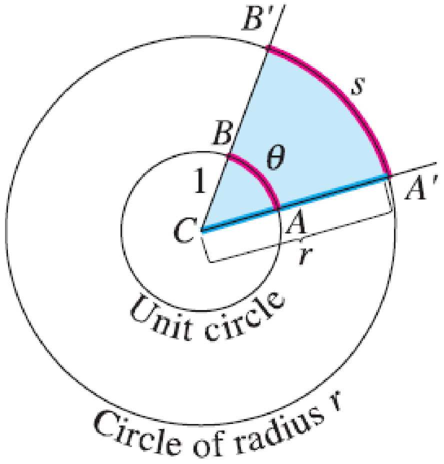
MAT1071 MATHEMATICS I
1. WEEK
PART 2

TRIGONOMETRIC FUNCTIONS

Asst. Prof. Dr. Nurten GÜRSES

Angles



The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius r is defined as the number of "radius units" contained in the arc s subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that $\theta = s/r$

If the circle is a unit circle having radius $r = 1$, we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180°$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

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Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$$

Degrees to radians: multiply by $\dfrac{\pi}{180}$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

Radians to degrees: multiply by $\dfrac{180}{\pi}$

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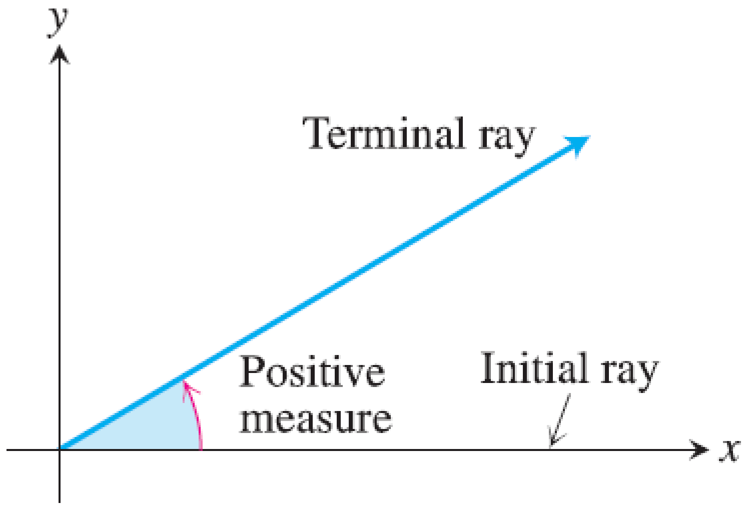
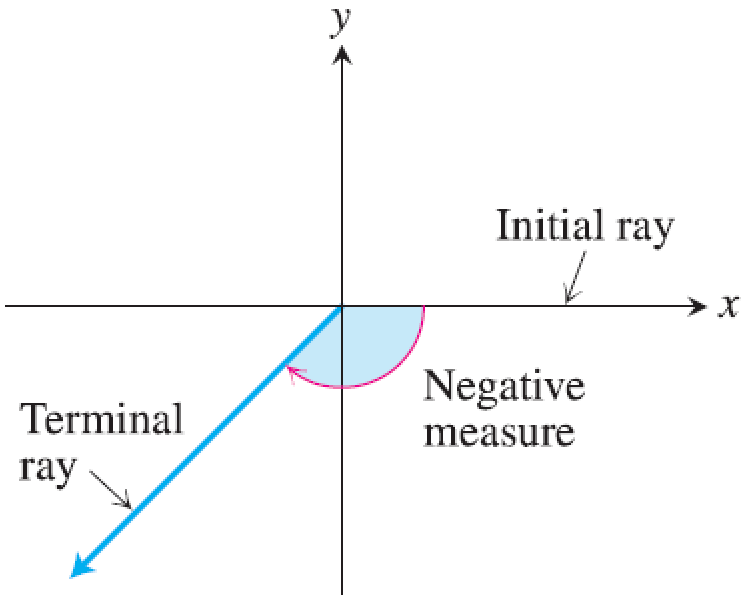
TABLE Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.





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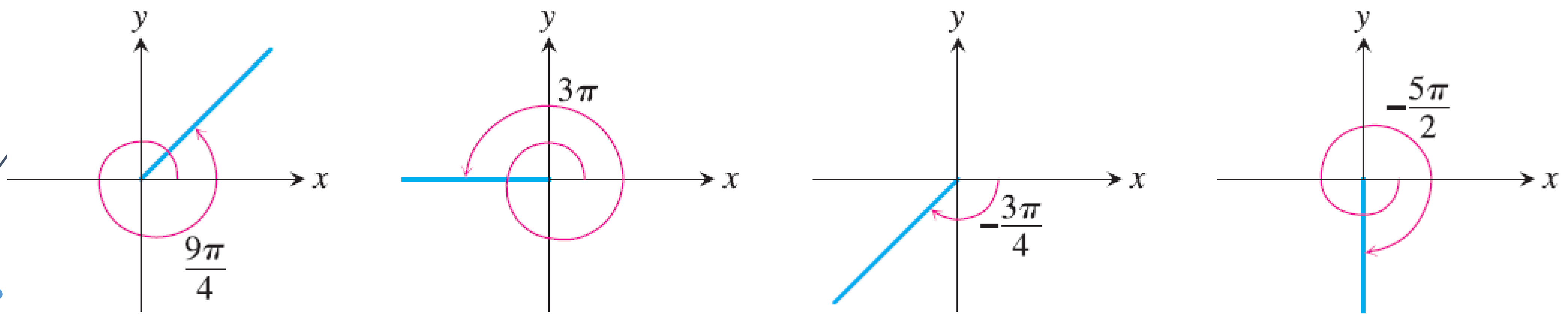
Angles describing counterclockwise rotations can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes



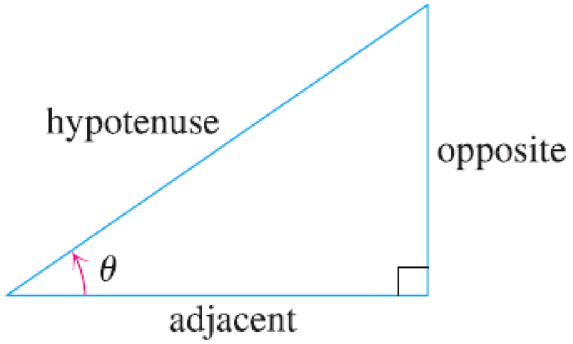
$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate once around counter clockwise)}$$

$$\frac{\pi}{6} + 4\pi = \frac{25\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice counter clockwise)}$$

$$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate once around clockwise)}$$

$$\frac{\pi}{6} - 4\pi = -\frac{23\pi}{6} \text{ (start at } \frac{\pi}{6} \text{ then rotate around twice clockwise)}$$

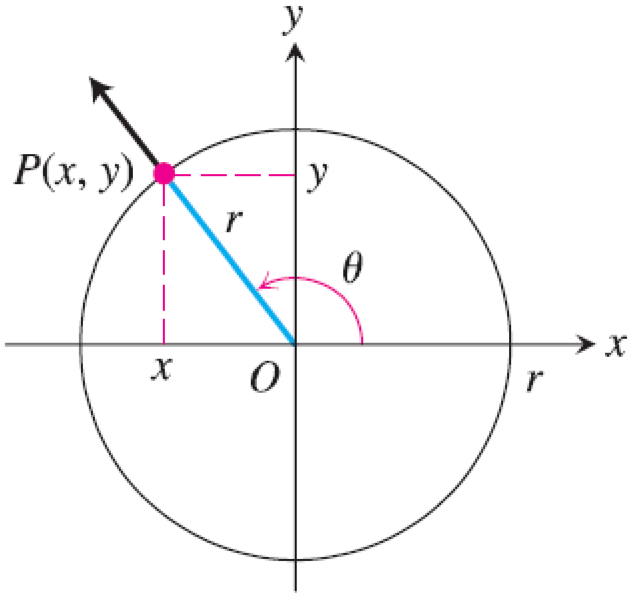
The Six Basic Trigonometric Functions



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$



sine: $\sin\theta = \dfrac{y}{r}$ **cosecant:** $\csc\theta = \dfrac{r}{y}$

cosine: $\cos\theta = \dfrac{x}{r}$ **secant:** $\sec\theta = \dfrac{r}{x}$

tangent: $\tan\theta = \dfrac{y}{x}$ **cotangent:** $\cot\theta = \dfrac{x}{y}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

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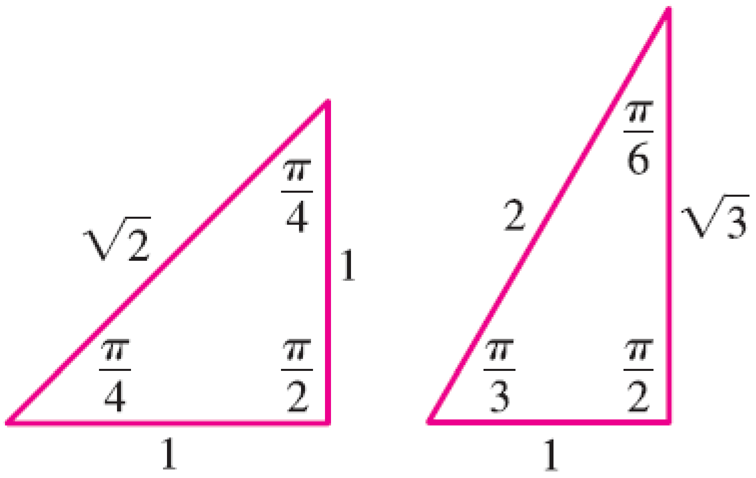

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

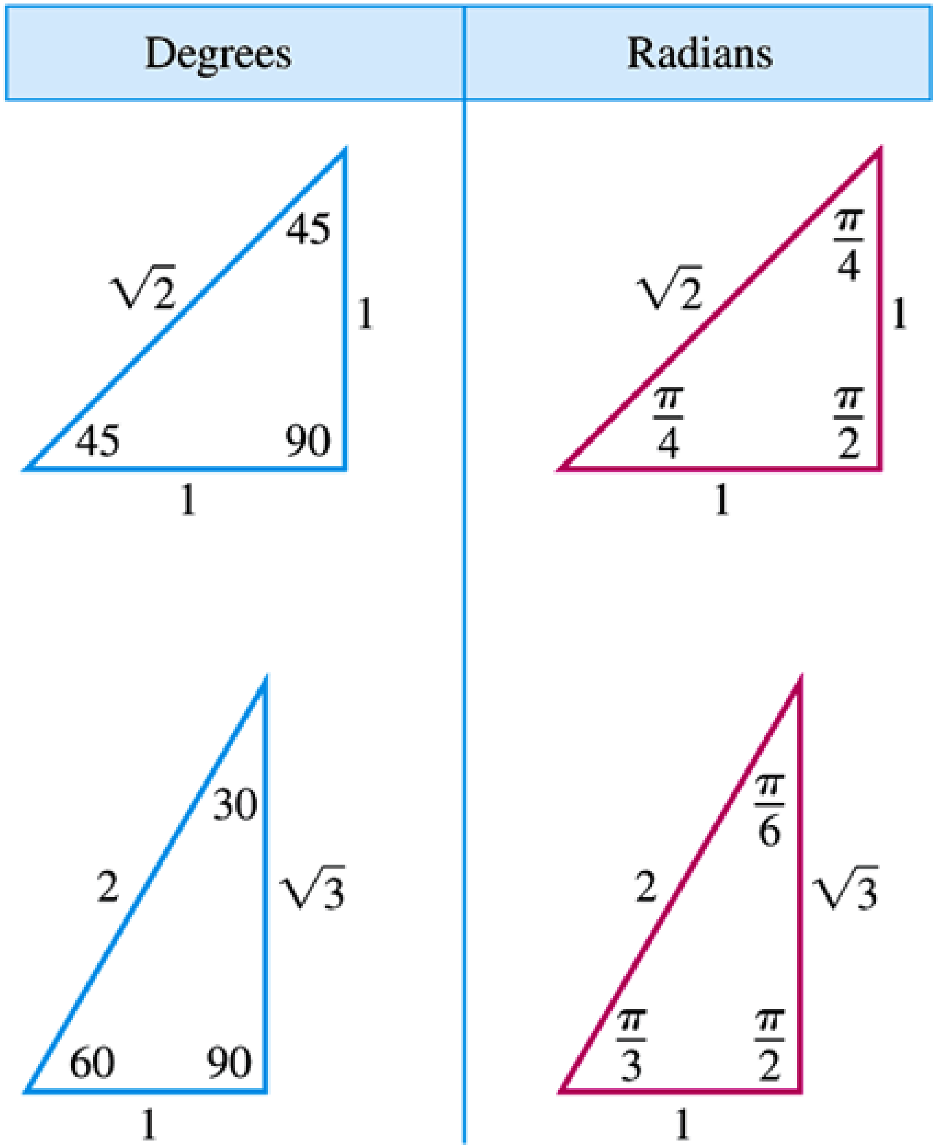

FIGURE The angles of two common triangles, in degrees and radians.

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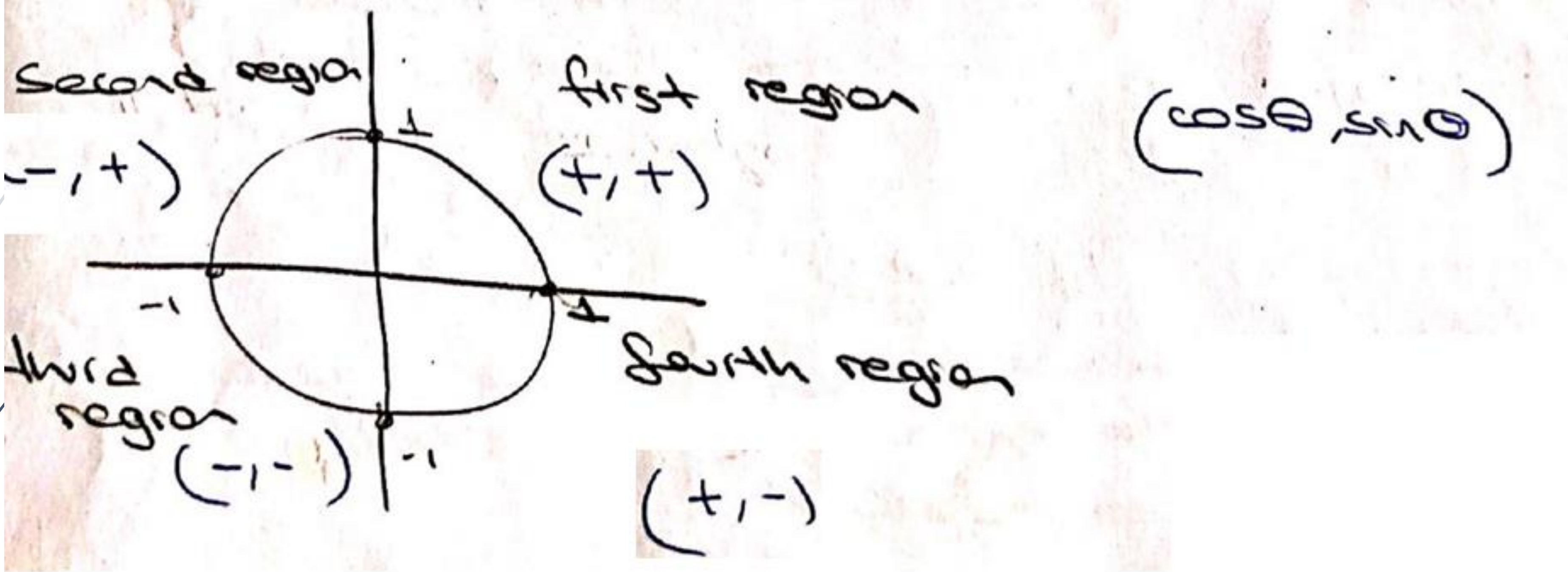
Second region
(-,+)
first region
(+,+)
1
-1
1
-1
third region
(-,-)
fourth region
(+,-)
(cosθ, sinθ)

TABLE Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

$\alpha+\beta = 90° \Rightarrow$

- $\sin\alpha = \cos\beta$
- $\tan\alpha = \cot\beta$

$$\sin\left(\tfrac{\pi}{2}-\theta\right) = \cos\theta \qquad \cos\left(\tfrac{\pi}{2}-\theta\right) = \sin\theta$$

$$\tan\left(\tfrac{\pi}{2}-\theta\right) = \cot\theta \qquad \cot\left(\tfrac{\pi}{2}-\theta\right) = \tan\theta$$

$$\sec\left(\tfrac{\pi}{2}-\theta\right) = \csc\theta \qquad \csc\left(\tfrac{\pi}{2}-\theta\right) = \sec\theta$$

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$$\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$$

$$\sin\left(\frac{\pi}{2}+\alpha\right) = +\cos\alpha$$

$$\sin\left(\frac{3\pi}{2}-\alpha\right) = -\sin\alpha$$

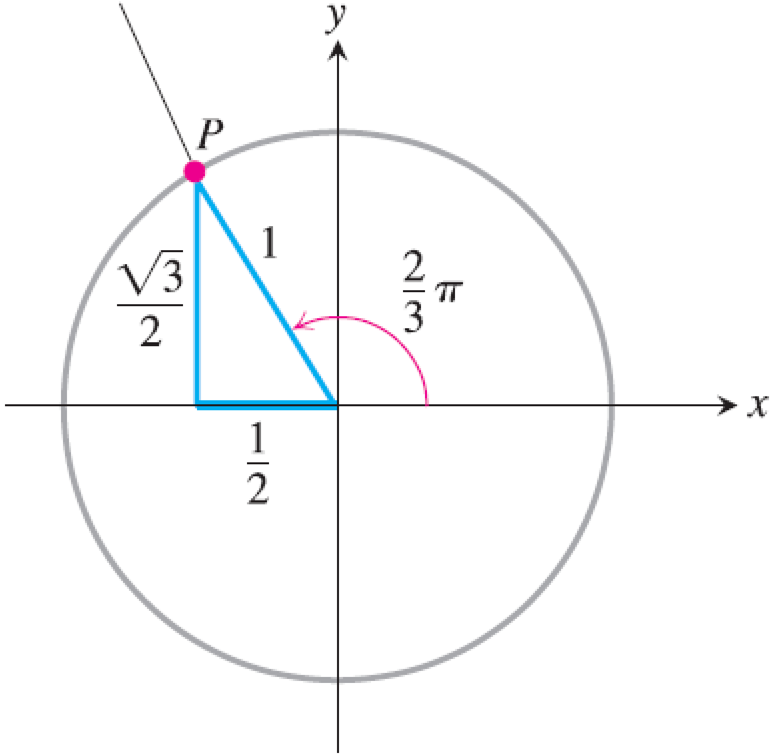
$$\tan\left(\frac{3\pi}{2}-\alpha\right) = \cot\alpha$$

$$\sin\left(\frac{3\pi}{2}+\alpha\right) = -\cos\alpha$$

$$\cos\left(\frac{3\pi}{2}+\alpha\right) = \sin\alpha$$

Example

$$\left(\cos\frac{2\pi}{3},\ \sin\frac{2\pi}{3}\right)=\left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$$



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Example

$$sin130 = sin(180-50) = sin30$$

$$cos150 = cos(180-30) = -cos30$$

$$tan110 = tan(180-70) = -tan70$$

$$cot95 = cot(180-85) = -cot85$$

$$sin220 = sin(180+40) = -sin40$$

$$cos260 = cos(180+80) = -cos80$$

$$tan190 = tan(180+10) = tan10$$

$$cot225 = cot(180+45) = cot45$$

$$sin300 = sin(360-60) = -sin60$$

$$cos330 = cos(360-30) = cos30$$

$$tan340 = tan(360-20) = -tan20$$

$$cot325 = cot(360-40) = -cot40$$

Periodicity and Graphs of the Trigonometric Functions

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Periods of Trigonometric Functions

Period π:

$$\tan(x + \pi) = \tan x$$
$$\cot(x + \pi) = \cot x$$

Period 2π:

$$\sin(x + 2\pi) = \sin x$$
$$\cos(x + 2\pi) = \cos x$$
$$\sec(x + 2\pi) = \sec x$$
$$\csc(x + 2\pi) = \csc x$$

Even
$\cos(-x) = \cos x$
$\sec(-x) = \sec x$

Odd
$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$
$\cot(-x) = -\cot x$

Even Functions and Odd Functions:

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

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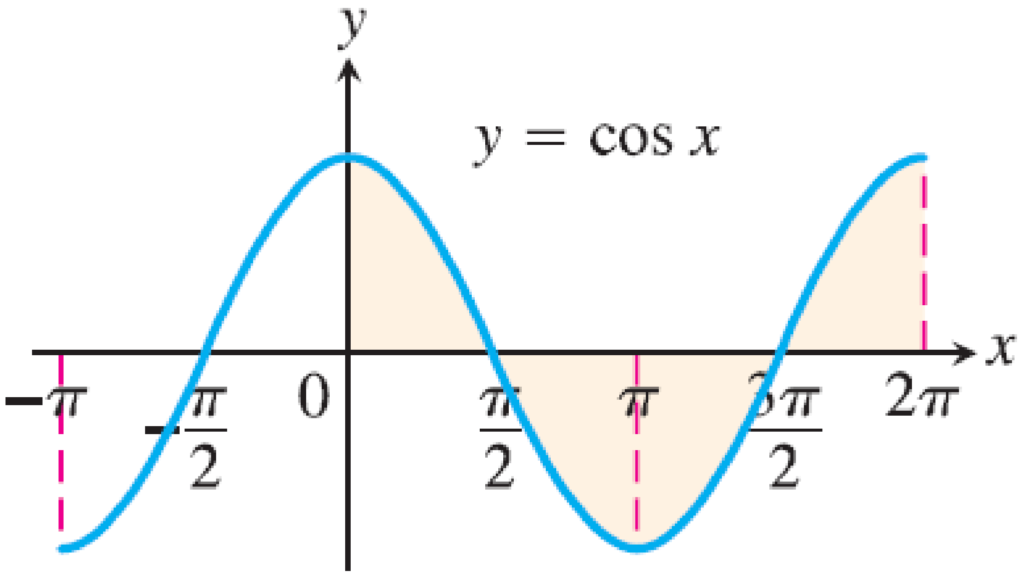
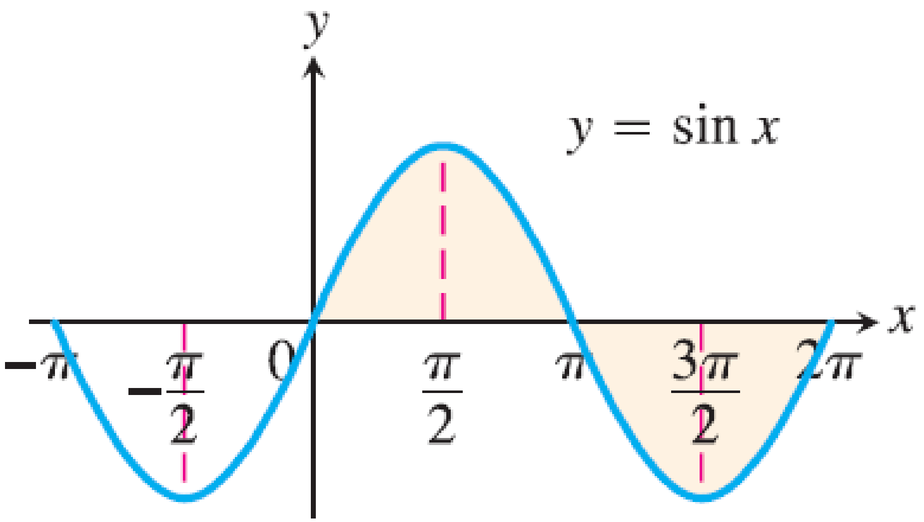
GRAPHS



Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π



Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π

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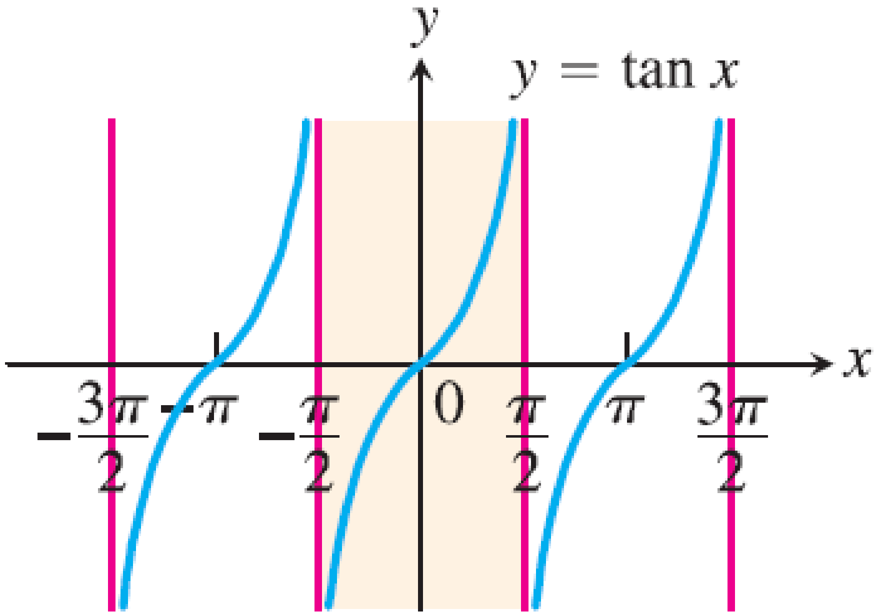
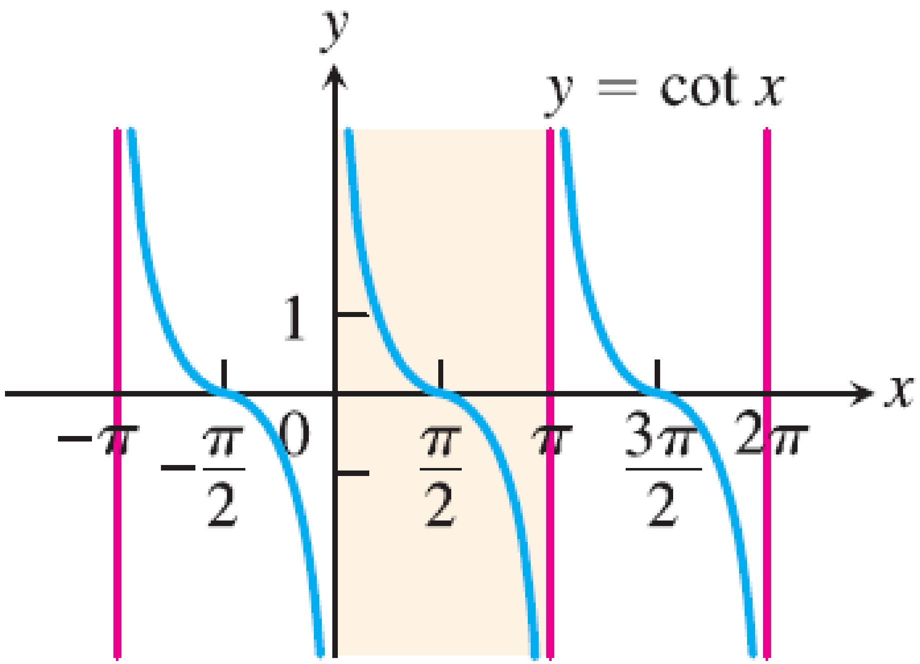


Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Range: $-\infty < y < \infty$

Period: π



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$

Range: $-\infty < y < \infty$

Period: π

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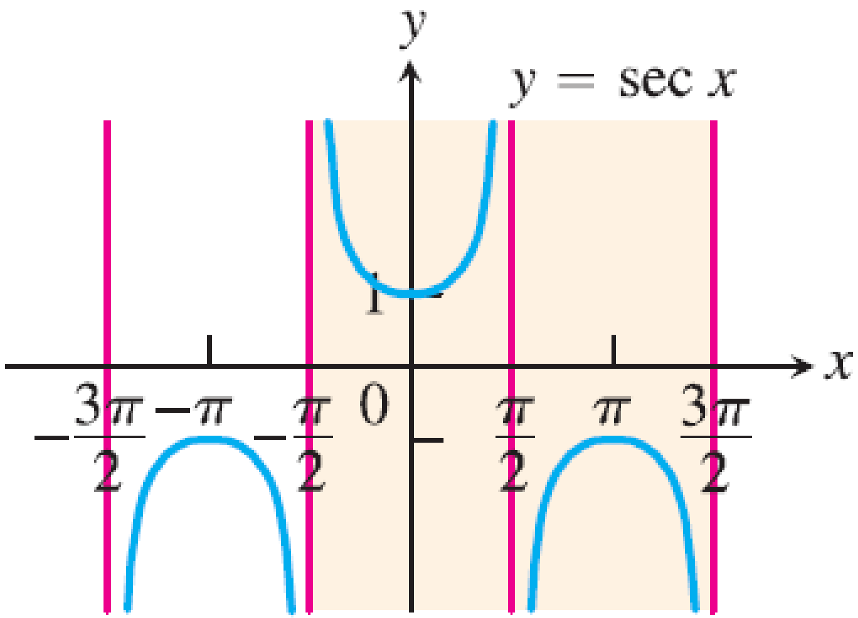
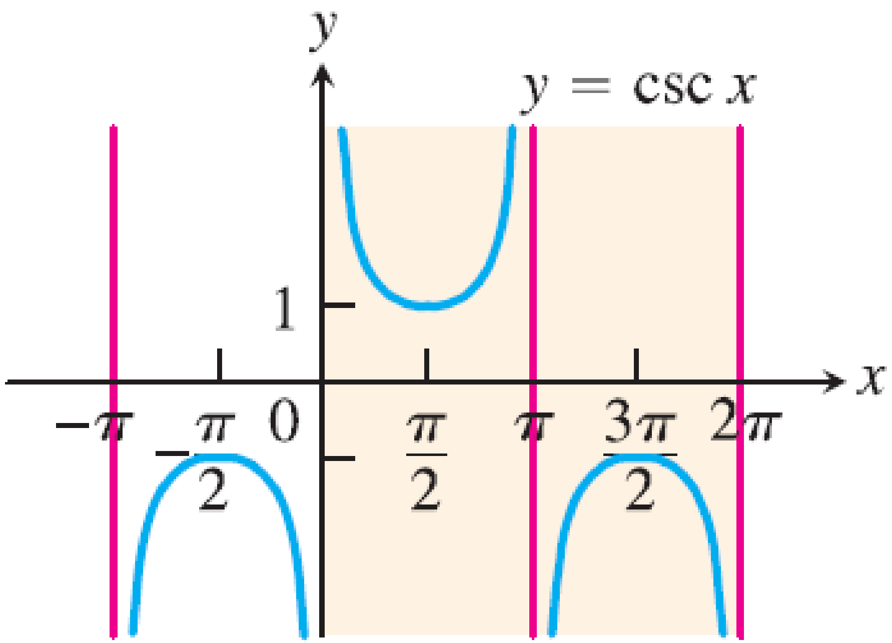


Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Range: $y \leq -1$ or $y \geq 1$

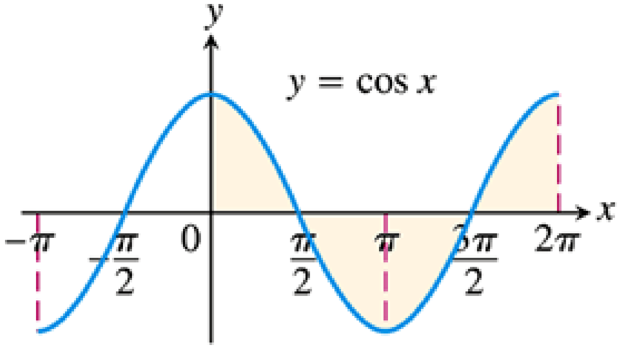
Period: 2π



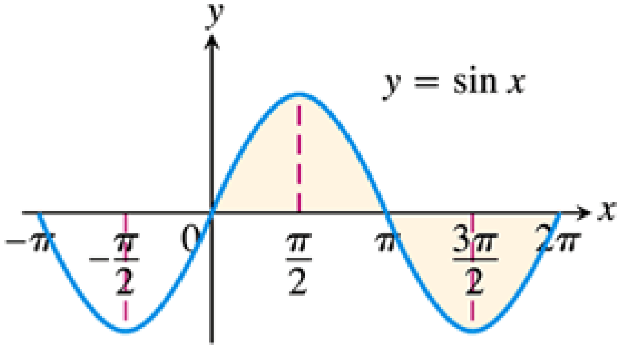
Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$

Range: $y \leq -1$ or $y \geq 1$

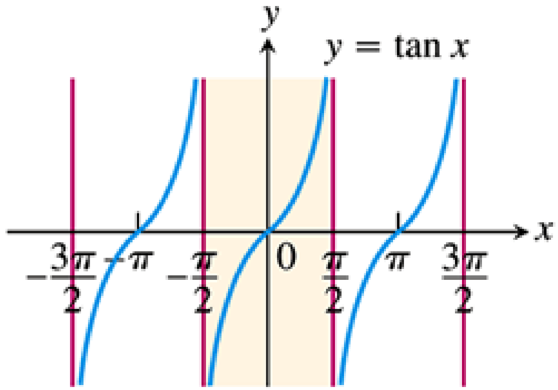
Period: 2π



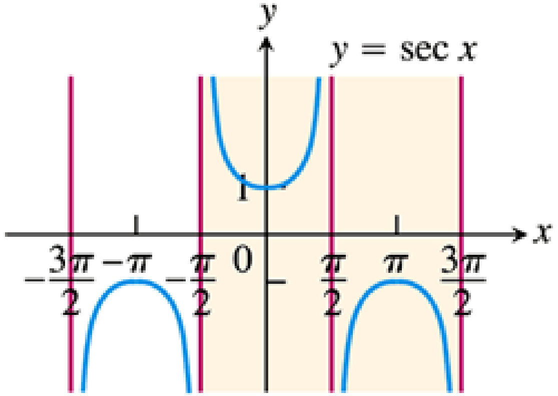
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



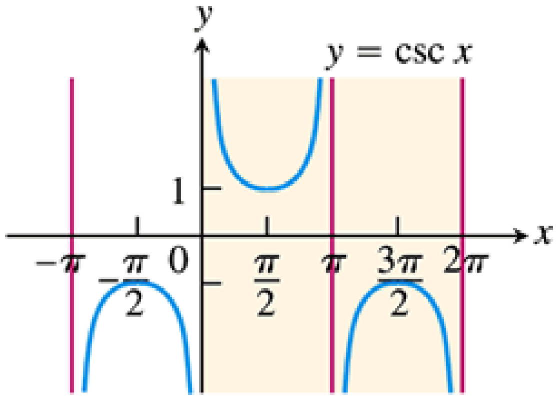
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



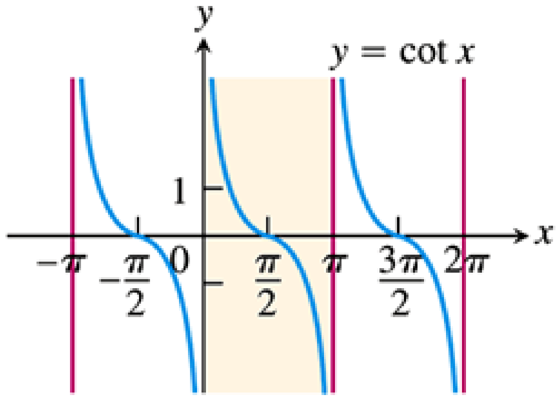
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π

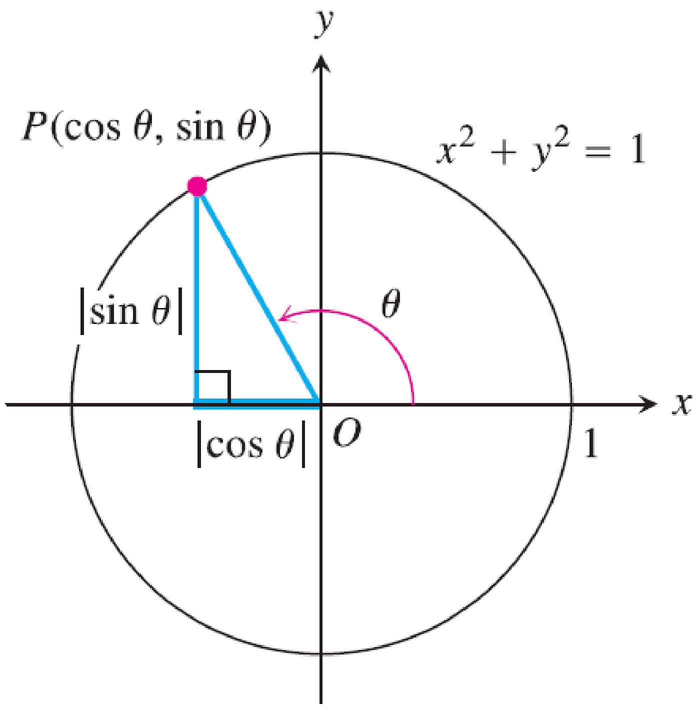


Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

Trigonometric Identities



$$x = r\cos\theta, \qquad y = r\sin\theta.$$

$$\cos^2\theta + \sin^2\theta = 1.$$

☆

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

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Addition Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$$

Double-Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\cot(2\theta) = \frac{\cot^2\theta - 1}{2\cot\theta}$$

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Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

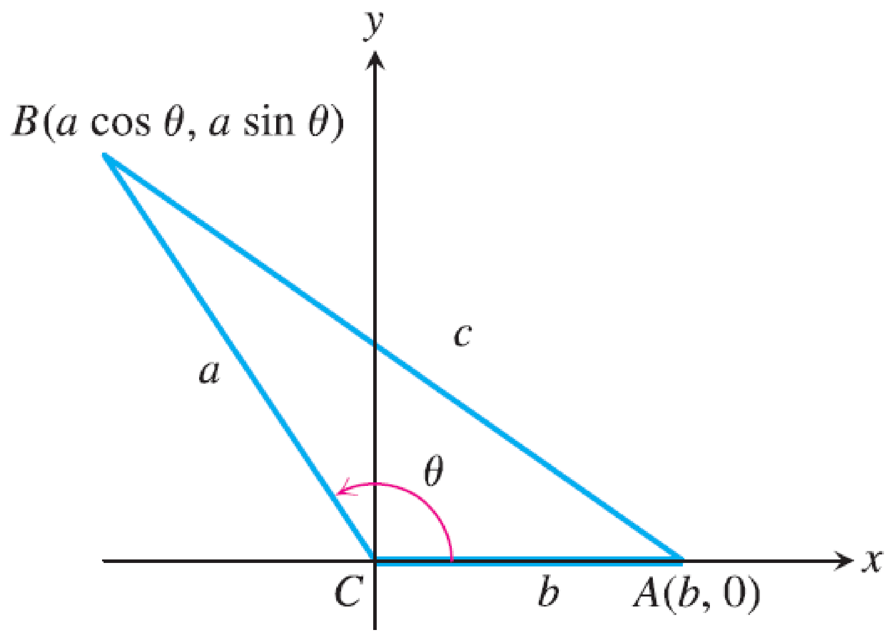
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

This equation is called the **law of cosines**.



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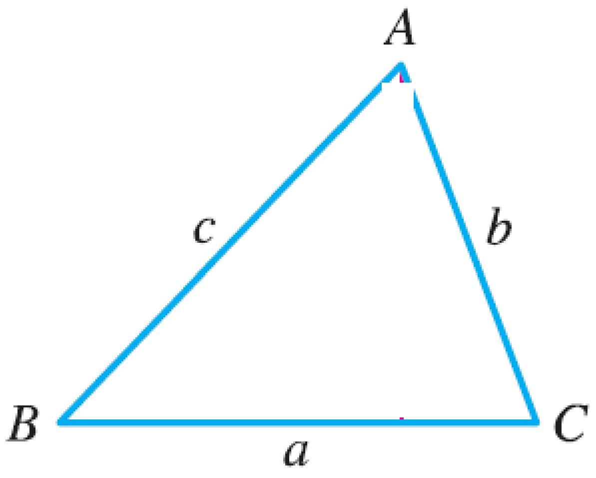
The law of sines *The law of sines* says that if a, b, and c are the sides opposite the angles A, B, and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



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Two Special Inequalities

For any angle θ measured in radians,

$$\boxed{-|\theta| \leq \sin\theta \leq |\theta| \qquad \text{and} \qquad -|\theta| \leq 1 - \cos\theta \leq |\theta|.}$$

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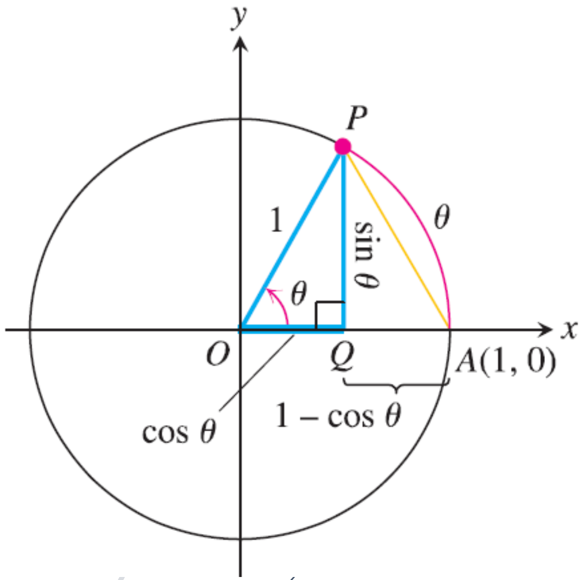

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \leq \theta^2.$$

The terms on the left-hand side are both positive, so each is smaller than their sum and hence is less than or equal to θ^2:

$$\sin^2\theta \leq \theta^2 \qquad \text{and} \qquad (1 - \cos\theta)^2 \leq \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin\theta| \leq |\theta| \qquad \text{and} \qquad |1 - \cos\theta| \leq |\theta|,$$

so

$$-|\theta| \leq \sin\theta \leq |\theta| \qquad \text{and} \qquad -|\theta| \leq 1 - \cos\theta \leq |\theta|.$$

These inequalities will be useful in the next chapter.

HW:

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \frac{3}{5}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

8. $\tan x = 2, \quad x \in \left[0, \frac{\pi}{2}\right]$

9. $\cos x = \frac{1}{3}, \quad x \in \left[-\frac{\pi}{2}, 0\right]$

10. $\cos x = -\frac{5}{13}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

11. $\tan x = \frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

12. $\sin x = -\frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

HW:

Using the Addition Formulas

Use the addition formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

32. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

34. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

HW:

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$

40. $\sin(2\pi - x)$

41. $\sin\left(\frac{3\pi}{2} - x\right)$

42. $\cos\left(\frac{3\pi}{2} + x\right)$

HW:

Using the Double-Angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2 \frac{\pi}{8}$

48. $\cos^2 \frac{5\pi}{12}$

49. $\sin^2 \frac{\pi}{12}$

50. $\sin^2 \frac{3\pi}{8}$

HW:

Solving Trigonometric Equations

For Exercises 51–54, solve for the angle θ, where $0 \leq \theta \leq 2\pi$.

51. $\sin^2 \theta = \dfrac{3}{4}$

52. $\sin^2 \theta = \cos^2 \theta$

53. $\sin 2\theta - \cos \theta = 0$

54. $\cos 2\theta + \cos \theta = 0$

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and F.R.Giordano, Addison-Wesley, 2012.