10 Sequences and Series

Q1 Find the limit of the sequence with general term $a_n = n - \ln(e^n + 1)$.

Solution:

$$a_n = n - \ln(e^n+1) = \ln(e^n) - \ln(e^n+1)$$
$$= \ln\left(\frac{e^n}{e^n+1}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{e^n}{e^n+1}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{1+\frac{1}{e^n}}\right) = \ln(1) = \boxed{0}$$

Q2 Calculate the limit of the sequence

$$a_n = \frac{n^2}{2n+1}\sin\frac{2}{n}$$

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n+1}\sin\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{n^2}{2n+1}\cdot\frac{2}{n}\cdot\frac{\sin\frac{2}{n}}{\frac{2}{n}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{2n^2+n}\cdot\lim_{n\to\infty}\frac{\sin\frac{2}{n}}{\frac{2}{n}} \qquad \left(t=\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \frac{2}{2+\frac{1}{n}}\cdot\lim_{t\to 0}\frac{\sin t}{t}$$

$$= 1\cdot 1 = \boxed{1}$$

Q3) Find the limit of the sequence

$$\left\{\frac{n^{5/2}}{2n^2+1}\cdot \sin\frac{1}{\sqrt{n}}\right\}_{n\geq 1}.$$

Solution:

$$\lim_{n\to\infty} \frac{n^{5/2}}{2n^2+1}\cdot \sin\frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{n^{5/2}}{2n^2+1}\cdot\frac{1}{\sqrt{n}}\cdot\frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{n^{5/2}}{2n^{5/2}+n^{1/2}}\cdot \lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \qquad \left(t=\frac{1}{\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2+\frac{1}{n^2}}\cdot \lim_{t\to 0}\frac{\sin t}{t}$$

$$= \frac{1}{2}\cdot 1 = \boxed{\frac{1}{2}}$$

Q4 A sequence $\{a_n\}$ is defined recursively as $a_1=\frac{1}{2}$ and for any natural number $n\geq 1$, $a_{n+1}=\sqrt{3+a_n}-1$. If it is known that $\lim_{n\to\infty} a_n=1$, find the limit of the sequence $\left\{\frac{a_{n+1}-1}{a_n-1}\right\}$.

Solution:

$$\lim_{n\to\infty}\frac{a_{n+1}-1}{a_n-1}=\lim_{n\to\infty}\frac{\left(\sqrt{3+a_n}-1\right)-1}{a_n-1}$$

$$=\lim_{n\to\infty}\frac{\sqrt{3+a_n}-2}{a_n-1}\qquad\left(\frac{0}{0}\right)$$

$$=\lim_{n\to\infty}\frac{\left(\sqrt{3+a_n}-2\right)}{(a_n-1)}\cdot\frac{\left(\sqrt{3+a_n}+2\right)}{\left(\sqrt{3+a_n}+2\right)}$$

$$=\lim_{n\to\infty}\frac{(3+a_n)-4}{(a_n-1)\cdot\left(\sqrt{3+a_n}+2\right)}$$

$$=\lim_{n\to\infty}\frac{(a_n-1)}{(a_n-1)\cdot\left(\sqrt{3+a_n}+2\right)}$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{3+a_n}+2}$$

$$=\frac{1}{\sqrt{3+1}+2}=\boxed{\frac{1}{4}}$$

In questions 5 – 16, determine whether the given series are convergent or divergent.

Q5 $\sum_{n=1}^{\infty} \frac{3^n}{1+n3^n}$.

Solution:

$$\frac{\frac{3^n}{1+n3^n}}{\frac{1}{n}} = \frac{n3^n}{1+n3^n} = \frac{1}{\frac{1}{n3^n}+1} \longrightarrow 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic series), by the Limit Comparison Test, the given series also diverges.

Q6 $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{(n+1)!}$

Solution:

Let $a_n = \dfrac{9^n}{(n+1)!}$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{9^{n+1}}{(n+2)!}}{\dfrac{9^n}{(n+1)!}} = \frac{9^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{9^n} = \frac{9}{n+2} \longrightarrow 0 < 1.$$

Therefore, the series <u>converges</u> by the Ratio Test.

Q7 $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Solution:

Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then,

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e} < 1$$

Therefore, the series converges by the Root Test.

Q8 $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(3^{-n})}{1+3^n}$

Solution:

For all $n \geq 1$, $0 < 3^{-n} = \frac{1}{3^n} \leq \frac{1}{3}$ and hence

$0 \leq \sin(3^{-n}) \leq 1$.

Therefore, $0 \leq \dfrac{\sin(3^{-n})}{1+3^n} \leq \dfrac{1}{1+3^n} \leq \dfrac{1}{3^n}$ for all $n \geq 1$.

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric series with $\left|r = \frac{1}{3}\right| < 1$),

by the Comparison Test, the given series also

converges.

Q9 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2 + n\sin n}$

Solution:

1

$$\frac{\frac{1}{2n^2+n\sin n}}{\frac{1}{n^2}} = \frac{n^2}{2n^2+n\sin n} = \frac{1}{2+\frac{\sin n}{n}} \longrightarrow \frac{1}{2}$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series with $p=2>1$),

by the Limit Comparison Test, the given series

also converges.

Q10 $\sum_{n=1}^{\infty} \frac{1}{2n+n\sin n}$

Solution:

For all $n \geqslant 1$, we have $-1 \leqslant \sin n \leqslant 1$,

$$-n \leqslant n\sin n \leqslant n,$$

$$2n-n \leqslant 2n+n\sin n \leqslant 2n+n,$$

$$0 \leqslant n \leqslant 2n+n\sin n \leqslant 3n.$$

Hence, for all $n \geqslant 1$, we get

$$0 \leqslant \frac{1}{3n} \leqslant \frac{1}{2n+n\sin n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \cdot \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic series), by the Comparison Test, the given series also diverges.

Q11 $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Solution:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{t\to 0} \frac{\sin t}{t} = 1.$$

$\boxed{t=\frac{1}{n}}$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic series),

by the Limit Comparison Test, the given

series also diverges.

Q12 $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \cdot \sin\frac{1}{n}}{\sqrt{n^2+n+1}}$

Solution:

$$\lim_{n\to\infty} \frac{n^2 \cdot \sin\frac{1}{n}}{\sqrt{n^2+n+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n+1}} \cdot \frac{\sin\frac{1}{n}}{\frac{1}{n}} \qquad \left(t=\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{n}{n\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} \cdot \lim_{t\to 0} \frac{\sin t}{t}$$

$$= 1 \cdot 1 = 1 \neq 0.$$

Therefore, the series diverges by the n^{th}-term Test.

$$\text{Q13} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^4+\ln n}$$

Solution:

Let $a_n = \dfrac{(-1)^n n^2}{n^4+\ln n}$. Then, $|a_n| = \dfrac{n^2}{n^4+\ln n}$ for all $n \geq 1$.

$$\frac{|a_n|}{\frac{1}{n^2}} = \frac{\frac{n^2}{n^4+\ln n}}{\frac{1}{n^2}} = \frac{n^4}{n^4+\ln n} = \frac{1}{1+\frac{\ln n}{n^4}} \longrightarrow 1.$$

$$\left(\lim_{n\to\infty} \frac{\ln n}{n^4} \underset{\frac{\infty}{\infty},\ \text{L'H}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{4n^3} = \lim_{n\to\infty} \frac{1}{4n^4} = 0\right).$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2>1$), by the Limit Comparison Test $\sum_{n=1}^{\infty} |a_n|$ also converges. Therefore, by the Absolute Convergence Test, $\sum_{n=1}^{\infty} a_n$ also <u>converges</u>.

Q14 $\sum_{n=2}^{\infty} n \cdot \ln(1-n^{-1})$.

Solution:

$$\lim_{n\to\infty} n \cdot \ln(1-n^{-1}) = \lim_{n\to\infty} n \cdot \ln\left(1-\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \ln\left(\left(1-\frac{1}{n}\right)^{n}\right)$$

$$= \ln\left(e^{-1}\right) = -1 \neq 0.$$

Therefore, the series <u>diverges</u> by the n^{th}-term Test.

Q15 $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n+e^{2n}}}$

Solution:

Let $b_n = \dfrac{n^2}{e^n}$. Then,

$$\sqrt[n]{b_n} = \sqrt[n]{\frac{n^2}{e^n}} = \frac{(\sqrt[n]{n})^2}{e} \longrightarrow \frac{1}{e} < 1.$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2}{e^n}$ converges by the Root Test.

Let $a_n = \dfrac{n^2}{\sqrt{n+e^{2n}}}$. Then,

$$0 \leq a_n = \frac{n^2}{\sqrt{n+e^{2n}}} \leq \frac{n^2}{\sqrt{e^{2n}}} = \frac{n^2}{e^n} = b_n.$$

Since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges.

Q16 $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$

Solution:

Let $f(x) = \frac{1}{x(1+\ln x)}$. The function f is continuous and positive for all $x \geqslant 1$. Also, as x increases, both x and $\ln x$ increase and hence $f(x) = \frac{1}{x(1+\ln x)}$ decreases. Hence $f(x)$ is decreasing for $x \geqslant 1$. We can apply the Integral Test.

$$\int_1^{\infty} \frac{1}{x(1+\ln x)}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x(1+\ln x)}\,dx = \lim_{t\to\infty} \int_1^{1+\ln t} \frac{1}{u}\,du$$

$$\boxed{\begin{array}{l} u = 1+\ln x \\ du = \frac{1}{x}\,dt \\ x=1 \Rightarrow u=1 \\ x=t \Rightarrow u=1+\ln t \end{array}}$$

$$= \lim_{t\to\infty} \Big[\ln|u|\Big]_1^{1+\ln t} = \lim_{t\to\infty} \Big[\ln(1+\ln t) - \ln(1)\Big]$$

$$= \infty.$$

Since the improper integral diverges, by the Integral Test, the given series also diverges.

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Q17 Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges, where the sequence $\{a_n\}$ is defined recursively as $a_1 = \frac{1}{2}$, $a_{n+1} = \frac{1}{2}\left(1+\frac{1}{n}\right)a_n$ for $n \geqslant 1$.

Solution:

Since $a_1 = \frac{1}{2} > 0$ and $\frac{1}{2}\left(1+\frac{1}{n}\right) > 0$ for all $n \geqslant 1$, we get $a_n > 0$ for all $n \geqslant 1$. Hence, we can apply the Ratio Test.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2}\left(1+\frac{1}{n}\right)a_n}{a_n} = \frac{1}{2}\left(1+\frac{1}{n}\right) \longrightarrow \frac{1}{2} < 1.$$

Therefore, the series <u>converges</u> by the Ratio Test.

Q18) Find the limit of the sequence $\{a_n\} = \left\{\left(\frac{n+1}{n+2}\right)^n\right\}$.

Determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)^n$$

$$= \lim_{n\to\infty}\left[\left(1-\frac{1}{n+2}\right)^{n+2}\right]^{\frac{n}{n+2}} = \left(e^{-1}\right)^1 = \frac{1}{e}.$$

$\left(\text{Letting } t=n+2,\ \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)^{n+2} = \lim_{t\to\infty}\left(1+\frac{-1}{t}\right)^t = e^{-1}.\right)$

Since $\lim_{n\to\infty} a_n = \frac{1}{e} \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ diverges by the n^{th}-term Test.

Find the sum of the series in questions 19 – 24

Q19 $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}$

Solution:

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n} = \sum_{n=1}^{\infty} \frac{2}{2^n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} + \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$$

$$= \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) + \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\right)$$

$$= \frac{1}{1-\frac{1}{2}} + \frac{-\frac{1}{2}}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{1}{\frac{1}{2}} + \frac{-\frac{1}{2}}{\frac{3}{2}} = 2 - \frac{1}{3} = \boxed{\frac{5}{3}}$$

Q20 $\sum_{n=1}^{\infty} (\sqrt{e})^{1-4n}$

Solution:

$$\sum_{n=1}^{\infty} (\sqrt{e})^{1-4n} = \sum_{n=1}^{\infty} \frac{\sqrt{e}}{(\sqrt{e})^{4n}} = \sum_{n=1}^{\infty} \frac{\sqrt{e}}{e^{2n}}$$

$$= \frac{\sqrt{e}}{e^2} + \frac{\sqrt{e}}{e^4} + \frac{\sqrt{e}}{e^6} + \cdots$$

$$= \frac{\frac{\sqrt{e}}{e^2}}{1 - \frac{1}{e^2}} = \frac{\frac{\sqrt{e}}{e^2}}{\frac{e^2-1}{e^2}} = \boxed{\frac{\sqrt{e}}{e^2-1}}$$

Q21 $\sum_{n=0}^{\infty} \frac{\pi^{-n}}{\cos(n\pi)}$

Solution:

$$\sum_{n=0}^{\infty} \frac{\pi^{-n}}{\cos(n\pi)} = \sum_{n=0}^{\infty} \frac{\pi^{-n}}{(-1)^n} = \sum_{n=0}^{\infty} \frac{1}{(-1)^n \cdot \pi^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{\pi}\right)^n$$

$$= 1 - \frac{1}{\pi} + \frac{1}{\pi^2} - \frac{1}{\pi^3} + \cdots$$

$$= \frac{1}{1-\left(-\frac{1}{\pi}\right)} = \frac{1}{1+\frac{1}{\pi}} = \frac{1}{\frac{\pi+1}{\pi}} = \boxed{\frac{\pi}{\pi+1}}$$

Q22 $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$

Solution:

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}\cdot\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}.$$

$$S_n = \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}} = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)$$

$$= \left(1-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$$

$$= 1 - \frac{1}{\sqrt{n+1}}.$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right) = \boxed{1}$$

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Q23 $$\sum_{n=1}^{\infty} \ln\left(1+\frac{2}{n(n+3)}\right)$$

Solution:

$$\ln\left(1+\frac{2}{n(n+3)}\right) = \ln\left(\frac{n(n+3)+2}{n(n+3)}\right) = \ln\left(\frac{n^2+3n+2}{n(n+3)}\right)$$

$$= \ln\left(\frac{(n+1)(n+2)}{n(n+3)}\right) = \ln(n+1)+\ln(n+2)-\ln(n)-\ln(n+3).$$

$$S_n = \sum_{k=1}^{n} \ln\left(1+\frac{2}{k(k+3)}\right) = \sum_{k=1}^{n}\left(\ln(k+1)+\ln(k+2)-\ln(k)-\ln(k+3)\right)$$

$$\begin{aligned} = &\left(\ln 2+\ln 3-\ln 1-\ln 4\right)\\ &+\left(\ln 3+\ln 4-\ln 2-\ln 5\right)\\ &+\left(\ln 4+\ln 5-\ln 3-\ln 6\right)\\ &\vdots\\ &+\left(\ln(n+1)+\ln(n+2)-\ln(n)-\ln(n+3)\right) = \ln(n+1)+\ln(3)-\ln(n+3)\\ &= \ln\left(\frac{3(n+1)}{n+3}\right) = \ln\left(\frac{3n+3}{n+3}\right). \end{aligned}$$

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{2}{n(n+3)}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\left(\frac{3n+3}{n+3}\right) \stackrel{!}{=} \boxed{\ln 3}$$

Q24 $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!\,9^n}$

Solution:

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x \in \mathbb{R}.$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!\,9^n} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!\,3^{2n}}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$$

$$= \cos\left(\frac{\pi}{3}\right) = \boxed{\frac{1}{2}}$$

Q25 If $n \geq 0$ is an integer, find the sum of the series $\sum_{k=n}^{\infty} 3^{n-k}$.

Solution:

$$\sum_{k=n}^{\infty} 3^{n-k} = 1 + 3^{-1} + 3^{-2} + 3^{-3} + \cdots$$

$$= 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$$

$$= \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \boxed{\frac{3}{2}}$$

Q26 Determine the interval of convergence and the sum of the series $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}}$.

Solution:

$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{x+2}{3}\right)^n$ is a geometric series whose first term is $a = \frac{1}{3}$ and ratio is $r = \frac{x+2}{3}$. Therefore, it converges if and only if $\left|\frac{x+2}{3}\right| < 1$, $-1 < \frac{x+2}{3} < 1$,

$-3 < x+2 < 3$, $-5 < x < 1$.

The interval of convergence is $(-5, 1)$.

The sum of the series is $\frac{\frac{1}{3}}{1 - \frac{x+2}{3}} = \frac{\frac{1}{3}}{\frac{3-(x+2)}{3}} = \frac{1}{1-x}$.

Q27 Determine the interval of convergence of the power series $\sum_{k=2}^{\infty} \frac{(3-2x)^k}{k^2 \ln k}$.

Solution:

Let $a_k = \frac{(3-2x)^k}{k^2 \ln k}$.

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left|\frac{(3-2x)^{k+1}}{(k+1)^2 \ln(k+1)}\right|}{\left|\frac{(3-2x)^k}{k^2 \ln k}\right|} = \frac{|3-2x|^{k+1}}{(k+1)^2 \ln(k+1)} \cdot \frac{k^2 \ln k}{|3-2x|^k}$$

$$= \left(\frac{k}{k+1}\right)^2 \cdot \frac{\ln k}{\ln(k+1)} \cdot |3-2x| \longrightarrow |3-2x|$$

$$\left(\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2 = \lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{k}}\right)^2 = 1, \quad \lim_{k\to\infty} \frac{\ln k}{\ln(k+1)} \underset{L'H}{=} \lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{k+1}} = \lim_{k\to\infty} \frac{k+1}{k} = 1\right)$$

Hence, the series converges absolutely if $|3-2x| < 1$,

$-1 < 3-2x < 1$, $\quad -4 < -2x < -2$, $\quad 1 < x < 2$.

When $x=1$, $\sum_{k=2}^{\infty} \frac{(3-2x)^k}{k^2 \ln k} = \sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$ converges by the Comparison Test since $0 < \frac{1}{k^2 \ln k} < \frac{1}{k^2}$ for $k \geq 3$ and $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges (p-series with $p = 2 > 1$).

When $x=2$, $\sum_{k=2}^{\infty} \frac{(3-2x)^k}{k^2 \ln k} = \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 \ln k}$ converges by the Absolute Convergence Test since $\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$ converges.

Therefore, the Interval of Convergence is $[1, 2]$.

Q28 For which values of $x \in \mathbb{R}$ does the power series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{2^n \cdot \sqrt[3]{n^2+5}}$

(i) absolutely converge?

(ii) conditionally converge?

(iii) diverge?

Solution:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|(-1)^{n+1} \frac{(x-3)^{n+1}}{2^{n+1} \cdot \sqrt[3]{(n+1)^2+5}}\right|}{\left|(-1)^n \frac{(x-3)^n}{2^n \cdot \sqrt[3]{n^2+5}}\right|} = \frac{|x-3|^{n+1}}{2^{n+1} \cdot \sqrt[3]{n^2+2n+6}} \cdot \frac{2^n \cdot \sqrt[3]{n^2+5}}{|x-3|^n}$$

$$= \frac{1}{2} \cdot \sqrt[3]{\frac{1+\frac{5}{n^2}}{1+\frac{2}{n}+\frac{6}{n^2}}} \cdot |x-3| \longrightarrow \frac{|x-3|}{2}.$$

The series converges absolutely if $\frac{|x-3|}{2} < 1$, $|x-3| < 2$,

$-2 < x-3 < 2$, $\quad 1 < x < 5$.

When $x=1$, $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{2^n \sqrt[3]{n^2+5}} = \sum_{n=1}^{\infty} (-1)^n \frac{(-2)^n}{2^n \sqrt[3]{n^2+5}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+5}}$

$$\frac{\frac{1}{\sqrt[3]{n^2+5}}}{\frac{1}{n^{2/3}}} = \frac{n^{2/3}}{\sqrt[3]{n^2+5}} = \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2+5}} = \sqrt[3]{\frac{1}{1+\frac{5}{n^2}}} \longrightarrow 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges (p-series with $p = \frac{2}{3} < 1$),

the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+5}}$ also diverges.

When $x=5$, $\sum_{n=1}^{\infty}(-1)^n \frac{(x-3)^n}{2^n\sqrt[3]{n^2+5}} = \sum_{n=1}^{\infty}(-1)^n \frac{2^n}{2^n \cdot \sqrt[3]{n^2+5}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2+5}}$

Let $u_n = \frac{1}{\sqrt[3]{n^2+5}}$.

(i) $u_n = \frac{1}{\sqrt[3]{n^2+5}} > 0$ for all $n \geq 1$.

(ii) As n increases, n^2+5 increases and hence $\sqrt[3]{n^2+5}$ increases, $\frac{1}{\sqrt[3]{n^2+5}}$ decreases. Hence, u_n is decreasing for $n \geq 1$.

(iii) $u_n = \frac{1}{\sqrt[3]{n^2+5}} \longrightarrow 0$ as $n \to \infty$.

Hence, by the Alternating Series Test, the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt[3]{n^2+5}}$ converges.

Therefore, the given power series

(i) converges absolutely for $1 < x < 5$

(ii) converges conditionally for $x = 5$

(iii) diverges for $x \leq 1$ or $x > 5$.

Q29 Find the value(s) of b for which the radius of convergence of the power series $\sum_{n=2}^{\infty} \frac{b^n x^n}{\ln n}$ is 3.

Solution:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{b^{n+1}\,x^{n+1}}{\ln(n+1)}\right|}{\left|\frac{b^n\,x^n}{\ln n}\right|} = \frac{|b|^{n+1}\cdot|x|^{n+1}}{\ln(n+1)}\cdot\frac{\ln(n)}{|b|^n\cdot|x|^n}$$

$$= \frac{\ln(n)}{\ln(n+1)}\cdot|b|\cdot|x| \longrightarrow |bx|$$

$$\left(\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)} \overset{\text{L'H}}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{n+1}{n} = 1\right)$$

The series converges absolutely if $|bx| < 1$, $|x| < \frac{1}{|b|}$.

Hence, its radius of convergence is $R = \frac{1}{|b|}$.

$$R = 3 \iff \frac{1}{|b|} = 3 \iff |b| = \frac{1}{3} \iff \boxed{b = \mp\frac{1}{3}}$$

Q30 By using the power series $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$, $|x|<1$, find a power series which represents the function $f(x)=\ln\left(\frac{1+x}{1-x}\right)$.

Solution:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots, \quad -1<x<1$$

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n = 1-x+x^2-x^3+\cdots, \quad -1<x<1$$

Integrating both sides, we get

$$\ln(1+x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = C + x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots, \quad -1<x<1.$$

For $x=0$, we get $\ln(1)=C$ and hence $C=0$. Hence,

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots, \quad -1<x<1.$$

$$\ln(1-x) = \sum_{n=0}^{\infty}(-1)^n \frac{(-x)^{n+1}}{n+1} = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots, \quad -1<x<1$$

$$f(x)=\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)+\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)$$

$$=2x+2\frac{x^3}{3}+2\frac{x^5}{5}+2\frac{x^7}{7}+\cdots=\sum_{n=0}^{\infty}2\cdot\frac{x^{2n+1}}{2n+1}, \quad -1<x<1$$

Q31 Find the Taylor series for $f(x)=\frac{x}{1+x}$ at $x=1$.

Solution:

$$f(x)=\frac{x}{1+x}=1-\frac{1}{1+x}$$

$$f'(x)=\frac{1}{(1+x)^2}$$

$$f''(x)=\frac{-2}{(1+x)^3}$$

$$f'''(x)=\frac{2\cdot 3}{(1+x)^4}$$

$$f^{(4)}(x)=\frac{-2\cdot 3\cdot 4}{(1+x)^5}$$

$$\vdots$$

$$f^{(n)}(x)=\frac{(-1)^{n+1}\cdot n!}{(1+x)^{n+1}} \text{ for } n\geq 1$$

$$f(1)=1-\frac{1}{2}=\frac{1}{2}.$$

For $n\geq 1$,

$$f^{(n)}(1)=\frac{(-1)^{n+1}\cdot n!}{2^{n+1}}. \text{ So,}$$

$$\frac{f^{(n)}(1)}{n!}=\frac{(-1)^{n+1}}{2^{n+1}}$$

Hence, the Taylor series of f at $x=1$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n=f(1)+\sum_{n=1}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}(x-1)^n$$

$$=\frac{1}{2}+\frac{1}{4}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3-\cdots.$$

Q32 Evaluate $\lim_{x\to 0} \frac{\ln(1+2x^2)}{x\sin x}$.

Solution:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x \leq 1$$

$$\ln(1+2x^2) = 2x^2 - \frac{(2x^2)^2}{2} + \frac{(2x^2)^3}{3} - \frac{(2x^2)^4}{4} + \cdots, \quad -1 < 2x^2 \leq 1$$

$$\ln(1+2x^2) = 2x^2 - 2x^4 + \frac{8x^6}{3} - 4x^8 + \cdots, \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad -\infty < x < \infty$$

$$\frac{\ln(1+2x^2)}{x\sin x} = \frac{2x^2 - 2x^4 + \frac{8x^6}{3} - 4x^8 + \cdots}{x^2 - \frac{x^4}{6} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots}, \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, \ x \neq 0$$

$$\frac{\ln(1+2x^2)}{x\sin x} = \frac{2 - 2x^2 + \frac{8x^4}{3} - 4x^6 + \cdots}{1 - \frac{x^2}{6} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}, \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, \ x \neq 0$$

$$\lim_{x\to 0} \frac{\ln(1+2x^2)}{x\sin x} = \frac{2 - 0 + 0 - 0 + \cdots}{1 - 0 + 0 - 0 + \cdots} = \boxed{2}$$

Q33 Evaluate $\lim\limits_{x\to 0} \dfrac{x\cos x - xe^{-x^2}}{\sin^3 x}$.

Solution:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad -\infty < x < \infty$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad -\infty < x < \infty$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots, \quad -\infty < x < \infty$$

$$\cos x - e^{-x^2} = \frac{1}{2}x^2 - \frac{11}{24}x^4 + \cdots, \quad -\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad -\infty < x < \infty$$

$$\sin^3 x = x^3 + 3\cdot x^2\left(-\frac{x^3}{3!}\right) + \cdots, \quad -\infty < x < \infty$$

$$\sin^3 x = x^3 - \frac{1}{2}x^5 + \cdots, \quad -\infty < x < \infty$$

$$\frac{x\cos x - xe^{-x^2}}{\sin^3 x} = \frac{x\left(\cos x - e^{-x^2}\right)}{\sin^3 x} = \frac{\frac{1}{2}x^3 - \frac{11}{24}x^5 + \cdots}{x^3 - \frac{1}{2}x^5 + \cdots}, \quad \text{for } \sin x \neq 0$$

$$\frac{x\cos x - xe^{-x^2}}{\sin^3 x} = \frac{\frac{1}{2} - \frac{11}{24}x^2 + \cdots}{1 - \frac{1}{2}x^2 + \cdots}, \quad \text{for } \sin x \neq 0$$

$$\lim_{x\to 0}\frac{x\cos x - xe^{-x^2}}{\sin^3 x} = \frac{\frac{1}{2} - 0 + 0 + \cdots}{1 - 0 + 0 + \cdots} = \boxed{\frac{1}{2}}$$

Q34 Evaluate $\lim_{x \to 0} \int_x^{2x} \frac{e^{2t}-1}{t^2} dt$.

Solution:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \quad -\infty < x < \infty$$

$$e^{2t} = 1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \cdots, \quad -\infty < t < \infty$$

$$e^{2t} - 1 = 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \cdots, \quad -\infty < t < \infty$$

$$\frac{e^{2t}-1}{t^2} = \frac{2}{t} + 2 + \frac{4}{3}t + \frac{2}{3}t^2 + \cdots, \quad \text{for } t \neq 0$$

$$\int_x^{2x} \frac{e^{2t}-1}{t^2} dt = \left[2\ln|t| + 2t + \frac{2}{3}t^2 + \frac{2}{9}t^3 + \cdots \right]_x^{2x}$$

$$= \left(2\ln|2x| + 4x + \frac{8}{3}x^2 + \frac{16}{9}x^3 + \cdots \right) - \left(2\ln|x| + 2x + \frac{2}{3}x^2 + \frac{2}{9}x^3 + \cdots \right)$$

$$= 2\left[\ln|2x| - \ln|x| \right] + 2x + 2x^2 + \frac{14}{9}x^3 + \cdots$$

$$= 2\ln 2 + 2x + 2x^2 + \frac{14}{9}x^3 + \cdots$$

$$\lim_{x \to 0} \int_x^{2x} \frac{e^{2t}-1}{t^2} dt = 2\ln 2 + 0 + 0 + 0 + \cdots = \boxed{2\ln 2}$$

Q35 Find the sum of the series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ by using power series.

Solution:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \quad -1 < x < 1$$

Differentiating both sides, we get

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \quad -1 < x < 1$$

Multiplying both sides by x, we get

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k x^k = x + 2x^2 + 3x^3 + 4x^4 + \cdots, \quad -1 < x < 1.$$

For $x = \frac{1}{2}$, we get $\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = \boxed{2}$