MAT1071 MATHEMATICS I
6. WEEK
PART 1

APPLICATIONS OF DERIVATIVES

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Applications of Derivatives

1. Extreme Values of Functions
2. Monotonic Functions and the First Derivative Test
3. The Mean Value Theorem
4. Concavity
5. Asymptotes of Graphs
6. Curve Sketching

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1. Extreme Values of Functions

a) Absolute Extreme Values

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

No greater value of f anywhere.

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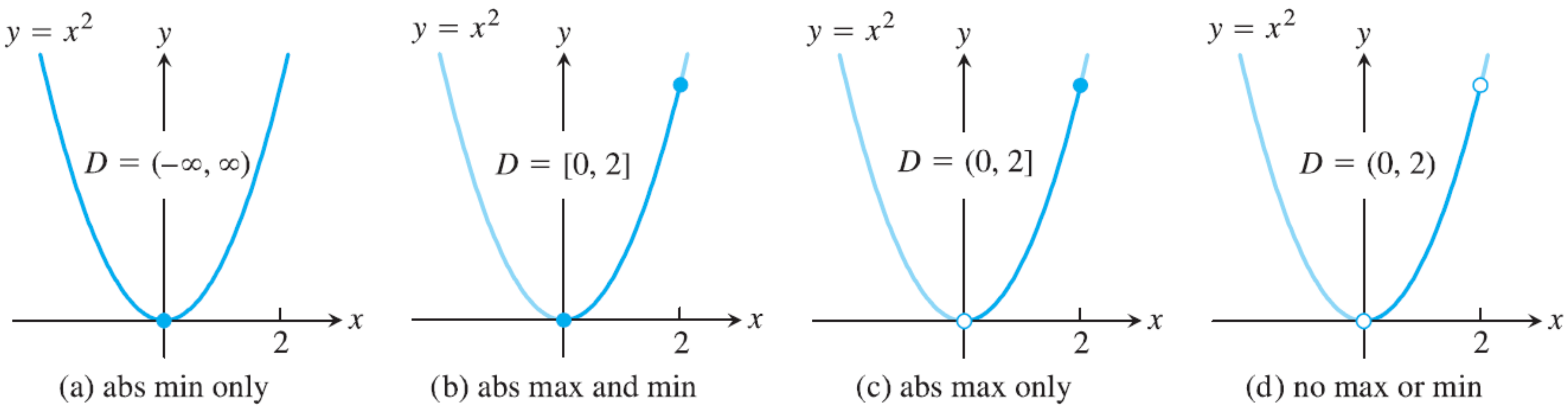
☆ Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as **global** maxima or minima.

EXAMPLE

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.



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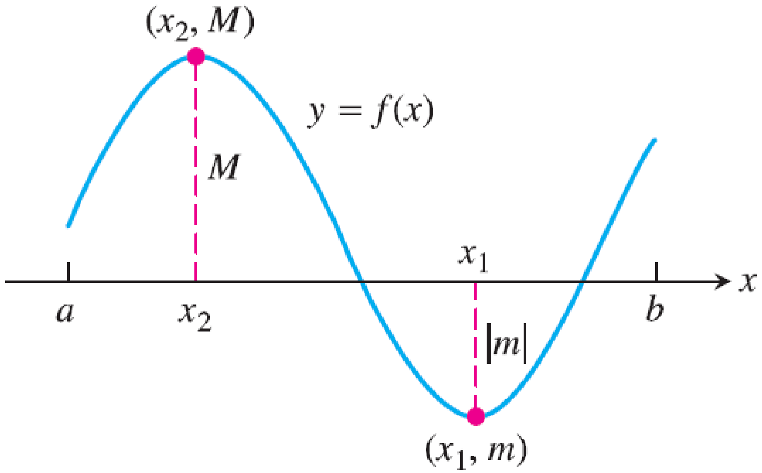
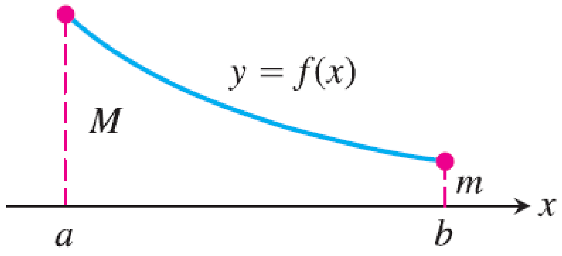
☆ The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$.

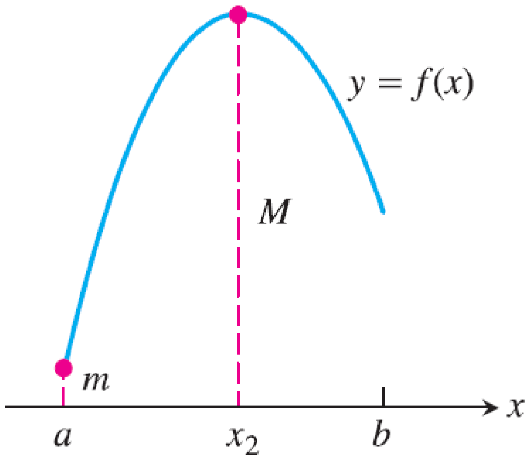
That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.



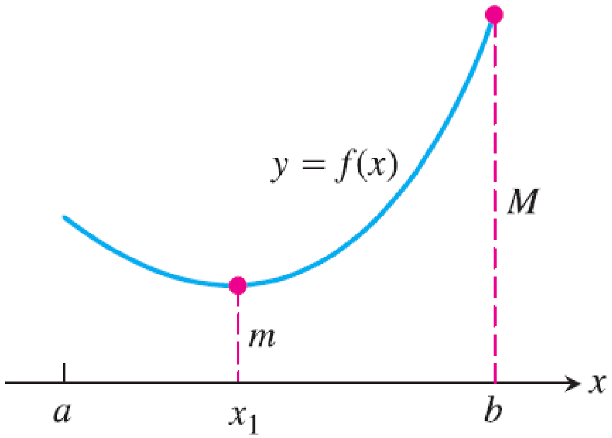
Maximum and minimum at interior points



Maximum and minimum at endpoints

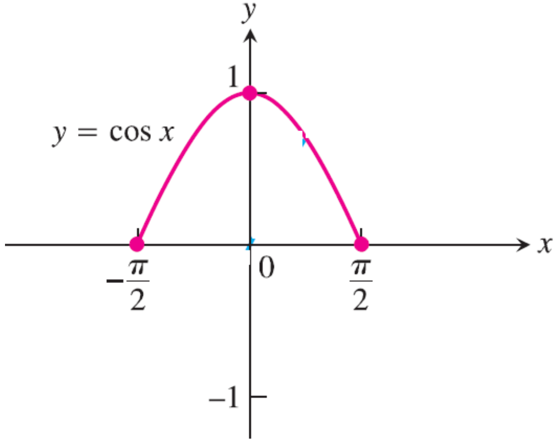


Maximum at interior point, minimum at endpoint

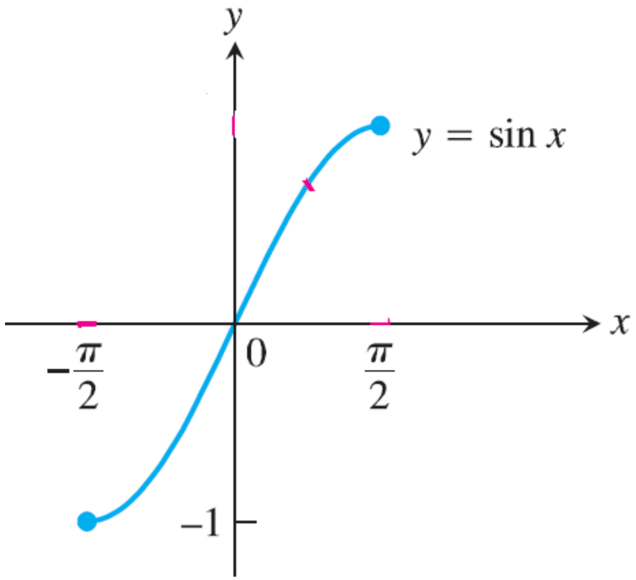


Minimum at interior point, maximum at endpoint

EXAMPLE on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice).



EXAMPLE on the closed interval $[-\pi/2, \pi/2]$ $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 .



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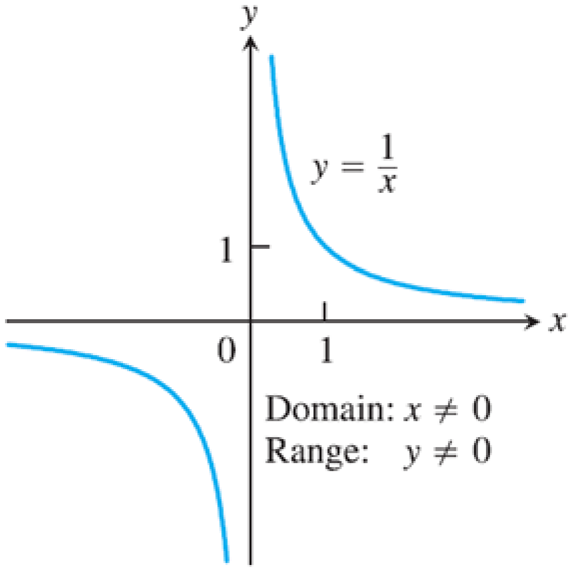
absolute maximum value 1

☆ **NOTE:** ① $f(x) = \sin x$ $\left(\frac{\pi}{2} + 2k\pi, 1\right)$ → absolute max point

② $f(x) = \frac{1}{x}$



no real absolute max or min point.

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b) Local Extreme Values

f has local max value at $c \in D$ if

interior

domain

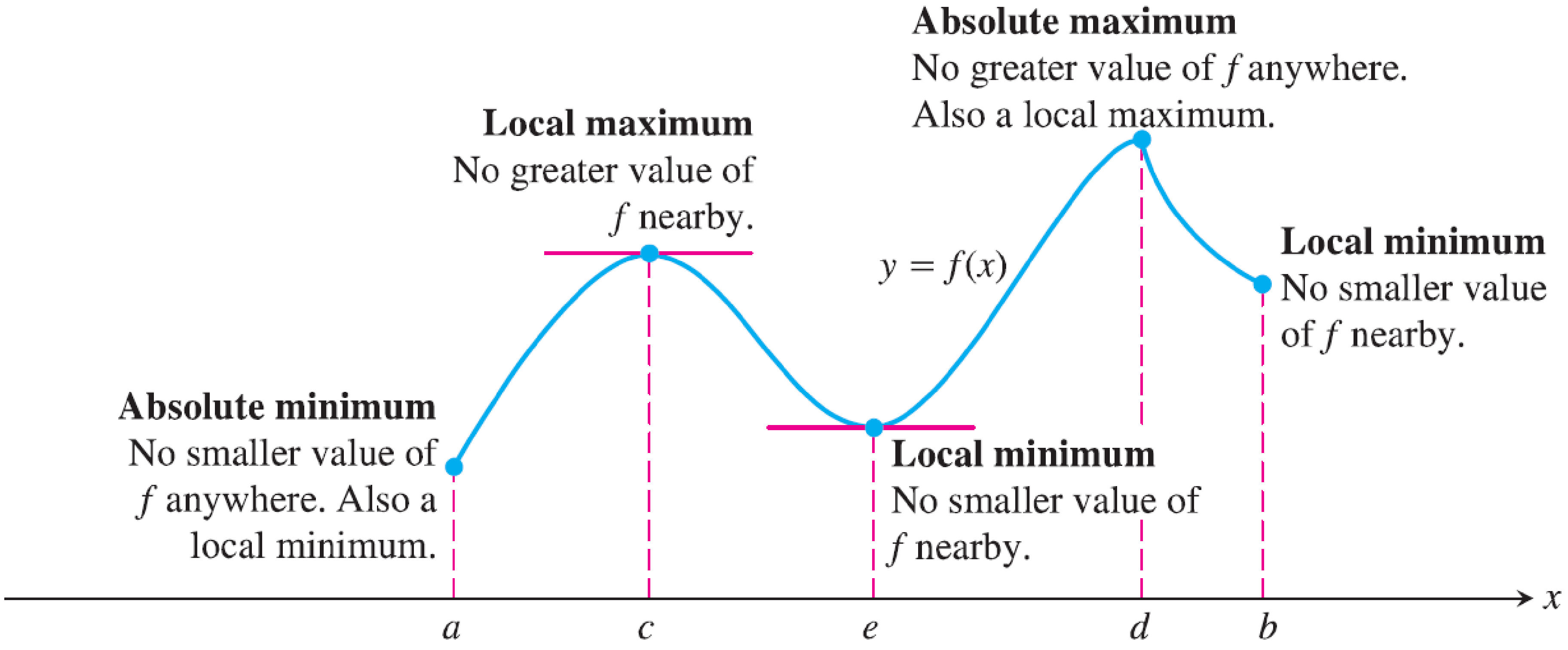
$$f(x) \le f(c)$$

for all open subinterval of D containing c.

f has local min value at $c \in D$ if

$$f(x) \ge f(c)$$

No greater value of f nearby.


Absolute maximum
No greater value of f anywhere.
Also a local maximum.
Local maximum
No greater value of
f nearby.
y = f(x)
Local minimum
No smaller value
of f nearby.
Absolute minimum
No smaller value of
f anywhere. Also a
local minimum.
Local minimum
No smaller value of
f nearby.
a
c
e
d
b
x

☆ An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood.

Hence,

a list of all local maxima will automatically include the absolute maximum if there is one

Similarly, *a list of all local minima will include the absolute minimum if there is one.*

Critical Points

An interior point of the domain of f where f' zero or undefined is a critical point of f.

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The only places where a function f can possibly have an extreme values (local or global) are

1. critical points → interior points where $f'=0$

   → interior points where f' is undefined

2. endpoints of the domain of f

Finding absolute extrema of a continuous function f on a finite closed interval:

1. Find all of the critical points and endpoints of f.
2. Evaluate the function at critical points and endpoints.
3. The greatest image → absolute max value

The smallest image → absolute min value

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EXAMPLE Find the absolute max and min values of $f(x)=x^2$ on $[-2,1]$.

Solution

① Critical points

→ $f'(x)=2x \Rightarrow \boxed{x=0}$

→ f' is undefined → no point

② endpoints

→ $\boxed{x=-2}$

→ $\boxed{x=1}$

		point
$f(0)=0$	→ absolute min value	→ $(0,0)$
$f(1)=1$		
$f(-2)=4$	→ absolute max value	→ $(-2,4)$

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EXAMPLE Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

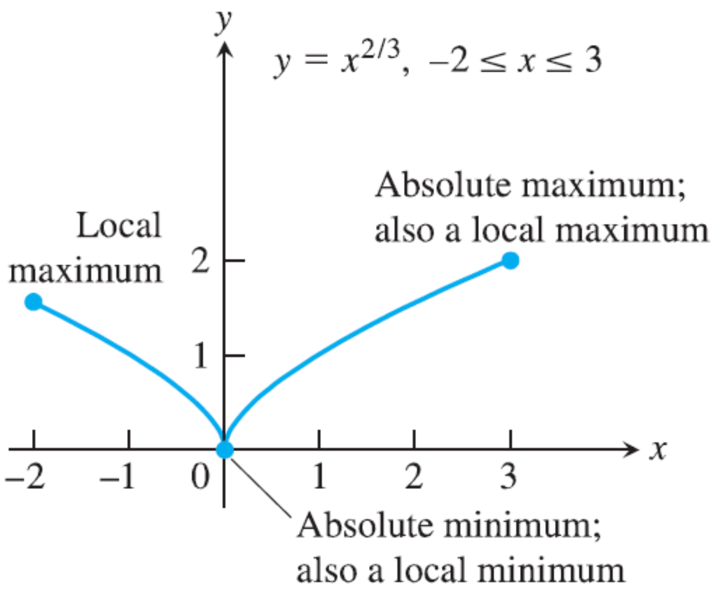
Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$g'(t) = 0 \quad \Rightarrow \quad 8 - 4t^3 = 0 \quad \Rightarrow \quad t = \sqrt[3]{2}$$

$t = \sqrt[3]{2} > 1$, a point not in the given domain.

The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum).

EXAMPLE Find the absolute max and min values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.



Solution

① Critical points $\rightarrow f'(x) = \dfrac{2}{3\sqrt[3]{x}} = 0 \Rightarrow$ no point

$\rightarrow f'$ is undefined $\Rightarrow \boxed{x=0}$

② endpoints $\rightarrow \boxed{x=-2}$

$\rightarrow \boxed{x=3}$

		Point
$f(0) = 0$	$\rightarrow$ absolute min value	$\rightarrow (0,0)$
$f(-2) = \sqrt[3]{4}$		
$f(3) = \sqrt[3]{9}$	$\rightarrow$ absolute max value	$\rightarrow (3, \sqrt[3]{9})$

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2. Monotonic Functions and the First Derivative Test

Recall: **DEFINITIONS** **Increasing, Decreasing Function**

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

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First Derivative Test for Local Extrema

Part I The test for critical points:

Let f be continuous at a critical point c in its domain ($f'(c)=0$). Let f be diff. at every point in some interval containing c except possibly at c itself.

① If f' changes from negative to positive at c, then f has a local minimum value at c

$f'<0$ at (a,x_0)
$f'>0$ at (x_0,b).

② If f' changes $\rightarrow$ local max

$f'>0$ at (a,x_0)
$f'<0$ at (x_0,b)

③ If the sign of f' does not change at "c" then f has no local ext. at c.

First Derivative Test for Local Extrema

Part 2 The test for endpoints

f cont on $\Rightarrow [a,b]$

- right cont.
  - If $f' > 0$ for $x > a$ $\Rightarrow$ local min at $x = a$.

    

  - If $f' < 0$ for $x > a$ $\Rightarrow$ local max at $x = a$

    

- left cont.
  - $f' > 0$ for $x < b$ $\Rightarrow$ local max at $x = b$

    

  - $f' < 0$ for $x < b$ $\Rightarrow$ local min at $x = b$

    

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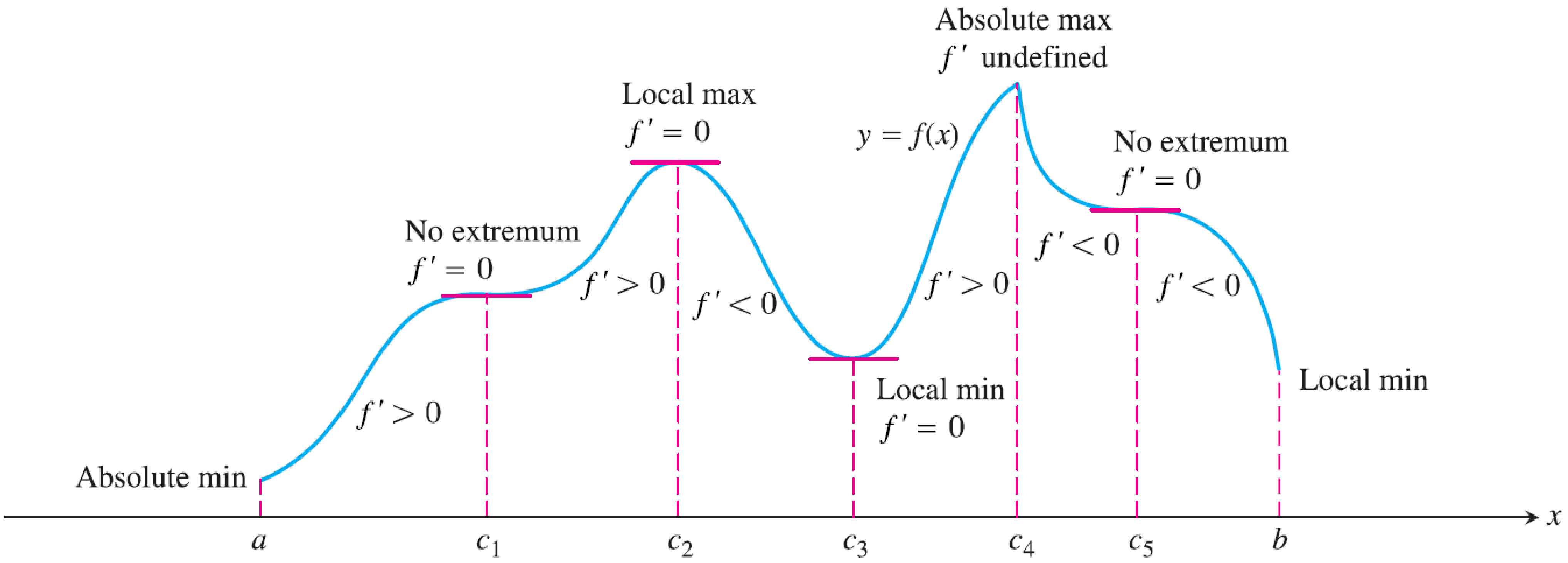

The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

☆ Note that there is no guarantee that the derivative will change signs, and therefore, it is essential to test each interval around a critical point.

EXAMPLE Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing. Find ext. values of f.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$
\begin{aligned}
f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\
&= 3(x + 2)(x - 2)
\end{aligned}
$$

$D_f = (-\infty, +\infty)$

$\downarrow$

no endpoints

is zero at $x = -2$ and $x = 2$.

x	$-\infty$		-2		2		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
f		$\nearrow$		$\searrow$		$\nearrow$	

$\rightarrow$ sign table for f'. to determine behaviour of f.

f is increasing on $(-\infty, -2) \cup (2, \infty)$

f is decreasing on $(-2, 2)$

local max value is $f(-2) = 11$

local min value is $f(2) = -21$

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EXAMPLE $y = f(x) = (x-1)^2(x+2)$

ⓐ Determine the intervals on which f is inc. and decreasing

ⓑ Find ext. values of f.

Solution

$D_f = (-\infty, +\infty)$
$\downarrow$
no endpoints

① Critical points

$f'(x) = (x-1)(3x+3) = 0$ ← $f' = 0$

$\boxed{x=1}$ $\boxed{x=-1}$

→ f' is undefined
no points

x	$-\infty$		-1		$+1$		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
f		$\nearrow$		$\searrow$		$\nearrow$	

→ sign table for f'.
to determine behaviour of f.

f is increasing on $(-\infty,-1) \cup (1,\infty)$

f is decreasing on $(-1,1)$

local max value is $f(-1)=4$

local min value is $f(1)=3$

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EXAMPLE $f(x) = x^{\frac{1}{3}}(x-4)$

identify the intervals on which f is increasing and decreasing.

Solution $D_f = \mathbb{R}$

$D_f = (-\infty, +\infty)$ $\downarrow$ no endpoints

$f'(x) = \dfrac{4(x-1)}{3\sqrt[3]{x^2}}$

Critical points $\rightarrow f' = 0 \Rightarrow \boxed{x=1}$

$\rightarrow f'$ is undef. $\Rightarrow \boxed{x=0 \text{ two fold}}$

x	$-\infty$		0		1		$+\infty$
f'		$-$	0	$-$	0	$+$	
f		$\searrow$	‖	$\searrow$		$\nearrow$	

f is increase on $(1, \infty)$

f is decreasing on $(-\infty, 0) \cup (0, 1)$

EXAMPLE $f(x) = x^{\frac{1}{3}}(x-4)$

a) Find critical points

b) Identify the intervals on which f is increasing and decreasing

c) Find the function's local and absolute extreme values

Solution

ⓐ $f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$ $D_f = (-\infty, +\infty)$ ⇓ no endpoints

$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$

$f'(x) = \frac{4}{3}x^{-2/3}(x-1)$

$\Rightarrow f'(x) = 0 \Rightarrow x = 1$

at $x = 0$ f' is undefined

çift katlı

$\left.\begin{matrix} x=1 \\ x=0 \end{matrix}\right\}$ critical points

ⓑ

	$-\infty$	0		1	∞
$x-1$	$-$		$-$	0	$+$
$x^{2/3}$	$+$	0	$+$		$+$
$f'(x)$	$-$	0	$-$	0	$+$
f	↘		↘	local min	↗

always positive

increasing on $(1, \infty)$

decreasing on $(-\infty, 1)$

ⓒ at $x = 1$ $f(x)$ has local min $f(1) = 1^{1/3}(1-4) = -3$

$(1, -3)$

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EXAMPLE $f(x) = x - 1 + \frac{1}{x+1}$

ⓐ Determine the intervals on which f is inc. and decreasing

ⓑ Find ext. values of f.

Solution

$D_f = \mathbb{R} - \{-1\}$ → no endpoint

Critical points

$$f'(x) = \frac{x^2+2x}{(x+1)^2}$$

$f' = 0$ ⇒ $\boxed{x=0}$ $\boxed{x=-2}$

f' is undefined $\boxed{x=-1}$

çift katlı

x	$-\infty$		-2		-1		0		$+\infty$
f'		$+$	0	$-$	‖	$-$	0	$+$	
f		↗		↘	‖	↘		↗	

local max $(-2,-4)$

local min $(0,0)$

f is increasing on $(-\infty,-2) \cup (0,\infty)$

f is decreasing on $(-2,-1) \cup (-1,0)$

↓

or

$(-2,0) - \{-1\}$

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3. The Mean Value Theorem

- We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero?

- If two functions have identical derivatives over an interval, how are the functions related?

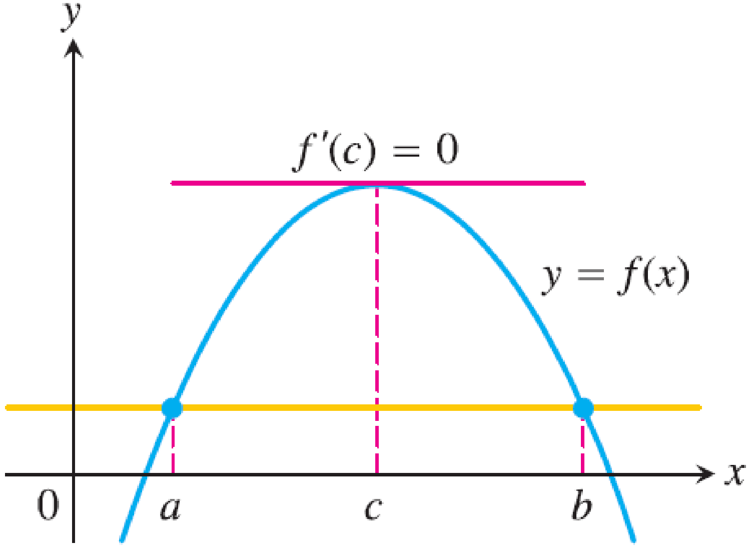
We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

☆ **Rolle's Theorem** Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b).

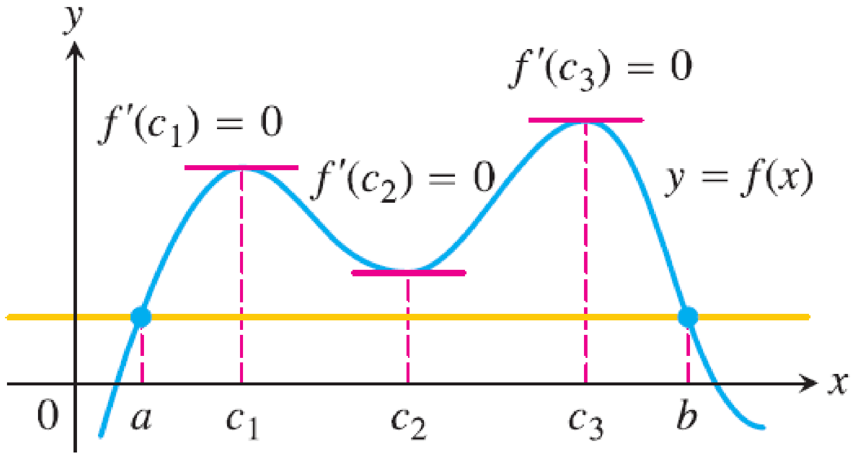
If $f(a) = f(b)$, then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

☆



(a)

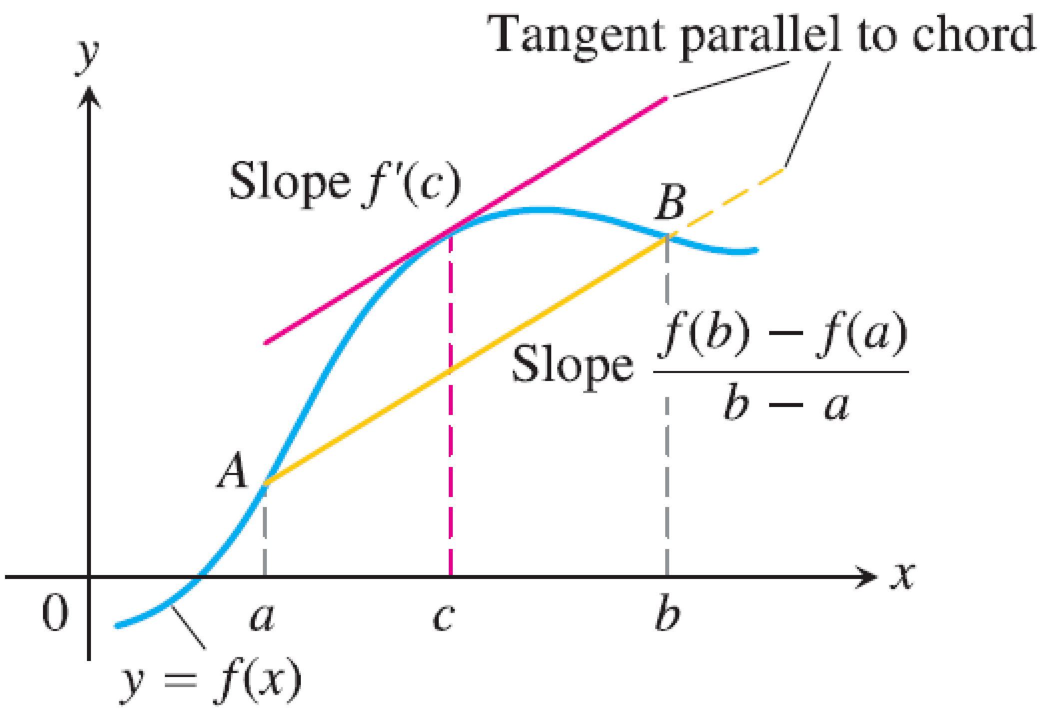


(b)

Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

☆ **The Mean Value Theorem** Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to chord AB.

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EXAMPLE Apply mean value theorem to $f(x)=x^2$ on $[0,2]$.

The function $f(x) = x^2$ is continuous for $0 \leq x \leq 2$ and differentiable for $0 < x < 2$.

Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$.

In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$.

However, it is not always easy to find c algebraically, even though we know it always exists.

EXAMPLE Let $f(x)=x^2-2x+3$. Apply the Rolle's theorem to f on $[0,2]$.

Solution

① f is continuous on $[0,2]$ ✓

② f is diff. on $(0,2)$ ✓

Because of the fact that $f(0)=f(2)=3$ from Rolle's theorem there is at least one number c in $(0,2)$ at which $f'(c)=0$

$$f'(c)=2c-2=0 \Rightarrow c=1$$

So $c=1 \in (0,2)$

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EXAMPLE Let $f(x)=\sqrt{4-x^2}$. Can we apply The Mean Value theorem to f on $[0,2]$?

Solution f is defined on $[0,2]$ and continuous

$4-x^2 \geq 0$ $\quad D_f=[-2,2]$

$x^2 \leq 4$

$-2 \leq x \leq 2$

(2) f is differentiable on $(0,2)$

$f'(x)=\dfrac{-2x}{2\sqrt{4-x^2}}$ $\quad 4-x^2>0$

diff. on $(-2,2)$

So $(0,2) \subset (-2,2)$ ✓

The Mean value theorem can be *applied*

So There is at least one point c in $(0,2)$ at which

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$f'(c)=\frac{f(2)-f(0)}{2-0}=-1=\frac{-c}{\sqrt{4-c^2}}$$

$\Rightarrow c=\sqrt{4-c^2} \Rightarrow c=\pm\sqrt{2}$ $\quad \sqrt{2}\in(0,2)$ ✓

$-\sqrt{2}\notin(0,2)$ —

EXAMPLE Let $f(0)=2$ and $f'(x) \leq 5$ for all x. Find the maximum value of $f(4)$.

Solution f is differentiable on $\mathbb{R}$ $\Rightarrow$ f is continuous on $[0,4]$

$\Rightarrow$ f is diff. also $(0,4)$.

So we can apply the mean value theorem to f on $[0,4]$.

From the statement of theorem there is a least one c in $(0,4)$ such that

$$f'(c) = \frac{f(4)-f(0)}{4-0}$$

$$f'(c) \leq 5 \Rightarrow \frac{f(4)-2}{4} \leq 5 \Rightarrow f(4) \leq 22.$$

the max value 22.

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EXAMPLE $f:[0,3]\to\mathbb{R}$ $\quad f(x)=\dfrac{x^2-x}{x+1}$ Apply mean value theorem

Solution $f'(x)=\dfrac{x^2+2x-1}{(x+1)^2}$ $\quad D_f=\mathbb{R}-\{-1\}$

$D_{f'}=\mathbb{R}-\{-1\}$

f is cont. on $[0,3]$ and dif. on $(0,3)$.

Then the mean value theorem can be applied to f.

$$f'(c)=\frac{f(\overset{3}{b})-f(\overset{0}{a})}{\underset{3}{b}-\underset{0}{a}}=\frac{c^2+2c-1}{(c+1)^2}=\frac{1}{2}$$

$$\Rightarrow c^2+2c-3=0$$

$$\Rightarrow c=-3\notin(0,3)$$

$$\boxed{c=1}\in(0,3)\quad //$$

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4. Concavity

The graph of a differentiable function $y=f(x)$ is

(a) concave up on an open interval I if f' is increasing on I. $\nearrow$ $(f''>0)$

(b) concave down on an open interval I if f' is decreasing on I $\searrow$ $(f''<0)$

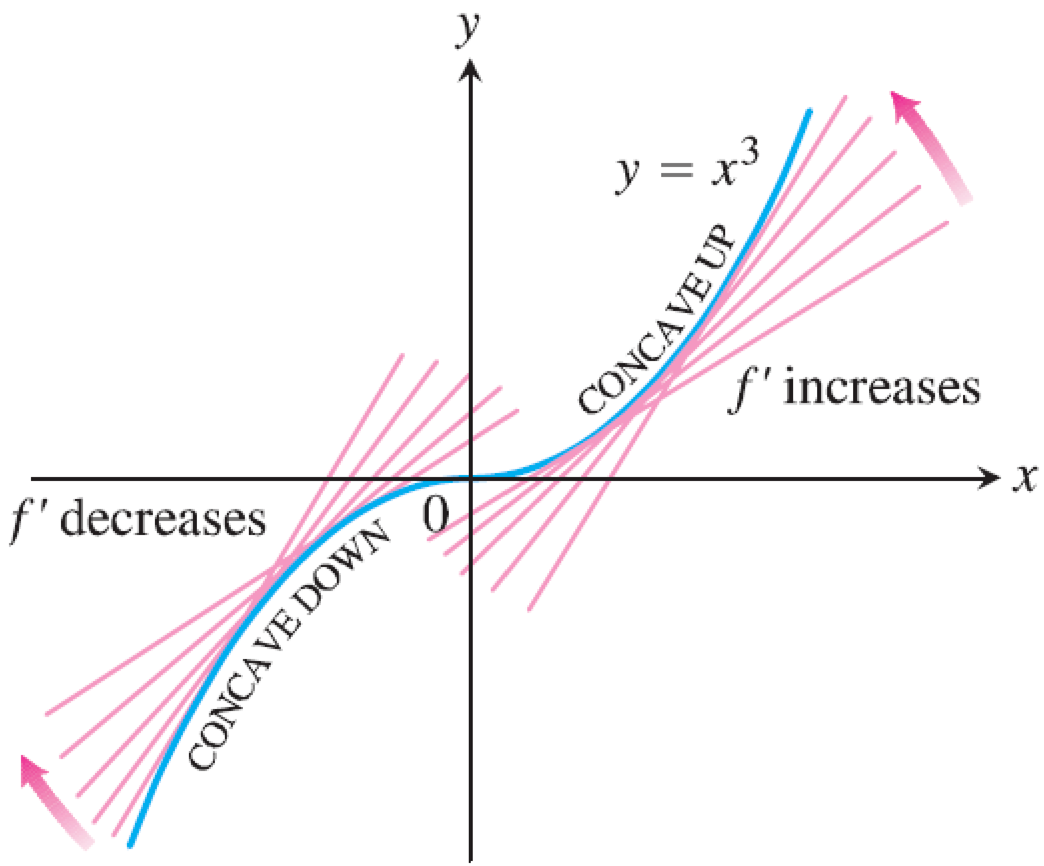
The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

EXAMPLE

The curve $y = x^3$ is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.



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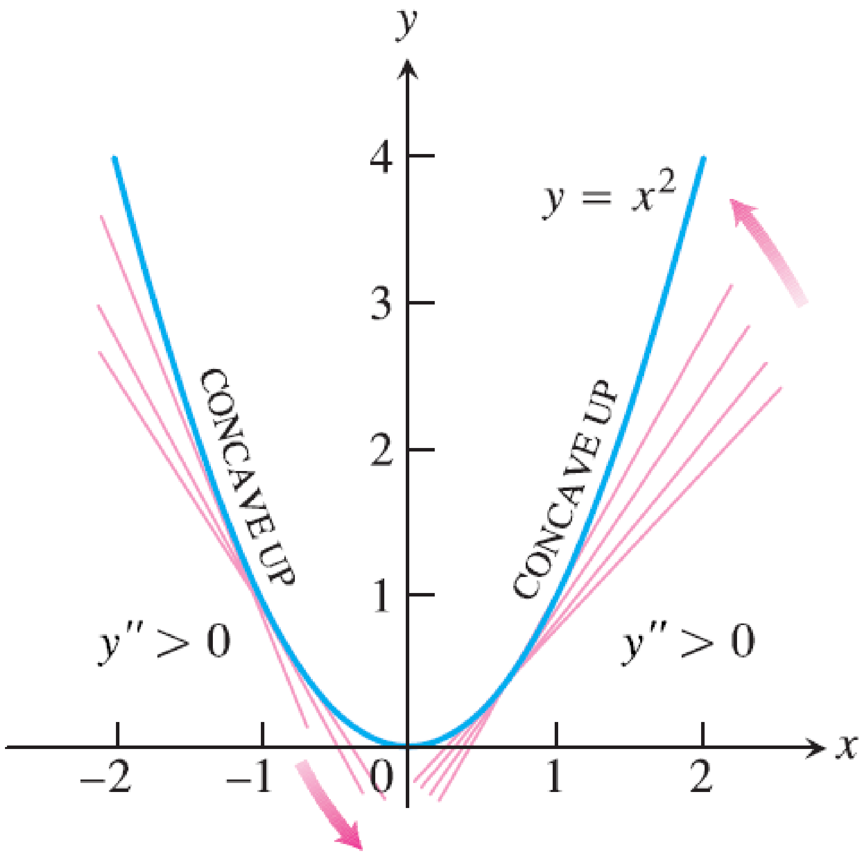
EXAMPLE

The curve $y = x^2$ is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive.



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☆ Points of Inflection

A point

where the graph of a function has a tangent line

and

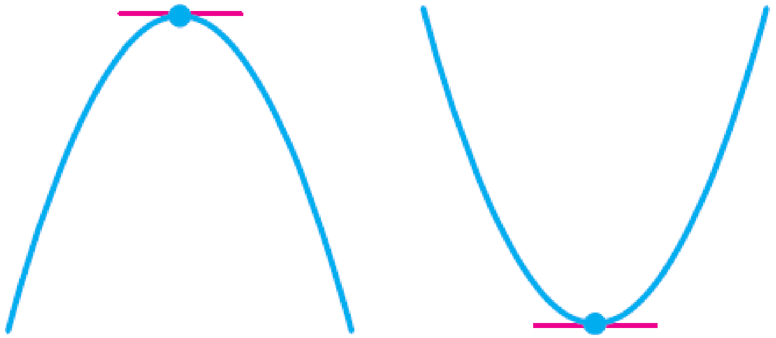
where the concavity changes

is a **point of inflection**.

☆ At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

☆ **Second Derivative Test for Local Extrema** Suppose f'' is continuous on an open interval that contains $x = c$.

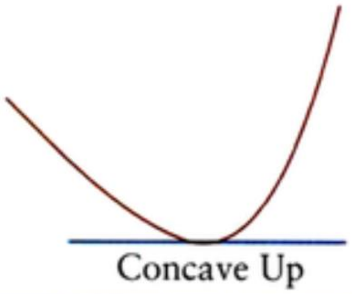
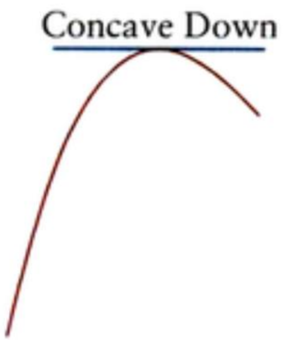
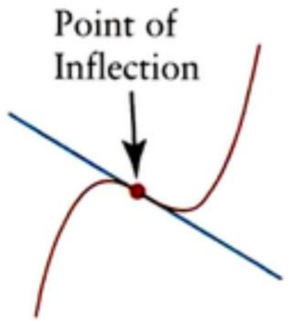
1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$ $\Rightarrow$ local max

$f' = 0, f'' > 0$ $\Rightarrow$ local min

Summary for second derivative test

The Second Derivative Test	
If $f'(a) = 0$ and $f''(a) > 0$, $f(x)$ is concave up. There is a local minimum at $(a, f(a))$.	
If $f'(a) = 0$ and $f''(a) < 0$, $f(x)$ is concave down. There is a local maximum at $(a, f(a))$.	
If $f''(x) = 0$ and $f''(x)$ changes sign at a, there is a point of inflection at $(a, f(a))$.	

http://ghcimcv4u.weebly.com/section-33.html

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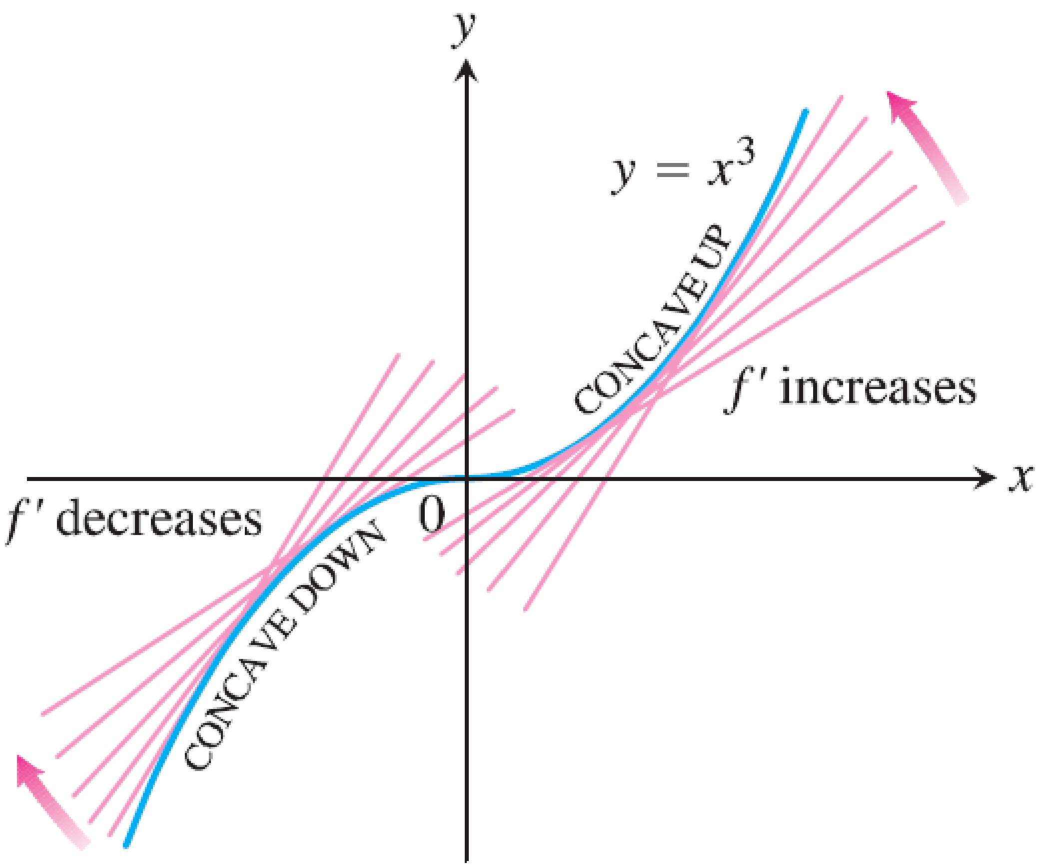
EXAMPLE

The curve $y = x^3$ is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.



$$at\ (0,0)$$

the graph of a function has a tangent line $\surd$

concavity changes $\surd$

point of inflection $\surd$

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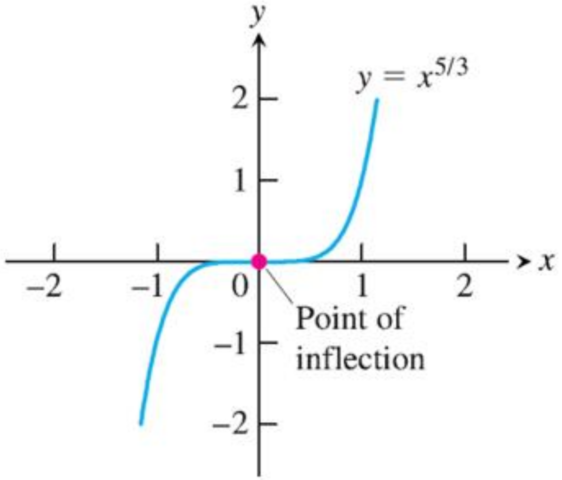
☆ example illustrates a function having a point of inflection where the first derivative exists, but the second derivative fails to exist.

EXAMPLE The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin.



The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$

at $(0,0)$

the graph of a function has a tangent line √

concavity changes √

---

point of inflection √

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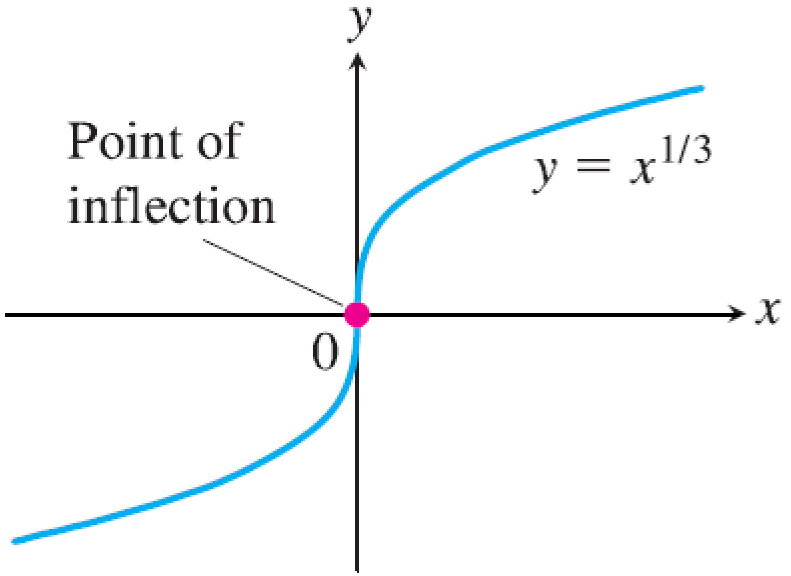
EXAMPLE The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there.



$at\ (0,0)$

the graph of a function has a tangent line $\surd$

concavity changes $\surd$

---

point of inflection $\surd$

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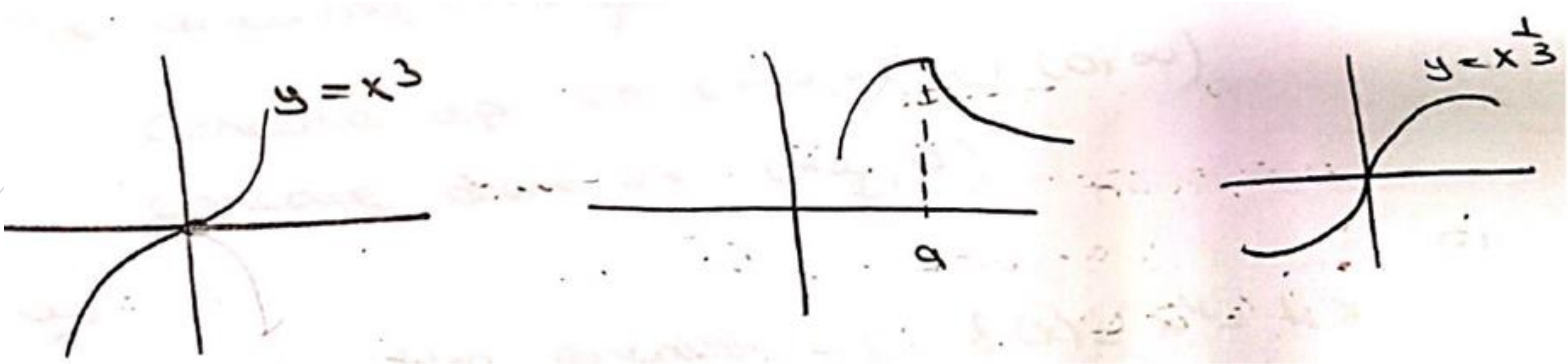
☆



at $(0,0)$

the graph of a function has a tangent line √

concavity changes √

point of inflection √

the graph of a function has a tangent line ✗

concavity changes √

point of inflection ✗

at $(0,0)$

the graph of a function has a tangent line √

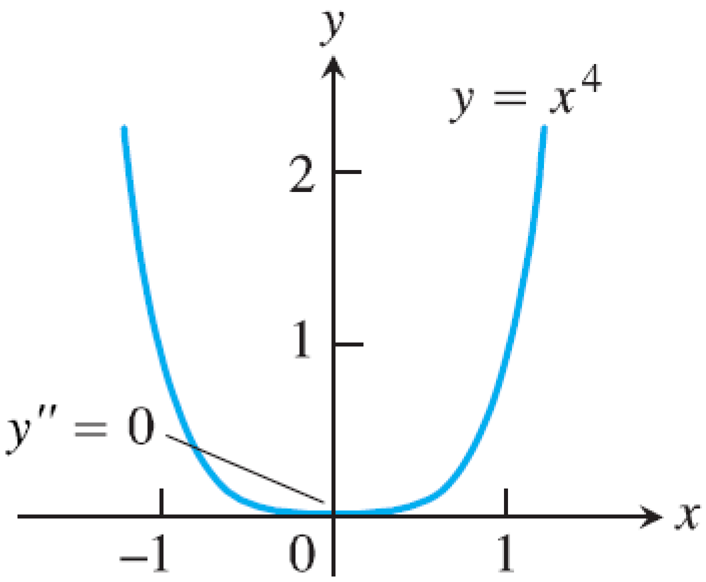
concavity changes √

point of inflection √

☆ Here is an example showing that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

EXAMPLE The curve $y = x^4$ has no inflection point at $x = 0$

Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign.



at $(0,0)$

the graph of a function has a tangent line ✓

concavity changes ✗

point of inflection ✗

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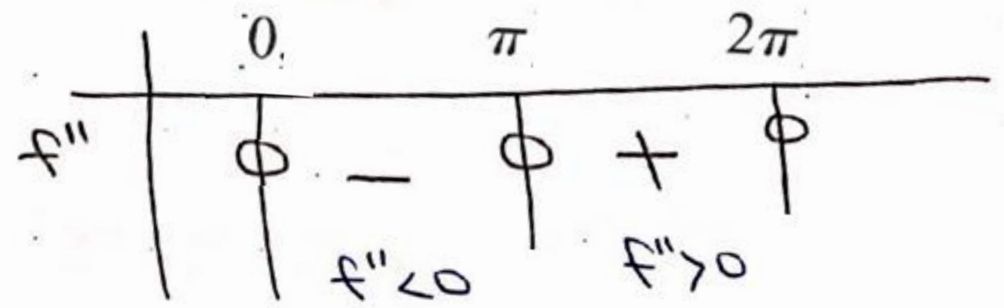
EXAMPLE Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution

$$y = 3 + \sin x$$
$$y' = \cos x$$
$$y'' = -\sin x.$$

The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$,

It is concave up on $(\pi, 2\pi)$,



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EXAMPLE Examine the concavity of $f(x)=x^4+x^3-2x$ and find inflection points.

Solution

$f'(x)=4x^3+3x^2-2$

$f''(x)=12x^2+6x=0 \Rightarrow 6x(2x+1)=0$

$\boxed{x=0}$ $\boxed{x=-\frac{1}{2}}$

	$-\infty$	$-1/2$	0	∞
f''	$+$	0	$-$	0 $+$
	$f''>0$		$f''<0$	$f''>0$

$\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right)$ $(0, f(0))$

the graph of a function has a tangent line √
concavity changes √

---

point of inflection √

the graph of f
concave up on $(-\infty,-\frac{1}{2})\cup(0,\infty)$
concave down on $(-\frac{1}{2},0)$

EXAMPLE Examine the concavity of $f(x)=x^4-4x^3+10$ and find inflection points.

Solution

$$f'(x)=4x^3-12x^2$$

$$f''(x)=12x^2-12x=12x(x-1)=0$$

$$\boxed{x=0} \quad \boxed{x=1}$$

	$-\infty$		0		1		$+\infty$
f''		$+$	0	$-$	0	$+$	
		$f''>0$		$f''<0$		$f''>0$	

$(1, f(1))$ $\quad$ $(0, f(0))$

the graph of a function has a tangent line √

concavity changes √

point of inflection √

the graph of f concave up on $(-\infty,0)\cup(1,\infty)$

concave down on $(0,1)$

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EXAMPLE Find the local extrema of the function

$$f(x) = x^3 - 9x^2 + 24x - 7$$

Solution

$$f'(x) = \left(x^3 - 9x^2 + 24x - 7\right)' = 3x^2 - 18x + 24.$$

$$f'(x) = 0, \quad \Rightarrow 3x^2 - 18x + 24 = 0, \quad \Rightarrow 3\left(x^2 - 6x + 8\right) = 0,$$

$$\Rightarrow 3(x-2)(x-4) = 0, \quad \Rightarrow x_1 = 2, x_2 = 4. \text{ critical points}$$

$$f''(x) = \left(3x^2 - 18x + 24\right)' = 6x - 18.$$

$f''(2) = 6 \cdot 2 - 18 = -6 < 0.$ $\quad f(2) = 13,$ $\quad$ local max at $(2, 13)$

$f''(4) = 6 \cdot 4 - 18 = 6 > 0.$ $\quad f(4) = 9.$ $\quad$ local min at $(4, 9)$

local max value of f is 13

local min value of f 9

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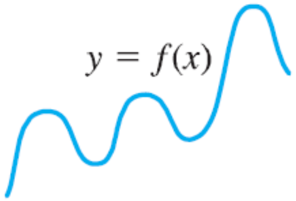
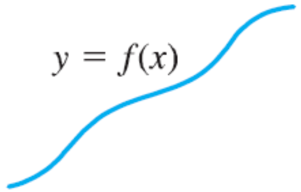
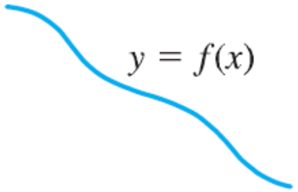
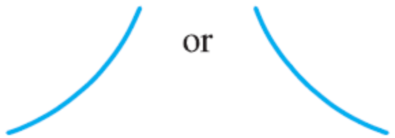
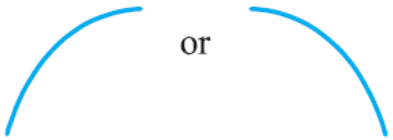
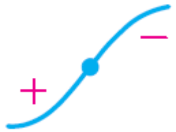
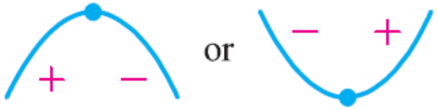
 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 y'' changes sign at an inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

HW:

Absolute Extrema on Finite Closed Intervals

In Exercises 21–36, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

21. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

22. $f(x) = -x - 4, \quad -4 \le x \le 1$

23. $f(x) = x^2 - 1, \quad -1 \le x \le 2$

24. $f(x) = 4 - x^2, \quad -3 \le x \le 1$

25. $F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$

26. $F(x) = -\frac{1}{x}, \quad -2 \le x \le -1$

HW:

Finding Critical Points

In Exercises 41–48, determine all critical points for each function.

41. $y = x^2 - 6x + 7$

42. $f(x) = 6x^2 - x^3$

43. $f(x) = x(4 - x)^3$

44. $g(x) = (x - 1)^2(x - 3)^2$

Finding Extreme Values

In Exercises 49–58, find the extreme values (absolute and local) of the function and where they occur.

49. $y = 2x^2 - 8x + 9$

50. $y = x^3 - 2x + 4$

51. $y = x^3 + x^2 - 8x + 5$

52. $y = x^3(x - 5)^2$

HW:

Local Extrema and Critical Points

In Exercises 59–66, find the critical points, domain endpoints, and local extreme values (absolute and local) for each function.

59. $y = x^{2/3}(x + 2)$

60. $y = x^{2/3}(x^2 - 4)$

61. $y = x\sqrt{4 - x^2}$

62. $y = x^2\sqrt{3 - x}$

HW:

Checking the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–6.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$

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HW:

For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \leq x \leq 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

HW:

Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of f?

b. On what intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$

2. $f'(x) = (x - 1)(x + 2)$

3. $f'(x) = (x - 1)^2(x + 2)$

4. $f'(x) = (x - 1)^2(x + 2)^2$

5. $f'(x) = (x - 1)(x + 2)(x - 3)$

6. $f'(x) = (x - 7)(x + 1)(x + 5)$

7. $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$

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Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.