MAT1320-Linear Algebra
Lecture Notes

Sarrus' Rule, Finding Inverse Matrices Using Adjoint Matrices

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Sarrus' Rule

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Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix of order 3. Then the determinant of $\mathbf{A}$ can be computed as follows:

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$$\begin{array}{c} \begin{array}{ccccc} + & + & + \; - & - & - \end{array} \\ \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} \end{array}$$

This method is called Sarrus' Rule.

Example

By using Sarrus' Rule, find the determinant of the matrix

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$$\begin{array}{ccccc} + & + & + \; - & - & - \\ \left[\begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{matrix}\right. & & \left.\right] & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

$$\begin{aligned} &\Rightarrow \quad (1.5.9 + 4.8.3 + 7.2.6) - (7.5.3 + 1.8.6 + 4.2.9) \\ &= \quad 45 + 96 + 84 - 105 - 48 - 72 = 0 \end{aligned}$$

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Theorem
Let A be any square matrix. Then

$$A(adjA) = (adjA)A = |A|I$$

where I is the identity matrix. Thus, if $|A| \neq 0$,

$$A^{-1} = \frac{1}{|A|}(adjA)$$

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$$A_{11} = +\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4, \quad A_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 2, \quad A_{13} = +\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

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$$A_{31} = +\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5, \quad A_{32} = -\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = -4, \quad A_{33} = +\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = 2$$

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Example

The transpose of the above matrix of cofactors yields the classical adjoint of A; that is,

$$adjA = \begin{pmatrix} -4 & 2 & 5 \\ 2 & -1 & -4 \\ -1 & -1 & 2 \end{pmatrix}$$

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Thus, A does have an inverse, and, by Theorem 8.9 ,

$$A^{-1} = \frac{1}{|A|}(adjA) = -\frac{1}{3}\begin{pmatrix} -4 & 2 & 5 \\ 2 & -1 & -4 \\ -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{5}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

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