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Gaussian Elimination

The fundamental idea is to add multiples of one equation to the others in order to eliminate a variable and to continue this process until only one variable is left.

Once this final variable is determined, its value is substituted back into the other equations in order to evaluate the remaining unknowns. This method, characterized by step-by-step elimination of the variables, is called Gaussian elimination.

$[A]\{x\}=\{C\}$

Where

[A] is the matrix of coefficient and

{x} is the vector of unknowns and

{C} is the vector of constants.

Gauss elimination method eliminate unknowns' coefficients of the equations one by one.

Therefore the matrix of coefficients of the system of linear equations is transformed to an upper triangular matrix.

The last transformed equation has only one unknown which can be determined easily.

This evaluated unknown can be used in the upper equation for determining the next unknown and so on.

Finally the system of linear equations can be solved by back substitution of evaluated unknowns.

Example 2: Solve this system:

$$x + y = 3$$
$$3x - 2y = 4$$

The first step is to write the coefficients of the unknowns in a matrix:

$$\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

This is called the **coefficient matrix** of the system. Next, the coefficient matrix is augmented by writing the constants that appear on the right-hand sides of the equations as an additional column:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -2 & 4 \end{array}\right]$$

This is called the **augmented matrix**, and each row corresponds to an equation in the given system.

The first row, $\mathbf{r}_1 = (1, 1, 3)$, corresponds to the first equation, $1x + 1y = 3$,
and the second row, $\mathbf{r}_2 = (3, -2, 4)$, corresponds to the second equation, $3x - 2y = 4$.

You may choose to include a vertical line—as shown above—to separate the coefficients of the unknowns from the extra column representing the constants.
Now, the counterpart of eliminating a variable from an equation in the system is changing one of the entries in the coefficient matrix to zero.

Likewise, the counterpart of adding a multiple of one equation to another is adding a multiple of one row to another row. Adding −3 times the first row of the augmented matrix to the second row yields

$$\begin{bmatrix} 1 & 1 & 3 \\ 3 & -2 & 4 \end{bmatrix} \xrightarrow{-3r_1 \text{ added to } r_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -5 & -5 \end{bmatrix}$$

The new second row translates into $-5y = -5$, which means $y = 1$.
Back-substitution into the first row (that is, into the equation that represents the first row) yields $x = 2$ and, therefore, the solution to the system: $(x, y) = (2, 1)$.

Gaussian elimination can be summarized as follows. Given a linear system expressed in matrix form, $A x = b$, first write down the corresponding augmented matrix:

Example 3: Solve the following system using Gaussian elimination:

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 0 \\ 2x & + & y & - & 3z & = & 5 \\ 4x & - & 7y & + & z & = & -1 \end{array}$$

The augmented matrix which represents this system is

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$

The first goal is to produce zeros below the first entry in the first column, which translates into eliminating the first variable, *x*, from the second and third equations. The row operations which accomplish this are as follows:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{bmatrix} \xrightarrow[-4r_1 \text{ added to } r_3]{-2r_1 \text{ added to } r_2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

The second goal is to produce a zero below the second entry in the second column, which translates into eliminating the second variable, *y*, from the third equation. One way to accomplish this would be to add $-1/5$ times the second row to the third row. However, to avoid fractions, there is another option: first interchange rows two and three. Interchanging two rows merely interchanges the equations, which clearly will not alter the solution of the system:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -5 & 5 \end{bmatrix}$$

Now, add -5 times the second row to the third row:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -5 & 5 \end{bmatrix} \xrightarrow{-5r_2 \text{ added to } r_3} \underbrace{\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 10 & 10 \end{bmatrix}}_{\text{echelon form}}$$

Since the coefficient matrix has been transformed into echelon form, the "forward" part of Gaussian elimination is complete. What remains now is to use the third row to evaluate the third unknown, then to back-substitute into the second row to evaluate the second unknown, and, finally, to back-substitute into the first row to evaluate the first unknwon.

The third row of the final matrix translates into 10 z = 10, which gives z = 1. Back-substitution of this value into the second row, which represents the equation $y - 3z = -1$, yields y = 2. Back-substitution of both these values into the first row, which represents the equation $x - 2y + z = 0$, gives x = 3. The solution of this system is therefore $(x, y, z) = (3, 2, 1)$.

Example 4: Solve the following system using Gaussian elimination:

$$\begin{aligned} 2x - 2y \qquad &= -6 \\ x - y + z &= 1 \\ 3y - 2z &= -5 \end{aligned}$$

For this system, the augmented matrix (vertical line omitted) is

$$\begin{bmatrix} 2 & -2 & 0 & -6 \\ 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & -5 \end{bmatrix}$$

First, multiply row 1 by 1/2:

$$\begin{bmatrix} 2 & -2 & 0 & -6 \\ 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & -5 \end{bmatrix} \xrightarrow{\text{Multiply } r_1 \text{ by } \frac{1}{2}} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & -5 \end{bmatrix}$$

Now, adding −1 times the first row to the second row yields zeros below the first entry in the first column:

$$\begin{bmatrix} 1 & -1 & 0 & -3 \\ 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & -5 \end{bmatrix} \xrightarrow{-r_1 \text{ added ro } r_2} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & -2 & -5 \end{bmatrix}$$

Interchanging the second and third rows then gives the desired upper-triangular coefficient matrix:

$$\begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & -2 & -5 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

The third row now says $z = 4$. Back-substituting this value into the second row gives $y = 1$, and back-substitution of both these values into the first row yields $x = -2$. The solution of this system is therefore $(x, y, z) = (-2, 1, 4)$.

Gauss-Jordan elimination. Gaussian elimination proceeds by performing elementary row operations to produce zeros below the diagonal of the coefficient matrix to reduce it to echelon form. (Recall that a matrix $A' = [a_{ij}']$ is in echelon form when $a_{ij}' = 0$ for $i > j$, any zero rows appear at the bottom of the matrix, and the first nonzero entry in any row is to the right of the first nonzero entry in any higher row.) Once this is done, inspection of the bottom row(s) and back-substitution into the upper rows determine the values of the unknowns. However, it is possible to reduce (or eliminate entirely) the computations involved in back-substitution by performing additional row operations to transform the matrix from echelon form to **reduced echelon** form. A matrix is in reduced echelon form when, in addition to being in echelon form, each column that contians a nonzero entry (usually made to be 1) has zeros not just below that entry but also above that entry. Loosely speaking, Gaussian elimination works from the top down, to produce a matrix in echelon form, whereas **Gauss Jordan elimination** continues where Gaussian left off by then working from the bottom up to produce a matrix in reduced echelon form. The technique will be illustrated in the following example.

Example 5: The height, y, of an object thrown into the air is known to be given by a quadratic function of t (time) of the form $y = at^2 + bt + c$. If the object is at height y = 23/4 at time t = 1/2, at y = 7 at time t = 1, and at y = 2 at t = 2, determine the coefficients a, b, and c.

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Since t = 1/2 gives y = 23/4,

$$\begin{aligned}\tfrac{23}{4} &= a(\tfrac{1}{2})^2 + b(\tfrac{1}{2}) + c\\ &= \tfrac{1}{4}a + \tfrac{1}{2}b + c\end{aligned}$$

while the other two conditions, $y(t = 1) = 7$ and $y(t = 2) = 2$, give the following equations for a, b, and c:

$$\begin{aligned}7 &= a + b + c\\ 2 &= 4a + 2b + c\end{aligned}$$

Therefore, the goal is solve the system

$$\begin{aligned}\tfrac{1}{4}a + \tfrac{1}{2}b + c &= \tfrac{23}{4}\\ a + b + c &= 7\\ 4a + 2b + c &= 2\end{aligned}$$

The augmented matrix for this system is reduced as follows:

$$\begin{bmatrix}\frac{1}{4} & \frac{1}{2} & 1 & \frac{23}{4}\\ 1 & 1 & 1 & 7\\ 4 & 2 & 1 & 2\end{bmatrix} \xrightarrow{4r_1} \begin{bmatrix}1 & 2 & 4 & 23\\ 1 & 1 & 1 & 7\\ 4 & 2 & 1 & 2\end{bmatrix}$$

$$\xrightarrow[-4r_1 \text{ added to } r_3]{-r_1 \text{ added to } r_2} \begin{bmatrix}1 & 2 & 4 & 23\\ 0 & -1 & -3 & -16\\ 0 & -6 & -15 & -90\end{bmatrix}$$

$$\xrightarrow[-r_2]{-6r_2 \text{ added to } r_3} \begin{bmatrix}1 & 2 & 4 & 23\\ 0 & 1 & 3 & 16\\ 0 & 0 & 3 & 6\end{bmatrix}$$

At this point, the forward part of Gaussian elimination is finished, since the coefficient matrix has been reduced to echelon form. However, to illustrate Gauss-Jordan elimination, the following additional elementary row operations are performed:

$$\begin{bmatrix}1 & 2 & 4 & 23\\ 0 & 1 & 3 & 16\\ 0 & 0 & 3 & 6\end{bmatrix} \xrightarrow[\frac{1}{3}r_3]{-r_3 \text{ added to } r_2} \begin{bmatrix}1 & 2 & 4 & 23\\ 0 & 1 & 0 & 10\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$$\xrightarrow{-4r_3 \text{ added to } r_1} \begin{bmatrix}1 & 2 & 0 & 15\\ 0 & 1 & 0 & 10\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$$\xrightarrow{-2r_2 \text{ added to } r_1} \begin{bmatrix}1 & 0 & 0 & -5\\ 0 & 1 & 0 & 10\\ 0 & 0 & 1 & 2\end{bmatrix}$$

This final matrix immediately gives the solution: $a = -5$, $b = 10$, and $c = 2$.

Example 6: Solve the following system using Gaussian elimination:

$$\begin{aligned} x + y - 3z &= 4 \\ 2x + y - z &= 2 \\ 3x + 2y - 4z &= 7 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & -4 & 7 \end{bmatrix}$$

Multiples of the first row are added to the other rows to produce zeros below the first entry in the first column:

$$\begin{bmatrix} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & -4 & 7 \end{bmatrix} \xrightarrow[-3r_1 \text{ added to } r_3]{-2r_1 \text{ added to } r_2} \begin{bmatrix} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & -1 & 5 & -5 \end{bmatrix}$$

Next, −1 times the second row is added to the third row:

$$\begin{bmatrix} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & -1 & 5 & -5 \end{bmatrix} \xrightarrow{-r_2 \text{ added to } r_3} \begin{bmatrix} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The third row now says $0\,x + 0\,y + 0\,z = 1$, an equation that cannot be satisfied by any values of x, y, and z. The process stops: this system has no solutions.

The previous example shows how Gaussian elimination reveals an inconsistent system. A slight alteration of that system (for example, changing the constant term "7" in the third equation to a "6") will illustrate a system with infinitely many solutions.

Example 7: Solve the following system using Gaussian elimination:

$$\begin{aligned} x + y - 3z &= 4 \\ 2x + y - z &= 2 \\ 3x + 2y - 4z &= 6 \end{aligned}$$

The same operations applied to the augment matrix of the system in Example 6 are applied to the augmented matrix for the present system:

$$\begin{bmatrix} 1 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & -4 & 6 \end{bmatrix} \xrightarrow[-3r_1 \text{ added to } r_3]{-2r_1 \text{ added to } r_2} \begin{bmatrix} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & -1 & 5 & -6 \end{bmatrix}$$

$$\xrightarrow{-r_2 \text{ added to } r_3} \begin{bmatrix} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the third row translates into $0\,x + 0\,y + 0\,z = 0$, an equation which is satisfied by *any* x, y, and z. Since this offer no constraint on the unknowns, there are not three conditions on the unknowns, only two (represented by the two nonzero rows in the final augmented matrix). Since there are 3 unknowns but only 2 constrants, $3 - 2 = 1$ of the unknowns, z say, is arbitrary; this is called a **free variable**. Let $z = t$, where t is any real number. Back-substitution of $z = t$ into the second row ($-y + 5z = -6$) gives

$$-y + 5t = -6 \;\Rightarrow\; y = 6 + 5t$$

Therefore, every solution of the system has the form

$$(x, y, z) = (-2 - 2t, 6 + 5t, t) = (-2t, 5t, t) + (-2, 6, 0) \quad (*)$$

where t is any real number. There are infinitely many solutions, since every real value of t gives a different particular solution. For example, choosing $t = 1$ gives $(x, y, z) = (-4, 11, 1)$, while $t = 3$ gives $(x, y, z) = (4, -9, -3)$, and so on. Geometrically, this system represents three planes in $\mathbf{R}^3$ that intersect in a line, and (*) is a parametric equation for this line.

Example 7 provided an illustration of a system with infinitely many solutions, how this case arises, and how the solution is written. Every linear system that possesses infinitely many solutions must contain at least one arbitrary **parameter** (free variable). Once the augmented matrix has been reduced to echelon form, the number of free variables is equal to the total number of unknowns minus the number of nonzero rows:

$$\boxed{\begin{array}{c} \#\text{ free variables} \\ = \#\text{ unknowns} - \#\text{ nonzero rows in echelon form} \end{array}}$$

This agrees with Theorem B above, which states that a linear system with fewer equations than unknowns, if consistent, has infinitely many solutions. The condition "fewer equations than unknowns" means that the number of rows in the coefficient matrix is less than the number of unknowns. Therefore, the boxed equation above implies that there must be at least one free variable. Since such a variable can, by definition, take on infinitely many values, the system will have infinitely many solutions.

Example 8: Find all solutions to the system

$$\begin{aligned} w - x + y - z &= 1 \\ 2w + x - 3y \quad &= 2 \\ 5w - 2x \quad - 3z &= 5 \end{aligned}$$

First, note that there are four unknwons, but only thre equations. Therefore, if the system is consistent, it is guaranteed to have infinitely many solutions, a condition characterized by at least one parameter in the general solution. After the corresponding augmented matrix is constructed, Gaussian elimination yields

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 2 & 1 & -3 & 0 & 2 \\ 5 & -2 & 0 & -3 & 5 \end{bmatrix} \xrightarrow[-5r_1 \text{ added to } r_3]{-2r_1 \text{ added to } r_2} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 3 & -5 & 2 & 0 \\ 0 & 3 & -5 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{-r_2 \text{ added to } r_3} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 3 & -5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The fact that only two nonzero rows remain in the echelon form of the augmented matrix means that 4 − 2 = 2 of the variables are free:

$$\begin{aligned} \#\text{ free variables} &= \#\text{ unknowns} - \#\text{ nonzero rows in echelon form} \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

Therefore, selecting y and z as the free variables, let $y = t_1$ and $z = t_2$. The second row of the reduced augmented matrix implies

$$3x - 5t_1 + 2t_2 = 0 \quad \Rightarrow \quad x = \tfrac{1}{3}(5t_1 - 2t_2)$$

and the first row then gives

$$w - \tfrac{1}{3}(5t_1 - 2t_2) + t_1 - t_2 = 1 \quad \Rightarrow \quad w = 1 + \tfrac{1}{3}(2t_1 + t_2)$$

Thus, the solutions of the system have the form

$$(w, x, y, z) = \left(1 + \tfrac{1}{3}(2t_1 + t_2),\ \tfrac{1}{3}(5t_1 - 2t_2),\ t_1,\ t_2\right)$$

where t_1 t_2 are allowed to take on any real values.