Método de los elementos finitos para el problema de transferencia de calor en una dimensión con funciones de forma lineales y con peso de Galerkin

Mallado:

Asumimos que la respuesta está en los nodos:

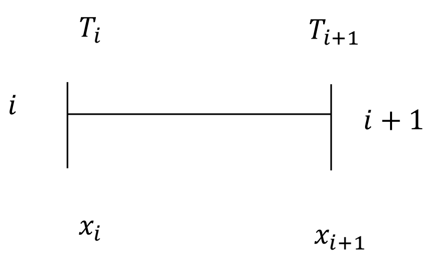


Interpolación:

Utilizaremos funciones de forma lineales para una dimensión.

$$T_i = N_i T_i + N_{i+1} T_{i+1}$$

Para el elemento i:



$$N_i = \frac{x_{i+1}-x}{x_{i+1}-x_i}, \qquad N_{i+1} = \frac{x-x_i}{x_{i+1}-x_i}$$

Elemento	$\boldsymbol{i}$	$\boldsymbol{i+1}$
1	1	2
2	2	3
3	3	4

Nuestros elementos tienen diferentes longitudes:

$$d = x_{i+1} - x_i$$

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = Q, \qquad k \wedge Q = kte$$

$$T = \begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$\hat{T} \approx \mathbf{NT}, \qquad \mathbf{N}_{(x)}$$

Discretización:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = Q$$

$$\frac{d}{dx}\left(k\frac{d\hat{T}}{dx}\right) \approx Q \;\equiv\; \frac{d}{dx}\left(k\frac{d\mathbf{N}_{(x)}}{dx}\right)\mathbf{T} \approx Q$$

Calculo Residual:

$$\frac{d}{dx}\left(k\frac{d\mathbf{N}_{(x)}}{dx}\right)\mathbf{T} - \mathrm{Q} = \xi$$

Método de los residuos ponderados:

$$\int_{\Omega} \xi_i w_i \, d\Omega = 0$$

$$\int_{\Omega} w\left[\frac{d}{dx}\left(k\frac{d\mathbf{N}}{dx}\right)\mathbf{T} - \mathrm{Q}\right] d\Omega = 0$$

$$\frac{d}{dx}\begin{bmatrix} N_i & N_{i+1} \end{bmatrix} = \begin{bmatrix} \frac{d}{dx}N_i & \frac{d}{dx}N_{i+1} \end{bmatrix}$$

$$\frac{d}{dx}N_i = \frac{-1}{x_{i+1} - x_i}$$

$$\frac{d}{dx}N_{i+1} = \frac{1}{x_{i+1} - x_i}$$

$$\int_{\Omega} w \left[\frac{d}{dx}\left[\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i}\right] \mathbf{T} - \mathrm{Q}\right] d\Omega = 0$$

$$\int_{\Omega} \boldsymbol{w} \left[\frac{d}{dx}\left[\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i}\right] \mathbf{T} - \mathrm{Q}\right] d\Omega = 0$$

$$\mathbf{w} = \begin{bmatrix} w_{x_i} \\ w_{x_i+1} \end{bmatrix}$$

$$\int_{\Omega} \begin{bmatrix} w_{x_i} \\ w_{x_i+1} \end{bmatrix} \left[\frac{d}{dx}\left[\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i}\right] \mathbf{T} - \mathrm{Q}\right] d\Omega = 0$$

Método de Galerkin

$$W_i = N_i$$

Forma Fuerte:

$$\int_{x_i}^{x_i+1} \mathbf{N}^{\mathbf{T}} \left[\frac{d}{dx}\left[\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i}\right] \mathbf{T} - \mathrm{Q}\right] dx = 0$$

Integración por partes:

$$\int_{x_i}^{x_i+1} \mathbf{N}^{\mathbf{T}} \frac{d}{dx}\left(\frac{-k}{x_{i+1} - x_i} \quad \frac{k}{x_{i+1} - x_i}\right) \mathbf{T}\, dx - \mathrm{Q} \int_{x_i}^{x_i+1} \mathbf{N}^{\mathbf{T}}\, dx = 0$$

$$\int u dv = uv - v\int du$$

$$u = \mathbf{N^T}$$

$$du = \frac{d}{dx}\mathbf{N^T} dx$$

$$dv = \ k\frac{d\mathbf{NT}}{dx}\,\mathrm{dx}$$

$$v = k\frac{d\mathbf{NT}}{dx}$$

$$\mathbf{N^T} k\frac{d\mathbf{NT}}{dx} - \int \frac{d}{dx}\mathbf{N^T}\, k\frac{d\mathbf{NT}}{dx}\ dx$$

Forma débil:

$$\mathbf{N^T} k\frac{d\mathbf{NT}}{dx}\bigg|_{\Gamma} - \int_{x_i}^{x_{i+1}} \frac{d}{dx}\mathbf{N^T}\, k\frac{d\mathbf{NT}}{dx}\ dx - Q\int_{x_i}^{x_i+1} \mathbf{N^T}\, dx = 0$$

$$\frac{d}{dx}\mathbf{N^T} = \frac{d}{dx}\begin{bmatrix} N_i \\ N_{i+1} \end{bmatrix} = \begin{bmatrix} \frac{dN_i}{dx} \\ \frac{dN_{i+1}}{dx} \end{bmatrix} = \frac{1}{x_{i+1} - x_i}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\frac{d}{dx}\mathbf{N} = \ \frac{1}{x_{i+1} - x_i}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\frac{d}{dx}\mathbf{N^T}\frac{d}{dx}\mathbf{N} = \frac{1}{(x_{i+1} - x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k\int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \mathbf{T}\mathrm{dx}$$

$$\frac{k}{(x_{i+1} - x_i)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(x_{i+1} - x_i)\ \mathbf{T} = \frac{k}{(x_{i+1} - x_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix}$$

$$-Q\int_{x_i}^{x_i+1} \mathbf{N}^{\mathbf{T}}\,dx = \frac{-Q}{(x_{i+1}-x_i)}\int_{x_i}^{x_i+1} \begin{bmatrix} x_{i+1}-x \\ x-x_i \end{bmatrix} dx = \frac{-Q}{(x_{i+1}-x_i)}\begin{bmatrix} \int_{x_i}^{x_{i+1}} x_{i+1}-x\,dx \\ \int_{x_i}^{x_{i+1}} x-x_i\,dx \end{bmatrix}$$

$$\frac{-Q}{(x_{i+1}-x_i)}\begin{bmatrix} \int_{x_i}^{x_{i+1}} x_{i+1}-x\,dx \\ \int_{x_i}^{x_{i+1}} x-x_i\,dx \end{bmatrix} = \frac{-Q}{(x_{i+1}-x_i)}\begin{bmatrix} \frac{(x_{i+1}-x_i)^2}{2} \\ \frac{(x_{i+1}-x_i)^2}{2} \end{bmatrix} = \frac{-Q(x_{i+1}-x_i)}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{k}{(x_{i+1}-x_i)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix} = \frac{-Q(x_{i+1}-x_i)}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mathbf{N}^{\mathbf{T}} k \frac{d\mathbf{NT}}{dx}\bigg|$$

$$\boldsymbol{kT} = \boldsymbol{b}$$

Para cada elemento i:

$$\boldsymbol{k} = \begin{bmatrix} k_{\boldsymbol{i1}} & k_{\boldsymbol{i2}} \\ k_{\boldsymbol{i3}} & k_{\boldsymbol{i4}} \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix}$$