Applied Optimization Exercise 2 - Convex Functions

Chenrui Fan

October 2023

Convex Functions (5 pts)

Simple functions (3 pt)

Are the following functions convex? Prove your statement.

1. $f(x) = x^2, x \in \mathbb{R}$

The first derivative of the function is: $f'(x) = 2x$
And the second derivative of the function is: $f''(x) = 2$
Due to $f''(x) = 2$ is a positive constant, this function is convex.

2. $f(x) = e^{x^2}, x \in \mathbb{R}$

The first derivative of the function is: $f'(x) = 2xe^{x^2}$
And the second derivative of the function is: $f''(x) = 2e^{x^2}(1 + 2x^2)$
Due to $2e^{x^2}$ and $1 + 2x^2$ are always positive when $x \in \mathbb{R}$, this function is convex.

3. $f(x, y) = x^2 + 3xy + 2y^2, x \in \mathbb{R}, y \in \mathbb{R}$

The first derivative of the function is:
$\frac{\partial f}{\partial x} = 2x + 3y$
$\frac{\partial f}{\partial y} = 3x + 4y$
And the second derivative of the function is:
$\frac{\partial^2 f}{\partial x^2} = 2$
$\frac{\partial^2 f}{\partial y^2} = 4$
$\frac{\partial^2 f}{\partial x \partial y} = 3$
$\frac{\partial^2 f}{\partial y \partial x} = 3$
Due to Hessian matrix, we can calculate that: $ad - bc = 8 - 9 = -1 < 0$
Therefore, this function is not convex.

Log-sum-exp (2 pt)

Show that the function
$f(x) = log(e^{x_1} + ... + e^{x_n})$
is convex on $\mathbb{R}^n$.
Hint: The proof is outlined in the Boyd's book p. 74. Develop by it yourself and provide the intermediate steps.

The first derivative of the function is:
$\frac{\partial f}{\partial x_i} = \frac{e^{x_i}}{e^{x_1} + ... + e^{x_n}}$
Due to quotient rule, when i=j, the second derivative of the function is:

$\frac{\partial^2 f}{\partial x_i^2} = \frac{e^{x_i}(e^{x_1}+...+e^{x_n})-(e^{x_i})^2}{(e^{x_1}+...+e^{x_n})^2} = \frac{e^{x_i}(e^{x_1}+...+e^{x_n}-e^{x_i})}{(e^{x_1}+...+e^{x_n})^2}$
when i≠j, the second derivative of the function is:
$\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$
Due to its Hessian matrix is positive semi-definite everywhere, this function is convex.

Log-sum-exp (2 pt)

Show that the geometric mean
$f(\mathbf{x}) = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$
is concave on $\mathbb{R}_{++}^n$**.**

let:
$f(\mathbf{x}) = g(h(\mathbf{x}))$
where:
$g(u) = u^{\frac{1}{n}}, \quad h(\mathbf{x}) = \prod_{i=1}^n x_i$
then:
$\frac{dg}{du} = \frac{1}{n} \cdot u^{\frac{1}{n}-1}$
$\frac{\partial h}{\partial x_i} = \prod_{j=1, j\neq i}^n x_j$
So the first derivative of the function is:
$\frac{\partial f}{\partial x_i} = \frac{1}{n} \cdot \left(\prod_{j=1}^n x_j\right)^{\frac{1}{n}-1} \cdot \left(\prod_{j=1, j\neq i}^n x_j\right) = \frac{1}{n} \cdot \left(\prod_{j=1}^n x_j\right)^{\frac{1}{n}} \cdot \frac{1}{x_i}$
And the second derivative of the function is:
when i=j:
$\frac{\partial^2 f}{\partial x_i^2} = -\frac{1}{n} \cdot x_i^{-\frac{1}{n}-2} \cdot \prod_{j=1, j\neq i}^n x_j^{\frac{1}{n}}$
when i≠j:
$\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$
Since the Hessian matrix is diagonal and the elements on the diagonal are negative, the matrix is negative definite.
Therefore, this function is concave on $\mathbb{R}_{++}^n$.

Programming Exercise: Convexity Test (5 pts)

Please check out the solution in the readme.txt.