SOLUTION OF INITIAL VALUE PROBLEMS WİTH LAPLACE AND INVERSE LAPLACE TRANSFORMS

1. $y' + 2y = e^{2t} \quad , \quad y(0) = 1$

$$L\{y' + 2y\} = sY - y(0) + 2Y$$

$$= sY - 1 + 2Y$$

$$L\{e^{2t}\} = \frac{1}{s-2}$$

$$sY + 2Y - 1 = \frac{1}{s-2}$$

$$Y(s+2) = \frac{1}{s-2} + 1 = \frac{1+s-2}{s-2} = \frac{s-1}{s-2}$$

$$Y(s) = \frac{s-1}{(s+2)(s-2)}$$

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{s-1}{(s+2)(s-2)}\right\}$$

$$y(t) = L^{-1}\left\{\frac{3/4}{s+2} + \frac{1/4}{s-2}\right\}$$

$$y(t) = \frac{3}{4}L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$y(t) = \frac{3}{4}e^{-2t} + \frac{1}{4}e^{2t}.$$

2. $y'' + 4y = 0$, $y(0) = 1 \, , \; y\left(\frac{\pi}{4}\right) = -1$

$$s^2 Y - s\,y(0) - y'(0) + 4Y = 0$$

$$y(t)=L^{-1}\left\{\frac{s}{s^2+4}\right\}+c\,L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$y(t)=Cos\,2t+c\frac{Sin\,2t}{2}$$

3. $y'''-3y''+3y'-y=t^2e^t\ ;\quad y(0)=1\ ,\ y'(0)=2\ ,\ y''(0)=3$

$$L\{y'''\}=s^3Y-s^2y(0)-s\,y'(0)-y''(0)=s^3Y-s^2\cdot 1-s\cdot 2-3$$

$$-3L\{y''\}=-3[s^2Y-s\,y(0)-y'(0)]=-3[s^2Y-s\cdot 1-2]$$

$$3L\{y'\}=3[sY-y(0)]=3[sY-1]$$

$$-L\{y\}=-Y$$

$$Y(s)=\frac{2}{(s-1)^6}+\frac{s(s-1)}{(s-1)^3}=\frac{2}{(s-1)^6}+\frac{s}{(s-1)^2}$$

$$L^{-1}\{Y\}=2\,L^{-1}\left\{\frac{1}{(s-1)^6}\right\}+L^{-1}\left\{\frac{s}{(s-1)^2}\right\},$$

$$y(t)=\frac{2}{5!}t^5e^t+t\,e^t+e^t$$

4. $y''+4y=Sin\,t\cdot Cos\,t\quad ;\ y(0)=0\ ,\ y'(0)=0$

$$L\{y''\}=s^2Y-s\,y(0)-y'(0)\ ,\ \ 4L\{y\}=4Y\ ,\ \ L\{Sin\,t\cdot Cos\,t\}=L\left\{\frac{1}{2}Sin\,2t\right\}=\frac{1}{s^2+4}\ .$$

$$Y(s)=\frac{1}{(s^2+4)(s^2+4)}$$

$$y(t)=\frac{1}{16}Sin\,(2t)-\frac{1}{8}t\,Cos\,(2t)$$

$y''' - y' = 0\ ,\ y(0) = 1\ , y'(0) = 0, y''(0) = 0\ .$

$$s^3 Y(s) - s^2 \underbrace{y(0)}_{1} - s\cancel{y'(0)} - \cancel{y''(0)} - (sY(s) - \underbrace{y(0)}_{1}) = 0$$

$$[s^3 - s]\, Y(s) = s^2 - 1$$

$$Y(s) = \frac{\cancel{s^2-1}}{s\cancel{(s^2-1)}} = \frac{1}{s}$$

$$\underbrace{L^{-1}\{Y(s)\}} = L^{-1}\left\{\frac{1}{s}\right\}$$

$$\boxed{y(t) = 1}$$

$y'' + 4y' + 4y = t^2 e^{-2t}\ ,\ y(0) = 0\ , y'(0) = 0\ .$

$$s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)} + 4(sY(s) - \cancel{y(0)}) + 4Y(s) = \frac{2}{(s+2)^3}$$

$$[\underbrace{s^2 + 4s + 4}_{(s+2)^2}]\, Y(s) = \frac{2}{(s+2)^3}$$

$$Y(s) = \frac{2}{(s+2)^5}$$

$$\underbrace{L^{-1}\{Y(s)\}} = 2L^{-1}\left\{\frac{1}{(s+2)^5}\right\}$$

$$y(t) = \frac{2t^4}{4!} e^{-2t}$$

$$\frac{1}{s^5}$$

$n+1=5$

$n=4$

$$s+2 = s - \underbrace{(-2)}_{a}$$

: $y'' + y = sin2t$, $y(0) = 2, y'(0) = 1.$

$$s^2Y(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{1} + Y(s) = \frac{2}{s^2+4}$$

$$[s^2+1]Y(s) = \frac{2}{s^2+4} + 2s + 1$$

$$\underbrace{L^{-1}\{Y(s)\}}_{y(t)} = L^{-1}\left\{\frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{s^2+1}\right\}$$

$$\Downarrow$$

$$\frac{2}{3}\underbrace{L^{-1}\left(\frac{1}{s^2+1}\right)}_{\sin t} - \frac{2}{3}\underbrace{L^{-1}\left(\frac{1}{s^2+4}\right)}_{\frac{\sin 2t}{2}}$$

$$L^{-1}\left\{\frac{2s+1}{s^2+1}\right\} = 2\underbrace{L^{-1}\left\{\frac{s}{s^2+1}\right\}}_{\cos t} + \underbrace{L^{-1}\left\{\frac{1}{s^2+1}\right\}}_{\sin t}$$

$$y(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t + 2\cos t + \sin t$$

Homework

$y'' + 3y' + 2y = e^t$, $y(0) = 0, y'(0) = 1$. **S:** $\boldsymbol{y(t) = \frac{1}{6}e^t + \frac{1}{2}e^{-t} - \frac{2}{3}e^{-2t}}$

: $y'' - 4y' + 4y = sint$, $y(0) = 0, y'(0) = 0$. **S:** $\boldsymbol{y(t) = -\frac{4}{15}e^{2t} + \frac{1}{5}te^{2t} + \frac{1}{15}sint}$

$y'' - y' - 3y = 0$, $y(0) = -2, y'(0) = 0$ **S:** $\boldsymbol{y(t) = -2e^{\frac{t}{2}}cosh\frac{\sqrt{13}}{2}t + \frac{2}{\sqrt{13}}e^{\frac{t}{2}}sinh\frac{\sqrt{13}}{2}t}$

$y'^{v} - y = 0$, $y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1$. **S:** $\boldsymbol{y(t) = sint}$