START: 9 15

SUMMARY LAST LECTURE

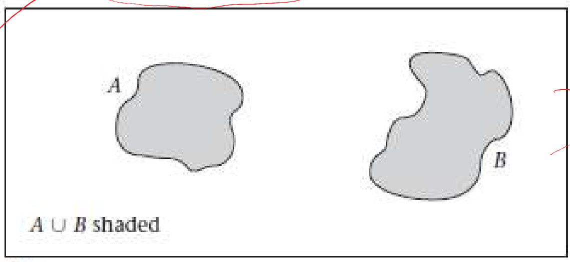
DEFINITION 3.2 Two events A and B are **mutually exclusive** if they cannot both happen at the same time.

DEFINITION 3.3 The symbol { } is used as shorthand for the phrase "the event."

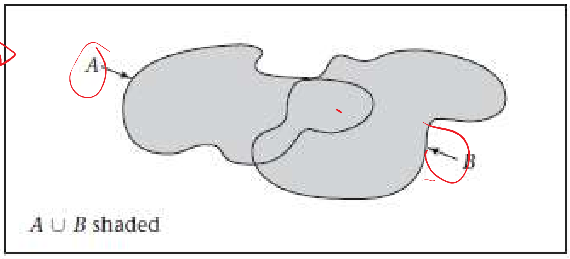
DEFINITION 3.4 $A \cup B$ is the event that either A or B occurs, or they both occur.

Figure 3.1 diagrammatically depicts $A \cup B$ both for the case in which A and B are and are not mutually exclusive.

FIGURE 3.1 Diagrammatic representation of $A \cup B$: (a) A, B mutually exclusive; (b) A, B not mutually exclusive



(a)



(b)

To formulate the postulates of probability, we shall follow the practice of denoting events by means of capital letters, and we shall write the probability of event A as P(A).

$P(A) =$

The following postulates of probability apply only to discrete sample spaces, S.

The Postulates of Probability

1. The probability of an event is a nonnegative real number; that is, $\boxed{P(A) \geq 0}$ for any subset A of S.
2. P(S) = 1.
3. If A1, A2, A3, · · · , is a finite or infinite sequence of mutually exclusive events of S, then P(A1 ∪ A2 ∪ A3 ∪ · · ·) = P(A1) + P(A2) + P(A3) + · · · .

Ex. If we twice flip a balanced coin, what is the probability of getting at least one head?

$S = [HH, TT, HT, TH]$

$P(A) = P(HH) + P(HT) + P(TH)$

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

Solution.
The sample space is S = {HH,HT,TH,TT}.
Since we assume that the coin is balanced, these outcomes are equally likely and we assign to each sample point the probability 1/4 .
Letting A denote the event that we will get at least one head, we get A = {HH,HT,TH} and
P(A) = P(HH) + P(HT) + P(TH)

$$P(A) = P(HH) + P(HT) + P(TH)$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

If an experiment can result in any one of **N** different equally likely outcomes, and

if **n** of these outcomes together constitute event **A**,
then the probability of event **A** is

$$P(A) = \frac{n}{N}.$$

If A and A′ are complementary events in a sample space S, then

$$P(A') = 1 - P(A).$$

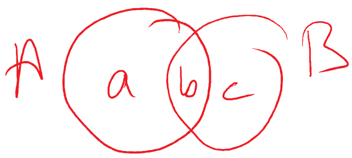
$$0 \le P(A) \le 1 \text{ for any event } A.$$

THE ADDITION LAW OF PROBABILITY

We have seen from the definition of probability that if A and B are mutually exclusive events, then $Pr(A \cup B) = Pr(A) + Pr(B)$. A more general formula for $Pr(A \cup B)$ can be developed when events A and B are not necessarily mutually exclusive. This formula, the *addition law of probability*, is stated as follows:

If A and B are any two events in a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



$P(A \cup B) = a + b + c$

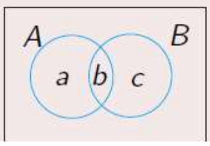
$P(A) = a + b$

$P(B) = b + c$

$P(A \cap B) = b$

Proof.

Assigning the probabilities a, b, and c to the mutually exclusive events $A \cap B$, $A \cap B'$, and $A' \cap B$ as in the figure we find that



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$a+b + b+c - b$

$= a+b+c$

Proof.

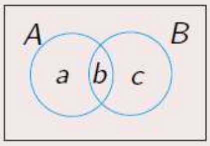
Assigning the probabilities a, b, and c to the mutually exclusive events $A \cap B$, $A \cap B'$, and $A' \cap B$ as in the figure we find that

$P(A \cap B) = b$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$a+b+b+c-b$

$= a+b+c$



$$\begin{aligned} P(A \cup B) &= a + b + c \\ &= (a + b) + (c + b) - b \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

In a large metropolitan area, the probabilities are 0.86, 0.35, and 0.29 that a family (randomly chosen for a sample survey) owns a LCDTV set, a HDTV set or both kinds of sets. What is the probability that a family owns either or both kinds of sets?

Solution. Let $A = \{$a family owns LCDTV$\}$, $B = \{$a family owns HDTV$\}$. Since $P(A) = 0.86$, $P(B) = 0.35$, and $P(A \cap B) = 0.29$, thus

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.86 + 0.35 - 0.29 \\ &= 0.92. \end{aligned}$$

$\boxed{\%92}$

$P(A) = 0.86$

$P(B) = 0.35$

$P(A \cap B) = 0,29$

$P(A \cup B) =$

You roll two dice. What is the probability of the events:

(a) They show the same?
(b) Their sum is seven or eleven?
(c) They have no common factor greater than unity?

Solution First, we must choose the sample space. A natural representation is as ordered pairs of numbers (i, j), where each number refers to the face shown by one of the dice. We require the dice to be distinguishable (one red and one green, say) so that $1 \le i \le 6$ and $1 \le j \le 6$. Because of the symmetry of a perfect die, we assume that these 36 outcomes are equally likely.

(a) Of these 36 outcomes, just six are of the form (i, i), so using (1.3.1) the required probability is

$$\frac{|A|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}.$$

$$\frac{|A|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}.$$

(b) There are six outcomes of the form $(i, 7 - i)$ whose sum is 7, so the probability that the sum is 7 is $\frac{6}{36} = \frac{1}{6}$.

There are two outcomes whose sum is 11—namely, (5, 6) and (6, 5)—so the probability that the sum is 11 is $\frac{2}{36} = \frac{1}{18}$.

Hence, using (1.4.3), the required probability is $\frac{1}{6} + \frac{1}{18} = \frac{2}{9}$.

(c) It is routine to list the outcomes that do have a common factor greater than unity. They are 13 in number, namely:

$$\{(i, i); i \geq 2\}, (2, 4), (4, 2), (2, 6), (6, 2), (3, 6), (6, 3), (4, 6), (6, 4).$$

This is the complementary event, so by (1.4.5) the required probability is

$$1 - \frac{13}{36} = \frac{23}{36}.$$

Doing it this way gives a slightly quicker enumeration than the direct approach.

1.12 Example: Family Planning

A woman planning her family considers the following schemes on the assumption that boys and girls are equally likely at each delivery:

(a) Have three children.
(b) Bear children until the first girl is born or until three are born, whichever is sooner, and then stop.
(c) Bear children until there is one of each sex or until there are three, whichever is sooner, and then stop.

Let B_i denote the event that i boys are born, and let C denote the event that more girls are born than boys. Find $\mathbf{P}(B_1)$ and $\mathbf{P}(C)$ in each of the cases (a) and (b).

Solution (a) If we do not consider order, there are four distinct possible families: *BBB, GGG, GGB,* and *BBG*, but these are not equally likely. With order included, there are eight possible families in this larger sample space:

(1) $$\{BBB; BBG; BGB; GBB; GGB; GBG; BGG; GGG\} = \Omega$$

and by symmetry they are equally likely. Now, by (1.3.1), $\mathbf{P}(B_1) = \frac{3}{8}$ and $\mathbf{P}(C) = \frac{1}{2}$. The fact that $\mathbf{P}(C) = \frac{1}{2}$ is also clear by symmetry.

Now consider (b). There are four possible families: $F_1 = G$, $F_2 = BG$, $F_3 = BBG$, and $F_4 = BBB$.

Once again, these outcomes are not equally likely, but as we have now done several times we can use a different sample space. One way is to use the sample space in (1), remembering that if we do this then some of the later births are fictitious. The advantage is that outcomes are equally likely by symmetry. With this choice, F_2 corresponds to $\{BGG \cup BGB\}$ and so $\mathbf{P}(B_1) = \mathbf{P}(F_2) = \frac{1}{4}$. Likewise, $F_1 = \{GGG \cup GGB \cup GBG \cup GBB\}$ and so $\mathbf{P}(C) = \frac{1}{2}$.

(2) **Exercise** Find $\mathbf{P}(B_1)$ and $\mathbf{P}(C)$ in case (c).

(3) **Exercise** Find $\mathbf{P}(B_2)$ and $\mathbf{P}(B_3)$ in all three cases.

(4) **Exercise** Let E be the event that the completed family contains equal numbers of boys and girls. Find $\mathbf{P}(E)$ in all three cases.