



MAT1320-Linear Algebra

Lecture Notes

Vectors

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Table of contents

1. Vectors
2. Vector Addition
3. Scalar Multiplication
4. Dot (Inner) Product
5. Dot (Inner) Product: Properties
6. The Angle Between Two Nonzero Vectors
7. Projection of A Vector onto Another
8. Cross Product
9. Cross Product: Properties
10. Mixed Product
11. Mixed Product: Properties
12. Two Fold Cross Product

Vectors

Vectors: Physical Point of View

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- On the other hand, there are also quantities, such as force and velocity, that possess both **magnitude** and **direction**.
- These quantities, which can be represented by arrows having appropriate lengths and directions and emanating from some given reference point A , are called **vectors**.

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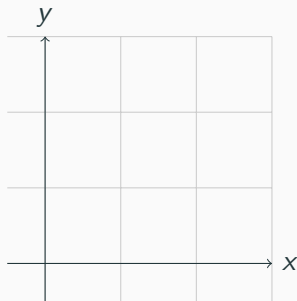
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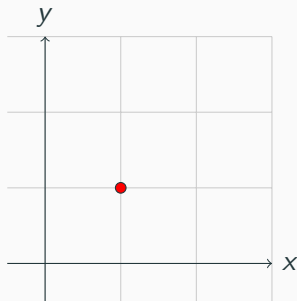


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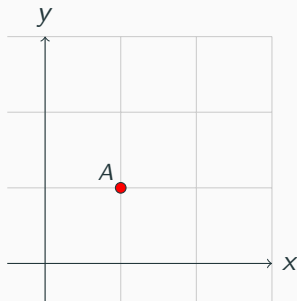


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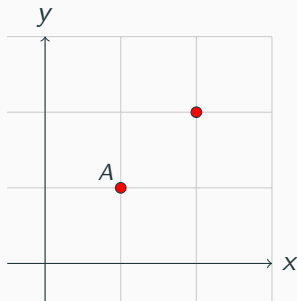


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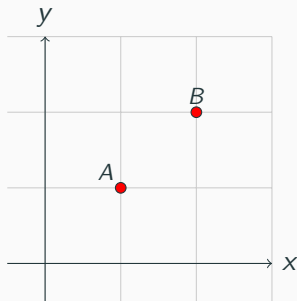


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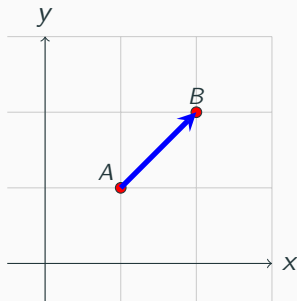


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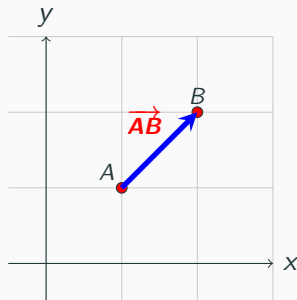


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Norm (Length) of a Vector

- Let $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ be two points in \mathbb{R}^3 . Then the vector with start point P_0 and end point P_1 is denoted by $\overrightarrow{P_0P_1}$ and defined as

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Note: For any nonzero vector \vec{v} , the vector $\frac{\vec{v}}{|\vec{v}|}$ is the unique unit vector in the same direction as \vec{v} .

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The **sum** of vectors $\vec{u} = (u_1, u_2, \dots, u_n)$, and $\vec{v} = (v_1, v_2, \dots, v_n)$ is denoted by $\vec{u} + \vec{v}$

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$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

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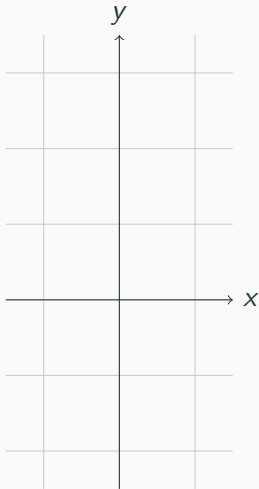
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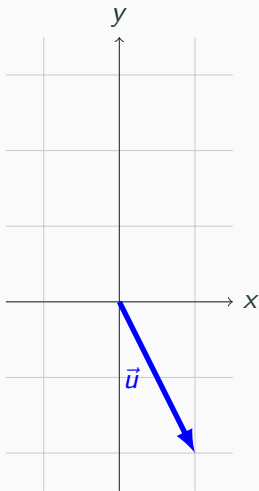


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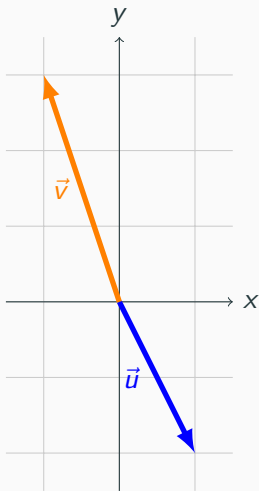


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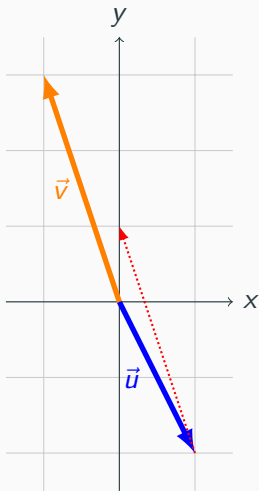


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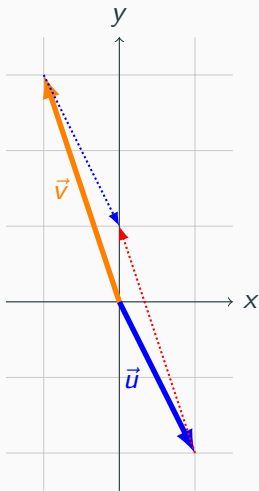


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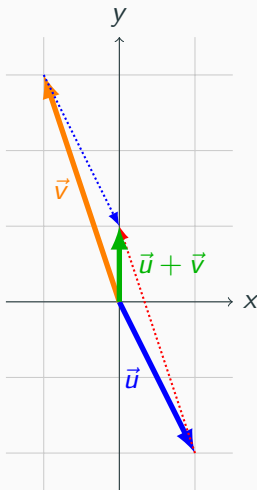


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4. If $\lambda < -1$ and $1 < \lambda$, then $|\vec{u}| < |\lambda \vec{u}|$.

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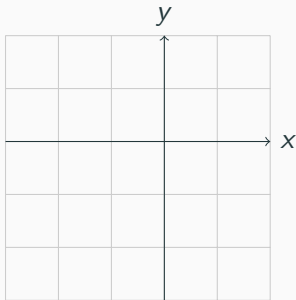
If $\vec{v} = (1, 1)$ and $\lambda = -2 \in \mathbb{R}$, then

$$\begin{aligned}\lambda \vec{v} &= -2 \vec{v} = ((-2) \cdot 1, (-2) \cdot 1) \\ &= (-2, -2)\end{aligned}$$

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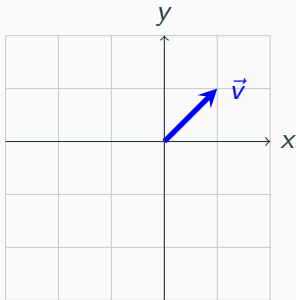
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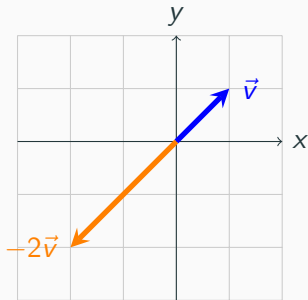
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The **dot product or inner product** of vectors $\vec{\mathbf{u}} = (u_1, u_2, \dots, u_n)$ and $\vec{\mathbf{v}} = (v_1, v_2, \dots, v_n)$ is denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ or $\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle$

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If $\vec{u} = (2, 1, 0, 1)$ and $\vec{v} = (-1, 1, 3, 2)$, then the dot product of \vec{u} and \vec{v} is

$$\begin{aligned}\vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 = \sum_{i=1}^4 u_i v_i \\ &= 2(-1) + 1 \cdot 1 + 0 \cdot 3 + 1 \cdot 2 = 1.\end{aligned}$$

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Note: The dot product of two vectors is a real number.

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6. $|\vec{u} \cdot \vec{v}| \leq |\vec{u}| |\vec{v}|$ (Schwartz Inequality)

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5. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
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The Angle Between Two Nonzero Vectors

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The cosine between vectors $\vec{\mathbf{u}} = (u_1, u_2, \dots, u_n)$ and $\vec{\mathbf{v}} = (v_1, v_2, \dots, v_n)$ is given by

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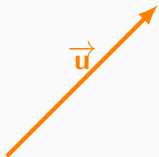
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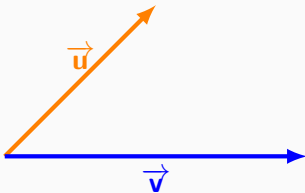
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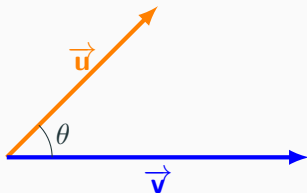
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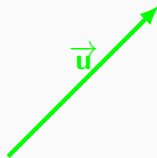
Projection of A Vector onto Another

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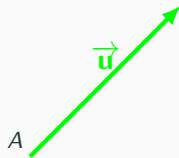


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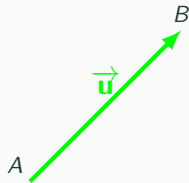


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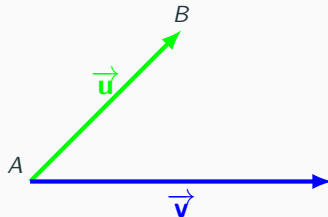


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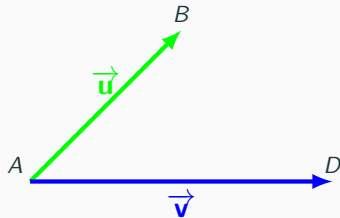


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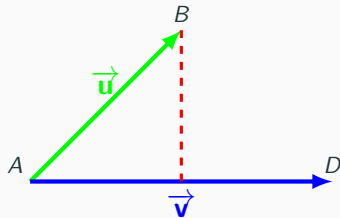


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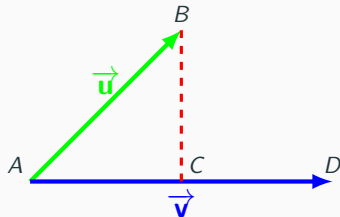


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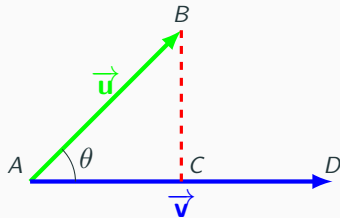


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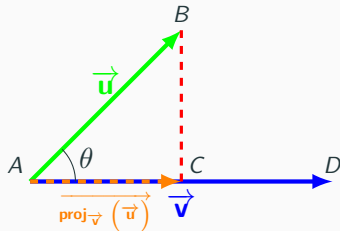


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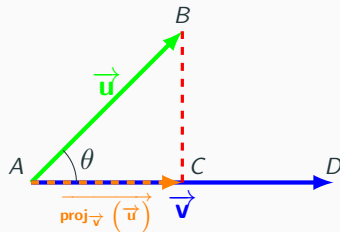


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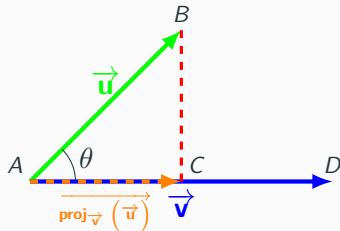
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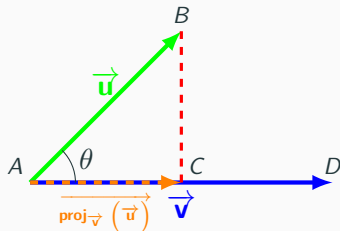
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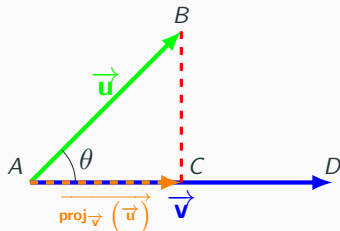


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$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{2}{6}(-1, 1, 0, 2) = \left(-\frac{1}{3}, \frac{1}{3}, 0, \frac{2}{3}\right)$$

Cross Product

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Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be two vectors in \mathbb{R}^3 . The **cross product** of the vectors \vec{u} and \vec{v} is denoted by $\vec{u} \times \vec{v}$ or $\vec{u} \wedge \vec{v}$ and defined as follows:

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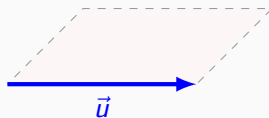
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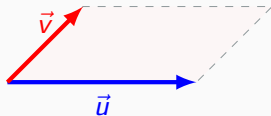


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Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be two vectors in \mathbb{R}^3 . The **cross product** of the vectors \vec{u} and \vec{v} is denoted by $\vec{u} \times \vec{v}$ or $\vec{u} \wedge \vec{v}$ and defined as follows:

Note: $|\vec{u} \times \vec{v}|$ is the area of the parallelogram having \vec{u} and \vec{v} as sides.

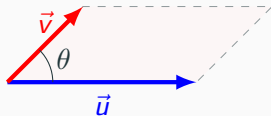


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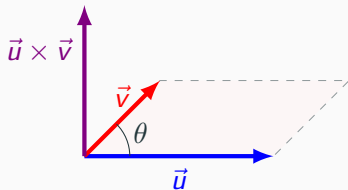


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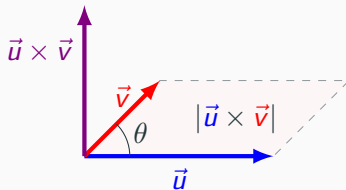


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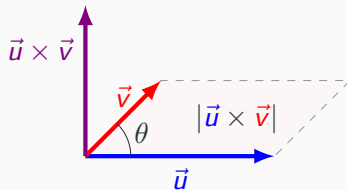
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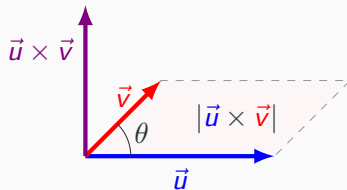


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Cross Product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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or

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Let $\vec{u} = (1, 2, -1)$ and $\vec{v} = (-2, 3, 4) \in \mathbb{R}^3$. Find the cross product of \vec{u} and \vec{v} .

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Mixed Product

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Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be three vectors in \mathbb{R}^3 . Then the **mixed product** of the vectors \vec{u} , \vec{v} and \vec{w} is denoted by $\vec{u} \cdot (\vec{v} \times \vec{w})$ or $(\vec{u}, \vec{v}, \vec{w})$

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Note: Mixed product is only defined over \mathbb{R}^3 .

Mixed Product: Geometric Interpretation

Geometrically, mixed product of the vectors \vec{u} , \vec{v} and \vec{w} is the volume of the parallelepiped having edges as \vec{u} , \vec{v} and \vec{w} .

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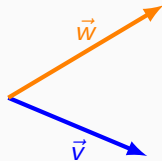


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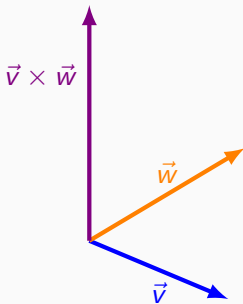


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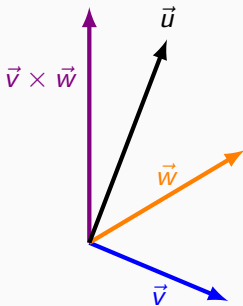


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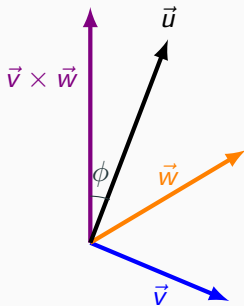
Mixed Product: Geometric Interpretation

Geometrically, mixed product of the vectors \vec{u} , \vec{v} and \vec{w} is the volume of the parallelepiped having edges as \vec{u} , \vec{v} and \vec{w} .

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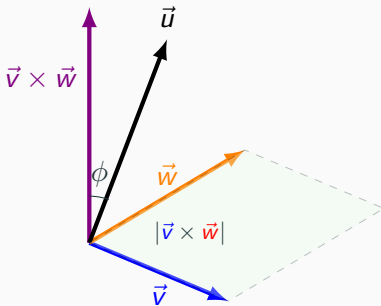
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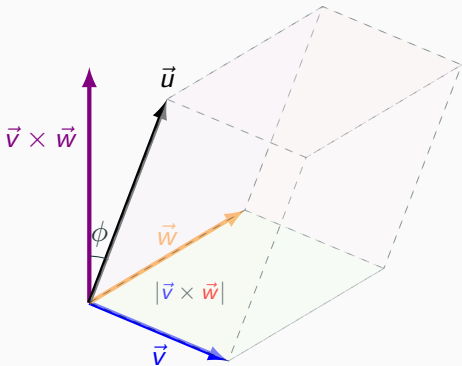
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Mixed Product

Example

The mixed product $\vec{u} \cdot (\vec{v} \times \vec{w})$, of the vectors $\vec{u} = (1, 2, -1)$, $\vec{v} = (-2, 3, 4)$ and $\vec{w} = (2, 1, 0) \in \mathbb{R}^3$ is

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix}$$

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5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0 \Leftrightarrow \vec{u} \parallel \vec{v} \text{ or } \vec{u} \parallel \vec{w} \text{ or } \vec{v} \parallel \vec{w} \text{ or } \vec{u} \parallel \vec{v} \parallel \vec{w}$

Two Fold Cross Product

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The **two fold cross product** of the vectors $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$ is defined by

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Note: The result of two fold cross product is a vector over \mathbb{R}^3 .

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Example

Let $\vec{u} = (1, 2, -1)$, $\vec{v} = (-2, 3, 4)$ and $\vec{w} = (2, 1, 0) \in \mathbb{R}^3$ given. Find $\vec{u} \times (\vec{v} \times \vec{w})$.

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$$\vec{u} \cdot \vec{w} = 1 \cdot 2 + 2 \cdot 1 + (-1) \cdot 0 = 4$$

$$\vec{u} \cdot \vec{v} = 1 \cdot (-2) + 2 \cdot 3 + (-1) \cdot 4 = 0$$

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?