START : 09 $\underline{\underline{15}}$

$$P(x) = \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} y_1 + \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} y_2 + \cdots + \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} y_n.$$

$$g(x) = f_o\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)} + f_1\frac{(x-x_o)(x-x_2)(x-x_3)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)} + f_2\frac{(x-x_o)(x-x_1)(x-x_3)}{(x_2-x_o)(x_2-x_1)(x_2-x_3)} + f_3\frac{(x-x_o)(x-x_1)(x-x_2)}{(x_3-x_o)(x_3-x_1)(x_3-x_2)}$$

***Example***

- Consider the following table of functional values (generated with $f(x) = \ln x$ )

| $i$ | $x_i$ | $f_i$ |
|---|---|---|
| 0 | 0.40 | -0.916291 |
| 1 | 0.50 | -0.693147 |
| 2 | 0.70 | -0.356675 |
| 3 | 0.80 | -0.223144 |

- Find $g(0.60)$ as:

$$g(x) = f_o\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)} + f_1\frac{(x-x_o)(x-x_2)(x-x_3)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)} + f_2\frac{(x-x_o)(x-x_1)(x-x_3)}{(x_2-x_o)(x_2-x_1)(x_2-x_3)} + f_3\frac{(x-x_o)(x-x_1)(x-x_2)}{(x_3-x_o)(x_3-x_1)(x_3-x_2)}$$

$$g(0.60) = -0.916291 \cdot \frac{(0.60-0.50)(0.60-0.70)(0.60-0.80)}{(0.40-0.50)(0.40-0.70)(0.40-0.80)}$$

$$-0.693147 \cdot \frac{(0.60-0.40)(0.60-0.70)(0.60-0.80)}{(0.50-0.40)(0.50-0.70)(0.50-0.80)}$$

$$-0.356675 \cdot \frac{(0.60-0.40)(0.60-0.50)(0.60-0.80)}{(0.70-0.40)(0.70-0.50)(0.70-0.80)}$$

$$-0.223144 \cdot \frac{(0.60-0.40)(0.60-0.50)(0.60-0.70)}{(0.80-0.40)(0.80-0.50)(0.80-0.70)}$$

$$\Rightarrow$$

$$g(0.60) = -0.509976$$

Example: By the knowledge of the points $(x, y) : (0,0), (2,4), (4,16)$ the Polynomial Lagrangian Interpolation method allows to find back the équation $y = x^2$. Calculations details step by step :

$$\begin{aligned} P(x) &= 0 \times \frac{(x-2)}{(0-2)}\frac{(x-4)}{(0-4)} + 4 \times \frac{(x-0)}{(2-0)}\frac{(x-4)}{(2-4)} + 16 \times \frac{(x-0)}{(4-0)}\frac{(x-2)}{(4-2)} \\ &= 4 \times \frac{x}{2}\frac{(x-4)}{(-2)} + 16 \times \frac{x}{4}\frac{(x-2)}{2} \\ &= -x(x-4) + 2x(x-2) \\ &= -x^2 + 4x + 2x^2 - 4x \\ &= x^2 \end{aligned}$$

Once deducted, the interpolating function $f(x) = x^2$ allows to estimate the value for $x = 3$, here $f(x) = 9$.

The **Lagrange interpolation** method allows a good approximation of polynomial functions.

## Lagrange Interpolation Formula

Lagrange polynomials are used for polynomial interpolation. For a given set of distinct points $x_j$ and numbers $y_j$. Lagrange's interpolation is also an $N^{th}$ degree polynomial approximation to f(x).

Find the **Lagrange Interpolation Formula** given below,

$$y = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)} y_1 + .. + \frac{(x-x_1)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_0-x_1).....(x_n-x_{n-1})} y_n$$

**Question:** Find the value of y at x = 0 given some set of values (-2, 5), (1, 7), (3, 11), (7, 34)?

**Solution:**

Given the known values are,

x = 0 ; $x_0$ = -2 ; $x_1$ = 1 ; $x_2$= 3 ; $x_3$ = 7 ; $y_0$ = 5 ; $y_1$ = 7 ; $y_2$ = 11 ; $y_3$ = 34

Using the interpolation formula,

$$y = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)} y_1 + .... + \frac{(x-x_1)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_0-x_1).....(x_n-x_{n-1})} y_n$$

$$y = \frac{(0-1)(0-3)(0-7)}{(-2-1)(-2-3)(-2-7)} \times 5 + \frac{(0+2)(0-3)(0-7)}{(1+2)(1-3)(1-7)} \times 7 + \frac{(0+2)(0-1)(0-7)}{(3+2)(3-1)(3-7)} \times 11 + \frac{(0+2)(0-1)(0-3)}{(7+2)(7-1)(7-3)} \times 34$$

$$y = \frac{21}{27} + \frac{49}{6} + \frac{-77}{20} + \frac{51}{54}$$

$$y = \frac{1087}{180}$$

Consider the same points as before, namely (1 , 1), (2 , 2) , (3 , 2). The three Lagrange polynomials are

$$L_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x^2 - 5x + 6),$$

$$L_2(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -x^2 + 4x - 3,$$

$$L_3(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x^2 - 3x + 2).$$

Equation 1.11.2 for the polynomial of degree $n-1$ that goes through the three points is, then,

$$y = 1 \times \frac{1}{2}(x^2 - 5x + 6) + 2 \times (-x^2 + 4x - 3) + 2 \times \frac{1}{2}(x^2 - 3x + 2);$$

that is

$$y = -\frac{1}{2}x^2 + \frac{5}{2}x - 1,$$

which agrees with what we got before.

Let us recall the example that we had in Section 1.10 on Besselian interpolation, in which we were asked to estimate the value of $\sin 51^\circ$ from the table

| $x^\circ$ | $\sin x$ |
|---|---|
| 0 | 0.0 |
| 30 | 0.5 |
| 60 | $\sqrt{3}/2 = 0.86603$ |
| 90 | 1.0 |

The four Lagrangian polynomials, evaluated at $x = 51$, are

$$L_1(51) = \frac{(51-30)(51-60)(51-90)}{(0-30)(0-60)(0-90)} = -0.0455,$$

$$L_2(51) = \frac{(51-0)(51-60)(51-90)}{(30-0)(30-60)(30-90)} = +0.3315,$$

$$L_3(51) = \frac{(51-0)(51-30)(51-90)}{(60-0)(60-30)(60-90)} = +0.7735,$$

$$L_4(51) = \frac{(51-0)(51-30)(51-60)}{(90-0)(90-30)(90-60)} = -0.0595.$$

Equation 1.11.2 then gives us

$$\sin 51^\circ = 0 \times (-0.0455) + 0.5 \times 0.3315 + 0.86603 \times 0.7735 + 1 \times (-0.0595)$$
$$= 0.776.$$

**Question:** Find the value of y at x = 0 given some set of values (-2, 5), (1, 7), (3, 11), (7, 34)?

**Solution:**

Given the known values are,

x = 0 ; $x_0$ = -2 ; $x_1$ = 1 ; $x_2$= 3 ; $x_3$ = 7 ; $y_0$ = 5 ; $y_1$ = 7 ; $y_2$ = 11 ; $y_3$ = 34

Using the interpolation formula,

$$y = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2).....(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2).....(x_1-x_n)} y_1 + .... + \frac{(x-x_1)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_0-x_1).....(x_n-x_{n-1})} y_n$$

$$y = \frac{(0-1)(0-3)(0-7)}{(-2-1)(-2-3)(-2-7)} \times 5 + \frac{(0+2)(0-3)(0-7)}{(1+2)(1-3)(1-7)} \times 7 + \frac{(0+2)(0-1)(0-7)}{(3+2)(3-1)(3-7)} \times 11 + \frac{(0+2)(0-1)(0-3)}{(7+2)(7-1)(7-3)} \times 34$$

$$y = \frac{21}{27} + \frac{49}{6} + \frac{-77}{20} + \frac{51}{54}$$

$$y = \frac{1087}{180}$$

**Example** → Compute the quadratic interpolating polynomial to $f(x) = \cos(x)$ with nodes $\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\}$ using Lagrange polynomials.

The Lagrange polynomials of degree 2 for these nodes are

$$\ell_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-\frac{\pi}{4})}{\frac{\pi}{4}\cdot\frac{\pi}{2}},$$
$$\ell_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+\frac{\pi}{4})(x-\frac{\pi}{4})}{-\frac{\pi}{4}\cdot\frac{\pi}{4}},$$
$$\ell_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{x(x+\frac{\pi}{4})}{\frac{\pi}{2}\cdot\frac{\pi}{4}}.$$

So the interpolating polynomial is

$$\begin{aligned} p_2(x) &= f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x) \\ &= \frac{1}{\sqrt{2}}\frac{8}{\pi^2}x(x-\frac{\pi}{4}) - \frac{16}{\pi^2}(x+\frac{\pi}{4})(x-\frac{\pi}{4}) + \frac{1}{\sqrt{2}}\frac{8}{\pi^2}x(x+\frac{\pi}{4}) = \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}}-1\right)x^2 + 1. \end{aligned}$$

The Lagrange polynomials and the resulting interpolant are shown below:

# Lagrange Interpolation

Lagrange polynomials are used for polynomial interpolation.

For a given set of distinct points $x_j$ and numbers $y_j$,
Lagrange's interpolation is also an $N^{th}$ degree polinomial approximation to $f(x)$,

$$f(x) = \boxed{\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\cdot f(x_0)} + \boxed{\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\, f(x_1)} + \boxed{\frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)}\, f(x_2)} + \cdots$$

## 1. Degree Langrange Polynomal
(used if 2 points are know)

$\boxed{x_0} \to f(x_0)$
$\boxed{x_1} \to f(x_1)$

$$f(x) = \underbrace{f(x_0)\cdot\frac{x-x_1}{x_0-x_1}}_{1.\ term} + \underbrace{f(x_1)\cdot\frac{x-x_0}{x_1-x_0}}_{2.\ term}$$

## 2. Degree Lagrange Polynomall
(used if 3 points are know)

$(x_0, f(x_0))$
$(x_1, f(x_1))$
$(x_2, f(x_2))$

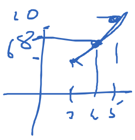
$$f(x) = \boxed{f(x_0)\cdot\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}} + \boxed{f(x_1)\cdot\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}} + \boxed{f(x_2)\cdot\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}$$

## 3. Degree Lagrange Polynomial
(used if 4 points are know)

$(x_0, f(x_0))$
$(x_1, f(x_1))$
$(x_2, f(x_2))$
$(x_3, f(x_3))$

$$\begin{aligned} f(x) = {} & f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \\ & f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + \\ & f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \\ & f(x_3)\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{aligned}$$

$(x_0, f(x_0))$
$(x_1, f(x_1))$
$(x_2, f(x_2))$
$(x_3, f(x_3))$




Ex By the knowledge of points
$(x, y) = (3, 6)\ (5, 10)$ — $x_0\ f(x_0)\quad x_1\ f(x_1)$
find equation using polynomal Lagrange Inter...
And find $(f_4) = ?$

$$f_1(x) = f(x_0)\,\frac{x-x_1}{(x_0-x_1)} + f(x_1)\cdot\frac{x-x_0}{x_1-x_0}$$

know: 3, 6 ($x$, $f$) and 5, 10 ($x_1$, $f(x_1)$)

$x_0 \quad f(x_0)$

$$f_1(x) = \cancel{6}^{(-3)}\frac{(x-5)}{\underbrace{3-5}_{-2}} + \cancel{10}^{5}\frac{(x-3)}{\underbrace{(5-3)}_{2}}$$

$$= -3x + \cancel{15} + 5x - \cancel{15}$$

$$\boxed{f_1(x) = 2x} \quad \text{equation}$$

$$f(4) = 2\cdot 4 = 8$$

Ex: (1,3) (2,6) (4,24)

find equation Lagrange

$(1,3)\ (2,6)\ (4,24)$

$x_0 \ f(x_0) \ x_1 \ f(x_1) \ x_2 \ f(x_2)$

2' Degree Lag. P-

$f(x) =$

1 term $\quad f(x_0)\cdot\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \cancel{3}\frac{(x-2)(x-4)}{\cancel{(1-2)(1-4)}} = x^2 - 6x + 8$

$+$

2 term $\quad f(x_1)\cdot\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \cancel{6}^{-3}\frac{(x-1)(x-4)}{\cancel{(2-1)(2-4)}}$

$-3(x^2-5x+4) = -3x^2+15x-12$

$+$

3 term $\quad f(x_2)\cdot\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \cancel{24}^{4}\frac{(x-1)(x-2)}{\underbrace{(4-1)}_{3}\underbrace{(4-2)}_{2}}$

$= 4(x^2-3x+2)$

$= 4x^2 - 12x + 8$

$$f(x) = x^2 - 6x + 8 + (-3x^2 + 15x - 12) + 4x^2 - 12x + 8$$

$$\boxed{f(x) = 2x^2 - 3x + 4}$$

(1,3)

(2,6)

(4,24)

$f(1) = 2 - 3 + 4 = 3$

$f(2) = 8 - 6 + 4 = 6$

$f(4) = 2\cdot 4^2 - 3\cdot 4 + 4 = 24$

$f(3) = 2\cdot 3^2 - 3\cdot 3 + 4 = 13$

$f(5) = 2\cdot 5^2 - 3\cdot 5 + 4 = 39$

$f(3)\,?$

$f(5) = ?$

3' degree Lagrange,

$(0,5)\ (1,1)\ (2,-1)\ (4,25)$

$x_0 \ f(x_0) \ x_1 \ f(x_1) \ x_2 \ f(x_2) \ x_3 \ f(x_3)$

$f(x) =$

1 term $\quad f(x_0)\cdot\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = 5\,\frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)} = \frac{5\cdot 2\cdot 1\cdot(-1)}{-8} = \frac{5}{4}$

$+$

2 term $\quad f(x_1)\cdot\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = 1\,\frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)} = \frac{1\cdot 3\cdot 1\cdot(-1)}{1\cdot(-1)\cdot(-3)} = -1$

$$\overline{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \qquad (1-0)(1-2)(1-4) \qquad 1.(-1).(-3)$$

$+$

3 ter

$$f(x_2)\cdot\frac{(x-x_0).(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = (-1)\frac{(\overset{3}{x}-0)(\overset{3}{x}-1)(\overset{3}{x}-4)}{(2-0)(2-1)(2-4)} = \frac{(-1).3.2.(-1)}{2\ 1.(-2)} = -\frac{3}{2}$$

$+$

4 ter

$$f(x_3)\cdot\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = 25\frac{(\overset{3}{x}-0)(\overset{3}{x}-1)(\overset{3}{x}-2)}{(4-0)(4-1)(4-2)} = \frac{25.3.2.1}{4.3.2} = \frac{25}{4}$$

$$f(3) = \frac{5}{4} - 1 - \frac{3}{2} + \frac{25}{4} = 5$$

---

| X | log x |
|---|---|
| 2 | 0.301 |
| 3 | 0.477 |
| 4 | 0.602 |
| 5 | ? |
| 6 | 0,778 |

$\log 5 = ?$

---

$f(x) =$

| x | f(x) |
|---|---|
| $x_0$ 2 | 0,301 |
| $x_1$ 3 | 0,477 |
| $x_2$ 4 | 0,602 |
| $x_3$ 6 | 0,778 |

1 ter

$$f(x_0)\cdot\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = 0,301\frac{(\overset{5}{x}-3)(\overset{5}{x}-4)(\overset{5}{x}-6)}{(2-3)(2-4)(2-6)} = 0,31.\frac{2.1.(-1)}{(-1)(-2)(-4)} = \frac{0,301}{4}$$

$+$

$$f(x_1)\cdot\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = 0,477\frac{(\overset{5}{x}-2)(\overset{5}{x}-4)(\overset{5}{x}-6)}{(3-2)(3-4)(3-6)} = -0,477$$

$+$

$$f(x_2)\cdot\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = 0,602\frac{(\overset{5}{x}-2)(\overset{5}{x}-3)(\overset{5}{x}-6)}{(4-2)(4-3)(4-6)} = 0,602.\frac{3}{2}$$

$+$

$$f(x_3)\cdot\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = 0,778\frac{(\overset{5}{x}-2)(\overset{5}{x}-3)(\overset{6}{x}-4)}{(6-2)(6-3)(6-4)} = \frac{0,778}{4}$$

$$f(5) = 0.07525 + -0,477 + 0.903 + 0.1545$$

$$= 0,69575$$

---

31. Newton Raphson
32 Secant methot
33. Jacobi Iteration
34: Gauss Seidel Iterati
35. Linear Interpolation mt
36. Quadratic interpoh mes
37. 1. Degree Lagrange Int
38. 2. " "
39. 3 " "

1. Question

ytubiomed@gmail.com

07.06.2021

PDF

Name
Surname
Numbe

Each page

19017067-UGUR-ER

4. Degree

$f(x)$

1. ter. $f(x_0)\dfrac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$

2ter $f(x_1)\dfrac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ \ldots}$

..

1. Degree $(x_0, f(x_0))$ $(x_1, f(x_1))$

$$f(x) = f(x_0)\cdot\frac{(x-x_1)}{(x_0-x_1)} + f(x_1)\frac{(x-x_0)}{(x_1-x_0)}$$