11 PARAMETRIC EQUATIONS AND POLAR COORDINATES

Q1) Find the length of the parametric curve
$x(t)=t^3, \quad y(t)=(1-t^2)^{3/2}$ where $-1\leqslant t\leqslant 1$.

Solution:

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$\frac{dx}{dt}=3t^2, \quad \frac{dy}{dt}=\frac{3}{2}(1-t^2)^{1/2}\cdot(-2t)=-3t(1-t^2)^{1/2}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{9t^4+9t^2(1-t^2)}=\sqrt{9t^4+9t^2-9t^4}=3|t|$$

$$L=\int_{-1}^{1}3|t|\,dt=2\cdot\int_0^1 3|t|\,dt=\int_0^1 6t\,dt=3t^2\Big|_0^1=\boxed{3}$$

Q2) Find the length of the curve
$x = 8\cos t + 8t\sin t$, $y = 8\sin t - 8t\cos t$ for $0 \leqslant t \leqslant \frac{\pi}{2}$.

Solution:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -8\sin t + 8\cdot\sin t + 8t\cdot\cos t = 8t\cos t$$

$$\frac{dy}{dt} = 8\cos t - \left(\cancel{8\cos t} + 8t\cdot(-\sin t)\right) = 8t\sin t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t} = \sqrt{64t^2(\cos^2 t + \sin^2 t)} = 8|t|$$

$$L = \int_0^{\pi/2} 8|t|\,dt = \int_0^{\pi/2} 8t\,dt = 4t^2\Big|_0^{\pi/2} = 4\cdot\frac{\pi^2}{4} = \boxed{\pi^2}$$

Q3) Assuming $x=x(t)$ and $y=y(t)$, find the equation of the line tangent to the curve given parametrically as $t^2 \sin x + x^3 = e^t$, $y = t\sin t - 2t$ at $t=0$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Differentiating $t^2 \sin x + x^3 = e^t$ with respect to t, we get

$$2t \cdot \sin x + t^2 \cdot \cos x \cdot \frac{dx}{dt} + 3x^2 \cdot \frac{dx}{dt} = e^t$$

At $t=0$, we get $0 + 0 + 3x^2 \cdot \left.\frac{dx}{dt}\right|_{t=0} = 1$

At $t=0$, $t^2 \sin x + x^3 = e^t$ becomes $x^3 = 1$ and hence $x = 1$.

Hence, we get $3 \cdot 1^2 \cdot \left.\frac{dx}{dt}\right|_{t=0} = 1$ and hence $\left.\frac{dx}{dt}\right|_{t=0} = \frac{1}{3}$.

From $y = t\sin t - 2t$, we get $\frac{dy}{dt} = 1 \cdot \sin t + t \cdot \cos t - 2$.

Hence, $\left.\frac{dy}{dt}\right|_{t=0} = 0 + 0 - 2 = -2$.

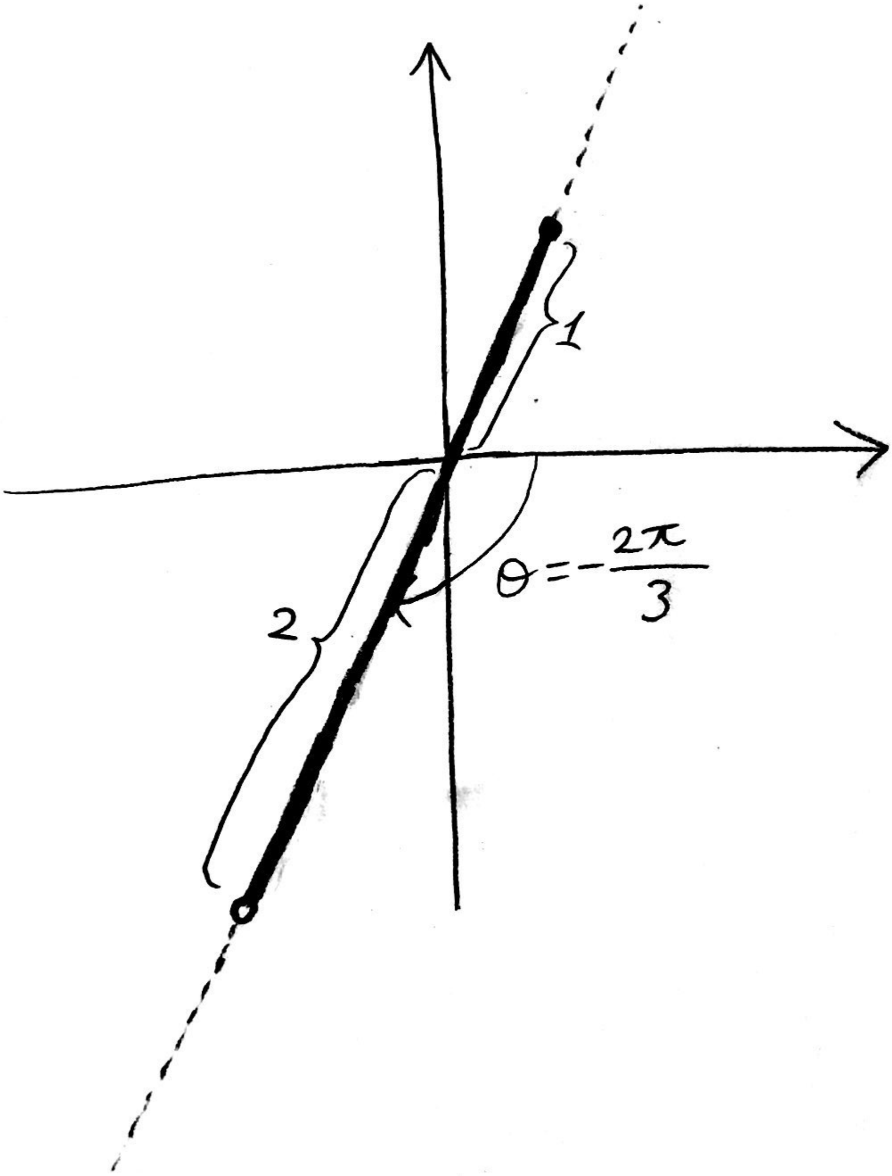
Therefore, slope of the tangent line at $t=0$ is

$$m = \left.\frac{dy}{dx}\right|_{t=0} = \frac{\left.\frac{dy}{dt}\right|_{t=0}}{\left.\frac{dx}{dt}\right|_{t=0}} = \frac{-2}{\frac{1}{3}} = -6.$$

At $t=0$, we have $y=0$ and hence the line passes through the point $(1,0)$. Its equation is $\boxed{y = -6x + 6}$.

Q4) Sketch the graph of the set of points which satisfy $\theta = -\frac{2\pi}{3}$ and $-1 \leq r < 2$ in polar coordinates.

Solution:



Q5 Replace the Cartesion equations $x^2+y^2=2x$ and $x^2+y^2=2y$ by equivalent polar equations and identify their graphs.

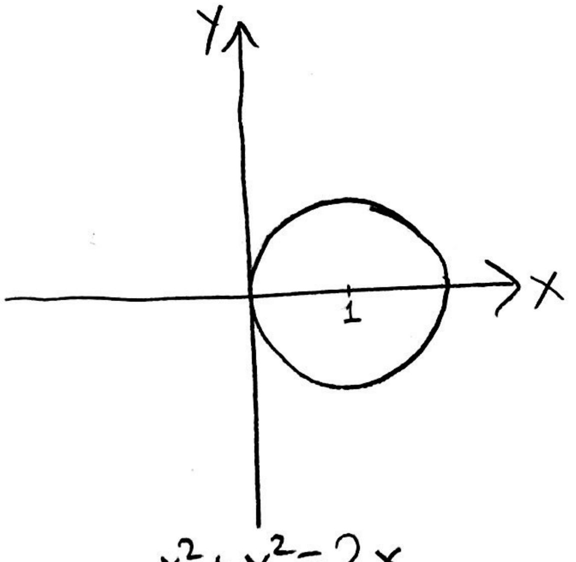
Solution:

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

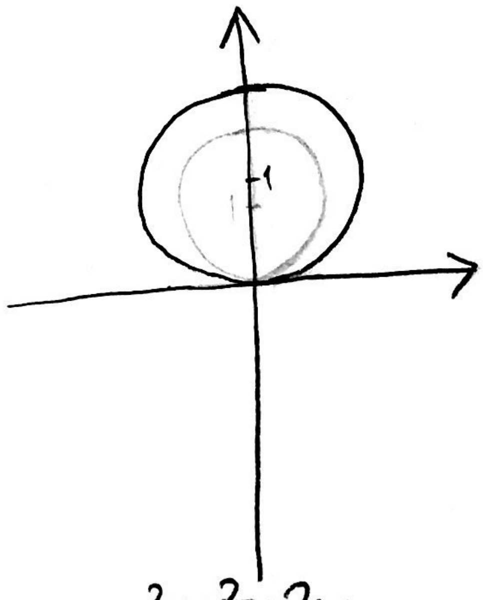
$$x^2+y^2=2x \Rightarrow r^2 = 2r\cos\theta \Rightarrow r^2 - 2r\cos\theta = 0$$
$$\Rightarrow r(r-2\cos\theta)=0 \Rightarrow r=0 \text{ or } r=2\cos\theta$$
$$\Rightarrow \boxed{r=2\cos\theta} \quad \begin{pmatrix}\text{includes}\\ r=0\end{pmatrix}$$

$$x^2+y^2=2y \Rightarrow r^2 = 2r\sin\theta \Rightarrow r=0 \text{ or } r=2\sin\theta$$
$$\Rightarrow \boxed{r=2\sin\theta}$$

$$x^2+y^2=2x \Rightarrow x^2-2x+y^2=0 \Rightarrow x^2-2x+1+y^2=1 \Rightarrow (x-1)^2+y^2=1^2$$
$$x^2+y^2=2y \Rightarrow x^2+y^2-2y=0 \Rightarrow x^2+y^2-2y+1=1 \Rightarrow x^2+(y-1)^2=1^2$$



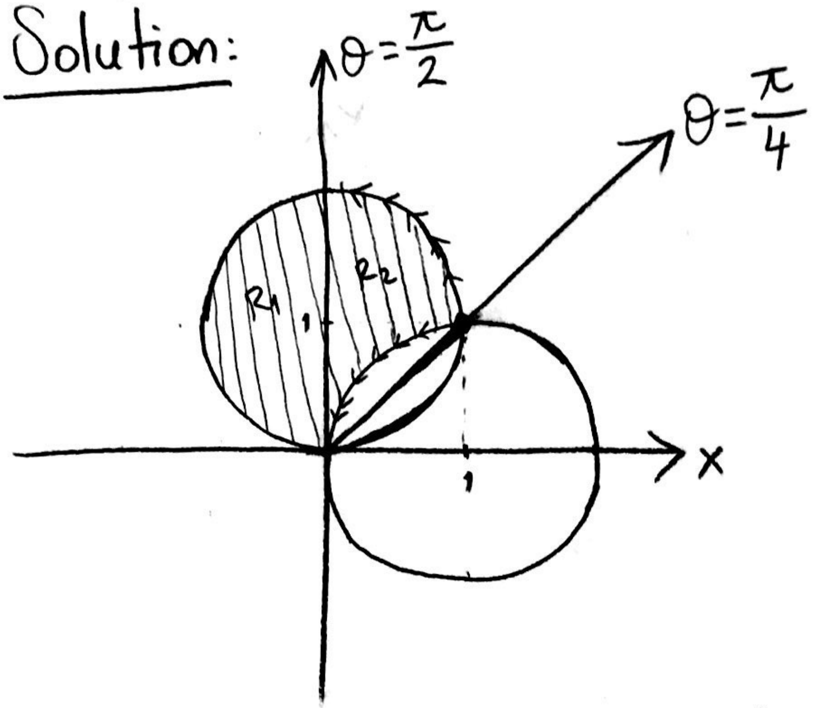
$x^2+y^2=2x$
$r=2\cos\theta$



$x^2+y^2=2y$
$r=2\sin\theta$

Q6) Calculate the area of the region that lies outside the circle $r=2\cos\theta$ and inside the circle $r=2\sin\theta$.

Solution:



$$\text{Area} = \text{Area of } R_1 + \text{Area of } R_2$$

$$\text{Area of } R_1 = \frac{\pi \cdot 1^2}{2} = \frac{\pi}{2}$$

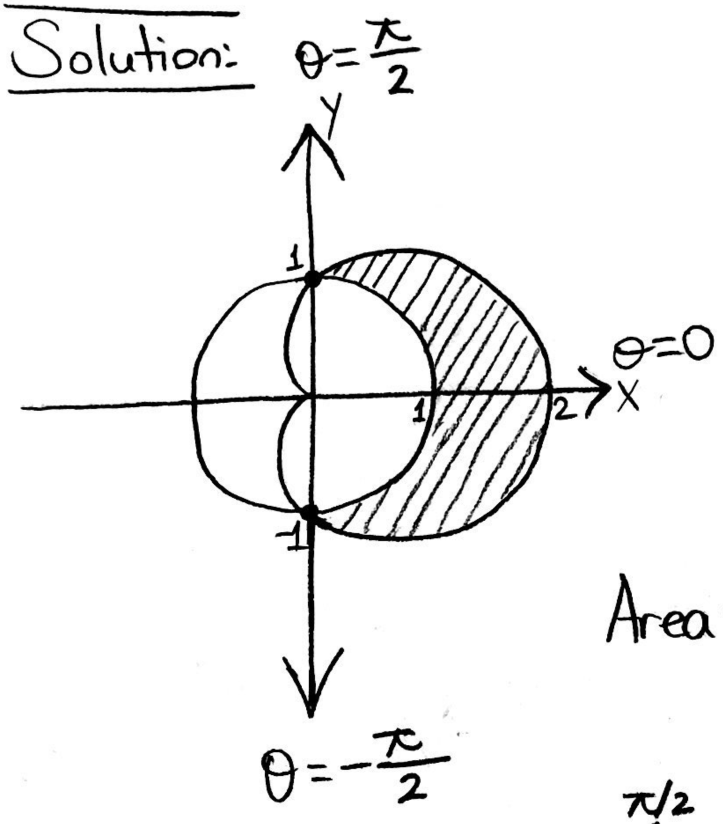
$$\text{Area of } R_2 = \int_{\pi/4}^{\pi/2} \frac{1}{2}\left((2\sin\theta)^2 - (2\cos\theta)^2\right)d\theta = \int_{\pi/4}^{\pi/2} 2\left(\sin^2\theta - \cos^2\theta\right)d\theta$$

$$= \int_{\pi/4}^{\pi/2} -2\cos(2\theta)\,d\theta = \Big[-\sin(2\theta)\Big]_{\pi/4}^{\pi/2} = -\sin\pi + \sin\frac{\pi}{2} = 1.$$

$$\text{Area} = \text{Area of } R_1 + \text{Area of } R_2 = \frac{\pi}{2} + 1.$$

Q7) Find the area of the region that lies outside the circle $r=1$ and inside the cardioid $r=1+\cos\theta$. (Sketch the region.)

Solution:



$$1+\cos\theta = 1$$
$$\cos\theta = 0$$
$$\theta = \mp\frac{\pi}{2}$$

$$\text{Area} = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left((1+\cos\theta)^2 - 1^2\right)d\theta$$

$$= 2\cdot\int_{0}^{\pi/2} \frac{1}{2}\left(2\cos\theta + \cos^2\theta\right)d\theta$$

$$= \int_{0}^{\pi/2}\left(2\cos\theta + \frac{1}{2} + \frac{\cos(2\theta)}{2}\right)d\theta$$

$$= \left[2\sin\theta + \frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{\pi/2}$$

$$= \left(2+\frac{\pi}{4}+0\right) - (0+0+0)$$

$$\boxed{= 2+\frac{\pi}{4}}$$

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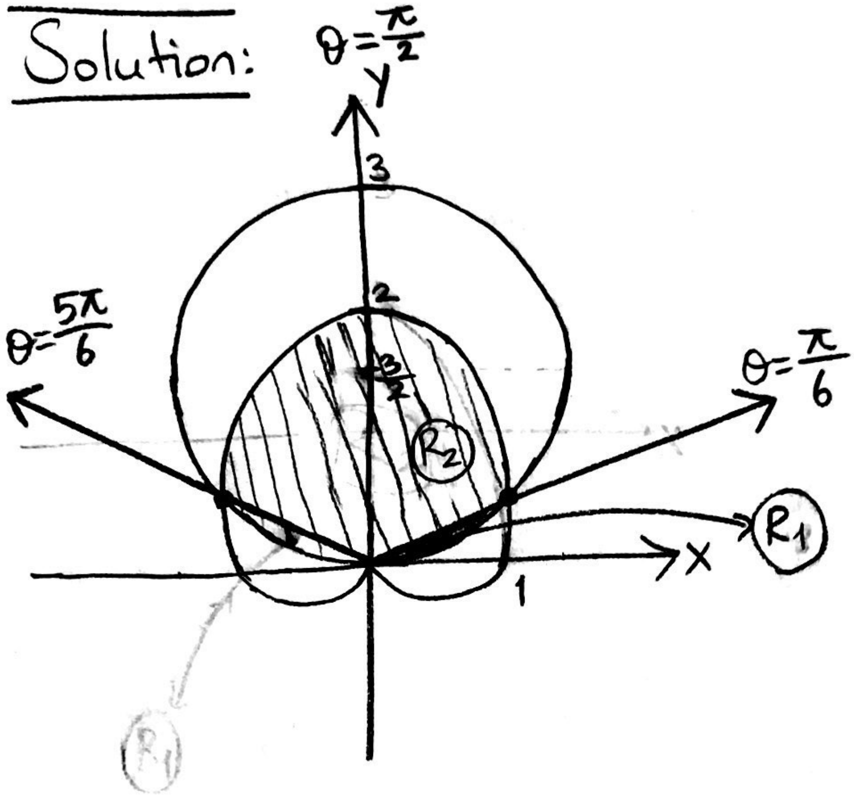
Q8) Find the area of the region enclosed by the curves $r=3\sin\theta$ and $r=1+\sin\theta$. (Sketch the region).

Solution:



$$3\sin\theta = 1+\sin\theta$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$
$$\theta=\frac{\pi}{6},\quad \theta=\frac{5\pi}{6}$$

$$\text{Area} = 2\cdot(\text{Area of } R_1 + \text{Area of } R_2)$$

$$\text{Area of } R_1 = \int_0^{\pi/6} \frac{1}{2}(3\sin\theta)^2 d\theta = \int_0^{\pi/6} \frac{9}{2}\sin^2\theta d\theta = \int_0^{\pi/6}\left(\frac{9}{4}-\frac{9\cos(2\theta)}{4}\right)d\theta$$

$$=\left[\frac{9}{4}\theta - \frac{9\sin(2\theta)}{8}\right]_0^{\pi/6} = \frac{3\pi}{8}-\frac{9\sqrt{3}}{16}$$

$$\text{Area of } R_2 = \int_{\pi/6}^{\pi/2}\frac{1}{2}(1+\sin\theta)^2 d\theta = \int_{\pi/6}^{\pi/2}\left(\frac{1}{2}+\sin\theta+\frac{1}{2}\sin^2\theta\right)d\theta = \int_{\pi/6}^{\pi/2}\left(\frac{1}{2}+\sin\theta+\frac{1}{4}-\frac{\cos(2\theta)}{4}\right)d\theta$$

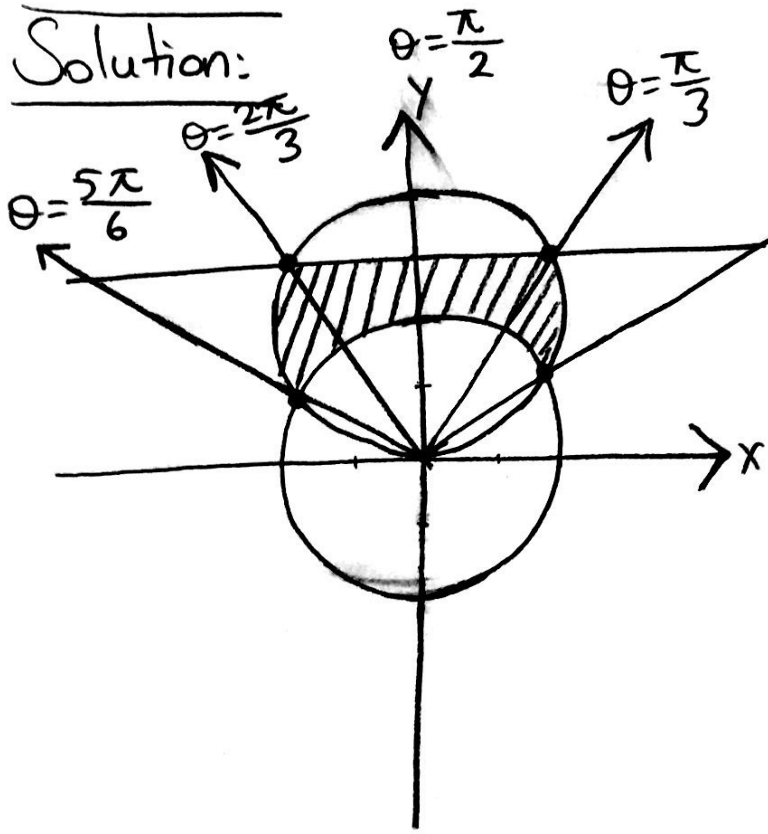
$$=\int_{\pi/6}^{\pi/2}\left(\frac{3}{4}+\sin\theta-\frac{\cos(2\theta)}{4}\right)d\theta = \left[\frac{3}{4}\theta-\cos\theta-\frac{\sin(2\theta)}{8}\right]_{\pi/6}^{\pi/2}$$

$$=\left(\frac{3\pi}{8}-0-0\right)-\left(\frac{\pi}{8}-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{16}\right)=\frac{\pi}{4}+\frac{9\sqrt{3}}{16}$$

$$\text{Area}=2\cdot\left(\frac{3\pi}{8}-\frac{9\sqrt{3}}{16}+\frac{\pi}{4}+\frac{9\sqrt{3}}{16}\right)$$
$$=\boxed{\frac{5\pi}{4}}$$

Q9 Write the definite integral that expresses the area of the region enclosed by $r=2$, $r=4\sin\theta$ and $r\sin\theta=3$. (Sketch the region.) (Do not evaluate the integral(s)).

Solution:



$r\sin\theta=3 \Rightarrow y=3$

$r=\frac{3}{\sin\theta}=3\csc\theta$

$\left.\begin{array}{l} r=2 \\ r=4\sin\theta \end{array}\right\} \sin\theta=\frac{1}{2}$

$\theta=\frac{\pi}{6}, \ \theta=\frac{5\pi}{6}$

$\left.\begin{array}{l} r=4\sin\theta \\ r\sin\theta=3 \end{array}\right\} 4\sin^2\theta=3$

$\sin^2\theta=\frac{3}{4}$

$\sin\theta=\frac{\sqrt{3}}{2}$

$\theta=\frac{\pi}{3}, \ \theta=\frac{2\pi}{3}$

$$\text{Area}=2\cdot\left(\int_{\pi/6}^{\pi/3}\frac{1}{2}\left((4\sin\theta)^2-2^2\right)d\theta+\int_{\pi/3}^{\pi/2}\frac{1}{2}\left((3\csc\theta)^2-2^2\right)d\theta\right)$$

Q10 Find the length of the curve $r = e^{a\theta}$, $-\pi \leq \theta \leq \pi$.

Solution:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\frac{dr}{d\theta} = a e^{a\theta}$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{e^{2a\theta} + a^2 e^{2a\theta}} = e^{a\theta}\sqrt{1 + a^2}$$

$$L = \int_{-\pi}^{\pi} \sqrt{1+a^2} \cdot e^{a\theta}\, d\theta = \left[\frac{\sqrt{1+a^2}}{a} \cdot e^{a\theta}\right]_{-\pi}^{\pi}$$

$$\boxed{= \frac{\sqrt{1+a^2}}{a}\left(e^{a\pi} - e^{-a\pi}\right)}$$