MAT1071 MATHEMATICS I

6.2. APPLICATIONS OF DERIVATIVES

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Applications of Derivatives

1. Extreme Values of Functions
2. Monotonic Functions and the First Derivative Test
3. The Mean Value Theorem
4. Concavity
5. Asymptotes of Graphs
6. Curve Sketching

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5. Asymptotes of Graphs

- ☆ Horizontal Asymptote
- ☆ Vertical Asymptote
- ☆ Oblique Asymptote

☆ Horizontal Asymptote

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

EXAMPLE $f(x) = \dfrac{5x^2+8x-3}{3x^2+2}$

$$\lim_{x \to \infty} \frac{5x^2+8x-3}{3x^2+2} = \frac{5}{3} \quad \text{or} \quad \lim_{x \to -\infty} f(x) = \frac{5}{3}$$

So $y = \frac{5}{3}$ is horizontal asymptote for $f(x)$

EXAMPLE Find the horizontal asymptotes of the graph of

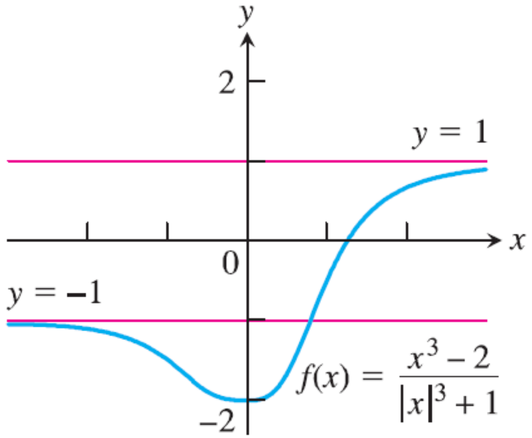
$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \to \pm\infty$.

For $x \geq 0$: $$\lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x\to\infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$$

For $x < 0$: $$\lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$.
Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x.



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EXAMPLE Find $\lim_{x \to \infty} \sin(1/x)$

Solution

We introduce the new variable $t = 1/x$. we know that $t \to 0^+$ as $x \to \infty$ Therefore,

$$\lim_{x \to \infty} \sin \frac{1}{x} = \lim_{t \to 0^+} \sin t = 0.$$

we see that the line $y = 0$ is a horizontal asymptote.

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EXAMPLE Find $\lim_{x \to \pm\infty} x \sin(1/x)$.

Solution

We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{t \to 0^+} \frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{x \to -\infty} x \sin \frac{1}{x} = \lim_{t \to 0^-} \frac{\sin t}{t} = 1.$$

we see that the line $y = 1$ is a horizontal asymptote.

EXAMPLE Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

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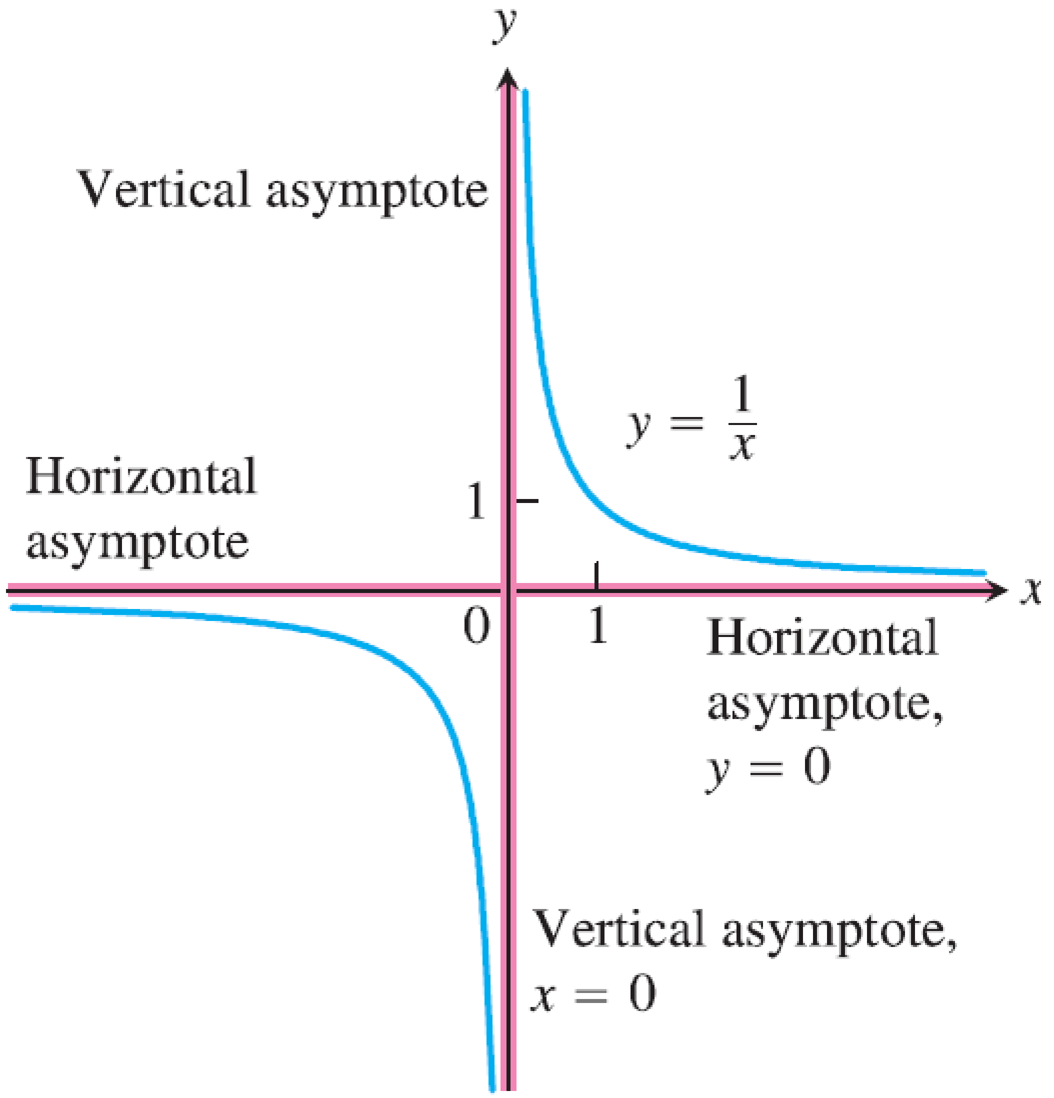
☆ Vertical Asymptotes

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

EXAMPLE



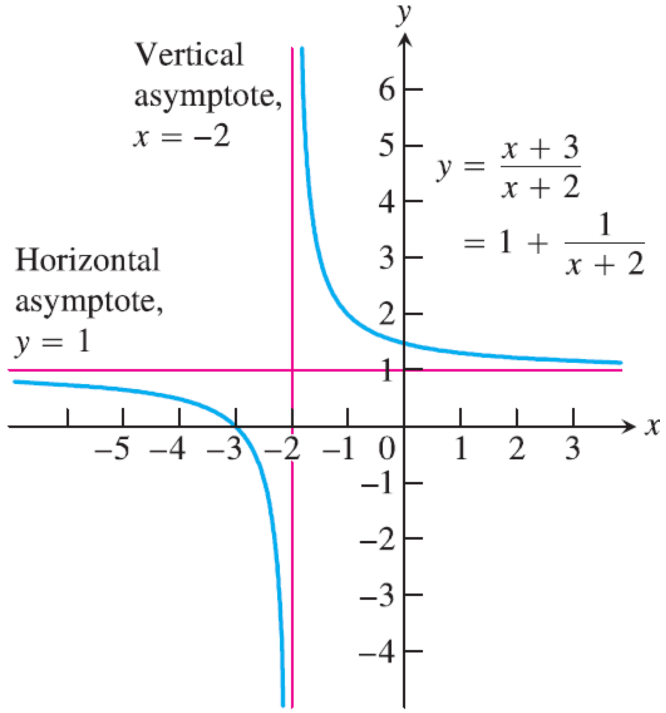
y
Vertical asymptote
$y = \frac{1}{x}$
Horizontal
asymptote
1
0
1
x
Horizontal
asymptote,
$y = 0$
Vertical asymptote,
$x = 0$

EXAMPLE Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and the behavior as $x \to -2$, where the denominator is zero.

As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$;



$$\lim_{x \to -2^+} \frac{x+3}{x+2} = +\infty$$

$$\lim_{x \to -2^-} \frac{x+3}{x+2} = -\infty$$

vertical asymptote $x = -2$.

EXAMPLE Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution

$y = \frac{-8}{x^2-4}$ Find vertical asymptote $x^2-4=0$

$x = \pm 2$

$$\left.\begin{array}{l} \lim_{x\to 2^+} \frac{-8}{x^2-4} = -\infty \\ \lim_{x\to 2^-} \frac{-8}{x^2-4} = +\infty \end{array}\right\} x = 2 \text{ vertical asymptote}$$

$$\left.\begin{array}{l} \lim_{x\to -2^+} \frac{-8}{x^2-4} = +\infty \\ \lim_{x\to -2^-} \frac{-8}{x^2-4} = -\infty \end{array}\right\} x = -2 \text{ vertical asymptote}$$

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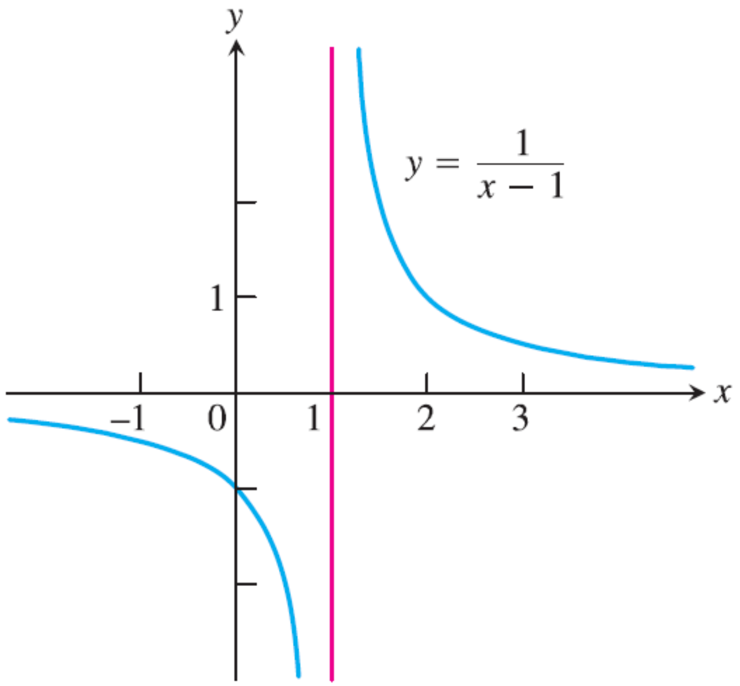
EXAMPLE Find $\lim_{x\to 1^+} \frac{1}{x-1}$ and $\lim_{x\to 1^-} \frac{1}{x-1}$.

Solution

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \qquad \text{vertical asymptote } x = 1$$

$$\lim_{x\to 1^-} \frac{1}{x-1} = -\infty \qquad \text{vertical asymptote } x = 1$$



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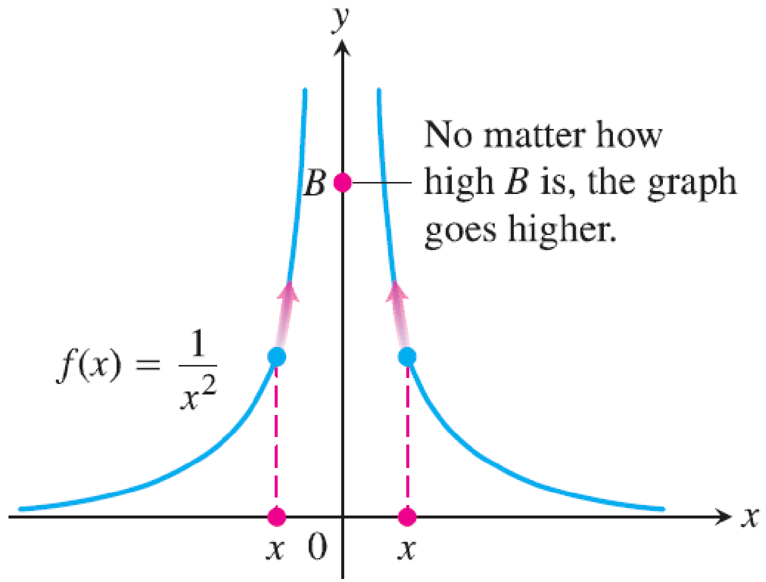
EXAMPLE Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{as} \quad x \to 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large . This means that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty.$$



$x = 0$

vertical asymptote

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EXAMPLE The curves

$$y = \sec x = \frac{1}{\cos x} \qquad \text{and} \qquad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$

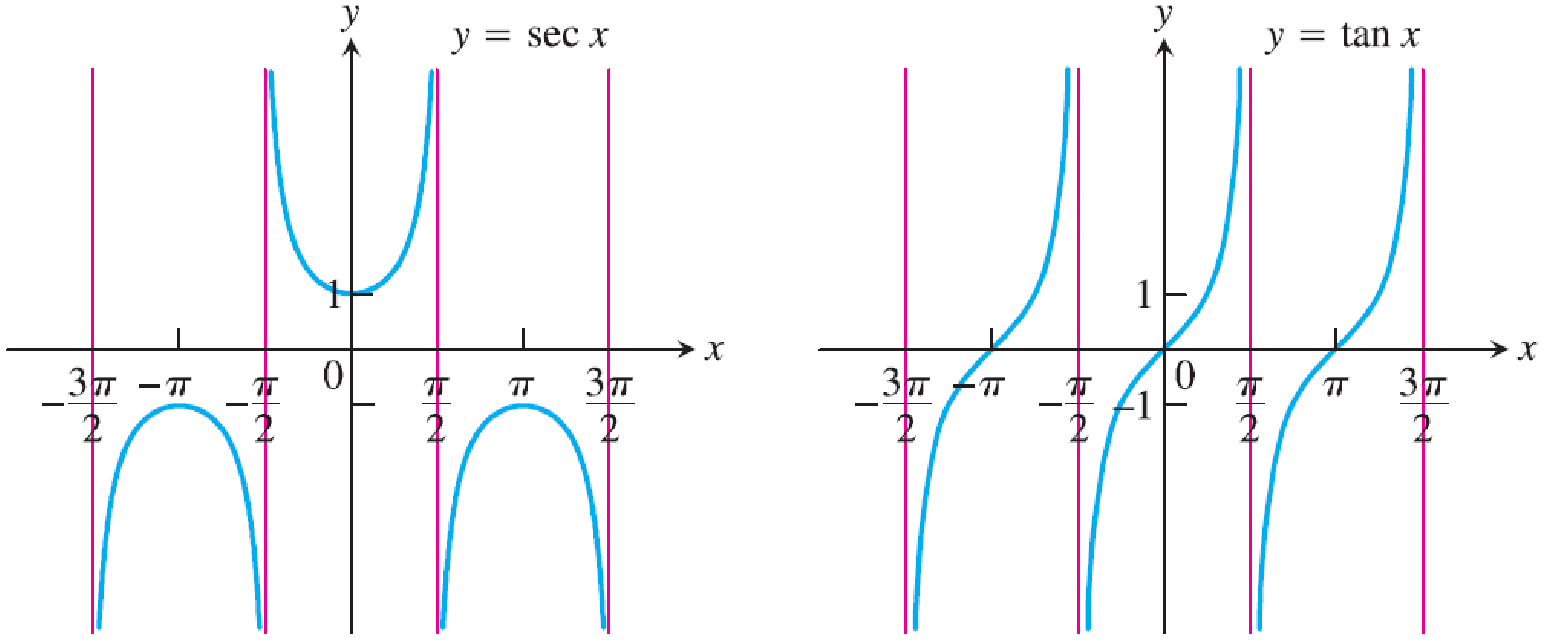

FIGURE The graphs of sec x and tan x have infinitely many vertical asymptotes

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EXAMPLE infinitely many vertical asymptotes

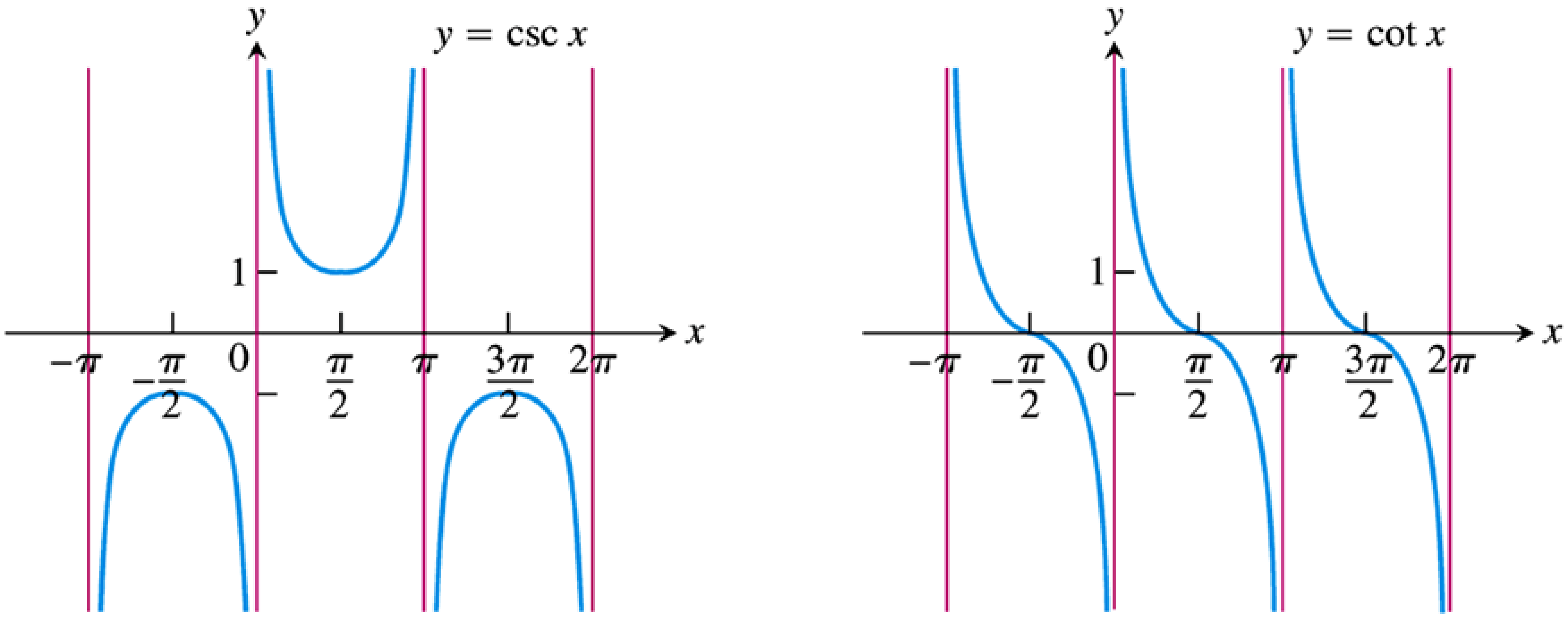


FIGURE The graphs of csc x and cot x

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

Note: The oblique asymptote can be obtained by the equations:

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x} \quad \text{and} \quad n = \lim_{x \to \pm\infty} \left(f(x) - mx\right)$$

$$\boxed{y = mx + n}$$

EXAMPLE Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

Solution We are interested in the behavior as $x \to \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r@{}l} & \frac{x}{2} + 1 \\ 2x - 4 \,\big) & x^2 - 3 \\ & \underline{x^2 - 2x} \\ & \quad 2x - 3 \\ & \quad \underline{2x - 4} \\ & \qquad\quad 1 \end{array}$$

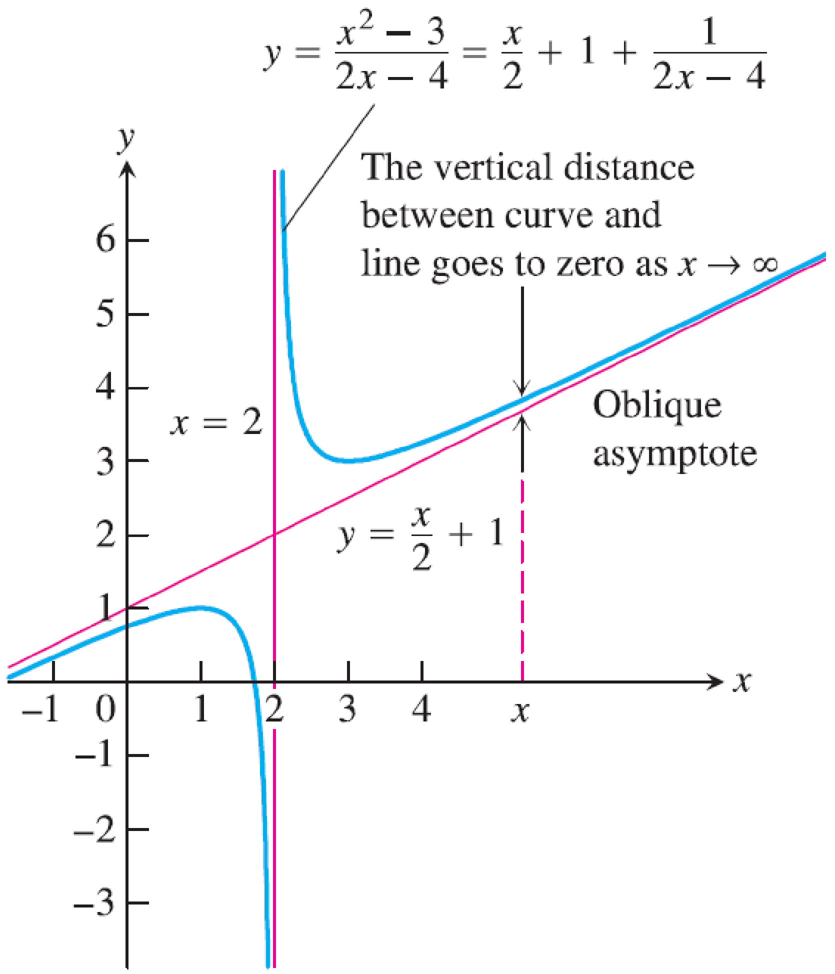
This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}.$$

As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

Oblique asymptote



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Second way

$$f(x)=\frac{x^2-3}{2x-4}$$

$$m=\lim_{x\to\infty}\frac{\frac{x^2-3}{2x-4}}{x}=\frac{1}{2}$$

$$n=\lim_{x\to\infty}\left(\frac{x^2-3}{2x-4}-\frac{1}{2}x\right)=\lim_{x\to\infty}\frac{2x-3}{2x-4}=1$$

$$\boxed{y=mx+n=\frac{1}{2}x+1}$$

EXAMPLE

$$f(x) = \sqrt{x^2 - 4x + 2}$$

Solution

$$\lim_{x \to -\infty} \frac{\sqrt{x^2 - 4x + 2}}{x}$$

$$= \lim_{x \to -\infty} \frac{x\sqrt{1 - \frac{4}{x} + \frac{2}{x^2}}}{x} = -1$$

and $\lim_{x \to -\infty} \left(\sqrt{x^2 - 4x + 2} + x\right) = 2$

$y = -x + 2$ oblique asym

$$\lim_{x \to \infty} \frac{f(x)}{x} = 1 \qquad n = \lim_{x \to \infty} \left(\sqrt{x^2 - 4x + 2} - x\right) = -2$$

$\boxed{y = x - 2}$ oblique asym.

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☆ Asymptotes of a Rational Function

$$f(x)=\frac{P(x)}{Q(x)}=\frac{ax^m+\dots}{bx^n+\dots}$$

degree of $P(x) \to m$

degree of $Q(x) \to n$

a) f has a vertical asym. at the points where $Q(x)=0$.

b) If $m<n$, then $y=0$ is horizontal asym.

c) If $m=n$, then $y=L$ is horizontal asym. $\left(\lim_{x\to\mp\infty} f(x)=L\right)$

$$L=\frac{a}{b}$$

d) If $m=n+1$, then f has an oblique asymp.

$$P(x) \;\Big|\; \frac{Q(x)}{\text{oblique asym.}}$$

e) If $m>n+1$, There is no oblique or horizontal asym.

6. Curve Sketching

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.

2. Find the intercepts

3. Identify any asymptotes that may exist

4. Find f'.

   Find the critical points of f, if any, and identify the function's behavior at each one. Find where the curve is increasing and where it is decreasing.

5. Find f''.

   Find the points of inflection, if any occur, and determine the concavity of the curve.

6. Construct the sign table for f' and f''.

7. Plot key points, such as the intercepts and the points found in Steps 2 –5, and sketch the curve together with any asymptotes that exist.

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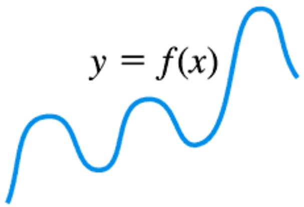
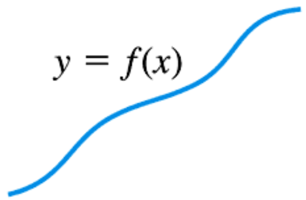
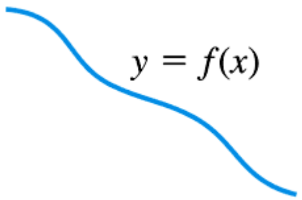
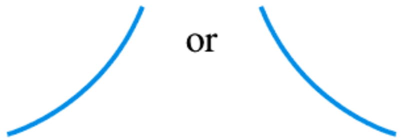
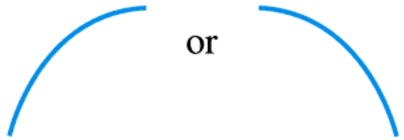
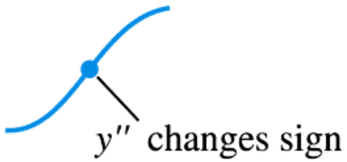
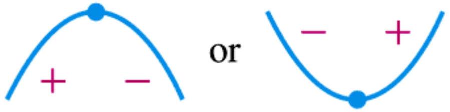
 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 Inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

EXAMPLE Sketch the graph of the function $f(x)=\frac{x}{x-1}$

Solution

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EXAMPLE Sketch the graph of the function $f(x)=\frac{x}{x-1}$

Solution

EXAMPLE Sketch the graph of the function $f(x)=\frac{x}{x-1}$

Solution

Domain

① $D_f=\mathbb{R}-\{1\}$ $\quad f(-x)=\frac{-x}{-x-1}\neq f(x) \Rightarrow f$ is not even

$\neq -f(x) \Rightarrow f$ is not odd

no symmetries

Intercepts

② $x=0 \Rightarrow y=0$

$y=0 \Rightarrow x=0$ $\quad \boxed{(0,0) \text{ intercept point}}$

③ Asymptotes

Horizontal asymptote

$\lim\limits_{x\to\infty}\frac{x}{x-1}=1$

$\lim\limits_{x\to-\infty}\frac{x}{x-1}=1$ $\quad$ line $\boxed{y=1}$

Vertical asymptote

$\lim\limits_{x\to1^+}\frac{x}{x-1}=\infty$

$\lim\limits_{x\to1^-}\frac{x}{x-1}=-\infty$ $\quad$ line $\boxed{x=1}$

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(4) $f'(x) = \dfrac{-1}{(x-1)^2}$. Critical points $\to f'=0 \Rightarrow$ no point

$\to f'$ is undefined $\Rightarrow \boxed{x=1 \text{ two fold}}$

(5) $f''(x) = \dfrac{2}{(x-1)^3}$ $\to f''=0 \Rightarrow$ no ~~point~~

$\to f''$ is undefined $\to \boxed{x=1}$

(6)

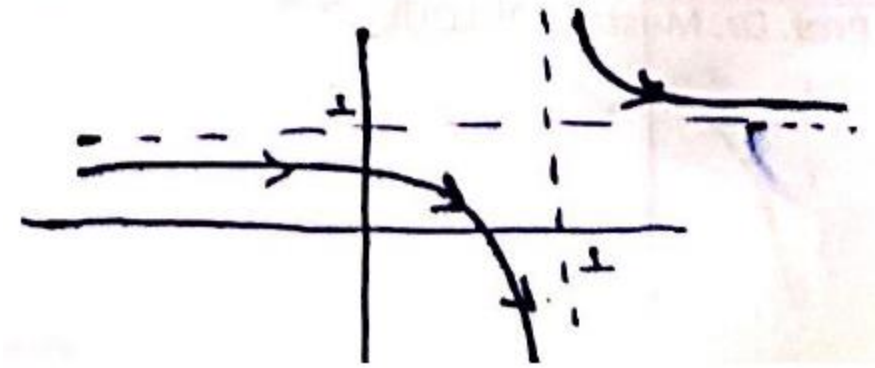
x	$-\infty$		0		1		$+\infty$
f'		$-$		$-$	$\circ$	$-$	
f''	1	$-$		$-$	$\circ$ ($-\infty$ | $+\infty$)	$+$	$+1$

$\downarrow$ is not inflection

Decreasing on $(-\infty,1)\cup(1,\infty)$

Concave up on $(1,\infty)$

Concave down on $(-\infty,1)$



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EXAMPLE $f(x) = \frac{8}{x^2-4}$ Sketch the graph

Solution

EXAMPLE $f(x) = \dfrac{8}{x^2 - 4}$ Sketch the graph

Solution

EXAMPLE $f(x) = \dfrac{8}{x^2-4}$ Sketch the graph

Solution

① Domain $D_f = \mathbb{R} - \{-2, +2\}$ sym.

② intercept points $x=0 \Rightarrow y=-2$ $(0,-2)$

$y=0 \Rightarrow$ no intersection on axis

③ Asymptotes

horizontal asym.

$\lim\limits_{x \to \pm\infty} \dfrac{8}{x^2-4} = 0 \Rightarrow \boxed{y=0}$

vertical asymptote

$\lim\limits_{x \to 2^+} f(x) = +\infty$

$\lim\limits_{x \to 2^-} f(x) = -\infty$

$\boxed{x=2}$

$\lim\limits_{x \to -2^+} f(x) = -\infty$

$\lim\limits_{x \to -2^-} f(x) = +\infty$

$\boxed{x=-2}$

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4) $f'(x) = \dfrac{-16x}{(x^2-4)^2}$ Critical points $\rightarrow f'=0 \Rightarrow \boxed{x=0}$

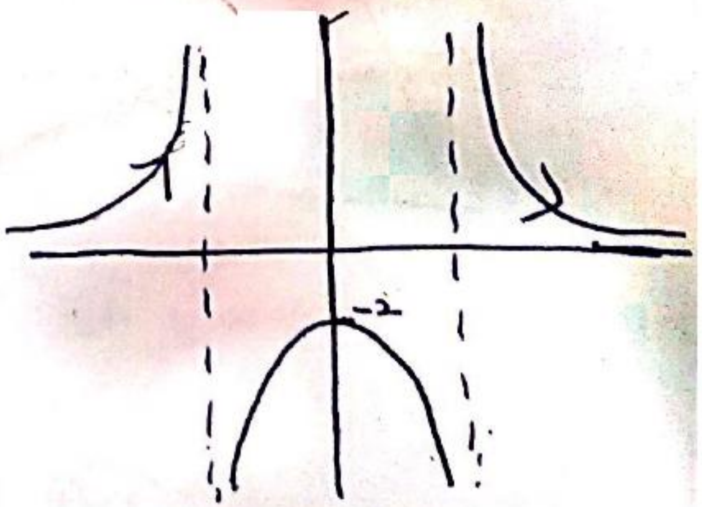
$\rightarrow f'$ is undefined $\Rightarrow \boxed{x=2 \text{ two fold}}$ $\boxed{x=-2 \text{ two fold}}$

5) $f''(x) = \dfrac{16(3x^2+4)}{(x^2-4)^3}$ $\rightarrow f''=0 \rightarrow$ no point

$\rightarrow f''$ is undefined $\Rightarrow \boxed{x=2}$ $\boxed{x=-2}$

6)

x	$-\infty$		-2		0		$+2$		$+\infty$
f'		$+$	O	$+$	O	$-$	O	$-$	
f''	0	$+$	O ($-\infty$, $+\infty$)	$-$	-2	$-$	O ($-\infty$, $+\infty$)	$+$	



$(-\infty,-2) \cup (-2,0) \rightarrow$ increasing

$(0,2) \cup (2,\infty) \rightarrow$ decreasing

$(-\infty,-2) \cup (2,\infty) \rightarrow$ concave up

$(-2,0) \cup (0,2) \rightarrow$ concave down

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HW:

Horizontal and Vertical Asymptotes

47. Use limits to determine the equations for all vertical asymptotes.

a. $y = \dfrac{x^2 + 4}{x - 3}$ **b.** $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x + 1}$

48. Use limits to determine the equations for all horizontal asymptotes.

a. $y = \dfrac{1 - x^2}{x^2 + 1}$ **b.** $f(x) = \dfrac{\sqrt{x} + 4}{\sqrt{x + 4}}$

HW:

Oblique Asymptotes

31. $y = \dfrac{2x^{3/2} + 2x - 3}{\sqrt{x} + 1}$

99. $y = \dfrac{x^2}{x - 1}$

100. $y = \dfrac{x^2 + 1}{x - 1}$

HW:

Graphing Equations

Use the steps of the graphing procedure to graph tl equations in Exercises 9–48. Include the coordinates of any local ar absolute extreme points and inflection points.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

Graphing Rational Functions

Graph the rational functions in Exercises 75–92.

75. $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$

76. $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

77. $y = \dfrac{x^4 + 1}{x^2}$

78. $y = \dfrac{x^2 + 4}{2x}$

79. $y = \dfrac{1}{x^2 - 1}$

80. $y = \dfrac{x^2}{x^2 - 1}$

Reference:

Thomas’ Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.