# Applied Optimization
# Exercise 5 - Line Search Methods

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## Exact line search for the convex quadratic function

$$f(x) = \frac{1}{2}x^TQx + q^Tx + c$$

$$t := \underset{s\geq 0}{\operatorname{argmin}}\, f(x + s\Delta x)$$

Let's start by defining and computing $g(s) := f(x + s\Delta x)$ :

$$\begin{aligned} g(s) := f(x+s\Delta x) &= \frac{1}{2}(x+s\Delta x)^T Q(x+s\Delta x) + q^T(x+s\Delta x) + c \\ &= f(x) + \frac{1}{2}s\Delta x^T Qx + \frac{1}{2}x^T Qs\Delta x + \frac{1}{2}s\Delta x^T Qs\Delta x + q^T s\Delta x \\ &= f(x) + s\left(\frac{1}{2}\Delta x^T Qx + \frac{1}{2}x^T Q\Delta x + q^T\Delta x\right) + s^2\frac{1}{2}\Delta x^T Q\Delta x \end{aligned}$$

As we are looking to minimize $g(s)$ for $s$, let's compute the derivative of $g$ and search $s$ such as $g'(s) = 0$ :

$$\begin{aligned} \frac{dg(s)}{ds} &= \left(\frac{1}{2}\Delta x^T Qx + \frac{1}{2}x^T Q\Delta x + q^T\Delta x\right) + s\Delta x^T Q\Delta x = 0 \\ &\Leftrightarrow \\ s\Delta x^T Q\Delta x &= -\left(\frac{1}{2}\Delta x^T Qx + \frac{1}{2}x^T Q\Delta x + q^T\Delta x\right) \\ &\Leftrightarrow \\ s &= -\frac{\frac{1}{2}\Delta x^T Qx + \frac{1}{2}x^T Q\Delta x + q^T\Delta x}{\Delta x^T Q\Delta x} \end{aligned}$$

Therefore, as $f(x)$ is convex and $s \geq 0$, for an arbitrary $x$

$$t = \max\left(-\frac{\Delta x^T Qx + x^T Q\Delta x + 2q^T\Delta x}{2\Delta x^T Q\Delta x}, 0\right)$$

# Gradient descent with exact line search

Given the unconstrained optimisation problem minimize $f(x) := \frac{1}{4}x_1^2 + x_2^2$, we want to perform one iteration of the gradient descent algorithm from the starting point $x^{(0)} = (2, 1)$ with exact line search. As $\Delta x$ is define for the gradient descent as $\Delta x := -\nabla f(x)$, let's start by computing the gradient $\nabla f(x)$ and the line search parameter $t$ :

$$\begin{aligned}
\nabla f(x) &= \begin{pmatrix} \frac{1}{2}x_1 \\ 2x_2 \end{pmatrix} \\
t &:= \operatorname*{argmin}_{s \geq 0} f(x + s\Delta x) \\
\frac{df(x + s\Delta x)}{ds} &= \frac{1}{2}(x_1 + s\Delta x_1)\Delta x_1 + 2(x_2 + s\Delta x_2)\Delta x_2 \\
&= s\left(\frac{\Delta x_1^2}{2} + 2\Delta x_2^2\right) + \frac{x_1}{2}\Delta x_1 + 2x_2\Delta x_2 \\
t &= \max\left\{-\frac{\frac{x_1}{2}\Delta x_1 + 2x_2\Delta x_2}{\frac{\Delta x_1^2}{2} + 2\Delta x_2^2}, 0\right\}
\end{aligned}$$

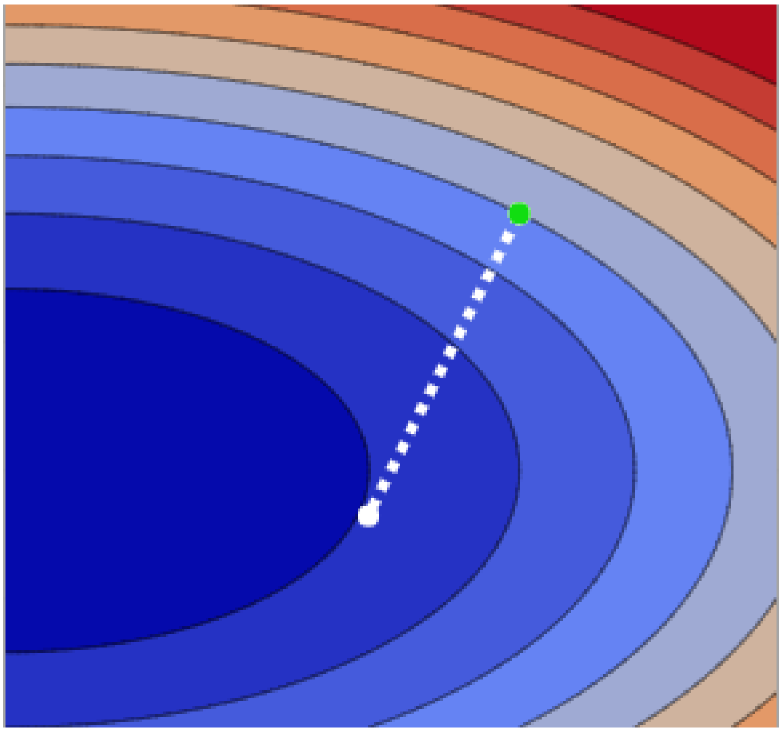

Figure 1 – First iteration of the gradient descent algorithm with exact line search for $f(x)$

This give us the plot 1 and following parameter for the first step :

$$\Delta x^{(0)} = -\nabla f(x^{(0)}) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$t^{(0)} = \max\left\{ -\frac{\frac{2}{2}(-1) + 2 \cdot 1(-2)}{\frac{(-1)^2}{2} + 2(-2)^2}, 0 \right\} = \frac{10}{17}$$

Therefore,

$$x^{(1)} := x^{(0)} + t^{(0)} \Delta x^{(0)} = \frac{3}{17} \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

$$\left|\left|\nabla f\left(x^{(0)}\right)\right|\right|^2 = \frac{1}{4} 2^2 + 4 \cdot 1^2 = 5$$

$$\left|\left|\nabla f\left(x^{(1)}\right)\right|\right|^2 = \frac{1}{4} \left(\frac{24}{17}\right)^2 + 4 \left(\frac{-3}{17}\right)^2 = \frac{180}{289} \approx 0.62$$