MAT1071 MATHEMATICS I

7.2 ANTIDERIVATIVES AND INDEFINITE INTEGRALS

Asst. Prof. Dr. Nurten GÜRSES

MAT1071 Ytu Bologna:

9	Asymptotes of Graphs, Curve Sketching, Antiderivatives, Indefinite Integrals, Integral Tables

Asst. Prof. Dr. Nurten GÜRSES

Antiderivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

☆ We need to think backward here: What function do we know has a derivative equal to the given function?

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ **(b)** $g(x) = \cos x$ **(c)** $h(x) = 2x + \cos x$

EXAMPLE Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ **(b)** $g(x) = \cos x$ **(c)** $h(x) = 2x + \cos x$

Solution

(a) $F(x) = x^2$ **(b)** $G(x) = \sin x$ **(c)** $H(x) = x^2 + \sin x$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$ and the derivative of $H(x) = x^2 + \sin x$ is $2x + \cos x$.

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C. Are there others?

☆ If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

EXAMPLE Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

EXAMPLE Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution Since the derivative of x^3 is $3x^2$, the general antiderivative

$$F(x) = x^3 + C$$

gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for C. Substituting $x = 1$ into $F(x) = x^3 + C$ gives

$$F(1) = (1)^3 + C = 1 + C.$$

Since $F(1) = -1$, solving $1 + C = -1$ for C gives $C = -2$. So

$$F(x) = x^3 - 2$$

is the antiderivative satisfying $F(1) = -1$.

Antiderivative formulas, k a nonzero constant

	Function	General antiderivative
1.	x^n	
2.	$\sin kx$	
3.	$\cos kx$	
4.	$\sec^2 kx$	
5.	$\csc^2 kx$	
6.	$\sec kx \tan kx$	
7.	$\csc kx \cot kx$	

Asst. Prof. Dr. Nurten GÜRSES

Antiderivative formulas, k a nonzero constant

	Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule*:	$kf(x)$	$kF(x) + C$, $\quad k$ a constant
2.	*Negative Rule*:	$-f(x)$	$-F(x) + C$
3.	*Sum or Difference Rule*:	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

EXAMPLE Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$ **(b)** $g(x) = \dfrac{1}{\sqrt{x}}$ **(c)** $h(x) = \sin 2x$ **(d)** $i(x) = \cos \dfrac{x}{2}$

EXAMPLE Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$ **(b)** $g(x) = \dfrac{1}{\sqrt{x}}$ **(c)** $h(x) = \sin 2x$ **(d)** $i(x) = \cos \dfrac{x}{2}$

Solution

(a) $F(x) = \dfrac{x^6}{6} + C$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

(c) $H(x) = \dfrac{-\cos 2x}{2} + C$

(d) $I(x) = \dfrac{\sin(x/2)}{1/2} + C = 2\sin\dfrac{x}{2} + C$

EXAMPLE Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$

EXAMPLE Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$

Solution $F(x) = 6\sqrt{x} - \frac{1}{2}\cos 2x + C$

where C is an arbitrary constant.

Indefinite Integrals

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Example $\int 2x\,dx =$

Example $\int \cos x\,dx =$

Example $\int (2x + \cos x)\,dx =$

Example $\int (x^2 - 2x + 5)\,dx =$

Example $$\int 2x\,dx = x^2 + C,$$

Example $$\int \cos x\,dx = \sin x + C,$$

Example $$\int (2x + \cos x)\,dx = x^2 + \sin x + C.$$

Example $$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

Asst. Prof. Dr. Nurten GÜRSES

Basic integration formulas

1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$

Asst. Prof. Dr. Nurten GÜRSES

Basic integration formulas

6. $\displaystyle\int \sin x\, dx = -\cos x + C$

7. $\displaystyle\int \cos x\, dx = \sin x + C$

8. $\displaystyle\int \sec^2 x\, dx = \tan x + C$

9. $\displaystyle\int \csc^2 x\, dx = -\cot x + C$

10. $\displaystyle\int \sec x \tan x\, dx = \sec x + C$

11. $\displaystyle\int \csc x \cot x\, dx = -\csc x + C$

Asst. Prof. Dr. Nurten GÜRSES

Basic integration formulas

12. $\int \tan x \, dx = -\ln|\cos x| + C$

13. $\int \cot x \, dx = \ln|\sin x| + C$

14. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x \, dx = \cosh x + C$

17. $\int \cosh x \, dx = \sinh x + C$

Asst. Prof. Dr. Nurten GÜRSES

Basic integration formulas

18. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

19. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

20. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

21. $$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$$

22. $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$$

Asst. Prof. Dr. Nurten GÜRSES

Example $\int (2x+5)\,dx =$

Example $\int x(x-3)\,dx =$

Example $\int \left(3x - \frac{x^3}{4}\right) dx =$

Example $\int (x^2+\sqrt{x})\,dx =$

Example $\int (2x+5)\,dx = 2\cdot\frac{x^2}{2} + 5x + C$

$x^2 - 3x$

Example $\int x(x-3)\,dx = \frac{x^3}{3} - 3\cdot\frac{x^2}{2} + C$

Example $\int \left(3x - \frac{x^3}{4}\right)dx = 3\cdot\frac{x^2}{2} - \frac{1}{4}\cdot\frac{x^4}{4} + C$

Example $\int (x^2 + \sqrt{x})\,dx = \frac{x^3}{3} + \frac{x^{1/2+1}}{1/2+1} + C$

$x^2 + x^{1/2}$

Example $\int 2\sec^2 x\, dx =$

Example $\int \csc\theta \cot\theta \, d\theta$

Example $\int \tan^2 x\, dx =$

Example $\int \sin 2x\, dx =$

Example $\int 2\sec^2 x\, dx = 2.\tan x + C$

Example $\int \csc\theta \cot\theta\, d\theta = -\csc\theta + C$

Example $\int \tan^2 x\, dx = \int (1+\tan^2 x - 1)\, dx$

$= \int (\sec^2 x - 1)\, dx = \tan x - x + C$

Example $\int \sin 2x\, dx = -\frac{1}{2}\cos 2x + C$

Example $\int (\sqrt{r}+1)^2 \, dr =$

Example $\int \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du =$

Exampl $\int \frac{s^2+\sqrt{3}}{s^2} \, ds =$

r^2+2r+1

Example $\int (r+1)^2 \, dr = \dfrac{r^3}{3} + \dfrac{2r^2}{2} + r + C$

u^{-5}

Example $\int \left(\dfrac{u^7}{2} - \dfrac{1}{u^5} \right) du = \dfrac{1}{2} \dfrac{u^8}{8} - \dfrac{u^{-5+1}}{-5+1} + C$

Exampl $\int \dfrac{s^2+\sqrt{s}}{s^2} \, ds = s + \dfrac{s^{-3/2+1}}{-\frac{3}{2}+1} + C$

$1 + s^{-3/2}$

Example $\int \left(x^3 + e^{3x} + 5^x + \tan\frac{x}{2}\right) dx =$

Example $\int \left(\cos 2x + \frac{x}{x^2-1} + \sec^2 3x\right) dx =$

Example $\int \left(\frac{2}{\sqrt{1-x^2}} + \frac{5}{x^2+4}\right) dx =$

Example $\int \left(\sec x + \cosh x\right) dx =$

Asst. Prof. Dr. Nurten GÜRSES

Example $\int \left(x^3 + e^{3x} + 5^x + \tan\frac{x}{2}\right) dx = \frac{x^4}{4} + \frac{e^{3x}}{3} + \frac{5^x}{\ln 5} - 2\ln\left|\cos\frac{x}{2}\right| + c$

Example $\int \left(\cos 2x + \frac{x}{x^2-1} + \sec^2 3x\right) dx = \frac{1}{2}\sin 2x + \ln|x^2-1| + \frac{1}{3}\tan 3x + c$

Example $\int \left(\frac{2}{\sqrt{1-x^2}} + \frac{5}{x^2+4}\right) dx = 2\arcsin x + 5.\frac{1}{2}\arctan\frac{x}{2} + c$

Example $\int (\sec x + \cosh x)\, dx = \ln|\sec x + \tan x| + \sinh x + c$

Example $\int (\sec x + \cosh x)\, dx =$

Example $\int \left(\frac{1}{\sqrt{x^2-9}} + \frac{1}{\sqrt{9-x^2}} \right) dx =$

Example $\int \left(\frac{1}{x\sqrt{x^2-4}} + \frac{1}{\sqrt{9+x^2}} \right) dx =$

Example $\int (\sec x + \cosh x)\,dx = \ln|\sec x + \tan x| + \sinh x + C$

Example $\int \left(\frac{1}{\sqrt{x^2-9}} + \frac{1}{\sqrt{9-x^2}} \right) dx = \cosh^{-1}\frac{x}{3} + \arcsin\frac{x}{3} + C$

Example $\int \left(\frac{1}{x\sqrt{x^2-4}} + \frac{1}{\sqrt{9+x^2}} \right) dx = \frac{1}{2}\operatorname{arcsec}\left|\frac{x}{2}\right| + \sinh^{-1}\frac{x}{3} + C$

HW:

Finding Antiderivatives

In Exercises 1–16, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$ **b.** x^2 **c.** $x^2 - 2x + 1$

2. a. $6x$ **b.** x^7 **c.** $x^7 - 6x + 8$

3. a. $-3x^{-4}$ **b.** x^{-4} **c.** $x^{-4} + 2x + 3$

4. a. $2x^{-3}$ **b.** $\dfrac{x^{-3}}{2} + x^2$ **c.** $-x^{-3} + x - 1$

5. a. $\dfrac{1}{x^2}$ **b.** $\dfrac{5}{x^2}$ **c.** $2 - \dfrac{5}{x^2}$

HW:

Finding Antiderivatives

In Exercises 1–16, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

6. a. $-\frac{2}{x^3}$ b. $\frac{1}{2x^3}$ c. $x^3 - \frac{1}{x^3}$
7. a. $\frac{3}{2}\sqrt{x}$ b. $\frac{1}{2\sqrt{x}}$ c. $\sqrt{x} + \frac{1}{\sqrt{x}}$
8. a. $\frac{4}{3}\sqrt[3]{x}$ b. $\frac{1}{3\sqrt[3]{x}}$ c. $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
9. a. $\frac{2}{3}x^{-1/3}$ b. $\frac{1}{3}x^{-2/3}$ c. $-\frac{1}{3}x^{-4/3}$
10. a. $\frac{1}{2}x^{-1/2}$ b. $-\frac{1}{2}x^{-3/2}$ c. $-\frac{3}{2}x^{-5/2}$
11. a. $-\pi \sin \pi x$ b. $3 \sin x$ c. $\sin \pi x - 3 \sin 3x$
12. a. $\pi \cos \pi x$ b. $\frac{\pi}{2} \cos \frac{\pi x}{2}$ c. $\cos \frac{\pi x}{2} + \pi \cos x$
13. a. $\sec^2 x$ b. $\frac{2}{3} \sec^2 \frac{x}{3}$ c. $-\sec^2 \frac{3x}{2}$

HW:

Finding Indefinite Integrals

In Exercises 17–54, find the most general antiderivative or indefinite integral. Check your answers by differentiation.

17. $\int (x + 1)\, dx$

18. $\int (5 - 6x)\, dx$

19. $\int \left(3t^2 + \frac{t}{2}\right) dt$

20. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$

21. $\int (2x^3 - 5x + 7)\, dx$

22. $\int (1 - x^2 - 3x^5)\, dx$

23. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$

24. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

25. $\int x^{-1/3}\, dx$

26. $\int x^{-5/4}\, dx$

27. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

Asst. Prof. Dr. Nurten GÜRSES

HW:

Checking Antiderivative Formulas

Verify the formulas in Exercises 55–60 by differentiation.

55. $\displaystyle\int (7x - 2)^3\, dx = \frac{(7x - 2)^4}{28} + C$

56. $\displaystyle\int (3x + 5)^{-2}\, dx = -\frac{(3x + 5)^{-1}}{3} + C$

57. $\displaystyle\int \sec^2 (5x - 1)\, dx = \frac{1}{5}\tan (5x - 1) + C$

58. $\displaystyle\int \csc^2 \left(\frac{x - 1}{3}\right) dx = -3 \cot \left(\frac{x - 1}{3}\right) + C$

59. $\displaystyle\int \frac{1}{(x + 1)^2}\, dx = -\frac{1}{x + 1} + C$

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.