MAT1071 MATHEMATICS I
3. WEEK
PART 1

DIFFERENTIATION

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Rates of Change and Tangents to Curves

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y,

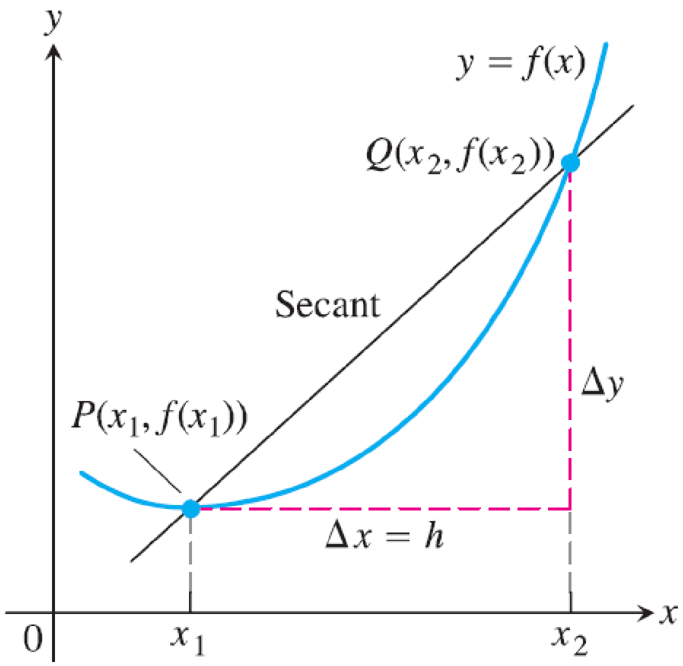

FIGURE A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

$$\Delta y = f(x_2) - f(x_1),$$

by the length $\Delta x = x_2 - x_1 = h$

of the interval over which the change occurs.

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DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

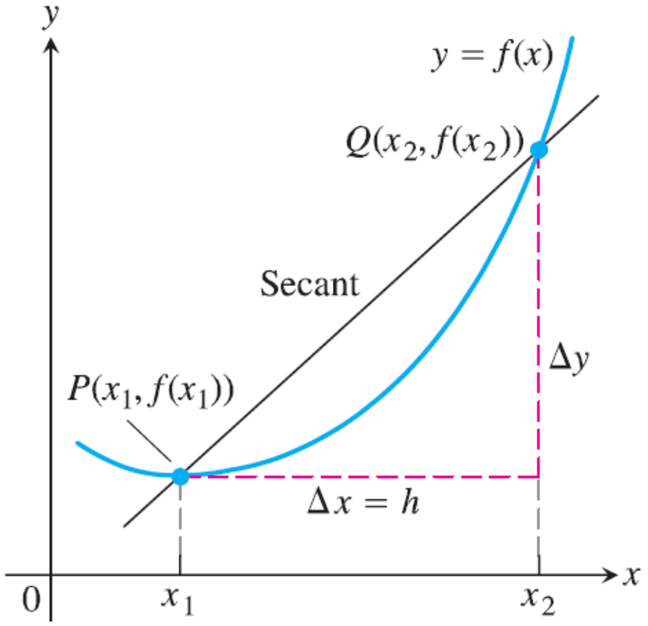

FIGURE A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$

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Let's consider what happens as the point Q approaches the point P along the curve, so the length $h \to 0$

$Q \to P \; , \; h \to 0$

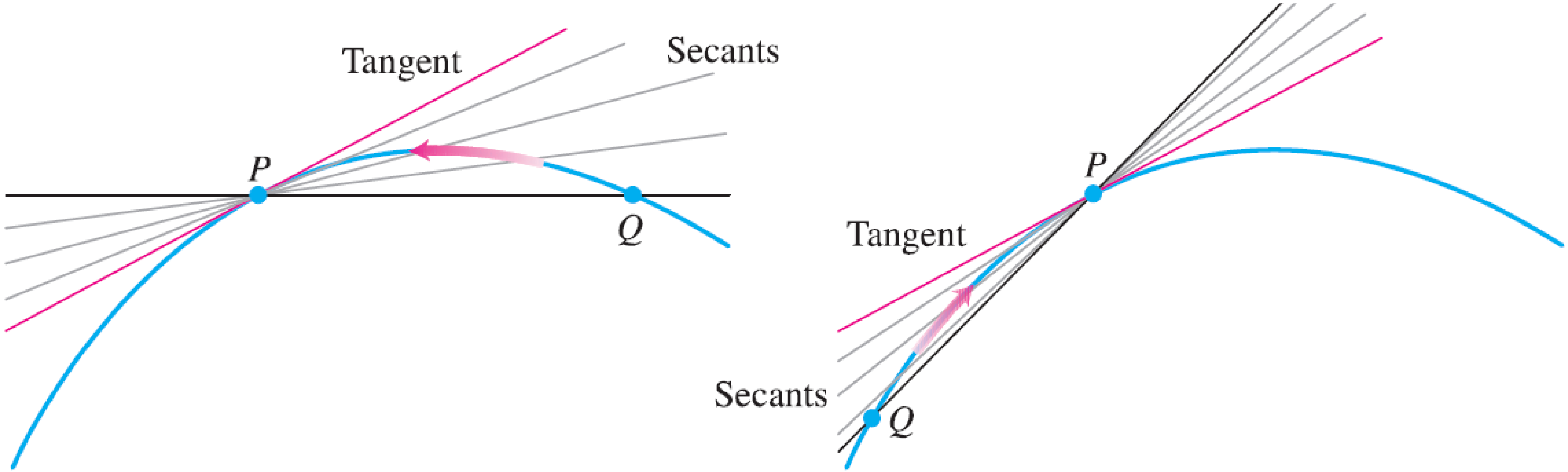

FIGURE The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

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Tangent to the Graph of a Function

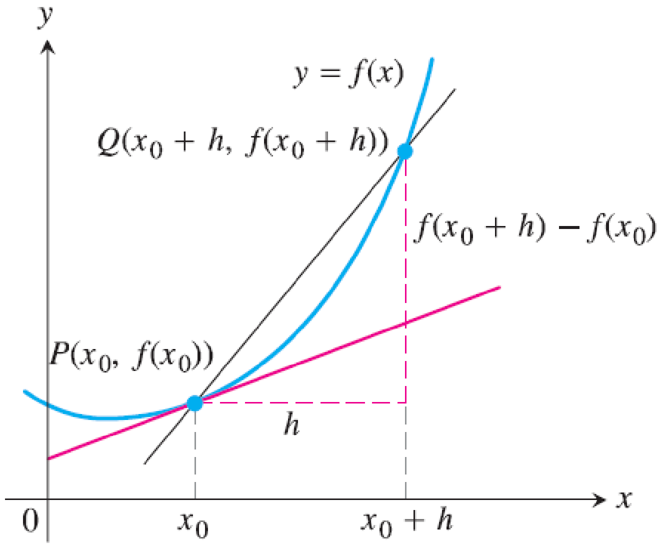
To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$,

We calculate the slope of the secant through P and a nearby point

$Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$

If the limit exists,

we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.



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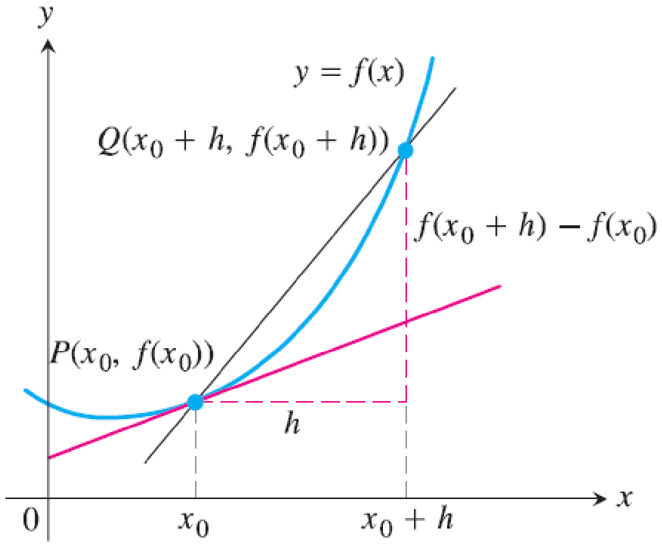

slope of secant $\frac{\Delta y}{\Delta x} = \frac{f(x_0+h)-f(x_0)}{h}$

when $Q \to P$ and $h \to 0$ we have

The slope of curve $y=f(x)$ at P

$$m_T = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

The tangent line to the curve at P is the line through P with this slope.

equation of tangent line $y - y_0 = m_T (x - x_0)$

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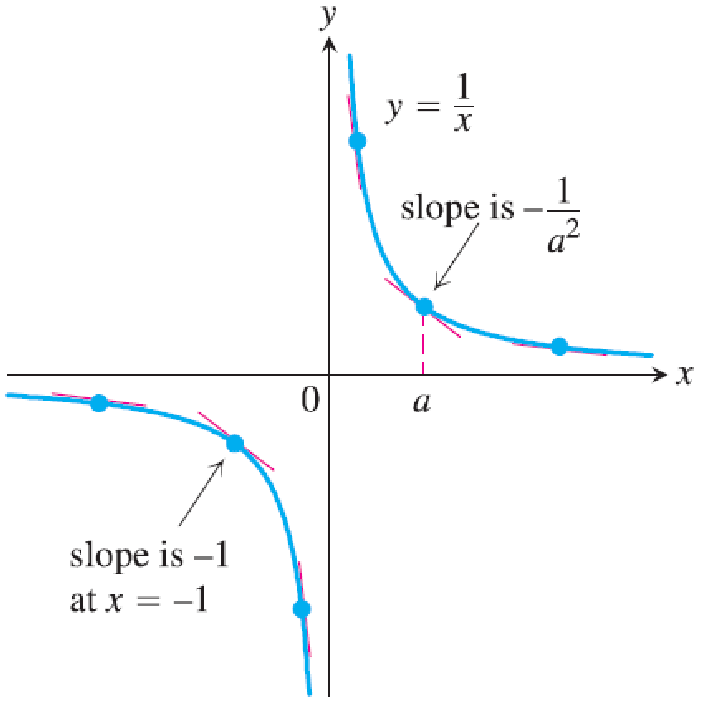

EXAMPLE

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-1/4$?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \frac{-h}{ha(a+h)} = \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$

EXAMPLE Find the slope of the graph of $f(x) = x^2+1$ at $(2,5)$
eq. of tang. line

Solution

$$\underset{x}{2},\ \underset{f(x)}{5}$$

$$m_t = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 4$$

$$\Rightarrow \boxed{y-5 = 4(x-2)}$$

eq of tangent

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at $x = x_0$

Slope of Tangent $\rightarrow m_T = \lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$

slope of normal $\rightarrow m_n = \dfrac{-1}{m_t}$

equation of tangent line $y - y_0 = m_T (x - x_0)$

equation of normal line $y - y_0 = m_n (x - x_0)$

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EXAMPLE Find the equation of normal at (1,1) of the $y=x^2$.

Solution

$$m_T = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2-1}{h} = 2$$

$m_n = -\frac{1}{2} \Rightarrow$ equation of normal $\boxed{y = -\frac{1}{2}(x-1)+1}$

$y = -\frac{x}{2} + \frac{3}{2}$

Derivative at a Point

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

provided this limit exists.

The derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$.

The derivative gives the function's instantaneous rate of change with respect to x at $x = x_0$.

$$\boxed{\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

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The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

The Derivative as a Function

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

If f' exists at a particular x, we say that f is **differentiable (has a derivative) at x**.

If f' exists at every point in the domain of f, we call f **differentiable**.

EXAMPLE

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{1}{2\sqrt{x}}$$

EXAMPLE Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{Definition} \\
&= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} && \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} && \text{Simplify} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. && \text{Cancel } h \neq 0
\end{aligned}$$

Notations $y = f(x)$ ↳ independent variable

↓ dependent variable

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

operation of differentiation.

The derivative of f with respect to x.

at $x = a$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}.$$

One-Sided Derivatives

Right hand derivative $f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$

Left hand derivative $f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$

$f'_+(a) = f'_-(a) = K$

$\Rightarrow f'(a)$ exists and equals K.

If

Right hand derivative = Left hand derivative

then the derivative of $f(x)$ exists at $x=a$

EXAMPLE Find the derivative of $f(x)=\begin{cases} x^2-2 & , x\leq 1 \\ 2x-3 & , x>1 \end{cases}$ at $x=1$

Solution

$$f'_+(1)=\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{2(1+h)-3+1}{h}=\lim_{h\to 0^+}\frac{2h}{h}=2$$

$$f'_-(1)=\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=2$$

$f'_+(1)=f'_-(1)\Rightarrow$ the derivative at $x=1$ exists at $f'(1)=2$.

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval.

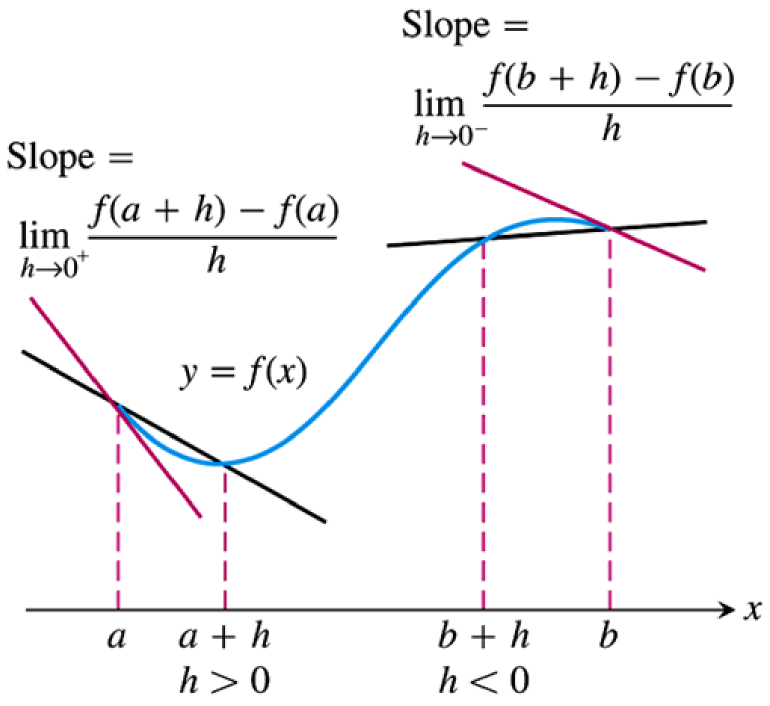

FIGURE Derivatives at endpoints are one-sided limits.

A function $y = f(x)$

☆ It is **differentiable on a closed interval** $[a, b]$

if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \quad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \quad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints

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Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

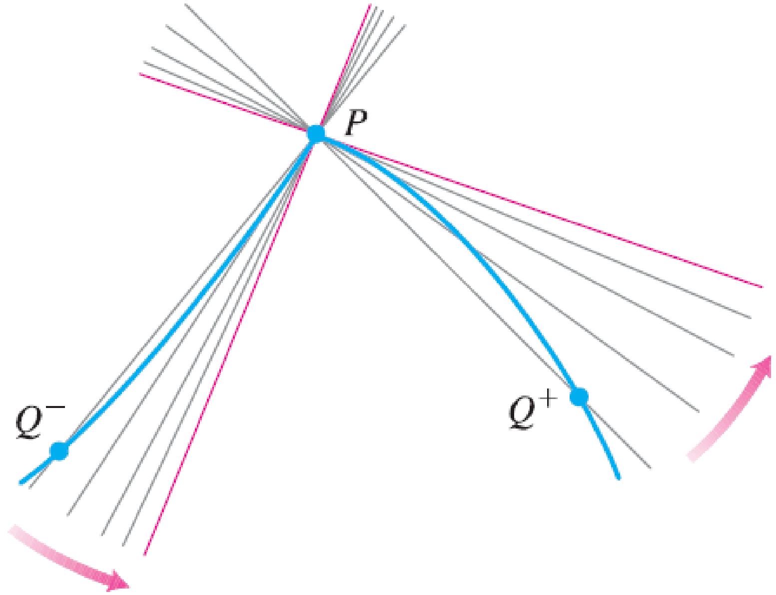
THEOREM —Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

Theorem says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there.

Caution The converse of Theorem is false. A function need not have a derivative at a point where it is continuous.

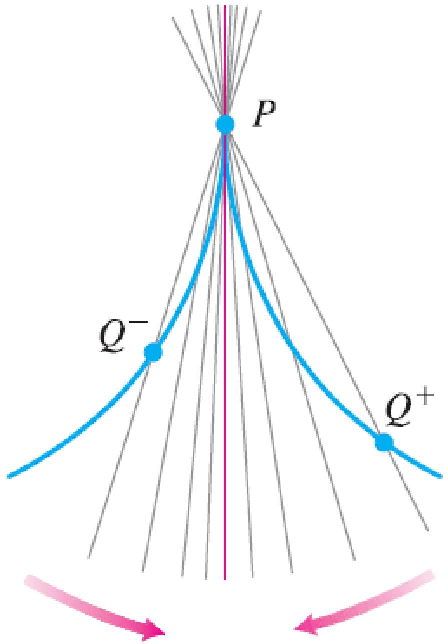
When Does a Function *Not* Have a Derivative at a Point?

differentiability is a "smoothness" condition on the graph of f.



1. a *corner*, where the one-sided derivatives differ.

$$f'_+(c) \neq f'_-(c)$$



2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.

$$f'_+(c) = +\infty \qquad \text{or} \qquad f'_+(c) = -\infty$$
$$f'_-(c) = -\infty \qquad \qquad f'_-(c) = +\infty$$

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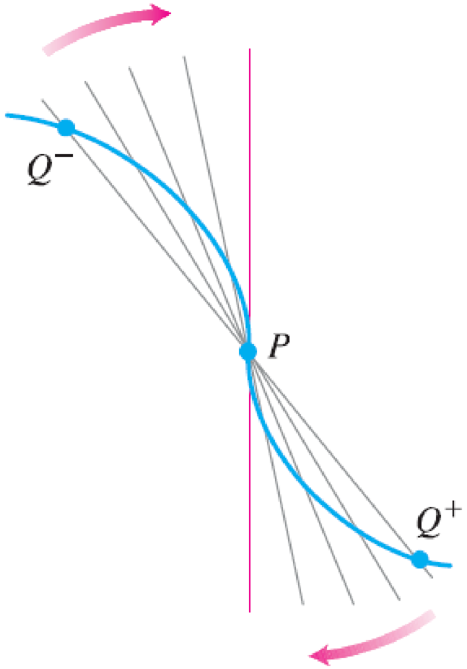
When Does a Function *Not* Have a Derivative at a Point?



$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \begin{matrix} +\infty \\ -\infty \end{matrix} \to \text{then}$$

$f(x)$ has vertical tangent at $x = x_0$

3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).

$$f'_+(c) = f'_-(c) = \mp\infty$$

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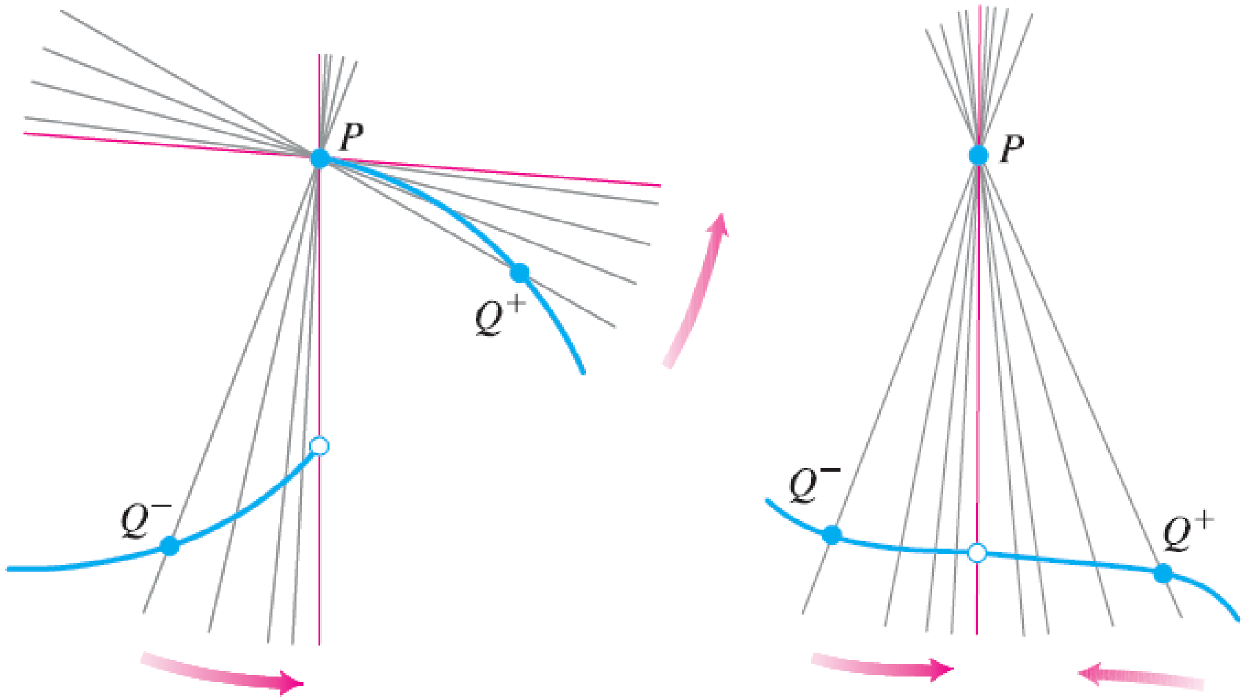
When Does a Function *Not* Have a Derivative at a Point?



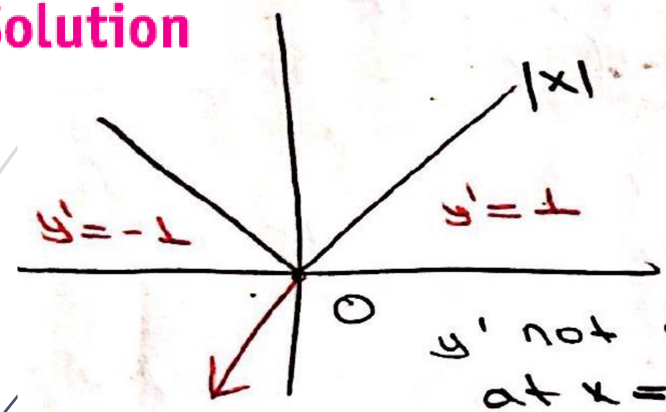
4. a *discontinuity* (two examples shown).

When Does a Function *Not* Have a Derivative at a Point?

☆ Another case in which the derivative may fail to exist occurs when the function's slope is oscillating rapidly near P, like $f(x) = \sin(1/x)$ near the origin, where it is discontinuous

EXAMPLE Show that the function $y=|x|$ is differentiable on $(-\infty,0)$ and $(0,\infty)$ but has no derivative at $x=0$

Solution



Corner point

y' not defined at $x = 0$

righthand derivative ≠ left-hand derivative

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$a = 0$

right-hand derivative $\lim_{x\to 0^+} \frac{x}{x} = 1$

left hand derivative $\lim_{x\to 0^-} \frac{-x}{x} = -1$

not equal to each other

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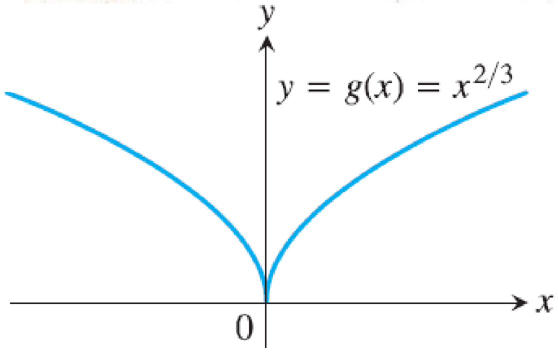
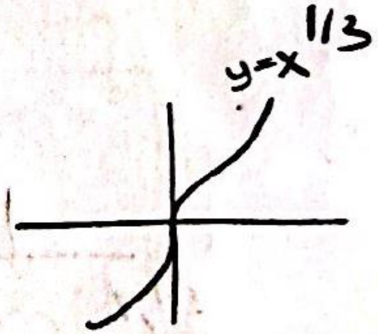
EXAMPLE $f(x)=x^{2/3}$ examine $f'(x)$ at $x=0$.

Solution

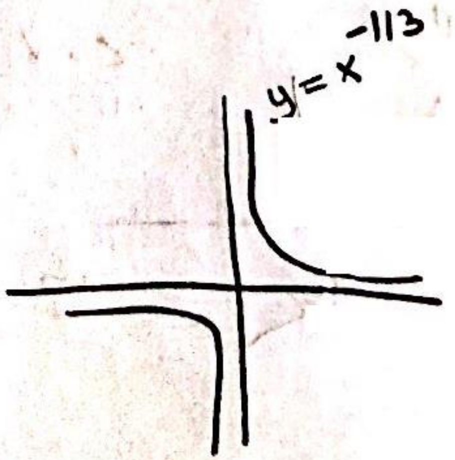


NO VERTICAL TANGENT AT ORIGIN



$$f'(0)=\lim_{h\to 0}\frac{1}{h^{1/3}}$$

$$=\lim_{h\to 0} h^{-1/3}$$



$$f'_+(0)=\lim_{h\to 0^+} h^{-1/3}=\infty \qquad f'_-(0)=\lim_{h\to 0^-} h^{-1/3}=-\infty$$

The derivative does not exist. Also the fnct. has no vertical tangent

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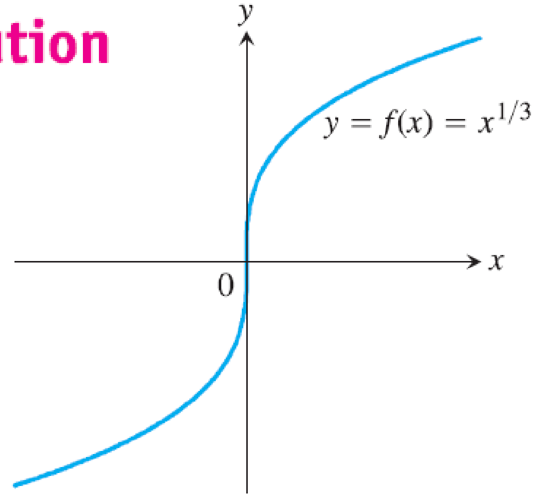
EXAMPLE $f(x) = x^{1/3}$ examine $f'(x)$ at $x=0$.

Solution

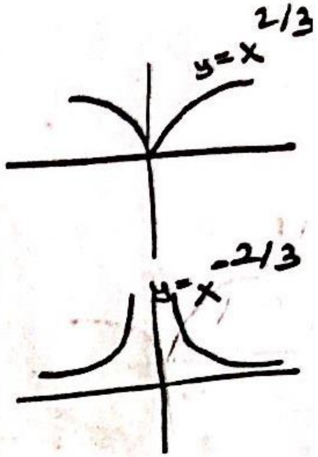


VERTICAL TANGENT AT ORIGIN

$$f'(0) = \lim_{h \to 0} \frac{(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h^{2/3}}$$

$$= \lim_{h \to 0} h^{-2/3}$$



$f'_+(0) = \lim_{h \to 0^+} h^{-2/3} = \infty$

$f'_-(0) = \lim_{h \to 0^-} h^{-2/3} = \infty$

$$\Rightarrow f'(0) = \infty$$

So $m_T = \infty$ $(\tan\theta = \infty)$ $\Rightarrow$ y-axis is vertical tangent

So $f(x) = x^{1/3}$ has not derivative at $x=0$.

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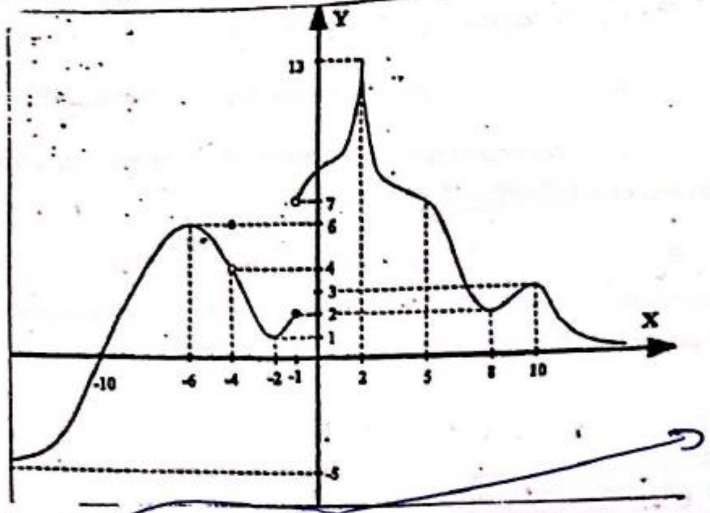
EXAMPLE

f fonksiyonu, grafiği aşağıda verilmiş olan bir fonksiyon olsun.



write the reasons!

Buna göre aşağıdaki soruları ***nedenleriyle* birlikte** cevaplayınız (15 Puan).

Solution

a) $\lim_{x \to 2} f(x) =$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 13$

b) $\lim_{x \to -1} f(x) =$

$\lim_{x \to -1^-} f(x) = 2 \neq \lim_{x \to -1^+} f(x) = 7$

limit does not exists

c) $\lim_{x \to 2^-} f'(x) = \infty$

d) $\lim_{x \to -\infty} f(x) = 0$

e) Eğer varsa, *f* 'nin süreksiz olduğu noktaları bulup sınıflandırınız.

$x = -1$ $\quad \lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$

jump discont.

$x = -4$'de $\lim_{x \to -4^+} f(x) = \lim_{x \to -4^-} f(x) \neq f(4) = 6$

removable disc.

f) Find the points where f is not diff.

at $x = -1$ ⇒ the function is not cont ⇒ is not diff.

at $x = -4$ ⇒ the funct is not cont ⇒ is not diff.

at $x = 2$ ⇒ the funct has an edge ⇒ is not diff.

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Differentiation Rules

If u and v are differentiable functions of x, and c is a constant, then

① $f(x) = c \qquad \dfrac{df}{dx} = \dfrac{d}{dx} c = 0 \qquad$ c; constant

② Power Rule: $\dfrac{d}{dx} x^n = n x^{n-1}$

③ Constant Multiple Rule: $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$

④ Sum Rule: $\dfrac{d}{dx}(u+v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$

⑤ Product Rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

⑥ Quotient Rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

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☆ Power Rule for Rational Powers

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

Second- and Higher-Order Derivatives

$f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x).$$

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the ***n*th derivative** of y with respect to x for any positive integer n.

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How to Read the Symbols for Derivatives

y'	"y prime"
y''	"y double prime"
$\dfrac{d^2y}{dx^2}$	"d squared y dx squared"
y'''	"y triple prime"
$y^{(n)}$	"y super n"
$\dfrac{d^ny}{dx^n}$	"d to the n of y by dx to the n"
D^n	"D to the n"

EXAMPLE The first four derivatives of $y = x^3 - 3x^2 + 2$ are

$$\begin{aligned} &\text{First derivative:} && y' = 3x^2 - 6x \\ &\text{Second derivative:} && y'' = 6x - 6 \\ &\text{Third derivative:} && y''' = 6 \\ &\text{Fourth derivative:} && y^{(4)} = 0. \end{aligned}$$

The function has derivatives of all orders, the fifth and later derivatives all being zero.

Derivatives of Trigonometric Functions

☆ $$\frac{d}{dx}(\sin x) = \cos x.$$

☆ $$\frac{d}{dx}(\cos x) = -\sin x.$$

☆ $$\frac{d}{dx}(\tan x) = \sec^2 x$$

☆ $$\frac{d}{dx}(\cot x) = -\csc^2 x$$

☆ $$\frac{d}{dx}(\sec x) = \sec x \tan x$$

☆ $$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$

$$= 2x - \cos x.$$

(b) $y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x \qquad \text{Product Rule}$$

$$= x^2 \cos x + 2x \sin x.$$

(c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x \cos x - \sin x}{x^2}.$$

EXAMPLE We find derivatives of the cosine function in combinations with other functions.

(a) $y = 5x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$

$$= 5 - \sin x.$$

(b) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x.$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}.$$

EXAMPLE Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

EXAMPLE Find y'' if $y = \sec x$.

Solution Finding the second derivative involves a combination of trigonometric derivatives.

$$
\begin{aligned}
y &= \sec x \\
y' &= \sec x \tan x && \text{Derivative rule for secant function} \\
y'' &= \frac{d}{dx}(\sec x \tan x) \\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Derivative Product Rule} \\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) && \text{Derivative rules} \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}
$$

The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

"Outside-Inside" Rule

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to think about the Chain Rule using functional notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

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☆ If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}\left(u^n\right) = nu^{n-1}$$

EXAMPLE Differentiate $\sin(x^2 + x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin\underbrace{(x^2+x)}_{\text{inside}} = \cos\underbrace{(x^2+x)}_{\substack{\text{inside} \\ \text{left alone}}} \cdot \underbrace{(2x+1)}_{\substack{\text{derivative of} \\ \text{the inside}}}.$$

EXAMPLE Find the equation of tangent and normal line of the curve $y=\tan\left(\frac{\pi x}{4}\right)$ at $(1,1)$.

Solution

$$\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{\pi}{4}\sec^2\left(\frac{\pi x}{4}\right)\right|_{x=1}$$

$$= \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{2} \Rightarrow m_T = \frac{\pi}{2}$$

Tangent line $\rightarrow$ $y = 1 + \frac{\pi}{2}(x-1) = \frac{\pi x}{2} - \frac{\pi}{2} + 1$

$$m_N = -\frac{2}{\pi}$$

Normal line $\rightarrow$ $y = 1 - \frac{2}{\pi}(x-1) = -\frac{2x}{\pi} + \frac{2}{\pi} + 1$

EXAMPLE Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

EXAMPLE

$$* \ \frac{d}{dx}\left(\sqrt{x^2+1}\right) =$$

$$* \ \frac{d}{dx}\left(3x + 6 + \frac{x}{2}\right) =$$

$$* \ \frac{d}{dx} \sin^7(x^3) = ?$$

EXAMPLE The Power Chain Rule simplifies computing the derivative of a power of an expression.

(a)
$$\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$$

Power Chain Rule with $u = 5x^3 - x^4,\ n = 7$

$$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3)$$
$$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$$

(b)
$$\frac{d}{dx}\left(\frac{1}{3x - 2}\right) = \frac{d}{dx}(3x - 2)^{-1}$$
$$= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2)$$

Power Chain Rule with $u = 3x - 2,\ n = -1$

$$= -1(3x - 2)^{-2}(3)$$
$$= -\frac{3}{(3x - 2)^2}$$

In part (b) we could also find the derivative with the Derivative Quotient Rule.

(c)
$$\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cdot \frac{d}{dx}\sin x$$

Power Chain Rule with $u = \sin x,\ n = 5$, because $\sin^n x$ means $(\sin x)^n$, $n \neq -1$.

$$= 5\sin^4 x \cos x$$

■

Derivative of the Absolute Value Function

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}, \quad x \neq 0$$

EXAMPLE $y = |x|$ is not differentiable at $x = 0$. However, the function *is* differentiable at all other real numbers as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula:

$$\begin{aligned}
\frac{d}{dx}(|x|) &= \frac{d}{dx}\sqrt{x^2} \\
&= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2) \\
&= \frac{1}{2|x|} \cdot 2x \\
&= \frac{x}{|x|}, \quad x \neq 0.
\end{aligned}$$

Power Chain Rule with $u = x^2, n = 1/2, x \neq 0$

$\sqrt{x^2} = |x|$

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HW:

In Exercises 5–10, find an equation for the tangent to the curve at the given point.

5. $y = 4 - x^2, \quad (-1, 3)$

6. $y = (x - 1)^2 + 1, \quad (1, 1)$

7. $y = 2\sqrt{x}, \quad (1, 2)$

8. $y = \dfrac{1}{x^2}, \quad (-1, 1)$

9. $y = x^3, \quad (-2, -8)$

10. $y = \dfrac{1}{x^3}, \quad \left(-2, -\dfrac{1}{8}\right)$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1, \quad (2, 5)$

12. $f(x) = x - 2x^2, \quad (1, -1)$

13. $g(x) = \dfrac{x}{x - 2}, \quad (3, 3)$

14. $g(x) = \dfrac{8}{x^2}, \quad (2, 2)$

15. $h(t) = t^3, \quad (2, 8)$

16. $h(t) = t^3 + 3t, \quad (1, 4)$

17. $f(x) = \sqrt{x}, \quad (4, 2)$

18. $f(x) = \sqrt{x + 1}, \quad (8, 3)$

HW:

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$
2. $F(x) = (x - 1)^2 + 1; \quad F'(-1), F'(0), F'(2)$
3. $g(t) = \dfrac{1}{t^2}; \quad g'(-1), g'(2), g'\left(\sqrt{3}\right)$
4. $k(z) = \dfrac{1 - z}{2z}; \quad k'(-1), k'(1), k'\left(\sqrt{2}\right)$
5. $p(\theta) = \sqrt{3\theta}; \quad p'(1), p'(3), p'(2/3)$
6. $r(s) = \sqrt{2s + 1}; \quad r'(0), r'(1), r'(1/2)$

HW:

In Exercises 19–22, find the values of the derivatives.

19. $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$

20. $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \dfrac{1}{x}$

21. $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4 - \theta}}$

22. $\left.\dfrac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

HW:

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$

2. $y = x^2 + x + 8$

3. $s = 5t^3 - 3t^5$

4. $w = 3z^7 - 7z^3 + 21z^2$

5. $y = \dfrac{4x^3}{3} - x$

6. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{4}$

7. $w = 3z^{-2} - \dfrac{1}{z}$

8. $s = -2t^{-1} + \dfrac{4}{t^2}$

9. $y = 6x^2 - 10x - 5x^{-2}$

10. $y = 4 - 2x - x^{-3}$

11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$

12. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$

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HW:

Find the derivatives of all orders of the functions in Exercises 29– 32.

29. $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$

30. $y = \frac{x^5}{120}$

31. $y = (x - 1)(x^2 + 3x - 5)$

32. $y = (4x^3 + 3x)(2 - x)$

53. a. Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

HW:

In Exercises 1–18, find dy/dx.

5. $y = \csc x - 4\sqrt{x} + 7$

6. $y = x^2 \cot x - \dfrac{1}{x^2}$

7. $f(x) = \sin x \tan x$

8. $g(x) = \csc x \cot x$

9. $y = (\sec x + \tan x)(\sec x - \tan x)$

10. $y = (\sin x + \cos x) \sec x$

HW:

In Exercises 19–22, find ds/dt.

19. $s = \tan t - t$

20. $s = t^2 - \sec t + 1$

21. $s = \dfrac{1 + \csc t}{1 - \csc t}$

22. $s = \dfrac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

23. $r = 4 - \theta^2 \sin \theta$

24. $r = \theta \sin \theta + \cos \theta$

25. $r = \sec \theta \csc \theta$

26. $r = (1 + \sec \theta) \sin \theta$

In Exercises 27–32, find dp/dq.

27. $p = 5 + \dfrac{1}{\cot q}$

28. $p = (1 + \csc q) \cos q$

29. $p = \dfrac{\sin q + \cos q}{\cos q}$

30. $p = \dfrac{\tan q}{1 + \tan q}$

HW:

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$

10. $y = (4 - 3x)^9$

11. $y = \left(1 - \frac{x}{7}\right)^{-7}$

12. $y = \left(\frac{x}{2} - 1\right)^{-10}$

13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

14. $y = \sqrt{3x^2 - 4x + 6}$

15. $y = \sec(\tan x)$

16. $y = \cot\left(\pi - \frac{1}{x}\right)$

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