1.2 Review of Taylor Series

Most students will have encountered infinite series (particularly Taylor series) in their study of calculus without necessarily having acquired a good understanding of this topic. Consequently, this section is particularly important for numerical analysis, and deserves careful study.

Once students are well grounded with a basic understanding of Taylor series, the Mean-Value Theorem, and alternating series (all topics in this section) as well as computer number representation (Section 2.2), they can proceed to study the fundamentals of numerical methods with better comprehension.

Taylor Series

Familiar (and useful) examples of Taylor series are the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad (|x| < \infty) \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad (|x| < \infty) \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad (|x| < \infty) \tag{3}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k \qquad (|x| < 1) \tag{4}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \qquad (-1 < x \leqq 1) \tag{5}$$

For each case, the series represents the given function and converges in the interval specified. Series (1)–(5) are Taylor series expanded about $c = 0$. A Taylor series expanded about $c = 1$ is

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k}$$

where $0 < x \leqq 2$. The reader should recall the **factorial** notation

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$$

for $n \geqq 1$ and the special definition of $0! = 1$.

Series of this type are often used to compute good approximate values of complicated functions at specific points.

EXAMPLE 1 Use five terms in Series (5) to approximate $\ln(1.1)$.

Solution Taking $x = 0.1$ in the first five terms of the series for $\ln(1 + x)$ gives us

$$\ln(1.1) \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} = 0.09531\,03333\ldots$$

where $\approx$ means "approximately equal." This value is correct to six decimal places of accuracy. ■

On the other hand, such good results are not always obtained in using series.

EXAMPLE 2 Try to compute e^8 by using Series (1).

Solution The result is

$$e^8 = 1 + 8 + \frac{64}{2} + \frac{512}{6} + \frac{4096}{24} + \frac{32768}{120} + \cdots$$

It is apparent that many terms will be needed to compute e^8 with reasonable precision. By repeated squaring, we find $e^2 = 7.38905\,6$, $e^4 = 54.59815\,00$, and $e^8 = 2980.95798\,7$. The first six terms given above yield $570.06666\,5$. ■

These examples illustrate a general rule:

A Taylor series converges rapidly near the point of expansion and slowly (or not at all) at more remote points.

A graphical depiction of the phenomenon can be obtained by graphing a few partial sums of a Taylor series. In Figure 1.2, we show the function

$$y = \sin x$$

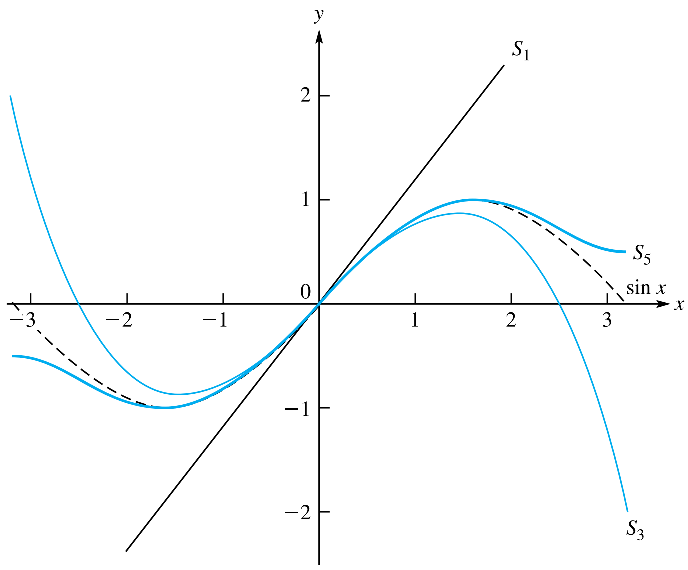

FIGURE 1.2 Approximations to $\sin x$

and the partial-sum functions

$$S_1 = x$$
$$S_3 = x - \frac{x^3}{6}$$
$$S_5 = x - \frac{x^3}{6} + \frac{x^5}{120}$$

which come from Series (2). While S_1 may be an acceptable approximation to $\sin x$ when $x \approx 0$, the graphs for S_3 and S_5 match that of $\sin x$ on larger intervals about the origin.

All of the series illustrated above are examples of the following general series:

■ **THEOREM 1** **FORMAL TAYLOR SERIES FOR f ABOUT c**

$$f(x) \sim f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \tag{6}$$

Here, rather than using $=$, we have written $\sim$ to indicate that we are not allowed to assume that $f(x)$ *equals* the series on the right. All we have at the moment is a formal series that can be written down provided that the successive derivatives f', f'', f''', ... exist at the point c. Series (6) is called the **"Taylor series of f at the point c."**

In the special case $c = 0$, Series (6) is also called a **Maclaurin series**:

$$f(x) \sim f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \tag{7}$$

The first term is $f(0)$ when $k = 0$.

EXAMPLE 3 What is the Taylor series of the function

$$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$$

at the point $c = 2$?

Solution To compute the coefficients in the series, we need the numerical values of $f^{(k)}(2)$ for $k \geqq 0$. Here are the details of the computation:

$$\begin{array}{llll}
f(x) &= 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5 & f(2) &= 207 \\
f'(x) &= 15x^4 - 8x^3 + 45x^2 + 26x - 12 & f'(2) &= 396 \\
f''(x) &= 60x^3 - 24x^2 + 90x + 26 & f''(2) &= 590 \\
f'''(x) &= 180x^2 - 48x + 90 & f'''(2) &= 714 \\
f^{(4)}(x) &= 360x - 48 & f^{(4)}(2) &= 672 \\
f^{(5)}(x) &= 360 & f^{(5)}(2) &= 360 \\
f^{(k)}(x) &= 0 & f^{(k)}(2) &= 0
\end{array}$$

for $k \geqq 6$. Therefore, we have

$$\begin{aligned} f(x) \sim 207 + 396(x-2) + 295(x-2)^2 \\ + 119(x-2)^3 + 28(x-2)^4 + 3(x-2)^5 \end{aligned}$$

In this example, it is not difficult to see that $\sim$ may be replaced by $=$. Simply expand all the terms in the Taylor series and collect them to get the original form for f. Taylor's Theorem, discussed soon, will allow us to draw this conclusion without doing any work! ■

Complete Horner's Algorithm

An application of Horner's algorithm is that of finding the Taylor expansion of a polynomial about any point. Let $p(x)$ be a given polynomial of degree n with coefficients a_k as in Equation (2) in Section 1.1, and suppose that we desire the coefficients c_k in the equation

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \\ &= c_n(x-r)^n + c_{n-1}(x-r)^{n-1} + \cdots + c_1(x-r) + c_0 \end{aligned}$$

Of course, Taylor's Theorem asserts that $c_k = p^{(k)}(r)/k!$, but we seek a more efficient algorithm. Notice that $p(r) = c_0$, so this coefficient is obtained by applying Horner's algorithm to the polynomial p with the point r. The algorithm also yields the polynomial

$$q(x) = \frac{p(x) - p(r)}{x - r} = c_n(x-r)^{n-1} + c_{n-1}(x-r)^{n-2} + \cdots + c_1$$

This shows that the second coefficient, c_1, can be obtained by applying Horner's algorithm to the polynomial q with point r, because $c_1 = q(r)$. (Notice that the first application of Horner's algorithm does not yield q in the form shown but rather as a sum of powers of x. (See Equations (3)–(4) in Section 1.1.) This process is repeated until all coefficients c_k are found.

We call the algorithm just described the **complete Horner's algorithm**. The pseudocode for executing it is arranged so that the coefficients c_k *overwrite* the input coefficients a_k.

```
integer n, k, j;   real r;   real array (a_i)_{0:n}
for k = 0 to n − 1 do
    for j = n − 1 to k do
        a_j ← a_j + r a_{j+1}
    end for
end for
```

This procedure can be used in carrying out Newton's method for finding roots of a polynomial, which we discuss in Chapter 3. Moreover, it can be done in complex arithmetic to handle polynomials with complex roots or coefficients.

EXAMPLE 4 Using the complete Horner's algorithm, find the Taylor expansion of the polynomial

$$p(x) = x^4 - 4x^3 + 7x^2 - 5x + 2$$

about the point $r = 3$.

Solution The work can be arranged as follows:

$$
\begin{array}{rrrrrr}
 & 1 & -4 & 7 & -5 & 2\\
3\,) & & 3 & -3 & 12 & 21\\ \cline{2-6}
 & 1 & -1 & 4 & 7 & 23\\
 & & 3 & 6 & 30 & \\ \cline{2-5}
 & 1 & 2 & 10 & 37 & \\
 & & 3 & 15 & & \\ \cline{2-4}
 & 1 & 5 & 25 & & \\
 & & 3 & & & \\ \cline{2-3}
 & 1 & 8 & & &
\end{array}
$$

The calculation shows that

$$p(x) = (x-3)^4 + 8(x-3)^3 + 25(x-3)^2 + 37(x-3) + 23$$

Taylor's Theorem in Terms of $(x - c)$

THEOREM 2 **TAYLOR'S THEOREM FOR $f(x)$**

If the function f possesses continuous derivatives of orders $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1} \tag{8}$$

where the error term E_{n+1} can be given in the form

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Here ξ is a point that lies between c and x and depends on both.

In practical computations with Taylor series, it is usually necessary to *truncate* the series because it is not possible to carry out an infinite number of additions. A series is said to be **truncated** if we ignore all terms after a certain point. Thus, if we truncate the exponential Series (1) after seven terms, the result is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

This no longer represents e^x except when $x = 0$. But the truncated series should *approximate* e^x. Here is where we need Taylor's Theorem. With its help, we can assess the difference between a function f and its truncated Taylor series.

The explicit assumption in this theorem is that $f(x), f'(x), f''(x), \ldots, f^{(n+1)}(x)$ are all continuous functions in the interval $I = [a, b]$. The final term E_{n+1} in Equation (8) is the **remainder** or **error term**. The given formula for E_{n+1} is valid when we assume only that $f^{(n+1)}$ exists at each point of the open interval (a, b). The error term is similar to the terms preceding it, but notice that $f^{(n+1)}$ must be evaluated at a point other than c. This point ξ depends on x and is in the open interval (c, x) or (x, c). Other forms of the remainder are possible; the one given here is **Lagrange's form**. (We do not prove Taylor's Theorem here.)

EXAMPLE 5 Derive the Taylor series for e^x at $c = 0$, and prove that it converges to e^x by using Taylor's Theorem.

Solution If $f(x) = e^x$, then $f^{(k)}(x) = e^x$ for $k \geqq 0$. Therefore, $f^{(k)}(c) = f^{(k)}(0) = e^0 = 1$ for all k. From Equation (8), we have

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^\xi}{(n+1)!}x^{n+1} \tag{9}$$

Now let us consider all the values of x in some symmetric interval around the origin, for example, $-s \leqq x \leqq s$. Then $|x| \leqq s$, $|\xi| \leqq s$, and $e^\xi \leqq e^s$. Hence, the remainder term satisfies this inequality:

$$\lim_{n\to\infty} \left| \frac{e^\xi}{(n+1)!}x^{n+1} \right| \leqq \lim_{n\to\infty} \frac{e^s}{(n+1)!}s^{n+1} = 0$$

Thus, if we take the limit as $n \to \infty$ on both sides of Equation (9), we obtain

$$e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

■

This example illustrates how we can establish, in specific cases, that a formal Taylor Series (6) actually represents the function. Let's examine another example to see how the formal series can *fail* to represent the function.

EXAMPLE 6 Derive the formal Taylor series for $f(x) = \ln(1 + x)$ at $c = 0$, and determine the range of positive x for which the series represents the function.

Solution We need $f^{(k)}(x)$ and $f^{(k)}(0)$ for $k \geqq 1$. Here is the work:

$$\begin{array}{llll}
f(x) &= \ln(1+x) & f(0) &= 0 \\
f'(x) &= (1+x)^{-1} & f'(0) &= 1 \\
f''(x) &= -(1+x)^{-2} & f''(0) &= -1 \\
f'''(x) &= 2(1+x)^{-3} & f'''(0) &= 2 \\
f^{(4)}(x) &= -6(1+x)^{-4} & f^{(4)}(0) &= -6 \\
\vdots & & \vdots & \\
f^{(k)}(x) &= (-1)^{k-1}(k-1)!(1+x)^{-k} & f^{(k)}(0) &= (-1)^{k-1}(k-1)!
\end{array}$$

Hence by Taylor's Theorem, we obtain

$$\begin{aligned}
\ln(1+x) &= \sum_{k=1}^{n} (-1)^{k-1} \frac{(k-1)!}{k!} x^k + \frac{(-1)^n n! (1+\xi)^{-n-1}}{(n+1)!} x^{n+1} \\
&= \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + \frac{(-1)^n}{n+1} (1+\xi)^{-n-1} x^{n+1}
\end{aligned} \tag{10}$$

For the *infinite* series to represent $\ln(1+x)$, it is necessary and sufficient that the error term converge to zero as $n \to \infty$. Assume that $0 \leqq x \leqq 1$. Then $0 \leqq \xi \leqq x$ (because zero is the point of expansion); thus, $0 \leqq x/(1+\xi) \leqq 1$. Hence, the error term converges to zero in this case. If $x > 1$, the terms in the series do not approach zero, and the series does not converge. Hence, the series represents $\ln(1+x)$ if $0 \leqq x \leqq 1$ but *not* if $x > 1$. (The series also represents $\ln(1+x)$ for $-1 < x < 0$ but not if $x \leqq -1$.) ■

Mean-Value Theorem

The special case $n = 0$ in Taylor's Theorem is known as the **Mean-Value Theorem**. It is usually stated, however, in a somewhat more precise form.

■ **THEOREM 3**

MEAN-VALUE THEOREM

If f is a continuous function on the closed interval $[a, b]$ and possesses a derivative at each point of the open interval (a, b), then

$$f(b) = f(a) + (b-a)f'(\xi)$$

for some ξ in (a, b).

Hence, the ratio $[f(b) - f(a)]/(b-a)$ is equal to the derivative of f at some point ξ between a and b; that is, for some $\xi \in (a, b)$,

$$f'(\xi) = \frac{f(b) - f(a)}{b-a}$$

The right-hand side could be used as an *approximation* for $f'(x)$ at any x within the interval (a, b). The approximation of derivatives is discussed more fully in Section 4.3.

Taylor's Theorem in Terms of *h*

Other forms of Taylor's Theorem are often useful. These can be obtained from the basic Formula (8) by changing the variables.

■ **COROLLARY 1**

TAYLOR'S THEOREM FOR $f(x+h)$

If the function f possesses continuous derivatives of order $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any x in I,

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1} \tag{11}$$

where h is any value such that $x + h$ is in I and where

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

for some ξ between x and $x + h$.

The form (11) is obtained from Equation (8) by replacing x by $x + h$ and replacing c by x. Notice that because h can be positive or negative, the requirement on ξ means $x < \xi < x + h$ if $h > 0$ or $x + h < \xi < x$ if $h < 0$.

The **error term** E_{n+1} depends on h in two ways: First, h^{n+1} is explicitly present; second, the point ξ generally depends on h. As h converges to zero, E_{n+1} converges to zero with essentially the same rapidity with which h^{n+1} converges to zero. For large n, this is quite rapid. To express this qualitative fact, we write

$$E_{n+1} = \mathcal{O}(h^{n+1})$$

as $h \to 0$. This is called **big O notation**, and it is shorthand for the inequality

$$|E_{n+1}| \leqq C|h|^{n+1}$$

where C is a constant. In the present circumstances, this constant could be any number for which $|f^{(n+1)}(t)|/(n+1)! \leqq C$, for all t in the initially given interval, I. Roughly speaking, $E_{n+1} = \mathcal{O}(h^{n+1})$ means that the behavior of E_{n+1} is similar to the much simpler expression h^{n+1}.

It is important to realize that Equation (11) corresponds to an entire sequence of theorems, one for each value of n. For example, we can write out the cases $n = 0, 1, 2$ as follows:

$$\begin{aligned}
f(x+h) &= f(x) + f'(\xi_1)h \\
&= f(x) + \mathcal{O}(h) \\
f(x+h) &= f(x) + f'(x)h + \frac{1}{2!} f''(\xi_2)h^2 \\
&= f(x) + f'(x)h + \mathcal{O}(h^2) \\
f(x+h) &= f(x) + f'(x)h + \frac{1}{2!} f''(x)h^2 + \frac{1}{3!} f'''(\xi_3)h^3 \\
&= f(x) + f'(x)h + \frac{1}{2!} f''(x)h^2 + \mathcal{O}(h^3)
\end{aligned}$$

The importance of the error term in Taylor's Theorem cannot be stressed too much. In later chapters, many situations require an estimate of errors in a numerical process by use of Taylor's Theorem. Here are some elementary examples.

EXAMPLE 7 Expand $\sqrt{1+h}$ in powers of h. Then compute $\sqrt{1.00001}$ and $\sqrt{0.99999}$.

Solution Let $f(x) = x^{1/2}$. Then $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$, and so on. Now use Equation (11) with $x = 1$. Taking $n = 2$ for illustration, we have

$$\sqrt{1+h} = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3\xi^{-5/2} \tag{12}$$

where ξ is an unknown number that satisfies $1 < \xi < 1 + h$, if $h > 0$. It is important to notice that the function $f(x) = \sqrt{x}$ possesses derivatives of all orders at any point $x > 0$.

In Equation (12), let $h = 10^{-5}$. Then

$$\sqrt{1.00001} \approx 1 + 0.5 \times 10^{-5} - 0.125 \times 10^{-10} = 1.00000\,49999\,87500$$

By substituting $-h$ for h in the series, we obtain

$$\sqrt{1-h} = 1 - \frac{1}{2}h - \frac{1}{8}h^2 - \frac{1}{16}h^3\xi^{-5/2}$$

Hence, we have

$$\sqrt{0.99999} \approx 0.99999\,49999\,87500$$

Since $1 < \xi < 1 + h$, the absolute error does not exceed

$$\frac{1}{16}h^3\xi^{-5/2} < \frac{1}{16}10^{-15} = 0.00000\,00000\,00000\,0625$$

and both numerical values are correct to all 15 decimal places shown. ■

Alternating Series

Another theorem from calculus is often useful in establishing the convergence of a series and in estimating the error involved in truncation. From it, we have the following important principle for alternating series:

> *If the magnitudes of the terms in an alternating series converge monotonically to zero, then the error in truncating the series is no larger than the magnitude of the first omitted term.*

This theorem applies only to **alternating series**—that is, series in which the successive terms are alternately positive and negative.

■ **THEOREM 4** **ALTERNATING SERIES THEOREM**

If $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq \cdots 0$ for all n and $\lim_{n\to\infty} a_n = 0$, then the alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

converges; that is,

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k = \lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}a_k = \lim_{n\to\infty} S_n = S$$

where S is its sum and S_n is the nth partial sum. Moreover, for all n,

$$|S - S_n| \leqq a_{n+1}$$

EXAMPLE 8 If the sine series is to be used in computing $\sin 1$ with an error less than $\frac{1}{2} \times 10^{-6}$, how many terms are needed?

Solution From Series (2), we have

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$$

If we stop at $1/(2n-1)!$, the error does not exceed the first neglected term, which is $1/(2n+1)!$. Thus, we should select n so that

$$\frac{1}{(2n+1)!} < \frac{1}{2} \times 10^{-6}$$

Using logarithms to base 10, we obtain $\log(2n+1)! > \log 2 + 6 = 6.3$. With a calculator, we compute a table of values for $\log n!$ and find that $\log 10! \approx 6.6$. Hence, if $n \geqq 5$, the error will be acceptable. ■

EXAMPLE 9 If the logarithmic Series (5) is to be used for computing $\ln 2$ with an error of less than $\frac{1}{2} \times 10^{-6}$, how many terms will be required?

Solution To compute $\ln 2$, we take $x = 1$ in the series, and using $\approx$ to mean approximate equality, we have

$$S = \ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} = S_n$$

By the Alternating Series Theorem, the error involved when the series is truncated with n terms is

$$|S - S_n| \leqq \frac{1}{n+1}$$

We select n so that

$$\frac{1}{n+1} < \frac{1}{2} \times 10^{-6}$$

Hence, more than two million terms would be needed! We conclude that this method of computing $\ln 2$ is not practical. (See Problems 1.2.10 through 1.2.12 for several good alternatives.) ■

A word of caution is needed about this technique of calculating the number of terms to be used in a series by just making the $(n+1)$st term less than some tolerance. This procedure is valid only for alternating series in which the terms decrease in magnitude to zero, although it is occasionally used to get rough estimates in other cases. For example, it can be used to identify a nonalternating series as one that converges slowly. When this technique cannot be used, a bound on the remaining terms of the series has to be established. Determining such a bound may be somewhat difficult.

EXAMPLE 10 It is known that

$$\frac{\pi^4}{90} = 1^{-4} + 2^{-4} + 3^{-4} + \cdots$$

How many terms should we take to compute $\pi^4/90$ with an error of at most $\frac{1}{2} \times 10^{-6}$?

Solution A naive approach is to take

$$1^{-4} + 2^{-4} + 3^{-4} + \cdots + n^{-4}$$

where n is chosen so that the next term, $(n+1)^{-4}$, is less that $\frac{1}{2} \times 10^{-6}$. This value of n is 37, but this is an erroneous answer because the partial sum

$$S_{37} = \sum_{k=1}^{37} k^{-4}$$

differs from $\pi^4/90$ by approximately 6×10^{-6}. What we should do, of course, is to select n so that *all* the omitted terms add up to less than $\frac{1}{2} \times 10^{-6}$; that is,

$$\sum_{k=n+1}^{\infty} k^{-4} < \frac{1}{2} \times 10^{-6}$$

By a technique familiar from calculus (see Figure 1.3), we have

$$\sum_{k=n+1}^{\infty} k^{-4} < \int_n^{\infty} x^{-4}\, dx = \left.\frac{x^{-3}}{-3}\right|_n^{\infty} = \frac{1}{3n^3}$$

Thus, it suffices to select n so that $(3n^3)^{-1} < \frac{1}{2} \times 10^{-6}$, or $n \geqq 88$. (A more sophisticated analysis will improve this considerably.)

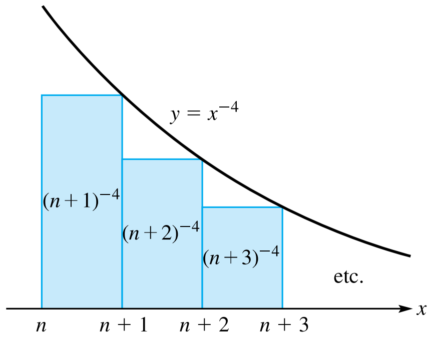

FIGURE 1.3 Illustrating Example 10

Summary

(1) The **Taylor series expansion** about c for $f(x)$ is

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}$$

with error term

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

A more useful form for us is the **Taylor series expansion** for $f(x+h)$, which is

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}h^k + E_{n+1}$$