MAT1320-Linear Algebra
Lecture Notes

Determinants and Properties

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Table of contents

Determinants

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- For $n = 1$, $A = [a]_{1\times 1}$ and $\det(A) = a$.
- For $n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$ and

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Determinants

- For $n = 3$, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3\times 3}$ and

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

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$$\begin{aligned} = \quad & a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) \\ & + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \end{aligned}$$

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In order to generalize this concept for $n > 3$, we need to give definition of the minors and cofactors.

Minors and Cofactors

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Definition
Given an $n \times n$ matrix $\mathbf{A}$, the (i, j)-th minor, denoted M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by deleting the i-th row and the j-th column.

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Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

we have

$$M_{23} = det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = 8 - 14 = -6$$

We also find $A_{23} = (-1)^{2+3}(-6) = 6$

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$$\begin{aligned} det\left(\mathbf{A}\right) &= \sum_{k=1}^{n} a_{ik}A_{ik} \text{ (LE along } i\text{-th row)} \\ det\left(\mathbf{A}\right) &= \sum_{k=1}^{n} a_{kj}A_{kj} \text{ (LE along } j\text{-th column)} \end{aligned}$$

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We find the determinant of

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}.$$

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Thus,

$$det\,(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 2(4) + (1) + 3(2) = 15$$

Example (1-cont.)

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Notice that

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a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = (-1)\,(4) + 2(1) + 1(2) = 0
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\end{aligned}$$

Notice that

$$\begin{aligned}
a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} &= (-1)(4) + 2(1) + 1(2) = 0 \\
a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} &= (-2)(4) + 2(1) + 3(2) = 0
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We compute the cofactors as:

$$\begin{aligned} A_{12} &= (-1)^{1+2}\, det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\ A_{22} &= (-1)^{2+2}\, det \begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \end{aligned}$$

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Thus,

$$det\,(A) = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = 1(1) + 2(12) + 2(-5) = 15$$

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a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} &= 3(1) + 1(12) + 3(-5) = 0
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- The determinant behaves like a linear function on the rows of the matrix:

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

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- If A has a row or column that is all zeros, then $det\,(A) = 0$.
- The determinant of a triangular matrix is the product of the diagonal entries (pivots) $d_{11}, d_{22}, \ldots, d_{nn}$.

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- $det\left(A^{2}\right) = \left(det\left(A\right)\right)^{2}$ and $det\left(2A\right) = 2^{n}det\left(A\right).$ We assume A is an $n \times n$ square matrix.
- $det\left(A^{T}\right) = det\left(A\right).$

Properties of Determinants

- $det\left(AB\right) = det\left(A\right) det\left(B\right).$
- $det\left(A^{-1}\right) = \frac{1}{det(A)}$. Because $A^{-1}A = I$. Note that if A is singular then A^{-1} does not exist and $det\left(A^{-1}\right)$ is undefined.
- $det\left(A^2\right) = \left(det\left(A\right)\right)^2$ and $det\left(2A\right) = 2^n det\left(A\right)$. We assume A is an $n \times n$ square matrix.
- $det\left(A^T\right) = det\left(A\right).$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

Properties of Determinants

Example

By using properties of determinants factorize the determinant

$$\begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix}.$$

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$$= \underbrace{(x+y+z+t)}_{r} \begin{vmatrix} 1 & z & t \\ 1 & x+y & t \\ 1 & t & x+y \end{vmatrix}$$

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$$= \underbrace{(x+y+z+t)}_{r} \begin{vmatrix} 1 & z & t \\ 1 & x+y & t \\ 1 & t & x+y \end{vmatrix} = r \begin{vmatrix} 1 & z & t \\ 0 & x+y-z & 0 \\ 0 & t-z & x+y-t \end{vmatrix}$$

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$$\begin{aligned}
\Rightarrow \begin{vmatrix} x+y & z & t \\ z & x+y & t \\ z & t & x+y \end{vmatrix} &= \begin{vmatrix} x+y+z+t & z & t \\ x+y+z+t & x+y & t \\ x+y+z+t & t & x+y \end{vmatrix} \\
&= \underbrace{(x+y+z+t)}_{r} \begin{vmatrix} 1 & z & t \\ 1 & x+y & t \\ 1 & t & x+y \end{vmatrix} = r \begin{vmatrix} 1 & z & t \\ 0 & x+y-z & 0 \\ 0 & t-z & x+y-t \end{vmatrix} \\
&= r \begin{vmatrix} x+y-z & 0 \\ t-z & x+y-t \end{vmatrix} \\
&= (x+y+z+t)(x+y-z)(x+y-t).
\end{aligned}$$

Example

By using properties of determinants find the determinant of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$.

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$$\begin{vmatrix} 1 & -2 & -1 \\ 0 & 5 & 5 \\ 0 & -2 & 1 \end{vmatrix} \begin{matrix} \frac{1}{5} r_2 \to r_2 \\ = \end{matrix} 5 \begin{vmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} \begin{matrix} 2r_2 + r_3 \to r_3 \\ = \end{matrix}$$

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$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{vmatrix} \begin{array}{c} r_1 \leftrightarrow r_2 \\ -r_1 \rightarrow r_1 \\ = \end{array} \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ -2 & 2 & 3 \end{vmatrix} \begin{array}{c} -2r_1 + r_2 \rightarrow r_2 \\ 2r_1 + r_3 \rightarrow r_3 \\ = \end{array}$$

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$$\Rightarrow \quad 5 \begin{vmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} \Rightarrow |A| = 5.(1.1.3) = 15.$$

?