MAT1320-Linear Algebra Lecture Notes

Linear Dependence and Independence of Vectors and Spanning Sets

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Definition

Let V be a real vector space, $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n \in V$.

Linear Dependence and Independence of Vectors

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Let V be a real vector space, $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n \in V$. We say vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n$ are linearly dependent if there exists scalars $x_1, x_2, \ldots, x_n \in \mathbb{R}$, not all of them zero, such that $x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n = \overrightarrow{\mathbf{0}}$. We say vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n$ are linearly independent if there exists scalars $x_1, x_2, \ldots, x_n \in \mathbb{R}$, all of them zero, such that $x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n = \overrightarrow{\mathbf{0}}$.

Note: If the only solution of the homogeneous system $x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n = \overrightarrow{\mathbf{0}}$ is zero solution, then we say vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n$ are linearly independent.

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Example
The vectors
$\{\overrightarrow{\mathbf{e}}_1 = (1,0,0), \overrightarrow{\mathbf{e}}_2 = (0,1,0), \overrightarrow{\mathbf{e}}_3 = (0,0,1)\} \subset \mathbb{R}^3$ are linearly independent.

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The vectors $\left\{\overrightarrow{\mathbf{e}}_1 = (1,0,0), \overrightarrow{\mathbf{e}}_2 = (0,1,0), \overrightarrow{\mathbf{e}}_3 = (0,0,1)\right\} \subset \mathbb{R}^3$ are linearly independent. For $x_1, x_2, x_3 \in \mathbb{R}$ we have

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$$\begin{aligned} &\Rightarrow \quad x_1\overrightarrow{\mathbf{e}}_1 + x_2\overrightarrow{\mathbf{e}}_2 + x_3\overrightarrow{\mathbf{e}}_3 = (0,0,0) \\ &\Rightarrow \quad x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1) = (0,0,0) \end{aligned}$$

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&\Rightarrow \quad x_1 \overrightarrow{\mathbf{e}}_1 + x_2 \overrightarrow{\mathbf{e}}_2 + x_3 \overrightarrow{\mathbf{e}}_3 = (0,0,0) \\
&\Rightarrow \quad x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1) = (0,0,0) \\
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&\Rightarrow \quad (x_1, x_2, x_3) = (0,0,0) \\
&\Rightarrow \quad x_1 = 0, x_2 = 0, x_3 = 0.
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&\Rightarrow \quad (x_1, x_2, x_3) = (0,0,0)\\
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Thus $\overrightarrow{\mathbf{e}}_1$, $\overrightarrow{\mathbf{e}}_2$, $\overrightarrow{\mathbf{e}}_3$ are linearly independent.

Example

Let's show that the vectors $\{\overrightarrow{\mathbf{v}}_1 = (1,2,0)\,, \overrightarrow{\mathbf{v}}_2 = (2,0,1)\,, \overrightarrow{\mathbf{v}}_3 = (3,2,1)\} \subset \mathbb{R}^3$ are linearly dependent.

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$$\Rightarrow \quad x_1(1,2,0) + x_2(2,0,1) + x_3(3,2,1) = (0,0,0)$$

$$\Rightarrow \quad \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

Linear Dependence and Independence of Vectors

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&\Rightarrow \quad x_1\overrightarrow{\mathbf{v}}_1 + x_2\overrightarrow{\mathbf{v}}_2 + x_3\overrightarrow{\mathbf{v}}_3 = (0,0,0) \\
&\Rightarrow \quad x_1(1,2,0) + x_2(2,0,1) + x_3(3,2,1) = (0,0,0) \\
&\Rightarrow \quad \left\{ \begin{array}{c} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{array} \right. \quad \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}
\end{aligned}$$

Example

Let's show that the vectors $\{\overrightarrow{\mathbf{v}}_1 = (1,2,0)\,, \overrightarrow{\mathbf{v}}_2 = (2,0,1)\,, \overrightarrow{\mathbf{v}}_3 = (3,2,1)\} \subset \mathbb{R}^3$ are linearly dependent. To do this, let $x_1, x_2, x_3 \in \mathbb{R}$. Then we have

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&\Rightarrow \quad x_1(1,2,0) + x_2(2,0,1) + x_3(3,2,1) = (0,0,0) \\
&\Rightarrow \quad \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \quad \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&\Rightarrow \quad x_1 + x_3 = 0,\ x_2 + x_3 = 0
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&\Rightarrow \quad x_1\overrightarrow{\mathbf{v}}_1 + x_2\overrightarrow{\mathbf{v}}_2 + x_3\overrightarrow{\mathbf{v}}_3 = (0,0,0)\\
&\Rightarrow \quad x_1(1,2,0) + x_2(2,0,1) + x_3(3,2,1) = (0,0,0)\\
&\Rightarrow \quad \left\{\begin{array}{c} x_1 + 2x_2 + 3x_3 = 0\\ 2x_1 + 2x_3 = 0\\ x_2 + x_3 = 0 \end{array}\right. \quad \Rightarrow \begin{pmatrix} 1 & 2 & 3\\ 2 & 0 & 2\\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0 \end{pmatrix}\\
&\Rightarrow \quad x_1 + x_3 = 0,\ x_2 + x_3 = 0 \Rightarrow \text{If } x_3 = r,\ \text{then } x_1 = x_2 = -r.
\end{aligned}$$

Example

Let's show that the vectors $\{\overrightarrow{\mathbf{v}}_1 = (1,2,0), \overrightarrow{\mathbf{v}}_2 = (2,0,1), \overrightarrow{\mathbf{v}}_3 = (3,2,1)\} \subset \mathbb{R}^3$ are linearly dependent. To do this, let $x_1, x_2, x_3 \in \mathbb{R}$. Then we have

$$\begin{aligned}
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&\Rightarrow \quad x_1(1,2,0) + x_2(2,0,1) + x_3(3,2,1) = (0,0,0) \\
&\Rightarrow \quad \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&\Rightarrow \quad x_1 + x_3 = 0,\ x_2 + x_3 = 0 \Rightarrow \text{If } x_3 = r,\ \text{then } x_1 = x_2 = -r.
\end{aligned}$$

This means that the system has infinitely many solution. So $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \overrightarrow{\mathbf{v}}_3$ are linearly dependent.

Linear Combination of Vectors

Definition
Let V be a real vector space, $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n \in V$.

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$$\overrightarrow{\mathbf{w}} = x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n,$$

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$$\overrightarrow{\mathbf{w}} = x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n,$$

then we say $\overrightarrow{\mathbf{w}}$ is a linear combination of the vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n$.

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$$\overrightarrow{\mathbf{w}} = x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n,$$

then we say $\overrightarrow{\mathbf{w}}$ is a linear combination of the vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n$. Here, again $\overrightarrow{\mathbf{w}} = x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n$ is an element of V.

Example
Show that $\overrightarrow{\mathbf{w}} = (9, 2, 7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1, 2, -1)$ and $\overrightarrow{\mathbf{v}} = (6, 4, 2)$, but $\overrightarrow{\mathbf{w}}' = (4, -1, 8)$ is not.

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Show that $\overrightarrow{\mathbf{w}} = (9, 2, 7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1, 2, -1)$ and $\overrightarrow{\mathbf{v}} = (6, 4, 2)$, but $\overrightarrow{\mathbf{w}}' = (4, -1, 8)$ is not.

$$\overrightarrow{\mathbf{w}} \quad = \quad x_1 \overrightarrow{\mathbf{u}} + x_2 \overrightarrow{\mathbf{v}}$$

Example

Show that $\overrightarrow{\mathbf{w}} = (9, 2, 7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1, 2, -1)$ and $\overrightarrow{\mathbf{v}} = (6, 4, 2)$, but $\overrightarrow{\mathbf{w}}' = (4, -1, 8)$ is not.

$$\begin{aligned} \overrightarrow{\mathbf{w}} &= x_1 \overrightarrow{\mathbf{u}} + x_2 \overrightarrow{\mathbf{v}} \\ (9, 2, 7) &= x_1 (1, 2, -1) + x_2 (6, 4, 2) \end{aligned}$$

Example

Show that $\overrightarrow{\mathbf{w}} = (9,2,7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1,2,-1)$ and $\overrightarrow{\mathbf{v}} = (6,4,2)$, but $\overrightarrow{\mathbf{w}}' = (4,-1,8)$ is not.

$$\begin{aligned}
\overrightarrow{\mathbf{w}} &= x_1\overrightarrow{\mathbf{u}} + x_2\overrightarrow{\mathbf{v}} \\
(9,2,7) &= x_1(1,2,-1) + x_2(6,4,2) \\
&\Rightarrow \begin{cases} x_1 + 6x_2 = 9 \\ 2x_1 + 4x_2 = 2 \\ -x_1 + 2x_2 = 7 \end{cases}
\end{aligned}$$

Example

Show that $\overrightarrow{\mathbf{w}} = (9,2,7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1,2,-1)$ and $\overrightarrow{\mathbf{v}} = (6,4,2)$, but $\overrightarrow{\mathbf{w}}' = (4,-1,8)$ is not.

$$\begin{aligned}
\overrightarrow{\mathbf{w}} &= x_1 \overrightarrow{\mathbf{u}} + x_2 \overrightarrow{\mathbf{v}} \\
(9,2,7) &= x_1 (1,2,-1) + x_2 (6,4,2) \\
&\Rightarrow \begin{cases} x_1 + 6x_2 = 9 \\ 2x_1 + 4x_2 = 2 \\ -x_1 + 2x_2 = 7 \end{cases} \Rightarrow \begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix}
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$$\begin{aligned}
\overrightarrow{\mathbf{w}} &= x_1 \overrightarrow{\mathbf{u}} + x_2 \overrightarrow{\mathbf{v}} \\
(9, 2, 7) &= x_1 (1, 2, -1) + x_2 (6, 4, 2) \\
&\Rightarrow \begin{cases} x_1 + 6x_2 = 9 \\ 2x_1 + 4x_2 = 2 \\ -x_1 + 2x_2 = 7 \end{cases} \Rightarrow \begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Example

Show that $\overrightarrow{\mathbf{w}} = (9, 2, 7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1, 2, -1)$ and $\overrightarrow{\mathbf{v}} = (6, 4, 2)$, but $\overrightarrow{\mathbf{w}}' = (4, -1, 8)$ is not.

$$
\begin{aligned}
\overrightarrow{\mathbf{w}} &= x_1 \overrightarrow{\mathbf{u}} + x_2 \overrightarrow{\mathbf{v}} \\
(9, 2, 7) &= x_1 (1, 2, -1) + x_2 (6, 4, 2) \\
&\Rightarrow \begin{cases} x_1 + 6x_2 = 9 \\ 2x_1 + 4x_2 = 2 \\ -x_1 + 2x_2 = 7 \end{cases} \Rightarrow \begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\
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$$
\begin{aligned}
\overrightarrow{\mathbf{w}} &= x_1 \overrightarrow{\mathbf{u}} + x_2 \overrightarrow{\mathbf{v}} \\
(9, 2, 7) &= x_1 (1, 2, -1) + x_2 (6, 4, 2) \\
&\Rightarrow \begin{cases} x_1 + 6x_2 = 9 \\ 2x_1 + 4x_2 = 2 \\ -x_1 + 2x_2 = 7 \end{cases} \Rightarrow \begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\
&\Rightarrow x_1 = -3, \; x_2 = 2.
\end{aligned}
$$

This means that the system has a unique solution and we have $(9, 2, 7) = -3 (1, 2, -1) + 2 (6, 4, 2)$.

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Show that $\overrightarrow{\mathbf{w}} = (9, 2, 7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1, 2, -1)$ and $\overrightarrow{\mathbf{v}} = (6, 4, 2)$, but $\overrightarrow{\mathbf{w}}' = (4, -1, 8)$ is not.

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$$\begin{aligned} \overrightarrow{\mathbf{w}}' &= x_1 \overrightarrow{\mathbf{u}} + x_2 \overrightarrow{\mathbf{v}} \\ (4, -1, 8) &= x_1 (1, 2, -1) + x_2 (6, 4, 2) \end{aligned}$$

Example
Show that $\overrightarrow{\mathbf{w}} = (9, 2, 7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1, 2, -1)$ and $\overrightarrow{\mathbf{v}} = (6, 4, 2)$, but $\overrightarrow{\mathbf{w}}' = (4, -1, 8)$ is not.

$$
\begin{aligned}
\overrightarrow{\mathbf{w}}' &= x_1 \overrightarrow{\mathbf{u}} + x_2 \overrightarrow{\mathbf{v}} \\
(4, -1, 8) &= x_1 (1, 2, -1) + x_2 (6, 4, 2)
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$$
\begin{cases}
x_1 + 6x_2 = 4 \\
2x_1 + 4x_2 = -1 \\
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\end{cases}
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Example
Show that $\overrightarrow{\mathbf{w}} = (9, 2, 7) \in \mathbb{R}^3$ is a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (1, 2, -1)$ and $\overrightarrow{\mathbf{v}} = (6, 4, 2)$, but $\overrightarrow{\mathbf{w}}' = (4, -1, 8)$ is not.

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This means that the system is inconsistent. Thus $\overrightarrow{\mathbf{w}}'$ can not be written as a combination of the vectors $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$.

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For which values of k, $\overrightarrow{\mathbf{w}} = (1, -2, k) \in \mathbb{R}^3$ can be written as a linear combination of the vectors $\overrightarrow{\mathbf{u}} = (3, 0, -2)$ and $\overrightarrow{\mathbf{v}} = (2, -1, 5)$.

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$$\Rightarrow k = -2(-1) + 5.2 = 12.$$

Theorem

Let $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_m$ be m linearly independent vectors in V. If $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_m, \overrightarrow{\mathbf{v}}_{m+1}$ are linearly dependent, then $\overrightarrow{\mathbf{v}}_{m+1}$ can be written as a linear combination of the vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_m$.

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For $r < m$, if r vectors among $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_m \in V$ are linearly dependent, then $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_m$ are also linearly dependent.

Theorem

Let V a vector space, and for $m \leqslant n$, $\overrightarrow{\mathbf{v}}_1 = (a_{11}, a_{12}, \ldots a_{1n})$, $\overrightarrow{\mathbf{v}}_2 = (a_{21}, a_{22}, \ldots a_{2n})$, $\ldots$, $\overrightarrow{\mathbf{v}}_m = (a_{m1}, a_{m2}, \ldots a_{mn})$ such that

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

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3. *If $n < m$, then $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_m$ are linearly dependent.*

Example

Let $\overrightarrow{\mathbf{a}} = (1, 0, 0, 1)$, $\overrightarrow{\mathbf{b}} = (0, -1, 2, 1)$, $\overrightarrow{\mathbf{c}} = (1, 2, 2, 1)$ and $\overrightarrow{\mathbf{d}} = (-2, 1, 0, 0) \in \mathbb{R}^4$ are given. Then

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1. Determine, whether or not the vectors $\overrightarrow{\mathbf{a}}$, $\overrightarrow{\mathbf{b}}$, $\overrightarrow{\mathbf{c}}$, $\overrightarrow{\mathbf{d}}$ are linearly independent or dependent?
2. Express $\overrightarrow{\mathbf{u}} = (1,-1,2,1)$ as a linear combination of $\overrightarrow{\mathbf{a}}$, $\overrightarrow{\mathbf{b}}$, $\overrightarrow{\mathbf{c}}$, $\overrightarrow{\mathbf{d}}$.

Solution (1)

$c_1 \overrightarrow{\mathbf{a}} + c_2 \overrightarrow{\mathbf{b}} + c_3 \overrightarrow{\mathbf{c}} + c_4 \overrightarrow{\mathbf{d}} = (0, 0, 0, 0)$

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is the unique solution of the system. Hence $\overrightarrow{\mathbf{a}}$, $\overrightarrow{\mathbf{b}}$, $\overrightarrow{\mathbf{c}}$, $\overrightarrow{\mathbf{d}}$ *are linearly independent.*

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So we have $\frac{6}{7} \overrightarrow{\mathbf{b}} + \frac{1}{7} \overrightarrow{\mathbf{c}} - \frac{3}{7} \overrightarrow{\mathbf{d}} = (1, -1, 2, 1).$

Spanning Sets

Definition
Le $S = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_r\} \subset V$ be given. The set spanned by S is denoted by $span\,(S)$ or $\langle S \rangle$ and defined as the set of possible all linear combinations of S.

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$$span\,(S) = \left\{ k_1 \overrightarrow{\mathbf{v}}_1 + k_2 \overrightarrow{\mathbf{v}}_2 + \ldots + k_r \overrightarrow{\mathbf{v}}_r \right| k_1, k_2, \ldots, k_r \in \mathbb{R} \left.\right\}.$$

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$$span\,(\{(1, -2, 1)\}) = \left\{ k\,(1, -2, 1) \right| k \in \mathbb{R} \right\}.$$

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$=\{\, a\,(1, -2, 1, 3) + b\,(0, 2, -1, 0) \mid a, b \in \mathbb{R}\}$.

?