Probability

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Sets

A set is a collection of things that are called the elements of the set.

The elements can be any kind of entity:
numbers, people, poems, blueberries, points, lines, and so on, endlessly.

A particular set S is well defined if it is possible to tell whether any given point belongs to it or not.

Upper case letters are always used to denote sets.

If the set S includes some element denoted by x, then we say x belongs to S and write x ∈ S.

$\boxed{x \in S}$

If x does not belong to S, then we write

$x \notin S.$

$x \notin S$

$\omega \in S; \quad \omega \notin S.$

There are essentially two ways of defining a set,
* by a list
* by a rule.

die =zar

Example:
If S is the set of numbers shown by a conventional die, then the rule is that S comprises the integers between 1 and 6 inclusive.

This may be written formally as follows:
S = {x : 1 ≤ x ≤ 6 and x is an integer}.

$S = \{x : 1 \leq x \leq 6$ and x is an integer$\}$

Alternatively, S may be given as a list:
S = {1, 2, 3, 4, 5, 6}.

$S = \{1, 2, 3, 4, 5, 6\}$

Let us then fix a sample space to be denoted by Ω, the capital Greek letter omega.

Sample space:

Example

Two coins are tossed.
Represent the sample space for this experiment by making a list, a table, and a tree diagram

H : Head T : Tail

List : HH, HT, TH, TT

Table

It may contain any number of points, possibly infinite but at least one.

In extreme cases a set may be Ω itself or the empty set, which has no point in it.

You may be surprised to hear that the empty set is an important entity and is given a special symbol Ø.

	H	T

The number of points in a set S will be called its symbol ∅.

The number of points in a set S will be called its size and denoted by |S|;

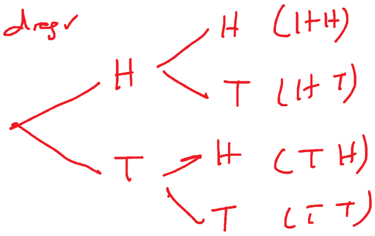
thus it is a nonnegative integer or ∞.

In particular |∅| = 0 .

Table

	H	T
H	HH	HT
T	TH	TT

Tree dagr



One important special case arises when the rule is impossible;
for example,
consider the set of elephants playing football on Mars.

This is impossible (there is no pitch on Mars) and the set therefore is empty;
we denote the empty set by ϕ.
We may write ϕ as { }.

Operations with sets

We learn about sets by operating on them,
just as we learn about numbers by operating on them.

In the latter case we also say that we compute with numbers: add, subtract, multiply, and so on.

These operations performed on given numbers produce other numbers, which are called their sum, difference, product, etc.

In the same way, operations performed on sets produce other sets with new names.

We are now going to discuss some of these and the laws governing them.

Included

If every point of A belongs to B then A is *contained* or *included* in B and is a *subset* of B, while B is a *superset* of A. We write this in one of the two ways below:

$$A \subset B, \quad B \supset A.$$

Two sets are *identical* if they contain exactly the same points, and then we write

$$A = B.$$

Combining Sets

Given any nonempty set, we can divide it up, and given any two sets, we can join them together.

These simple observations are important enough to warrant definitions and notation.

Definition:
Let A and B be sets.

Their union, denoted by **A ∪ B**, is the set of elements that are in A or B, or in both.

Their intersection, denoted by A ∩ B, is the set of elements in both A and B.

Definition:
If A ∩ B = ϕ,
then A and B are said to be disjoint.

We can also remove bits of sets, giving rise to set differences, as follows.

Definition:
Let A and B be sets.

That part of A that is not also in B is denoted by **A\B**, called the difference of A from B.

Finally, we can combine sets in a more complicated way by taking elements in pairs, one from each set.

Definition Let A and B be sets, and let
C = {(a, b) : a ∈ A, b ∈ B}
be the set of ordered pairs of elements from A and B.

Then C is called the product of A and B and denoted by
A × B.

Venn Diagrams

The above ideas are attractively and simply expressed in terms of Venn diagrams.

These provide very expressive pictures, which are often so clear that they make algebra redundant

In probability problems, all sets of interest A lie in a universal set Ω, so that $A \subset \Omega$ for all A. That part of Ω that is not in A is called the *complement* of A, denoted by A^c.

Complement. The complement of a set A is denoted by A^c and is the set of points that do not belong to A. Remember we are talking only about points in a fixed Ω! We write this symbolically as follows:

$A^c = \{w \mid w \notin A\}$

Complement. The complement of a set A is denoted by A^c and is the set of points that do not belong to A. Remember we are talking only about points in a fixed Ω! We write this symbolically as follows:

$$\boxed{A^c = \{\omega \mid \omega \notin A\},}$$

$A^c = \{w \mid w \notin A\},$

which reads: "A^c is the set of ω that does not belong to A." In particular $\Omega^c = \emptyset$ and $\emptyset^c = \Omega$. The operation has the property that if it is performed twice in succession on A, we get A back:

$$(A^c)^c = A. \tag{1.2.1}$$

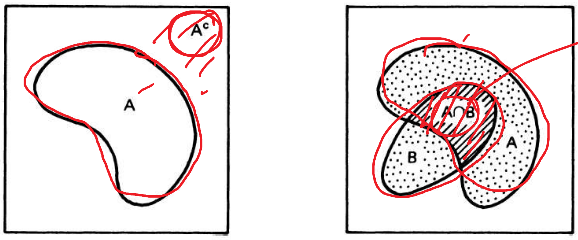
Union. The union $A \cup B$ of two sets A and B is the set of points that belong to at least one of them. In symbols:

$$\boxed{A \cup B = \{\omega \mid \omega \in A \text{ or } \omega \in B\}}$$

where "or" means "and/or" in pedantic [legal] style and will always be used in this sense.

Intersection. The intersection $A \cap B$ of two sets A and B is the set of points that belong to both of them. In symbols:

$$A \cap B = \{\omega \mid \omega \in A \text{ and } \omega \in B\}.$$



$A \cap B$

Figure 1

Commutative Law. $\boxed{A \cup B = B \cup A,}$ $\boxed{A \cap B = B \cap A.}$

Associative Law. $(A \cup B) \cup C = A \cup (B \cup C),$
$(A \cap B) \cap C = A \cap (B \cap C).$

But observe that these relations are instances of identity of sets mentioned above, and are subject to proof. They should be compared, but not confused, with analogous laws for sum and product of numbers:

$$a + b = b + a, \quad a \times b = b \times a$$
$$(a + b) + c = a + (b + c), \quad (a \times b) \times c = a \times (b \times c).$$

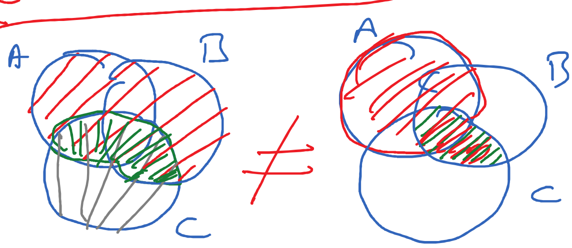
Brackets are needed to indicate the order in which the operations are to be performed. Because of the associative laws, however, we can write

$$A \cup B \cup C, \quad A \cap B \cap C \cap D$$

without brackets. But a string of symbols like $A \cup B \cap C$ is ambiguous, therefore not defined; indeed $(A \cup B) \cap C$ is not identical with $A \cup (B \cap C)$. You should be able to settle this easily by a picture.

$\boxed{A \cup B \cap C}$

$\boxed{(A \cup B) \cap C \neq A \cup (B \cap C)}$



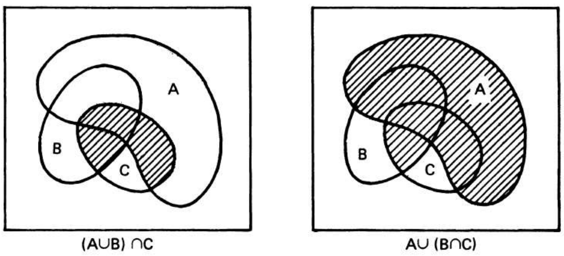
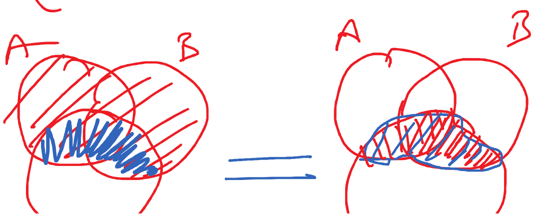


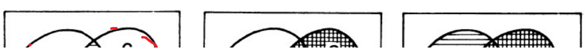
Figure 2

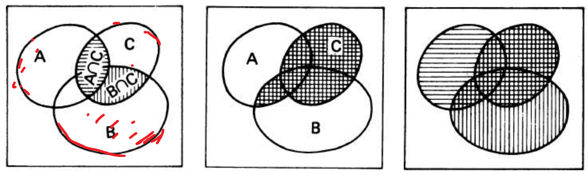
$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

The next pair of *distributive laws* connects the two operations as follows:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C); \tag{D$_1$}$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C). \tag{D$_2$}$$

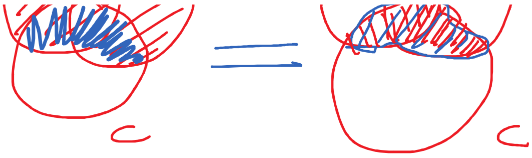






(A∪B)∩C = (A∩C)∪(B∩C)   (A∩B)∪C = (A∪C)∩(B∪C)

Figure 3



$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C); \qquad (D_1)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C). \qquad (D_2)$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C); \qquad (D_1)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C). \qquad (D_2)$$

you have probably already discovered the use of diagrams to prove or disprove assertions about sets.

It is also a good practice to see the truth of such formulas as (D1) and (D2) by well-chosen examples.

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C); \qquad (D_1)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C). \qquad (D_2)$$

Suppose then that
A = inexpensive things,
B = really good things,
C = food [edible things].
Then (A∪B)∩C means
"(inexpensive or really good) food,"

$(A \cup B) \cap C =$

while (A∩C)∪ (B ∩ C) means
"(inexpensive food) or (really good food)."

$\underbrace{(A \cap C)}_{i.F} \cup \underbrace{(B \cap C)}_{R.G.F}$

So they are the same thing all right.
This does not amount to a proof, as one swallow does not make a summer, but if one is convinced that whatever logical structure or thinking process involved above in no way depends on the precise nature of the three things A, B, and C, so much so that they can be anything, then one has in fact landed a general proof.

Now it is interesting that the same example applied to (D2) somehow does not make it equally obvious (at least to the author).

Why? Perhaps because some patterns of logic are in more common use in our everyday experience than others.
This last remark becomes more significant if one notices an obvious duality between the two distributive laws.
Each can be obtained from the other by switching the two symbols ∪ and ∩.

Indeed each can be deduced from the other by making use of this duality (Exercise 11).

Finally, since (D2) comes less naturally to the intuitive mind, we will avail ourselves of this opportunity to demonstrate the roundabout method of identifying sets mentioned above by giving a rigorous proof of the formula.

According to this method, we must show: (i) each point on the left side of (D2) belongs to the right side; (ii) each point on the right side of (D2) belongs to the left side.
(i)

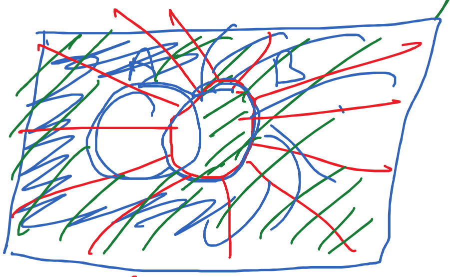
1.3. Various relations

The three operations so far defined: complement, union, and intersection obey two more laws called *De Morgan's laws:*

$$(A \cup B)^c = A^c \cap B^c; \qquad (C_1)$$
$$(A \cap B)^c = A^c \cup B^c. \qquad (C_2)$$

A : inexpensive
B : Really good

$$\boxed{(A \cup B)^c = A^c \cap B^c}$$



Same

They are dual in the same sense as (D1) and (D2) are. Let us check these by our previous example.
If A = inexpensive, and
B = really good,
then clearly (A ∪ B)c = not inexpensive nor really good, namely high-priced junk, which is the same as $A^c \cap B^c$ = inexpensive and not really good.

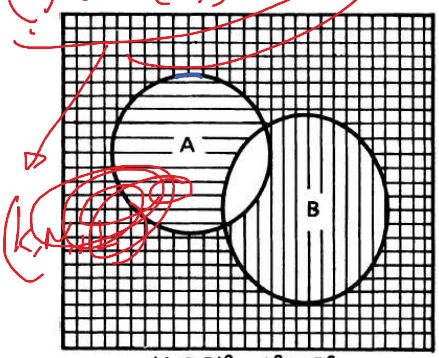
Similarly we can check (C2).
Logically, we can deduce either (C1) or (C2) from the other; let us show it one way.

Suppose then (C1) is true, then since A and B are arbitrary sets we can substitute their complements and get

$(K,N,L)^c = M$

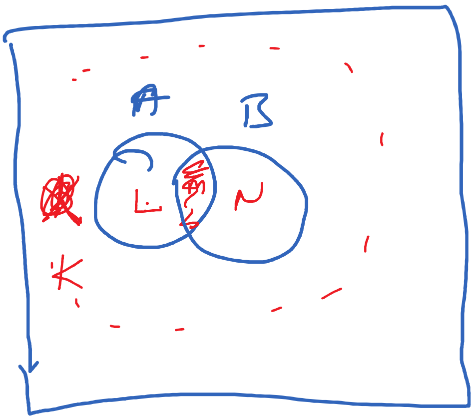
$$(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c = A \cap B$$

(K,N) (K,L)



$(A \cap B)^c = A^c \cup B^c$

Figure 4



where we have also used (1.2.1) for the second equation. Now taking the complements of the first and third sets in (1.3.1) and using (1.2.1) again we get

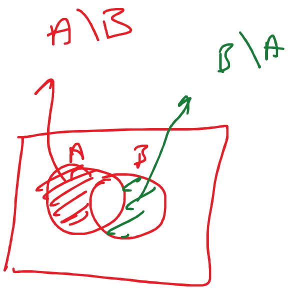
$$A^c \cup B^c = (A \cap B)^c.$$

This is (C_2). Q.E.D.

It follows from De Morgan's laws that if we have complementation, then either union or intersection can be expressed in terms of the other. Thus we have

$$A \cap B = (A^c \cup B^c)^c,$$
$$A \cup B = (A^c \cap B^c)^c;$$

and so there is redundancy among the three operations. On the other hand,



Difference. The set $A \setminus B$ is the set of points that belong to A and (but) not to B. In symbols:

$$A \setminus B = A \cap B^c = \{\omega \mid \omega \in A \text{ and } \omega \notin B\}.$$

This operation is neither commutative nor associative. Let us find a *counterexample* to the associative law, namely, to find some A, B, C for which

$$(A \setminus B) \setminus C \neq A \setminus (B \setminus C). \tag{1.3.2}$$

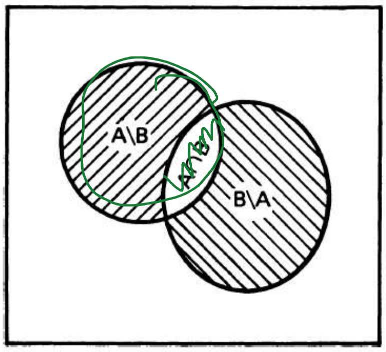


Figure 5

$A \setminus B = A - (A \cap B)$

Note that in contrast to a proof of identity discussed above, a single instance of falsehood will destroy the identity. In looking for a counterexample one usually begins by specializing the situation to reduce the "unknowns." So try $B = C$. The left side of (1.3.2) becomes $A \setminus B$, while the right side becomes $A \setminus \emptyset = A$. Thus we need only make $A \setminus B \neq A$, and that is easy.

In case $A \supset B$ we write $A - B$ for $A \setminus B$. Using this new symbol we have

$$A \setminus B = A - (A \cap B)$$

and

$$A^c = \Omega - A.$$

The operation "$-$" has some resemblance to the arithmetic operation of subtracting, in particular $A - A = \emptyset$, but the analogy does not go very far. For instance, there is no analogue to $(a + b) - c = a + (b - c)$.