SORULAR

1. Aşağıda karakteristik denklemlerinin kökleri verilen sabit katsayılı lineer diferansiyel denklemlerin homojen kısımlarının çözümlerini (y_h) yazınız.

a. $r_1 = 1$, $r_2 = -1$, $r_3 = \dfrac{2}{3}$, $r_4 = -\dfrac{2}{3}$, $r_5 = 5$, $r_6 = -6$.

$y_h = C_1 e^{x} + C_2 e^{-x} + C_3 e^{2/3 x} + C_4 e^{-\frac{2}{3}x} + C_5 e^{5x} + C_6 e^{-6x}$

b. $r_1 = r_2 = 0$, $r_3 = r_4 = -2$, $r_5 = 3i$, $r_6 = -3i$.

$y_h = C_1 + C_2 x + C_3 e^{-2x} + C_4 x e^{-2x} + C_5 \cos 3x + C_6 \sin 3x$

c. $r_1 = r_2 = 2i$, $r_3 = r_4 = -2i$, $r_5 = i$, $r_6 = -i$.

$y_h = (C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x + C_5 \cos x + C_6 \sin x$

d. $r_1 = i$, $r_2 = -i$, $r_3 = -2i$, $r_4 = 2i$, $r_5 = -3i$, $r_6 = 3i$.

$y_h = C_1 \cos x + C_2 \sin x + C_3 \cos 2x + C_4 \sin 2x + C_5 \cos 3x + C_6 \sin 3x$

e. $r_1 = r_2 = r_3 = 2$, $r_4 = -2$, $r_5 = 2 - 3i$, $r_6 = 2 + 3i$.

$y_h = (C_1 + C_2 x + C_3 x^2)e^{2x} + (C_4 + C_5 x + C_6 x^2)e^{-2x} + e^{2x}(C_7 \cos 3x + C_8 \sin 3x)$

f. $r_1 = r_2 = r_3 = i$, $r_4 = r_5 = r_6 = -i$.

$y_h = (C_1 + C_2 x + C_3 x^2)\cos x + (C_4 + C_5 x + C_6 x^2)\sin x$

g. $r_1 = r_2 = -1 + 3i$, $r_3 = r_4 = -1 - 3i$, $r_5 = 2$, $r_6 = -2$.

$y_h = e^{-x}(C_1 \cos 3x + C_2 \sin 3x) + C_3 e^{2x} + C_4 e^{-2x}$

h. $r_1 = r_2 = r_3 = 2 - i$, $r_4 = r_5 = r_6 = 2 + i$.

$y_h = e^{2x}\left[(C_1 + C_2 x + C_3 x^2)\cos x + (C_4 + C_5 x + C_6 x^2)\sin x\right]$

i. $r_1 = 1$, $r_2 = -1$, $r_3 = i$, $r_4 = -i$, $r_5 = 1 + i$, $r_6 = 1 - i$.

$y_h = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x + e^{x}(C_5 \cos x + C_6 \sin x)$

j. $r_1 = \sqrt{2}$, $r_2 = r_3 = 2i$, $r_4 = r_5 = -2i$ $r_6 = \sqrt{3}$.

$y_h = C_1 e^{2x} + (C_2 + C_3 x)\cos 2x + (C_4 + C_5 x)\sin 2x + C_6 e^{\sqrt{3}x}$

$$y''' + 3y'' + 3y' + y = 0$$

$$\underline{r^3} + \underline{3r^2 + 3r} + \underline{1} = 0$$

$$3r(r+1) + (r+1)(r^2 - r + 1) = 0$$

$$(r+1)\left[\underline{3r + r^2 - r + 1}\right] = 0$$

$$(r+1)(r^2 + 2r + 1) = 0$$

$$(r+1)^3 = 0 \quad r_1 = r_2 = r_3 = -1$$

$$y_h = (c_1 + c_2 x + c_3 x^2) e^{-x}$$

I. way

$-1+3-3+1=0 \quad (r+1)$

$$
\begin{array}{l|l}
r^3+3r^2+3r+1 & r+1 \\
\cline{2-2}
-\underline{\;r^3+r^2\;} & r^2+2r+1 \\
\quad 2r^2+3r+1 & \\
-\underline{\;2r^2+2r\;} & \\
\qquad r+1 &
\end{array}
$$

$(r+1)^2=0$

$$y^{IV} - 4y''' + 29y'' = 0$$

$$r^4 - 4r^3 + 29r^2 = 0$$

$$r^2\left[r^2 - 4r + 29\right] = 0$$

$$\Delta = (-4)^2 - 4.1.29$$

$$r_1 = r_2 = 0$$

$$r_{3,4} = \frac{4 \mp \sqrt{\Delta}}{2.1} \quad (i = \sqrt{-1})$$

$$\Delta = 16 - 116 = -100$$

$$r_{3,4} = \frac{4 \mp 10i}{2} = 2 \mp 5i$$

$$y_h = c_1 + c_2 x + e^{2x}(c_3 \cos 5x + c_4 \sin 5x)$$

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$$r_1 = r_2 = -\frac{1}{2} \qquad r_3 = r_4 = r_5 = \frac{1}{2}i$$

$$r_6 = r_7 = r_8 = -\frac{1}{2}i$$

$$y_h = (c_1 + c_2 x)e^{-\frac{1}{2}x} +$$

$$(c_3 + c_4 x + c_5 x^2)\cos\frac{1}{2}x +$$

$$(c_6 + c_7 x + c_8 x^2)\sin\frac{1}{2}x$$

7\. a. (x) d^2y ... dy

$$r_{1,2} = \mp i \quad , \quad r_{3,4} = -1 \mp 2i$$

$$r_5 = r_6 = r_7 = 1$$

$$y_h = c_1 \cos x + c_2 \sin x + e^{-x}\left[c_3 \cos 2x + c_4 \sin 2x\right] + (c_5 + c_6 x + c_7 x^2) e^{x}$$

$$r_{1,2} = 3i \quad r_{3,4} = -3i$$

$$r_5 = r_6 = 0 \quad , \quad r_{7,8} = 1 \mp i$$

$$y_h = (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x + c_5 + c_6 x + e^{x}(c_7 \cos x + c_8 \sin x)$$

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$\frac{d^2}{dx^2} \quad \frac{d}{dx} + a_2(x)y = 0$ diferansiye

$$y''' - 2y'' + 4y' - 8y = 0$$

$$r^3 - 2r^2 + 4r - 8 = 0$$

$$\begin{array}{r|l} r^3 - 2r^2 + 4r - 8 & r-2 \\ \cline{2-2} -\;\; r^3 - 2r^2 \qquad\qquad & r^2 + 4 \\ \cline{1-1} 4r - 8 & \\ 4r - 8 & \end{array}$$

$$(r-2)(r^2+4) = 0$$

$$r_1 = 2 \;,\; r_{2,3} = \mp 2i$$

$$y_h = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$$

$$y''' - 3y'' - y' + 3y = 0$$

$$\underbrace{r^3 - 3r^2} \underbrace{- r + 3} = 0$$

$$r^2(r-3) - (r-3) = 0$$

$$(r-3)(r^2-1) = 0$$

$$r_1 = 3, \quad r_2 = 1, \quad r_3 = -1$$

$$y_h = c_1 e^{3x} + c_2 e^{x} + c_3 e^{-x}$$

Questions

1) $y' - \frac{y}{x} = y^{-2}\frac{x^3}{3}\sin x \quad (x \neq 0, y \neq 0)$

Bernoulli D.

$y^{1-(-2)} = y^3 = u$

$3y^2y' = u'$

$y'y^2 - \frac{y^3}{x} = \frac{x^3}{3}\sin x$

$\frac{u'}{3} - \frac{u}{x} = \frac{x^3}{3}\sin x \quad$ Linear D.

$u' - \frac{3u}{x} = x^3 \sin x$

$\lambda(x) = e^{-\int\frac{3}{x}dx} = e^{-3\ln x} = \frac{1}{x^3}$

$u(x) = x^3\left[\int \frac{1}{\cancel{x^3}}\cdot \cancel{x^3}\sin x + c\right] = x^3(-\cos x + c)$

$y = \sqrt[3]{u} = \sqrt[3]{x^3(-\cos x + c)}$

2) $(4xy-3)y' + 2(y^2+x) = 0 \quad$ D.E

$\underbrace{(4xy-3)}_{N}dy + \underbrace{2(y^2+x)}_{m}dx = 0$

$\frac{\partial m}{\partial y} = 4y = \frac{\partial N}{\partial x} = 4y \quad$ Exact (Tam) D.

$\frac{\partial f}{\partial x} = m = 2y^2 + 2x$

$\frac{\partial f}{\partial y} = N = 4xy - 3$

$f(x,y) = 2xy^2 + x^2 + h(y)$

$\frac{\partial f}{\partial y} = 4xy + \frac{dh}{dy} = 4xy - 3$

$\frac{dh}{dy} = -3 \rightarrow \int dh = -3\int dy$

$h = -3y + c$

$f(x,y) = 2xy^2 + x^2 - 3y + c = k$

5. $x^2y' = x^2e^{y/x} + xy + x^2$ (Homogeneous D) (Homojen D.)

$y = ux$

$dy = udx + xdu$

$x^2\,dy = (x^2e^{y/x} + xy + x^2)\,dx$

$x^2(udx + xdu) = (x^2e^u + ux^2 + x^2)\,dx$

$(\cancel{ux^2} - x^2e^u - \cancel{ux^2} - x^2)\,dx + x^3\,du = 0$

$-x^2(e^u+1)dx + x^3du = 0 \implies -\int\frac{dx}{x} + \int\frac{du}{e^u+1} = \int 0$

$\left(\int\frac{e^{-u}du}{(e^u+1).e^{-u}}\right)$

$-\ln x + \int\frac{e^{-u}du}{1+e^{-u}} = \ln c$

$-\ln x - \ln(1+e^{-u}) = \ln c \rightarrow x(1+e^{-u}) = \frac{1}{c}$

$x\left(1+e^{-\frac{y}{x}}\right) = \frac{1}{c}$

6. $2yy'(2+x) + 2x - x^2 + y^2 = 0$

$2y'(2+x) + \frac{2x-x^2}{y} + y = 0 \Rightarrow 2y'(2+x) + y = \frac{x^2-2x}{y}$ Bernoulli

$y^{1-(-1)} = y^2 = u$

$2yy' = u'$

$2y'y(2+x) + y^2 = x^2 - 2x$

$u'(2+x) + u = x^2 - 2x$ Linear D.

$u' + \frac{u}{2+x} = \frac{x^2-2x}{2+x}$

$\lambda(x) = e^{\int\frac{dx}{2+x}} = e^{\ln(x+2)} = (x+2)$

$u = \frac{1}{(x+2)}\left[\int \cancel{(x+2)}\frac{(x^2-2x)}{\cancel{(x+2)}}dx + c\right] = \frac{1}{x+2}\left(\frac{x^3}{3} - x^2 + c\right)$

$y = \sqrt{u} = \sqrt{\frac{1}{(x+2)}\left(\frac{x^3}{3} - x^2 + c\right)}$

7. $a_0(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_2(x)y = 0$ diferansiyel denkleminde a_0, a_1 ve a_2 fonksiyonları $a \leq x \leq b$ reel aralığında sürekli ve bu aralıktaki her x için $a_0(x) \neq 0$ olsun. f_1 ve f_2 fonksiyonları verilen diferansiyel denklemin lineer bağımsız iki çözümü ve A_1, A_2, B_1, B_2 de $A_1B_2 - A_2B_1 \neq 0$ denklemini sağlayan sabitler olsun. Bu durumda verilen diferansiyel denklemin $A_1f_1 + A_2f_2$ ve $B_1f_1 + B_2f_2$ çözümlerinin de $a \leq x \leq b$ de lineer bağımsız olduğunu göster.

For $a \leq x \leq b$ real numbes, the functions a_0, a_1 and a_2 in the D.E $a_0(x)y'' + a_1(x)y' + a_2y = 0$ are continuous and for $\forall x$, $a_0(x) \neq 0$. If the functions f_1 and f_2 are linearly indepadent solution of the D.E and if A_1, A_2, B_1 and B_2 are constats which provide $A_1B_2 - A_2B_1 \neq 0$, then show that $A_1f_1 + A_2f_2$ and $B_1f_1 + B_2f_2$ are linearly indepadent solutions of the D.E for $a \leq x \leq b$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} \neq 0 \Rightarrow f_1f_2' - f_2f_1' \neq 0$$

$$W = \begin{vmatrix} A_1f_1 + A_2f_2 & B_1f_1 + B_2f_2 \\ A_1f_1' + A_2f_2' & B_1f_1' + B_2f_2' \end{vmatrix} = \cancel{A_1B_1f_1f_1'} + A_1B_2f_1f_2' + A_2B_1f_2f_1' + \cancel{A_2B_2f_2f_2'}$$
$$- \cancel{A_1B_1f_1f_1'} - A_1B_2f_1'f_2 - A_2B_1f_1f_2' - \cancel{A_2B_2f_2f_2'}$$

$$\Rightarrow (A_1B_2 - A_2B_1)f_1f_2' + (A_2B_1 - A_1B_2)f_1'f_2$$
$$= \underbrace{(A_1B_2 - A_2B_1)}_{\neq 0}\underbrace{(f_1f_2' - f_1'f_2)}_{\neq 0}$$

$$\Rightarrow W \neq 0$$

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8. $\left(\frac{1}{x-y}+\frac{x}{x^2+y^2}\right)dx+\left(\frac{1}{y-x}+\frac{y}{x^2+y^2}\right)dy=0$

$$\left.\begin{array}{l} y=ux \\ dy=u\,dx+x\,du \end{array}\right\}\left(\frac{1}{x-ux}+\frac{x}{x^2+u^2x^2}\right)dx+\left(\frac{1}{ux-x}+\frac{ux}{x^2+u^2x^2}\right)(u\,dx+x\,du)=0$$

$$\Bigg[\underbrace{\frac{1}{x(1-u)}+\frac{u}{x(u-1)}}_{\frac{1}{x}}+\underbrace{\frac{1}{x(1+u^2)}+\frac{u^2}{1+u^2}}_{\frac{1}{x}}\Bigg]dx+\left(\frac{1}{u-1}+\frac{u}{1+u^2}\right)du=0$$

$$\int\frac{2}{x}\,dx+\int\left(\frac{1}{u-1}+\frac{u}{1+u^2}\right)du=\int 0$$

$$2\ln x+\ln(u-1)+\frac{1}{2}\ln(1+u^2)=\ln c$$

$$x^2(u-1)\sqrt{1+u^2}=c \;\longrightarrow\; x^2\left(\frac{y}{x}-1\right)\sqrt{1+\frac{y^2}{x^2}}=c$$

9. $y=2xy'+(y')^3$

$$y'=p \;\rightarrow\; y=2xp+p^3$$

$$\underbrace{y'}_{p}=2p+2xp'+3p^2p' \;\Rightarrow\; -p=\overset{\frac{dp}{dx}}{p'}\left[2x+3p^2\right]$$

$$\frac{dx}{dp}=\frac{2x+3p^2}{-p} \;\Rightarrow\; \frac{dx}{dp}+\frac{2x}{p}=-3p$$

$$\lambda(p)=e^{\int\frac{2}{p}dp}=e^{2\ln p}=p^2$$

$$x=\frac{1}{p^2}\left[\int p^2(-3p)\,dp+c\right]=\frac{1}{p^2}\left(-3\frac{p^4}{4}+c\right)=-\frac{3p^2}{4}+\frac{c}{p^2}$$

$$y=-\frac{3}{2}p^3+\frac{2c}{p}+p^3=-\frac{1}{2}p^3+\frac{2c}{p}$$

10. $\left(x - y\cos\left(\frac{y}{x}\right)\right)dx + x\cos\left(\frac{y}{x}\right)dy = 0 \qquad f(e) = e\frac{\pi}{2}$

$$y = ux \qquad \left(x - ux\cos u\right)dx + x\cos u\,(u\,dx + x\,du) = 0$$

$$dy = u\,dx + x\,du \qquad (x - \cancel{ux\cos u} + \cancel{ux\cos u})\,dx + x^2\cos u\,du = 0$$

$$\int \frac{dx}{x} + \int \cos u\,du = \int 0 \Rightarrow \ln x + \sin u = c$$

$$\Rightarrow \ln x + \sin\left(\frac{y}{x}\right) = c \qquad \Rightarrow f(e) = e.\frac{\pi}{2}$$

$$\ln e + \sin\left(\frac{\pi}{2}\right) = c \rightarrow c = 2 \qquad \Rightarrow \ln x + \sin\left(\frac{y}{x}\right) = 2$$

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12. $y' + y^2 = \frac{y}{x} - \frac{1}{x^2}$, $y_1(x) = \frac{1}{x}$

$$y = \frac{1}{x} + \frac{1}{u}$$
$$y' = -\frac{1}{x^2} - \frac{u'}{u^2}$$

$$\cancel{-\frac{1}{x^2}} - \frac{u'}{u^2} + \cancel{\frac{1}{x^2}} + \frac{1}{u^2} + \frac{2}{ux} = \cancel{\frac{1}{x^2}} + \frac{1}{ux} - \cancel{\frac{1}{x^2}}$$

$$-\frac{u'}{u^2} + \frac{1}{u^2} + \frac{1}{ux} = 0 \Rightarrow u' - 1 - \frac{u}{x} = 0 \Rightarrow u' - \frac{u}{x} = 1$$

$$\lambda(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$u = x\left[\int \frac{1}{x} dx + c\right] = x(\ln x + c) \Rightarrow y = \frac{1}{x} + \frac{1}{x(\ln x + c)}$$

13. $y = 2xy' + (y')^2$

dp/dx

$$y' = p \Rightarrow y = 2xp + p^2$$
$$\underbrace{y'}_{p} = 2p + 2xp' + 2pp'$$

$$-p = 2p'(x+p)$$

$$\frac{dx}{dp} = \frac{2(x+p)}{-p} = -\frac{2x}{p} - 2$$

$$\frac{dx}{dp} + \frac{2x}{p} = -2$$

$$\lambda(p) = e^{\int \frac{2dp}{p}} = e^{2\ln p} = p^2$$

$$x = \frac{1}{p^2}\left[\int p^2(-2)dp + c\right] = \frac{1}{p^2}\left(-\frac{2p^3}{3} + c\right) = \frac{-2p}{3} + \frac{c}{p^2}$$

$$y = -\frac{4p^2}{3} + \frac{2c}{p} + p^2 = \frac{2c}{p} - \frac{p^2}{3}$$

14. $(x - 2y) = (x - 2y + 1)y' \Rightarrow (x - 2y)dx = (x - 2y + 1)dy$

$$x - 2y = u$$
$$dx - 2dy = du$$
$$dy = \frac{dx - du}{2}$$

$$u\,dx = (u+1)\left(\frac{dx - du}{2}\right)$$

$$\left(u - \frac{u}{2} - \frac{1}{2}\right)dx + \frac{(u+1)}{2}du = 0$$

$$\frac{1}{2}(u-1)dx + \frac{(u+1)}{2}du = 0 \Rightarrow \int dx + \int \frac{u+1}{u-1}du = \int 0$$

$$x + \int\left(1 + \frac{2}{u-1}\right)du = c \Rightarrow x + u + 2\ln(u-1) = c \Rightarrow 2x - 2y + 2\ln(x - 2y - 1) = c$$

15. $y' = \frac{y^2}{x^2} - \frac{y}{x} + 1 \qquad y_1 = x$

$$\left.\begin{array}{l} y = x + \frac{1}{u} \\ y' = 1 - \frac{u'}{u^2} \end{array}\right\} \begin{array}{l} \cancel{1} - \frac{u'}{u^2} = \cancel{1} + \frac{2}{ux} + \frac{1}{u^2x^2} \cancel{-1} - \frac{1}{ux} \cancel{+1} \\ -\frac{u'}{u^2} = \frac{1}{ux} + \frac{1}{u^2x^2} \quad \Rightarrow \quad u' + \frac{u}{x} = -\frac{1}{x^2} \end{array}$$

$$\lambda(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$u = \frac{1}{x}\left[\int x\left(-\frac{1}{x^2}\right)dx + c\right] = \frac{1}{x}\left(-\ln x + c\right)$$

$$y = x + \frac{x}{c - \ln x}$$

16. $\underbrace{(xy^2+y)}_{m}dx - \underbrace{(x+y)}_{N}dy = 0$

$$\frac{\partial m}{\partial y} = 2xy + 1 \neq \frac{\partial N}{\partial x} = -1$$

$$\ln\lambda = \int \frac{2xy+1+1}{-(x+y)}dx \neq \lambda(x)$$

$$\ln\lambda = \int \frac{-1-2xy-1}{y(xy+1)}dy = \int \frac{-2dy}{y} = -2\ln y \Rightarrow \lambda = \frac{1}{y^2}$$

$$\left(x + \frac{1}{y}\right)dx - \left(\frac{x}{y^2} + \frac{1}{y}\right)dy = 0$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = x + \frac{1}{y} \\ \frac{\partial f}{\partial y} = -\frac{x}{y^2} - \frac{1}{y} \end{array}\right\} \begin{array}{l} f(x,y) = \frac{x^2}{2} + \frac{x}{y} + h(y) \\ \frac{\partial f}{\partial y} = \cancel{-\frac{x}{y^2}} + \frac{dh}{dy} = \cancel{-\frac{x}{y^2}} - \frac{1}{y} \end{array}$$

$$\int dh = -\int \frac{1}{y}dy \Rightarrow h = -\ln y + c$$

$$f(x,y) = \frac{x^2}{2} + \frac{x}{y} - \ln y + c = k$$

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