Applied Optimization Exercise 3 - Convex Optimization Problems

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QCQP to SOCP (2 pts)

1. **Show that a quadratic constraint of the form**

   $$x^T A^T A x + b^T x + c \leq 0$$

   is equivalent to the second order cone constraint

   $$\left\| \begin{bmatrix} \frac{1+b^T x+c}{2} \\ Ax \end{bmatrix} \right\|_2 \leq \frac{1 - b^T x - c}{2}$$

Because:
$\frac{(1+b^Tx+c)^2}{4} = \frac{(1+2(b^Tx+c)+(b^Tx+c)^2)}{4}$
$\frac{(1-b^Tx-c)^2}{4} = \frac{(1-2(b^Tx+c)+(b^Tx+c)^2)}{4}$
for the first form, add $\frac{1}{4} - \frac{-(b^Tx+c)}{2} + \frac{(b^Tx+c)^2}{4}$ for both sides:
$x^T A^T A x + b^T x + c + \frac{1}{4} - \frac{-(b^Tx+c)}{2} + \frac{(b^Tx+c)^2}{4} \leq \frac{1}{4} - \frac{-(b^Tx+c)}{2} + \frac{(b^Tx+c)^2}{4}$
then we have:
$x^T A^T A x + \frac{(1+2(b^Tx+c)+(b^Tx+c)^2)}{4} \leq \frac{(1-2(b^Tx+c)+(b^Tx+c)^2)}{4}$
due to: $\|Ax\|_2^2 = x^T A^T A x$
then we have:
$\|Ax\|_2^2 + (\frac{1+b^Tx+c}{2})^2 \leq (\frac{1-b^Tx-c}{2})^2$
which is equivalent to the second order cone constraint.

2. **Convert the following QCQP into SOCP.**

$$\begin{aligned} \textbf{minimize } & x_1^2 + 4x_1x_2 + 4x_2^2 \\ \textbf{subject to } & 9x_1^2 + 16x_2^2 \leq 25 \\ & x_1 - x_2 = 1 \end{aligned}$$

for the constraint part 1:
$9x_1^2 + 16x_2^2 \leq 25$ is equal to: $(3x_1)^2 + (4x_2)^2 \leq 5^2$
apply the second-order cone representation, this constraint can be written as:
$\left\| \begin{bmatrix} 3x_1 \\ 4x_2 \end{bmatrix} \right\|_2 \leq 5$
for the constraint part 2:
$x_1 - x_2 = 1$ is a linear equation, satisfy the SOCP.
for the minimize part:
$x_1^2 + 4x_1x_2 + 4x_2^2$ is equal to: $(x_1 + 2x_2)^2$
here we introduce t, where:
$t \geq (x_1 + 2x_2)^2$
and we get:
$\left\| \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} \right\|_2 \leq \sqrt{t}$
then, we can convert the QCQP into SOCP:

$$\begin{array}{rl} \text{minimize} & t \\ \text{subject to} & \left\| \begin{bmatrix} 3x_1 \\ 4x_2 \end{bmatrix} \right\|_2 \leq 5 \\ & x_1 - x_2 = 1 \\ & \left\| \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} \right\|_2 \leq \sqrt{t} \end{array}$$

Linear Programming

Transform (2 pts)

For the following optimization problem

$$\begin{array}{rl} \textbf{minimize} & \left\| (2x_1 + 3x_2, -3x_1)^T \right\|_\infty \\ \textbf{subject to} & |x_1 - 2x_2| \leq 3 \end{array}$$

(1) Express the problem as a linear program.

Here we introduce t, where:
$t \geq 2x_1 + 3x_2$ and $t \geq -2x_1 - 3x_2$
and:
$t \geq -3x_1$ and $t \geq 3x_1$

then we get the linear program:

$$
\begin{aligned}
\text{minimize} \quad & t \\
\text{subject to} \quad & t \geq 2x_1 + 3x_2 \\
& t \geq -2x_1 - 3x_2 \\
& t \geq 3x_1 \\
& t \geq -3x_1 \\
& x_1 - 2x_2 \leq 3 \\
& -x_1 + 2x_2 \leq 3
\end{aligned}
$$

(2) Convert the LP so that all variables are in $\mathbb{R}_+$, and there is no other inequality constraints that each individual variable is ≥ 0.

We can add slack variables to this LP program, however, because we don't know whether the x_1 and x_2 is less than or equal to zero, so we have to replace it by the difference between two positive real numbers.
here, we let:
$x_1 = x_{11} - x_{12}$
$x_2 = x_{21} - x_{22}$
then:

$$
\begin{aligned}
\text{minimize} \quad & t \\
\text{subject to} \quad & t = 2(x_{11} - x_{12}) + 3(x_{21} - x_{22}) + x_3 \\
& t = -2(x_{11} - x_{12}) - 3(x_{21} - x_{22}) + x_4 \\
& t = 3(x_{11} - x_{12}) + x_5 \\
& t = -3(x_{11} - x_{12}) + x_6 \\
& (x_{11} - x_{12}) - 2(x_{21} - x_{22}) + x_7 = 3 \\
& -(x_{11} - x_{12}) + 2(x_{21} - x_{22}) + x_8 = 3 \\
& x_{11}, x_{12}, x_{21}, x_{22}, x_3, x_4, x_5, x_6, x_7, x_8 > 0
\end{aligned}
$$

Transform general LP to standard form (Bonus 2 pts)

A general linear program has the form

$$
\begin{aligned}
\textbf{minimize} \ & c^T x + d \\
\textbf{subject to} \ & Gx \preceq h \\
& Ax = b
\end{aligned}
$$

where $G \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{p \times n}$. Transform the general LP to its standard form:

$$\begin{aligned} \textbf{minimize } & p^T x' \\ \textbf{subject to } & Bx' = e \\ & x' \succeq 0 \end{aligned}$$

Explain in detail the relation between the feasible sets, the optimal solutions, and the optimal values of the standard form LP and the original LP.

This is simply like the question we've just done. There several steps to do that.

First, convert inequality constraints to equality constraints. Here the inequality constraint is $Gx \preceq h$. Then we add a set of slack variables. Here we introduce x_3, where $x_3 \succeq 0$. Then:

$Gx + x_3 = h$
$x_3 \succeq 0$

Second, we want to ensure all variables are non-negative, but we do not know the value of x is negative or not. Therefore, we introduce $x = x_1 - x_2$ to replace the x. Like this:

$G(x_1 - x_2) + x_3 = h$
$x_3, x_1, x_2 \succeq 0$

Now we have the program like this:

$$\begin{aligned} \text{minimize } & c^T(x_1 - x_2) + d \\ \text{subject to } & G(x_1 - x_2) + x_3 = h \\ & Ax = b \\ & x_1, x_2, x_3 \succeq 0 \end{aligned}$$

About feasible sets:
The feasible set of the standard form LP encompasses the feasible set of general LP plus the additional constraints on the slack variables and the representations of unrestricted variables.

Optimal Solutions:
If x_1 is an optimal solution to the original LP, then there exists a x_1' which is an optimal solution to the standard form LP, and vice versa. This is because transformed LP is simply another representation of the original problem.

Optimal Values:
The optimal value of the original LP will be the same as the optimal value of the standard form LP, since transformed LP just represented it in a different way but did not change the basic problem.