3. Haga una formulación conjunta tomando en cuenta al mismo tiempo las modificaciones a la ecuación de transferencia de calor de los ejercicios 1 y 2

Mallado:

$$\frac{d}{dx}\left(k(x)\frac{dT}{dx}\right) = -Q(x)$$

$$T = \begin{bmatrix} N_i & N_{i+1} \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix}$$

$$\hat{T} \approx \mathbf{NT}, \qquad \mathbf{N}_{(x)}$$

Discretización:

$$\frac{d}{dx}\left(k(x)\frac{dT}{dx}\right) = -Q(x)$$

$$\frac{d}{dx}\left(k(x)\frac{d\hat{T}}{dx}\right) \approx -Q(x) \equiv \frac{d}{dx}\left(k(x)\frac{d\mathbf{N}_{(x)}}{dx}\right)\mathbf{T} \approx -Q(x)$$

Residuos:

$$\frac{d}{dx}\left(k(x)\frac{d\mathbf{N}_{(x)}}{dx}\right)\mathbf{T} + Q(x) = \xi$$

Método de los residuos ponderados:

$$\int_{\Omega} \xi_i w_i \, d\Omega = 0$$

$$\int_{\Omega} w\left[\frac{d}{dx}\left(k(x)\frac{d\mathbf{N}}{dx}\right)\mathbf{T} + \mathrm{Q(x)}\right] d\Omega = 0$$

$$\frac{d}{dx}\begin{bmatrix} N_i & N_{i+1} \end{bmatrix} = \begin{bmatrix} \frac{d}{dx}N_i & \frac{d}{dx}N_{i+1} \end{bmatrix}$$

$$\frac{d}{dx}N_i = \frac{-1}{x_{i+1} - x_i}$$

$$\frac{d}{dx}N_{i+1} = \frac{1}{x_{i+1} - x_i}$$

$$\int_{\Omega} w\left[\frac{d}{dx}k(x)\begin{bmatrix} \frac{-1}{x_{i+1} - x_i} & \frac{1}{x_{i+1} - x_i} \end{bmatrix}\mathbf{T} + \mathrm{Q(x)}\right] d\Omega = 0$$

$$\int_{\Omega} \boldsymbol{w}\left[\frac{d}{dx}k(x)\begin{bmatrix} \frac{-1}{x_{i+1} - x_i} & \frac{1}{x_{i+1} - x_i} \end{bmatrix}\mathbf{T} + \mathrm{Q(x)}\right] d\Omega = 0$$

$$\mathbf{w} = \begin{bmatrix} w_{x_i} \\ w_{x_{i+1}} \end{bmatrix}$$

$$\int_{\Omega} \begin{bmatrix} w_{x_i} \\ w_{x_{i+1}} \end{bmatrix}\left[\frac{d}{dx}k(x)\begin{bmatrix} \frac{-1}{x_{i+1} - x_i} & \frac{1}{x_{i+1} - x_i} \end{bmatrix}\mathbf{T} + \mathrm{Q(x)}\right] d\Omega = 0$$

Método de Galerkin

$$W_i = N_i$$

Forma Fuerte:

$$\int_{x_i}^{x_i+1} \mathbf{N^T} \left[\frac{d}{dx} k(x) \left[\frac{-1}{x_{i+1} - x_i} \quad \frac{1}{x_{i+1} - x_i}\right] \mathbf{T} + \mathrm{Q(x)}\right] dx = 0$$

Integración por partes:

$$\int_{x_i}^{x_i+1} \mathbf{N^T} \frac{d}{dx} k(x) \left(\frac{-1}{x_{i+1} - x_i} \quad \frac{1}{x_{i+1} - x_i}\right) \mathbf{T}\, dx + \int_{x_i}^{x_i+1} \mathbf{N^T} Q(x)\, dx = 0$$

$$\int u dv = uv - v \int du$$

$$u = \mathbf{N^T}$$

$$du = \frac{d}{dx} \mathbf{N^T} dx$$

$$dv = \frac{d}{dx} \left(\frac{d}{dx} (k(x) \mathbf{NT})\right) \mathrm{dx}$$

$$v = \frac{d}{dx} (k(x) \mathbf{NT})$$

$$\mathbf{N^T} \frac{d}{dx} (k(x) \mathbf{NT}) \ - \int \frac{d}{dx} \mathbf{N^T} \ \frac{d}{dx} (k(x) \mathbf{NT}) \ dx$$

Forma débil:

$$\mathbf{N^T}\frac{d}{dx}(k(x)\mathbf{NT})\Big|_{\mathbf{\Gamma}} - \int_{x_i}^{x_{i+1}} \frac{d}{dx}\mathbf{N^T}\ \frac{d}{dx}(k(x)\mathbf{NT})\,dx + \int_{x_i}^{x_i+1} \mathbf{N^T}Q(x)\,dx = 0$$

$$\frac{d}{dx}\mathbf{N^T} = \frac{d}{dx}\begin{bmatrix} N_i \\ N_{i+1}\end{bmatrix} = \begin{bmatrix} \dfrac{dN_i}{dx} \\ \dfrac{dN_{i+1}}{dx}\end{bmatrix} = \frac{1}{x_{i+1}-x_i}\begin{bmatrix} -1 \\ 1\end{bmatrix}$$

$$\frac{d}{dx}\mathrm{k(x)}\mathbf{N} = \begin{bmatrix} \left(e^x - \frac{1}{2}\right)\frac{(x_{i+1}-x)}{x_{i+1}-x_i} - \frac{\left(e^x-\frac{x}{2}\right)}{x_{i+1}-x_i} & \frac{x_i(2e^x-1)+2x-2e^x(x+1)}{-2(x_{i+1}-x_i)}\end{bmatrix}$$

$$\frac{d}{dx}\mathbf{N^T}\frac{d}{dx}\mathrm{k(x)}\mathbf{N} = \frac{1}{x_{i+1}-x_i}\begin{bmatrix} -1 \\ 1\end{bmatrix}\begin{bmatrix} \left(e^x - \frac{1}{2}\right)\frac{(x_{i+1}-x)}{x_{i+1}-x_i} - \frac{\left(e^x-\frac{x}{2}\right)}{x_{i+1}-x_i} & \frac{x_i(2e^x-1)+2x-2e^x(x+1)}{-2(x_{i+1}-x_i)}\end{bmatrix}$$

$$= \frac{1}{x_{i+1}-x_i}\begin{bmatrix} -\left(e^x - \frac{1}{2}\right)\frac{(x_{i+1}-x)}{x_{i+1}-x_i} - \frac{\left(e^x-\frac{x}{2}\right)}{x_{i+1}-x_i} & -\frac{x_i(2e^x-1)+2x-2e^x(x+1)}{-2(x_{i+1}-x_i)} \\ \left(e^x - \frac{1}{2}\right)\frac{(x_{i+1}-x)}{x_{i+1}-x_i} - \frac{\left(e^x-\frac{x}{2}\right)}{x_{i+1}-x_i} & \frac{x_i(2e^x-1)+2x-2e^x(x+1)}{-2(x_{i+1}-x_i)}\end{bmatrix}$$

$$\int_{x_i}^{x_{i+1}} \frac{1}{x_{i+1}-x_i}\begin{bmatrix} -\left(e^x - \frac{1}{2}\right)\frac{(x_{i+1}-x)}{x_{i+1}-x_i} - \frac{\left(e^x-\frac{x}{2}\right)}{x_{i+1}-x_i} & -\frac{x_i(2e^x-1)+2x-2e^x(x+1)}{-2(x_{i+1}-x_i)} \\ \left(e^x - \frac{1}{2}\right)\frac{(x_{i+1}-x)}{x_{i+1}-x_i} - \frac{\left(e^x-\frac{x}{2}\right)}{x_{i+1}-x_i} & \frac{x_i(2e^x-1)+2x-2e^x(x+1)}{-2(x_{i+1}-x_i)}\end{bmatrix}\mathbf{T}\mathrm{dx}$$

$$\int_{x_i}^{x_i+1} \mathbf{N}^{\mathbf{T}} Q(x)\, dx = -\int_{x_i}^{x_i+1} \begin{bmatrix} N_i \\ N_{i+1} \end{bmatrix} (x^2 - 3x)\, dx = \int_{x_i}^{x_i+1} \begin{bmatrix} N_i(x^2 - 3x) \\ N_{i+1}(x^2 - 3x) \end{bmatrix} dx$$

$$= \begin{bmatrix} \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{x_{i+1} - x_i}(x^2 - 3x) \\ \int_{x_i}^{x_{i+1}} \frac{x - x_i}{x_{i+1} - x_i}(x^2 - 3x) \end{bmatrix} = \begin{bmatrix} -\frac{(x_i - x_{i+1})\left(3({x^2}_i) + 2(x_i)(x_{i+1} - 6) + (x_{i+1} - 6)(x_{i+1})\right)}{12} \\ \frac{(x_i - x_{i+1})\left(({x^2}_i) + 2(x_i)(x_{i+1} - 3) + 3(x_{i+1} - 4)(x_{i+1})\right)}{12} \end{bmatrix}$$

$$\int_{x_i}^{x_{i+1}} \frac{1}{x_{i+1} - x_i} \begin{bmatrix} -\left(e^x - \frac{1}{2}\right)\frac{(x_{i+1} - x)}{x_{i+1} - x_i} - \frac{\left(e^x - \frac{x}{2}\right)}{x_{i+1} - x_i} & -\frac{x_i(2e^x - 1) + 2x - 2e^x(x + 1)}{-2(x_{i+1} - x_i)} \\ \left(e^x - \frac{1}{2}\right)\frac{(x_{i+1} - x)}{x_{i+1} - x_i} - \frac{\left(e^x - \frac{x}{2}\right)}{x_{i+1} - x_i} & \frac{x_i(2e^x - 1) + 2x - 2e^x(x + 1)}{-2(x_{i+1} - x_i)} \end{bmatrix} \mathbf{T}\mathrm{dx} = -\frac{(x_i - x_{i+1})}{12} \begin{bmatrix} -\left(3({x^2}_i) + 2(x_i)(x_{i+1} - 6) + (x_{i+1} - 6)(x_{i+1})\right) \\ \left(({x^2}_i) + 2(x_i)(x_{i+1} - 3) + 3(x_{i+1} - 4)(x_{i+1})\right) \end{bmatrix} + \mathbf{N}^{\mathbf{T}} k \frac{d\mathbf{NT}}{dx}\Big|$$

$$\boldsymbol{kT} = \boldsymbol{b}$$

Para cada elemento $i$:

$$\boldsymbol{k} = \begin{bmatrix} k_{\boldsymbol{i1}} & k_{\boldsymbol{i2}} \\ k_{\boldsymbol{i3}} & k_{\boldsymbol{i4}} \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix}$$

Ensamblaje:

| **Elemento** | $\boldsymbol{i}$ | $\boldsymbol{i+1}$ |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 3 |
| 3 | 3 | 4 |

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{13} & k_{14}+k_{21} & k_{22} & 0 \\ 0 & k_{23} & k_{24}+k_{31} & k_{32} \\ 0 & 0 & k_{33} & k_{34} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12}+b_{21} \\ b_{22}+b_{31} \\ b_{32} \end{bmatrix}$$

$$\boldsymbol{KT} = \boldsymbol{B}$$

Condiciones de contorno:

Condiciones de Neumann: $\frac{dT}{dx} = T_0$ en $\Gamma_N, \Gamma_N \subseteq \Gamma$

$$\mathbf{N^T} k \frac{d\mathbf{NT}}{dx}\bigg|_{\mathbf{\Gamma}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{13} & k_{14}+k_{21} & k_{22} & 0 \\ 0 & k_{23} & k_{24}+k_{31} & k_{32} \\ 0 & 0 & k_{33} & k_{34} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12}+b_{21} \\ b_{22}+b_{31} \\ b_{32} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

Condiciones de Dirichlet: $T = T_0$ en $\Gamma_D, \Gamma_D \subseteq \Gamma$

$$\begin{bmatrix} 10 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$