05.04.2021

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Some of the questions below will be solved during the lesson as an example.

The remaining questions that cannot be solved are homework.

Very similar questions to the questions solved in the course and given as homework will be asked in the exam.

START: 09:15

Need:
1. Notebook,
2. Calculator
3. Excel

1. Bisection Method
2. False Position Method
3. Newton-Raphson Method

$f(x)=x^3+2x-2$ has a root in [0,1] and use the bisection method, False Position and Newton-Raphson, to determine on approximation to the root that is accurate to at least within 0.01.

1. x^3+2x-2 [0,1] (Bisection, False Position and Newton-Raphson)
2. $X^3-\sin(x)$ [1,2] (Bisection, False Position and Newton-Raphson)
3. $2-e^x$ [0,1] (Bisection, False Position and Newton-Raphson)
4. x^3+4x^2-3 ($x_o=0,5$) (Newton-Raphson)
5. X^3-x-1 [1,2] (Bisection, False Position and Newton-Raphson)
6. x^3+2x-4 [1,2] ($x_o=1$) (Bisection, False Position and Newton-Raphson)
7. X^3-2x^2-5 [2,3] ($x_o=2$) (Bisection, False Position and Newton-Raphson)
8. $X\sin(x)-1$ [0,2] (error 0.0001) (Bisection, False Position and Newton-Raphson)
9. $e^{-x}-x$ $x_o=0$ (Newton-Raphson)
10. $X^3-10x^2+34x-40$ $x_o=1$ (Newton-Raphson)
11. $2e^x-x^2-1$ $x_o=3$ (Newton-Raphson)
12. $\sin(x)+x^2-1$ $x_o=3$ (Newton-Raphson)
13. $e^{-x}-\sin x$ [0,1] (Bisection, False Position and Newton-Raphson)

Homework number will start from 14

Bisection Algorithm and Pseudocode

The **bisection method** exploits this property of continuous functions. At each step in this algorithm, we have an interval $[a, b]$ and the values $u = f(a)$ and $v = f(b)$. The numbers u and v satisfy $uv < 0$. Next, we construct the midpoint of the interval, $c = \frac{1}{2}(a + b)$, and compute $w = f(c)$. It can happen fortuitously that $f(c) = 0$. If so, the objective of the algorithm has been fulfilled. In the usual case, $w \neq 0$, and either $wu < 0$ or $wv < 0$. (Why?) If $wu < 0$, we can be sure that a root of f exists in the interval $[a, c]$. Consequently, we store the value of c in b and w in v. If $wu > 0$, then we cannot be sure that f has a root in $[a, c]$, but since $wv < 0$, f must have a root in $[c, b]$. In this case, we store the value of c in a and w in u. In either case, the situation at the end of this step is just like that at the beginning except that the final interval is half as large as the initial interval. This step can now be repeated until the interval is satisfactorily small, say $|b - a| < \frac{1}{2} \times 10^{-6}$. At the end, the best estimate of the root would be $(a + b)/2$, where $[a, b]$ is the last interval in the procedure.

False Position (Regula Falsi) Method and Modifications

The **false position method** retains the main feature of the bisection method: that a root is trapped in a sequence of intervals of decreasing size. Rather than selecting the midpoint of each interval, this method uses the point where the secant lines intersect the x-axis.

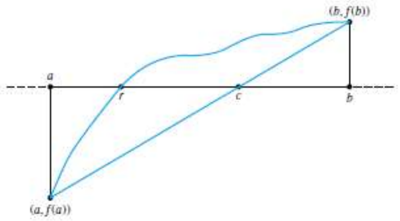


FIGURE 3.2 False position method

In Figure 3.2, the secant line over the interval $[a, b]$ is the chord between $(a, f(a))$ and $(b, f(b))$. The two right triangles in the figure are *similar*, which means that

$$\frac{b - c}{f(b)} = \frac{c - a}{-f(a)}$$

It is easy to show that

$$c = b - f(b)\left[\frac{a - b}{f(a) - f(b)}\right] = a - f(a)\left[\frac{b - a}{f(b) - f(a)}\right] = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

We then compute $f(c)$ and proceed to the next step with the interval $[a, c]$ if $f(a)f(c) < 0$ or to the interval $[c, b]$ if $f(c)f(b) < 0$.

In the general case, the **false position method** starts with the interval $[a_0, b_0]$ containing a root: $f(a_0)$ and $f(b_0)$ are of opposite signs. The false position method uses intervals $[a_k, b_k]$ that contain roots in almost the same way that the bisection method does. However, instead of finding the midpoint of the interval, it finds where the secant line joining $(a_k, f(a_k))$ and $(b_k, f(b_k))$ crosses the x-axis and then selects it to be the new endpoint.

Interpretations of Newton's Method

In Newton's method, it is assumed at once that the function f is differentiable. This implies that the graph of f has a definite *slope* at each point and hence a unique tangent line. Now let us pursue the following simple idea. At a certain point $(x_0, f(x_0))$ on the graph of f, there is a tangent, which is a rather good approximation to the curve in the vicinity of that point. Analytically, it means that the linear function

$$l(x) = f'(x_0)(x - x_0) + f(x_0)$$

is close to the given function f near x_0. At x_0, the two functions l and f agree. We take the zero of l as an approximation to the zero of f. The zero of l is easily found:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Thus, starting with point x_0 (which we may interpret as an approximation to the root sought), we pass to a new point x_1 obtained from the preceding formula. Naturally, the process can be repeated (iterated) to produce a sequence of points:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \qquad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \qquad \text{etc.}$$

Under favorable conditions, the sequence of points will approach a zero of f.

The geometry of Newton's method is shown in Figure 3.4. The line $y = l(x)$ is tangent to the curve $y = f(x)$. It intersects the x-axis at a point x_1. The slope of $l(x)$ is $f'(x_0)$.

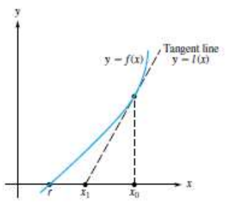


FIGURE 3.4 Newton's method

There are other ways of interpreting Newton's method. Suppose again that x_0 is an initial approximation to a root of f. We ask: *What correction h should be added to x_0 to obtain the root precisely?* Obviously, we want

$$f(x_0 + h) = 0$$

If f is a sufficiently well-behaved function, it will have a Taylor series at x_0 [see Equation (11) in Section 1.2]. Thus, we could write

$$f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \cdots = 0$$

Determining h from this equation is, of course, not easy. Therefore, we give up the expectation of arriving at the true root in one step and seek only an approximation to h. This can be obtained by ignoring all but the first two terms in the series:

$$f(x_0) + hf'(x_0) = 0$$

The h that solves this is *not* the h that solves $f(x_0 + h) = 0$, but it is the easily computed number

$$h = -\frac{f(x_0)}{f'(x_0)}$$

Our new approximation is then

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

and the process can be repeated. In retrospect, we see that the Taylor series was not needed after all because we used only the first two terms. In the analysis to be given later, it is assumed that f'' is continuous in a neighborhood of the root. This assumption enables us to estimate the errors in the process.

If Newton's method is described in terms of a sequence $x_0, x_1, \ldots$, then the following **recursive** or **inductive** definition applies:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Naturally, the interesting question is whether

$$\lim_{n\to\infty} x_n = r$$

where r is the desired root.

You may be given a starting point to get started.
If you are not given a starting point, you can start iteration from any point you want.

But my advice is to start from the lower point of the given interval.

Summary

4 information will be given to you.
(1) Function (2) interval (3) Error (4) methot

Solution methot

1. Check Continuity
2. Check if the function has a root in given interval (range)
3. Identify a new candidate root
4. Compare the error with the new root
5. Stop, if new root is less than error
6. If the new root is more than the error, go to step 2 or 3

	Bisection metods	False Position method.
New root	$X_n = \frac{a+b}{2}$	$X_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

a : smallest number in interval
b : largest " " "

a : smallest number in interval
b : largest " " "

stop condition

① $\frac{b-a}{2^n} < \varepsilon$ error

or

② $|X_n - X_{n-1}| < \varepsilon$

① $X_n - a_n < \varepsilon$, $b_n - X_n < \varepsilon$

or

② $|X_n - X_{n-1}| < \varepsilon$

You can use this stop condition when solving your own special function.

In our lesson. We will not use this stop condition in my homework and exam.

stop $|X_n - X_{n-1}| < \varepsilon$

$f(x)=x^3+2x-2$ has a root in [0,1] and use the bisection method, False Position and Newton-Raphson, to determine on approximation to the root that is accurate to at least within 0.01.

1. $X^3-\sin(x)$ [1,2] (Bisection, False Position and Newton-Raphson)
2. $2-e^x$ [0,1] (Bisection, False Position and Newton-Raphson)

$f(x) = x^3+2x-2$ [0,1]

Bisection method [0,1]

$X_n = \frac{a+b}{2}$

$f(0) = -2$ (−)

$f(1) = 1$ (+)

$f(0{,}5) = -0{,}875$ (−) [0,5, 1]

$X_n = \frac{1+0{,}5}{2} = 0{,}75$

$f(0{,}75) = -0{,}0781$

$X_n = \frac{1+0{,}75}{2} = 0{,}875$

$f(0{,}875) = 0{,}619.....$

False Position Method.

$X_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$a = 0$
$b = 1$
$f(a) = -2$
$f(b) = 1$

$X_n = \frac{0 \cdot 1 - 1 \cdot (-2)}{1 - (-2)}$

$X_n = 0{,}6667$

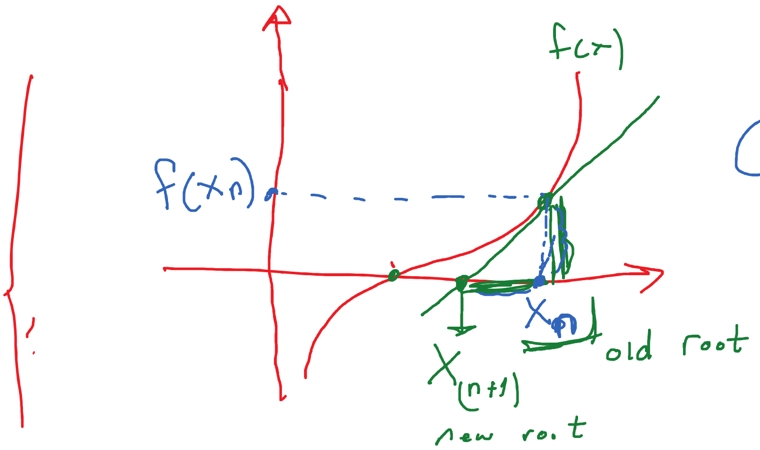
$f(0{,}6667) =$

1. $X^3 - \sin(x)$ [1,2] (Bisection, False Position and Newton-Raphson)
2.

10^{30} START

Newton-Raphson Method

① Check conti....
② You can chose a root from any point you want
③ Calculate a new root using the root of your choise
④ Compare the error with new root
5. Stop if new root is less then error.
6. If the new root is more then the error go to step 3



$$slop = f'(x_n)$$

$$slop = \frac{f(x_n) - 0}{X_n - X_{(n+1)}}$$

$$\boxed{f'(x_n) = \frac{f(x_n) - 0}{X_n - X_{(n+1)}}}$$

$$X_n - X_{(n+1)} = \frac{f(x_n)}{f'(x_n)}$$

$$\boxed{X_{(n+1)} = X_n - \frac{f(x_n)}{f'(x_n)}}$$

$$f(x) = x^3 + 2x - 2 \qquad [0,1]$$

$$f'(x) = 3x^2 + 2$$

starting point $X_0 = 0$

$$\boxed{X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)}}$$

① $X_1 = X_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_1 = 0 - \frac{-2}{2} = \underline{1}$$

$f(1) = 1$
$f'(1) = 5$

② $X_2 = X_1 - \dfrac{f(x_1)}{f'(x_1)}$ $\Big\}$ $X_2 = 1 - \dfrac{1}{5} = 0{,}8$

$f(0.8) = 0{,}112$
$f'(0.8) = 3.92$

③ $X_3 = X_2 - \dfrac{f(x_2)}{f'(x_2)}$ $\Rightarrow$ $X_3 = 0{,}8 - \dfrac{0.112}{3.92} = \underline{0{,}771429}$

④ $X_4 = X_3 - \dfrac{f(x_3)}{f'(x_3)} = \underline{\underline{0{,}770917}}$

$f(0.7714) = f(x_3) = 0{,}0019$
$f'(x_3) = 3{,}7853$

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14 ~~3~~. $2-e^x$ [0,1] (Bisection, False Position and Newton-Raphson)
15 ~~4~~. x^3+4x^2-3 $(x_o=0{,}5)$ (Newton-Raphson)
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