# MAT1071 MATHEMATICS I 5. WEEK

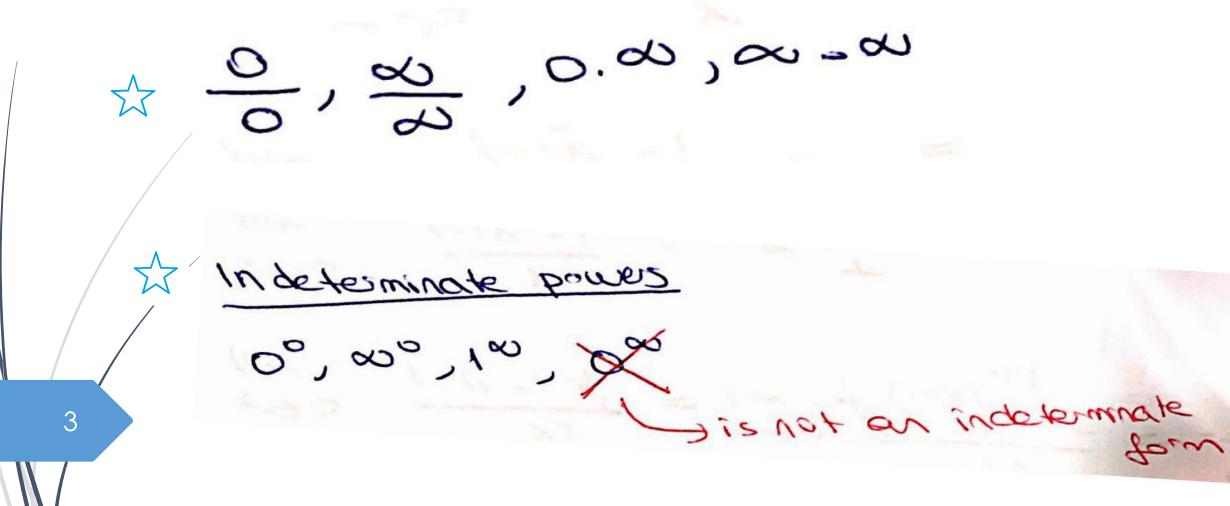
INDETERMINATE FORMS AND L HOPITAL RULE

# **Indeterminate Forms**

If the continuous functions f(x) and g(x) are both zero at x = a, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting x = a. The substitution produces 0/0, a meaningless expression, which we cannot evaluate. We use 0/0 as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as  $\infty/\infty$ ,  $\infty \cdot 0$ ,  $\infty - \infty$ ,  $0^0$ , and  $1^\infty$ , which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations.



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Numerators and denominates

Numerators and deposition of the both approach sees or the both approach

# L'Hôpital's Rule

L'Hôpital's Rule Suppose that f(a) = g(a) = 0, that f and g are differentiable on an open interval I containing a, and that  $g'(x) \neq 0$  on I if  $x \neq a$ . Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

# Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form 0/0 at x = a. But as soon as one or the other of these derivatives is different from zero at x = a we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

\$ 0.00,00=00 -> trons form to 0,00

#### L'Hôpital's Rule

# **EXAMPLE**

$$\sqrt{1+x}$$
 \_ \(\sigma\)

#### L'Hôpital's Rule

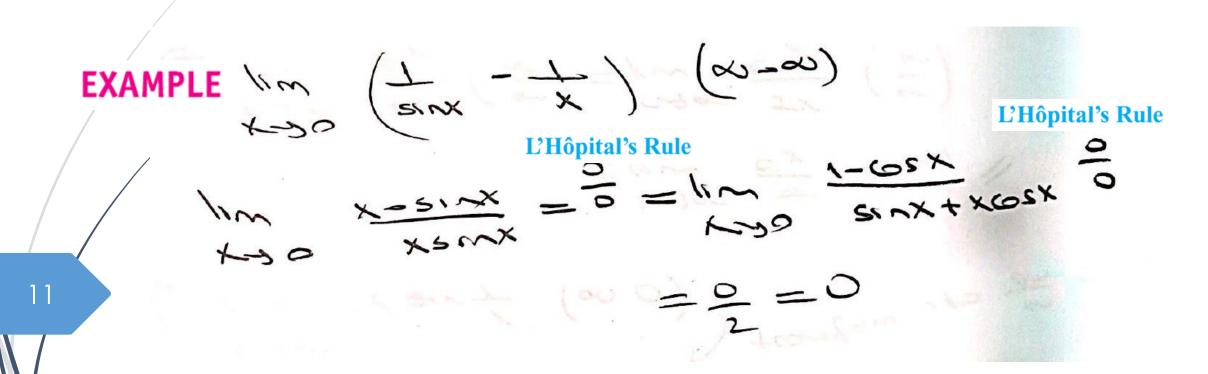
#### L'Hôpital's Rule

$$\sqrt{1+x} - 1 - \frac{x^2}{x^2} = \frac{1}{x^3} - \frac{1}{2} (1+x)^{-1/2} - \frac{1}{2} = -\frac{1}{2}$$

L'Hôpital's Rule

L'Hôpital's Rule

L'Hôpital's Rule



Solution 
$$\lim_{x \to -\infty} (2x + ((x^2 + 3x^2)) (\infty - \infty)$$

$$\lim_{x \to -\infty} (2x + ((x^2 + 3x^2)) (2x - ((x^2 + 3x^2)))$$

$$= \lim_{x \to -\infty} (2x - ((x^2 + 3x^2)) (2x - ((x^2 + 3x^2)))$$

$$= \lim_{x \to -\infty} (2x + ((x^2 + 3x^2)) (\infty - \infty)$$

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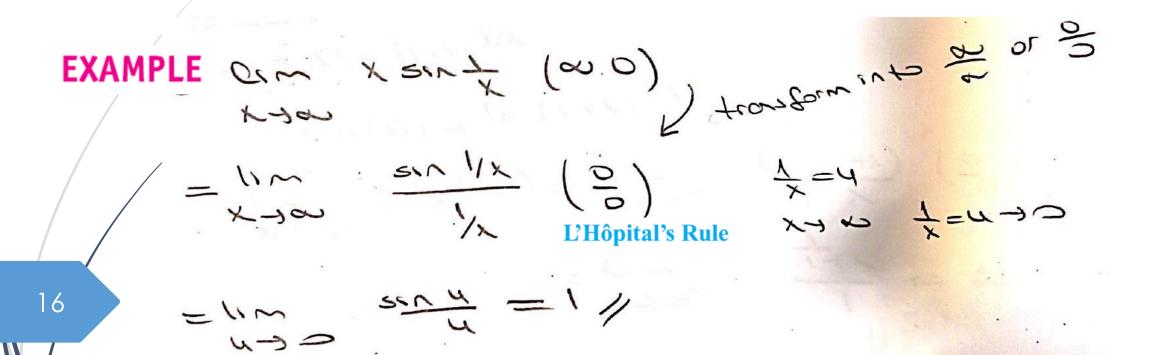
$$= \lim_{x \to -\infty} (2x + ((x^2 + 3x^2)) (\infty - \infty)$$

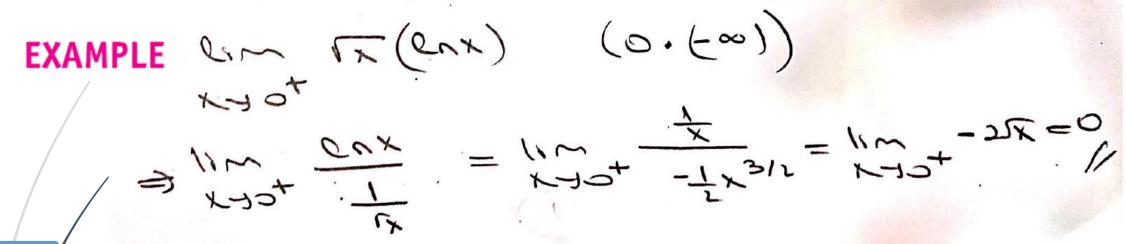
$$= \lim_{x \to -\infty} (2x + ((x^2 + 3x^2)) (\infty - \infty)$$

Or 
$$\frac{2\pi x}{2\pi x} \left(\frac{\infty}{\alpha}\right) = \frac{1}{x} \frac{1/x}{x}$$

#### L'Hôpital's Rule

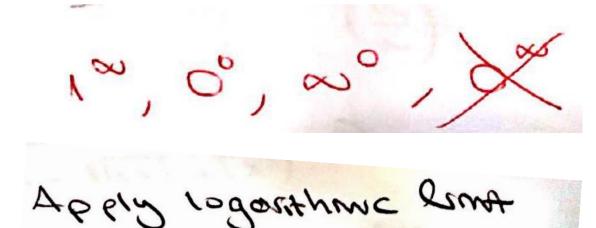
$$\frac{\text{Unifical's Rule}}{\text{Not }} = \frac{1}{1} = \frac$$





L'Hôpital's Rule

### **Indeterminate Powers**



 $\stackrel{'}{\Longrightarrow}$ 

If  $\lim_{x\to a} \ln f(x) = L$ , then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^{L}.$$

Here a may be either finite or infinite.

# EXAMPLE Apply L'Hoptal's Rule to show that lim (1+x) 1/x = e x+0+

Solution the and leads the indeterminate form in

$$f(x) = f(+x) = f(x) = f(x)$$

$$f(x) = f(x) = f(x)$$

$$f(x) = f(x) = f(x)$$

$$f(x) = f(x)$$

EXAMPLE 
$$lim x = ? (\infty^{\circ})$$

Solution F(x)=x1/x

$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right)$$

$$= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right)$$

Exam Q. Im (2-erx) 1/x, 100

Solution en ( 15mg + (2-em) /x = en L

# EXAMPLE lim ent x2 000

#### **Solution**

$$= \lim_{x \to 0} \frac{-2 \cdot x^{2}}{x^{3}} = \lim_{x \to 0} \frac{x}{x \cdot s \cdot c}$$

$$= \lim_{x \to 0} \frac{x}{x \cdot s \cdot c} = \lim_{x \to 0} \frac{x}{x \cdot c}$$

um >1 x-1 +an 12

Answer:  $-\frac{1}{\kappa}$ 

EXAMPLE Lin 
$$\left(1-\frac{3}{n}\right)^{n} = 11^{\infty}$$
 believely)

Solution 
$$y = \left(1 - \frac{3}{x}\right)^n = 1$$
 by  $= x \ln \left(1 - \frac{3}{x}\right)$  (so O Berszligi)

$$= \frac{\int_{-\infty}^{\infty} \left(\frac{1-3}{2}\right)}{\frac{1}{2}} \quad \left(\frac{0}{0} \text{ Behalzhigh}\right) \quad \text{L'Hôpital's Rule}$$

$$\lim_{N\to\infty} \frac{\ln\left(1-\frac{3}{N}\right)}{\frac{1}{N}} = \lim_{N\to\infty} \frac{3\ln^2/1-\frac{3}{N}}{-\frac{1}{N^2}} = \lim_{N\to\infty} \frac{\frac{3}{N^2(1-\frac{3}{N})}}{-\frac{1}{N^2}}$$

$$= \lim_{n \to \infty} \frac{-3}{1 - \frac{3}{n}} = -3$$

$$= \lim_{n \to \infty} \frac{-3}{1 - \frac{3}{n}} = -3$$

$$\lim_{n \to \infty} \ln y = -3 \iff \lim_{n \to \infty} \frac{-3}{n} = \frac{1}{e^3}$$

$$\lim_{n \to \infty} \ln y = -3 \iff \lim_{n \to \infty} \frac{-3}{n} = \frac{1}{e^3}$$

Solution 
$$y = x^{shx} = 1$$
 by  $= shx$ . by  $= shx$ . by  $= \frac{1}{x}$  by  $= \frac{1}{x}$ 

$$=\lim_{N\to0^+} \left(-\frac{\sin N}{N} \cdot \frac{\sin N}{\cos N}\right) = 0$$

$$\lim_{N\to0^+} \lim_{N\to0^+} \lim_{N\to0^+} \frac{\sin N}{\cos N} = 0$$

$$\lim_{N\to0^+} \lim_{N\to0^+} \frac{\sin N}{\cos N} = 0$$

$$\lim_{N\to0^+} \frac{\sin N}{\cos N} = 0$$
And Duck Duck Duck

EXAMPLE 
$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = ? \left(1^{\infty}\right)$$

#### **Solution**

$$y = (1+\frac{1}{\lambda})^{\lambda} \implies hy = x \cdot h(1+\frac{1}{\lambda}) \quad (\infty.0)$$

$$= \frac{h(1+\frac{1}{\lambda})}{\frac{1}{\lambda}} \quad (\stackrel{\bigcirc}{\circ})$$

S2. a) 
$$\lim_{x\to 0^+} x^{\frac{1}{\ln(e^x-1)}}$$
 limitini hesaplayınız. (15p) Indeterminate form

#### Solution

$$y = x \frac{1}{\ln(e^x - 1)}$$
  $\Rightarrow$   $\ln y = \frac{1}{\ln(e^x - 1)} \cdot \ln x$  (2)

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{\ln (e^x-1)} \stackrel{\text{lim}}{=} \lim_{x\to 0^+} \frac{1}{e^x-1} \stackrel{\text{lim}}{=} \frac{1}{e^x-1} \stackrel{\text{lim}}{=}$$

$$= \lim_{x\to 0^{+}} \frac{e^{x} - 1}{x = x} = \lim_{x\to 0^{+}} \frac{e^{x}}{e^{x} + x = x} = \frac{1}{1+0} = 1$$

$$\ln\left(\lim_{x\to 0^+} x^{\frac{1}{\ln(e^x-1)}}\right) = 1 \implies \lim_{x\to 0^+} x^{\frac{1}{\ln(e^x-1)}} = e \quad (2)$$

Solution 
$$\lim_{x \to \frac{\pi}{2}^{+}} \left[ \ln \left( x - \frac{\pi}{2} \right) \cdot \cos x \right] \text{ limitini hesaplayınız. (13p)}$$

Solution  $\lim_{x \to \frac{\pi}{2}^{+}} \left[ \ln \left( x - \frac{\pi}{2} \right) \cdot \cos x \right] = \lim_{x \to \frac{\pi}{2}^{+}} \frac{\ln \left( x - \frac{\pi}{2} \right)}{\operatorname{Sec} x} \xrightarrow{\infty} \frac{$ 

3. a) 
$$\lim_{x\to\infty} \frac{e^{\arctan x} - x}{\ln(1+x^2) + x} = ?$$
 (10 puan)

$$\frac{\frac{1}{1+x^{2}}e^{\arctan x}-1}{\frac{2x}{1+x^{2}}+1}$$

$$= 0.e^{\pi/2} - 1 \ 2$$

Solution 
$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{e^x - 1}\right)$$
 limitini hesaplayınız. (12p)

Solution  $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{e^x - 1}\right) = 0$   $\lim_{x\to 0} \lim_{x\to 0} \frac{e^x - \sin x}{\sin x} \left(\frac{0}{0}\right)$ 
 $\lim_{x\to 0} \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{e^x - 1}\right) = 0$   $\lim_{x\to 0} \lim_{x\to 0} \frac{e^x - \cos x}{\cos x} \left(\frac{e^x - 1}{e^x}\right) + e^x \sin x$ 
 $\lim_{x\to 0} \frac{e^x + \sin x}{\cos x} \left(\frac{0}{0}\right)$ 
 $\lim_{x\to 0} \lim_{x\to 0} \frac{e^x - \cos x}{\cos x} \left(\frac{e^x - 1}{e^x}\right) + e^x \cos x + e^x \sin x$ 
 $\lim_{x\to 0} \frac{e^x + \sin x}{\cos x} \left(\frac{0}{0}\right)$ 

2-b) Find 
$$\lim_{x\to 0} \frac{1-\cos(\sin x)}{2x^2}$$
. (Do NOT use the L'Hôpital's Rule) (12 Points)

## **Solution**

$$\lim_{x\to 0} \frac{1-\cos(\sin x)}{2x^2} = \frac{0}{0} \text{ bt.}$$

$$= \lim_{x\to 0} \frac{1-\cos(\sin x)}{2x^2}$$

$$= \lim_{x\to 0} \frac{1-\cos(\sin x)}{2x^2} \cdot \frac{\left[1+\cos(\sin x)\right]}{\left[1+\cos(\sin x)\right]} = \lim_{x\to 0} \frac{\sin^2(\sin x)}{1+\cos(\sin x)}$$

$$\frac{\sin^2(\sin x)}{\sin^2 x}$$

$$\frac{\sin^2 x}{x^2} \cdot \frac{1}{2[\text{H-Cos(sinx)}]}$$

$$=\lim_{x\to 0} \frac{\sin^2(\sin x)}{\sin^2 x} \cdot \frac{\sin^2 x}{\frac{x^2}{1}} \cdot \frac{1}{2[H\cos(\sin x)]} = 1 - \frac{1}{2} \cdot \frac{1}{2}$$



Solution 
$$y = \left(\frac{a^{x} + b^{x}}{2}\right)^{\frac{2}{x}} \Rightarrow lny = \frac{2}{x} \cdot ln\left(\frac{a^{x} + b^{x}}{2}\right)^{\frac{1}{x}}$$

$$\lim_{x \to 0^{+}} lny = ln\left(\lim_{x \to 0^{+}} y\right) = \lim_{x \to 0^{+}} \frac{2}{x} \cdot ln\left(\frac{a^{x} + b^{x}}{2}\right) = \infty \cdot 0$$

$$\lim_{x \to 0^{+}} lny = ln\left(\lim_{x \to 0^{+}} y\right) = \lim_{x \to 0^{+}} \frac{2}{x} \cdot ln\left(\frac{a^{x} + b^{x}}{2}\right) = \infty \cdot 0$$

$$\lim_{x \to 0^{+}} \frac{2 \cdot ln\left(\frac{a^{x} + b^{x}}{2}\right)}{x} = \frac{0}{0} \text{ id.} = \lim_{x \to 0^{+}} \frac{2 \cdot \frac{a^{x} \cdot lna + b^{x} \cdot lnb}{2} \cdot \frac{2}{(a^{x} + b^{x})}$$

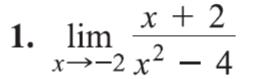
$$= lna + lnb = ln(a \cdot b)$$

$$ln\left(\lim_{x \to 0^{+}} y\right) = ln(a \cdot b) \Rightarrow \lim_{x \to 0^{+}} y = a \cdot b$$

32

Good Luck ...

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit.



$$3. \lim_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1}$$

$$2. \lim_{x \to 0} \frac{\sin 5x}{x}$$

4. 
$$\lim_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3}$$

## Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. 
$$\lim_{x \to 2} \frac{x-2}{x^2-4}$$

9. 
$$\lim_{t \to -3} \frac{t^3 - 4t + 15}{t^2 - t - 12}$$

11. 
$$\lim_{x \to \infty} \frac{5x^3 - 2x}{7x^3 + 3}$$

$$13. \lim_{t\to 0} \frac{\sin t^2}{t}$$

8. 
$$\lim_{x \to -5} \frac{x^2 - 25}{x + 5}$$

10. 
$$\lim_{t \to 1} \frac{t^3 - 1}{4t^3 - t - 3}$$

12. 
$$\lim_{x \to \infty} \frac{x - 8x^2}{12x^2 + 5x}$$

$$14. \lim_{t \to 0} \frac{\sin 5t}{2t}$$

**25.** 
$$\lim_{x \to (\pi/2)^{-}} \left( x - \frac{\pi}{2} \right) \sec x$$
 **26.**  $\lim_{x \to (\pi/2)^{-}} \left( \frac{\pi}{2} - x \right) \tan x$ 

**26.** 
$$\lim_{x \to (\pi/2)^{-}} \left( \frac{\pi}{2} - x \right) \tan x$$

$$27. \lim_{\theta \to 0} \frac{3^{\sin \theta} - 1}{\theta}$$

**28.** 
$$\lim_{\theta \to 0} \frac{(1/2)^{\theta} - 1}{\theta}$$

#### **Indeterminate Powers and Products**

Find the limits in Exercise 51–66.

**51.** 
$$\lim_{x \to 1^+} x^{1/(1-x)}$$

**52.** 
$$\lim_{x \to 1^+} x^{1/(x-1)}$$

$$\mathbf{53.} \ \lim_{x \to \infty} (\ln x)^{1/x}$$

**54.** 
$$\lim_{x \to e^+} (\ln x)^{1/(x-e)}$$

**55.** 
$$\lim_{x \to 0^+} x^{-1/\ln x}$$

$$56. \lim_{x \to \infty} x^{1/\ln x}$$

57. 
$$\lim_{x \to \infty} (1 + 2x)^{1/(2 \ln x)}$$

**58.** 
$$\lim_{x\to 0} (e^x + x)^{1/x}$$

#### Reference:

Thomas' Calculus, 12th Edition, G.B Thomas, M.D.Weir, J.Hass and F.R.Giordano, Addison-Wesley, 2012.