MAT1071 MATHEMATICS I
4. WEEK
PART 1

TRANSCENDENTAL FUNCTIONS

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Transcendental Functions

In this chapter

we investigate the calculus of important transcendental functions, including

1. the logarithmic, exponential,
2. inverse trigonometric,
3. hyperbolic functions.
4. inverse hyperbolic functions.

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Inverse Functions and Their Derivatives

One-to-One Functions

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Some functions are one-to-one on their entire natural domain. Other functions are not one-to-one on their entire domain, but by restricting the function to a smaller domain we can create a function that is one-to-one. The original and restricted functions are not the same functions, because they have different domains. However, the two functions have the same values on the smaller domain, so the original function is an extension of the restricted function from its smaller domain to the larger domain.

☆ The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, it assumes the same y-value for at least two different x-values and is therefore not one-to-one

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

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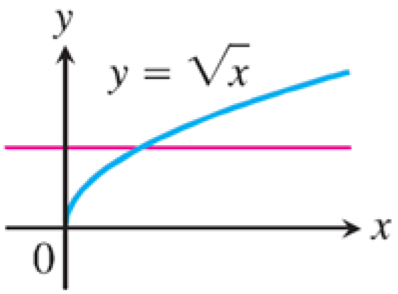
EXAMPLE

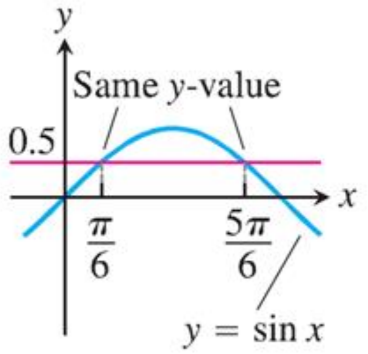
$f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.



One-to-one: Graph meets each horizontal line at most once.

EXAMPLE

$g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ giving distinct outputs for distinct inputs.



Not one-to-one: Graph meets one or more horizontal lines more than once.

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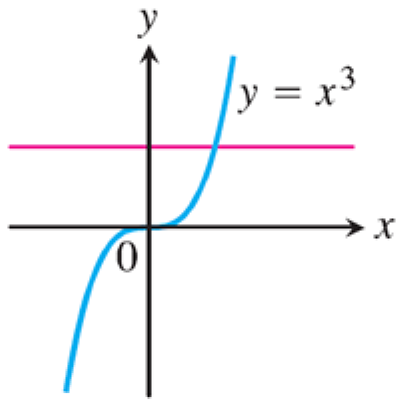
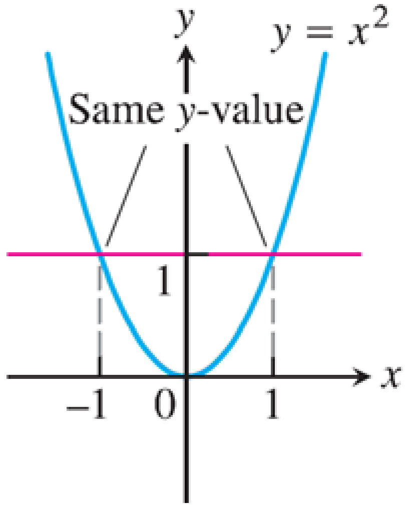
EXAMPLE



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

Inverse Functions

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

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☆ Only a one-to-one function can have an inverse.

☆ A function that is increasing on an interval so it satisfies the inequality $f(x_2) > f(x_1)$ when $x_2 > x_1$, is one-to-one and has an inverse. Decreasing functions also have an inverse.

☆ Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse.

☆ The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.

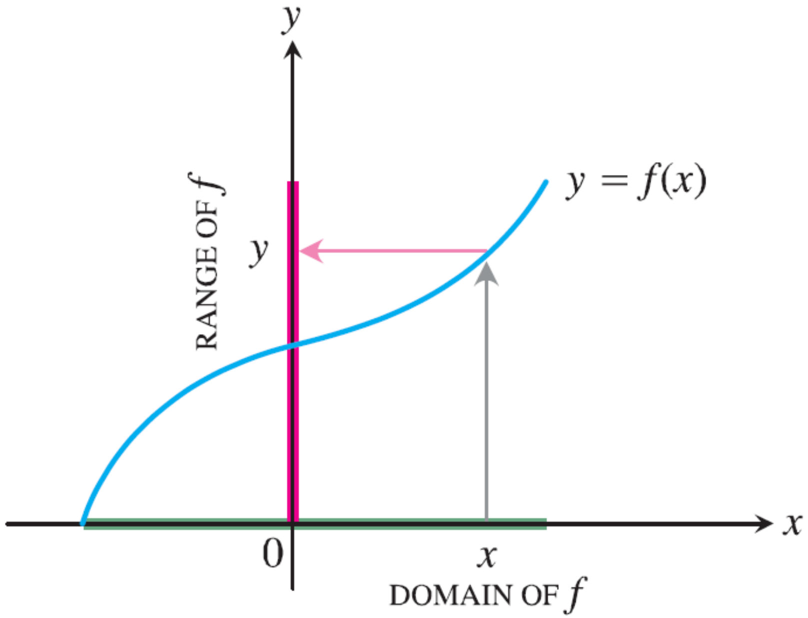
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE Suppose a one-to-one function $y = f(x)$ is given by a table of values

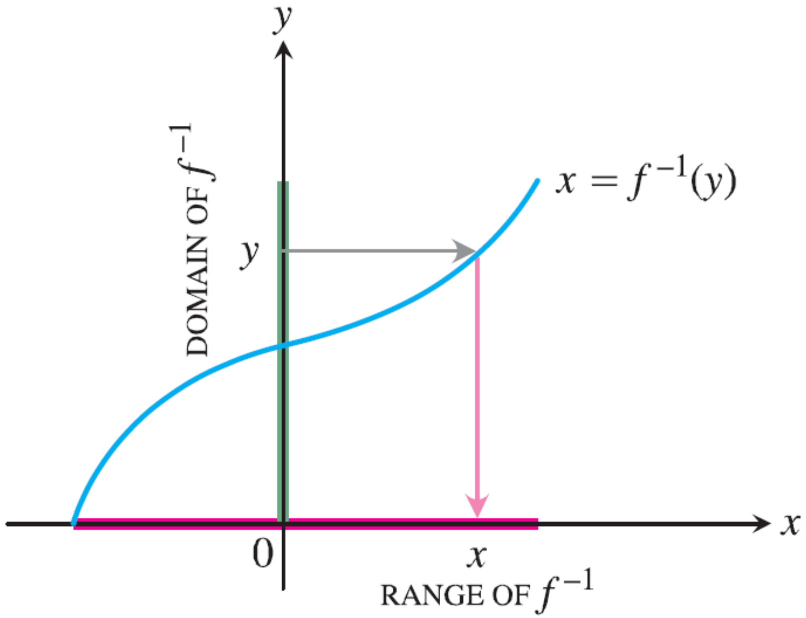
x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

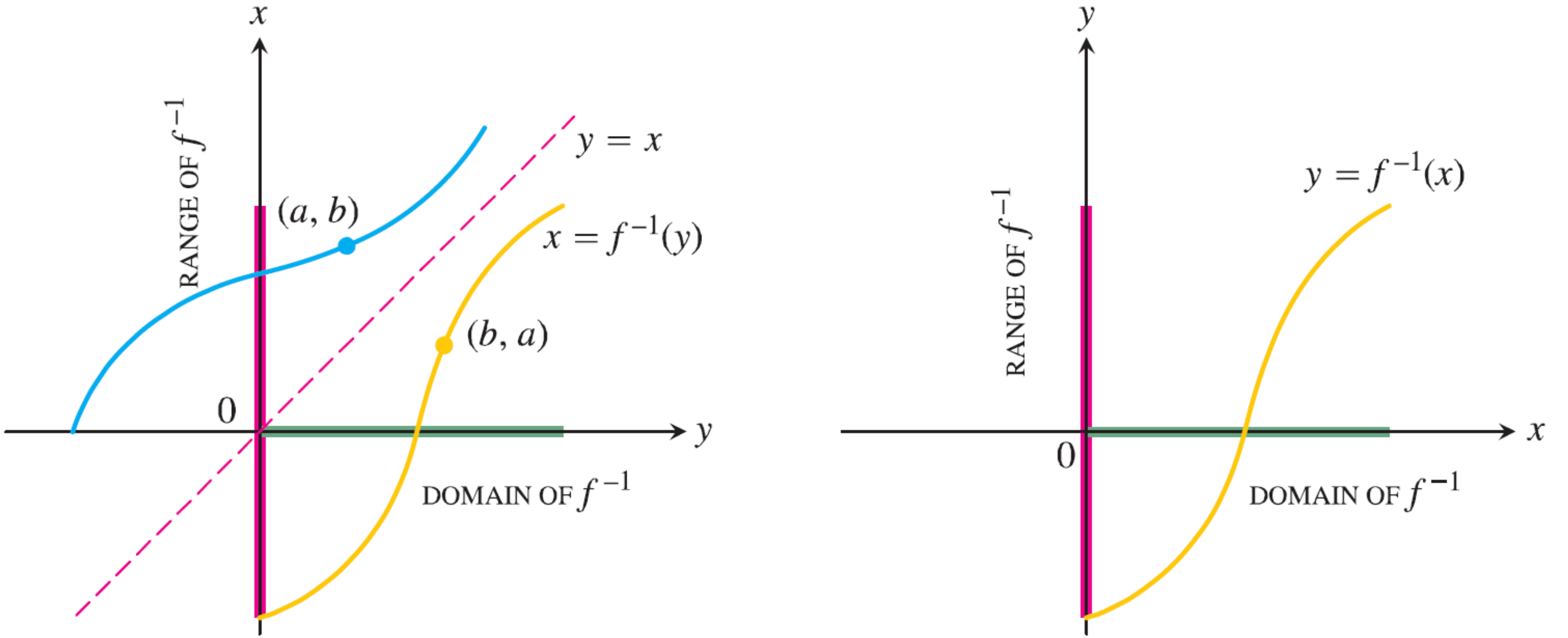


(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

☆ The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

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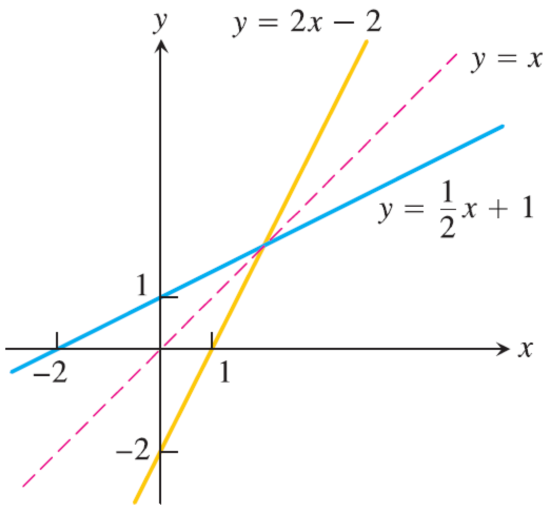

FIGURE Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$

EXAMPLE Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:*

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

2. *Interchange x and y:* $y = 2x - 2.$

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$.
To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$
$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

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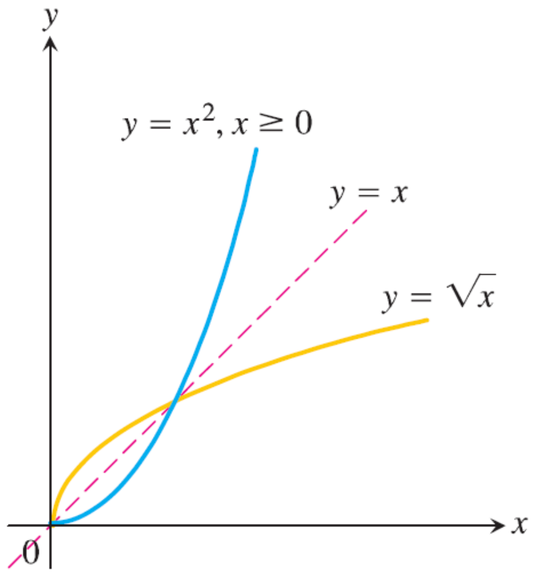

FIGURE The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another

EXAMPLE Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, *is* one-to-one and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, *is not* one-to-one and therefore has no inverse.

Derivatives of Inverses of Differentiable Functions

THEOREM —The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}}$$

If we set $b = f(a)$, then

$$\boxed{(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.}$$

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Derivatives of Inverses of Differentiable Functions

$y = f(x)$

$(a,b) = (a, f(a))$
$= (f^{-1}(b), b)$

Slope $\rightarrow f'(a)$
at
$x = a$

$y = f^{-1}(x)$

$(b,a) = (b, f^{-1}(b))$
$= (f(a), a)$

Slope $\rightarrow (f^{-1})'(b)$
$= (f^{-1})'(f(a))$

If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

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Derivatives of Inverses of Differentiable Functions

Proof: $(f \circ f^{-1})(x) = x$

$$\frac{d}{dx}\left((f \circ f^{-1})(x)\right) = 1$$

$$f'(f^{-1}(x))\,\frac{d}{dx}\left(f^{-1}(x)\right) = 1 \qquad \text{(chain rule)}$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad //$$

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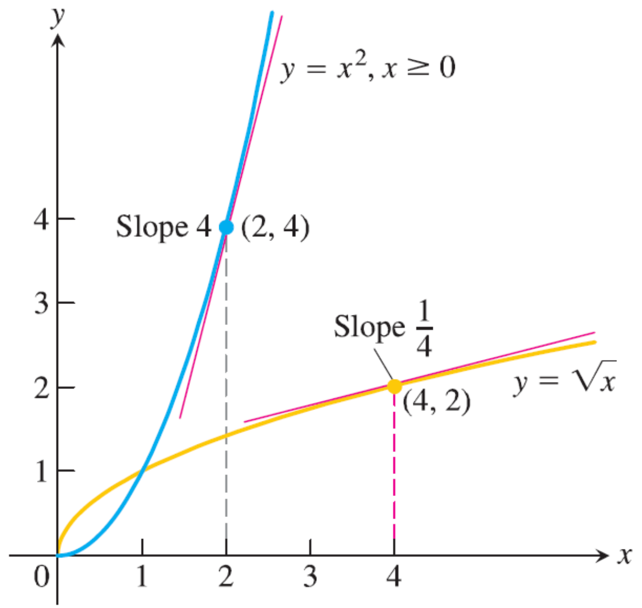

FIGURE The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4)

EXAMPLE The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Let's verify that gives the same formula for the derivative of $f^{-1}(x)$:

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{2(f^{-1}(x))} \qquad f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x) \\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

Let's examine at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b).

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x}\bigg|_{x=2} = \frac{1}{4}.$$

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EXAMPLE Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution at $x = 6$:

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

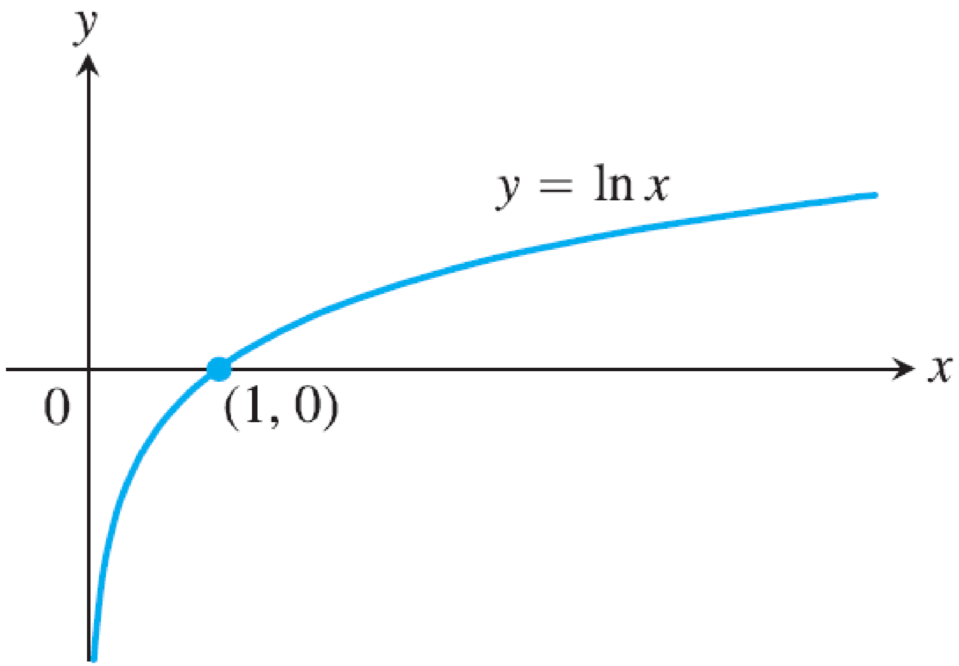
$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=2}} = \frac{1}{12}$$

1. Logarithm and Exponential Functions

Natural Logarithm

The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$



$$Domain = (0, \infty)$$

$$Range = (-\infty, \infty)$$

$$\ln(e) = 1.$$

$$e \approx 2.718281828459045$$

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General Logarithm

☆ General logarithm function

$f(x) = \log_a x \quad (\ln x = \log_e x)$

For any positive number $a \neq 1$,

$$Domain = (0, \infty)$$

$$Range = (-\infty, \infty)$$

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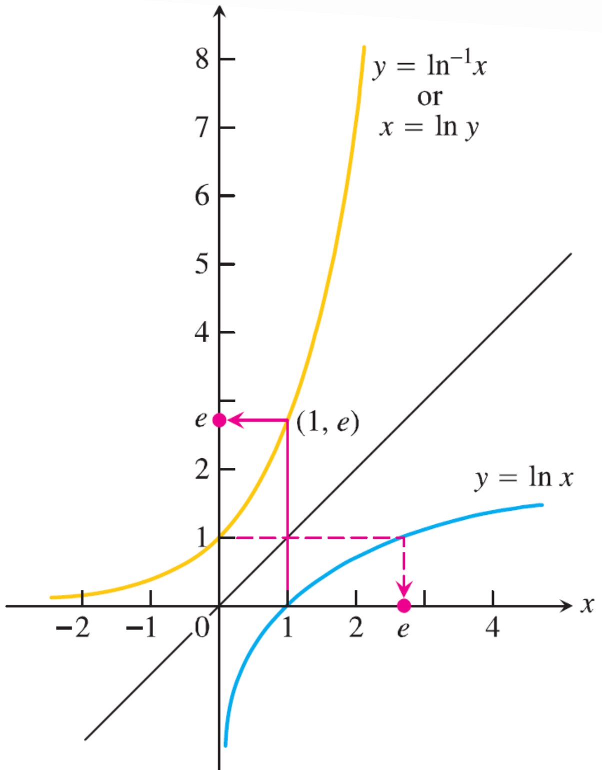
Natural Exponential

For every real number x, we define the **natural exponential function** to be $e^x = \exp x$.

$\ln^{-1} x = e^x \rightarrow e^x$ is inverse of natural logarithm



$$Domain = (-\infty, \infty)$$

$$Range = (0, \infty)$$

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General Exponential

☆ $f(x) = a^x$ $a > 0$ $a \neq 1$ → General exp. function

☆ For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

$$Domain = (-\infty, \infty)$$
$$Range = (0, \infty)$$

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☆ $\log_a x = y \Leftrightarrow a^y = x$

$\ln x = y \Leftrightarrow x = e^y$

Properties

① $\begin{cases} \log_a a = 1 \\ \ln e = 1 \end{cases}$

② $\begin{cases} \log_a xy = \log_a x + \log_a y \\ \ln xy = \ln x + \ln y \end{cases}$

③ $\begin{cases} \log_a \frac{x}{y} = \log_a x - \log_a y \\ \ln \frac{x}{y} = \ln x - \ln y \end{cases}$

④ $\begin{cases} \log_a x^r = r \log_a x \\ \ln x^r = r \ln x \end{cases}$

⑤ $\begin{cases} \log_a a^x = x \\ \ln e^x = x \end{cases}$

⑥ $\begin{cases} e^{\ln x} = x \\ a^{\log_a x} = x \end{cases}$

⑦ $\begin{cases} a^x = e^{x \ln a} \\ x^n = e^{n \ln x} \end{cases}$

⑧ $\log_a x = \dfrac{\ln x}{\ln a}$

⑨ $\begin{cases} \ln 1 = 0 \\ \log_a 1 = 0 \end{cases}$

⑩ $\begin{cases} \log_a \frac{1}{x} = -\log_a x \\ \ln \frac{1}{x} = -\ln x \end{cases}$

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EXAMPLE

(a) $\ln 4 + \ln \sin x = \ln (4 \sin x)$ — Product

(b) $\ln \dfrac{x+1}{2x-3} = \ln (x+1) - \ln (2x-3)$ — Quotient

(c) $\ln \dfrac{1}{8} = -\ln 8$ — Reciprocal

$= -\ln 2^3 = -3 \ln 2$ — Power

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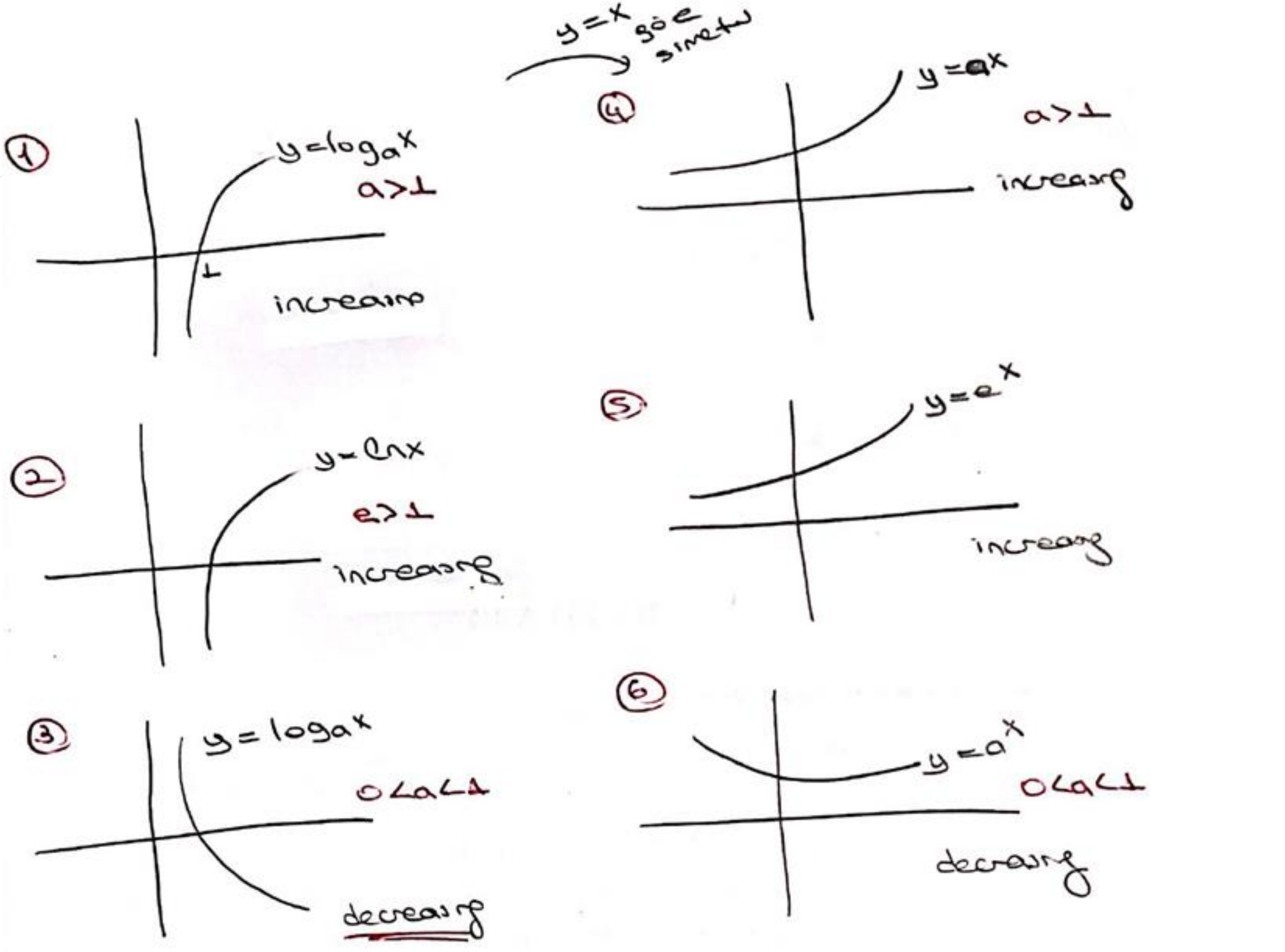
Graphs



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Limits

① $\lim_{x \to \infty} \log_a x = \infty$ $\quad a>1$

$\lim_{x \to 0^+} \log_a x = -\infty$

② $\lim_{x \to \infty} \ln x = \infty$ $\quad e>1$

$\lim_{x \to 0^+} \ln x = -\infty$

③ $\lim_{x \to \infty} \log_a x = -\infty$ $\quad 0<a<1$

$\lim_{x \to 0^+} \log_a x = \infty$

④ $\lim_{x \to \infty} a^x = \infty$ $\quad a>1$

$\lim_{x \to -\infty} a^x = 0$

⑤ $\lim_{x \to \infty} e^x = \infty$ $\quad e>1$

$\lim_{x \to -\infty} e^x = 0$

⑥ $\lim_{x \to \infty} a^x = 0$ $\quad 0<a<1$

$\lim_{x \to -\infty} a^x = \infty$

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EXAMPLE

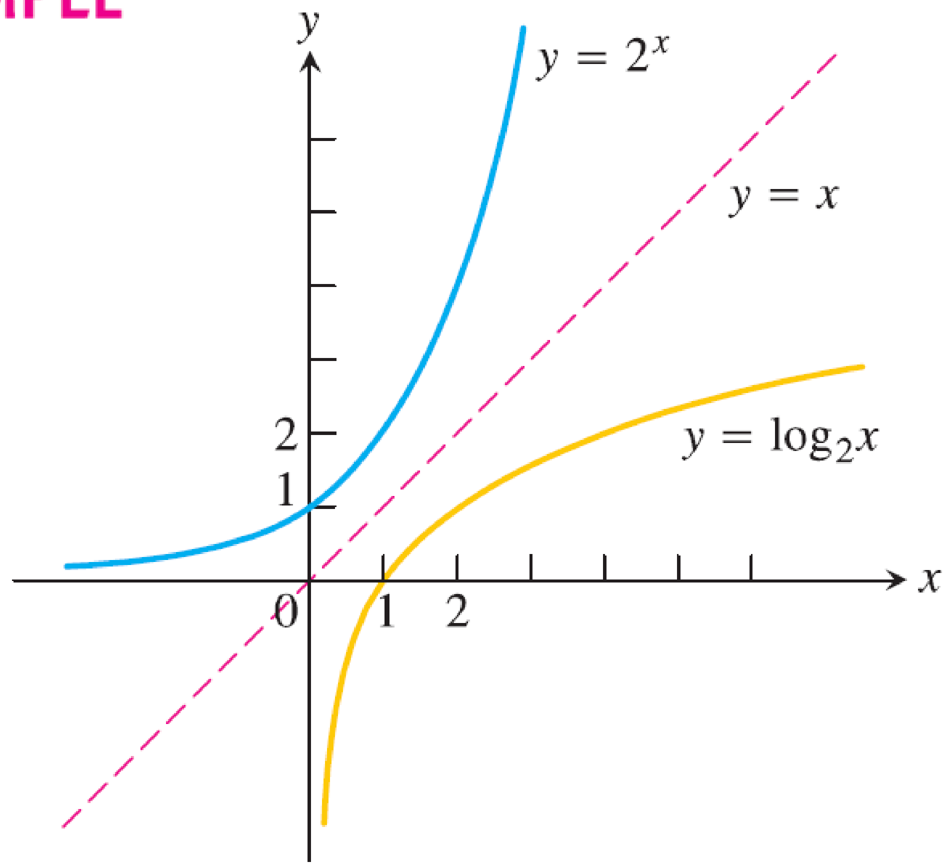

FIGURE The graph of 2^x and its inverse, $\log_2 x$.

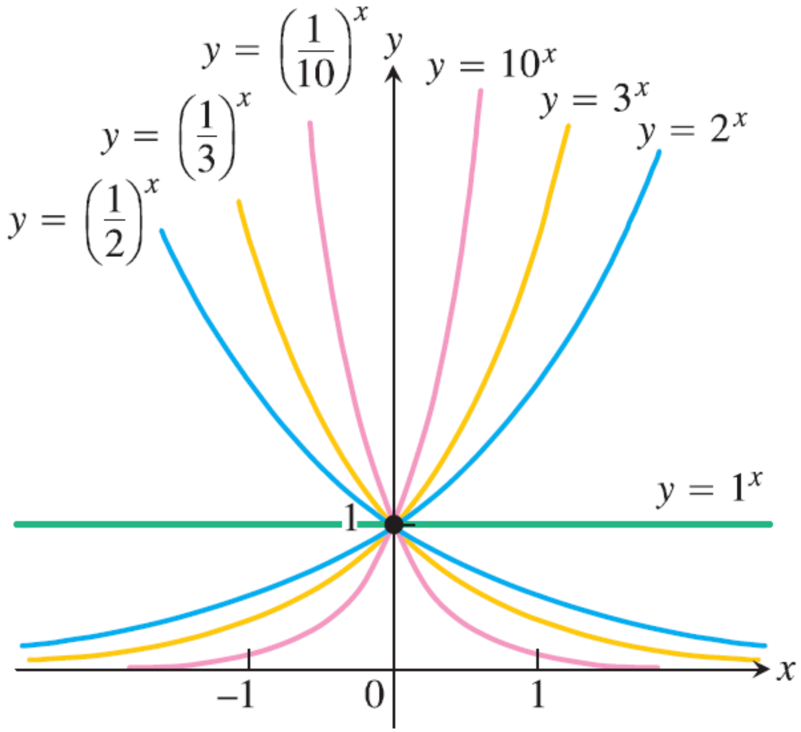

FIGURE Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

Derivatives

1. $$\left(\log_a f(x)\right)' = \frac{f'(x)}{f(x)\ln a}$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$\ln(f(x))' = \frac{f'(x)}{f(x)}$$

$$(\ln x)' = \frac{1}{x}$$

2. $$\left(a^{f(x)}\right)' = f'(x)\, a^{f(x)} \ln a$$

$$(a^x)' = a^x \ln a$$

$$\left(e^{f(x)}\right)' = f'(x)\, e^{f(x)}$$

$$(e^x)' = e^x$$

EXAMPLE

(a) $$\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$$

(b) $u = x^2 + 3$

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$

EXAMPLE

(a) $$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

EXAMPLE Solve the equation $e^{2x-6} = 4$ for x.

Solution We take the natural logarithm of both sides of the equation and use the second inverse equation:

$$\ln(e^{2x-6}) = \ln 4$$

$$2x - 6 = \ln 4 \qquad \text{Inverse relationship}$$

$$2x = 6 + \ln 4$$

$$x = 3 + \frac{1}{2}\ln 4 = 3 + \ln 4^{1/2}$$

$$x = 3 + \ln 2$$

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EXAMPLE

(a) $\dfrac{d}{dx}(5e^x) = 5\dfrac{d}{dx}e^x = 5e^x$

(b) $\dfrac{d}{dx}e^{-x} = e^{-x}\dfrac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$

(c) $\dfrac{d}{dx}e^{\sin x} = e^{\sin x}\dfrac{d}{dx}(\sin x) = e^{\sin x}\cdot\cos x$

(d) $\dfrac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}}\cdot\dfrac{d}{dx}\left(\sqrt{3x+1}\right)$

$$= e^{\sqrt{3x+1}}\cdot\frac{1}{2}(3x+1)^{-1/2}\cdot 3 = \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}$$

EXAMPLE

(a) $$\frac{d}{dx}3^x = 3^x \ln 3$$

(b) $$\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3$$

(c) $$\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$$

The Number e as a Limit

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x} = e^{a}$$

$$\lim_{x \to 0^{+}} (1 + ax)^{\frac{1}{x}} = e^{a}$$

1^{∞}

Proof: $e = \lim_{x \to 0^+} (1+x)^{\frac{1}{x}}$

$f(x) = \ln x$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(1+h) - \overset{\to 0}{\ln 1}}{h}$$

$$= \lim_{h \to 0} \ln(1+h)^{1/h} = 1$$

$$\Rightarrow 1 = \ln\left(\lim_{h \to 0} (1+h)^{1/h}\right) \Rightarrow e = \lim_{x \to 0} (1+x)^{1/x}$$

EXAMPLE $\lim_{x \to \infty} \left(\frac{x+7}{x+3}\right)^{2x+3} = ?$

Solution $\lim_{x \to \infty} \left[\underbrace{\left(1 + \frac{4}{x+3}\right)^{x+3}}_{e^4}\right]^{\underbrace{\frac{2x+3}{x+3}}_{2}} = (e^4)^2 = e^8$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**.

EXAMPLE Find dy/dx if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1} \\
&= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) \\
&= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) \\
&= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1).
\end{aligned}$$

We then take derivatives of both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y from the original equation:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

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☆ $f(x) = g(x)^{h(x)} \Rightarrow \ln f(x) = h(x) \ln g(x)$

$$\Rightarrow \frac{f'(x)}{f(x)} = h'(x) \ln g(x) + h(x) \frac{g'(x)}{g(x)}$$

$$- - - -$$

EXAMPLE Differentiate $f(x) = x^x,\ x > 0.$

Solution

☆ $x^n = e^{n \ln x}$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(e^{x \ln x}\right) \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) \\ &= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right) \\ &= x^x(\ln x + 1). \end{aligned}$$

$x > 0$

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Ex: $f(x)=\dfrac{(\ln x)^x}{x^{\ln x}} \Rightarrow f'(x)=?$

$$\ln(f(x))=\ln\frac{(\ln x)^x}{x^{\ln x}}$$

$$\ln(f(x))=\ln(\ln x)^x-\ln\left(x^{\ln x}\right)$$

$$=x\ln(\ln x)-\ln x\ln x$$

$$\frac{f'(x)}{f(x)}=\ln(\ln x)+x\cdot\frac{\frac{1}{x}}{\ln x}-2\ln x\cdot\frac{1}{x}$$

$$\Rightarrow f'(x)=\left[\frac{(\ln x)^x}{x^{\ln x}}\right]\left(\ln(\ln x)+\frac{1}{\ln x}-\frac{2\ln x}{x}\right)$$

2. Inverse Trigonometric Functions

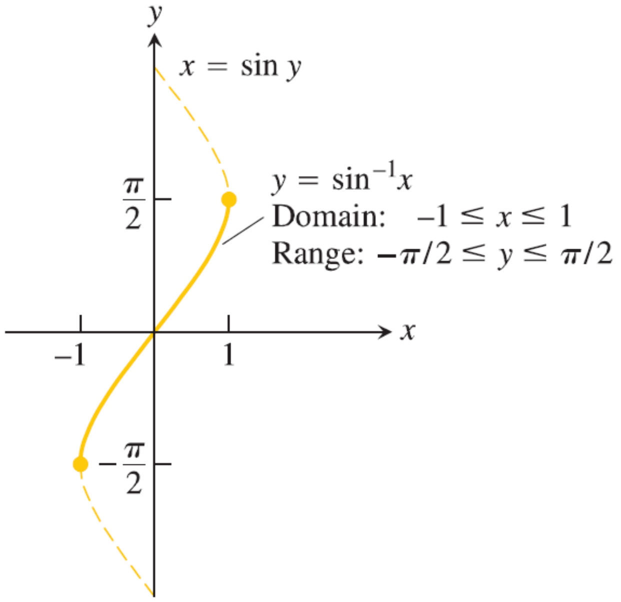

FIGURE The graph of $y = \sin^{-1} x$.

sinx, cosx, tanx, cotx, secx, cosecx — The six basic trigonometric functions are not one-to-one (their values repeat periodically. However we can restrict their domains to intervals on which they are one-to-one.

$-1 \le \sin x \le 1$

$x = \frac{\pi}{2}$ $\quad$ $x = \frac{\pi}{2}$

By restricting its domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we make it one-to-one, so that it has an inverse $\sin^{-1}x$ in $[-\frac{\pi}{2}, \pi/2]$. Similar domain restrictions can be applied to all six trigonom. functions

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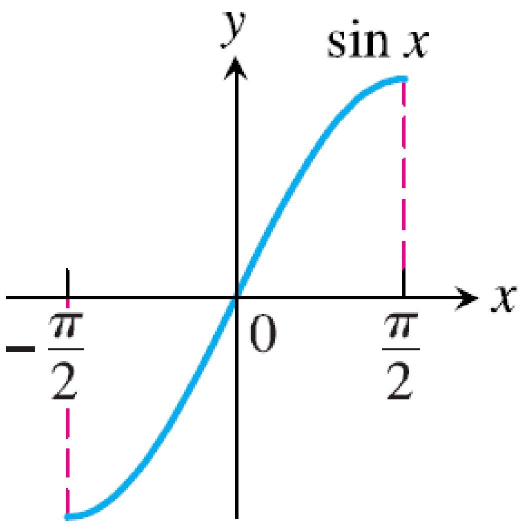
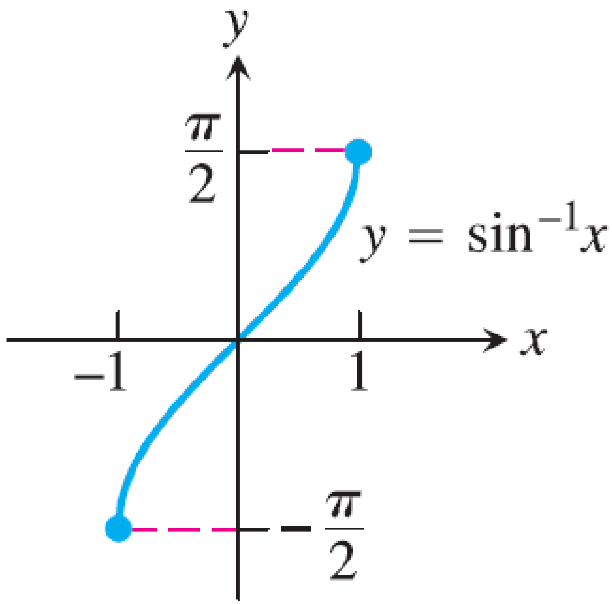
Domain restrictions that make the trigonometric functions one-to-one

$$y = \sin^{-1} x \qquad \text{or} \qquad y = \arcsin x$$

$\boldsymbol{y = \sin^{-1} x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.



$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

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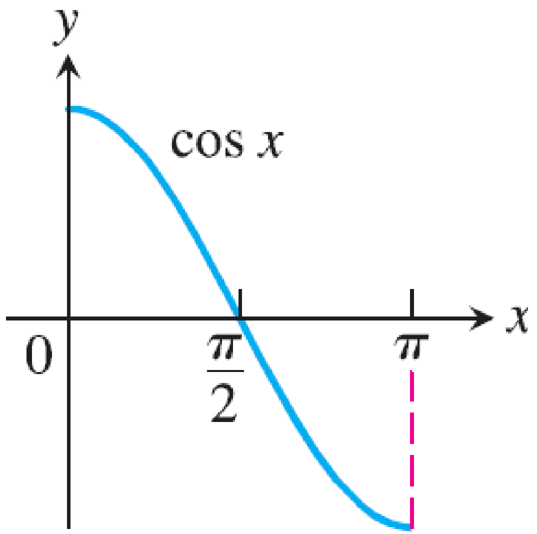
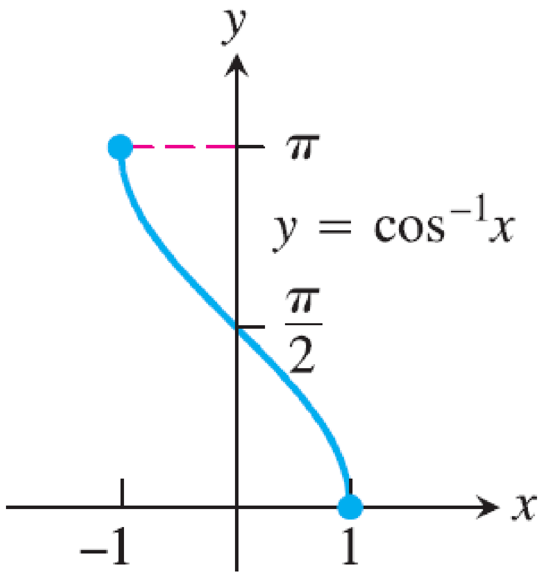
Domain restrictions that make the trigonometric functions one-to-one

$$y = \cos^{-1} x \qquad \text{or} \qquad y = \arccos x$$

$\boldsymbol{y = \cos^{-1} x}$ is the number in $[0, \pi]$ for which $\cos y = x$.



$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$

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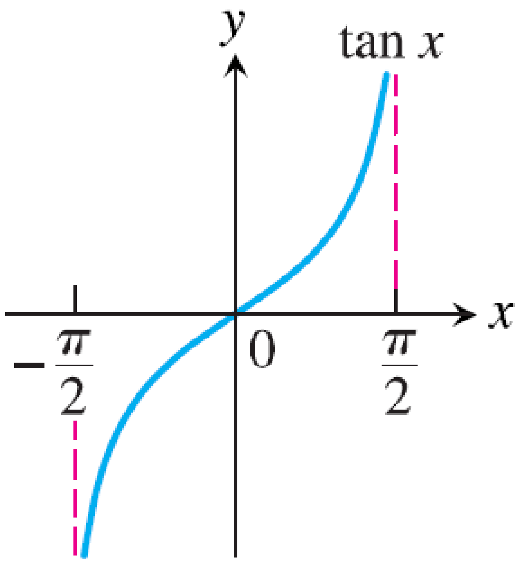
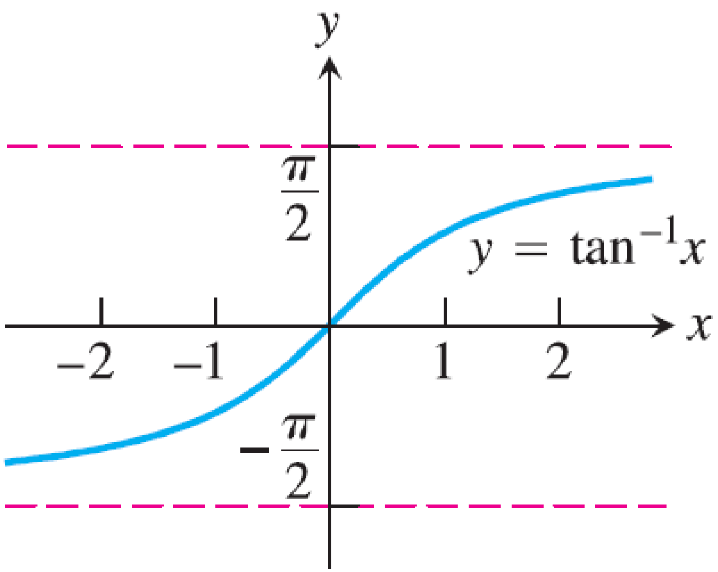
Domain restrictions that make the trigonometric functions one-to-one

$$y = \tan^{-1} x \qquad \text{or} \qquad y = \arctan x$$

$\boldsymbol{y = \tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.



$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

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x	$\arcsin x$	$\arccos x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$1/\sqrt{2}$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-1/\sqrt{2}$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/4$

x	$\arctan x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$1/\sqrt{3}$	$\pi/6$
$-1/\sqrt{3}$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Domain of $\arcsin x$ and $\arccos x$ $[-1,1]$

Range $\parallel$ $[-\frac{\pi}{2},\frac{\pi}{2}]$

Range $[0,\pi]$

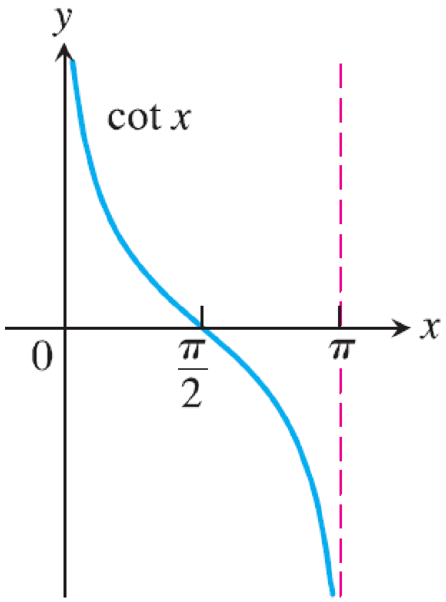
Domain of $\arctan x$ $=\mathbb{R}$

Range $(-\frac{\pi}{2},\frac{\pi}{2})$

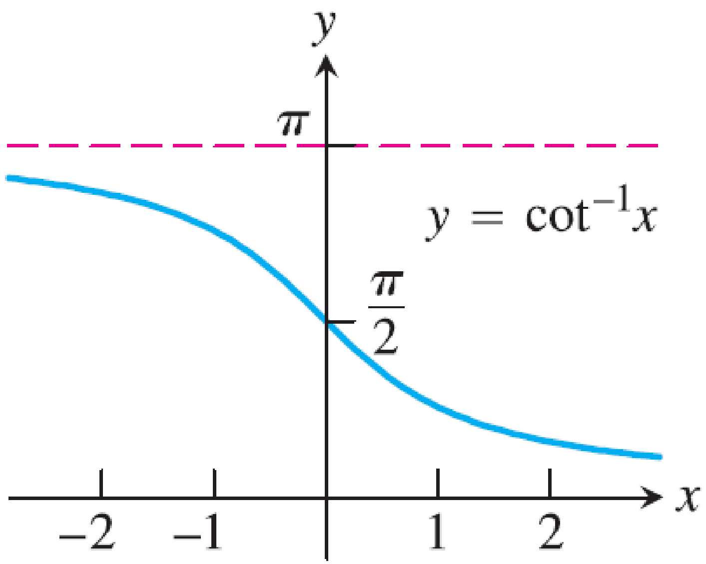
Domain restrictions that make the trigonometric functions one-to-one

$$y = \cot^{-1} x \qquad \text{or} \qquad y = \operatorname{arccot} x$$

$\boldsymbol{y = \cot^{-1} x}$ is the number in $(0, \pi)$ for which $\cot y = x$.



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$

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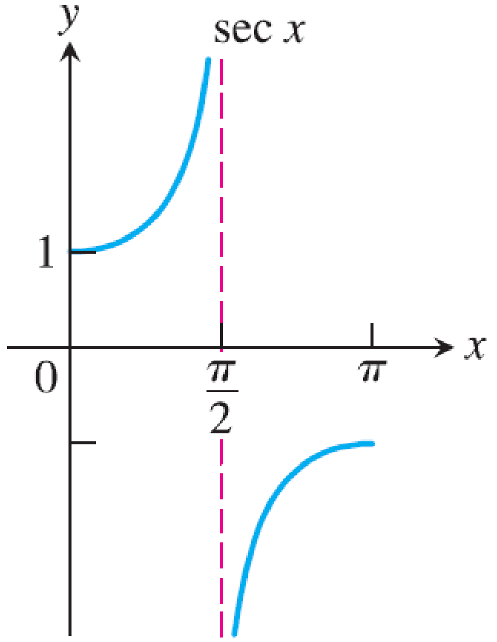
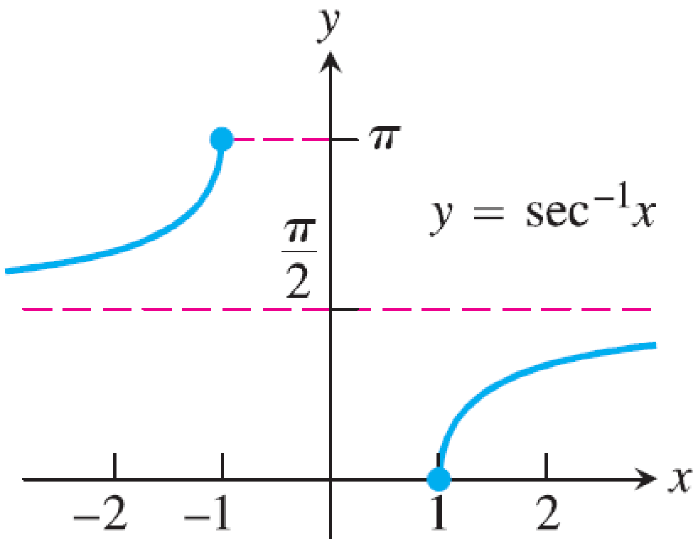
Domain restrictions that make the trigonometric functions one-to-one

$$y = \sec^{-1} x \qquad \text{or} \qquad y = \operatorname{arcsec} x$$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \leq -1$ or $x \geq 1$
Range: $0 \leq y \leq \pi,\ y \neq \dfrac{\pi}{2}$

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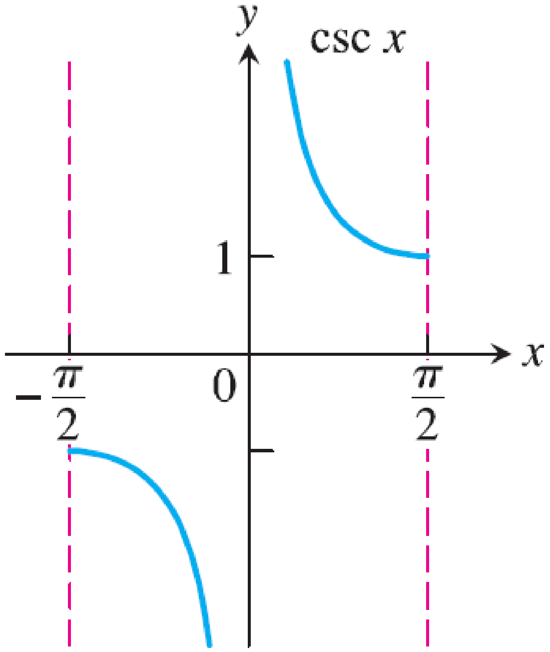
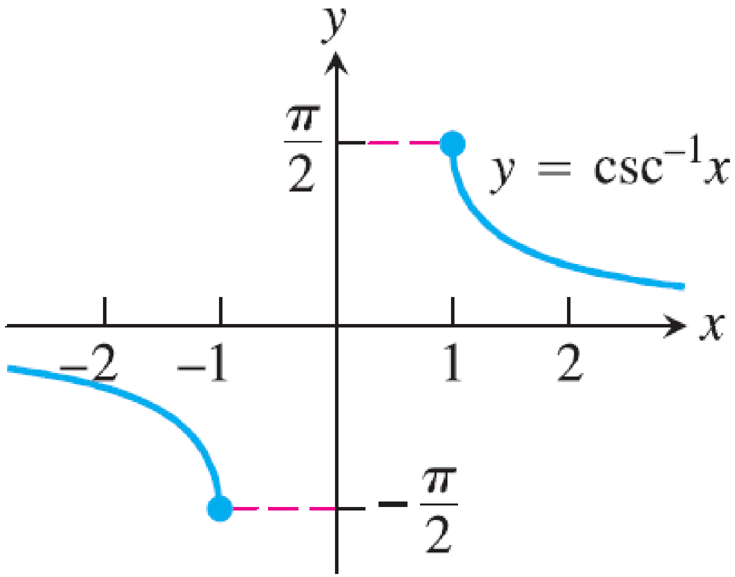
Domain restrictions that make the trigonometric functions one-to-one

$$y = \csc^{-1} x \qquad \text{or} \qquad y = \operatorname{arccsc} x$$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

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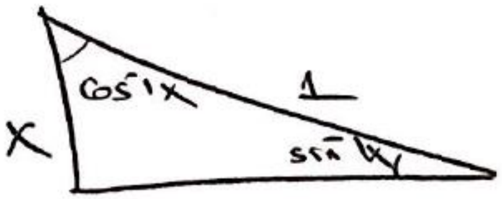
Identities Involving Inverse Trigonom. Functions

① $\arccos x + \arcsin x = \frac{\pi}{2}$

$\operatorname{arccot} x + \arctan x = \frac{\pi}{2}$

$\operatorname{arccsc} x + \operatorname{arcsec} x = \frac{\pi}{2}$



$\cos\alpha = x \Rightarrow \cos^{-1}x = \alpha$

$\sin(\frac{\pi}{2}-\alpha) = x \Rightarrow \sin^{-1}x = \frac{\pi}{2}-\alpha$

$\pi/2$

② $\arccos x + \arccos(-x) = \pi$

③ $\arcsin(-x) = -\arcsin x$ $\Rightarrow$ odd

$\arctan(-x) = -\arctan x$ sym. about orgin

④ $\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right) = \frac{\pi}{2} - \operatorname{arcsec}\left(\frac{1}{x}\right)$

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EXAMPLE Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function.

(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function.

Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

☆ $$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

$y = \sin^{-1}x \Rightarrow y' = ?$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1}x)}$$

$f(x) = \sin x$
$f'(x) = \cos x$

$$= \frac{1}{\sqrt{1-\sin^2(\sin^{-1}x)}}$$

$(\sin^2 x + \cos^2 x = 1)$

$\sqrt{1-[\sin(\sin^{-1}x)]^2}$

$$= \frac{1}{\sqrt{1-x^2}}$$

$(\sin(\sin^{-1}x) = x)$

$$\Rightarrow \boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad |x|<1}$$

veya $y = \sin^{-1}x \Rightarrow x = \sin y$

$$\frac{d}{dx}x = \frac{d}{dx}\sin y$$

$$1 = \cos y \cdot y'$$

$$y' = \frac{1}{\cos y}$$

$$y' = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}}$$

Buradan $$\boxed{\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \quad |u|<1}$$

EXAMPLE

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}.$$

EXAMPLE

$$\frac{d}{dx}\sec^{-1}(5x^4) = \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}}\frac{d}{dx}(5x^4)$$

$$= \frac{1}{5x^4\sqrt{25x^8-1}}(20x^3) \qquad 5x^4 > 1 > 0$$

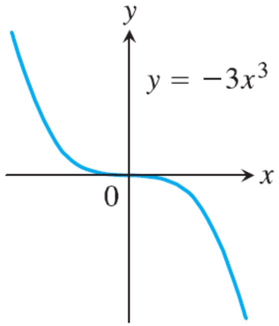
$$= \frac{4}{x\sqrt{25x^8-1}}.$$

HW:

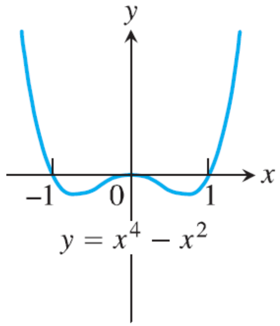
Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, an which are not?

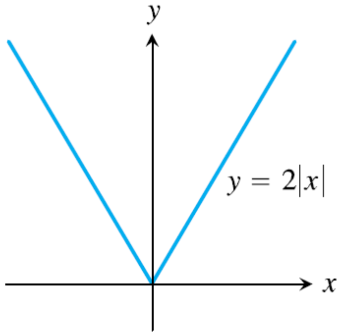
1.



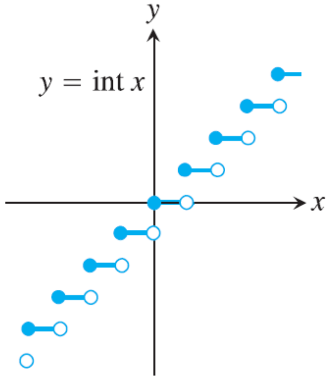
2.



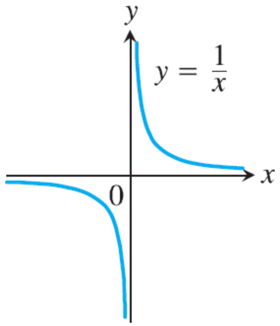
3.



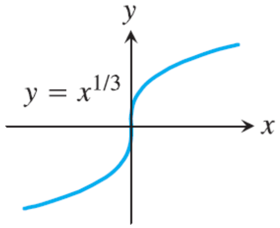
4.



5.



6.



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HW:

Derivatives of Inverse Functions

Each of Exercises 25–34 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

25. $f(x) = x^5$

26. $f(x) = x^4, \quad x \geq 0$

27. $f(x) = x^3 + 1$

28. $f(x) = (1/2)x - 7/2$

29. $f(x) = 1/x^2, \quad x > 0$

30. $f(x) = 1/x^3, \quad x \neq 0$

HW:

Finding Derivatives

In Exercises 5–36, find the derivative of y with respect to x, t, or θ, as appropriate.

5. $y = \ln 3x$

6. $y = \ln kx$, k constant

7. $y = \ln (t^2)$

8. $y = \ln (t^{3/2})$

9. $y = \ln \dfrac{3}{x}$

10. $y = \ln \dfrac{10}{x}$

11. $y = \ln (\theta + 1)$

12. $y = \ln (2\theta + 2)$

13. $y = \ln x^3$

14. $y = (\ln x)^3$

15. $y = t(\ln t)^2$

16. $y = t\sqrt{\ln t}$

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HW:

Logarithmic Differentiation

In Exercises 55–68, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

55. $y = \sqrt{x(x+1)}$

56. $y = \sqrt{(x^2+1)(x-1)^2}$

57. $y = \sqrt{\dfrac{t}{t+1}}$

58. $y = \sqrt{\dfrac{1}{t(t+1)}}$

59. $y = \sqrt{\theta + 3}\sin\theta$

60. $y = (\tan\theta)\sqrt{2\theta+1}$

61. $y = t(t+1)(t+2)$

62. $y = \dfrac{1}{t(t+1)(t+2)}$

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HW:

Finding Derivatives

In Exercises 5–24, find the derivative of y with respect to x, t, or θ, as appropriate.

5. $y = e^{-5x}$

6. $y = e^{2x/3}$

7. $y = e^{5-7x}$

8. $y = e^{(4\sqrt{x}+x^2)}$

9. $y = xe^x - e^x$

10. $y = (1 + 2x)e^{-2x}$

11. $y = (x^2 - 2x + 2)e^x$

12. $y = (9x^2 - 6x + 2)e^{3x}$

13. $y = e^{\theta}(\sin\theta + \cos\theta)$

14. $y = \ln(3\theta e^{-\theta})$

HW:

Differentiation

In Exercises 55–82, find the derivative of y with respect to the given independent variable.

55. $y = 2^x$

56. $y = 3^{-x}$

57. $y = 5^{\sqrt{s}}$

58. $y = 2^{(s^2)}$

59. $y = x^{\pi}$

60. $y = t^{1-e}$

61. $y = (\cos\theta)^{\sqrt{2}}$

62. $y = (\ln\theta)^{\pi}$

HW:

Logarithmic Differentiation

In Exercises 111–118, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

111. $y = (x + 1)^x$

112. $y = x^2 + x^{2x}$

113. $y = (\sqrt{t})^t$

114. $y = t^{\sqrt{t}}$

115. $y = (\sin x)^x$

116. $y = x^{\sin x}$

117. $y = \sin x^x$

118. $y = (\ln x)^{\ln x}$

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HW:

Finding Derivatives

In Exercises 21–42, find the derivative of y with respect to the appropriate variable.

21. $y = \cos^{-1}(x^2)$

22. $y = \cos^{-1}(1/x)$

23. $y = \sin^{-1}\sqrt{2}\,t$

24. $y = \sin^{-1}(1 - t)$

25. $y = \sec^{-1}(2s + 1)$

26. $y = \sec^{-1} 5s$

27. $y = \csc^{-1}(x^2 + 1), \quad x > 0$

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.