MAT1320-Linear Algebra Lecture Notes

Basis, Dimension and Coordinates

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Fall 2020

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Basis and Dimension

Definition

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3. $\mathcal{B} = \{\overrightarrow{\mathbf{e}}_1 = (1,0,0), \overrightarrow{\mathbf{e}}_2 = (0,1,0), \overrightarrow{\mathbf{e}}_3 = (0,0,1)\} \subset V = \mathbb{R}^3 \Rightarrow \mathbb{R}^3 = \langle \mathcal{B} \rangle$ and $\dim(V) = 3$.

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Note: All vector spaces given above are of finite dimension.

Example

1. $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset V = \mathcal{M}_{2\times 2}(\mathbb{R}) \Rightarrow \mathcal{M}_{2\times 2}(\mathbb{R}) = \langle \mathcal{B} \rangle$ and $\dim(\mathcal{M}_{2\times 2}(\mathbb{R})) = 4$.

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2. $\mathcal{B}=\{1,x,x^2,\ldots,x^{n-1}\}\subset V=\mathcal{P}_{n-1}\left(x\right)=\left\{a_0+a_1x+\ldots+a_{n-1}x^{n-1}\,\middle|\,a_i\in\mathbb{R}\right\}\Rightarrow\mathcal{P}_{n-1}\left(x\right)=\langle\mathcal{B}\rangle$ and $\dim\left(\mathcal{P}_{n-1}\left(x\right)\right)=n.$

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2. $\mathcal{B} = \{1, x, x^2, \ldots, x^{n-1}\} \subset V = \mathcal{P}_{n-1}(x) = \{ a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} \mid a_i \in \mathbb{R}\} \Rightarrow \mathcal{P}_{n-1}(x) = \langle \mathcal{B} \rangle$ and $\dim(\mathcal{P}_{n-1}(x)) = n$.
3. $\mathcal{B} = \{1, x, x^2, \ldots\} \subset V = \mathcal{P}(x) =$ {the set of all polynomials} $\Rightarrow \mathcal{P}(x) = \langle \mathcal{B} \rangle$ and $\dim(\mathcal{P}(x)) = \infty$.

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4. Let $V=\left\{\overrightarrow{\mathbf{0}}\right\}$ be the zero space, then $V=\left\langle\overrightarrow{\mathbf{0}}\right\rangle$ and $\dim\left(V\right)=0$.

Example
Determine whether the set
$\mathcal{B} = \{\overrightarrow{\mathbf{v}}_1 = (1, 2, 0)\,, \overrightarrow{\mathbf{v}}_2 = (2, 0, 1)\,, \overrightarrow{\mathbf{v}}_3 = (1, 2, 1)\}$ is a basis for $\mathbb{R}^3$.

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1. Linear Independence: For $x_1, x_2, x_3 \in \mathbb{R}$

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Thus $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are linearly independent.

2. Span: For all $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + x_3 \overrightarrow{\mathbf{v}}_3$$

2. **Span:** For all $(x, y, z) \in \mathbb{R}^3$

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Hence, we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$

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Hence, we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$ and

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Hence, we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$ and

$$(x, y, z) = x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1).$$

Consequently, $\mathbb{R}^3 = \langle \mathcal{B} \rangle$.

Theorem
Let V be a vector space of dimension n and $\mathcal{B} = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\} \subset V$. Then the following statements hold:

1. If $\mathcal{B}$ is a linearly independent set, then $\langle \mathcal{B} \rangle = V$.

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Note: By the above Theorem, in order to check that $\mathcal{B} = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\} \subset V$ is a basis of the finite dimensional vector space V, it is just sufficient to check linearly independence.

Basis and Dimension

Example

Determine whether the set $\mathcal{B} = \{\overrightarrow{\mathbf{v}}_1 = (1, 2), \overrightarrow{\mathbf{v}}_2 = (2, 0)\}$ is a basis for $\mathbb{R}^2$.

Example

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Therefore, the vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2$ are linearly independent. Since $\dim\left(\mathbb{R}^2\right) = 2$ we have $\langle \mathcal{B} \rangle = \mathbb{R}^2$ and we conclude that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

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Since, $\mathbb{R}^2 = \langle \mathcal{B} \rangle$ the vectors $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2$ are linearly independent. As a result, $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

Example

Let $\mathcal{U} = \{\,(x, y, z) |\, x + 2z = 0\} \subset \mathbb{R}^3$.

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3. $\dim(\mathcal{U}) = ?$

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This says that $\mathcal{U}$ is a subspace of $\mathbb{R}^3$.

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It can be easily seen that the vectors $(-2, 0, 1)$ and $(0, 1, 0)$ are linearly independent and spans $\mathcal{U}$. Thus, $\mathcal{B} = \{(-2, 0, 1), (0, 1, 0)\}$ is a basis for $\mathcal{U}$.

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3. Dimension: $\dim(\mathcal{U}) = 2$.

Example

Let $\mathcal{U} = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & a+b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\} \subset \mathcal{M}_{2\times 3}\left(\mathbb{R}\right).$

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Coordinates

Theorem
Let V be vector space of dimension n and with the ordered basis $\mathcal{B} = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\}$. Then any vector $\overrightarrow{\mathbf{w}} \in V$ can be expressed uniquely as a linear combination of basis vectors in $\mathcal{B}$, say

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Note: These n scalars $x_1, x_2, \ldots, x_n$ are called the coordinates of $\overrightarrow{\mathbf{w}}$ relative to the basis $\mathcal{B}$, and they form a vector $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ called the coordinate vector of $\overrightarrow{\mathbf{w}}$ relative to $\mathcal{B}$. We denote this vector by $[\overrightarrow{\mathbf{w}}]_{\mathcal{B}}$, or simply $[\overrightarrow{\mathbf{w}}]$, when $\mathcal{B}$ is understood. Thus,

$$[\overrightarrow{\mathbf{w}}]_{\mathcal{B}} = (x_1, x_2, \ldots, x_n).$$

Proof.
Let $\mathcal{B} = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\}$ be an ordered basis of n-space V and $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_n \in \mathbb{R}$.

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Let $\mathcal{B} = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\}$ be an ordered basis of n-space V and $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_n \in \mathbb{R}$. Assume that $\overrightarrow{\mathbf{w}}$ has two expressions as follows:

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Since basis elements are linearly independent we have

$$\Rightarrow \quad (x_1 - y_1)\overrightarrow{\mathbf{v}}_1 + (x_2 - y_2)\overrightarrow{\mathbf{v}}_2 + \ldots + (x_n - y_n)\overrightarrow{\mathbf{v}}_n = \overrightarrow{\mathbf{0}}$$

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Let $\mathcal{B} = \left\{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\right\}$ be an ordered basis of n-space V and $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_n \in \mathbb{R}$. Assume that $\overrightarrow{\mathbf{w}}$ has two expressions as follows:

$$\begin{aligned}\overrightarrow{\mathbf{w}} &= x_1\overrightarrow{\mathbf{v}}_1 + x_2\overrightarrow{\mathbf{v}}_2 + \ldots + x_n\overrightarrow{\mathbf{v}}_n \\ \overrightarrow{\mathbf{w}} &= y_1\overrightarrow{\mathbf{v}}_1 + y_2\overrightarrow{\mathbf{v}}_2 + \ldots + y_n\overrightarrow{\mathbf{v}}_n\end{aligned}$$

Since basis elements are linearly independent we have

$$\begin{aligned}&\Rightarrow \quad (x_1 - y_1)\overrightarrow{\mathbf{v}}_1 + (x_2 - y_2)\overrightarrow{\mathbf{v}}_2 + \ldots + (x_n - y_n)\overrightarrow{\mathbf{v}}_n = \overrightarrow{\mathbf{0}} \\ &\Rightarrow \quad x_1 - y_1 = 0, x_2 - y_2 = 0, \ldots, x_n - y_n = 0\end{aligned}$$

Proof.
Let $\mathcal{B} = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\}$ be an ordered basis of n-space V and $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_n \in \mathbb{R}$. Assume that $\overrightarrow{\mathbf{w}}$ has two expressions as follows:

$$\begin{aligned}\overrightarrow{\mathbf{w}} &= x_1\overrightarrow{\mathbf{v}}_1 + x_2\overrightarrow{\mathbf{v}}_2 + \ldots + x_n\overrightarrow{\mathbf{v}}_n\\ \overrightarrow{\mathbf{w}} &= y_1\overrightarrow{\mathbf{v}}_1 + y_2\overrightarrow{\mathbf{v}}_2 + \ldots + y_n\overrightarrow{\mathbf{v}}_n\end{aligned}$$

Since basis elements are linearly independent we have

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$\square$

Example

Find coordinate vector $[\overrightarrow{\mathbf{w}}]_{\mathcal{B}}$ of $\overrightarrow{\mathbf{w}} = (1, 2, 3) \in \mathbb{R}^3$ relative to ordered basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$.

Coordinates

Example

Find coordinate vector $[\overrightarrow{\mathbf{w}}]_{\mathcal{B}}$ of $\overrightarrow{\mathbf{w}} = (1, 2, 3) \in \mathbb{R}^3$ relative to ordered basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$. For all $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1)$$

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$$(x,y,z) = x_1(1,2,0) + x_2(2,0,1) + x_3(1,2,1)$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned}$$

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$$\begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 2 & 0 & 2 & y \\ 0 & 1 & 1 & z \end{array}\right)$$

Coordinates

Example

Find coordinate vector $[\overrightarrow{\mathbf{w}}]_{\mathcal{B}}$ of $\overrightarrow{\mathbf{w}} = (1,2,3) \in \mathbb{R}^3$ relative to ordered basis $\mathcal{B} = \{(1,2,0), (2,0,1), (1,2,1)\}$. For all $(x,y,z) \in \mathbb{R}^3$

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Coordinates

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Then we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$

Example

Find coordinate vector $[\overrightarrow{\mathbf{w}}]_{\mathcal{B}}$ of $\overrightarrow{\mathbf{w}} = (1,2,3) \in \mathbb{R}^3$ relative to ordered basis $\mathcal{B} = \{(1,2,0),(2,0,1),(1,2,1)\}$. For all $(x,y,z) \in \mathbb{R}^3$

$$(x,y,z) \quad = \quad x_1(1,2,0) + x_2(2,0,1) + x_3(1,2,1)$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 2 & 0 & 2 & y \\ 0 & 1 & 1 & z \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2x+y-4z}{4} \\ 0 & 1 & 0 & \frac{2x-y}{4} \\ 0 & 0 & 1 & \frac{-2x+y+4z}{4} \end{array}\right).$$

Then we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$ and hence,

$$(1,2,3) = -2.(1,2,0) + 0.(2,0,1) + 3(1,2,1).$$

?