MAT1071 MATHEMATICS I
5. WEEK

INDETERMINATE FORMS AND L HOPITAL RULE

Asst. Prof. Dr. Nurten GÜRSES

Indeterminate Forms

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞, which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations.

Asst. Prof. Dr. Nurten GÜRSES

- $\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, \infty-\infty$

- Indeterminate powers

$0^0, \infty^0, 1^\infty, \cancel{0^\infty}$ is not an indeterminate form

☆ $\frac{0}{0}$, $\frac{\infty}{\infty}$ $\longrightarrow$ We use derivatives to calculate the limits of fractions whose numerators and denominators both approach zero or ∞

L'Hôpital's Rule

L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Asst. Prof. Dr. Nurten GÜRSES

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

☆ $0 \cdot \infty$, $\infty - \infty$ $\longrightarrow$ transform to $\frac{0}{0}$, $\frac{\infty}{\infty}$

Asst. Prof. Dr. Nurten GÜRSES

EXAMPLE

L'Hôpital's Rule

$$\lim_{x \to 0} \frac{3x - \sin x}{x} \quad \frac{0}{0} \quad = 2$$

EXAMPLE

L'Hôpital's Rule

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} \quad \frac{0}{0} \quad = \frac{1}{2}$$

EXAMPLE

L'Hôpital's Rule   L'Hôpital's Rule

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} \quad \frac{0}{0} = \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x} \quad \frac{0}{0} = -\frac{1}{8}$$

EXAMPLE

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{1-\cos x}{3x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\sin x}{6x} \overset{\frac{0}{0}}{=} \frac{1}{6}$$

L'Hôpital's Rule L'Hôpital's Rule L'Hôpital's Rule

EXAMPLE $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$ $(\infty - \infty)$

L'Hôpital's Rule

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \frac{0}{0} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \quad \frac{0}{0}$$

L'Hôpital's Rule

$$= \frac{0}{2} = 0$$

Asst. Prof. Dr. Nurten GÜRSES

Exam Q.

$$\lim_{x \to -\infty} \left(2x + \sqrt{4x^2+3x}\right) \quad (\infty - \infty)$$

Solution

$$\lim_{x \to -\infty} \frac{\left(2x+\sqrt{4x^2+3x}\right)\left(2x-\sqrt{4x^2+3x}\right)}{\left(2x-\sqrt{4x^2+3x}\right)}$$

$$= \lim_{x \to -\infty} \frac{-3x}{2x-\sqrt{4x^2+3x}} = \lim_{x \to -\infty} \frac{-3x}{2x-\underbrace{|x|}_{=-x}\sqrt{4+\frac{3}{x}}} \quad \frac{\infty}{\infty}$$

$$= -\frac{3}{4}$$

L'Hôpital's Rule

Asst. Prof. Dr. Nurten GÜRSES

EXAMPLE

L'Hôpital's Rule

$$\lim_{x \to \pi/2} \frac{\sec x}{1+\tan x} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \pi/2} \frac{\cancel{\sec x} \cdot \tan x}{\sec^{\cancel{2}} x}$$

$$= \lim_{x \to \pi/2} \sin x = 1$$

EXAMPLE

L'Hôpital's Rule

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \,//$$

EXAMPLE

L'Hôpital's Rule $\qquad\qquad$ L'Hôpital's Rule

$$\lim_{x\to\infty} \frac{e^x}{x^2} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{e^x}{2x} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

EXAMPLE $\lim_{x \to \infty} x \sin\frac{1}{x}$ $(\infty \cdot 0)$

transform into $\frac{\infty}{\infty}$ or $\frac{0}{0}$

$$= \lim_{x \to \infty} \frac{\sin 1/x}{1/x} \quad \left(\frac{0}{0}\right)$$

L'Hôpital's Rule

$\frac{1}{x} = u$

$x \to \infty \quad \frac{1}{x} = u \to 0$

$$= \lim_{u \to 0} \frac{\sin u}{u} = 1 \;//$$

Asst. Prof. Dr. Nurten GÜRSES

EXAMPLE $\lim_{x \to 0^+} \sqrt{x}(\ln x) \quad (0 \cdot (-\infty))$

$$\Rightarrow \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x \to 0^+} -2\sqrt{x} = 0$$

L'Hôpital's Rule

Indeterminate Powers

1^{∞}, 0^{0}, ∞^{0}, $\cancel{0^{\infty}}$

Apply logarithmic limit

☆ If $\lim_{x \to a} \ln f(x) = L$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^{L}.$$

Here a may be either finite or infinite.

EXAMPLE Apply L'Hopital's Rule to show that

$$\lim_{x \to 0^+} (1+x)^{1/x} = e$$

Solution the limit leads the indeterminate form 1^∞

$$f(x) = (1+x)^{1/x}$$

$$\ln(f(x)) = \ln(1+x)^{1/x}$$

$$\lim_{x \to 0} \ln(f(x)) = \lim_{x \to 0^+} \frac{\ln(1+x)}{x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1} = 1$$

$$\Rightarrow \lim_{x \to 0^+} (1+x)^{1/x} = e^1 = e \; //$$

Asst. Prof. Dr. Nurten GÜRSES

EXAMPLE $\lim_{x \to \infty} x^{\frac{1}{x}} = ? \quad (\infty^0)$

Solution $f(x) = x^{1/x}$

$$\ln\left(\lim_{x \to \infty} x^{1/x}\right) = \lim_{x \to \infty} \ln x^{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{1}{x} = 0$$

$$\Rightarrow \boxed{\lim_{x \to \infty} x^{\frac{1}{x}} = e^0 = 1} \; //$$

Asst. Prof. Dr. Nurten GÜRSES

Exam Q. $\lim\limits_{x \to 0^+} (2-e^{\sqrt{x}})^{1/x}$, 1^{∞}

Solution $\ln\Big(\lim\limits_{x \to 0^+} (2-e^{\sqrt{x}})^{1/x} = \ln L$

$$\Rightarrow \lim_{x \to 0^+} \ln(2-e^{\sqrt{x}})^{\frac{1}{x}} = \ln L$$

$$\Rightarrow \lim_{x \to 0^+} -\frac{1}{2\sqrt{x}}\, e^{\sqrt{x}} = \ln L$$

$$-\infty = \ln L \Rightarrow L = 0$$

Asst. Prof. Dr. Nurten GÜRSES

EXAMPLE $\lim_{x \to 0} \dfrac{\ln\left(\frac{1}{x^2}\right)}{\cot x^2}$ $\qquad \dfrac{\infty}{\infty}$

Solution

$$= \lim_{x \to 0} \frac{-\frac{2}{x^3} \cdot x^2}{-2x \cdot \operatorname{cosec}^2 x} = \lim_{x \to 0} \frac{x^{-1}}{x \operatorname{cosec}^2 x}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} = 1 \,//$$

Exam Q.

$$\lim_{x \to 1} \sin\frac{x-1}{2} \tan\frac{\pi x}{2} \longrightarrow 0 \cdot \infty$$

Answer : $-\frac{1}{\pi}$ //

EXAMPLE $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x =$ (1^∞ belirsizliği)

Solution $y=\left(1-\frac{3}{x}\right)^x \Rightarrow \ln y = x\ln\left(1-\frac{3}{x}\right)$ ($\infty \cdot 0$ Belirsizliği)

$$=\frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}}$$ ($\frac{0}{0}$ Belirsizliği) **L'Hôpital's Rule**

$$\lim_{x\to\infty}\frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{3/x^2 \big/ 1-\frac{3}{x}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\frac{3}{\cancel{x^2}\left(1-\frac{3}{x}\right)}}{-\frac{1}{\cancel{x^2}}}$$

$$=\lim_{x\to\infty}\frac{-3}{1-\frac{3}{x}} = -3$$

$$\lim_{x\to\infty}\ln y = -3 \Leftrightarrow \lim_{x\to\infty} y = e^{-3} = \frac{1}{e^3}$$ //

EXAMPLE

$$\lim_{x\to 0^+} x^{\sin x} = (0^0 \text{ Belirsizliği})$$

Solution

$$y = x^{\sin x} \Rightarrow \ln y = \sin x \cdot \ln x \quad (0 \cdot -\infty \text{ Belirsizliği})$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0^+} -\frac{\sin^2 x}{x\cos x}$$

$$\left(\frac{\infty}{\infty}\right)$$

(türev aldıkça böyle devam edecek. düzenleyelim...)

$$= \lim_{x\to 0^+} \left(-\underbrace{\frac{\sin x}{x}}_{1} \cdot \underbrace{\frac{\sin x}{\cos x}}_{0} \right) = 0$$

$$\lim_{x\to 0^+} \ln y = 0 \Leftrightarrow \lim_{x\to 0^+} x^{\sin x} = e^0 = 1$$

Asst. Prof. Dr. Nurten GÜRSES

EXAMPLE $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = ?\quad (1^\infty)$

Solution

$$y=\left(1+\frac{1}{x}\right)^x \Rightarrow \ln y = x.\ln\left(1+\frac{1}{x}\right) \quad (\infty.0)$$

$$=\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{\cancel{-1/x^2}}{1+1/x}}{\cancel{-1/x^2}} = 1$$

$$\lim_{x\to\infty}\ln y = 1 \Leftrightarrow \lim_{x\to\infty} y = e^1 = e$$

Asst. Prof. Dr. Nurten GÜRSES

Exam Q.

S2. a) $\lim_{x\to 0^+} x^{\frac{1}{\ln(e^x-1)}}$ limitini hesaplayınız. (15p) Indeterminate form 0^0

Solution 0^0 belirsizliği var. (1)

$$y = x^{\frac{1}{\ln(e^x-1)}} \Rightarrow \ln y = \frac{1}{\ln(e^x-1)} \cdot \ln x \quad (2)$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{\ln(e^x-1)} \ (3) \overset{\frac{\infty}{\infty}}{\underset{L}{=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{e^x}{e^x-1}} \quad (2)$$

$$= \lim_{x\to 0^+} \frac{e^x-1}{xe^x} \ (2) \overset{\frac{0}{0}}{\underset{L}{=}} \lim_{x\to 0^+} \frac{e^x}{e^x+xe^x} \ (2) = \frac{1}{1+0} = 1 \quad (1)$$

$$\ln\left(\lim_{x\to 0^+} x^{\frac{1}{\ln(e^x-1)}}\right) = 1 \Rightarrow \lim_{x\to 0^+} x^{\frac{1}{\ln(e^x-1)}} = e \quad (2)$$

Asst. Prof. Dr. Nurten GÜRSES

Exam Q.

S 3.a) $\lim\limits_{x \to \frac{\pi}{2}^+} \left[\ln\left(x - \frac{\pi}{2}\right) \cdot \cos x \right]$ limitini hesaplayınız. (13p)

Solution $\lim\limits_{x \to \frac{\pi}{2}^+} \left[\ln\left(x - \frac{\pi}{2}\right) \cdot \cos x \right]^{(0 \cdot \infty)} = \lim\limits_{x \to \frac{\pi}{2}^+} \frac{\ln\left(x - \frac{\pi}{2}\right)}{\sec x} \quad \left(\frac{\infty}{\infty}\right)$

$$\overset{L}{=} \lim_{x \to \frac{\pi}{2}^+} \frac{\frac{1}{x - \frac{\pi}{2}}}{\frac{\sin x}{\cos^2 x}} = \lim_{x \to \frac{\pi}{2}^+} \frac{\cos^2 x}{\sin x \left(x - \frac{\pi}{2}\right)}$$

$$\overset{\frac{0}{0}}{\underset{L}{=}} \lim_{x \to \frac{\pi}{2}^+} \frac{2 \cos x \sin x}{\cos x \left(x - \frac{\pi}{2}\right) + \sin x} = 0$$

Exam Q.

Solution

3. a) $\lim_{x\to\infty} \dfrac{e^{\arctan x}-x}{\ln(1+x^2)+x} = ?$ (10 *puan*)

$$\lim_{x\to\infty} \frac{e^{\arctan x}-x}{\ln(1+x^2)+x} \overset{\frac{\infty}{\infty}}{\underset{L}{=}} (1) \lim_{x\to\infty} \frac{\frac{1}{1+x^2}\cdot e^{\arctan x}-1}{\frac{2x}{1+x^2}+1} \quad (6)$$

$$= \frac{0.e^{\pi/2}-1}{0+1} \quad (2)$$

$$= -1 \quad (1)$$

Asst. Prof. Dr. Nurten GÜRSES

Exam Q.

S1. a) $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{e^x-1}\right)$ limitini hesaplayınız. (12p)

Solution

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{e^x-1}\right) \overset{\infty-\infty}{=} ① \lim_{x\to 0}\frac{e^x-1-\sin x}{\sin x\,(e^x-1)} \quad ③ \quad \left(\frac{0}{0}\right)$$

$$\overset{L}{=} \lim_{x\to 0}\frac{e^x-\cos x}{\cos x\,(e^x-1)+e^x\sin x} \quad ③ \quad \left(\frac{0}{0}\right)$$

$$\overset{L}{=} \lim_{x\to 0}\frac{e^x+\sin x}{-\sin x\,(e^x-1)+e^x\cos x+e^x\cos x+e^x\sin} \quad ③$$

$$=\frac{1}{2} \quad ②$$

Asst. Prof. Dr. Nurten GÜRSES

Exam Q.

2-b) Find $\lim_{x\to 0}\frac{1-\cos(\sin x)}{2x^2}$. (Do NOT use the L'Hôpital's Rule) (12 Points)

Solution

$$\lim_{x\to 0}\frac{1-\cos(\sin x)}{2x^2} = \frac{0}{0} \text{ bl.}$$

$$= \lim_{x\to 0}\frac{1-\cos(\sin x)}{2x^2}\cdot\frac{[1+\cos(\sin x)]}{[1+\cos(\sin x)]} = \lim_{x\to 0}\frac{\sin^2(\sin x)}{2x^2}\cdot\frac{1}{1+\cos(\sin x)}$$

$$= \lim_{x\to 0}\underbrace{\frac{\sin^2(\sin x)}{\sin^2 x}}_{1}\cdot\underbrace{\frac{\sin^2 x}{x^2}}_{1}\cdot\frac{1}{2\underbrace{[1+\cos(\sin x)]}_{2}} = 1\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= \boxed{\frac{1}{4}}$$

Asst. Prof. Dr. Nurten GÜRSES

Exam Q.

1-b) Let $a, b \in \mathbb{R}^+$. Then evaluate $\lim_{x\to 0^+}\left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}}$. (12 Points) 1^∞ bl.

Solution

$$y=\left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}} \Rightarrow \ln y = \frac{2}{x}\cdot\ln\left(\frac{a^x+b^x}{2}\right)$$

$$\lim_{x\to 0^+}\ln y = \ln\left[\lim_{x\to 0^+} y\right] = \lim_{x\to 0^+}\frac{2}{x}\ln\left(\frac{a^x+b^x}{2}\right) = \infty . 0$$

$$\Rightarrow \lim_{x\to 0^+}\frac{2.\ln\left(\frac{a^x+b^x}{2}\right)}{x} = \frac{0}{0} \text{ bl.} \overset{L}{=} \lim_{x\to 0^+}\frac{2.\frac{a^x.\ln a+b^x.\ln b}{2}.\frac{2}{(a^x+b^x)}}{1}$$

$$= \ln a+\ln b = \ln(a.b)$$

$$\ln\left(\lim_{x\to 0^+} y\right) = \ln(a.b) \Rightarrow \boxed{\lim_{x\to 0^+} y = a.b}$$

Good Luck...

HW:

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit.

1. $\lim_{x \to -2} \dfrac{x + 2}{x^2 - 4}$

2. $\lim_{x \to 0} \dfrac{\sin 5x}{x}$

3. $\lim_{x \to \infty} \dfrac{5x^2 - 3x}{7x^2 + 1}$

4. $\lim_{x \to 1} \dfrac{x^3 - 1}{4x^3 - x - 3}$

HW:

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

8. $\lim_{x \to -5} \frac{x^2 - 25}{x + 5}$

9. $\lim_{t \to -3} \frac{t^3 - 4t + 15}{t^2 - t - 12}$

10. $\lim_{t \to 1} \frac{t^3 - 1}{4t^3 - t - 3}$

11. $\lim_{x \to \infty} \frac{5x^3 - 2x}{7x^3 + 3}$

12. $\lim_{x \to \infty} \frac{x - 8x^2}{12x^2 + 5x}$

13. $\lim_{t \to 0} \frac{\sin t^2}{t}$

14. $\lim_{t \to 0} \frac{\sin 5t}{2t}$

HW:

25. $\lim_{x \to (\pi/2)^-} \left(x - \frac{\pi}{2}\right) \sec x$

26. $\lim_{x \to (\pi/2)^-} \left(\frac{\pi}{2} - x\right) \tan x$

27. $\lim_{\theta \to 0} \frac{3^{\sin\theta} - 1}{\theta}$

28. $\lim_{\theta \to 0} \frac{(1/2)^{\theta} - 1}{\theta}$

HW:

Indeterminate Powers and Products

Find the limits in Exercise 51–66.

51. $\lim_{x \to 1^+} x^{1/(1-x)}$

52. $\lim_{x \to 1^+} x^{1/(x-1)}$

53. $\lim_{x \to \infty} (\ln x)^{1/x}$

54. $\lim_{x \to e^+} (\ln x)^{1/(x-e)}$

55. $\lim_{x \to 0^+} x^{-1/\ln x}$

56. $\lim_{x \to \infty} x^{1/\ln x}$

57. $\lim_{x \to \infty} (1 + 2x)^{1/(2 \ln x)}$

58. $\lim_{x \to 0} (e^x + x)^{1/x}$

Asst. Prof. Dr. Nurten GÜRSES

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.