Laplace mechanism

Group 1

October 2023

Question 2

Objective: The goal is to prove that under the utility function $U(a, x) = -|a - f(x)|$, the exponential mechanism is equivalent to the Laplace mechanism.

Expression for the Exponential Mechanism:

$$\pi(a|x) = \frac{\exp\left(-\frac{\epsilon|a-f(x)|}{2L}\right)}{\int_{-\infty}^{\infty} \exp\left(-\frac{\epsilon|a'-f(x)|}{2L}\right) da'}$$

Expression for the Laplace Mechanism:

$$p(a) = \frac{1}{2L} \exp\left(-\frac{\epsilon|a - f(x)|}{L}\right)$$

Analysis: Firstly, we attempt to analytically solve the integral in the denominator and compare it with the probability density function (PDF) of the Laplace mechanism.

For the integral, we have:

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\epsilon|a' - f(x)|}{2L}\right) da'$$

Considering the properties of absolute values, we can split the above integral into two parts, one positive and one negative:

$$\int_{-\infty}^{f(x)} \exp\left(\frac{\epsilon(a' - f(x))}{2L}\right) da' + \int_{f(x)}^{\infty} \exp\left(-\frac{\epsilon(a' - f(x))}{2L}\right) da'$$

Since these two integrals are similar in form, we only need to solve one of them and then multiply by 2. Solving one of the integrals yields:

$$-\frac{2L}{\epsilon} \exp\left(\frac{\epsilon(a' - f(x))}{2L}\right)$$

At $a' = f(x)$, the value of the above expression is $-2L/\epsilon$ and, being an exponential function, as a' approaches $-\infty$, its value tends towards 0.

Substituting these values into the two integrals and multiplying by 2, we get:

$$\frac{4L}{\epsilon}$$

Now, we can rewrite the probability density function for the exponential mechanism as:

$$\pi(a|x) = \frac{1}{2L} \exp\left(-\frac{\epsilon|a - f(x)|}{L}\right)$$

This is exactly the probability density function for the Laplace mechanism.

Conclusion: This completes the proof, showing that under the given utility function $U(a, x) = -|a - f(x)|$, the exponential mechanism is equivalent to the Laplace mechanism.