MAT1320-Linear Algebra Lecture Notes

Vector Spaces, Subspaces

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Table of contents

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Example

The set of n-tuples $V = \mathbb{R}^n = \left\{ \overrightarrow{\mathbf{x}} = (x_1, x_2, \ldots, x_n) \middle| \, x_i \in \mathbb{R} \right\}$ together with the following binary operations

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is a vector space over $\mathbb{R}$.

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Note that (1) $\neq$ (2).

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$$((a,b)+(c,d))+(e,f) = (b,d)+(e,f)=(d,f) \quad (3)$$
$$(a,b)+((c,d)+(e,f)) = (a,b)+(d,f)=(b,f) \quad (4)$$

Note that (3) $\neq$ (4).

Example

The set of column vectors $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} x + y + z = 0 \\ x, y, z \in \mathbb{R} \end{array} \right\}$

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is a vector space.

Example

Let $\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 | a_0, \ldots, a_3 \in \mathbb{R}\}$ be the set of all polynomials of degree at most three in one variable.

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scalar multiplication.

Example
The set of 2×2 square matrices
$\mathcal{M}_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$ is a vector space with respect to binary operations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix}$$

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Definition
Let V be a real vector space and $\varnothing \neq U \subseteq V$. Then U is a subspace of V, if U itself is a vector space over $\mathbb{R}$ with respect to the operations of vector addition and scalar multiplication on V.

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1. *The zero vector* 0 *belong to* U.
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Note: The three conditions given in above Theorem can be combined as a single one. That is, U is a subspace of V if for each $\overrightarrow{\mathbf{v}}, \overrightarrow{\mathbf{w}} \in U$, and $r, s \in \mathbb{R}$, $r\overrightarrow{\mathbf{v}} + s\overrightarrow{\mathbf{w}} \in U$.

Example

Let $\mathcal{B} = \left\{ \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right) \middle| \, c = a + b \text{ and } a, b \in \mathbb{R} \right\} \subset \mathcal{M}_{2\times 2}\left(\mathbb{R}\right)$. Is $\mathcal{B}$ a subspace of $\mathcal{M}_{2\times 2}\left(\mathbb{R}\right)$?

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Also the zero of $\mathcal{M}_{2\times 2}(\mathbb{R})$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}$. Then $\mathcal{B}$ is a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

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Also the zero of $\mathbb{R}^2$, $(0, 0) \in \mathcal{A}$. Then $\mathcal{A}$ is a subspace of $\mathbb{R}^2$.

Subspaces

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$\mathcal{W} = \{\, (a, b, c) \mid a \geqslant 0 \text{ and } a, b, c \in \mathbb{R}\} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$.

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$$r\,(a, b, c) = \underbrace{(ra, rb, rc)}_{r<0 \Rightarrow ra<0} \notin \mathcal{W}.$$

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$\mathcal{W}' = \{\,(a, b, c)|\; a^2 + b^2 + c^2 \leqslant 1 \text{ and } a, b, c \in \mathbb{R}\} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$.

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$$\overrightarrow{\mathbf{u}} = (1, 0, 0) \, , \overrightarrow{\mathbf{v}} = (0, 1, 0) \in \mathcal{W}'$$

Example
$\mathcal{W}' = \{(a, b, c) \mid a^2 + b^2 + c^2 \leqslant 1$ and $a, b, c \in \mathbb{R}\} \subset \mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$. Because $\mathcal{W}'$ is not closed under addition. For example,

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= (1,0,0)\,,\ \overrightarrow{\mathbf{v}} = (0,1,0) \in \mathcal{W}' \\ \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} &= \underbrace{(1,1,0)}_{1^2+1^2=2>1} \notin \mathcal{W}'. \end{aligned}$$

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