MAT1071 MATHEMATICS I

EXAMPLES 2

Asst. Prof. Dr. Nurten GÜRSES

EXAMPLE

$$\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x} = ? \quad \frac{0}{0} \text{ b}$$

Solution

$$\lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \lim_{x\to 0} \frac{\sin x - \sin x \cos x}{\cos x . \sin^3 x}$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{\cos x . \sin^2 x}$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{\cos x . (1 - \cos^2 x)}$$

$$= \lim_{x\to 0} \frac{1}{\cos x (1 + \cos x)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

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EXAMPLE Prove that $\lim_{x \to 10} (3x+5) = 35$.

Solution

. Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x-10| < \delta$, then $|f(x)-35| < \epsilon$. Begin with $|f(x)-35| < \epsilon$ and ``solve for'' $|x-10|$. Then,

$$|f(x)-35| < \epsilon \implies |(3x+5)-35| < \epsilon$$
$$\implies |3x-30| < \epsilon$$
$$\implies |3(x-10)| < \epsilon$$
$$\implies |3|\,|x-10| < \epsilon$$
$$\implies |x-10| < \frac{\epsilon}{3}.$$

Now choose $\delta = \frac{\epsilon}{3}$. Thus, if $0 < |x-10| < \frac{\epsilon}{3}$, it follows that $|f(x)-35| < \epsilon$. This completes the proof.

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EXAMPLE $y = \sin(x+y)$ olduğuna göre $y'' = ?$

Solution

$$y' = \cos(x+y) \cdot (1+y')$$

$$y' = \cos(x+y) + y' \cos(x+y)$$

$$y' = \frac{\cos(x+y)}{1-\cos(x+y)}$$

$$y'' = \frac{\left[-(1+y')\sin(x+y)(1-\cos(x+y))\right] - \left[\cos(x+y)\cdot(1+y')\cdot\sin(x+y)\right]}{\left[1-\cos(x+y)\right]^2}$$

$$y'' = \frac{-(1+y')\sin(x+y) + (1+y')\sin\cancel{(x+y)\cos(x+y)} - \cancel{\cos(x+y)(1+y')}\sin(x+y)}{\left[1-\cos(x+y)\right]^2}$$

$$y'' = \frac{-(1+y')\sin(x+y)}{\left[1-\cos(x+y)\right]^2} = \frac{-\left(1+\dfrac{\cos(x+y)}{1-\cos(x+y)}\right)\cdot\sin(x+y)}{\left[1-\cos(x+y)\right]^2}$$

$$y'' = -\frac{\sin(x+y)}{\left[1-\cos(x+y)\right]^2}$$ bulunur.

EXAMPLE $y = \sin x$ Find a formula for $y^{(n)}$

Solution $y = \sin x \Rightarrow y' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$ "$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos\alpha$"

$$y'' = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + \frac{2\pi}{2}\right)$$

$$y''' = \cos\left(x + \frac{2\pi}{2}\right) = \sin\left(x + \frac{2\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + \frac{3\pi}{2}\right)$$

$$\vdots$$

$$y^{(n-1)} = \sin\left(x + (n-1)\frac{\pi}{2}\right)$$

$$y^{(n)} = \cos\left(x + (n-1)\frac{\pi}{2}\right) = \sin\left(x + \frac{(n-1)\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + n\frac{\pi}{2}\right)$$

O halde; $y^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)$ bulunur.

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Use chain rule to calculate $\frac{dy}{dt}$

EXAMPLE Zincir kuralını kullanarak, eğer $x \neq -1$ ve $t \neq -1$ olmak üzere,

$$y = \frac{x}{x+1} \quad \text{ve} \quad x = t^2 + 2t \quad \text{ise} \quad \frac{dy}{dt} \text{ türevini bulunuz.}$$

Solution

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{(x+1)-x}{(x+1)^2}(2t+2) = \frac{1}{(x+1)^2}.(2t+2) = \frac{1}{(t^2+2t+1)^2}.(2t+2)$$

$$= \frac{1}{(t+1)^4}.2.(t+1)$$

$$= \frac{2}{(t+1)^3} \text{ bulunur.}$$

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EXAMPLE $f(x)=\dfrac{\sec x}{1+\tan x}$ Find the x values for which $f(x)$ has horizontal tangent.

Solution

$$f'(x)=\frac{(\sec x)'.(1+\tan x)-\sec x(1+\tan x)'}{(1+\tan x)^2}$$

$$=\frac{\sec x(\tan x-1)}{(1+\tan x)^2}=0$$

$$\Rightarrow \underbrace{\sec x}_{\frac{1}{\cos x}\neq 0}(\tan x-1)=0 \Rightarrow \tan x=1$$

$$x=\frac{\pi}{4}+n\pi$$

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EXAMPLE Prove that $\lim_{x \to 2} x^3 = 8$. (This proves that x^3 is continuous at $x = 2$.)

Solution Let $\varepsilon > 0$. We want $\delta > 0$ such that for any $x \in \mathbb{R}$, $0 < |x-2| < \delta$ forces $|x^3 - 8| < \varepsilon$.

$$|x^3 - 8| = |x-2||x^2 + 2x + 4| \quad \text{(factoring)}$$

let chose $0 < \delta < 1 \Rightarrow |x-2| < \delta < 1 \Rightarrow 1 < x < 3$

$$\begin{aligned} 1 < x < 3 &\Rightarrow 1 < x^2 < 9 \\ 1 < x < 3 &\Rightarrow 2 < 2x < 6 \end{aligned} \quad \Rightarrow \quad |x^2 + 2x + 4| < 19$$

$$\begin{aligned} |x^3 - 8| &= |x-2||x^2 + 2x + 4| \\ &< 19\,\delta \qquad (|x^2 + 2x + 4| < 19 \text{ if } |x-2| < 1) \end{aligned}$$

$$\delta = \frac{\varepsilon}{19}$$

$$\boxed{\delta = \min\left\{\frac{\varepsilon}{19}, 1\right\}}$$

gets the job done.

It follows that $\lim_{x \to 2} x^3 = 8$.

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Exam Q.

Find the domain $f(x)=\dfrac{\ln(18-2x^2)}{|2x-5|}+\arcsin(x-3)$

Solution

$18-2x^2>0$

$18>2x^2$

$9>x^2$

$(-3,3)$

$2x-5\neq 0$

$x\neq \frac{5}{2}$

$-1\leq x-3\leq 1$

$2\leq x\leq 4$

$[2,4]$

T.K : $[2,\frac{5}{2})\cup(\frac{5}{2},3)$

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Exam Q.

1. b) $f(x)=\dfrac{|x^2-9|}{x^2-4x+3}$ ile tanımlı f fonksiyonunun süreksiz olduğu tüm noktaları bulunuz. Bulduğunuz bu süreksizlik noktalarını sınıflandırınız. *(12 Puan)*

Find the points where f is disc. and classify these points.

Solution $x^2-4x+3=0 \Rightarrow x=3,\ x=1$ için süreksiz,

$x=1$ için: $\lim\limits_{x\to1^-} \dfrac{-\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x-1)}=+\infty$, $\lim\limits_{x\to1^+} \dfrac{-\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x-1)}=-\infty$

$\emptyset 2$

$\Rightarrow x=1$, infinite disc.

$x=3$ için: $\lim\limits_{x\to3^+} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x-1)}=3$, $\lim\limits_{x\to3^-} \dfrac{-\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x-1)}=-3$

$\emptyset 2$

$\lim\limits_{x\to3^+} f(x) \neq \lim\limits_{x\to3^-} f(x) \Rightarrow x=3$

jump disc. at $x=3$

Exam Q.

2. a) g ve h fonksiyonları, $\boxed{g(1) = h'(1) = 1, \quad g'(1) = h(1) = 2}$ şartlarını sağlayan pozitif de[ğerli] ve türevlenebilen birer fonksiyon olmak üzere, f fonksiyonu da $f(x) = [g(x^2)]^{h(x)}$ ile tanımlı olsun. Buna göre $f'(1)$ değerini bulunuz. *(12 Puan)*

calculate

Solution

$$f(x) = [g(x^2)]^{h(x)}$$

$$\ln f(x) = \ln [g(x^2)]^{h(x)}$$

$$\ln f(x) = h(x).\ln[g(x^2)]$$

$$\frac{f'(x)}{f(x)} = h'(x).\ln[g(x^2)] + \frac{2x.g'(x^2)}{g(x^2)}.h(x)$$

$$f(1) = [g(1)]^{h(1)} = 1^2 = 1$$

$$x=1 \Rightarrow \frac{f'(1)}{f(1)} = h'(1).\ln[g(1)] + \frac{2g'(1)}{g(1)}.h(1)$$

$$\frac{f'(1)}{1} = 1.\ln 1 + \frac{2.2}{1}.2$$

$$\boxed{f'(1) = 8}$$

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Exam Q.

4.b) State the Intermediate Value Theorem. Using this theorem, prove that the equation $\cos x = x$ has a solution in the interval $\left[0, \frac{\pi}{2}\right]$. *(10 Pts)*

Solution

Intermediate value theorem

Let $f(x)$ be a function which is cont. on $[a,b]$ and let y_0 be a real number such that $f(a) \le y_0 \le f(b)$. Then there exists at least one $c \in [a,b]$ such that $y_0 = f(c)$.

If $f(a)$ and $f(b)$ has opposite signs then $f(c)=0$

— o —

$f(x) = \cos x - x = 0$ $\quad f(0) = 1$, $\quad f(\pi/2) = -\pi/2$ } opposite signs

then there exist at least c which is solution of the eq. $\cos x = x$ such that $f(c) = \cos c - c = 0$.

$\Rightarrow \cos c = c$ //

Exam Q.

Show that f is a constant function and find this constant value

2. b) $x \geq 0$ için $f(x) = \frac{\pi}{4} + \arctan\sqrt{e^{2x}-1} - \arccos e^{-x}$ ile verilen f fonksiyonunun bir sabit fonksiyon olduğunu gösteriniz ve değerini bulunuz. *(12 Puan)*

Solution

$$f'(x) = \frac{\frac{\cancel{2}e^{2x}}{\cancel{2}\sqrt{e^{2x}-1}}}{\cancel{1}+e^{2x}-\cancel{1}} - \frac{-(-e^{-x})}{\sqrt{1-e^{-2x}}} = \frac{\cancel{e^{2x}}}{\cancel{e^{2x}}\sqrt{e^{2x}-1}} - \frac{e^{-x}}{\sqrt{1-e^{-2x}}}$$

$$= \frac{1}{\sqrt{e^{2x}-1}} - \frac{e^{-x}}{\sqrt{e^{-2x}(e^{2x}-1)}} = \frac{1}{\sqrt{e^{2x}-1}} - \frac{\cancel{e^{-x}}}{\cancel{e^{-x}}\sqrt{e^{2x}-1}} = 0$$

olup $f(x) = C$ dir.

$$f(x) = \frac{\pi}{4} + \arctan\sqrt{e^{2x}-1} - \arccos e^{-x} = C$$

$$f(0) = \frac{\pi}{4} + \underbrace{\arctan 0}_{0} - \underbrace{\arccos 1}_{0} = C \Rightarrow C = \frac{\pi}{4}$$

$$f(x) = \frac{\pi}{4}$$

(Show that $f(x) = \frac{\pi}{4} + \arctan\sqrt{e^{2x}-1} - \arccos e^{-x}$ is a const funct and find its value)

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Exam Q.

3-b) Let g be a continuous but not differentiable function at $x = 0$ and $g(0) = 8$. Consider the function f represented by $f(x) = x.g(x)$. Then find the value of $f'(0)$. (12 Points)

Solution

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h\,g(h) - 0}{h} = g(0) = 8$$

$$\boxed{f'(0) = 8}$$

Caution!

$$f'(x) = g(x) + x\,g'(x)$$

g is not differentiable at $x=0$!

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Exam Q.

4-a) By finding the linearization of $f(x)=(x^3+x-1)^7+\arctan(x^4-1)$ at $x=1$. Calculate the approximate value of $f(1.02)$. (13 Points)

Solution $L(x)=f(1)+f'(1)(x-1)$

$$f'(x)=7(x^3+x-1)^6(3x^2+1)+\frac{4x^3}{1+(x^4-1)^2}$$

$$f(1)=1$$

$$f'(1)=28+4=32$$

$$\boxed{L(x)=1+32(x-1)}$$

$$f(1.02)\approx L(1.02)=1+32(1.02-1)$$

$$=1+0{,}64$$

$$\boxed{f(1.02)\approx 1{,}64}$$

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Exam Q.

4-b) The equation of the normal line to the curve $y=f(x)$ the at $x=1$ is $2x+y-1=0$. There exists inverse $f^{-1}(x)$ at $x=-1$. Then, find $\left(f^{-1}\right)'(-1)$. (12 Points)

Solution $2x+y-1=0 \Rightarrow y=-2x+1$

$m_N=-2 \Rightarrow m_N \cdot m_T=-1$

$x=1$ için $2.1+y-1=0$

$y=-1$

$$\boxed{m_T=\frac{1}{2}=f'(1)}$$

$f(1)=-1$

$$\left(f^{-1}\right)'(-1)=\frac{1}{f'(1)}=\frac{1}{\frac{1}{2}}=2$$

$$\boxed{\left(f^{-1}\right)'(-1)=2}$$

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Exam Q.

2-a) $\boxed{f\left(\frac{\pi}{2}\right)=6 \text{ ve } f'\left(\frac{\pi}{2}\right)=3}$ olacak şekilde bir f fonksiyonu verilsin. Eğer g fonksiyonu

$$g(x)=[f(x)]^{\sin x},$$

ise, $g'(x)$ 'in $x=\frac{\pi}{2}$ 'deki değerini bulunuz (13 Puan).

Find $g'\left(\frac{\pi}{2}\right)$

Solution

$$\ln g(x)=\sin x.\ln(f(x))$$

$$\frac{g'(x)}{g(x)}=\cos x.\ln(f(x))+\sin x\,\frac{f'(x)}{f(x)}$$

$x=\frac{\pi}{2}$ için

$$\frac{g'\left(\frac{\pi}{2}\right)}{g\left(\frac{\pi}{2}\right)}=\left(\cos\frac{\pi}{2}\right)\ln\left(f\left(\frac{\pi}{2}\right)\right)+\left(\sin\frac{\pi}{2}\right).\frac{f'\left(\frac{\pi}{2}\right)}{f\left(\frac{\pi}{2}\right)}$$

$$\frac{g'\left(\frac{\pi}{2}\right)}{6}=0+1.\frac{3}{6} \Rightarrow g'\left(\frac{\pi}{2}\right)=3$$

Exam Q.

approximate value

1.b) $\sqrt[4]{18}$ sayısının yaklaşık değerini lineer yaklaşım veya diferansiyel hesap kullanarak hesaplayınız. (12P)

Solution

$$f(x)=\sqrt[4]{x} \quad , \quad a=16 \quad , \quad f(16)=2$$

$$f'(x)=\frac{1}{4}x^{-\frac{3}{4}} \quad , \quad f'(16)=\frac{1}{32}$$

$$f(x)\approx L(x)=f(16)+f'(16)(x-16)$$

$$L(x)=2+\frac{1}{32}(x-16)$$

$$f(18)\approx L(18)=2+\frac{1}{32}(18-16)=2+\frac{1}{16}=\frac{33}{16}$$

$$f(18)\approx\frac{33}{16}$$

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Exam Q.

2) $f(x)=\begin{cases} \dfrac{1-\cos x^2}{x^4} & , x<0 \\ \dfrac{2}{\pi}\arcsin x+\dfrac{3}{\pi}\arccos x & , 0\le x\le 1 \\ (x-1)\sin\dfrac{1}{x-1} & , x>1 \end{cases}$ fonksiyonu için:

a) $\lim\limits_{x\to 0} f(x)$ ve $\lim\limits_{x\to 1} f(x)$ Examine the limits (do not use Lhopıtal rule) (17P)

Solution

$$\lim_{x\to 0^-}\frac{1-\cos x^2}{x^4}=\lim_{x\to 0^-}\frac{(1-\cos x^2)}{x^4}\cdot\frac{(1+\cos x^2)}{(1+\cos x^2)}=\lim_{x\to 0^-}\underbrace{\frac{\sin^2 x^2}{x^4}}_{1}\cdot\frac{1}{1+\cos x^2}=\frac{1}{2}$$

$$\lim_{x\to 0^+}\left[\frac{2}{\pi}\underbrace{\arcsin x}_{0}+\frac{3}{\pi}\underbrace{\arccos x}_{\frac{\pi}{2}}\right]=\frac{3}{2}$$

$\lim\limits_{x\to 0^+} f(x)\neq\lim\limits_{x\to 0^-} f(x)\Rightarrow$ does not exist

$$\lim_{x\to 1^-}\left[\frac{2}{\pi}\underbrace{\arcsin x}_{\frac{\pi}{2}}+\frac{3}{\pi}\underbrace{\arccos x}_{0}\right]=1$$

$$\lim_{x\to 1^+}\underbrace{(x-1)}_{0}\underbrace{\sin\frac{1}{x-1}}_{\substack{A \\ -1\le A\le 1}}=0$$

$\lim\limits_{x\to 1^+} f(x)\neq\lim\limits_{x\to 1^-} f(x)\Rightarrow$ does not exist

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b) $x = 0$ ve $x = 1$ noktalarında f fonksiyonunun sürekliliğini araştırıp, süreksizlik olması halinde türünü belirleyiniz. (8P)

$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) \Rightarrow$ jump discontinuity at x=0

$\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x) \Rightarrow$ jump discontinuity at x= 1

Exam Q.

By using implicit differentiation, find the normal line

3.a) <u>**Kapalı Türetme Yöntemini kullanarak,**</u> $2x+\cos(x+y)=y^2-\pi$ ile kapalı olarak tanımlı $y=f(x)$ eğrisinin $\left(-\frac{\pi}{2},0\right)$ noktasındaki normal doğrusunun denklemini bulunuz. **(12P)**

Solution

$$2-(1+y')\sin(x+y)=2y\cdot y'$$

$x=-\frac{\pi}{2}$ ve $y=0$ için;

$$2+(1+y')=0$$

$$m_T=y'=-3 \Rightarrow \underline{m_N=\frac{1}{3}}$$

N.D.D. $\quad y-0=\frac{1}{3}\left(x-\left(-\frac{\pi}{2}\right)\right)$

$$\underline{y=\frac{1}{3}x+\frac{\pi}{6}} \;//$$

Exam Q.

$f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=x.\sec x$ Show that the inverse of f exist and calculate $(f^{-1})'\left(\frac{2\pi}{3}\right)$

Solution

$f'(x)=\sec x+x.\sec x.\tan x>0 \quad \left(\forall x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$

olduğundan f artandır $\Rightarrow$ f , 1-1 $\Rightarrow$ f 'in tersi mevcuttur.

$(f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}$, $(f^{-1})'\left(\frac{2\pi}{3}\right)=\frac{1}{f'(\underbrace{f^{-1}(\frac{2\pi}{3})}_{a})}$

$f^{-1}\left(\frac{2\pi}{3}\right)=a \Rightarrow f(a)=\frac{2\pi}{3}$

$a.\sec a=\frac{2\pi}{3}$

$a=\frac{\pi}{3}$,

$=\frac{1}{f'(\frac{\pi}{3})}$

$=\frac{3}{6+2\sqrt{3}\pi}$ //

$f'\left(\frac{\pi}{3}\right)=\sec\frac{\pi}{3}+\frac{\pi}{3}\sec\frac{\pi}{3}\tan\frac{\pi}{3}$

$=2+\frac{\pi}{3}\cdot 2\cdot\sqrt{3}$

$=\frac{6+2\sqrt{3}\pi}{3}$

Exam Q.

4.a) $\lim\limits_{x\to 0}\left(1+\arctan x\right)^{\frac{1}{x^2+2x}}$ limitini hesaplayınız. **(14P)**

Solution $y=\left(1+\arctan x\right)^{\frac{1}{x^2+2x}}$

$$\ln y=\frac{1}{x^2+2x}\cdot\ln\left(1+\arctan x\right)$$

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln\left(1+\arctan x\right)}{x^2+2x}=\left(\frac{0}{0}\text{ bl.}\right)$$

$$\overset{L'H}{=}\lim_{x\to 0}\frac{\dfrac{\frac{1}{1+x^2}}{1+\arctan x}}{2x+2}=\frac{1}{2}$$

$$\lim_{x\to 0}\ln y=\frac{1}{2}$$

$$\ln\left(\lim_{x\to 0}y\right)=\frac{1}{2}\Rightarrow\lim_{x\to 0}\left(1+\arctan x\right)^{\frac{1}{x^2+2x}}=e^{1/2}\ //$$

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Exam Q.

Show that there is a root of the equation $\sqrt[3]{x-1} - 3x = 0$ in $\left[-\frac{1}{2}, 0\right]$

Solution

$f(x) = \sqrt[3]{x-1} - 3x$ fonksiyonu $\left[-\frac{1}{2}, 0\right]$ aralığında süreklidir.

$f(0) = -1$

$f(-\frac{1}{2}) = \sqrt[3]{-\frac{3}{2}} + \frac{3}{2} = \frac{3}{2} - \sqrt[3]{\frac{3}{2}} = a > 0$

olayısıyla f $[-1, a]$ aralığında
er değeri alır. O halde $0 \in [-1, a]$
eğerini de alır. Yani $f(c) = 0$
lacak şekilde $c \in \left[-\frac{1}{2}, 0\right]$ vardır.

veya

$f(0) = -1 < f(c) = 0 < f(-\frac{1}{2}) = a$
olduğundan Ara Değer Teoremine göre
$f(c) = 0$ olacak şekilde $c \in \left[-\frac{1}{2}, 0\right]$
vardır.

mean value theorem

Exam Q.

$f(x) = x^{\sin x}$ $\qquad f'\left(\frac{\pi}{2}\right) = ?$

Solution

$$f(x) = x^{\sin x}$$

$$2 \quad \ln f(x) = \sin x \cdot \ln x \qquad , \; f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad 1$$

$$2 \quad \frac{f'(x)}{f(x)} = \overset{1}{\cos x \cdot \ln x} + \overset{1}{\sin x \cdot \frac{1}{x}}$$

$$f'(x) = f(x) \cdot \left[\cos x \cdot \ln x + \frac{\sin x}{x}\right]$$

$$f'\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) \cdot \left[\underbrace{\cos\frac{\pi}{2} \cdot \ln\frac{\pi}{2}}_{0 \quad 1} + \underbrace{\sin\frac{\pi}{2}}_{1} \cdot \frac{2}{\pi}\right] \quad 1$$

$$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cdot \frac{2}{\pi} = 1 \quad 1$$

Exam Q.

Show that f is a constant function and find this constant value

$$x>0 \text{ için } f(x)=\arcsin\frac{x-1}{x+1}-2\arctan\sqrt{x}$$

Solution

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\cdot\frac{x+1-(x-1)}{(x+1)^2}-2\,\frac{1}{2\sqrt{x}}\cdot\frac{1}{1+(\sqrt{x})^2}$$

2 2

$$=\frac{\cancel{x+1}}{\sqrt{(x+1)^2-(x-1)^2}}\cdot\frac{2}{(x+1)^{\cancel{2}}}-\frac{1}{(1+x)\sqrt{x}}$$

1

$$=\frac{1}{\cancel{2}\sqrt{x}}\cdot\frac{\cancel{2}}{(x+1)}-\frac{1}{(1+x)\sqrt{x}}=0 \Rightarrow f(x) \text{ sobit fonksiyon}$$

1 1

$f(1)=c$ olmalı 1

$$f(1)=\arcsin 0-2\arctan 1=0-2\cdot\frac{\pi}{4}=-\frac{\pi}{2}$$ 3

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Exam Q.

5.) $f(x)=\begin{cases} x.e^{-\frac{1}{x^2}} & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$ Find $f'(0)$ by using definition of derivative.

hesaplayınız. **(10P)**

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \quad 4$$

$$=\lim_{h\to 0}\frac{h.e^{-\frac{1}{h^2}}}{h}=\lim_{h\to 0}\frac{1}{e^{1/h^2}}=0 \;//$$

$$2 \qquad 2 \qquad 2$$

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Exam Q.

Exam & Ex: $f(x) = (2+\arcsin x)^{\tan x}$ $|x| < 1 \Rightarrow f'(0) = ?$

log. diff.

$\ln f(x) = \ln (2+\arcsin x)^{\tan x}$

$\frac{f'(x)}{f(x)} = [\tan x \cdot \ln (2+\arcsin x)]'$

$= \sec^2 x \cdot \ln(2+\arcsin x) + \tan x \cdot \frac{\frac{1}{\sqrt{1-x^2}}}{2+\arcsin x}$

$\Rightarrow \boxed{f'(0) = \ln 2}$ //

Exam Q.

S.S SORU $\lim_{x\to 0^+} (\cos x)^{\cot x} = ?$ (1^∞) (Exam Q)

$$\ln \lim_{x\to 0^+} (\cos x)^{\cot x} = \ln A$$

$$\Rightarrow \lim_{x\to 0^+} \cot x \,\ln(\cos x) \quad (0\cdot\infty)$$

$$\Rightarrow \lim_{x\to 0^+} \frac{\ln(\cos x)}{\tan x} \quad \frac{0}{0}$$

$$\Rightarrow \lim_{x\to 0^+} \frac{-\sin x}{\cos x\,(1+\tan^2 x)} = \frac{0}{1} = 0$$

$\ln A = 0$

$e^0 = A$

$A = 1$

Exam Q.

S.S

SORU $\lim\limits_{x\to 1^+} \left(\tan \frac{\pi x}{4}\right)^{\frac{3}{x-1}}$ limitini hesaplayınız. 1^{∞}

(Exam Q)

$$\lim_{x\to 1^+} \frac{3}{x-1} \ln\left(\tan \frac{\pi x}{4}\right) \quad \frac{0}{0}$$

$$3 \lim_{x\to 1^+} \frac{\dfrac{1+\tan^2 \frac{\pi x}{4}}{\tan \frac{\pi x}{4}} \cdot \dfrac{\pi}{4}}{1} = \frac{3\pi}{2}$$

$$e^{3\pi/2} \Rightarrow \text{cevap} \; //$$

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Exam Q.

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{\cot^2 x}$$ Calculate without using LHopital rule.

Solution

$$\lim_{x\to\frac{\pi}{2}} \frac{(\sin x - 1)(\sin x + 1)}{\cot^2 x \cdot (\sin x + 1)}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin^2 x - 1}{\underbrace{\cot^2 x}_{\cos^2 x} (\sin x + 1)}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{-\sin^2 x}{\sin x + 1} = -\frac{1}{2}$$

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Exam Q.

$$f(x)=\begin{cases}\sin 2x & ,x\le 0\\ mx & ,x>0\end{cases}$$

Find the value of m for which f is differentiable at x=0.

Solution

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{mx-0}{x}=m$$

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{\sin 2x-0}{x}=$$

$m=2$

Exam Q.

$$f(x) = \begin{cases} ax+b & , x<0 \\ 2\sin x + 3\cos x & ; x \geq 0 \end{cases}$$

Find the values of a and b for which f is differentiable at x=0.

Solution

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{2\cosh - 3\sinh}{1} = 2$$

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{ah+b-3}{h} = a$$

$f'_+(0) = f'_-(0)$

a=2

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2\sin x + 3\cos x = 3$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} ax+b = b$$

$b = 3$

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Exam Q.

$$f(x)=\begin{cases}\frac{1}{4}x^3-\frac{1}{2}x^2, & x\geq 2\\ -\frac{6x+6}{x^2+2}, & x<2\end{cases}$$

Examine the derivative of f at $x=2$

Solution

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\left(\frac{1}{4}x^3-\frac{1}{2}x^2\right)=0,$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\left(-\frac{6x+6}{x^2+2}\right)=-3$$

the limit does not exist $\Rightarrow$ f is not cont $\Rightarrow$ f is not differentiable

Wrong Calculation: $x>2$ için $f'(x)=\frac{3}{4}x^2-x \Rightarrow \lim_{x\to 2^+} f'(x)=1$ ve

$x<2$ için $f'(x)=\frac{6x^2+12x-1}{(x^2+2)^2} \Rightarrow \lim_{x\to 2^-} f'(x)=1$ buradan

$f'(2)=1$ dir.

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Exam Q.

$$\lim_{x\to 1^+}\left(\tan\frac{\pi x}{4}\right)^{\frac{1}{x-1}}$$

Solution

$$e^{\frac{3\pi}{2}}$$

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Exam Q.

$\lim\limits_{x\to 0} x^2 \cos(x+\frac{1}{x^3}) = ?$ Hint:Use the sandwich theorem

Solution

- $\forall\theta\in\mathbb{R}$ için $-1\leq\cos\theta\leq 1$ olduğundan $-1\leq\cos(x+\frac{1}{x^3})\leq 1,\ x\neq 0.$
- $-x^2\leq x^2\cos(x+\frac{1}{x^3})\leq x^2,\ x\neq 0$
- $\lim\limits_{x\to 0}(-x^2)=\lim\limits_{x\to 0}(x^2)=0$ olduğundan Sandviç teoremi gereğince
- $\lim\limits_{x\to 0} x^2 \cos(x+\frac{1}{x^3})=0.$

Exam Q.

$$f(x) = \begin{cases} x^2 - 2, & x \le 1 \\ 2x - 3, & x > 1 \end{cases}$$

determine differentiability of f

Solution

$$\begin{aligned} f'_-(1) &= \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} \qquad [f(x) = x^2 - 2] \\ &= \lim_{h\to 0^-} \frac{[(1+h)^2 - 2] - (1-2)}{h} \\ &= \lim_{h\to 0^-} (2+h) = 2, \end{aligned}$$

$$\begin{aligned} f'_+(1) &= \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} \qquad [f(x) = 2x - 3] \\ &= \lim_{h\to 0^+} \frac{[2(1+h) - 3] - (2-3)}{h} \\ &= \lim_{h\to 0^+} 2 = 2 \end{aligned}$$

$$f'_-(1) = 2 = f'_+(1)$$

differentiable for all x

Exam Q.

$$\lim_{x\to -1} \frac{1+x}{1+x\sqrt{2+x}}$$ Calculate without using LHopital rule.

Solution

$$\lim_{x\to -1} \frac{1+x}{1+x\sqrt{2+x}} = \lim_{x\to -1} \frac{1+x}{1+x\sqrt{2+x}} \frac{1-x\sqrt{2+x}}{1-x\sqrt{2+x}}$$

$$= \lim_{x\to -1} \frac{(1+x)(1-x\sqrt{2+x})}{1-2x^2-x^3}$$

$$= \lim_{x\to -1} \frac{(1+x)(1-x\sqrt{2+x})}{(1+x)(1-x-x^2)}$$

$$= \lim_{x\to -1} \frac{1-x\sqrt{2+x}}{1-x-x^2}$$

$$= 2$$

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Exam Q.

$$f(x)=\begin{cases}\dfrac{\pi}{2} & ,x=0\\ \sin\left(\dfrac{x}{3}\right) & ,0<x<3\\ 2^{\frac{1}{x-4}} & ,3\le x<4 \text{ ve } 4<x\le 5\\ \dfrac{\pi}{2} & ,x=4\end{cases}$$

Determine the types of discont. at x=0,x=3, x=4.

Solution

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\sin\left(\frac{x}{3}\right)=0 \qquad f(0)=\frac{\pi}{2}$$

$\Longrightarrow$ removable discontinuity at x=0

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} 2^{\frac{1}{x-4}}=\frac{1}{2},\quad \lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}\sin\left(\frac{x}{3}\right)=\sin 1 \Rightarrow \text{ jump disc at x=3}$$

$$\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+} 2^{\frac{1}{x-4}}=\infty,\quad \lim_{x\to 4^-} f(x)=\lim_{x\to 4^-} 2^{\frac{1}{x-4}}=0 \Rightarrow \text{ infinite disc at x=4}$$

Exam Q.

$$f(x) = e^{\frac{3x+4}{x-2}}$$

Examine the limit and cont of f at x=2

Solution

$$\lim_{x \to 2^+} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f(2+h) = \lim_{h \to 0} e^{\frac{10+h}{h}} = \infty$$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} e^{\frac{10-h}{-h}} = 0$$

the limit does not exist $\Rightarrow$ f is not cont

EXAMPLE

Find the domain of the function

$$f(x) = \arcsin\left(\log \frac{x}{5}\right)$$

Solution

$$-1 \leq \log \frac{x}{5} \leq 1 \quad \text{and} \quad \frac{x}{5} > 0$$

$$10^{-1} \leq \frac{x}{5} \leq 10 \qquad \boxed{x > 0}$$

$$\boxed{\frac{1}{2} \leq x \leq 50}$$

intersection $D_f = \left[\frac{1}{2}, 50\right]$

EXAMPLE

$$f_{\alpha,\beta} = \begin{cases} x^{\alpha} \sin x^{\beta}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$

$\alpha \geq 0 \quad \beta \in \mathbb{R}$

Is $f_{0,-1}$ cont. on $\mathbb{R}$ or not?

Solution

$$f_{0,-1} = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

for $x=0$ $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

it's not cont. on $\mathbb{R}$.

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EXAMPLE $\lim\limits_{x\to 0} \dfrac{1-\sqrt{2-\cos x}}{\sin x} = ?$

Solution $\dfrac{0}{0} \overset{LH}{\Rightarrow} \lim\limits_{x\to 0} \dfrac{-\dfrac{\sin x}{2\sqrt{2-\cos x}}}{\cos x} = \lim\limits_{x\to 0} \dfrac{-\sin x}{2\sqrt{2-\cos x}\cdot\cos x} = 0$

Without using LH

$$\lim_{x\to 0} \frac{(1-\sqrt{2-\cos x})}{\sin x} \cdot \frac{(1+\sqrt{2-\cos x})}{(1+\sqrt{2-\cos x})}$$

$$\lim_{x\to 0} \frac{1-2+\cos x}{\underbrace{\sin x}_{\sqrt{1-\cos^2 x}}\,(1+\sqrt{2-\cos x})} = \lim_{x\to 0} \frac{\overset{-1}{\cancel{\sqrt{\cos x-1}}}\,\sqrt{\cos x-1}}{\cancel{\sqrt{1-\cos x}}\,\sqrt{1+\cos x}\,(1+\sqrt{2-\cos x})}$$

$$= \lim_{x\to 0} \frac{-\sqrt{\cos x-1}}{\underbrace{\sqrt{1+\cos x}\,(1+\sqrt{2-\cos x})}_{2\sqrt{2}}} = 0$$

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EXAMPLE $f(x) = x^{\frac{1}{3}}(x-4)$

identify the intervals on which f is increasing and decreasing.

Solution $D_f = \mathbb{R}$

$D_f = (-\infty, +\infty)$

$\downarrow$

no endpoints

$$f'(x) = \frac{4(x-1)}{3\sqrt[3]{x^2}}$$

Critical points

$\rightarrow f' = 0 \Rightarrow \boxed{x = 1}$

$\rightarrow f'$ is undef. $\Rightarrow$ $\boxed{x = 0 \text{ two fold}}$

x	$-\infty$		0		1		$+\infty$
f'		$-$	‖	$-$	0	$+$	
f		↘	‖	↘		↗	

f is increase on $(1, \infty)$

f is decreasing on $(-\infty, 0) \cup (0, 1)$

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EXAMPLE $\lim_{x\to\infty}\left(1+2^x+3^x\right)^{\frac{1}{x}}=?$ $\qquad \infty^0$

Solution $y=\left(1+2^x+3^x\right)^{\frac{1}{x}}$

$$\ln y=\frac{1}{x}\ln\left(1+2^x+3^x\right)$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(1+2^x+3^x\right)}{x}\quad\left(\frac{\infty}{\infty}\right)$$

$$\overset{LH}{=}\lim_{x\to\infty}\frac{2^x.\ln 2+3^x.\ln 3}{1+2^x+3^x}$$

$$=\lim_{x\to\infty}\frac{\cancel{3^x}\left[\left(\frac{2}{3}\right)^{x\to 0}.\ln 2+\ln 3\right]}{\cancel{3^x}\left[\frac{1}{3^x}+\underbrace{\left(\frac{2}{3}\right)^x}_{0}+1\right]}=\ln 3$$

$$\lim_{x\to\infty}\left(1+2^x+3^x\right)^{\frac{1}{x}}=\lim_{x\to\infty}e^{\ln\left(1+2^x+3^x\right)^{\frac{1}{x}}}=e^{\ln 3}$$

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EXAMPLE $f(x) = (\cos x^4)^{\arctan x^2}$, Calculate $f'(0)$.

Solution

$$\ln(f(x)) = \ln\left[(\cos x^4)^{\arctan x^2}\right]$$

$$\ln(f(x)) = \arctan x^2 \cdot \ln(\cos x^4)$$

$$\frac{f'(x)}{f(x)} = \frac{2x}{1+x^4} \cdot \ln\cos(x^4) + \arctan x^2 \cdot \frac{-4x^3 \cdot \sin x^4}{\cos x^4}$$

$$\Rightarrow \frac{f'(0)}{f(0)} = 0 \Rightarrow \boxed{f'(0) = 0}$$

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EXAMPLE $\lim_{x\to 0} (e^x - x)^{\frac{2}{\sin x^2}} = ? \quad 1^\infty$

Solution

$$y = (e^x - x)^{\frac{2}{\sin x^2}}$$

$$\ln y = \frac{2}{\sin x^2} \cdot \ln(e^x - x)$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{2\ln(e^x - x)}{\sin x^2} \quad \left(\frac{0}{0}\right)$$

$$\stackrel{LH}{=} \lim_{x\to 0} \frac{\cancel{2}\,\frac{e^x - 1}{e^x - x}}{\cancel{2}x\cos x^2}$$

$$= \lim_{x\to 0} \underbrace{\frac{e^x - 1}{x}}_{1} \cdot \underbrace{\frac{1}{(e^x - x)\cos x^2}}_{1} = 1$$

<u>Note:</u> $\lim_{x\to 0} \frac{e^x - 1}{x} \stackrel{LH}{=} \lim_{x\to 0} \frac{e^x}{1} = 1$

$$\lim_{x\to 0} (e^x - x)^{\frac{2}{\sin x^2}} = e^1 = e$$

EXAMPLE $f(x) = e^{\operatorname{arccot}(\sqrt{2+\ln x})}$ Show that f has an inverse and calculate $(f^{-1})'(e^{\pi/6})$

Solution

$f(x)$ is defined for $2+\ln x \geq 0 \Rightarrow \ln x \geq -2$
$\Rightarrow \boxed{x \geq e^{-2}}$

f is cont. for $x \geq e^{-2}$

$$f'(x) = \left[\operatorname{arccot}(\sqrt{2+\ln x})\right]' e^{\operatorname{arccot}(\sqrt{2+\ln x})}$$

$$f'(x) = \frac{-(\sqrt{2+\ln x})'}{1+(\sqrt{2+\ln x})^2} \cdot e^{\operatorname{arccot}(\sqrt{2+\ln x})}$$

$$f'(x) = -\frac{\frac{1}{x}}{2\sqrt{2+\ln x}} \cdot \frac{1}{3+\ln x} \cdot e^{\operatorname{arccot}(\sqrt{2+\ln x})} < 0$$

$f' < 0 \Rightarrow f$ is decreasing $\Rightarrow f$ is 1-1 $\Rightarrow f$ has an inverse

$f^{-1}(e^{\pi/6}) = a \Rightarrow f(a) = e^{\operatorname{arccot}(\sqrt{2+\ln a})} = e^{\pi/6}$

$\Rightarrow \frac{\pi}{6} = \operatorname{arccot}(\sqrt{2+\ln a})$

$\Rightarrow \cot\frac{\pi}{6} = \sqrt{2+\ln a}$

$\Rightarrow \sqrt{3} = \sqrt{2+\ln a} \Rightarrow \ln a = 1 \Rightarrow \boxed{a = e}$

$$(f^{-1})'(e^{\pi/6}) = \frac{1}{f'(e)} = \frac{-8e\sqrt{3}}{e^{\pi/6}}$$

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EXAMPLE $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin\sqrt{x}} & , x>0 \\ 1-a & , x=0 \\ \dfrac{x-b}{2x-1} & , x<0 \end{cases}$

for which values of a, and b f is cont. at $x=0$?

Solution

$$\lim_{x\to 0^+} \frac{\sin 3x}{\sin\sqrt{x}} = \lim_{x\to 0^+} \underbrace{\frac{\sin 3x}{3x}}_{1} \cdot \underbrace{\frac{\sqrt{x}}{\sin\sqrt{x}}}_{1} \cdot \underbrace{3\sqrt{x}}_{0} = 0$$

$$\lim_{x\to 0^-} \frac{x-b}{2x-1} = b$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) \Rightarrow \boxed{b=0}$$

$$1-a=0 \Rightarrow \boxed{a=1}$$