MAT1071 MATHEMATICS I
2. WEEK
PART 2

CONTINUITY

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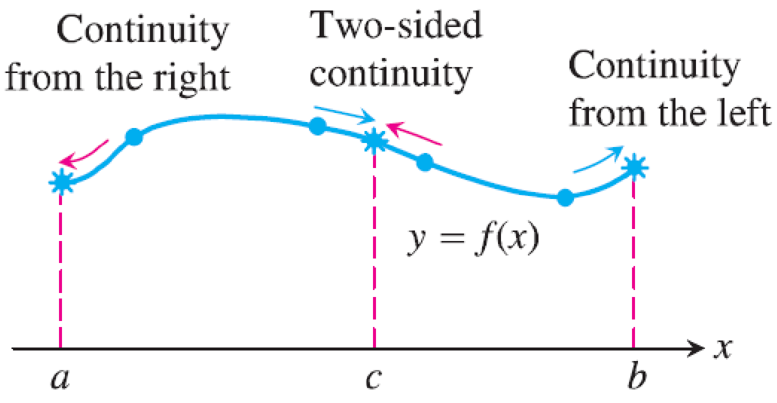
CONTINUITY



DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

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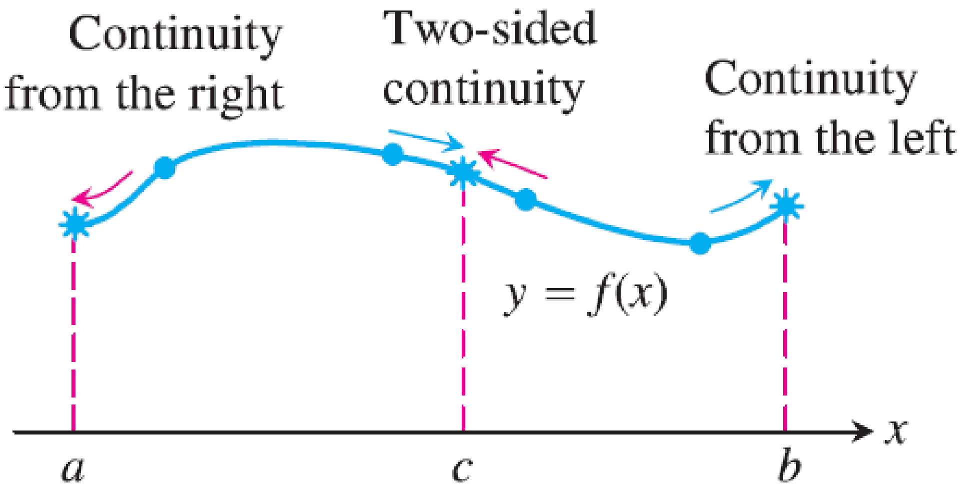
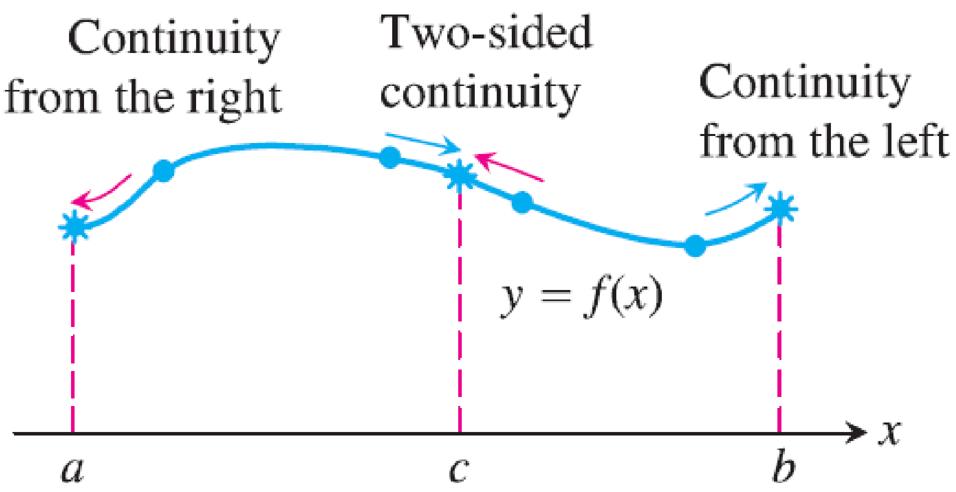


If a function f is not continuous at a point c, we say that f is **discontinuous** at c and that c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

A function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x \to c^-} f(x) = f(c)$.



☆ Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b.

☆ A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c

EXAMPLE The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain $[-2, 2]$, including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous.

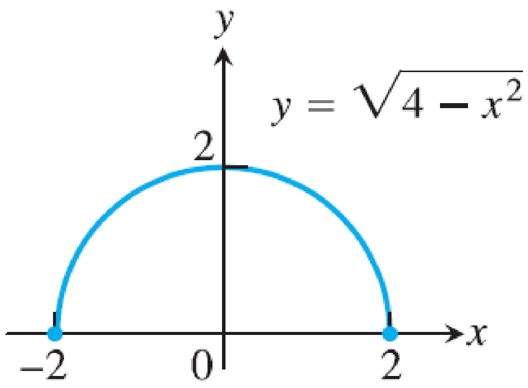


FIGURE A function that is continuous at every domain point

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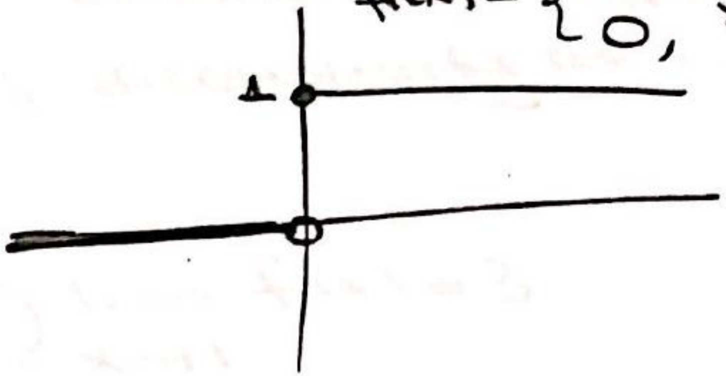
Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

EXAMPLE The unit step function

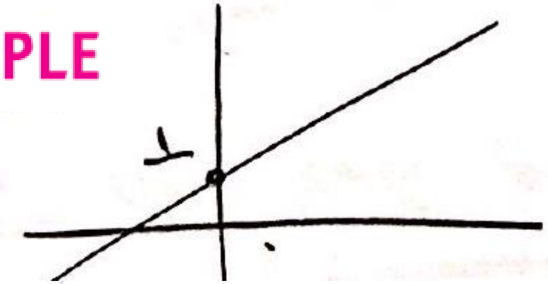
$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$$\left.\begin{aligned} \lim_{x \to 0^-} f(x) &= 0 \\ \lim_{x \to 0^+} f(x) &= 1 \end{aligned}\right\} \lim_{x \to 0} f(x) \text{ does not exist} \Rightarrow \text{is not continuous.}$$

This function $\lim_{x \to 0^+} f(x) = f(1) = 1$ is right continuous

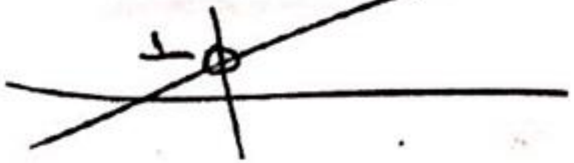
EXAMPLE



$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 1 = f(0)$

continuous at $x=0$.

EXAMPLE



$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 1$ but $f(0)$ is not defined

is not continuous at $x=0$

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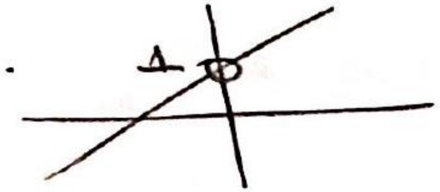
Types of Discontinuities

① Removable Discontinuity:

$\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$

EXAMPLE In above ex.



If we define $f(0)=1$, the function would be continuous at $x=0$.

EXAMPLE $f(x) = \begin{cases} x+2, & x>1 \\ 2, & x=1 \\ 4x-1, & x<1 \end{cases}$

Let examine the type of discontinuity at $x=1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x+2 = 3$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 4x-1 = 3$

$\Big\} \lim_{x \to 1} f(x) = 3$

but $f(1) = 2 \neq 3$

Removable discontinuity

② Jump Discontinuity:

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

The limit does not exist

EXAMPLE

$$f(x) = \begin{cases} x-1, & x < 1 \\ 1, & x = 1 \\ 2, & x > 1 \end{cases}$$

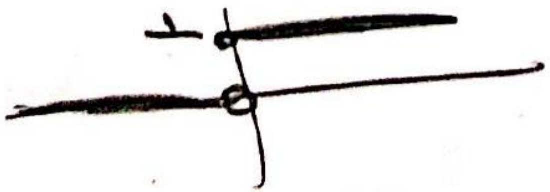
examine the type of discontinuity at $x = 1$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} x - 1 = 0$$

jump discontinuity

EXAMPLE Unit step function



$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = 0$$

jump discontinuity

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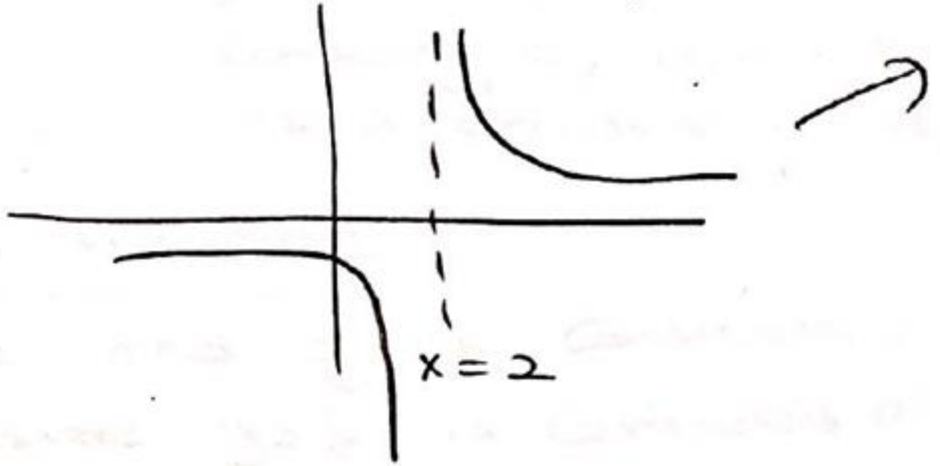
③ Infinite Discontinuity

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

EXAMPLE $f(x) = \dfrac{1}{x-2}$ Examine $x=2$

$$\left.\begin{array}{l}\lim\limits_{x\to 2^+} f(x) = +\infty \\ \lim\limits_{x\to 2^-} f(x) = -\infty\end{array}\right\}$$ infinite discontinuity at $x=2$



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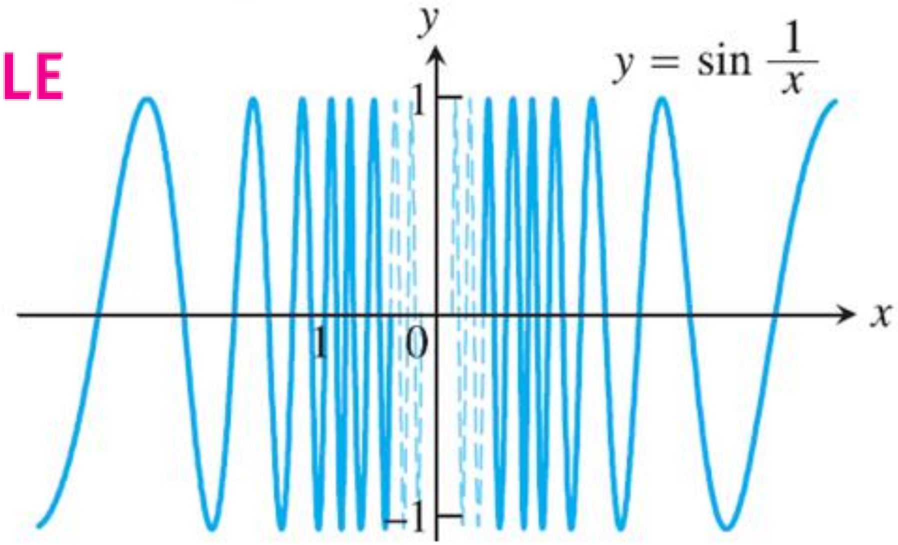
④ Oscillating Discontinuity

EXAMPLE



It oscillates too much to have a limit as $x \to 0$

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval.

EXAMPLE

(a) The function $y = 1/x$ is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there; that is, it is discontinuous on any interval containing $x = 0$.

(b) The identity function $f(x) = x$ and constant functions are continuous everywhere

THEOREM —Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:*	$f + g$	
2. *Differences:*	$f - g$	
3. *Constant multiples:*	$k \cdot f$,	for any number k
4. *Products:*	$f \cdot g$	
5. *Quotients:*	f/g,	provided $g(c) \neq 0$
6. *Powers:*	f^n,	n a positive integer
7. *Roots:*	$\sqrt[n]{f}$,	provided it is defined on an open interval containing c, where n is a positive integer

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THEOREM —Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

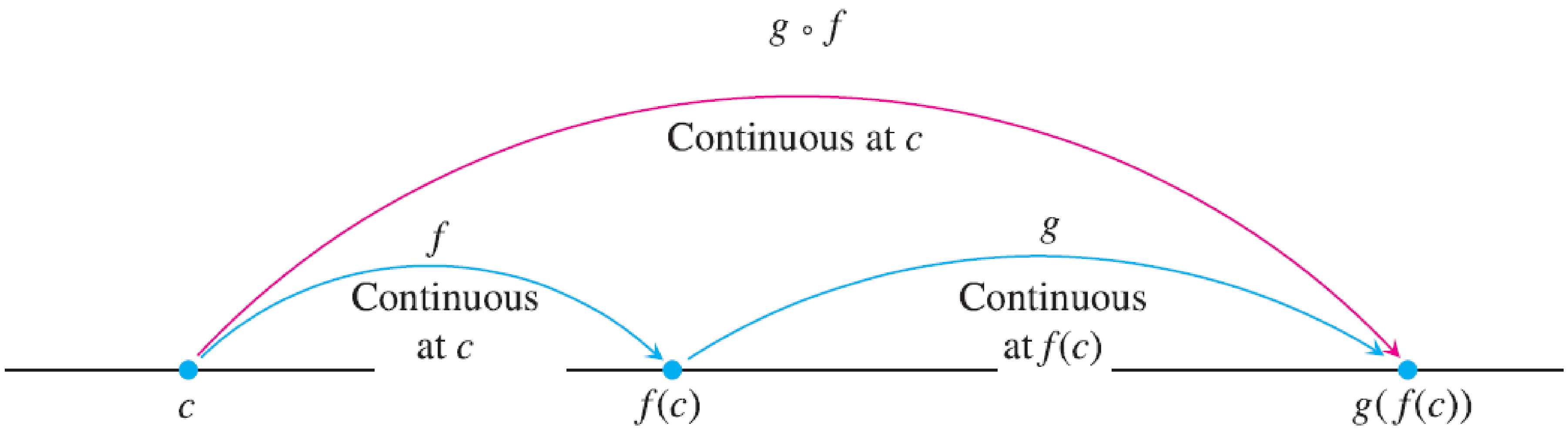

FIGURE Composites of continuous functions are continuous.

THEOREM —Limits of Continuous Functions If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

EXAMPLE

(a) Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \to c} P(x) = P(c)$

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined $(Q(c) \neq 0)$ ■

EXAMPLE The function $f(x) = |x|$ is continuous at every value of x. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \to 0} |x| = 0 = |0|$. ■

☆ The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$.
Both functions are, in fact, continuous everywhere

☆ It follows that all six trigonometric functions are then continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

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EXAMPLE Show that the following functions are continuous everywhere on their respective domains.

(a) $y = \sqrt{x^2 - 2x - 5}$ **(b)** $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ **(d)** $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its domain.

(b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function

x

(d) Because the sine function is everywhere-continuous , the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function ■

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EXAMPLE

$$\lim_{x\to\pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x\to\pi/2} 2x + \lim_{x\to\pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right)$$

$$= \cos(\pi + \sin 2\pi) = \cos \pi = -1.$$

THEOREM —Limits of Continuous Functions If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

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EXAMPLE $y = f(x) = \frac{\sin x}{x}$ is continuous at every point except $x=0$

$$F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

The function $F(x)$ is continuous at $x=0$ because

$$\left.\begin{aligned} \lim_{x\to 0^+} F(x) &= \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \\ \lim_{x\to 0^-} F(x) &= \lim_{x\to 0^-} 1 = 1 \end{aligned}\right\} = F(0)$$

The function F is continuous at $x=0$. So it is called the "continuous extension of f" to $x=0$.

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EXAMPLE Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \qquad x \neq 2$$

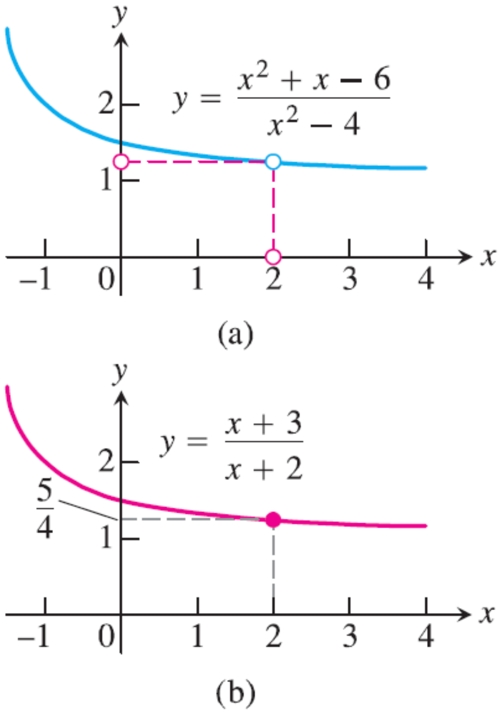
has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$

The new function

$$F(x) = \frac{x+3}{x+2}$$



is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f with its point of discontinuity at $x = 2$ removed.

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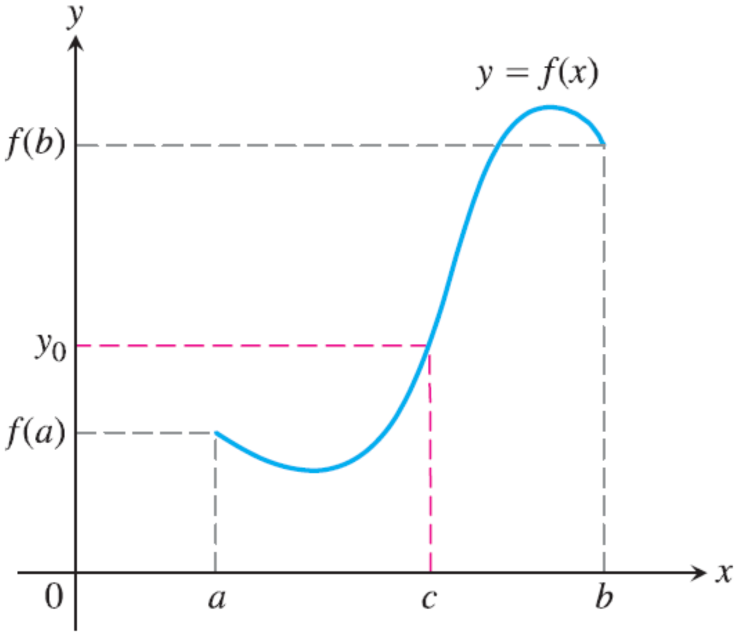
THEOREM —The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

The continuity of f on the interval is essential . If f is discontinuous at even one point of the interval, the theorem's conclusion may fail.

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EXAMPLE Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. f is continuous

$$f(1) = 1 - 1 - 1 = -1$$
$$f(2) = 2^3 - 2 - 1 = 5$$

we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$.

$$\boxed{f(1) = -1} < 0 < \boxed{f(2) = 5}$$

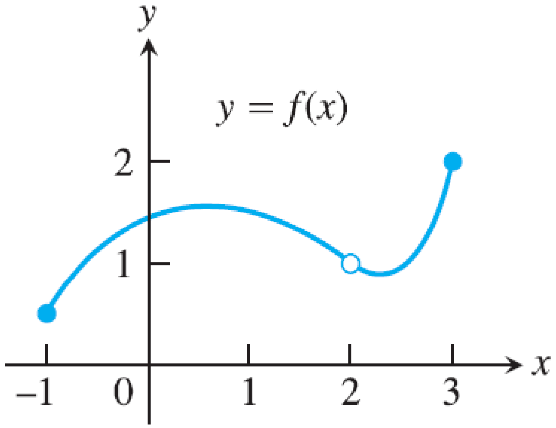
the Intermediate Value Theorem says there is a zero of f between 1 and 2.

HW:

Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

1.



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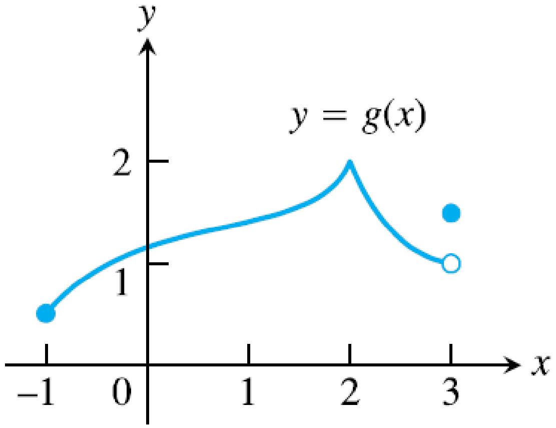

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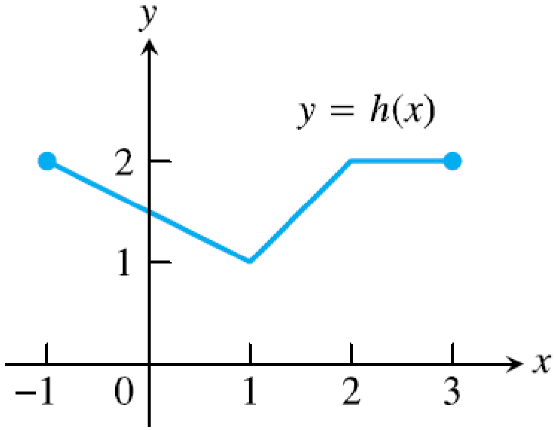

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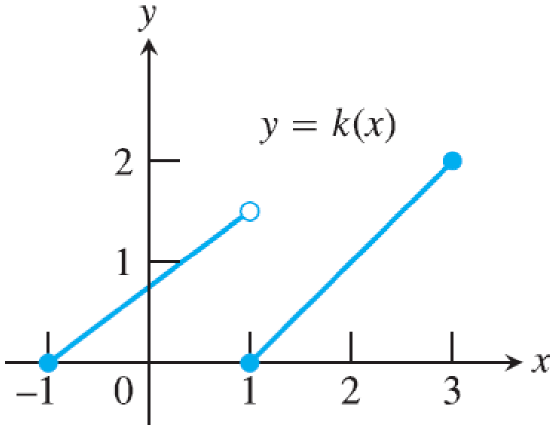

HW:

At what points are the functions in Exercises 13–30 continuous?

13. $y = \dfrac{1}{x - 2} - 3x$

14. $y = \dfrac{1}{(x + 2)^2} + 4$

15. $y = \dfrac{x + 1}{x^2 - 4x + 3}$

16. $y = \dfrac{x + 3}{x^2 - 3x - 10}$

HW:

Find the limits in Exercises 31–36. Are the functions continuous at the point being approached?

31. $\lim_{x \to \pi} \sin(x - \sin x)$

32. $\lim_{t \to 0} \sin\left(\frac{\pi}{2} \cos(\tan t)\right)$

33. $\lim_{y \to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

HW:

Use the Intermediate Value Theorem in Exercises 69–76 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

69. $x^3 - 3x - 1 = 0$

70. $2x^3 - 2x^2 - 2x + 1 = 0$

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.