Applied Optimization
Exercise 4 - Duality

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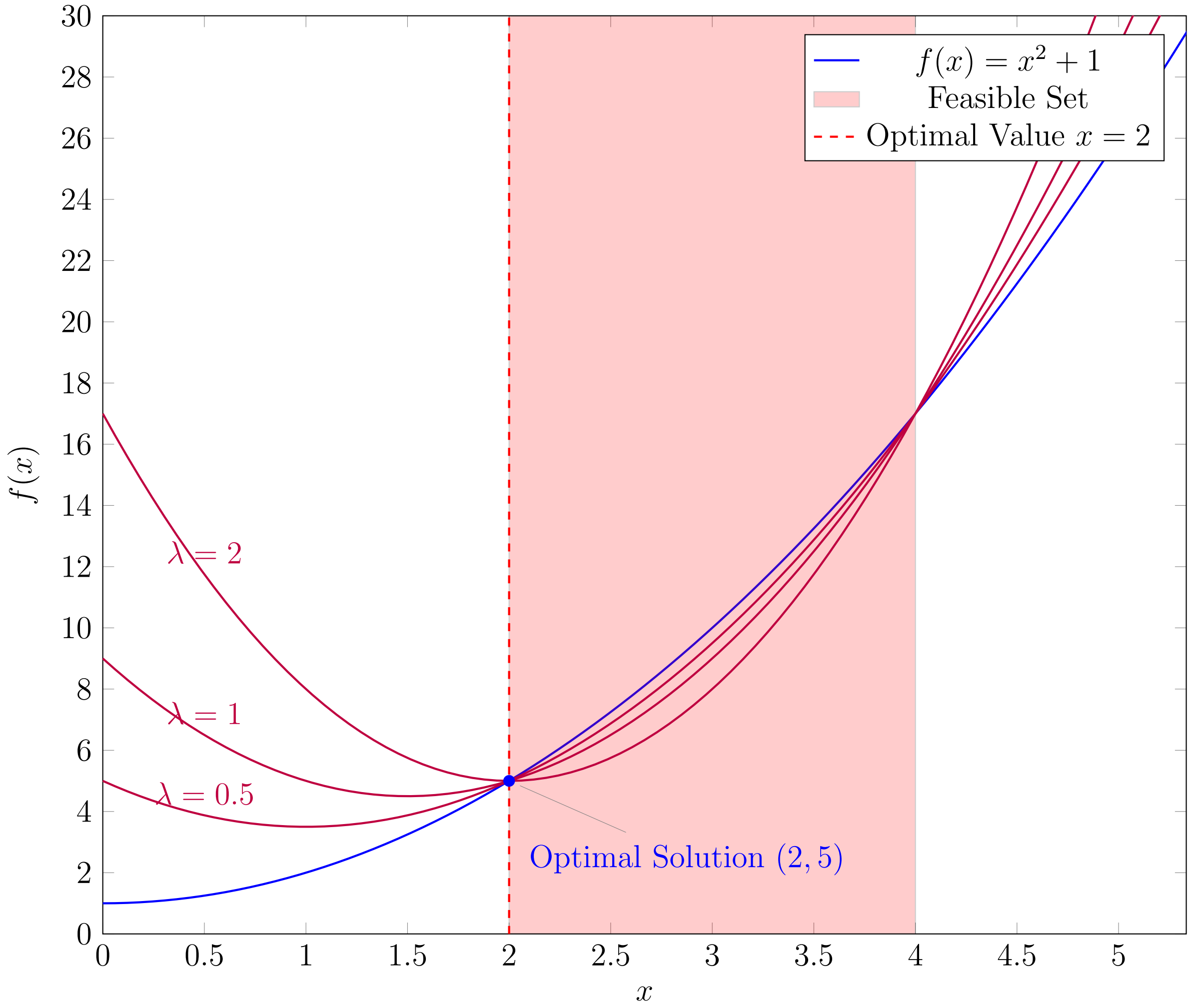
1 Lagrange Duality

a) For this problem we can easily see that the feasible set is $F = \{x|x \in [2:4]\}$, the optimal value is 2 and the optimal solution is 5.

b) Lagrange and dual function



The Lagrangian is defined as:

$$\mathcal{L}(x, \lambda) = x^2 + 1 + \lambda(x - 2)(x - 4)$$

Since $g(\lambda) = \inf_{x \in D} \mathcal{L}(x, \lambda)$ To find the dual function, we first differentiate the Lagrangian with respect to x:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x(1 + \lambda) - 6\lambda$$

Setting this to zero to get the minimum value, we get:

$$x^* = \frac{3\lambda}{1 + \lambda}$$

Substituting x^* into $\mathcal{L}(x, \lambda)$ gives us the Lagrange dual function, $g(\lambda)$:

$$g(\lambda) = \left(\frac{3\lambda}{1+\lambda}\right)^2 + 1 + \lambda \left(\frac{3\lambda}{1+\lambda} - 2\right)\left(\frac{3\lambda}{1+\lambda} - 4\right)$$

Simplifying we find

$$g(\lambda) = -\frac{\lambda^2 - 9\lambda - 1}{\lambda + 1}$$

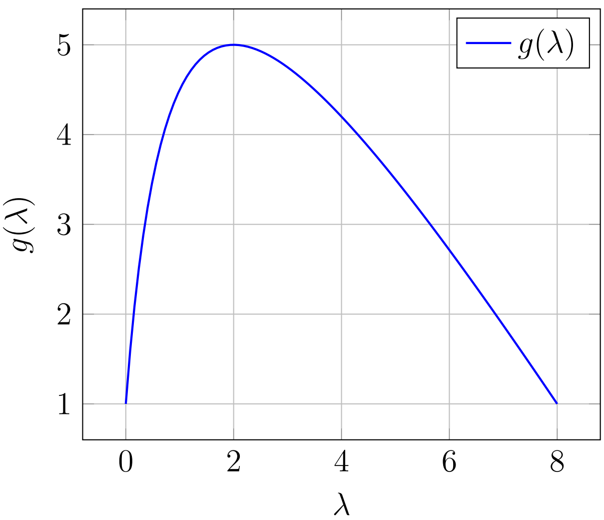
And by derivating $g(\lambda)$

$$\frac{\partial g}{\partial \lambda} = -\frac{(\lambda - 2)(\lambda + 4)}{(\lambda + 1)^2}$$

We find a local maximum of $g(2) = 5$, we also find that $g(0) = 1$ and that $\lim_{\lambda \to +\infty} g(\lambda) = -\infty$. Which verifies the lower-bound property.

And if we compute the Lagrange dual function, we get:

Lagrange Dual Function $g(\lambda)$



c) The dual problem aims to maximize $g(\lambda)$ subject to $\lambda \geq 0$. We can therefor state the dual problem as

$$\begin{aligned} &\text{maximize} \quad -\frac{\lambda^2 - 9\lambda - 1}{\lambda + 1} && (1) \\ &\text{subject to} \quad \lambda \geq 0 && (2) \end{aligned}$$

To check that $g(\lambda)$ is concave we can verify that the second derivative is negative over the domain $\lambda \geq 0$

$$\frac{\partial g}{\partial \lambda} = -\frac{(\lambda - 2)(\lambda + 4)}{(\lambda + 1)^2} \tag{3}$$

$$\frac{\partial^2 g}{\partial \lambda^2} = -\frac{18}{(\lambda + 1)^3} \tag{4}$$

And we can see that for any $\lambda \geq 0$, $\frac{\partial^2}{\partial \lambda^2} g(\lambda)$ will always be negative, verifying that $g(\lambda)$ is concave.
And as we have seen earlier $g(2) = 5$ is the global maximum on the domain $\lambda \geq 0$ making $\lambda = 2$ the optimal value and $g(2) = 5$ the optimal solution.
We can conclude that the strong duality does hold since $p*$ and $d*$ are equal.