Digital Image Processing

Lesson 6: Geometric Transformations

Master Course

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Outline

- Definition of geometrical transformations
- Applications of geometrical transformations
- Classification of geometrical transformations
- Homogeneous coordinates
- Matrixial representation of affine transformations
- Projective transformations
- Inverse warping
- Interpolation methods

Geometrical Transforms: Definition

- Geometric transformations are used to modify the geometric properties of an image, such as its size, shape, position, and orientation
- Mathematically it is defined by a function that maps the original coordinate space to another one
- Geometric transformations include operations such as
  - **Translations**
  - **Rotations**
  - **Scaling**
  - **Shearing**
  - **Projections**

Applications of Geometrical Transformations

- **Image registration:** aligning two or more images to enable comparison or analysis.
- **Image rectification:** removing distortions from an image caused by the projection of a three-dimensional scene onto a two-dimensional image plane
- **Image resizing:** changing the size of an image while preserving or not its aspect ratio
- **Image rotation:** rotating an image by a specified angle
- **Image morphing:** producing a metamorphosis from one image to another

Categories of Geometric Transforms

- **Euclidean transformations** (also called **rigid transformations**) include reflexions, translations, rotations, and any combinations of them
  - Distances (between two points) and (non-oriented) angles are preserved
- **Similarity transformations** additionally include isotropic scaling
  - Angles and proportions (ratio of distances) are preserved
  - Distances are no more preserved
- **Affine transformations** include alle the transformations mentioned above plus generalized (non-uniform) scaling, shearing
  - Alignments (collinearity) and proportions (ratio of distances) are preserved
  - Angles and distances are no more preserved!
- **Projective transformations** include any projection from a 3D space and models perspective
  - Only alignments and cross-ration are preserved

Affine Transformations

- This lesson is focused on **affine transformations**, which are defined by linear combinations of coordinates (x, y)

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

- In matrix form, it is expressed

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

  - c and f define translations
  - a and e define scaling
  - the combination of a, b, d, and e define rotations and shears

Homogeneous Coordinates

- **Homogeneous coordinates** use a 3D vector to represent a points of the 2D Euclidean space

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Thanks to homogeneous coordinates, affine transforms can be be represented with 3D matrices and matrix calculation can be applied

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Globally the equation can be written $P' = \mathbf{T}P$ where P and P' represent the original point and its transform and $\mathbf{T}$ represents the transform matrix

Matrix Representations of Basic Affine Transformations

- A **translation** by a vector (dx, dy) is represented as $\mathrm{T} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$

- A **scale** operation by sx (horizontally) and sy (vertically) is represented as $S = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- A **rotation** by an angle θ around the origin is represented as $\mathrm{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- A **horizontal shear** by an angle ϕ can be represented as $\mathrm{H} = \begin{bmatrix} 1 & Tan(\phi) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Composite Affine Transforms

- Any combination of affine transforms $\mathbf{T_1}$, $\mathbf{T_2}$, and $\mathbf{T_3}$ is also an affine transform and can be computed as

$$\mathbf{T} = \mathbf{T_3}\,\mathbf{T_2}\,\mathbf{T_1}$$

- Affine transforms computed in this way is inversible and its inverse transform can be computed as

$$\mathbf{T^{-1}} = \mathbf{T_1^{-1}}\,\mathbf{T_2^{-1}}\,\mathbf{T_3^{-1}}$$

Example: Central Image Rotation

- Let an image be defined inside the unit square coordinate system
- The central rotation can be computed by combining 3 operations
  - Translation to center the image on the origin

$$T_1 = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Rotation with a specified angle (example: 30 degrees)

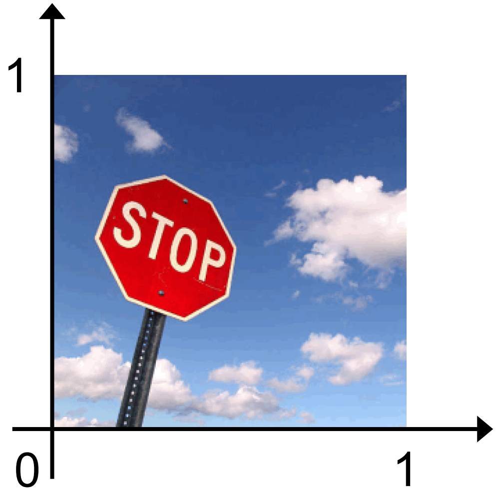
$$T_2 = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) & 0 \\ \sin(\pi/6) & \cos(\pi/6) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

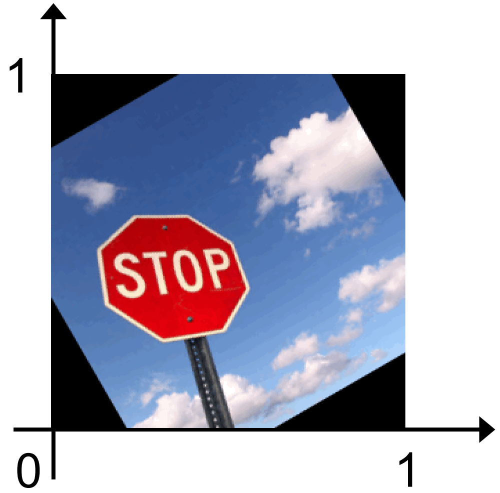
  - Back-translation to the original coordinate system

$$T_3 = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

- The global transformation is obtained with $T = T_3\ T_2\ T_1$

$$T = T_3 T_2 T_1 = \begin{bmatrix} 0.866 & -0.5 & 0.317 \\ 0.5 & 0.866 & -0.183 \\ 0 & 0 & 1 \end{bmatrix}$$





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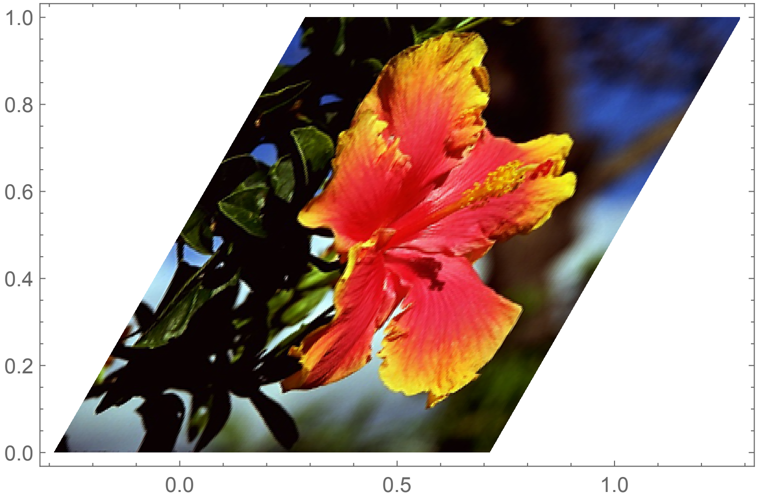
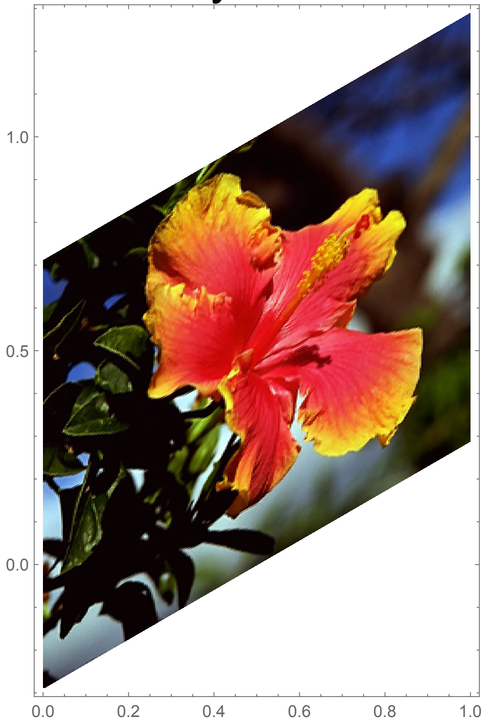
Shearing Transformations

- Shearing transformations are characterized by matrices of the form $\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$



Horizontal shear (30°)

$$\begin{bmatrix} 1 & 0.577 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Vertical shear (30°)

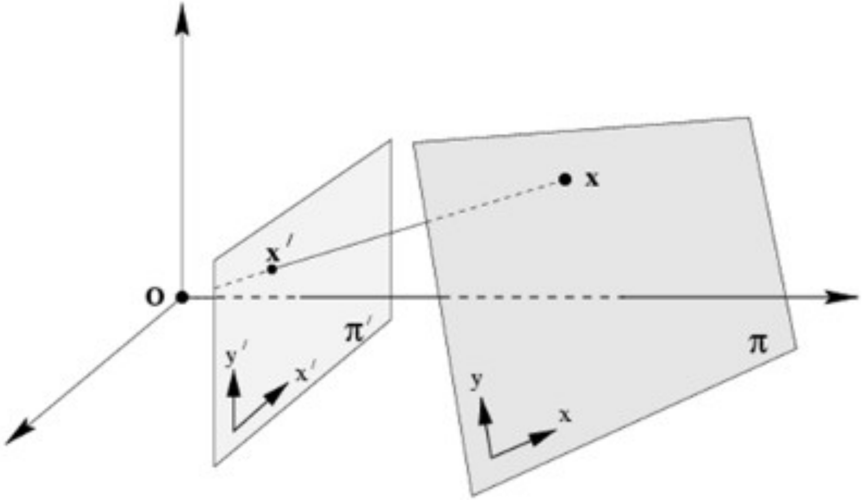
$$\begin{bmatrix} 1 & 0 & 0 \\ 0.577 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation (30°) followed by horizontal shear (30°)

$$\begin{bmatrix} 0.866 & 0 & 0.067 \\ 0.5 & 1.155 & -0.327 \\ 0 & 0 & 1 \end{bmatrix}$$

Projective Transformations

- **Projective transformations** (also called **homographies**) correspond to the projection of an arbitrary plane to another 2D surface
  - It models perspective



- It can be expressed as a linear transform of homogeneous coordinates
  - A vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ represents the point $(\frac{x}{z}, \frac{y}{z})$

$$\xrightarrow{}$$

$$\begin{bmatrix} 0.9 & 0.2 & -0.2 \\ -0.1 & 1.2 & -0.1 \\ 0.1 & 0.3 & 1 \end{bmatrix}$$

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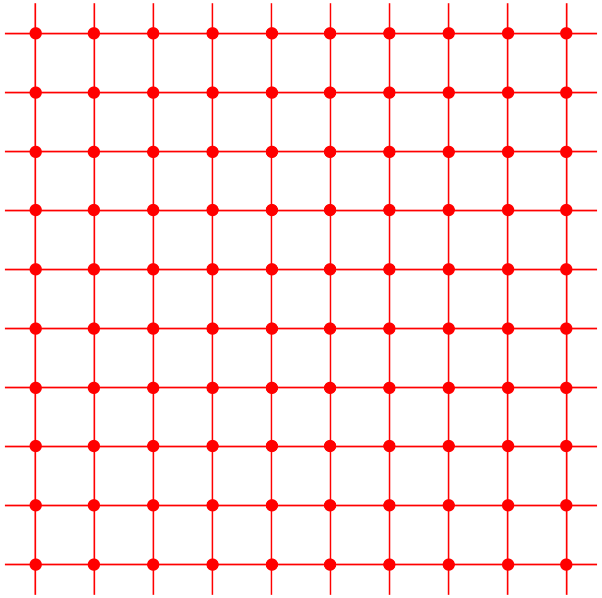
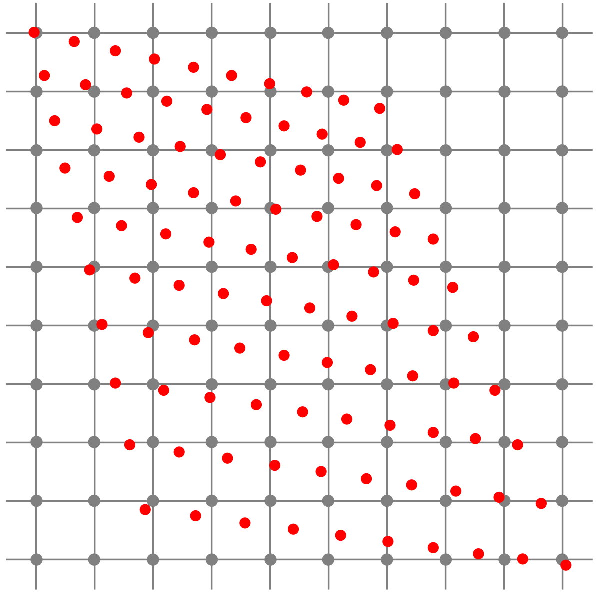
Direct Transform

- The mapping T locates the grid points of the initial coordinate system to the final coordinates, but those points are not on a regular grid!



$$\xrightarrow{\quad T \quad}$$

$$\begin{bmatrix} 0.9 & 0.2 & -0.2 \\ -0.1 & 1.2 & -0.1 \\ 0.1 & 0.3 & 1 \end{bmatrix}$$



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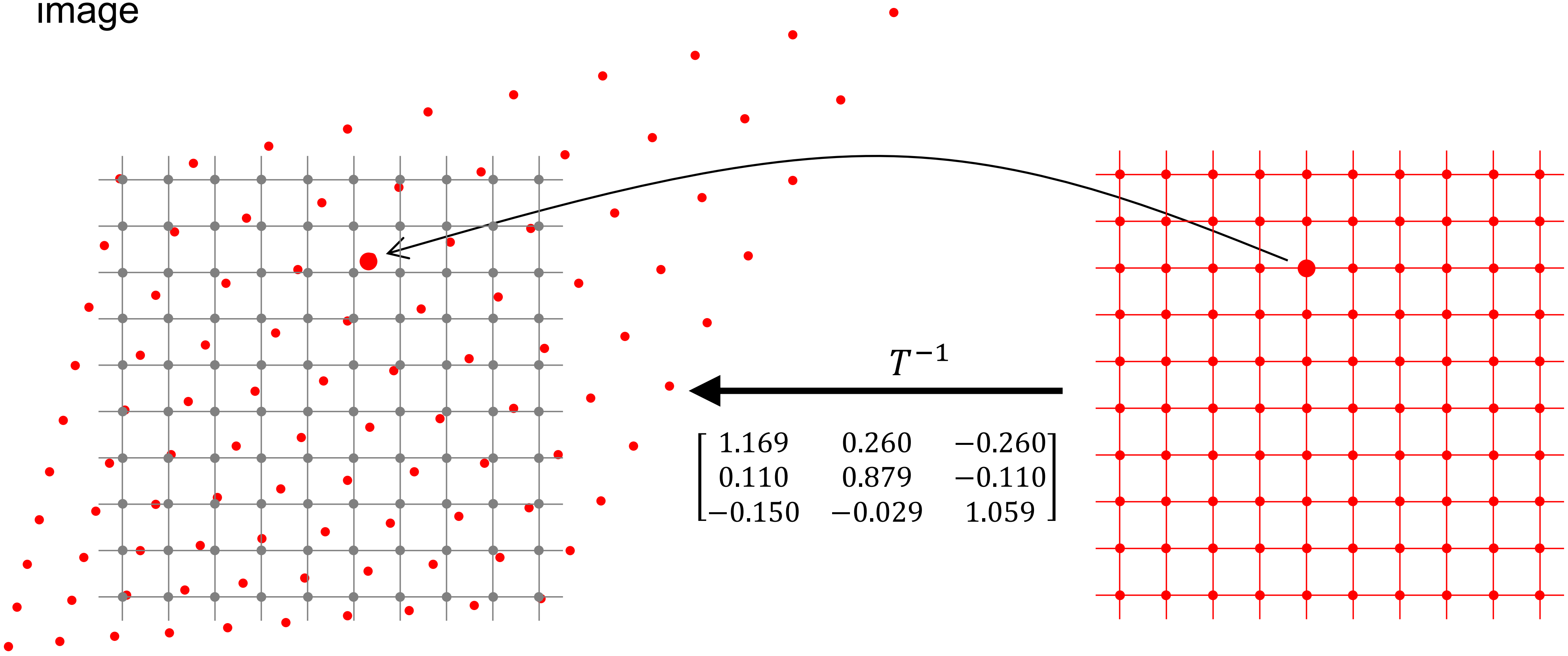
Inverse Transform

- Instead of mapping the original pixel values to the new coordinate system, a better approach consist of resampling the new pixels according to its corresponding location in the original image



$$T^{-1}\quad \begin{bmatrix} 1.169 & 0.260 & -0.260 \\ 0.110 & 0.879 & -0.110 \\ -0.150 & -0.029 & 1.059 \end{bmatrix}$$

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Interpolation

- Several interpolation methods can be used
  - **Nearest neighbor:** taking the value from the nearest pixel (in the discrete domain)
  - **Bilinear interpolation**: using a linear combination of a 2x2 neighborhood
  - **Bicubic interpolation**: using a 4x4 neighborhood and approximating the surface by a cubic function
  - **Binomial filter**
  - **Fourier based methods**

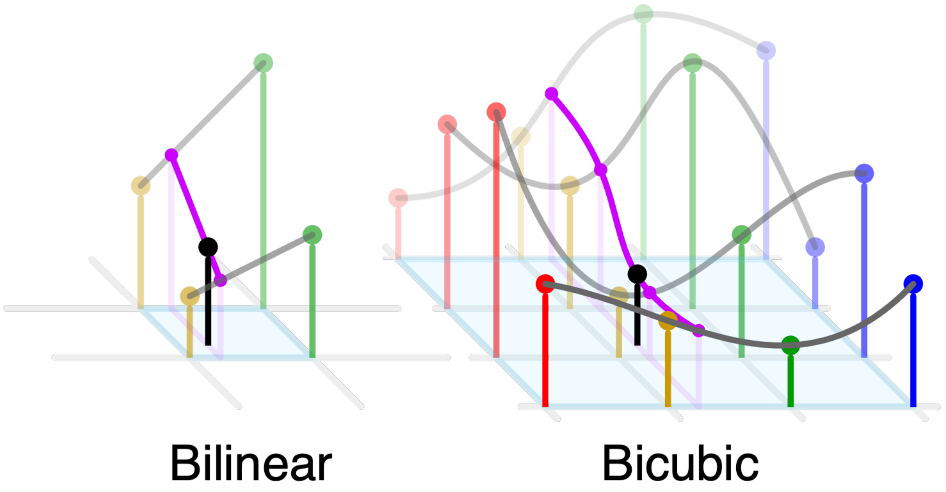
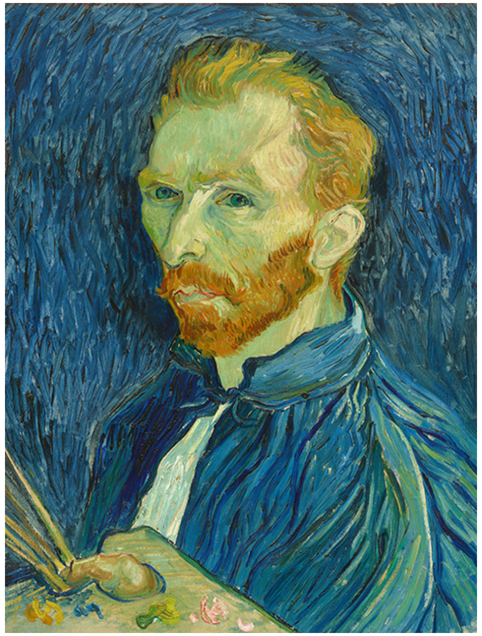


Illustration of Interpolation

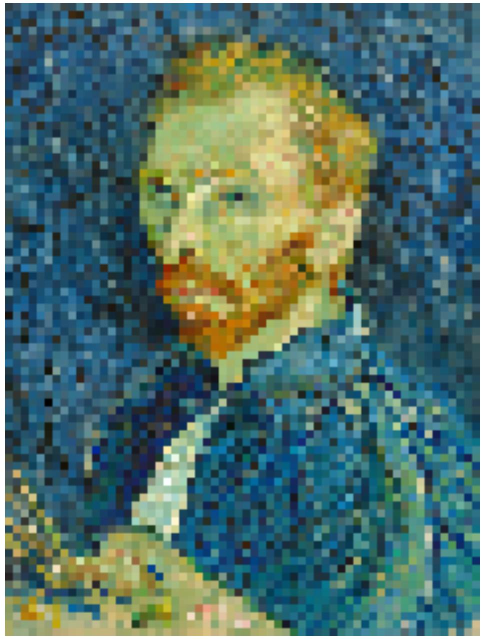
Nearest neighbor

Bilinear interpolation

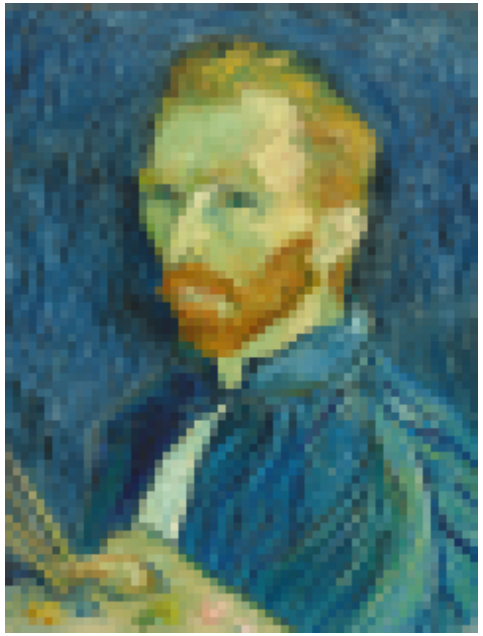
Illustration of Interpolation on Down Sampling



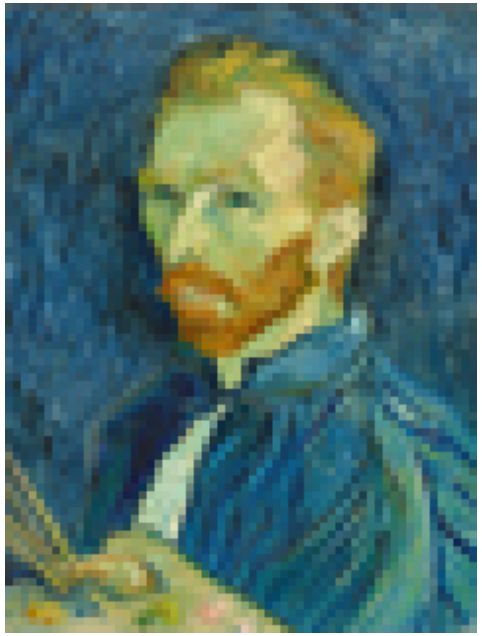
Original image



Nearest neighbor



Bilinear interpolation



Bicubic interpolation

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