MAT1320-Linear Algebra
Lecture Notes

Change of Basis and Coordinate Transformations

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Table of contents

Coordinates

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Theorem
Let V be vector space of dimension n and with basis $\mathcal{B} = \left\{ \overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n \right\}$. Then any vector $\overrightarrow{\mathbf{w}} \in V$ can be expressed uniquely as a linear combination of basis vectors in $\mathcal{B}$, say

$$\overrightarrow{\mathbf{w}} = x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n.$$

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$$\overrightarrow{\mathbf{w}} = x_1 \overrightarrow{\mathbf{v}}_1 + x_2 \overrightarrow{\mathbf{v}}_2 + \ldots + x_n \overrightarrow{\mathbf{v}}_n.$$

Note: These n scalars $x_1, x_2, \ldots, x_n$ are called the coordinates of $\overrightarrow{\mathbf{w}}$ relative to the basis $\mathcal{B}$, and they form a vector $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ called the coordinate vector of $\overrightarrow{\mathbf{w}}$ relative to $\mathcal{B}$. We denote this vector by $[\overrightarrow{\mathbf{w}}]_{\mathcal{B}}$, or simply $[\overrightarrow{\mathbf{w}}]$, when $\mathcal{B}$ is understood. Thus,

$$[\overrightarrow{\mathbf{w}}]_{\mathcal{B}} = (x_1, x_2, \ldots, x_n).$$

Example
Find coordinate vector of $\overrightarrow{\mathbf{w}} = (3, 2, 1) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Coordinates

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Find coordinate vector of $\overrightarrow{\mathbf{w}} = (3, 2, 1) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. For all $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)$$

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we have $x_1 = x$, $x_2 = y$, $x_3 = z$. Then

$$(3, 2, 1) = 3.(1,0,0) + 2.(0,1,0) + 1.(0,0,1)$$

and so $[\overrightarrow{\mathbf{w}}]_{\mathcal{B}} = (3, 2, 1)$.

Example
Find coordinate vector of $\overrightarrow{\mathbf{w}} = (1, 2, 3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$.

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Example
Find coordinate vector of $\overrightarrow{\mathbf{w}} = (1,2,3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1,2,0),(2,0,1),(1,2,1)\}$. For all $(x,y,z) \in \mathbb{R}^3$

$$(x,y,z) = x_1(1,2,0) + x_2(2,0,1) + x_3(1,2,1)$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned}$$

Example

Find coordinate vector of $\overrightarrow{\mathbf{w}} = (1, 2, 3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1, 2, 0), (2, 0, 1), (1, 2, 1)\}$. For all $(x, y, z) \in \mathbb{R}^3$

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$$\begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 2 & 0 & 2 & y \\ 0 & 1 & 1 & z \end{array}\right)$$

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$$\begin{aligned} x_1 + 2x_2 + x_3 &= x \\ 2x_1 + 2x_3 &= y \\ x_2 + x_3 &= z \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 2 & 0 & 2 & y \\ 0 & 1 & 1 & z \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2x+y-4z}{4} \\ 0 & 1 & 0 & \frac{2x-y}{4} \\ 0 & 0 & 1 & \frac{-2x+y+4z}{4} \end{array}\right)$$

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$$(x, y, z) = x_1(1, 2, 0) + x_2(2, 0, 1) + x_3(1, 2, 1)$$

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we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$.

Example
Find coordinate vector of $\overrightarrow{\mathbf{w}} = (1,2,3) \in \mathbb{R}^3$ relative to basis $\mathcal{B} = \{(1,2,0),(2,0,1),(1,2,1)\}$. For all $(x,y,z) \in \mathbb{R}^3$

$$(x,y,z) = x_1(1,2,0) + x_2(2,0,1) + x_3(1,2,1)$$

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we have $x_1 = \frac{2x+y-4z}{4}$, $x_2 = \frac{2x-y}{4}$, $x_3 = \frac{-2x+y+4z}{4}$. Thus

$$(1,2,3) = -2.(1,2,0) + 0.(2,0,1) + 3(1,2,1)$$

and $[\overrightarrow{\mathbf{w}}]_{\mathcal{B}} = (-2,0,3)$.

Change of Basis and Coordinate Transformations Matrix

Change of Basis and Coordinate Transformations

Definition
Let V be an n-space with two ordered basis
$\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\}$ and $\mathcal{B}_2 = \{\overrightarrow{\mathbf{w}}_1, \overrightarrow{\mathbf{w}}_2, \ldots, \overrightarrow{\mathbf{w}}_n\}$.

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$$\overrightarrow{\mathbf{v}}_1 \quad = \quad a_{11}\overrightarrow{\mathbf{w}}_1 + a_{21}\overrightarrow{\mathbf{w}}_2 + \ldots + a_{n1}\overrightarrow{\mathbf{w}}_n$$

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$$\begin{aligned}
\overrightarrow{\mathbf{v}}_1 &= a_{11}\overrightarrow{\mathbf{w}}_1 + a_{21}\overrightarrow{\mathbf{w}}_2 + \ldots + a_{n1}\overrightarrow{\mathbf{w}}_n \\
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$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

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The matrix A is called coordinates transformation matrix from basis $\mathcal{B}_1$ to basis $\mathcal{B}_2$ and denoted by $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$.

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$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}$$

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The matrix B is called coordinates transformation matrix from basis $\mathcal{B}_2$ to basis $\mathcal{B}_1$ and denoted by $B = [M]_{\mathcal{B}_2}^{\mathcal{B}_1}$.

Theorem

Let V be an n-space with two ordered basis

$\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \ldots, \overrightarrow{\mathbf{v}}_n\}$ *and* $\mathcal{B}_2 = \{\overrightarrow{\mathbf{w}}_1, \overrightarrow{\mathbf{w}}_2, \ldots, \overrightarrow{\mathbf{w}}_n\}$.

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1. $AB = I_n$ *or* $B = A^{-1}$ *i.e.,* $\left([M]_{\mathcal{B}_1}^{\mathcal{B}_2}\right)\left([M]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) = I_n$ *or* $\left([M]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) = \left([M]_{\mathcal{B}_1}^{\mathcal{B}_2}\right)^{-1}$,

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2. *For all* $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} \in V$ *we have* $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2}$ *and* $[\overrightarrow{\mathbf{v}}]_{\mathcal{B}_2} = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\overrightarrow{\mathbf{v}}]_{\mathcal{B}_1}$.

Example
Let

$$\begin{aligned} \mathcal{B}_1 &= \left\{ \overrightarrow{\mathbf{v}}_1 = (1,2,0)\,, \overrightarrow{\mathbf{v}}_2 = (2,0,1)\,, \overrightarrow{\mathbf{v}}_3 = (1,2,1) \right\}, \\ \mathcal{B}_2 &= \left\{ \overrightarrow{\mathbf{w}}_1 = (0,2,1)\,, \overrightarrow{\mathbf{w}}_2 = (-1,0,1)\,, \overrightarrow{\mathbf{w}}_3 = (-1,3,0) \right\} \end{aligned}$$

be two ordered bases of V and $\overrightarrow{\mathbf{u}} = (2,3,5)$ be the coordinates with respect to standard basis. Find each of the following.

1. $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$,

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1. $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$,
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be two ordered bases of V and $\overrightarrow{\mathbf{u}} = (2, 3, 5)$ be the coordinates with respect to standard basis. Find each of the following.

1. $A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2}$,
2. $B = [M]_{\mathcal{B}_2}^{\mathcal{B}_1}$,
3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$,
4. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2}$.

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\overrightarrow{\mathbf{v}}_1 = a_{11}\overrightarrow{\mathbf{w}}_1 + a_{21}\overrightarrow{\mathbf{w}}_2 + a_{31}\overrightarrow{\mathbf{w}}_3$$

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$$\begin{aligned} \overrightarrow{\mathbf{v}}_1 &= a_{11}\overrightarrow{\mathbf{w}}_1 + a_{21}\overrightarrow{\mathbf{w}}_2 + a_{31}\overrightarrow{\mathbf{w}}_3 \\ (1,2,0) &= a_{11}(0,2,1) + a_{21}(-1,0,1) + a_{31}(-1,3,0) \end{aligned}$$

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$$\begin{aligned} -a_{21} - a_{31} &= 1 \\ 2a_{11} + 3a_{31} &= 2 \\ a_{11} + a_{21} &= 0 \end{aligned}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned} \overrightarrow{\mathbf{v}}_1 &= a_{11}\overrightarrow{\mathbf{w}}_1 + a_{21}\overrightarrow{\mathbf{w}}_2 + a_{31}\overrightarrow{\mathbf{w}}_3 \\ (1,2,0) &= a_{11}(0,2,1) + a_{21}(-1,0,1) + a_{31}(-1,3,0) \end{aligned}$$

$$\begin{aligned} -a_{21} - a_{31} &= 1 \\ 2a_{11} + 3a_{31} &= 2 \\ a_{11} + a_{21} &= 0 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{array}\right)$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned} \overrightarrow{\mathbf{v}}_1 &= a_{11}\overrightarrow{\mathbf{w}}_1 + a_{21}\overrightarrow{\mathbf{w}}_2 + a_{31}\overrightarrow{\mathbf{w}}_3 \\ (1,2,0) &= a_{11}(0,2,1) + a_{21}(-1,0,1) + a_{31}(-1,3,0) \end{aligned}$$

$$\begin{aligned} -a_{21} - a_{31} &= 1 \\ 2a_{11} + 3a_{31} &= 2 \\ a_{11} + a_{21} &= 0 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned}
\overrightarrow{\mathbf{v}}_1 &= a_{11}\overrightarrow{\mathbf{w}}_1 + a_{21}\overrightarrow{\mathbf{w}}_2 + a_{31}\overrightarrow{\mathbf{w}}_3 \\
(1,2,0) &= a_{11}\,(0,2,1) + a_{21}\,(-1,0,1) + a_{31}\,(-1,3,0) \\
\begin{array}{r} -a_{21} - a_{31} = 1 \\ 2a_{11} + 3a_{31} = 2 \\ a_{11} + a_{21} = 0 \end{array} &\Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \\
a_{11} &= 1,\, a_{21} = -1,\, a_{31} = 0
\end{aligned}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\overrightarrow{\mathbf{v}}_2 = a_{12}\overrightarrow{\mathbf{w}}_1 + a_{22}\overrightarrow{\mathbf{w}}_2 + a_{32}\overrightarrow{\mathbf{w}}_3$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned} \overrightarrow{\mathbf{v}}_2 &= a_{12}\overrightarrow{\mathbf{w}}_1 + a_{22}\overrightarrow{\mathbf{w}}_2 + a_{32}\overrightarrow{\mathbf{w}}_3 \\ (2,0,1) &= a_{12}(0,2,1) + a_{22}(-1,0,1) + a_{32}(-1,3,0) \end{aligned}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned} \overrightarrow{\mathbf{v}}_2 &= a_{12}\overrightarrow{\mathbf{w}}_1 + a_{22}\overrightarrow{\mathbf{w}}_2 + a_{32}\overrightarrow{\mathbf{w}}_3 \\ (2,0,1) &= a_{12}(0,2,1) + a_{22}(-1,0,1) + a_{32}(-1,3,0) \end{aligned}$$

$$\begin{aligned} -a_{22} - a_{32} &= 2 \\ 2a_{12} + 3a_{32} &= 0 \\ a_{12} + a_{22} &= 1 \end{aligned}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned}
\overrightarrow{\mathbf{v}}_2 &= a_{12}\overrightarrow{\mathbf{w}}_1 + a_{22}\overrightarrow{\mathbf{w}}_2 + a_{32}\overrightarrow{\mathbf{w}}_3 \\
(2,0,1) &= a_{12}\,(0,2,1) + a_{22}\,(-1,0,1) + a_{32}\,(-1,3,0)
\end{aligned}$$

$$\begin{aligned}
-a_{22} - a_{32} &= 2 \\
2a_{12} + 3a_{32} &= 0 \\
a_{12} + a_{22} &= 1
\end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned} \overrightarrow{\mathbf{v}}_2 &= a_{12}\overrightarrow{\mathbf{w}}_1 + a_{22}\overrightarrow{\mathbf{w}}_2 + a_{32}\overrightarrow{\mathbf{w}}_3 \\ (2,0,1) &= a_{12}(0,2,1) + a_{22}(-1,0,1) + a_{32}(-1,3,0) \end{aligned}$$

$$\begin{aligned} -a_{22} - a_{32} &= 2 \\ 2a_{12} + 3a_{32} &= 0 \\ a_{12} + a_{22} &= 1 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{9}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & -\frac{6}{5} \end{array}\right)$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned}
\overrightarrow{\mathbf{v}}_2 &= a_{12}\overrightarrow{\mathbf{w}}_1 + a_{22}\overrightarrow{\mathbf{w}}_2 + a_{32}\overrightarrow{\mathbf{w}}_3 \\
(2,0,1) &= a_{12}\,(0,2,1) + a_{22}\,(-1,0,1) + a_{32}\,(-1,3,0) \\
\begin{array}{r} -a_{22} - a_{32} = 2 \\ 2a_{12} + 3a_{32} = 0 \\ a_{12} + a_{22} = 1 \end{array} &\Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{9}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & -\frac{6}{5} \end{array}\right) \\
a_{12} &= \frac{9}{5}, a_{22} = -\frac{4}{5}, a_{32} = -\frac{6}{5}
\end{aligned}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\overrightarrow{\mathbf{v}}_3 = a_{13}\overrightarrow{\mathbf{w}}_1 + a_{23}\overrightarrow{\mathbf{w}}_2 + a_{33}\overrightarrow{\mathbf{w}}_3$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned} \overrightarrow{\mathbf{v}}_3 &= a_{13}\overrightarrow{\mathbf{w}}_1 + a_{23}\overrightarrow{\mathbf{w}}_2 + a_{33}\overrightarrow{\mathbf{w}}_3 \\ (1,2,1) &= a_{13}(0,2,1) + a_{23}(-1,0,1) + a_{33}(-1,3,0) \end{aligned}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned} \overrightarrow{\mathbf{v}}_3 &= a_{13}\overrightarrow{\mathbf{w}}_1 + a_{23}\overrightarrow{\mathbf{w}}_2 + a_{33}\overrightarrow{\mathbf{w}}_3 \\ (1,2,1) &= a_{13}\,(0,2,1) + a_{23}\,(-1,0,1) + a_{33}\,(-1,3,0) \end{aligned}$$

$$\begin{aligned} -a_{23} - a_{33} &= 1 \\ 2a_{13} + 3a_{33} &= 2 \\ a_{13} + a_{23} &= 1 \end{aligned}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned}
\overrightarrow{\mathbf{v}}_3 &= a_{13}\overrightarrow{\mathbf{w}}_1 + a_{23}\overrightarrow{\mathbf{w}}_2 + a_{33}\overrightarrow{\mathbf{w}}_3 \\
(1,2,1) &= a_{13}\,(0,2,1) + a_{23}\,(-1,0,1) + a_{33}\,(-1,3,0)
\end{aligned}$$

$$\begin{array}{r} -a_{23} - a_{33} = 1 \\ 2a_{13} + 3a_{33} = 2 \\ a_{13} + a_{23} = 1 \end{array} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$\begin{aligned} \overrightarrow{\mathbf{v}}_3 &= a_{13}\overrightarrow{\mathbf{w}}_1 + a_{23}\overrightarrow{\mathbf{w}}_2 + a_{33}\overrightarrow{\mathbf{w}}_3 \\ (1,2,1) &= a_{13}(0,2,1) + a_{23}(-1,0,1) + a_{33}(-1,3,0) \end{aligned}$$

$$\begin{aligned} -a_{23} - a_{33} &= 1 \\ 2a_{13} + 3a_{33} &= 2 \\ a_{13} + a_{23} &= 1 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{8}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & -\frac{2}{5} \end{array}\right)$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$
\begin{aligned}
\overrightarrow{\mathbf{v}}_3 &= a_{13}\overrightarrow{\mathbf{w}}_1 + a_{23}\overrightarrow{\mathbf{w}}_2 + a_{33}\overrightarrow{\mathbf{w}}_3 \\
(1,2,1) &= a_{13}(0,2,1) + a_{23}(-1,0,1) + a_{33}(-1,3,0)
\end{aligned}
$$

$$
\begin{aligned}
-a_{23} - a_{33} &= 1 \\
2a_{13} + 3a_{33} &= 2 \\
a_{13} + a_{23} &= 1
\end{aligned}
\quad \Rightarrow \quad
\left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{8}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & -\frac{2}{5} \end{array}\right)
$$

$$
a_{13} = \frac{8}{5}, a_{23} = -\frac{3}{5}, a_{33} = -\frac{2}{5}
$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & \frac{9}{5} & \frac{8}{5} \\ -1 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{6}{5} & -\frac{2}{5} \end{pmatrix}$$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & \frac{9}{5} & \frac{8}{5} \\ -1 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{6}{5} & -\frac{2}{5} \end{pmatrix}$$

2. $\mathbf{B} = [\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}$:

$$B = \left([M]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) = \left([M]_{\mathcal{B}_1}^{\mathcal{B}_2}\right)^{-1} = A^{-1}$$

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$A = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & \frac{9}{5} & \frac{8}{5} \\ -1 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{6}{5} & -\frac{2}{5} \end{pmatrix}$$

2. $\mathbf{B} = [\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}$:

$$\begin{aligned} B &= \left([M]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) = \left([M]_{\mathcal{B}_1}^{\mathcal{B}_2}\right)^{-1} = A^{-1} \\ &= \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{4} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \end{pmatrix}. \end{aligned}$$

3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1 = (1,2,0), \overrightarrow{\mathbf{v}}_2 = (2,0,1), \overrightarrow{\mathbf{v}}_3 = (1,2,1)\}$

$$\overrightarrow{\mathbf{u}} = y_1\overrightarrow{\mathbf{v}}_1 + y_2\overrightarrow{\mathbf{v}}_2 + y_3\overrightarrow{\mathbf{v}}_3$$

3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1 = (1,2,0), \overrightarrow{\mathbf{v}}_2 = (2,0,1), \overrightarrow{\mathbf{v}}_3 = (1,2,1)\}$

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= y_1 \overrightarrow{\mathbf{v}}_1 + y_2 \overrightarrow{\mathbf{v}}_2 + y_3 \overrightarrow{\mathbf{v}}_3 \\ (2,3,5) &= y_1 (1,2,0) + y_2 (2,0,1) + y_3 (1,2,1) \end{aligned}$$

3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1 = (1,2,0), \overrightarrow{\mathbf{v}}_2 = (2,0,1), \overrightarrow{\mathbf{v}}_3 = (1,2,1)\}$

$$\begin{aligned}\overrightarrow{\mathbf{u}} &= y_1\overrightarrow{\mathbf{v}}_1 + y_2\overrightarrow{\mathbf{v}}_2 + y_3\overrightarrow{\mathbf{v}}_3 \\ (2,3,5) &= y_1(1,2,0) + y_2(2,0,1) + y_3(1,2,1)\end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 + y_3 &= 2 \\ 2y_1 + 2y_3 &= 3 \\ y_2 + y_3 &= 5 \end{aligned}$$

3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1 = (1,2,0), \overrightarrow{\mathbf{v}}_2 = (2,0,1), \overrightarrow{\mathbf{v}}_3 = (1,2,1)\}$

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= y_1 \overrightarrow{\mathbf{v}}_1 + y_2 \overrightarrow{\mathbf{v}}_2 + y_3 \overrightarrow{\mathbf{v}}_3 \\ (2,3,5) &= y_1 (1,2,0) + y_2 (2,0,1) + y_3 (1,2,1) \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 + y_3 &= 2 \\ 2y_1 + 2y_3 &= 3 \\ y_2 + y_3 &= 5 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 5 \end{array}\right)$$

3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1 = (1,2,0), \overrightarrow{\mathbf{v}}_2 = (2,0,1), \overrightarrow{\mathbf{v}}_3 = (1,2,1)\}$

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= y_1 \overrightarrow{\mathbf{v}}_1 + y_2 \overrightarrow{\mathbf{v}}_2 + y_3 \overrightarrow{\mathbf{v}}_3 \\ (2,3,5) &= y_1 (1,2,0) + y_2 (2,0,1) + y_3 (1,2,1) \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 + y_3 &= 2 \\ 2y_1 + 2y_3 &= 3 \\ y_2 + y_3 &= 5 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{13}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{19}{4} \end{array}\right)$$

3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$: $\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1 = (1,2,0), \overrightarrow{\mathbf{v}}_2 = (2,0,1), \overrightarrow{\mathbf{v}}_3 = (1,2,1)\}$

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= y_1 \overrightarrow{\mathbf{v}}_1 + y_2 \overrightarrow{\mathbf{v}}_2 + y_3 \overrightarrow{\mathbf{v}}_3 \\ (2,3,5) &= y_1 (1,2,0) + y_2 (2,0,1) + y_3 (1,2,1) \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 + y_3 &= 2 \\ 2y_1 + 2y_3 &= 3 \\ y_2 + y_3 &= 5 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{13}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{19}{4} \end{array}\right)$$

$$y_1 = -\frac{13}{4}, y_2 = \frac{1}{4}, y_3 = \frac{19}{4} \Rightarrow [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} = \left(-\frac{13}{4}, \frac{1}{4}, \frac{19}{4}\right).$$

4. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2}$:

$\mathcal{B}_2 = \{\overrightarrow{\mathbf{w}}_1 = (0, 2, 1), \overrightarrow{\mathbf{w}}_2 = (-1, 0, 1), \overrightarrow{\mathbf{w}}_3 = (-1, 3, 0)\}$

$$\overrightarrow{\mathbf{u}} = x_1\overrightarrow{\mathbf{w}}_1 + x_2\overrightarrow{\mathbf{w}}_2 + x_3\overrightarrow{\mathbf{w}}_3$$

4. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2}$:

$\mathcal{B}_2 = \{\overrightarrow{\mathbf{w}}_1 = (0,2,1), \overrightarrow{\mathbf{w}}_2 = (-1,0,1), \overrightarrow{\mathbf{w}}_3 = (-1,3,0)\}$

$$
\begin{aligned}
\overrightarrow{\mathbf{u}} &= x_1 \overrightarrow{\mathbf{w}}_1 + x_2 \overrightarrow{\mathbf{w}}_2 + x_3 \overrightarrow{\mathbf{w}}_3 \\
(2,3,5) &= x_1 (0,2,1) + x_2 (-1,0,1) + x_3 (-1,3,0)
\end{aligned}
$$

4. $\left[\overrightarrow{\mathbf{u}}\right]_{\mathcal{B}_2}$:

$\mathcal{B}_2 = \left\{\overrightarrow{\mathbf{w}}_1 = (0,2,1), \overrightarrow{\mathbf{w}}_2 = (-1,0,1), \overrightarrow{\mathbf{w}}_3 = (-1,3,0)\right\}$

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= x_1\overrightarrow{\mathbf{w}}_1 + x_2\overrightarrow{\mathbf{w}}_2 + x_3\overrightarrow{\mathbf{w}}_3 \\ (2,3,5) &= x_1(0,2,1) + x_2(-1,0,1) + x_3(-1,3,0) \end{aligned}$$

$$\begin{aligned} -x_2 - x_3 &= 1 \\ 2x_1 + 3x_3 &= 3 \\ x_1 + x_2 &= 5 \end{aligned}$$

4. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2}$:

$\mathcal{B}_2 = \{\overrightarrow{\mathbf{w}}_1 = (0, 2, 1), \overrightarrow{\mathbf{w}}_2 = (-1, 0, 1), \overrightarrow{\mathbf{w}}_3 = (-1, 3, 0)\}$

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= x_1 \overrightarrow{\mathbf{w}}_1 + x_2 \overrightarrow{\mathbf{w}}_2 + x_3 \overrightarrow{\mathbf{w}}_3 \\ (2, 3, 5) &= x_1 (0, 2, 1) + x_2 (-1, 0, 1) + x_3 (-1, 3, 0) \end{aligned}$$

$$\begin{aligned} -x_2 - x_3 &= 1 \\ 2x_1 + 3x_3 &= 3 \\ x_1 + x_2 &= 5 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 3 \\ 1 & 1 & 0 & 5 \end{array}\right)$$

4. $\left[\overrightarrow{\mathbf{u}}\right]_{\mathcal{B}_2}$:

$\mathcal{B}_2 = \left\{\overrightarrow{\mathbf{w}}_1 = (0,2,1), \overrightarrow{\mathbf{w}}_2 = (-1,0,1), \overrightarrow{\mathbf{w}}_3 = (-1,3,0)\right\}$

$$\begin{aligned} \overrightarrow{\mathbf{u}} &= x_1\overrightarrow{\mathbf{w}}_1 + x_2\overrightarrow{\mathbf{w}}_2 + x_3\overrightarrow{\mathbf{w}}_3 \\ (2,3,5) &= x_1(0,2,1) + x_2(-1,0,1) + x_3(-1,3,0) \end{aligned}$$

$$\begin{aligned} -x_2 - x_3 &= 1 \\ 2x_1 + 3x_3 &= 3 \\ x_1 + x_2 &= 5 \end{aligned} \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 3 \\ 1 & 1 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{24}{5} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{11}{5} \end{array}\right)$$

4. $\left[\overrightarrow{\mathbf{u}}\right]_{\mathcal{B}_2}$:

$\mathcal{B}_2 = \left\{\overrightarrow{\mathbf{w}}_1 = (0,2,1)\,, \overrightarrow{\mathbf{w}}_2 = (-1,0,1)\,, \overrightarrow{\mathbf{w}}_3 = (-1,3,0)\right\}$

$$
\begin{aligned}
\overrightarrow{\mathbf{u}} &= x_1\overrightarrow{\mathbf{w}}_1 + x_2\overrightarrow{\mathbf{w}}_2 + x_3\overrightarrow{\mathbf{w}}_3 \\
(2,3,5) &= x_1(0,2,1) + x_2(-1,0,1) + x_3(-1,3,0)
\end{aligned}
$$

$$
\begin{aligned}
-x_2 - x_3 &= 1 \\
2x_1 + 3x_3 &= 3 \\
x_1 + x_2 &= 5
\end{aligned}
\quad \Rightarrow \quad
\left(\begin{array}{ccc|c} 0 & -1 & -1 & 2 \\ 2 & 0 & 3 & 3 \\ 1 & 1 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{24}{5} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{11}{5} \end{array}\right)
$$

$$
x_1 = \frac{24}{5}, x_2 = \frac{1}{5}, x_3 = -\frac{11}{5} \Rightarrow \left[\overrightarrow{\mathbf{u}}\right]_{\mathcal{B}_2} = \left(\frac{24}{5}, \frac{1}{5}, -\frac{11}{5}\right).
$$

Notice that

$$[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} \quad = \quad [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2}$$

Change of Basis and Coordinate Transformations

Notice that

$$[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} \quad = \quad [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{4} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \frac{24}{5} \\ \frac{1}{5} \\ -\frac{11}{5} \end{pmatrix} = \begin{pmatrix} -\frac{13}{4} \\ \frac{1}{4} \\ \frac{19}{4} \end{pmatrix}$$

Notice that

$$[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{4} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \frac{24}{5} \\ \frac{1}{5} \\ -\frac{11}{5} \end{pmatrix} = \begin{pmatrix} -\frac{13}{4} \\ \frac{1}{4} \\ \frac{19}{4} \end{pmatrix}$$

$$[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$$

Notice that

$$
\begin{aligned}
[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} &= [M]_{\mathcal{B}_2}^{\mathcal{B}_1}[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{4} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} \frac{24}{5} \\ \frac{1}{5} \\ -\frac{11}{5} \end{pmatrix} = \begin{pmatrix} -\frac{13}{4} \\ \frac{1}{4} \\ \frac{19}{4} \end{pmatrix} \\
[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} &= [M]_{\mathcal{B}_1}^{\mathcal{B}_2}[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} = \begin{pmatrix} 1 & \frac{9}{5} & \frac{8}{5} \\ -1 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{6}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} -\frac{13}{4} \\ \frac{1}{4} \\ \frac{19}{4} \end{pmatrix} = \begin{pmatrix} \frac{24}{5} \\ \frac{1}{5} \\ -\frac{11}{5} \end{pmatrix}
\end{aligned}
$$

Example
Let $\mathcal{B}_1 = \{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \overrightarrow{\mathbf{v}}_3\}$ and $\mathcal{B}_2 = \{\overrightarrow{\mathbf{w}}_1, \overrightarrow{\mathbf{w}}_2, \overrightarrow{\mathbf{w}}_3\}$ be two ordered bases of V such that

$$\overrightarrow{\mathbf{v}}_1 = \overrightarrow{\mathbf{w}}_1 + \overrightarrow{\mathbf{w}}_2 - \overrightarrow{\mathbf{w}}_3$$

Example

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\overrightarrow{\mathbf{v}}_1 &= \overrightarrow{\mathbf{w}}_1 + \overrightarrow{\mathbf{w}}_2 - \overrightarrow{\mathbf{w}}_3 \\
\overrightarrow{\mathbf{v}}_2 &= 2\overrightarrow{\mathbf{w}}_1 - \overrightarrow{\mathbf{w}}_2 + \overrightarrow{\mathbf{w}}_3
\end{aligned}$$

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1. $[M]_{\mathcal{B}_1}^{\mathcal{B}_2} = ?$

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1. $[M]_{\mathcal{B}_1}^{\mathcal{B}_2} =?$
2. $[M]_{\mathcal{B}_2}^{\mathcal{B}_1} =?$

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1. $[M]_{\mathcal{B}_1}^{\mathcal{B}_2} =?$
2. $[M]_{\mathcal{B}_2}^{\mathcal{B}_1} =?$
3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} = (1, 0, 3) \Rightarrow [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} =?$

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3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} = (1, 0, 3) \Rightarrow [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} =?$
4. $[\overrightarrow{\mathbf{v}}]_{\mathcal{B}_2} = (2, -1, -2) \Rightarrow [\overrightarrow{\mathbf{v}}]_{\mathcal{B}_1} =?$

Change of Basis and Coordinate Transformations

1. $\mathbf{A} = [\mathbf{M}]_{\mathcal{B}_1}^{\mathcal{B}_2}$:

$$[M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

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$$[M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

2. $\mathbf{B} = [\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}$:

$$\left([M]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) \quad = \quad \left([M]_{\mathcal{B}_1}^{\mathcal{B}_2}\right)^{-1}$$

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$$[M]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

2. $\mathbf{B} = [\mathbf{M}]_{\mathcal{B}_2}^{\mathcal{B}_1}$:

$$\begin{aligned} \left([M]_{\mathcal{B}_2}^{\mathcal{B}_1}\right) &= \left([M]_{\mathcal{B}_1}^{\mathcal{B}_2}\right)^{-1} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2}$:

$$[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1}$$

3. $[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2}$:

$$[\overrightarrow{\mathbf{u}}]_{\mathcal{B}_2} = [M]_{\mathcal{B}_1}^{\mathcal{B}_2} [\overrightarrow{\mathbf{u}}]_{\mathcal{B}_1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$$

Change of Basis and Coordinate Transformations

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$$[\overrightarrow{\mathbf{v}}]_{\mathcal{B}_1} = [M]_{\mathcal{B}_2}^{\mathcal{B}_1} [\overrightarrow{\mathbf{v}}]_{\mathcal{B}_2}$$

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?