12

Laplace Transforms And Their Applications

12.1 INTRODUCTION

This subject was enunciated first by English Engineer Oliver Heaviside (1850–1925) from operational methods while studying some electrical engineering problem. However, Heaviside`s treatment was not very systematic and lacked rigour which later on attended to and recapitulated by Bromwich and Carson.

Laplace transform constitutes an important tool in solving linear ordinary and partial differential equations with constant coefficients under suitable initial and boundary conditions with first finding the general solution and then evaluating from it the arbitrary constants.

Laplace transforms when applied to any single or a system of linear ordinary differential equations, converts it into mere algebraic manipulations. In case of partial differential equations involving two independent variables, laplace transform is applied to one of the variables and the resulting differential equation in the second variable is then solved by the usual method of ordinary differential equations. Thereafter, inverse Laplace transform of the resulting equation gives the solution of the given p.d.e.

Another important application of Laplace Transform is in finding the solution of Mathematical Model of physical problem where in the right hand of the differential equation involves driving force which is either discontinuous or acts for short time only.

Definition: Let $f(t)$ be a function defined for all $t \geq 0$. Then the integral, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ if exists, is called **Laplace Transform** of $f(t)$. 's' is a parameter, may be *real or complex* number. Clearly $L\{f(t)\}$ being a function of s is briefly written as $\bar{f}(s)$ *i.e.* $L\{f(t)\} = \bar{f}(s)$. Here the symbol **L** which transforms $f(t)$ into $\bar{f}(s)$ is called **Laplace Transform Operator**. Then $f(t)$ is called inverse Laplace transform of $\bar{f}(s)$ or simply inverse transform of $\bar{f}(s)$ *i.e.* $L^{-1}\{\bar{f}(s)\}$.

***Note*:** *There are two types of laplace transforms. The above form of integral is known as one sided or unilateral transform. However, if the transform is defined as* $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ *, where s is complex variable, called two sided or bilateral laplace of f(t), provided the integral exists. If in the first type transform, variable s is a complex number with a result that* *laplace transform is defined over a portion of complex plane. If L{f(t)} exists for s real and then L{f(t)} exists in half of the complex plane in which Re s>a (Fig.12.1). The transform* $\overline{f}(s)$ *is an analytic function with properties:*

(*i*) $\underset{\text{Re. } s\to\infty}{Lt}\ \overline{f}(s) = 0$ viz. the necessary condition for $\overline{f}(s)$ to be a transform.

(*ii*) $\underset{s\to\infty}{Lt.}\ s\overline{f}(s) = A$, if the original function has a limit $\underset{t\to\infty}{Lt.}\ f(t) = A$

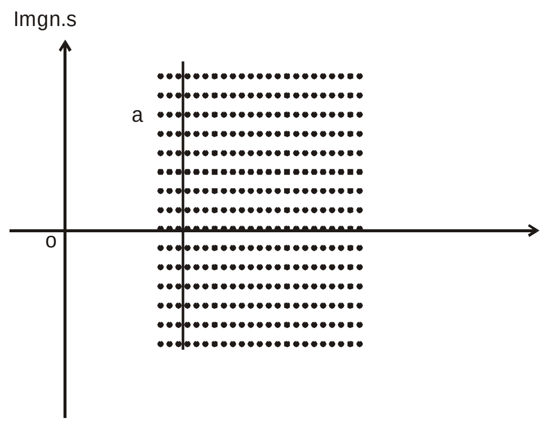

Fig. 12.1: Real axis

12.2 EXISTENCE CONDITIONS

The laplace transform does not exist for all functions. If it exists, it is uniquely determined. For existence of laplace, the given function has to be continuous on every finite interval and of exponential order i.e. if there exist positive constants M and 'a' such that $|f(t)| \le Me^{at}$ for all $t \ge 0$. The function $f(t)$ is some times termed as object function defined for all $t \ge 0$ and $\overline{f}(s)$ is termed as the resultant image function. Here the parameter should be sufficiently large to make the integral convergent. In the above discussion, the condition for existence of $\overline{f}(s)$ is sufficient but not necessary, which precisely means that if the above conditions are satisfied, the laplace transform of $f(t)$ must exist.

But if these conditions are not satisfied, the laplace transform may or may not exist. For e.g. in case of $f(t) = \frac{1}{\sqrt{t}}$, $f(t) \to \infty$ as $t \to 0$, precisely means $f(t) = \frac{1}{\sqrt{t}}$ is not piecewise continuous on every finite interval in the range $t \ge 0$. However, $f(t)$ is integrable from 0 to any positive value, say t_0. Also $|f(t)| \le Me^{at}$ for all $t > 1$ with $M = 1$ and $a = 0$. Thus, $Lf(t) = \sqrt{\frac{\pi}{s}},\ \ s > 0$ exists even if $\frac{1}{\sqrt{t}}$ is not piecewise continuous in the range $t \ge 0$.

12.3 EXISTENCE THEOREM ON LAPLACE TRANSFORM

If $f(t)$ is a function which is piecewise* continuous on every finite interval in the range $t \ge 0$ and satisfy $|f(t)| \le Me^{at}$ for all $t \ge 0$ and for some positive constants 'a' and M means, $f(t)$ is of exponential order 'a', then the laplace transform of $f(t)$ i.e. $\int_0^\infty e^{-st} f(t)\,dt$ exists.

Proof: We have $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{t_0} e^{-st} f(t)\,dt + \int_{t_0}^\infty e^{-st} f(t)\,dt$ … (1)

Here $\int_0^{t_0} e^{-st} f(t)\,dt$ exists since $f(t)$ is piecewise continuous on every finite interval $0 \geq t \geq t_0$.

Now $\left| \int_{t_0}^\infty e^{-st} f(t) dt \right| \leq \int_{t_0}^\infty \left| e^{-st} f(t) \right| dt \leq \int_{t_0}^\infty e^{-st} M e^{at} dt,$ since $|f(t)| \leq Me^{at}$

$$= \int_{t_0}^\infty e^{-(s-a)t} M\,dt = M \frac{e^{-(s-a)t_0}}{(s-a)};\ s > a \qquad \text{… (2)}$$

But $M \frac{e^{-(s-a)t_0}}{(s-a)}$ can be made as small as we please by making t_0 sufficiently large. Thus, from (1), we conclude that $L\{f(t)\}$ exists for all $s > a$.

****Note*:** *A function is said to be piecewise (sectionally) continuous on a closed interval [a, b], if this closed interval can be divided into a finite number of subintervals in each of which f(t) is continuous and has finite left hand and right hand limits.*

A function is said to be of exponential order 'a' (a > 0) as $t \to \infty$, if there exist finite positive constants t_0 and M such that $|f(t)| \leq Me^{at}$ *or* $|e^{-at} f(t) \leq M|$ *for all* $t \geq t_0$.

12.4 TRANSFORMS OF ELEMENTARY FUNCTIONS

By direct use of definition, we find the laplace transform of some of the simple functions:

1. $L(1) = \frac{1}{s}$, $(s > 0)$
2. $L(e^{at}) = \frac{1}{s-a}$, $(s > a)$
3. $L(t^n) = \frac{n\ !}{s^{n+1}}$, where $n = 0, 1, 2, 3, \ldots$ otherwise $\frac{\Gamma(n+1)}{s^{n+1}}$
4. $L(\sin at) = \frac{a}{s^2 + a^2}$, $(s > 0)$
5. $L(\cos at) = \frac{s}{s^2 + a^2}$, $(s > 0)$
6. $L(\sinh at) = \frac{a}{s^2 - a^2}$, $(s > |a|)$
7. $L(\cosh at) = \frac{s}{s^2 - a^2}$, $(s > |a|)$

Proofs:

1. $L(1) = \int_0^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$, $\quad s > 0$

2. $L\left(e^{at}\right)=\int_0^\infty e^{-st}e^{at}dt=\int_0^\infty e^{-(s-a)t}dt=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty=\frac{1}{(s-a)},\ s>a$

Remarks: Here the condition $s > a$ is necessary to ensure the convergence of the integral since otherwise divergent for $s \le a$.

3. $L\left(t^n\right)=\int_0^\infty e^{-st}t^n dt=\int_0^\infty e^{-p}\left(\frac{p}{s}\right)^n\frac{dp}{s}$ $\left(\text{on taking } st=p,\ dt=\frac{dp}{s}\right)$

$$=\frac{1}{s^{n+1}}\int_0^\infty e^{-p}p^n dp=\frac{1}{s^{n+1}}\int_0^\infty e^{-p}p^{(n+1)-1}dp=\frac{\Gamma(n+1)}{s^{n+1}},\ \text{if } n>-1 \text{ and } s>0$$

4. $L(\sin at)=\int_0^\infty e^{-st}\sin at\,dt=\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at-a\cos at)\right]_0^\infty=\frac{a}{s^2+a^2},\ s>0$

5. $L(\cos at)=\int_0^\infty e^{-st}\cos at\,dt=\left[\frac{e^{-st}}{s^2+a^2}(-s\cos at-a\sin at)\right]_0^\infty=\frac{s}{s^2+a^2},\ s>0$

6. $L(\sinh at)=\int_0^\infty e^{-st}\sinh at\ dt=\int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt=\frac{1}{2}\int_0^\infty\left[e^{-(s-a)t}-e^{-(s+a)t}\right]dt$

$$=\frac{1}{2}\left[-\frac{1}{s-a}+\frac{1}{s+a}\right]=\frac{a}{s^2-a^2}\quad s>|a|.$$

7. $L(\cosh at)=\int_0^\infty e^{-st}\cosh at\ dt=\int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt=\frac{1}{2}\int_0^\infty\left[e^{-(s-a)t}+e^{-(s+a)t}\right]dt$

$$=\frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]=\frac{s}{s^2-a^2},\quad s>|a|.$$

12.5 PROPERTIES OF LAPLACE TRANSFORMS

Sr. No.	*Property*	*f(t), function*	$\bar{f}(s)$, *laplace transform*
1.	Linearity	$af(t)+bg(t)-ch(t)$	$a\bar{f}(s)+b\bar{g}(s)-c\bar{h}(s)$
2.	Change of scale	$f(at)$	$\frac{1}{a}\bar{f}\left(\frac{s}{a}\right),\ a>0$
3.	Initial value	$\underset{t\to 0}{Lt}\ f(t)$	$\underset{s\to\infty}{Lt}\ s\bar{f}(s)$
4.	Final value	$\underset{t\to\infty}{Lt}\ f(t)$	$\underset{s\to 0}{Lt}\ s\bar{f}(s)$
5.	First shifting	$e^{at}f(t)$	$\bar{f}(s-a)$
6.	Second shifting	$f(t-a)u(t-a)$	$e^{-as}\bar{f}(s)$

7.	Derivatives	$\frac{d}{dt} f(t)$	$s\bar{f}(s) - f(0),\ \ s > 0$
		$\frac{d^2}{dt^2} f(t)$	$s^2\bar{f}(s) - s\,f(0) - f'(0),\ \ s > 0$
		$\frac{d^n}{dt^n} f(t)$	$s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - f^{n-1}(0),\ s > 0$
8.	Integral	$\int_0^t f(u)\,du$	$\frac{\bar{f}(s)}{s},\ \ s > 0$
		$\int_0^t \int_0^t f\left(du\right)^2$	$\frac{1}{s^2}\bar{f}(s)$
		$\int_0^t \int_0^t \ldots \int_0^t f\left(du\right)^n$	$\frac{1}{s^n}\bar{f}(s)$, $s > 0$ for any positive integer n.
9.	Multiplication by t	$t\,f(t)$	$-\frac{d}{ds}\bar{f}(s)$
		$t^2 f(t)$	$(-1)^2 \frac{d^2}{ds^2}\bar{f}(S)$
		$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\bar{f}(s)$
10.	Division by t	$\frac{f(t)}{t}$	$\int_s^\infty \bar{f}(s)\,ds$, provided the integral exists.
		$\frac{f(t)}{t^2}$	$\int_s^\infty \int_s^\infty \bar{f}(s)(ds)^2$, provided the entegral exists.
		$\frac{f(t)}{t^n}$	$\int_s^\infty \int_s^\infty \ldots \int_s^\infty \bar{f}(s)(ds)^n$, provided the integral exists
11.	Convolution Theorem $\left(\text{where } f(t)*g(t) = \int_0^t f(u)\,g\left(t-u\right)du,\right)$	$f(t)*g(t)$,	$\bar{f}(s)\bar{g}(s)$
12.	Periodic Function	$f(t + T) = f(t)$	$\frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$

Proofs of Some of the properties are taken in subsequent discussions.

P 1 Linearity Property:

Let $f(t)$, $g(t)$, $h(t)$ be any functions whose laplace transforms exist, then for any constants a, b, c and k, we have

(*i*) $L[af(t) + bg(t) - hf(t)] = aL[f(t)] + bL[g(t)] - cL[h(t)]$

(*ii*) $L[kf(t)] = kL[f(t)]$.

Proofs:

(*i*) $LHS = \int_0^\infty e^{-st}\left[a\,f(t) + b\,g(t) - c\,h(t)\right]dt.$

$$a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt - c\int_0^\infty e^{-st} h(t)\,dt = aL\left[f(t)\right] + bL\left[g(t)\right] - cL\left[h(t)\right]$$

This result can easily be generalized.

(*ii*) $LHS = L\left[kf(t)\right] = \int_0^\infty e^{-st} kf(t)\,dt = k\int_0^\infty e^{-st} f(t)\,dt = kL\left[f(t)\right]$

In view of above discussion, Laplace operator is a linear in nature.

P 2 First Shift Property: **[KUK, 2010]**

If $L\left[f(t)\right] = \bar{f}(s)$ then $L\left[e^{at} f(t)\right] = \bar{f}(s-a)$

$$L\left[e^{at} f(t)\right] = \int_0^\infty e^{-st}\left[e^{at} f(t)\right]dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-pt} f(t)\,dt = \bar{f}(p), \quad \text{where} \quad p = (s-a)$$

Thus, if we know the transform of $f(t)$, we can write the transform of $e^{at}f(t)$ simply replacing s by $(s-a)$ in $\bar{f}(s)$

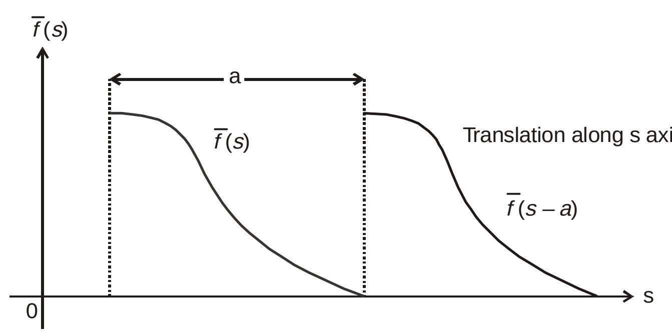

Fig. 12.2

Application of this property give arise to following simple results:

1. $L\left(e^{at}\right) = \dfrac{1}{s-a}$,	as $L(1) = \dfrac{1}{s}$

2. $L\left(e^{at}t^n\right)=\dfrac{n!}{(s-a)^{n+1}}$, as $L\left(t^n\right)=\dfrac{n!}{s^{n+1}}$

3. $L\left(e^{at}\sin bt\right)=\dfrac{b}{(s-a)^2+b^2}$, as $L\sin(bt)=\dfrac{b}{s^2+b^2}$

4. $L\left(e^{at}\cos t\right)=\dfrac{s-a}{(s-a)^2+b^2}$, as $L(\cos bt)=\dfrac{s}{s^2+b^2}$

5. $L\left(e^{at}\sinh bt\right)=\dfrac{b}{(s-a)^2-b^2}$, as $L(\sinh bt)=\dfrac{b}{s^2-b^2}$

6. $L\left(e^{at}\cosh bt\right)=\dfrac{s-a}{(s-a)^2-b^2}$, as $L(\cosh bt)=\dfrac{s}{s^2-b^2}$

where in each case $s > a$.

P 3 Change of Scale:

If $Lf(t)=\bar{f}(s)$ then $L\left[f(at)\right]=\dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right)$

$$L\left[f(at)\right]=\int_0^\infty e^{-st}f(at)\,dt=\int_0^\infty e^{-s\frac{p}{a}}f(p)\frac{dp}{a}\qquad\left(\text{on putting } at=p,\quad dt=\frac{dp}{a}\right)$$

$$=\frac{1}{a}\int_0^\infty e^{-\left(s/a\right)p}f(p)\,dp=\frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

Note: *Changing a to* $\left(\dfrac{1}{a}\right)$, *the result changes to* $L\left[f\left(\dfrac{t}{a}\right)\right]=a\bar{f}(as)$

P 4 Change of Scale with Shifting:

If $L\left[f(t)\right]=\bar{f}(s)$, then $L\left(e^{bt}f(at)\right)=\dfrac{1}{a}\bar{f}\left(\dfrac{s-b}{a}\right)$

$$L\left[e^{bt}f(at)\right]=\int_0^\infty e^{-st}e^{bt}f(at)\,dt=\int_0^\infty e^{-(s-b)t}f(at)\,dt\qquad\left(\text{on taking at}=p,\ dt=\frac{dp}{a}\right)$$

$$=\frac{1}{a}\int_0^\infty e^{-\left(\frac{s-b}{a}\right)p}f(p)\,dp=\frac{1}{a}\bar{f}\left(\frac{s-b}{a}\right)$$

Example 1: Find the laplace transform of the followings:

(*i*) $\sin 2t\cos 3t$ **(*ii*)** $\sin^3 2t$ **(*iii*)** $\sin\sqrt{t}$

(*iv*) $e^{-t}\sin^2 t$ **(*iv*)** $\cosh at\sin bt$ **(*v*)** $\sin at\sin bt$

Solution:

(i) $L(\sin 2t\cos 3t) = \frac{1}{2}L(\sin 5t - \sin t) = \frac{1}{2}L(\sin 5t) - \frac{1}{2}L\sin t$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1}\right], \quad s > 0$$

$$= \frac{2s^2-10}{(s^2+1)(s^2+25)} = \frac{2(s^2-5)}{(s^2+1)(s^2+25)}$$

(ii) $L(\sin^3 2t) = \frac{1}{4}L(3\sin 2t - \sin 6t)$, Using $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

$$= \frac{3}{4}L(\sin 2t) - \frac{1}{4}L(\sin 6t) = \frac{3}{4}\left(\frac{2}{s^2+2^2}\right) - \frac{1}{4}\left(\frac{6}{s^2+6^2}\right)$$

$$= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right] = \frac{48}{(s^2+4)(s^2+36)}, \quad s > 0$$

(iii) $L(\sin\sqrt{t}) = L\left(\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \dots\right)$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}} - \dots\dots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}} - \frac{1}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{\frac{5}{2}}} + \frac{1}{120}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{\frac{7}{2}}} \dots\dots$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \dots\right] = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\cdot e^{-\frac{1}{4s}}$$

(iv) $L(e^{-t}\sin^2 t) = L\left(e^{-t}\frac{1-\cos 2t}{2}\right)$ which comparable to $L(e^{at}f(t)) = \bar{f}(s-a)$

where $L\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t) = \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4}$

$\therefore$ $$L(e^{-t}\sin^2 t) = \frac{1}{2(s+1)} - \frac{1}{2}\frac{(s+1)}{(s+1)^2+4} = \frac{1}{2}\left[\frac{(s^2+2s+5)-(s+1)(s+1)}{(s+1)(s^2+2s+5)}\right]$$

$$= \frac{2}{(s+1)(s^2+2s+5)}$$

(v) $L(\cosh at \sin at) = L\left(\frac{e^{at}+e^{-at}}{2}\sin at\right) = \frac{1}{2}\left(L(e^{at}\sin at) + \frac{1}{2}L(e^{-at}\sin at)\right)$

But $\quad L(\sin at) = \frac{a}{s^2+a^2}$, $s > 0$; then by 1st shift property,

$$L(\cosh at \sin at) = \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{a}{2}\left[\frac{1}{(s^2+2a^2)-2as} + \frac{1}{(s^2+2a^2)+2as}\right]$$

$$= \frac{a}{2}\left[\frac{\{(s^2+2a^2)+2as\}+\{(s^2+2a^2)-2as\}}{\{(s^2+2a^2)-2as\}\{(s^2+2a^2)+2as\}}\right]$$

$$= \frac{a}{2}\frac{(2s^2+4a^2)}{\left[(s^2+2a^2)^2-4a^2s^2\right]} = \frac{a(s^2+2a^2)}{s^4+4a^4}$$

Alternately: Finding the imaginary part of $L\{e^{iat}\cosh at\}$ when $L(\cosh at) = \frac{s}{s^2-a^2}$, $|s| > |a|$.

(vi) $L(\sin at \sin bt) = L\left(\frac{e^{iat}-e^{-iat}}{2i}\sin bt\right) = \frac{1}{2i}\left[L(e^{iat}\sin bt) - L(e^{-iat}\sin bt)\right]$

But $L(\sin bt) = \frac{b}{s^2+b^2}$, $\quad s > 0$, then by 1st shift property,

$$L(\sin at \sin bt) = \frac{1}{2i}\left[\frac{b}{(s-ia)^2+b^2} - \frac{b}{(s+ia)^2+b^2}\right]$$

$$= \frac{b}{2i}\left[\frac{1}{s^2-a^2-2ias+b^2} - \frac{1}{s^2-a^2+2ias+b^2}\right]$$

$$= \frac{b}{2i}\left[\frac{1}{\{s^2+(b^2-a^2)\}-2ias} - \frac{1}{\{s^2+(b^2-a^2)\}+2ias}\right]$$

$$= \frac{b}{2i}\left[\frac{(\{s^2+(b^2-a^2)\}+2ias)-(\{s^2+(b^2-a^2)\}-2ias)}{\{s^2+(b^2-a^2)\}^2-4i^2a^2s^2}\right]$$

$$= \frac{b}{2i}\left[\frac{4ias}{\{s^4+(b^2-a^2)^2+2s^2(b^2-a^2)\}+4a^2s^2}\right]$$

$$= \left[\frac{2abs}{s^4 + \left(b^2 - a^2\right)^2 + 2s^2\left(b^2 + a^2\right)} \right]$$

$$= \frac{2abs}{\left(s^2 + (b+a)^2\right)\left(s^2 + (b-a)^2\right)}$$

Alternately: $L\left(\sin at \sin bt\right) = \frac{1}{2}\left[L\cos(a-b)t - L\cos(a+b)t\right]$, using $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

Example 2: Find the Laplace transform of

(*i*) $f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$ (*ii*) $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$ **[Madras, 2005]**

(*iii*)* $f(t) = \begin{cases} t/T, & \text{when } 0 < t < T \\ 1, & \text{when } t > T \end{cases}$ (*iv*) $f(t) = \begin{cases} \cos\left(t - 2\pi/3\right), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$

***[Kerala Tech. 2005, NIT Kurukshetra, 2010]**

(*v*) $f(t) = |t - 1| + |t + 1|, \ t \geq o$ **[NIT Kurukshetra, 2003]**

Solution:

(*i*) For function $f(t)$ defined for all $t \geq 0$, laplace transform is defined as:

$$L\left[f(t)\right] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt, \text{ Provided that this integral exists.}$$

$$= \int_0^1 e^{-st} e^t dt + \int_1^\infty e^{-st} o\,dt = \int_0^1 e^{(1-s)t} dt = \left.\frac{e^{(1-s)t}}{(1-s)}\right|_0^1 = \frac{1}{(1-s)}\left(e^{(1-s)} - 1\right)$$

s is any parameter, may have real or complex value.

(*ii*) $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

By definition, $L\left(f(t)\right) = \int_0^\infty e^{-st} f(t)\,dt = I(say)$

$$\Rightarrow \qquad I = \int_0^\pi e^{-st} \sin t\,dt + \int_\pi^\infty e^{-st} 0\,dt = \int_0^\pi e^{-st} \sin t\,dt$$

$$= \left\{ \left(e^{-st}. - \cos t\right)_0^\pi - \int_0^\pi \left(-se^{-st}\right)(-\cos t)\,dt \right\}$$

$$= \left\{ \left(e^{-s\pi} + 1\right) - s\left[\left(e^{-st} \sin t\right)\Big|_{o}^{\pi} - \int_0^{\pi} \left(e^{-st}. - s\right) \sin t \, dt \right] \right\}$$

$$I = \left\{ \left(e^{-s\pi} + 1\right) - s^2 \int_0^{\pi} e^{-st} \sin t \, dt \right\}$$

(Since value of the expression e^{-st} sin t is zero at both the limits)

$I = (e^{-s\pi} + 1) - s^2 I$ or $I(1 + s^2) = (1 + e^{-s\pi})$

$$I = \left(\frac{1 + e^{-\pi s}}{1 + s^2} \right) = L\left[f(t) \right]$$ Hence the result.

(*iii*) Here $f(t) = \begin{cases} \dfrac{t}{T}, & \text{when } 0 < t < T \\ 1, & \text{when } t > T \end{cases}$

$$Lf(t) = \int_0^{\infty} e^{-st} f(t) dt = \int_0^{T} e^{-st} f(t) dt + \int_T^{\infty} e^{-st} f(t) dt = \int_0^{T} e^{-st} \frac{t}{T} dt + \int_T^{\infty} e^{-st} 1 dt$$

$$= \frac{1}{T}\left[\left(t \frac{e^{-st}}{-s} \right)\Big|_o^T - \int_o^T 1 \frac{e^{-st}}{-s} dt \right] + \left(\frac{e^{-st}}{-s} \right)_T^{\infty} \Bigg| \quad \text{(Integration by parts)}$$

$$= \frac{1}{T}\left[\left(\frac{Te^{-sT}}{-s} - 0 \right) + \frac{1}{s}\left(\frac{e^{-st}}{-s} \right)_0^T \right] - \frac{1}{s}\left(0 - e^{-sT}\right)$$

$$= \frac{1}{T}\left[\frac{Te^{-sT}}{-s} - \frac{1}{s^2}\left(e^{-sT} - 1\right) \right] + \frac{e^{-sT}}{s} = \frac{1}{Ts^2}\left(1 - e^{-sT}\right)$$ Hence the result

(*iv*) Here $f(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right), & t > \dfrac{2\pi}{3} \\ 0, & t < \dfrac{2\pi}{3} \end{cases}$

$$\therefore \quad L\left(f(t)\right) = \int_0^{\infty} e^{-st} f(t) \, dt = \int_0^{\frac{2\pi}{3}} e^{-st} f(t) \, dt + \int_{\frac{2\pi}{3}}^{\infty} e^{-st} f(t) \, dt = \int_{\frac{2\pi}{3}}^{\infty} e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt$$

Now integrate by parts taking cos $(t - 2\pi/3)$ as 2nd function and e^{-st} as 1st function.

$$I = \left[\left(e^{-st} \sin\left(t - \frac{2\pi}{3}\right) \right)_{\frac{2\pi}{3}}^{\infty} - \int_{\frac{2\pi}{3}}^{\infty} \left(-se^{-st}\right) \sin\left(t - \frac{2\pi}{3}\right) dt \right] = s \int_{\frac{2\pi}{3}}^{\infty} e^{-st} \sin\left(t - \frac{2\pi}{3}\right) dt$$

(Since the value of the expression $e^{-st}\sin\left(t - \dfrac{2\pi}{3}\right)$ is zero at both the limits).

$$I = s\left[e^{-st} \cdot -\cos\left(t - \frac{2\pi}{3}\right)_{\frac{2\pi}{3}}^{\infty} - \int_{\frac{2\pi}{3}}^{\infty} \left(-se^{-st}\right)\left(-\cos\left(t - \frac{2\pi}{3}\right)\right) dt \right]$$

$$\Rightarrow \qquad I = s\left(0 + e^{-\frac{s2\pi}{3}}\cos 0 - s^2\int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt\right)$$

$$\Rightarrow \qquad I = se^{-\frac{2\pi s}{3}} - s^2 I \quad \text{or} \quad I(1+s^2) = se^{-\frac{2\pi s}{3}}$$

$$I = Lf(t) = \frac{se^{-\frac{2\pi s}{3}}}{\left(1+s^2\right)} \quad \text{Hence the result.}$$

(v) Here $f(t) = |t-1| + |t+1|,\ t \ge 0$

Clearly $f(t) = 2$ for $0 \le t \le 1$

$= 2t$ for $t > 1$

Say, if $t = 0$, $|t-1| + |t+1| = |0-1| + |0+1| = 2$

if $t = \frac{1}{2}, |t-1| + |t+1| = \left|\frac{1}{2}-1\right| + \left|\frac{1}{2}+1\right| = \frac{1}{2} + \frac{3}{2} = 2$

if $t = 1,\ |t-1| + |t+1| = |1-1| + |1+1| = 0 + 2 = 2$

i.e. for all value of t varying from 0 to 1, value of $|t-1| + |t+1|$ is always 2. Likewise for $t > 1$, it is 2.

$$\text{Hence,} \quad L\left[f(t)\right] = \int_0^{\infty} e^{-st} f(t)dt = \int_0^1 e^{-st} 2dt + \int_1^{\infty} e^{-st} 2t\,dt$$

$$= 2\left.\frac{e^{-st}}{-s}\right|_0^1 + 2\left\{t\left.\frac{e^{-st}}{-s}\right|_1^{\infty} - \int_1^{\infty}\frac{e^{-st}}{-s}1\cdot dt\right\}$$

$$= -\frac{2}{s}\left(e^{-s} - e^0\right) + 2\left\{-\frac{1}{s}\left(0 - 1\cdot e^{-s}\right) + \frac{1}{s}\int_1^{\infty} e^{-st}dt\right\}$$

$$= -\frac{2}{s}\left(e^{-s} - 1\right) + \frac{2}{s}e^{-s} + \frac{2}{s}\left.\frac{e^{-st}}{-s}\right|_1^{\infty}$$

$$= \frac{2}{s} - \frac{2}{s^2}\left(e^{-\infty} - e^{-s}\right) = \frac{2}{s} + \frac{2e^{-s}}{s^2} = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right) \quad \text{Hence the result.}$$

P 5. Transforms of Functions Multiplied by t^n

If $f(t)$ is a function of class A and if $L\left[f(t)\right] = \bar{f}(s)$, then

$$L\left(t\, f(t)\right) = -\frac{d}{ds}\bar{f}(s) \quad \text{and} \quad L\left(t^n f(t)\right) = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

By definition of laplace of $f(t)$, we have $\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$ … (1)

Differentiating (1) with respect to s, $\frac{d}{ds}\int_0^\infty e^{-st} f(s)dt = \frac{d}{ds}\bar{f}(s)$ … (2)

By Leibnitz rule of differentiation under integral sign, (2) reduces to

$$\int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)\,dt = \frac{d}{ds}\bar{f}(s)$$

$$\int_0^\infty \left(-te^{-st}\right) f(t)\,dt = \frac{d}{ds}\bar{f}(s) \quad \text{or} \quad \int_0^\infty e^{-st}\left(tf(t)\right)dt = -\frac{d}{ds}\bar{f}(s) \qquad \ldots(3)$$

This proves the theorem for $n = 1$ (i.e. first result)

Now assume the theorem is true for $n = m$ (say), so that (3) gives

$$\int_0^\infty e^{-st}\left(t^m f(t)\right)dt = (-1)^m \frac{d^m}{ds^m}\bar{f}(s) \qquad \ldots (4)$$

then
$$\frac{d}{ds}\int_0^\infty e^{-st}\left(t^m f(t)\right)dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s) \qquad \ldots (5)$$

Again by Leibnitz rule, (5) reduces to

$$\int_0^\infty \left(-te^{-st}\right)\left(t^m f(t)\right)dt = -\int_0^\infty e^{-st}\left(t^{m+1} f(t)\right)dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s)$$

or
$$\int_0^\infty e^{-st}\left(t^{m+1} f(t)\right)dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s) \qquad \ldots (6)$$

This clearly shows that if the theorem is true for $n = m$, it is true for $n = m + 1$ i.e. for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, so on. Hence the theorem is true for all positive integer value n.

Note: *A function which is piecewise or sectionally continuous on every finite interval in the range $t \ge 0$ is of exponential order s* $\left(\text{i.e. } \underset{t\to\infty}{Lt.}\, e^{-st} f(t) = 0\right)$ *is known as a function of class A.*

Example 3: Find Laplace of

(*i*)* $t(\sin^2 t)$ (*ii*) $t^2 \cos at$ (*iii*) ** $\sinh 3t \cos^2 t$

[* KUK, 2000; ** NIT Kurukshetra, 2005]

Solution:

(*i*) $L\left(t\sin^2 t\right) = L\left[\frac{t}{2}(1-\cos 2t)\right] = \frac{1}{2}L(t) - \frac{1}{2}L\left(t\cdot\cos 2t\right)$

As $L\left(t^n\right) = \frac{n!}{s^{n+1}}$ and $L\left(t\cdot f(t)\right) = -\frac{d}{ds}\bar{f}(s)$

$$\therefore \quad L\left(t\sin^2 t\right) = \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\left\{-\frac{d}{ds}\cdot\frac{s}{s^2+2^2}\right\} = \frac{1}{2s^2} - \frac{1}{2}\frac{s^2-4}{\left(s^2+4\right)^2} = \frac{2\left(3s^2+4\right)}{s^2\left(s^2+4\right)^2}.$$

(*ii*) $L\left(t^2\cos at\right)=(-1)^2\dfrac{d^2}{ds^2}\bar{f}(s),$ where $\bar{f}(s)=L(\cos at)=\dfrac{s}{s^2+a^2},\ s>0$

$$\therefore\ L\left(t^2\cos at\right)=\frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)=\frac{d}{ds}\left[\frac{\left(s^2+a^2\right)-s.2s}{\left(s^2+a^2\right)^2}\right]=\frac{d}{ds}\left[\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right]$$

$$=\frac{\left(s^2+a^2\right)^2(-2s)-\left(a^2-s^2\right)\cdot 2\left(s^2+a^2\right)\cdot 2s}{\left(s^2+a^2\right)^4}=\frac{2s\left(s^2-3a^2\right)}{\left(s^2+a^2\right)^3}.$$

(*iii*) $L\left(\sinh 3t\cos^2 t\right)=L\left[\dfrac{e^{3t}-e^{-3t}}{2}\cdot\left(\dfrac{1+\cos 2t}{2}\right)\right]=\dfrac{1}{4}\left[\left(e^{3t}-e^{-3t}\right)+\left(e^{3t}-e^{-3t}\right)\cos 2t\right]$

$$=\frac{1}{4}\left[L\left(e^{3t}-e^{-3t}\right)+L\left(e^{3t}\cos 2t-e^{-3t}\cos 2t\right)\right]$$

$$=\frac{1}{4}\left[\left(\frac{1}{s-3}-\frac{1}{s+3}\right)+\left(\frac{(s-3)}{(s-3)^2+2^2}-\frac{(s+3)}{(s+3)^2+2^2}\right)\right]$$

(using first shift property, replace s by $s-a$)

$$=\frac{1}{4}\left[\frac{6}{s^2-9}+\frac{(s-3)\left\{(s+3)^2+2^2\right\}-(s+3)\left\{(s-3)^2+2^2\right\}}{\left\{(s-3)^2+2^2\right\}\left\{(s+3)^2+2^2\right\}}\right]$$

$$=\frac{3}{2}\left[\frac{1}{s^2-9}+\frac{s^2-13}{s^4-10s^2+169}\right]$$

P 6 Transforms of Functions Divided By *t*

If $L[f(t)]=\bar{f}(s)$ then $L\left[\dfrac{f(t)}{t}\right]=\displaystyle\int_s^\infty \bar{f}(s)\,ds$, provided the integral exists.

Proof: We have $\bar{f}(s)=L[f(t)]=\displaystyle\int_0^\infty e^{-st}f(t)\,dt$

Integrating the above equation with respect to s between the limits s and ∞, we get

$$\int_s^\infty \bar{f}(s)\,ds=\int_s^\infty\left[\int_o^\infty e^{-st}f(t)\,dt\right]ds=\int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt$$

(On assumption that the change of order of integration exists).

$$=\int_0^\infty f(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt=\int_s^\infty e^{-st}\frac{f(t)}{t}dt\qquad\left(\text{Since }\lim_{x\to\infty}\left[\frac{e^{-st}}{t}\right]=0,\ s>o\right)$$

$$\therefore\qquad \int_s^\infty \bar{f}(s)\,ds=L\left[\frac{f(t)}{t}\right]$$

Remarks: Since $L\left(\frac{f(t)}{t}\right)$ corresponds to the integration of laplace transform of $f(t)$ with respect to s between the limits s and ∞, therefore the repeated application of the result gives $L\left[\frac{f(t)}{t^n}\right] = \underbrace{\int_s^\infty \int_s^\infty \dots \int_s^\infty}_{n\text{ times}} \bar{f}(s)(ds)^n$ provided for positive integer n, $\lim\limits_{x\to\infty}\left[\frac{f(t)}{t^n}\right]$ exists.

Example 4: Find the laplace transform of $\frac{1-\cos t}{t^2}$. [Osmania, 2003]

Solution: Here, $L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$, since $L(1) = \frac{1}{s}$, $L\cos t = \frac{s}{s^2+1}$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty = \left[\log\sqrt{\frac{s^2}{s^2+1}}\right]_s^\infty = \left[\log\sqrt{\frac{1}{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \left[\log 1 - \log\sqrt{\frac{s^2}{s^2+1}}\right], \quad \left(As\ \frac{1}{1+\frac{1}{s^2}} \to \frac{1}{1+\frac{1}{\infty}} = 1, \quad \text{when} \quad s\to\infty\right)$$

$$= \log\left(\frac{\sqrt{s^2+1}}{s}\right)$$

Again $L\left(\frac{1-\cos t}{t^2}\right) = L\left[\frac{f(t)}{t}\right] = \int_s^\infty \log\frac{\sqrt{s^2+1}}{s}\,ds = \int_s^\infty \log\left(\frac{\sqrt{s^2+1}}{s}\right)\cdot 1\,ds$

$$= \left[\left\{\left(\log\frac{\sqrt{s^2+1}}{s}\right)s\right\}_s^\infty - \int_s^\infty \frac{s}{\sqrt{s^2+1}}\frac{d}{ds}\left(\sqrt{\frac{s^2+1}{s^2}}\right)s\,ds\right], \text{ integration by parts}$$

$$= -s\log\sqrt{\frac{s^2+1}{s^2}}\Bigg|_s^\infty - \int_s^\infty \frac{s}{\sqrt{s^2+1}} \times \frac{s\cdot\frac{1}{2}\frac{2s}{\sqrt{s^2+1}} - 1\cdot\sqrt{s^2+1}}{s^2}\, s\cdot ds$$

$$= -s\log\sqrt{1+s^{-2}}\Big|_s^\infty + \int_s^\infty \frac{1}{s^2+1}ds = -s\log\sqrt{1+s^{-2}}\Big|_s^\infty + \left(\tan^{-1}s\right)_s^\infty$$

$$= -\frac{1}{2}s\log\left(1+s^{-2}\right) + \left(\frac{\pi}{2} - \tan^{-1}s\right) = \cot^{-1}s - \frac{1}{2}s\log\left(1+s^{-2}\right)$$

Example 5: Prove that $L\left(\frac{\sin t}{t}\right) = \tan^{-1}\frac{1}{s}$ and hence find $L\left(\frac{\sin at}{t}\right)$. Does the laplace transform of $\frac{\cos at}{t}$ exist?

Solution: For $f(t) = \sin t$, $\underset{t \to o}{lt} \dfrac{f(t)}{t} = \underset{t \to o}{lt} \dfrac{\sin t}{t} = 1$ and $L(\sin t) = \dfrac{1}{s^2+1} = \bar{f}(s)(say)$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds = \left(\tan^{-1} s\right)_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = \tan^{-1}\frac{1}{s}$$

Now $L\left(\dfrac{\sin at}{t}\right) = aL\left(\dfrac{\sin at}{at}\right) = a\left(\dfrac{1}{a}\tan^{-1}\dfrac{1}{\left(\frac{s}{a}\right)}\right) = \tan^{-1}\left(\dfrac{a}{s}\right)$

(Using change of scale property, $L\big(f(at)\big) = \dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right)$, $a > 0$)

Again, $L(\cos at) = \dfrac{s}{s^2+a^2} = \bar{f}(s),(say)$, then

$$L\left(\frac{\cos at}{t}\right) = \int_s^\infty \frac{s}{s^2+a^2}\,ds = \frac{1}{2}\left(\log\left(s^2+a^2\right)\right)_s^\infty = \frac{1}{2}\left(\underset{s\to\infty}{lt.} \log\left(s^2+a^2\right) - \log\left(s^2+a^2\right)\right)$$

Which does not exist as $\underset{s\to\infty}{lt}$ log (s^2 + a^2) is infinite. Hence $L\left(\dfrac{\cos at}{t}\right)$ does not exist.

Example 6: Using Laplace Transform, show that $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Solution: We know that $L[\sin(at)] = \dfrac{a}{s^2+a^2}$, $s > 0$

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right) \qquad \dots (1)$$

However by definition, $L\left(\dfrac{\sin at}{t}\right) = \displaystyle\int_0^\infty e^{-st}\left(\frac{\sin at}{t}\right)dt$ ... (2)

From (1) and (2), $\displaystyle\int_0^\infty e^{-st}\left(\frac{\sin at}{t}\right)dt = \cot^{-1}\left(\frac{s}{a}\right)$

On taking a = 1 and s = 0 in the above relation, we get $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1} 0 = \frac{\pi}{2}$

P 7 Laplace Transform of Integral of *f*(*t*)

$$L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\bar{f}(s) \quad \text{where} \quad Lf(t) = \bar{f}(s)$$

Let $\phi(t) = \int_0^t f(u)\,du$ and $\phi(0) = 0$ then $\phi'(t) = f(t)$

We know that, $L[\phi'(t)] = sL(\phi(t)) - \phi(0)$

i.e. $L[\phi'(t)] = sL(\phi(t)),\quad \text{since}\quad \phi(0) = 0$

i.e. $L[\phi(t)] = \frac{1}{s}L(\phi'(t))$

On putting the values of $\phi(t)$ and $\phi'(t)$, we get

$$L\left[\int_o^t f(u)\,du\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}\bar{f}(s)$$

Remarks: If the above preposition holds, then the inverse transforms

i.e. $L^{-1}\left[L\int_0^t f(u)\,du\right] = L^{-1}\left[\frac{1}{s}\bar{f}(s)\right]$ or $\int_0^t f(u)\,du = L^{-1}\left[\frac{1}{s}\bar{f}(s)\right]$ also holds.

P 8 Laplace Transforms of Derivatives

Let $f(t)$ be a continuous function for all $t \geq 0$ and be of exponential order as $t \to \infty$, and $f'(t)$ if also be piece wise continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$, then $L[(f'(t)] = sL(f(t)) - f(0)$.

$$L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\Big|_o^\infty - \int_0^\infty (-s)e^{-st}.f(t)\,dt\right] = \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\bar{f}(s)$$

Now, assuming $f(t)$ to be such that $\underset{t\to\infty}{Lt}\, e^{-st} f(t) = 0$ (When this condition is satisfied, $f(t)$ is said to be of exponential order s).

Therefore, we conclude that $L(f'(t))$ exists and $L(f'(t)) = sL(f(t)) - f(0)$

Remarks: By successive application of the above result, we obtain the laplace transform of $f^n(t)$ as follows: $L\,f^n(t) = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots\ldots - f^{n-1}(0)$, if $f(t), f'(t), \ldots\ldots, f^{n-1}(t)$ are continuous, $f^n(t)$ is piecewise continuous for all $t \geq 0$ and if $f(t), f'(t), \ldots, f^n(t)$ all are of exponential order, when $t \to \infty$.

Example 7: Find $L(\cos at)$ and deduce from it $L(\sin at)$ **[KUK, 2002]**

Solution: By def^n., $L(\cos at) = \int_0^\infty e^{-st}\left(\frac{e^{iat} + e^{-iat}}{2}\right)dt$, as $\cos at = \frac{e^{iat} + e^{-iat}}{2}$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-ia)t}dt + \int_0^\infty e^{-(s+ia)t}dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\Bigg|_o^\infty + \frac{e^{-(s+ia)t}}{-(s+ia)}\Bigg|_o^\infty\right]$$

$$= \frac{1}{2}\left[\left(0 - \frac{1}{-(s-ia)}\right) + \left(0 - \frac{1}{-(s+ia)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2 - i^2a^2}\right] = \frac{s}{s^2+a^2}, \quad i^2 = -1 \qquad \ldots (1)$$

Now to deduce L (sin at) using L (cos at), write, $\sin at = a\int_0^t \cos at\,dt$ … (2)

Take laplace on both sides, $L(\sin at) = aL\left(\int_0^t \cos at\,dt\right)$ … (3)

By formula, $L\left[\int_0^t f(t)\,dt\right] = \frac{\bar{f}(s)}{s}$, where $\bar{f}(s) = L\,f(t) = \left(\frac{s}{s^2+a^2}\right)$

$$L(\sin at) = a\frac{\left(\frac{s}{s^2+a^2}\right)}{s} = \left(\frac{a}{s^2+a^2}\right) \qquad \ldots (4)$$

Alternately: Write, $\sin at = \frac{-1}{a}\frac{d}{dt}(\cos at) = -\frac{1}{a}f'(t)$

Take laplace on both sides, $L(\sin at) = -\frac{1}{a}Lf'(t)$ … (5)

Now by the formula,

$$L[f'(t)] = [s\,\bar{f}(s) - f(0)], \text{ where } \bar{f}(s) = \frac{s}{s^2+a^2} \text{ and } f(0) = (\cos at)_{t=0} = 1$$

Using above, (5) becomes

$$\therefore \quad L(\sin at) = -\frac{1}{a}\left[s.\frac{s}{s^2+a^2} - 1\right] = -\frac{1}{a}\left[\frac{s^2 - (s^2+a^2)}{s^2+a^2}\right] = \frac{a}{(s^2+a^2)} \qquad \ldots (6)$$

Hence the result.

Example 8: Find $\sin\sqrt{t}$ and hence deduce $\frac{\cos\sqrt{t}}{\sqrt{t}}$. [KUK, 2010]

Solution: Let $\sin\sqrt{t} = f(t)$, then $f'(t) = \frac{d}{dt}(\sin\sqrt{t}) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$

But $Lf'(t) = s\,\bar{f}(s) - f(0)$, where $\bar{f}(s) = L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2\ s^{\frac{3}{2}}}e^{-\frac{1}{4s}}$

$$\therefore \quad L\left(\frac{\cos\sqrt{t}}{2\sqrt{t}}\right) = s.\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}.e^{-\frac{1}{4s}} - 0 \quad \text{as} \quad f(0) = \sin 0 = 0$$

or $$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}\,e^{-\frac{1}{4s}}$$

Example 9: Evaluate $L\left(\int_0^t e^t \frac{\sin t}{t}\, dt\right)$

Solution: $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \bar{f}(s)\, ds,$ where $\bar{f}(s) = L(\sin t) = \frac{1}{s^2+1}$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds = \tan^{-1} s\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$\therefore$ $L\left(e^t \frac{\sin t}{t}\right) = \cot^{-1}(s-1),$ by first shift property

Now $L\left[\int_0^t e^t \frac{\sin t}{t}\, dt\right] = \frac{\bar{f}(s)}{s} = \frac{\cot^{-1}(s-1)}{s}$

P 9 Evaluation of Integrals by Laplace Transforms

Example 10: Evaluate (*i*) $\int_0^\infty t e^{-3t} \sin t\, dt$ (*ii*) $\int_0^\infty t^3 e^{-t} \sin t\, dt$

OR

Find laplace of (*i*) $t \sin t$ at $s = 3$ (*ii*) $t^3 e^{-t} \sin t$ at $s = 1$

[MDU, 2000; KUK, 2002, 2004, 2005, 2006; VTU, 2007]

Solution:

(*i*) The Integral $\int_0^\infty t e^{-3t} \sin t\, dt$ is comparable to $\int_0^\infty e^{-st} f(t)\, dt$ with $s = 3$ and $f(t) = t \sin t$.

Now $L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s), \quad s > 0$

$$\therefore \quad L(t \sin t) = -\frac{d}{ds}\bar{f}(s) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

Hence the integral

$$\int_0^\infty e^{-3t} t \sin t\, dt = L(t \sin t) = \frac{2s}{(s^2+1)^2} = \frac{2.3}{(3^2+1)^2} = \frac{3}{50}, \quad \text{when} \quad s = 3.$$

(*ii*) The integral $\int_0^\infty t^3 e^{-t} \sin t\, dt$ is comparable to $\int_0^\infty e^{-st} f(t)\, dt$

Hence, $f(t) = t^3 \sin t$

Now $L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s), \quad (s > 0)$

$$\therefore \qquad L(t^3 \sin t) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right) = \frac{d^2}{ds^2}\left(\frac{2s}{\left(s^2+1\right)^2}\right) = 2\frac{d}{ds}\left(\frac{1-3s^2}{(s^2+1)^3}\right) = \frac{24s(s^2-1)}{(s^2+1)^4}$$

Hence the integral $\int t^3 e^{-t} \sin t\, dt = L\left(t^3 \sin t\right) = \dfrac{24s(s^2-1)}{(s^2+1)^4} = 0$ when $s = 1$

Example 11: Evaluate $\displaystyle\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt$ **[KUK, 2005]**

Solution: Write $\displaystyle\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = \int_0^\infty e^{-t}\left(\frac{1-e^{-2t}}{t}\right)dt$ comparable to $\displaystyle\int_0^\infty e^{-st} f(t)\,dt$, when $s = 1$

$$= \int_0^\infty e^{-st}\left(\frac{1}{t}\right)dt - \int_0^\infty e^{-st}\left(\frac{e^{-2t}}{t}\right)dt$$

$$= L\left(\frac{f(t)}{t}\right) - L\left(\frac{g(t)}{t}\right), \text{ where } f(t) = 1,\ g(t) = e^{-2t}$$

$$= \int_s^\infty \frac{1}{s}ds - \int_s^\infty \frac{1}{s+2}ds = \log s\Big|_s^\infty - \log(s+2)\Big|_s^\infty \quad \left(\text{using } L\left(\frac{f(t)}{t}\right) = \int_s^\infty \bar{f}(s)\,ds\right)$$

$$= \log\left(\frac{s}{s+2}\right)\Bigg|_s^\infty = 0 - \log\left(\frac{1}{1+\frac{2}{s}}\right) = -\log 1 + \log 3 = (\log 3), \text{ when } s = 1$$

Alternately:

$$\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = \int_0^\infty \frac{e^{-t}}{t}dt - \int_0^\infty \frac{e^{-3t}}{t}dt$$

$$= \underbrace{\int_0^\infty e^{-st}\frac{f(t)}{t}dt}_{for\ s=1} - \underbrace{\int_0^\infty e^{-st}\left(\frac{f(t)}{t}\right)dt}_{for\ s=3} \quad (\text{def}^n\text{. By } de^{fn}\text{. when } f(t) = 1)$$

$$= \underbrace{\int_s^\infty \frac{1}{s}ds}_{for\ s=1} - \underbrace{\int_s^\infty \frac{1}{s}ds}_{for\ s=3} = \underbrace{\log s\Big|_s^\infty}_{for\ s=1} - \underbrace{\log s\Big|_s^\infty}_{for\ s=3} = \underbrace{-\log\frac{1}{s}\Bigg|_s^\infty}_{for\ s=1} + \underbrace{\log\frac{1}{s}\Bigg|_s^\infty}_{for\ s=3} \quad \text{(On reformatting)}$$

$$= \left(-0 + \log 1\right) + \left(0 - \log\frac{1}{3}\right) = \log 3 \quad \left(as \quad s \to \infty, \quad \frac{1}{s} \to 0\right)$$

ASSIGNMENT 1

Find Laplace Transforms of followings:

1\.
(*i*) $\cosh at \cos at$ (*ii*) $^*\cosh at \sin at$

(*iii*) $\sin h\, at \cos at$ (*iv*) $\sinh at \sin at$

(*v*) $\sin at \cosh at + \cos at \sinh at$ (*vi*) $\sin at \cosh at - \cos at \sinh at$

*[NIT Kurukshetra; Delhi 2002, 2003]

2.

(*i*) $\sin at \cos at$ (*ii*) $\dfrac{at - \sin at}{a^3}$

3.

(*i*) $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t^2 - 1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$ (*ii*) $f(t) = \begin{cases} \sin\left(t - \dfrac{2\pi}{3}\right), & t > \dfrac{2\pi}{3} \\ 0, & t < \dfrac{2\pi}{3} \end{cases}$

4.

(*i*) $t \sin at$ [Raipur 2005] (*ii*) $t \sinh at$

(*iii*) $t \cos at$ (*iv*) $t \cosh at$

5.

(*i*) $\sin at + at \cos at$ (*ii*) $\sinh at + at \cosh at$

(*iii*) $\sin at - at \cos at$ (*iv*) $\sinh at - at \cosh at$

6.

(*i*) $\cos at - \dfrac{1}{2} at \sin at$ (*ii*) $\cosh at + \dfrac{1}{2} at \sinh at$

7.

*(*i*) $\dfrac{\cos at - \cos bt}{t}$ (*ii*) $\dfrac{1 - \cos 2t}{t}$

*[VTU 2006; INTU 2005; UPTU 2005]

(*iii*) $\dfrac{\sin at}{t}$ [WBTU, 2005] **(*iv*) $\dfrac{e^{-t} \sin t}{t}$ [KUK,*2003-04, **2006]

8. Find the laplace transform $\sin at$ and hence deduce $L(\cos at)$. using transforms of differentiation.
9. Find the laplace transform of $\cosh at$ and hence deduce $L(\sinh at)$. using the transforms of derivatives.
10. Find the laplace transform of the following:

*(*i*) $\int_0^t e^{-t} \cos t\, dt$ (*ii*) $\int_0^t \dfrac{\cos at - \cos bt}{t}\, dt$

*[Punjabi Univ., 2003]

11. Show that

(*i*) $\int_0^\infty t e^{-2t} \cos t\, dt = \dfrac{3}{25}$, * (*ii*) $\int_0^\infty \dfrac{e^{-t} \sin^2 t}{t}\, dt = \dfrac{1}{4} \log 5$,

(*iii*) $\int_0^\infty \left(\dfrac{\cos at - \cos bt}{t}\right) dt = \dfrac{1}{2} \log \dfrac{b^2}{a^2}$ *[KUK, 2006, 2010]

12.6 INVERSE LAPLACE TRANSFORMS

To find the inverse laplace transform of given functions, we try to recognize the given function either in the given form comparable to some standard expression whose inverse function is a known standard function or split the given function of s into number of expressions in s (*with the help of partial fractions*) comparable again to some standard functions of s of which inverse functions are known.

Let us list first inverse laplace transform of some standard elementary functions and associated functions.

$L^{-1}\left(\frac{1}{s}\right) = 1$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$	$L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{e^{at}t^{n-1}}{n-1!}$
$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
$L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}e^{at}\sin bt$	$L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$
$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a}t\sin at$	$L^{-1}\left(\frac{s}{(s^2-a^2)^2}\right) = \frac{1}{2a}t\sinh at$
$L^{-1}\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right) = t\cos at$	$L^{-1}\left(\frac{s^2+a^2}{(s^2-a^2)^2}\right) = t\cosh at$
$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}(\sin at - at\cos at)$	$L^{-1}\left(\frac{1}{(s^2-a^2)^2}\right) = -\frac{1}{2a^3}(\sinh at - at\cosh at)$
$L^{-1}\left(\frac{s^2}{(s^2+a^2)^2}\right) = \frac{1}{2a}(\sin at + at\cos at)$	$L^{-1}\left(\frac{s^2}{(s^2-a^2)^2}\right) = \frac{1}{2a}(\sinh at + at\cosh at)$
$L^{-1}\left(\frac{s^3}{(s^2+a^2)^2}\right) = (\cos at - \frac{1}{2}at\sin at)$	$L^{-1}\left(\frac{s^3}{(s^2-a^2)^2}\right) = (\cosh at + \frac{1}{2}at\sinh at)$
$L^{-1}\left(\frac{s}{s^4+4a^4}\right) = \frac{1}{2a^2}\sinh at.\sin at$	$L^{-1}\left(\frac{s^3}{s^4+4a^4}\right) = \cosh at.\cos at$
$L^{-1}\left(\frac{s^2+2a^2}{s^4+4a^4}\right) = \frac{1}{a}\cosh at.\sin at$	$L^{-1}\left(\frac{s^2-2a^2}{s^4+4a^4}\right) = \frac{1}{a}\sinh at.\cos at$

$$L^{-1}\left(\frac{1}{s^4+4a^4}\right)=\frac{1}{4a^3}(\cosh at\sin at-\sinh at\cos at)$$

$$L^{-1}\left(\frac{s^2}{s^4+4a^4}\right)=\frac{1}{2a}(\cosh at\sin at+\sinh at\cos at)$$

$$L^{-1}\left(\frac{1}{\sqrt{s^2+a^2}}\right)=J_0(at) \qquad L^{-1}\left(\frac{1}{\sqrt{s^2-a^2}}\right)=I_0(at)$$

$$L^{-1}\left(\frac{s}{\left(s^2+a^2\right)^{\frac{3}{2}}}\right)=tJ_0(at) \qquad L^{-1}\left(\frac{s}{\left(s^2-a^2\right)^{\frac{3}{2}}}\right)=t\,I_0(at)$$

Inverse transforms tabulated above will be frequently used as ready reckner, particularly for solving differential equations using laplace transforms. Without going into details of derivation of these, we may discuss some of them as under:

$$\frac{s^3}{(s^2+a^2)^2}=\frac{s(s^2+a^2-a^2)}{(s^2+a^2)^2}=\frac{s}{(s^2+a^2)}-a^2\frac{s}{(s^2+a^2)^2}$$

$$\therefore \qquad L^{-1}\left[\frac{s^3}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{s}{(s^2+a^2)}\right]-a^2L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$=\cos at-a^2\frac{t\sin at}{2a}=\cos at-\frac{1}{2}at\sin at$$

Likewise, $$L^{-1}\left[\frac{s^3}{(s^2-a^2)^2}\right]=L^{-1}\left[\frac{s}{(s^2-a^2)}\right]+a^2L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$$

$$=\cosh at+a^2\frac{t\sinh at}{2a}=\cosh at+\frac{1}{2}at\sinh at$$

As already stated, in deducing these results, we need to make use of partial fractions. Some of the properties of inverse laplace transforms and Convolution Theorem are taken up in subsequent discussions.

Tips for Partial Fractions: In any general expression $\dfrac{a_0s^m+a_1s^{m-1}+\ldots+\ a_m}{b_0s^n+b_1s^{n-1}+\ldots+\ b_n}$

Corresponding to a non-repeated linear factor (s – a), write $\dfrac{A}{(s-a)}$

Corresponding to a non-repeated linear factor (s – a), write $\dfrac{A_1}{(s-a)}+\dfrac{A_2}{(s-a)^2}+\ldots+\dfrac{A_r}{(s-a)^r}$

Corresponding to a non–repeated quadratic factor (s^2 + as + b), write $\dfrac{As+B}{(s^2+as+b)}$

Corresponding to repeated quadratic factors (s^2 + as + b), write

$$\frac{A_1 s + B_1}{(s^2 + as + b)} + \frac{A_2 s + B_2}{(s^2 + as + b)^2} + \ldots + \frac{A_r s + B_r}{(s^2 + as + b)^r}$$

Then determine unknown constants $A, B, C, \ldots A_1, A_2, A_3, \ldots$ by equating the coefficients of equal powers of s on both sides.

Example 12: Find the inverse Laplace transform of

(*i*) $\dfrac{1}{(s+a)^n}$, **(*ii*)** $\dfrac{1}{(As+B)^n}$,

where *A*, *B* and '*a*' are constants and *n* is a positive integer.

Solution:

(*i*) $L^{-1}\left[\dfrac{1}{(s+a)^n}\right] = e^{-at} L^{-1}\left(\dfrac{1}{s^n}\right) = e^{-at} \dfrac{t^{n-1}}{n-1!}$, using first shift property.

(*ii*) $L^{-1}\left[\dfrac{1}{(As+B)^n}\right] = \dfrac{1}{A^n} L^{-1}\left[\dfrac{1}{\left(s+\dfrac{B}{A}\right)^n}\right] = \dfrac{1}{A^n} e^{-\frac{B}{A}t} L^{-1}\left(\dfrac{1}{s^n}\right) = \dfrac{1}{A^n} e^{-\frac{B}{A}t} \dfrac{t^{n-1}}{n-1!}$

Example 13: Find the inverse laplace transform of $\dfrac{As+B}{Cs^2+Ds+E}$ **and** $\dfrac{As+B}{(Cs^2+Ds+E)^2}$.

Solution:

(*i*) $$\frac{As+B}{Cs^2+Ds+E} = \frac{A}{C}\frac{s}{s^2+\frac{D}{C}s+\frac{E}{C}} + \frac{B}{C}\cdot\frac{1}{s^2+\frac{D}{C}s+\frac{E}{C}}$$

$$= \frac{A}{C}\frac{s}{s^2+Fs+G} + \frac{B}{C}\frac{1}{s^2+Fs+G}, \text{ where } F = \frac{D}{C} \text{ and } G = \frac{E}{C}$$

$$= \frac{A}{C}\cdot\frac{s}{\left(S+\frac{F}{2}\right)^2 + \left(G-\frac{F^2}{4}\right)} + \frac{B}{C}\cdot\frac{1}{\left(S+\frac{F}{2}\right)^2 + \left(G-\frac{F^2}{4}\right)}$$

(By completing squares) … (1)

Case I: When $G > \dfrac{F^2}{4}$, then the expression $(s^2 + Fs + G)$ takes the form $(s + a)^2 + b^2$, where $a = \dfrac{F}{2}$ and $b^2 = \left(G - \dfrac{F^2}{4}\right)$, then

$$L^{-1}\left[\frac{As+B}{Cs^2+Ds+E}\right] = \frac{A}{C}L^{-1}\left[\frac{s}{(s+a)^2+b^2}\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$$

$$= \frac{A}{C}L^{-1}\left[\frac{(s+a)-a}{(s+a)^2+b^2}\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$$

$$= \frac{A}{C}\left[L^{-1}\left(\frac{(s+a)}{(s+a)^2+b^2}\right) - L^{-1}\left(\frac{a}{(s+a)^2+b^2}\right)\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$$

$$= \frac{A}{C}e^{-at}\left[L^{-1}\left(\frac{s}{s^2+b^2}\right) - aL^{-1}\left(\frac{1}{s^2+b^2}\right)\right] + \frac{B}{C}e^{-at}L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$= \frac{A}{C}e^{-at}\left[\cos bt - \frac{a}{b}\sin bt\right] + \frac{B}{C}e^{-at}\frac{\sin bt}{b}.$$

$$= e^{-at}\left[\frac{A}{C}\cos bt + \frac{B-aA}{bC}\sin bt\right] \qquad \ldots (2)$$

Case II: when $G < \frac{F^2}{4}$, then

$$L^{-1}\left[\frac{As+B}{Cs^2+Ds+E}\right] = \frac{A}{C}L^{-1}\left[\frac{s}{(s+a)^2-b^2}\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2-b^2}\right]$$

$$= \frac{A}{C}L^{-1}\left[\frac{(s+a)-a}{(s+a)^2-b^2}\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2-b^2}\right]$$

$$= \frac{A}{C}\left[L^{-1}\left(\frac{(s+a)}{(s+a)^2-b^2}\right) - aL^{-1}\left(\frac{1}{(s+a)^2-b^2}\right)\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2-b^2}\right]$$

$$= \frac{A}{C}e^{-at}\left[L^{-1}\left(\frac{1}{s^2-b^2}\right) - aL^{-1}\left(\frac{1}{s^2-b^2}\right)\right] + \frac{B}{C}e^{-at}L^{-1}\left[\frac{1}{s^2-b^2}\right]$$

$$= \frac{A}{C}\left[e^{-at}\cosh bt - \frac{a}{b}\sinh bt\right] + \frac{B}{C}e^{-at}\frac{\sinh bt}{b}.$$

$$\therefore\ L^{-1}\left[\frac{As+B}{Cs^2+Ds+E}\right] = e^{-at}\left[\frac{A}{C}\cosh bt + \frac{B-aA}{bC}\sinh bt\right] \qquad \ldots (3)$$

Case III: when $G = \frac{F^2}{4}$, then

$$L^{-1}\left[\frac{As+B}{Cs^2+Ds+E}\right] = \frac{A}{C}L^{-1}\left(\frac{s}{\left(s+\frac{F}{2}\right)^2 - \left(G+\frac{F^2}{4}\right)}\right) + \frac{B}{C}\left(\frac{1}{\left(s+\frac{F}{2}\right)^2 - \left(G-\frac{F^2}{4}\right)}\right)$$

$$= \frac{A}{C}L^{-1}\left[\frac{s}{(s+a)^2}\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2}\right]$$

$$= \frac{A}{C}\left[L^{-1}\left(\frac{(s+a)-a}{(s+a)^2}\right)\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2}\right]$$

$$= \frac{A}{C}\left[L^{-1}\left(\frac{1}{s+a}\right) - aL^{-1}\left(\frac{1}{(s+a)^2}\right)\right] + \frac{B}{C}L^{-1}\left[\frac{1}{(s+a)^2}\right]$$

$$= \frac{A}{C}e^{-at}\left[L^{-1}\left(\frac{1}{s}\right) - aL^{-1}\frac{1}{s^2}\right] + \frac{B}{C}e^{-at}L^{-1}\left(\frac{1}{s^2}\right).$$

$$= \frac{A}{C}e^{-at}\left[(1-at)\right] + \frac{B}{C}e^{-at}\,t$$

$$\therefore\ L^{-1}\left[\frac{As+B}{Cs^2+Ds+E}\right] = e^{-at}\left[\frac{A}{C}(1-at) + \frac{B}{C}t\right]$$

Example 14: Find inverse laplace of $\dfrac{\mathbf{2s+1}}{\mathbf{(s+2)^2(s-1)^2}}$.

Solution: Write $\dfrac{2s+1}{(s+2)^2(s-1)^2} = \dfrac{A}{s+2} + \dfrac{B}{(s+2)^2} + \dfrac{C}{(s-1)} + \dfrac{D}{(s-1)^2}$

Implying

$$(2s+1) = A(s+2)(s-1)^2 + B(s-1)^2 + C(s+2)^2(s-1) + D(s+2)^2 \quad \ldots (1)$$

$$= A(s+2)(s^2-2s+1) + B(s^2-2s+1) + C(s-1)(s^2+4s+4) + D(s^2+4s+4)$$

$$= A(s^3-3s+2) + B(s^2-2s+1) + C(s^3+3s^2-4) + D(s^2+4s+4)$$

$$(2s+1) = s^3(A+C) + s^2(B+3C+D) + s(-3A-2B+4D) + (2A+B-4C+4D) \quad \ldots (2)$$

From (1), if $s = 1$; then $(2+1) = D(1+2)^2$ implying $D = \dfrac{1}{3}$ … (3)

if $s = 2$; then $(-4+1) = B(-2-1)^2$ implying $B = -\dfrac{1}{3}$ … (4)

Further on equating coefficients of s^3 and constant terms on both sides of equation (2), we get,

$$A + C = 0 \quad \text{and} \quad 2A + B - 4C + 4D = 1 \quad \ldots (5)$$

Thus on putting $D = \dfrac{1}{3}$ and $B = -\dfrac{1}{3}$; equations (5) results in $A = 0 = C$

$$\therefore\quad L^{-1}\left[\frac{2s+1}{(s+2)^2(s-1)^2}\right] = L^{-1}\left[-\frac{1}{3(s+2)^2} + \frac{1}{3(s-1)^2}\right] = -\frac{-1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

Example 15: Find $L^{-1}\dfrac{s}{s^4+2s^2+1}$

Solution: Write $s^4 + s^2 + 1 = (s^4 + 2s^2 + 1) - s^2 = (s^2 + 1)^2 - s^2 = (s^2 + 1 + s)(s^2 + 1 - s)$

Thus, $$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+1+s)(s^2+1-s)} = \frac{As+B}{(s^2+1-s)} + \frac{Cs+D}{(s^2+1+s)}$$

or $$s = (As + B)(s^2 + 1 + s) + (Cs + D)(s^2 + 1 - s)$$
$$= s^3(A + C) + s^2(A + B - C + D) + s(A + B + C - D) + (B + D)$$

On comparing coefficients of equals powers of s on both sides,

$$\left.\begin{array}{llll} For & s^3, & A+C=0 & \ldots(1) \\ For & s^2, & A+B-C+D=0 & \ldots(2) \\ For & s, & A+B+C-D=1 & \ldots(3) \\ \text{Constant}, & & B+D=0 & \ldots(4) \end{array}\right\}$$

All these equation together give $B=\frac{1}{2},\ D=-\frac{1}{2}$

$$\therefore \quad L^{-1}\left[\frac{s}{s^4+s^2+1}\right] = L^{-1}\left[\frac{1}{2}\,\frac{1}{(s^2+1-s)} - \frac{1}{2}\,\frac{1}{s^2+1+s}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{\left(s^2+\frac{1}{4}-s\right)+\frac{3}{4}}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{\left(s^2+\frac{1}{4}+s\right)+\frac{3}{4}}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \frac{1}{2}e^{\frac{1}{2}t}L^{-1}\left[\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] - \frac{1}{2}e^{\frac{-1}{2}t}L^{-1}\left[\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] \quad \text{By Ist shift property}$$

$$= \frac{1}{2}e^{\frac{1}{2}t}\cdot\frac{\sin\left(\frac{\sqrt{3}}{2}\right)t}{\left(\frac{\sqrt{3}}{2}\right)} - \frac{1}{2}e^{-\frac{1}{2}t}\frac{\sin\left(\frac{\sqrt{3}}{2}\right)t}{\left(\frac{\sqrt{3}}{2}\right)} = \left(\frac{2}{\sqrt{3}}\right)\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{2}{\sqrt{3}}\sinh\frac{t}{2}\,\sin\frac{\sqrt{3}}{2}t$$

Example 16: Find inverse Laplace transform of $\dfrac{s^3}{s^4 - a^4}$ **[NIT Jalandhar, 2007]**

Solution: write $\dfrac{s^3}{s^4 - a^4} = \dfrac{s^3}{(s^2 - a^2)(s^2 + a^2)} = \dfrac{As + B}{(s^2 - a^2)} + \dfrac{Cs + D}{(s^2 + a^2)}$

Implying $s^3 = (As + B)(s^2 + a^2) + (Cs + D)(s^2 - a^2)$

$$= s^3(A + C) + s^2(B + D) + s(a^2A - a^2C) + a^2(B - D)$$

On comparing coefficients of equal powers of s^3, s^2, s and constant on both sides we get,

$$\left.\begin{array}{l} A + C = 1 \\ B + D = 0 \\ a^2(A - C) = 0 \\ a^2(B - D) = 0 \end{array}\right\} \text{All together implying} \quad A = \frac{1}{2} = C, \quad B = 0 = D.$$

$$\therefore \qquad L^{-1}\left(\frac{s^3}{s^4 - a^4}\right) = L^{-1}\left[\frac{1}{2}\frac{s}{s^2 - a^2} + \frac{1}{2}\frac{s}{s^2 + a^2}\right] = \frac{1}{2}[\cosh at + \cos at]$$

Example 17: Find inverse transforms of

(*i*) $\dfrac{1}{s^4 + 4a^4}$, **(*ii*)** $\dfrac{s}{s^4 + 4a^4}$, **(*iii*)*** $\dfrac{s^2}{s^4 + 4a^4}$,

(*iv*) $\dfrac{s^3}{s^4 + 4a^4}$, **(*v*)**** $\dfrac{a(s^2 - 2a^2)}{s^4 + 4a^4}$ **[*KUK, 2001, 2002; **NIT Kurukshetra, 2010]**

Solution:

(*i*) Write, $s^4 + 4a^4 = s^4 + 4a^4 + 4a^2s^2 - 4a^2s^2 = (s^2 + 2a^2)^2 - (2as)^2$

$$= (s^2 + 2a^2 - 2as)(s^2 + 2a^2 + 2as)$$

so that $$\frac{1}{s^4 + 4a^4} = \frac{As + B}{s^2 + 2a^2 + 2as} + \frac{Cs + D}{s^2 + 2a^2 - 2as} = \frac{1}{8a^3}\left[\frac{s + 2a}{s^2 + 2as + 2a^2} - \frac{s - 2a}{s^2 - 2as + 2a^2}\right]$$

$$\left(\text{By partial fractions, } A = \frac{1}{8a^3} = -C, B = \frac{1}{4a^2} = D\right)$$

Now $$L^{-1}\left[\frac{1}{s^4 + 4a^4}\right] = \frac{1}{8a^3}L^{-1}\left[\frac{(s + a) + a}{(s + a)^2 + a^2} - \frac{(s - a) - a}{(s - a)^2 + a^2}\right]$$

$$= \frac{1}{8a^3}\left[L^{-1}\frac{s + a}{(s + a)^2 + a^2} + aL^{-1}\frac{1}{(s + a)^2 + a^2} - L^{-1}\frac{s - a}{(s - a)^2 + a^2} + aL^{-1}\frac{1}{(s - a)^2 + a^2}\right]$$

$$= \frac{1}{8a^3}\left[e^{-at}L^{-1}\frac{s}{s^2 + a^2} + e^{-at}L^{-1}\frac{a}{s^2 + a^2} - e^{at}L^{-1}\frac{s}{s^2 + a^2} + e^{at}L^{-1}\frac{a}{s^2 + a^2}\right]$$

(By first shift property)

$$= \frac{1}{8a^3}\left[e^{-at}\cos at + e^{-at}\sin at - e^{at}\cos at + ae^{at}\sin at\right]$$

$$= \frac{1}{4a^3}\left[\sin at\,\frac{\left(e^{at}+e^{-at}\right)}{2} + \cos at\,\frac{\left(e^{-at}-e^{at}\right)}{2}\right]$$

$$= \frac{1}{4a^3}\left[\sin at\cosh at - \cos at\sinh at\right]$$

(*ii*) $$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{4a}L^{-1}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$ By partial fraction

$$= \frac{1}{4a}\left[L^{-1}\left(\frac{1}{(s-a)^2+a^2}\right) - L^{-1}\left(\frac{1}{(s+a)^2+a^2}\right)\right]$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\frac{1}{s^2+a^2} - e^{-at}L^{-1}\frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{4a}\left[e^{at}\frac{\sin at}{a} - e^{-at}\frac{\sin at}{a}\right] = \frac{1}{2a^2}\sin at\,\frac{\left(e^{at}-e^{-at}\right)}{2} = \frac{1}{2a^2}\sin at\sinh at$$

(*iii*) $$L^{-1}\left(\frac{s^2}{s^4+4a^4}\right) = \frac{1}{4a}L^{-1}\left[\frac{s}{s^2-2as+2a^2} - \frac{s}{s^2+2as+2a^2}\right],$$ (By partial fractions)

$$= \frac{1}{4a}L^{-1}\left[\frac{s}{(s-a)^2+a^2} - \frac{s}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{4a}L^{-1}\left[\frac{(s-a)+a}{(s-a)^2+a^2} - \frac{(s+a)-a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{4a}\left[L^{-1}\frac{(s-a)}{(s-a)^2+a^2} + L^{-1}\frac{a}{(s-a)^2+a^2} - L^{-1}\frac{(s+a)}{(s+a)^2+a^2} + L^{-1}\frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{4a}\left[L^{-1}\frac{(s-a)}{(s-a)^2+a^2} + L^{-1}\frac{a}{(s-a)^2+a^2} - L^{-1}\frac{(s+a)}{(s+a)^2+a^2} + L^{-1}\frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\frac{s}{s^2+a^2} + e^{at}L^{-1}\frac{a}{s^2+a^2} - e^{-at}L^{-1}\frac{s}{s^2+a^2} + e^{-at}L^{-1}\frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{4a}\left[e^{at}\cos at + e^{at}\sin at - e^{-at}\cos at + e^{-at}\sin at\right]$$

$$= \frac{1}{2a}\left[\cos at\left(\frac{e^{at}-e^{-at}}{2}\right)+\sin at\left(\frac{e^{at}+e^{-at}}{2}\right)\right]$$

$$= \frac{1}{2a}\left[\cos at \sinh at + \sin at \cosh at\right]$$

(*iv*) $$L^{-1}\left[\frac{s^3}{s^4+4a^4}\right] = \frac{1}{2}L^{-1}\left[\frac{s-a}{(s-a)^2+a^2}+\frac{s+a}{(s-a)^2+a^2}\right] \text{ (By partial fractions)}$$

$$= \frac{1}{2}e^{at}\cos at + \frac{1}{2}e^{-at}\cos at = \cosh at \cos at$$

(*v*) Here $$\frac{a(s^2-2a^2)}{(s^4-4a^4)} = \frac{As+B}{(s^2+2a^2-2as)}+\frac{Cs+D}{s^2+2a^2+2as}$$

On resolving partial fractions we get $A = \frac{1}{2} = C$ and $B = \frac{-a}{2} = D$

$$\therefore \quad L^{-1}\left[\frac{a(s^2-a^2)}{(s^4+4a^4)}\right] = \frac{1}{2}L^{-1}\left[\frac{(s-a)}{(s-a)^2+a^2}-\frac{(s+a)}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[e^{at}L^{-1}\frac{s}{s^2+a^2}-e^{-at}L^{-1}\frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{2}e^{at}\cos at - \frac{1}{2}e^{-at}\cos at.$$

$$= \frac{e^{at}-e^{-at}}{2}\cos at = \sinh at \cos at.$$

12.7 SOME IMPORTANT PROPERTIES ON INVERSE LAPLACE TRANSFORMS:

If $L^{-1}\left(\bar{f}(s)\right) = f(t)$, then

(*i*) $L^{-1}\left(\bar{f}(s-a)\right) = e^{at}f(t)$, (*ii*) $L^{-1}\left(s\,\bar{f}(s)\right) = \frac{d}{dt}f(t)$, provided $f(0) = 0$

(*iii*) $L^{-1}\left(\frac{\bar{f}(s)}{s}\right) = \int_0^t f(u)\,du$, (*iv*) $L^{-1}\left(-\frac{d}{ds}\bar{f}(s)\right) = t\,f(t)$,

(*v*) $L^{-1}\left(\int_s^\infty \bar{f}(s)\,ds\right) = \frac{f(t)}{t}$

(*vi*) $L^{-1}\left(e^{-as}\bar{f}(s)\right) = F(t)$, where $F(t) = \begin{cases} f(t-a), & t > a \\ 0 & t < a \end{cases}$

Example 18: Find the inverse laplace of

$$(i)\ \frac{(s+2)}{(s^2+4s+5)^2}, \quad (ii)\ \left(\frac{1}{s}\cos\frac{1}{s}\right) \quad (iii)\ \frac{1}{s^3(s^2+1)}$$

Solution:

(*i*) Write $\dfrac{(s+2)}{(s^2+4s+5)^2} = -\dfrac{1}{2}\dfrac{d}{ds}\left[\dfrac{1}{(s^2+4s+5)}\right] = -\dfrac{1}{2}\dfrac{d}{ds}\bar{f}(s)$ ...(1)

Now on using property; if $L^{-1}\bar{f}(s) = f(t)$ then $L^{-1}\left(-\dfrac{d}{ds}\bar{f}(s)\right) = t\,f(t)$

Here $f(t) = L^{-1}\dfrac{1}{(s^2+4s+5)} = L^{-1}\left[\dfrac{1}{(s^2+4s+4)+1}\right] = L^{-1}\left[\dfrac{1}{(s+2)^2+1^2}\right] = e^{-2t}\sin t$

$$\therefore\ L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \frac{1}{2}t\,f(t) = \frac{1}{2}te^{-2t}\sin t$$

(*ii*) We know that

$$\cos\frac{1}{s} = 1 - \frac{1}{s^2\,2!} + \frac{1}{s^4\,4!} + \ldots\ldots\left(\text{using, } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots\right)$$

$$\therefore\quad L^{-1}\left[\frac{1}{s}\cos\frac{1}{s}\right] = L^{-1}\left[\frac{1}{s}\left(1 - \frac{1}{2!s^2} + \frac{1}{4!s^2} - \ldots\right)\right]$$

$$= L^{-1}\left[\frac{1}{s} - \frac{1}{2!s^3} + \frac{1}{4!s^5} - \ldots\right]$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \ldots$$

(*iii*) As $L^{-1}\left[\dfrac{1}{(s^2+1)}\right] = \sin t$, therefore $L^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,du = (1-\cos t)$

implying $L^{-1}\left[\dfrac{1}{s^2(s^2+1)}\right] = \int_0^t (1-\cos u)\,du = (t - \sin t)$

and $L^{-1}\left[\dfrac{1}{s^3(s^2+1)}\right] = \int_0^t (u - \sin u)\,du = \left(\dfrac{t^2}{2} + \cos t - 1\right)$

Example 19: Find inverse laplace transform of $s.\log\dfrac{s-1}{s+1}$

[NIT Kurukshetra 2003]

Solution: Here, $-\dfrac{d}{ds}\bar{f}(s) = -\dfrac{d}{ds}\left(s\log\dfrac{s-1}{s+1}\right) = \dfrac{d}{ds}\left(s\log(s+1) - s\log(s-1)\right)$

$$= \left[\left\{\log(s+1) + \left(\frac{s}{s+1}\right)\right\} - \left\{\log(s-1) + \left(\frac{s}{s-1}\right)\right\}\right]$$

$$= \left[\left\{\log(s+1) - \log(s-1)\right\} + \left(\frac{s}{s+1} - \frac{s}{s-1}\right)\right]$$

$$= \bar{\phi}(s) - \frac{2s}{s^2-1}, \text{ where } \bar{\phi}(s) = \left\{\log(s+1) - \log(s-1)\right\} \qquad \ldots(1)$$

Taking inverses laplace on both sides,

$$t.f(t) = L^{-1}\bar{\phi}(s) - 2\cosh t \qquad \ldots(2)$$

Again, $$t\phi(t) = L^{-1}\left[-\frac{d}{ds}\bar{\phi}(s)\right] = L^{-1}\frac{d}{ds}\left[\log(s-1) - \log(s+1)\right]$$

$$= L^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = L^{-1}\frac{2}{s^2-1} = 2\sinh t$$

or $$\phi(t) = \frac{2\sinh t}{t} \qquad \ldots(3)$$

Using result (3) in (2), we get

$$t\,f(t) = \frac{2\sinh t}{t} - 2\cosh t \quad \text{or} \quad f(t) = 2\left(\frac{\sinh t - t\cosh t}{t^2}\right)$$

Example 20: Find inverse laplace transform of

(*i*) $\dfrac{1}{\left(s^2+a^2\right)^2}$, **(*ii*)** $\dfrac{s}{\left(s^2+a^2\right)^2}$ **(SVTU 2007)** **(*iii*)*** $\dfrac{s^2}{\left(s^2+a^2\right)^2}$

[Madrass 2000; *KUK, June 2004]

Solution: $\dfrac{1}{\left(s^2+a^2\right)^2} = \dfrac{1}{s}\dfrac{s}{\left(s^2+a^2\right)^2}$ is comparable to $\dfrac{\bar{f}(s)}{s}$ … (1)

Now $\dfrac{s}{\left(s^2+a^2\right)^2} = -\dfrac{1}{2}\dfrac{d}{ds}\left[\dfrac{1}{\left(s^2+a^2\right)}\right]$ is comparable to $-\dfrac{d}{ds}\bar{\phi}(s)$

$$\therefore \quad L^{-1}\left(\frac{s}{\left(s^2+a^2\right)^2}\right)=\frac{1}{2}t\,\phi(t), \quad \text{where} \quad \phi(t)=L^{-1}\left[\frac{1}{\left(s^2+a^2\right)}\right]=\frac{\sin at}{a}$$

Implying $L^{-1}\left(\dfrac{s}{\left(s^2+a^2\right)^2}\right)=\dfrac{1}{2}t\dfrac{\sin at}{a}$, which proves result (*ii*) ...(2)

For (i),

$$L^{-1}\left(\frac{1}{\left(s^2+a^2\right)^2}\right)=L^{-1}\left(\frac{1}{s}\frac{s}{\left(s^2+a^2\right)^2}\right)=L^{-1}\left(\frac{\bar{f}(s)}{s}\right)$$

$$=\int_0^t f(u)\,du=\int_0^t \frac{1}{2a}u\sin au\,du, \quad (\text{using } (2))$$

$$=\frac{1}{2a}\left(u\cdot\frac{-\cos au}{a}\right)_0^t-\int_0^t 1\left(-\frac{\cos au}{a}\right)du$$

$$=\frac{1}{2a}\left[-\frac{t\cos at}{a}+\frac{\sin at}{a^2}\right]=\frac{1}{2a^3}(\sin at-at\cos at)$$

For (*ii*), Alternately $\displaystyle\int_s^\infty \bar{f}(s)\,ds=L\left(\frac{f(t)}{t}\right)$

$$\text{or} \quad L\left(\frac{f(t)}{t}\right)=\int_s^\infty \frac{s}{\left(s^2+a^2\right)^2}ds=\frac{1}{2}\int_s^\infty \frac{2s}{\left(s^2+a^2\right)^2}ds=-\frac{1}{2}\left[\frac{1}{s^2+a^2}\right]_s^\infty=\frac{1}{2}\left(\frac{1}{\left(s^2+a^2\right)}\right)$$

Taking inverses, $\dfrac{f(t)}{t}=\dfrac{1}{2}\dfrac{\sin at}{a}$ or $f(t)=\dfrac{t\sin at}{2a}$

For (*iii*) $\dfrac{s^2}{\left(s^2+a^2\right)^2}=s\cdot\dfrac{s}{\left(s^2+a^2\right)^2}$ comparable to $s\,\bar{f}(s)$

$$\therefore \quad L^{-1}\left(\frac{s^2}{\left(s^2+a^2\right)^2}\right)=L^{-1}\left(s\,\bar{f}(s)\right)=\frac{d}{dt}f(t), \quad \text{where} \quad f(t)=L^{-1}\left(\bar{f}(s)\right) \text{ and } f(0)=0$$

Here $L^{-1}\bar{f}(s)=L^{-1}\left(\dfrac{s}{\left(s^2+a^2\right)^2}\right)=\dfrac{t\sin at}{2a}$, result (*ii*)

Implying $L^{-1}\left(\dfrac{s^2}{\left(s^2+a^2\right)^2}\right)=\dfrac{d}{dt}\left(\dfrac{t\sin at}{2a}\right)=\dfrac{1}{2a}\left(\sin at+at\cos at\right)$

12.8 CONVOLUTION THEOREM

Convolution: Let $f(t)$ and $g(t)$ be two functions of class '*A*', then the convolution of the two functions $f(t)$ and $g(t)$ denoted by $f\otimes g$ is defined as:

$f \otimes g = \int_0^t f(u)g(t-u)du$, where $f \otimes g$ is also known as falting of $f(t)$ and $g(t)$.

The convolution $f \otimes g$ is

(*i*) Commutative i.e. $f \otimes g = g \otimes f$
(*ii*) Associative i.e. $(f \otimes g) \otimes h = f \otimes (g \otimes h)$
(*iii*) Distributive with respect to addition i.e. $f \otimes (g + h) = f \otimes g + f \otimes h$

Statement: If $L^{-1}\bar{f}(s) = f(t)$ and $L^{-1}\bar{g}(s) = g(t)$, then $L^{-1}\left(\bar{f}(s)\bar{g}(s)\right) = \int_0^t f(u)g(t-u)\,du$

Proof: $$L(f \otimes g) = \int_0^\infty e^{-st}(f \otimes g)\,dt = \int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)\,du\right]dt \qquad \ldots (1)$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt \qquad \ldots (2)$$

(Expression (1) clearly shows that the integration is Ist taken along the vertical strip PQ from P to Q, P resting on the curve $u = 0$ and Q resting on the curve $u = t$ and finally the strip PQ slides between $t = 0$ to ∞ to cover the full dotted region)

On changing the order of integration, we get

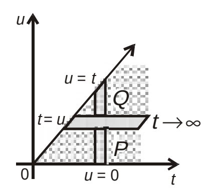

Fig: 12.3

$$L(f \otimes g) = \int_0^\infty \left(\int_u^\infty \left(e^{-st} f(u)\,g(t-u)\,dt\right)\right)du \qquad \ldots (3)$$

$$= \int_0^\infty \left(\int_u^\infty \left(e^{-s(t-u+u)} f(u)\,g(t-u)\,dt\right)\right)du$$

$$= \int_0^\infty e^{-su} f(u)\left(\int_u^\infty \left(e^{-s(t-u)} g(t-u)\,dt\right)\right)du$$

$$= \int_0^\infty e^{-su} f(u)\left(\int_0^\infty e^{-sp} g(p)\,dp\right)du, \text{ (on putting } t - u = p \text{ so that } dt = dp)$$

$$= \int_0^\infty e^{-su} f(u)\left(\bar{g}(s)\right)du = \left(\int_0^\infty e^{-su} f(u)\,du\right)\bar{g}(s) = \bar{f}(s)\bar{g}(s) \text{ (Using def}^n\text{. of laplace)}$$

Whence the desired result.

Example 21: Using Convolution theorem, find

(*i*) $L^{-1}\left[\dfrac{s}{(s^2 + a^2)}\right]$ **(*ii*)*** $L^{-1}\left[\dfrac{s^2}{(s^4 + a^4)}\right]$

***[KUK, 2002, 03, NIT Kurukshetra, 2005]**

Solution:

(*i*) Write, $L^{-1}\left[\dfrac{s}{(s^2+a^2)}\right] = L^{-1}\left[\dfrac{s}{(s^2+a^2)}\dfrac{1}{(s^2+a^2)}\right]$, where in $L^{-1}\left[\dfrac{s}{(s^2+a^2)}\right] = \cos at$ and

$$L^{-1}\left[\frac{s}{(s^2+a^2)}\right]=\frac{\sin at}{a}$$

$$\therefore \quad L^{-1}\left[\left(\frac{s}{s^2-a^2}\right)\left(\frac{1}{s^2+a^2}\right)\right]=\int_0^{\infty}(\cos au)\left(\frac{\sin a(t-u)}{a}\right)du$$

$$=\frac{1}{a}\int_0^t \cos au\cdot(\sin at\cos au-\cos at\sin au)\,du$$

$$=\frac{\sin at}{a}\int_0^t \cos^2 au\,du-\frac{\cos at}{2a}\int_0^t \sin 2au\;du$$

$$=\frac{\sin at}{2a}\int_0^t (1+\cos 2au)\,du-\frac{\cos at}{2a}\int_0^t \sin 2au\,du$$

$$=\frac{\sin at}{2a}\left[t+\frac{\sin 2at}{2a}\right]-\frac{\cos at}{2a}\left[-\frac{\cos 2au}{2a}\right]_0^t$$

$$=\frac{t\sin at}{2a}+\frac{(2\sin^2 at)(\cos at)}{2a\,2a}+\frac{\cos at}{2a}\frac{1}{2a}(\cos 2at-1)$$

$$=\frac{t\sin at}{2a}$$

(*ii*) $L^{-1}\left[\frac{s^2}{(s^4-a^4)}\right]=L^{-1}\left[\frac{s}{(s^2-a^2)}\frac{s}{(s^2+a^2)}\right]=L^{-1}\left[\bar{f}(s)\bar{g}(s)\right]=f(t)\otimes g(t)$

where, $\quad f(t)=L^{-1}\bar{f}(s)=L^{-1}\left(\frac{s}{s^2-a^2}\right)=\cosh at$ and $g(t)=L^{-1}\bar{g}(s)=L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$

$$\therefore \quad L^{-1}\left[\frac{s^2}{(s^4-a^4)}\right]=\int_0^t \cosh au\,\cos a(t-u)\,du=I,$$

Integrate by parts, taking cos*hau* as Ist function and $\cos a(t-u)$ as second function,

$$I=\left[\left\{\cosh au.\frac{\sin a(t-u)}{-a}\right\}_0^t-\int_0^t a\sinh at\frac{\sin a(t-u)}{-a}du\right]$$

$$=\left[\left\{\cos au\frac{\sin a(t-u}{-a}\right\}_0^t+\int_0^t \sinh au\,\sin a(t-u)\,du\right]$$

$$=\left[-\frac{1}{a}(0-\sin at)+\left(\sinh au.\frac{-\cos(t-u)}{-a}\Bigg|_0^t-\int_0^t a\cosh au.\frac{-\cos a(t-u)}{-a}du\right)\right]$$

$$I = \left[\frac{\sin at}{a} + \left(\frac{\sinh au \cos a(t-u)}{a}\right)_0^t - I\right]$$

implying $2I = \left(\frac{\sin at}{a} + \frac{\sinh at}{a}\right)$ or $I = \frac{1}{2a}(\sin at + \sinh at)$

Hence $L^{-1}\left[\frac{s^2}{s^4 + a^4}\right] = \frac{1}{2a}[\sin at + \sinh at]$

Alternately, may take $\cosh au = \frac{e^{au} + e^{-au}}{2}$ in integral I.

Example 22: Using convolution theorem, find $L^{-1}\frac{1}{s\left(s^2+4\right)}$. **[KUK, 2004]**

Solution: Write $\frac{1}{s\left(s^2+4\right)} = \frac{1}{s}\frac{1}{s^2+2^2}$ comparable to $\bar{f}(s)\bar{g}(s)$ with

$$\bar{f}(s) = \frac{1}{s} \text{ and } L^{-1}\left(\bar{f}(s)\right) = 1;\ g(s) = \frac{1}{s^2+2^2} \text{ and } L^{-1}\left(\frac{1}{s^2+2^2}\right) = \frac{\sin 2t}{2}$$

χ $$L^{-1}\left(\frac{1}{s\left(s^2+4\right)}\right) = \frac{1}{2}\int_0^t 1\cdot\sin 2(t-u)\,du = \frac{1}{2}\left(\frac{-\cos 2(t-u)}{-2}\right)_0^t$$

$$= \frac{1}{4}\cos(2t-2u)\Big|_0^t = \frac{1}{4}(1-\cos 2t)$$

Alternately: $\frac{1}{s\left(s^2+4\right)} = \frac{1}{s^2}\frac{s}{s^2+2^2}$; $\bar{f}(s) = \frac{1}{s^2}$ and $\bar{g}(s) = \frac{\cos 2t}{2}$

so that $L^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \frac{1}{2}\int_0^t u\cos 2(t-u)\,du = \frac{1}{4}(1-\cos 2t)$

ASSIGNMENT 2

Find the inverse laplace transform of

1. (*i*) $\frac{s}{\left(s^2-1\right)^2}$, (*ii*) $\frac{s}{(s-3)\left(s^2-4\right)}$, (*iii*) $\frac{1}{s^3-a^3}$,

   (*iv*) $\frac{s^2+s-2}{s(s+3)(s-2)}$ (*v*)* $\frac{s}{(s+1)^2\left(s^2+1\right)}$ *[NIT Kurukshetra, 2005]

2. $\frac{As+B}{\left(Cs^2+Ds+E\right)^2}$ where A, B, C, D, are constants.

3. (*i*) $\dfrac{2s-3}{s^2+4s+13}$, (*ii*) $\dfrac{3s+7}{s^2-2s-3}$, (*iii*) $\dfrac{3s}{s^2+2s-8}$

4. (*i*)* $\dfrac{s+2}{\left(s^2+4s+5\right)^2}$, (*ii*) $\dfrac{s+3}{\left(s^2+6s+13\right)^2}$, (*iii*)** $\dfrac{s}{\left(s^2-1\right)^2}$

[*Madrass, 2003; PTU, 2005; **KUK, 2005]

5. (*i*)* $\log\dfrac{s^2+1}{s(s+1)}$ (*ii*) $\cot^{-1}\left(\dfrac{s}{2}\right)$ or $\tan^{-1}\left(\dfrac{2}{s}\right)$ (*iii*)** $\tan^{-1}\left(\dfrac{2}{s^2}\right)$

[*VTU, 2004; **Bombay, 2005]

(*iv*)* $\log\dfrac{(s+a)}{(s+b)}$ (*v*) $\log\left(\dfrac{(s+1)}{(s+2)(s+3)}\right)$ (*vi*)* $\log\left(1-\dfrac{a^2}{s^2}\right)$

[*Anna 2003, UP Tech 2003]

(*vii*)* $\log\left(\dfrac{\left(s^2+1\right)}{(s-1)^2}\right)$ (*viii*) $\cot^{-1}(s+1)$ (*ix*)** $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$

[*Madras, 2005; **VTU, 2007]

6. (*i*)* $\dfrac{s}{a^2s^2+b^2}$ (*ii*) $\dfrac{s}{(s+a)^2}$ (*iii*)** $\dfrac{s}{(s+a)^3}$

[*Madras, 2005; **KUK, 2004]

7. (*i*) $\dfrac{1}{s(s+2)^3}$ (*ii*) $\dfrac{1}{s^2(s+2)}$ (*iii*)* $\dfrac{1}{s^2\left(s^2+a^2\right)}$

[*NIT Kurukshetra, 2010]

8. Show that $L^{-1}\left(\dfrac{1}{s}\sin\dfrac{1}{s}\right)=t-\dfrac{t^3}{(3!)^3}+\dfrac{t^5}{(5!)^2}-\ldots\ldots\infty$

9. Use Convolution theorem to find inverse of the following:

(*i*) $\dfrac{1}{(s+a)(s+b)}$ (*ii*) $\dfrac{1}{\left(s^2+a^2\right)^2}$ (*iii*)$^{\Delta}$ $\dfrac{1}{s^2(s+1)^2}$ (*iv*) $\dfrac{1}{s^2\left(s^2+a^2\right)^2}$

(*v*)* $\dfrac{1}{(s+1)(s+9)^2}$ (*vi*)** $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)}$ (*vi*)*** $\dfrac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}$

[KUK, *2004, 2005-06; **2004-05 *** JNTU, 05, SVTU, 07; $^{\Delta}$KUK, 2010]

Note: *Most of the above problems can be handled either by partial fractions or convolution theorem. However, for easy handling of problems 4 & 5, apply property* $L^{-1}\left(-\frac{d}{ds}\bar{f}(s)\right) = tf(t)$; *in problems (6), apply property* $L^{-1}\left(s\bar{f}(s)\right) = \frac{d}{dt}f(t)$; *in problems 7 & 8, apply property* $L^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(u)du.$

12.9 APPLICATION TO DIFFERENTIAL EQUATIONS

Example 23: Using laplace transforms, solve $(D^4 - K^4)y = 0$
where $y(0) = 1, y'(0) = y''(0) = y'''(0) = 0$ [NIT Kurukshetra, 2004]

Solution: Taking laplace transform of each individual term of the given equation,

$$L(D^4y) - K^4L(y) = 0$$

Implying $s^4\bar{y}(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - K^4\bar{y}(s) = 0$

i.e. $\left(s^4 - K^4\right)\bar{y}(s) = s^3$ implying $\bar{y}(s) = \dfrac{s^3}{(s-K)(s+K)\left(s^2+K^2\right)}$ … (1)

Then by partial fractions,

$$\frac{s^3}{(s-K)(s+K)\left(s^2+K^2\right)} = \frac{A}{(s-K)} + \frac{B}{(s+K)} + \frac{Cs+D}{s^2+K^2} \quad \dots (2)$$

implying $s^3 = A(s + K)(s^2 + K^2) + B(s - K)(s^2 + K^2) + (Cs + D)(s^2 - K^2)$

On comparing coefficients of equal powers of s on both sides

$1 = (A + B + C)$ … (*i*)

$0 = (A - B) K + D$ … (*ii*)

$0 = (A + B - C)K^2$ … (*iii*)

$0 = (AK - BK - D)K^2$ … (*iv*)

Subtracting (*iii*) from (*i*), we get $C = \frac{1}{2}$

Subtracting (*iv*) from (*ii*), we get $D = 0$

With values of C and D, equations (*i*) and (*iv*) becomes

$$\left.\begin{aligned} A + B &= \frac{1}{2} \\ A - B &= 0 \end{aligned}\right\rangle \text{ implying } A = \frac{1}{4} \text{ and } B = \frac{1}{4}$$

Putting the values of A, B, C and D in equation (2) or more precisely in (1), we get

$$\bar{y}(s) = \frac{1}{4(s-K)} + \frac{1}{4(s+K)} + \frac{1}{2}\frac{s}{\left(s^2+K^2\right)} \quad \dots (3)$$

Taking inverse transforms of each individual terms on both sides,

$$y(t) = \frac{1}{4}e^{Kt} + \frac{1}{4}e^{-Kt} + \frac{1}{2}\cos Kt = \frac{1}{2}\left(\frac{e^{Kt}+e^{-Kt}}{2}\right) + \frac{1}{2}\cos Kt = \frac{1}{2}(\cosh Kt + \cos Kt)$$

Example 24: Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ **when** $y = 1, \quad \frac{dy}{dx} = \frac{d^2y}{dx^2} = 2 \quad$ ***at*** $\quad x = 0$

Solution: Taking laplace transforms on both sides of the given equation

$$\left[s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0)\right] + 2\left[s^2\bar{y}(s) - sy(0) - y'(0)\right] - \left[s\bar{y}(s) - y(0)\right] - 2\bar{y}(s) = 0 \qquad \dots (1)$$

Making use of the given conditions,

$$\left[s^3\bar{y} - s^2\cdot 1 - s\cdot 2 - 2\right] + 2\left[s^2\bar{y} - s\cdot 1 - 2\right] - \left[s\bar{y} - 1\right] - 2\bar{y} = 0$$

i.e.
$$\left(s^3 + 2s^2 - s - 2\right)\bar{y}(s) = \left(s^2 + 4s + 5\right) \qquad \dots (2)$$

$$\bar{y}(s) = \frac{\left(s^2+4s+5\right)}{\left(s^3+2s^2-s-2\right)} = \frac{\left(s^2+4s+5\right)}{(s-1)(s+1)(s+2)} \qquad \dots (3)$$

$$\bar{y}(s) = \frac{5}{3}\left(\frac{1}{s-1}\right) - \left(\frac{1}{s+1}\right) + \frac{1}{3}\left(\frac{1}{s+2}\right) \quad \text{(By partial fractions)} \qquad \dots (4)$$

Taking inverse laplace on both sides of equation (4), we get

$$y(x) = \frac{5}{3}e^{x} - e^{-x} + \frac{1}{3}e^{-2x} \qquad \dots (5)$$

which is the desired solution.

Example 25: Solve the equation $(D^2 + 1)x = t\cos 2t$; $x = 0 = Dx$ ***at*** $t = 0$ **[KUK, 2010]**

Solution: Taking laplace on both sides,

$$\left[s^2\bar{x} - sx(0) - x'(0)\right] + \bar{x}(s) = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

Using the given conditions $(s^2+1)\bar{x}(s) = \dfrac{s^2-4}{\left(s^2+4\right)^2}$

i.e.
$$\bar{x}(s) = \frac{\left(s^2-4\right)}{\left(s^2+4\right)^2\left(s^2+1\right)} \qquad \dots (1)$$

By partial fractions,

$$\frac{\left(s^2-4\right)}{\left(s^2+4\right)^2\left(s^2+1\right)} = \frac{As+B}{\left(s^2+1\right)} + \frac{Cs+D}{\left(s^2+4\right)} + \frac{Es+F}{\left(s^2+4\right)^2}$$

or
$$(s^2 - 4) = (As + B)(s^2 + 4)^2 + (Cs + D)(s^2 + 1)(s^2 + 4) + (Es + F)(s^2 + 1)$$

On comparing coefficients of equal powers of ‘s’ on both sides, we get

$$\left.\begin{array}{lll} s^5, & A+C=0 & \ldots(i) \\ s^4, & B+D=0 & \ldots(ii) \\ s^3, & 8A+5C+E=0 & \ldots(iii) \\ s^2, & F+8B+5D=1 & \ldots(iv) \\ s^1, & 16A+4C+E=0 & \ldots(v) \\ s^0, & 16B+4D+F=-4 & \ldots(vi) \end{array}\right\} \Rightarrow \left.\begin{array}{l} A=0=C=E \\ B=-\dfrac{5}{9} \\ D=\dfrac{5}{9} \\ F=\dfrac{8}{3} \end{array}\right\} \quad \ldots (2)$$

Thus, $$\bar{x}(s) = -\frac{5}{9}\frac{1}{(s^2+1)} + \frac{5}{9}\frac{1}{(s^2+4)} + \frac{8}{3}\frac{1}{(s^2+4)^2} \quad \ldots (3)$$

Taking inverse laplace transform on both sides,

$$x(t) = -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}\frac{1}{2.2^3}(\sin 2t - 2t\cos 2t)$$

$$\left[\text{using } \; L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}(\sin at - at\cos at) \quad \text{where} \quad a = 2 \text{ in this case}\right]$$

$$x(t) = -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{6}(\sin 2t - 2t\cos 2t)$$

$$x(t) = -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{1}{3}t\cos 2t. \text{ Hence the result.}$$

Example 26: Solve $\dfrac{d^2x}{dt^2} + 9x = \cos 2t$ if $x(0) = 1$ and $x(\pi/2) = -1$. [UPTech 2002; Raipur 2004]

Solution: Taking laplace on both sides of the given equation,

$$\left[s^2 \cdot \bar{x}(s) - sx(0) - x'(0)\right] + 9\bar{x}(s) = \frac{s}{s^2+4}$$

i.e. $$s^2\bar{x}(s) - s.1 - a + 9\bar{x}(s) = \frac{s}{s^2+4}, \text{ taking } \quad x'(0) = a, \text{ say}$$

$$(s^2+9)\bar{x} = \frac{s}{s^2+4} + s + a$$

or $$\bar{x}(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{a}{s^2+9}$$

$$\bar{x}(s) = \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{a}{s^2+9} \text{ (By partial fractions)} \quad \ldots (1)$$

Taking inverse laplace transforms on both sides

$$x(t) = \frac{1}{5}[\cos 2t - \cos 3t] + \cos 3t + \frac{a}{3}\sin 3t \quad \ldots (2)$$

At $t = \frac{\pi}{2}$, $-1 = \frac{1}{5}[-1-0] + 0 + \frac{a}{3}(-1)$ or $a = \frac{12}{5}$ … (3)

$$\therefore \qquad x = \frac{1}{5}(\cos 2t + 4\cos 3t + 4\sin 3t)$$

Example 27: Solve $\boldsymbol{\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(0 = 1)}$.

Solution: Taking laplace on both sides,

$$(s\bar{y}(s) - y(0)) + 2\bar{y}(s) + \frac{\bar{y}(s)}{s} = \frac{1}{s^2+1}, \quad \text{using} \quad \left[L\int_0^t f(t)\,dt\right] = \frac{\bar{f}(s)}{s}$$

On using the given condition,

$$\left(s + 2 + \frac{1}{s}\right)\bar{y}(s) = \frac{s^2+2}{(s^2+1)} \quad \text{or} \quad \bar{y}(s) = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

By partial fractions, $\frac{s(s^2+2)}{(s+1)^2(s^2+1)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{Cs+D}{(s^2+1)}$

i.e. $s(s^2 + 2) = A(s + 1)(s^2 + 1) + B(s^2 + 1) + (Cs + D)(s + 1)^2$

On comparing the coefficients of equal powers of 's' and solving for A, B, C, D.

$$A = 1, \quad B = -\frac{3}{2}, \quad C = 0, \quad D = \frac{1}{2}$$

$$\therefore \qquad \bar{y}(s) = \frac{1}{(s+1)} - \frac{3}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s^2+1}$$

Taking inverses, $y(t) = e^{-t} + \frac{3}{2}te^{-t} + \frac{1}{2}\sin t$

Example 28: Solve $\boldsymbol{y'(t) = t + \int_0^t y(t-u)\cos u\,du}$ **if** $\boldsymbol{y(0) = 4}$.

Solution: Taking laplace of each individual term on both sides of the given equation,

$$L(y'(t)) = L(t) + L\left[\int_0^t y(t-u)\cos u\,du\right]$$

or $$(s\bar{y}(s) - y(0)) = \frac{1}{s^2} + L[y \otimes \cos t]; \quad \text{using} \quad def^n.\ f \otimes g = \int_0^t f(u)g(t-u)\,du$$

$$s\bar{y}(s) - 4 = \frac{1}{s^2} + \bar{y}(s)\frac{s}{s^2+1}; \quad L(f \otimes g) = \bar{f}(s)\bar{g}(s)$$

implying $\left(s - \frac{s}{s^2+1}\right)\bar{y}(s) = \frac{1}{s^2} + 4$ or $\bar{y}(s) = \frac{(4s^2+1)(s^2+1)}{s^2\ s^3} = \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}$

Taking inverses, $y(t) = 4 + \frac{5}{2}t^2 + \frac{1}{24}t^4$

Example 29: Show that the solution of the equation $y(t) = \phi(t) + \int_0^t y(u)\, g(t-u)\, du$ can be represented as $y = L^{-1}\left[\frac{\bar{\phi}(s)}{1-\bar{g}(s)}\right]$, where $\bar{\phi}(s) = L(\phi(t))$ and $\bar{g}(s) = L(g(t))$. Hence, solve $y(t) = \frac{t^2}{2} - \int_0^t u\, y(t-u)\, du.$

Solution: Laplace transform of the given integral equation results in

$$\bar{y}(s) = \bar{\phi}(s) + L\left(\int_0^t y(u)\, g(t-u)\, du\right) = \bar{\phi}(s) + L(y \otimes g) = \bar{\phi}(s) + \bar{y}(s)\bar{g}(s)$$

On simplification, $(1-\bar{g}(s))\bar{y}(s) = \bar{\phi}(s)$ or $\bar{y}(s) = \left[\frac{\bar{\phi}(s)}{1-\bar{g}(s)}\right]$

Implying, $$y(t) = L^{-1}\left[\frac{\bar{\phi}(s)}{1-\bar{g}(s)}\right]$$

or $$y(t) = L^{-1}\left[\bar{\phi}(s)\frac{1}{1-\bar{g}(s)}\right] = \phi(t) \otimes \psi(t), \quad \text{where} \quad L^{-1}\left[\frac{1}{1-\bar{g}(s)}\right] = \psi(t)$$

$$= \int_0^t \phi(u)\,\psi(t-u)\, du$$

Now, for hence part

$$\bar{y}(s) = \frac{1}{s^3} - L(t \otimes y) = \frac{1}{s^3} - \frac{1}{s^2}\bar{y}(s)$$

or $$\left(1 + \frac{1}{s^2}\right)\bar{y}(s) = \frac{1}{s^3} \text{ or } \bar{y}(s) = \frac{1}{s(s^2+1)} \text{ or } \bar{y}(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

Taking inverses, $y = (1 - \cos t)$.

Example 30: Solve the equation $t\frac{d^2x}{dt^2} + \frac{dx}{dt} + t\,x = 0,\ x(0) = 1,\ x'(0) = 0.$

[NIT Kurukshetra, 2006]

Solution: Taking laplace of each individual term on both sides of the given equation,

$$L\left(t\frac{d^2x}{dt^2}\right) + L\left(\frac{dx}{dt}\right) + L(t\ \ x) = 0 \qquad \dots (1)$$

$$\Rightarrow \qquad -\frac{d}{ds}\left(s^2\,\bar{x}(s) - s\,x(0) - x'(0)\right) + \left(s\,\bar{x}(s) - x(0)\right) - \frac{d}{ds}\left(\bar{x}(s)\right) = 0$$

$$\left(\text{using formula,} \quad L\left(t\,f(t)\right) = -\frac{d}{ds}\,\bar{f}(s)\right)$$

$$\Rightarrow \qquad \frac{d}{ds}\left(s^2\bar{x} - s - 0\right) - \left(s\bar{x} - 1\right) + \frac{d}{ds}\bar{x} = 0, \quad \text{(Taking } x(0) = 1, \text{ and } x'(0) = a\text{(say))}$$

$$\Rightarrow \qquad \left(s^2\frac{d\bar{x}}{ds} + 2s\bar{x} - 1\right) - \left(s\bar{x} - 1\right) + \frac{d\bar{x}}{ds} = 0$$

$$\left(s^2 + 1\right)\frac{d\bar{x}}{ds} + s\bar{x} = 0 \qquad \Rightarrow \qquad \frac{d\bar{x}}{\bar{x}} + \frac{1}{2}\left(\frac{2s}{s^2 + 1}\right)ds = 0$$

On integrating, $\log\bar{x} + \log\left(s^2 + 1\right)^{1/2} = \log C \quad \Rightarrow \quad \bar{x} = \dfrac{C}{\sqrt{s^2 + 1}}$ ... (2)

Taking inverse laplace transforms,

$x(t) = CJ_0(t)$ (It is a standard laplace of Bessel Function) ...(3)

On using given condition, $x(0) = 1$; $\quad 1 = CJ_0(0)$ or $C = 1$ as $J_0(0) = 1$.

Hence $\qquad x(t) = J_0(t)$

Example 31: Solve the equation $\boldsymbol{x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0; \quad y(0) = 2, \;\; y'(0) = 0.}$

[NIT Kurukshetra, 2010]

Solution: Taking inverse laplace on both sides of the given equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0, \qquad \text{... (1)}$$

We get, $-\dfrac{d}{ds}\left[s^2\,\bar{y} - s\,y(0) - y'(0)\right] + \left(s\,\bar{y} - y(0)\right) - \dfrac{d}{ds}\left(\bar{y}\right) = 0$

$$-\frac{d}{ds}\left(s^2\,\bar{y} - 2s\right) + \left(s\,\bar{y} - 2\right) - \frac{d\bar{y}}{ds} = 0, \text{ using the conditions } y(0) = 2,\ y'(0) = 0$$

implying $\qquad \dfrac{d\bar{y}}{ds} + \left(\dfrac{s}{s^2 + 1}\right)\bar{y} = 0 \quad \text{or} \quad \dfrac{d\bar{y}}{\bar{y}} + \dfrac{1}{2}\left(\dfrac{2s}{2\left(s^2 + 1\right)}\right)ds = 0$

On integration, we get

$$\log\bar{y} + \log\left(s^2 + 1\right)^{1/2} = \log C \quad \text{i.e.} \quad \bar{y} = \frac{C}{\sqrt{s^2 + 1}} \qquad \text{... (2)}$$

Taking Inverse laplace on both sides of equation (2), we get $y(x) = CJ_0(x)$,

(Since $L^{-1}\left(\dfrac{1}{\sqrt{s^2 + 1}}\right)$ is a standard transform of Bessel Function $J_0(x)$

Whence the result.

Example 32: Solve the equation $\frac{d^2x}{dt^2} - t\frac{dx}{dt} + x = 1,\ x(0) = 1,\ x'(0) = 2$

Solution: Taking Laplace Transform of an each individual terms of the given equation,

$$L\left(\frac{d^2x}{dt^2}\right) - L\left(t\frac{dx}{dt}\right) + L(x) = L(1) \qquad \dots (1)$$

$$\Rightarrow \quad \left[s^2\bar{x}(s) - sx(0) - x'(0)\right] + \frac{d}{ds}\left[s\bar{x}(s) - x(0)\right] + \bar{x}(s) = \frac{1}{s}$$

$$\Rightarrow \quad \left[s^2\bar{x}(s) - s - 2\right] + \frac{d}{ds}\left[s\bar{x}(s) - 1\right] + \bar{x}(s) = \frac{1}{s}$$

$$\Rightarrow \quad s^2\bar{x}(s) - s - 2 + s\frac{d\bar{x}}{ds} + \bar{x} + \bar{x} = \frac{1}{s}$$

$$\Rightarrow \quad s\frac{d\bar{x}}{ds} + \left(s^2 + 2\right)\bar{x} = \left(\frac{1}{s} + s + 2\right)$$

$$\Rightarrow \quad \frac{d\bar{x}}{ds} + \left(\frac{s^2+2}{s}\right)\bar{x} = \left(\frac{1}{s^2} + 1 + \frac{2}{s}\right) \qquad \dots (2)$$

Now this equation can be treated as Leibnitz linear differential equation in $\bar{x}(s)$ and s.

Here, $\quad P = \left(\frac{s^2+2}{s}\right), \quad Q = \left(1 + \frac{2}{s} + \frac{1}{s^2}\right)$

$$\therefore \quad I.F. = e^{\int P\,ds} = e^{\int\left(\frac{s^2+2}{s}\right)ds} = e^{\frac{s^2}{2}+2\log s} = e^{\frac{s^2}{2}}e^{\log s^2} = e^{\frac{s^2}{2}}\cdot s^2 \qquad \dots (3)$$

Now for solution, $\quad \bar{x}(s).I.F. = \int (I.F.)\,Q\,ds + c \qquad \dots (4)$

$$= \int\left(e^{\frac{s^2}{2}}s^2\right)\left(1 + \frac{2}{s} + \frac{1}{s^2}\right)ds + c$$

$$= \left(\int e^{\frac{s^2}{2}}s^2 ds + \int e^{\frac{s^2}{2}}\cdot 2s\,ds + \int e^{\frac{s^2}{2}}ds\right) + c$$

$$= \int\left(se^{\frac{s^2}{2}}\right)sds + 2\int\left(se^{\frac{s^2}{2}}\right)ds + \int e^{\frac{s^2}{2}}ds + c$$

$$= \int d\left(e^{\frac{s^2}{2}}\right)sds + 2\int\left(de^{\frac{s^2}{2}}\right)ds + \int e^{\frac{s^2}{2}}ds + c$$

$$= \left[se^{\frac{s^2}{2}} - \int e^{\frac{s^2}{2}}1ds\right] + 2e^{\frac{s^2}{2}} + \int e^{\frac{s^2}{2}}ds + c$$

$$\bar{x}(s)\cdot e^{\frac{s^2}{2}}s^2 = (s+2)e^{\frac{s^2}{2}} + c \quad \text{or} \quad \bar{x} = \frac{1}{s} + \frac{2}{s^2} + \frac{ce^{-\frac{s^2}{2}}}{s^2} \qquad \dots (5)$$

Taking Inverse Laplace Transforms on both sides, we get

$$x(t) = 1 + 2t + c.F(t), \text{ where } F(t) \text{ is some function of } t. \qquad \dots (6)$$

On using boundary conditions,

When $x(0) = 1$, $x(0) = 1 + 2.0 + c.F(0)$ i.e. $c.F(0) = 0$ … (7)

when $x'(0) = 2$, $c.F'(0) = 0$ … (8)

(7) & (8) together imply $c = 0$

Whence $x(t) = 1 + 2t$.

Example 33: $t = y'' + 2y' + ty = \sin t$, $y(0) = 1$ $y'(0) = 0$

[Punjab Univ., 2003; NIT Kurukshetra, 2005]

Solution: $L(ty'') + 2L(y') + L(ty) = L(\sin t)$,

$$-\frac{d}{ds}\left(s^2\bar{y}(s) - sy(0) - y'(0)\right) + 2s\bar{y}(s) - 2y(0) - \frac{d}{ds}\bar{y}(s) = \frac{1}{s^2+1}$$

$$-2s\bar{y}(s) - s^2\frac{d}{ds}\bar{y}(s) + y(0) + 2s\bar{y}(s) - 2y(0) - \frac{d}{ds}\bar{y}(s) = \frac{1}{s^2+1}$$

$$-\left(s^2+1\right)\frac{d}{ds}\bar{y}(s) - y(0) = \frac{1}{s^2+1},$$

$$-\left(s^2+1\right)\frac{d}{ds}\bar{y}(s) = \frac{1}{s^2+1} + 1, \quad y(0) = 0$$

$$-\frac{d}{ds}\bar{y}(s) = \frac{1}{s^2+1} + \frac{1}{(s^2+1)^2} \quad \dots (1)$$

Taking Inverse Laplace on both sides of (1),

$$ty(t) = \sin t + \frac{1}{2}(\sin t - t\cos t), \text{ using } L^{-1}\frac{1}{(s^2+1)^2} = \frac{1}{2a^3}(\sin at - at\cos at) \quad \dots (2)$$

$$y(t) = \frac{3}{2}\frac{\sin t}{t} - \frac{\cos t}{2}$$

Alternately: $$\bar{y}(s) = -\int\frac{1}{s^2+1}ds - \int\frac{1}{(s^2+1)^2}ds = -\tan^{-1}s - \int\frac{1}{(s^2+1)^2}ds \quad \dots (3)$$

Now for $\int\frac{1}{\left(s^2+1\right)^2}ds = I$, take $s = \tan\theta$ so that $ds = \sec^2\theta$

Implying $$I = \int\frac{\sec^2\theta}{\left(1+\tan^2\theta\right)^2}d\theta = \int\cos^2\theta\, d\theta = \int\frac{1+\cos 2\theta}{2}d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}$$

$$I = \frac{\tan^{-1}s}{2} + \frac{2s}{4\left(s^2+1\right)}, \quad \text{as} \quad \sin 2\theta = \frac{2\tan\theta}{\left(1+\tan^2\theta\right)} \quad \dots (4)$$

Putting (4) into (3), $\bar{y}(s) = -\tan^{-1}s - \frac{1}{2}\tan^{-1}s - \frac{s}{2(s^2+1)}$

$$\bar{y}(s) = -\frac{3}{2}\tan^{-1}s - \frac{1}{2}\frac{s}{s^2+1}$$

$$\Rightarrow \quad y(t) = -\frac{3}{2}L^{-1}\left(\tan^{-1}s\right) - \frac{1}{2}L^{-1}\frac{s}{\left(s^2+1\right)}$$

$$\Rightarrow \qquad y(t) = -\frac{3}{2}L^{-1}(\tan^{-1} s) - \frac{1}{2}\cos t \qquad \ldots (5)$$

Now let $g(t) = L^{-1}(\tan^{-1}s)$ so that $L\big(t\,g(t)\big) = -\frac{d}{ds}\tan^{-1} s = -\frac{1}{1+s^2}$ …(6)

Taking Inverse Laplace on both sides,

$$t\,g(t) = -\sin t \quad \text{or} \quad g(t) = -\frac{\sin t}{t} \qquad \ldots (7)$$

Putting (7) in (5), $f(t) = \frac{-3}{2}\left[\frac{-\sin t}{t}\right] - \frac{\cos t}{2}$

$$f(t) = \frac{3}{2}\frac{\sin t}{t} - \frac{\cos t}{2}$$

Example 34: A voltage Ee^{-at} is applied $at\ t = 0$ to a circuit of inductance, L and Resistance R. Show by the transformation method that current at any time t is $\frac{E}{R - aL}\left[e^{-at} - e^{\frac{-R}{L}t}\right]$.

[VTU, 2000]

Solution: Equation governing the current flow in L-R circuit is

$$L\frac{di}{dt} + Ri = Ee^{-at} \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$$

Taking laplace on both sides.

$$\left[s\bar{i}(s) - i(0)\right] + \frac{R}{L}\bar{i}(s) = \frac{E}{L}\frac{1}{(s+a)}$$

or

$$\bar{i}(s) = \frac{E}{L}\frac{1}{(s+a)\left(s+\frac{R}{L}\right)} = \frac{E}{L}\left(\frac{L}{R-aL}\frac{L}{(s+a)} - \frac{L}{RaL}\frac{L}{\left(s+\frac{R}{L}\right)}\right)$$

Taking inverse laplace transform on both sides, $i(t) = \frac{E}{R-aL}\left[e^{-at} - e^{-\frac{R}{L}t}\right]$

Example 35: Use laplace transform method to obtain the charge at any instant of a capacitor which is discharged in R–C–L circuit, after the switch is closed if R = 2.25 ohms, self inductance L = 1 Henry, capacitance, C = 2 farads, and the capacitor has initially a charge of 100 coulombs. Initially the switch is open and therefore, no current is flowing.

Solution: The desired equation in $L - C - R$ circuit is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E. \qquad \ldots (1)$$

Here given, $L = 1$, $R = 9/4$, $C = 2$, $q(0) = 100$, $q'(0) = \left(\frac{dq}{dt}\right)_{t=0} = 0$, $E = 0$

With above values, equation (1) becomes

$$\frac{d^2q}{dt^2}+\frac{9}{4}\frac{dq}{dt}+\frac{q}{2}=0 \qquad \ldots (2)$$

Taking laplace on both sides of the above equation

$$\left[s^2\bar{q}(s)-sq(0)-q'(0)\right]+\frac{9}{4}\left[s\bar{q}(s)-q(0)\right]+\frac{1}{2}\bar{q}(s)=0$$

i.e. $$\left[s^2+\frac{9}{4}s+\frac{1}{2}\right]\bar{q}(s)-100s-\frac{9}{4}\times 100=0$$

i.e. $$\bar{q}(s)=\frac{100(4s+9)}{(4s^2+9s+2)}=\frac{100}{7}\left[\frac{32}{(4s+1)}-\frac{1}{(s+2)}\right] \quad \text{(By partial fractions)}$$

$$\bar{q}(s)=\frac{100}{7}\left[8\cdot\frac{1}{\left(s+\frac{1}{4}\right)}-\frac{1}{s+2}\right] \qquad \ldots (3)$$

Taking inverse laplace transforms on both sides, $q(t)=\dfrac{100}{7}\left(8e^{\frac{-t}{4}}-e^{-2t}\right)$

Example 36: Alternating voltage 200 sin(100*t*) is applied at *t* = 0 to a circuit with an inductance 50 mH (millihenry), Capacitance 2000μ*f* (micro farad) and resistance 10 Ω (ohms). Find the current *i* at any time *t* seconds if the initial current *i* and charge *q* are zero.

Solution: Equation of current flow in $R-L-C$ circuit is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E, \quad \text{where} \quad \frac{dq}{dt}=i \qquad \ldots (1)$$

Here we are given $R = 10$ *ohms*

$L = 50 \times 10^{-3}$ Henrys

$C = 2000 \times 10^{-6}$ Farads

$E = 200 \sin(100t)$

With above values, equation (1) becomes

$$50\times 10^{-3}\frac{d^2q}{dt^2}+10\frac{dq}{dt}+\frac{q}{2000\times 10^{-6}}=200\sin(100t)$$

or $$\frac{d^2q}{dt^2}+200\frac{dq}{dt}+10^4q=4000\sin(100t) \qquad \ldots (2)$$

Taking Laplace on both sides,

$$\left[s^2\bar{q}(s)-sq(0)-q'(0)\right]+200\left[s\bar{q}(s)-q(0)\right]+10^4\bar{q}(s)=4000\frac{100}{s^2+100^2}$$

$$\left[s^2+200s+10^4\right]\bar{q}(s)=400000\frac{1}{(s^2+100^2)}$$

or $$\bar{q}(s)=4\times 10^5\frac{1}{(s^2+100^2)(s+100)^2}=(2\times 10^3)\left[\frac{1}{(s^2+100^2)}-\frac{1}{(s+100)^2}\right] \qquad \ldots (3)$$

Taking Inverse Laplace Transform,

$$q(t)=(2\times 10^3)\left[100\sin(100t)-t\,e^{-100t}\right] \qquad \ldots (4)$$

Also $$\frac{dq}{dt} = i = (2 \times 10^3)\left[\cos(100t) - e^{-100t}(1 - 100t)\right] \quad \ldots (5)$$

Example 37: Obtain the equation for the forced oscillations of mass m attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is k, when the driving force is $F_0 \sin at$. Solve this equation (Using the Laplace Transforms) when $a^2 \neq k/m$, given that initially velocity and displacement (from equilibrium position) are zero.

Solution: The problem lies under forced oscillations *without damping*.

(If the point of support of the spring is also vibrating with some external periodic force, then the resulting motion is called forced oscillatory motion, otherwise the motion is termed as forced oscillations without damping)

Taking the external periodic force to be F_0 sin at, the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(e + x) + F_0 \sin at \quad \ldots (1)$$

where,

x is the length of the stretched portion of the spring (*displacement*) after time t,
e is the elongation produced in the spring by the mass m,
k is the restoring force per unit stretch of the spring due to elasticity;
a is any arbitrary constant and p is any scalar.

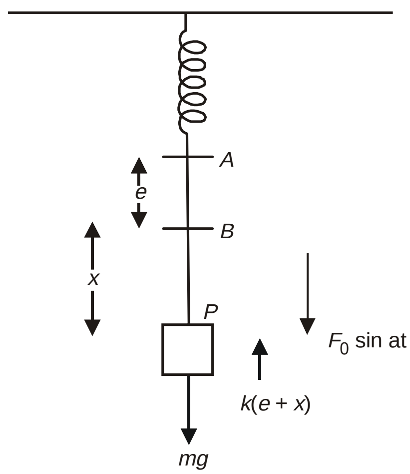

Fig. 12.4

In this particular problem, tension $mg = ke$, therefore equation (1), becomes

$$m\frac{d^2x}{dt^2} = -kx + F_0 \sin at \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_0}{m}\sin at$$

$$\frac{d^2x}{dt^2} + n^2x = \frac{F_0}{m}\sin at \quad \ldots (2)$$

Taking laplace on both sides of (2), we get

$$\left[s^2\bar{x}(s) - sx(0) - x'(0)\right] + n^2\bar{x}(s) = \frac{F_0}{m}\frac{a}{s^2 + a^2}$$

$$\left(s^2 + \frac{k}{m}\right)\bar{x}(s) = \frac{F_0}{m}\frac{a}{s^2 + a^2}, \text{ using } x(0) = 0, \quad x'(0) = 0$$

$$\bar{x}(s) = \frac{aF_0}{m} \cdot \frac{1}{\left(s^2 + a^2\right)\left(s^2 + \left(\sqrt{\frac{k}{m}}\right)^2\right)} = \frac{aF_0}{m}\frac{1}{(s^2 + a^2)(s^2 + n^2)}, \mu = \sqrt{\frac{k}{m}} \qquad \ldots (3)$$

Now by partial fractions, $\dfrac{1}{\left(s^2 + a^2\right)\left(s^2 + n^2\right)} = \dfrac{As + B}{\left(s^2 + a^2\right)} + \dfrac{Cs + D}{\left(s^2 + n^2\right)}$

$$1 = (As + B)\left(s^2 + n^2\right) + (Cs + D)\left(s^2 + a^2\right)$$

$$\Rightarrow \qquad 1 = s^3(A + C) + s^2(B + D) + s\left(n^2A + a^2C\right) + \left(n^2B + a^2D\right)$$

On solving for A, B, C, D; we get $A = 0, \quad B = \dfrac{1}{n^2 - a^2}, \quad C = 0, \quad D = -\dfrac{1}{\left(n^2 - a^2\right)}$

Hence $\qquad \bar{x}(s) = \dfrac{aF_0}{m\left(n^2 - a^2\right)}\left[\dfrac{1}{\left(s^2 + a^2\right)} - \dfrac{1}{\left(s^2 + n^2\right)}\right]$

Taking Laplace Inverse on both sides,

$$x(t) = \frac{aF_0}{m\left(n^2 - a^2\right)}\left[\frac{1}{a} \sin at - n \sin nt\right],$$

$$= \frac{F_0}{mn\left(n^2 - a^2\right)}\left[n \sin at - a \sin nt\right], \qquad \text{where } n = \sqrt{\frac{k}{m}} \text{ and } \frac{k}{m} \neq a^2$$

ASSIGNMENT 3

Solve the following Linear Differential equations:

1. $\dfrac{dx}{dt} + x = \sin wt, \quad x(0) = 2$ [KUK, 2001]
2. $\dfrac{d^2y}{dx^2} + w^2y = 0,$ where $y(0) = A, \quad \left(\dfrac{dy}{dx}\right)_{x=0} = B$
3. $y'' - 3y' + 2y = 4t + e^{3t}$, when $y(0) = 1$ and $y'(0) = -1$
   [Andhra, 2000; NIT Kurukshetra, 2008]
4. $(D^2 + n^2)x = a\sin(nt + \alpha)$, $x = Dx = 0$ at $t = 0$
5. $y''' + 2y'' - y' - 2y = 0$, given $y(0) = y'(0) = 0$ and $y''(0) = 6$
6. $(D^3 - 3D^2 + 3D - 1)y = t^2e^t$, given $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$
7. $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = e^t$ with $x = 2$, $x = -1$ at $t = 0$ [KUK, 2004]
8. $y^{ir}(t) + 2y'' + y = \sin t$, when $y(0) = y'(0) = y''(0) = y'''(0) = 0$

9. $\dfrac{d^2x}{dt^2}+2\dfrac{dy}{dx}+5y=e^t\sin t,$ where $y(0)=0,\quad y'(0)=1$

[KUK, 2002, 2004; Bombay, 2005; PTU, 2005]

10. $ty'' + 2y' + ty = \cos t,$ given that $y(0) = 1$

11. $ty'' + (1 - 2t)y' - 2y = 0,$ when $y(0) = 1, y'(0) = 2$ [Punjabi Univ. 2003]

12. $f(t)=\cos t+\int_0^t e^{-u} f(t-u)\,du$

12.10: APPLICATIONS TO SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Example 38: Solve the simultaneous equations

$$\frac{dx}{dt}-y=e^t,\quad \frac{dy}{dt}+x=\sin t,\ \text{given } x(0)=1,\quad y(0)=0$$ **[Delhi, 2002]**

Solution: Taking Laplace Transforms of the given simultaneous equations,

$$\left[s\bar{x}(s)-x(0)\right]-\bar{y}(s)=\frac{1}{s-1}\quad\text{or}\quad s\bar{x}(s)-1-\bar{y}(s)=\frac{1}{s-1}\quad\text{or}\quad s\bar{x}-\bar{y}=\frac{s}{s-1}\qquad \ldots(1)$$

and $$\left[s\bar{y}(s)-y(0)\right]+\bar{x}(s)=\frac{1}{s^2+1}\quad\text{or}\quad \bar{x}+s\bar{y}=\frac{1}{s^2+1}\qquad \ldots(2)$$

On solving (1) and (2) for $\bar{x}(s)$ and $\bar{y}(s)$,

$$\bar{x}(s)=\frac{s^2}{(s-1)(s^2+1)}+\frac{1}{(s^2+1)^2}$$

$$=\frac{1}{2}\left(\frac{1}{s-1}+\frac{1}{s^2+1}+\frac{s}{s^2+1}\right)+\frac{1}{(s^2+1)^2}\qquad \ldots(3)$$

and $$\bar{y}(s)=\frac{s}{(s^2+1)^2}-\frac{s}{(s-1)(s^2+1)}$$

$$=\frac{s}{(s^2+1)^2}=\frac{1}{2}\left(\frac{1}{s-1}+\frac{1}{s^2+1}-\frac{s}{s^2+1}\right)\qquad \ldots(4)$$

Taking Inverse Laplace Transforms on both sides of equations (3) and (4), we get

$$x=\frac{1}{2}L^{-1}\left(\frac{1}{s-1}+\frac{1}{s^2+1}+\frac{s}{s^2+1}\right)+L^{-1}\left(\frac{1}{(s^2+1)^2}\right)$$

$$=\frac{1}{2}\left(e^t+\sin t+\cos t\right)+\frac{1}{2}\left(\sin t-t\cos t\right)$$

$$=\frac{1}{2}\left(e^t+2\sin t+\cos t-t\cos t\right)\qquad \ldots(5)$$

and
$$y = L^{-1}\left(\frac{s}{(s^2+1)^2}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{s-1} + \frac{1}{s^2+1} - \frac{s}{s^2+1}\right)$$
$$= \frac{1}{2}t\sin t - \frac{1}{2}\left(e^t + \sin t - \cos t\right)$$
$$= \frac{1}{2}\left(t\sin t - \sin t - e^t - \cos t\right) \qquad \dots (6)$$

Hence,
$$\left.\begin{aligned} x &= \frac{1}{2}\left(e^t + 2\sin t + \cos t - t\cos t\right) \\ y &= \frac{1}{2}\left(t\sin t - \sin t - e^t - \cos t\right) \end{aligned}\right\}$$

Example 39: The coordinates (x, y) of a particle moving along a plane curve at any time t are given by $\frac{dy}{dt} + 2x = \sin 2t$, $\frac{dx}{dt} - 2y = \cos t$, $(t > 0)$. If at $t = 0$, $x = 1$ and $y = 0$, show by using transforms that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

[UPTech, 2003]

Solution: On taking Laplace on both sides of the given simultaneous equations,
$$\left[s\bar{y} = y(0)\right] + 2\bar{x} = \frac{2}{s^2+4} \quad \text{or} \quad 2\bar{x} + s\bar{y} = \frac{2}{s^2+4}, \quad \text{using } (y(0) = 0) \qquad \dots (1)$$

Similarly
$$\left[s\bar{x} = x(0)\right] + 2\bar{y} = \frac{2}{s^2+4} \quad \text{or} \quad s\bar{x} + 2\bar{y} = \frac{s}{s^2+4} + 1, \quad \text{using } (x(0) = 1) \qquad \dots (2)$$

For solving $\bar{x}$ and $\bar{y}$, multiply throughout (1) by s, (2) by 2 and take the difference of the two,
$$\bar{y} = -\frac{2}{s^2+4} \qquad \text{or} \qquad y(t) = -\sin 2t \qquad \dots (3)$$

Further
$$\bar{x} = \frac{1}{2}\left(\frac{2}{s^2+4} + 2\cdot\frac{s}{s^2+4}\right) \quad \text{or} \quad x(t) = \frac{1}{2}(\sin 2t + \cos 2t) \qquad \dots (4)$$

Equation (4) implies, $2x = \sin 2t + 2\cos 2t$ i.e. $2x + y = 2\cos 2t$, using (3)
$$(2x + y)^2 = 4(1 - \sin^2 2t) \quad \text{or} \quad 4x^2 + y^2 + 4xy = 4 - 4y^2, \text{ using (3)}$$
$$4x^2 + 4xy + 5y^2 = 4$$

Hence the desired particle (x, y) moves along the curve $4x^2 + 4xy + 5y^2 = 4$

Example 40: Solve $\frac{d^2x}{dt^2} + \frac{dy}{dt} + \sin t = 0$, $\frac{dx}{dt} - \frac{d^2y}{dt^2} + \cos t = 0$ with the initial conditions

$x = 0$, $\frac{dx}{dt} = 1$; $y = 1$, $\frac{dy}{dt} = 0$ when $t = 0$.

Solution: On applying Laplace transform on each individual terms of the two given equations

$$L\left(\frac{d^2x}{dt^2}\right)+L\left(\frac{dy}{dt}\right)+L\sin(t)=0 \quad \Rightarrow \quad (s^2\bar{x}-1)+(s\bar{y}-1)+\frac{1}{(s^2+1)}=0 \qquad \ldots (1)$$

and
$$L\left(\frac{dx}{dt}\right)-L\left(\frac{d^2y}{dt^2}\right)+L(\cos t)=0 \quad \Rightarrow \quad s\bar{x}-(s^2\bar{y}-s)+\frac{s}{s^2+1}=0 \qquad \ldots (2)$$

On simplification of (1) and (2),

$$s^2\bar{x}+s\bar{y}=\frac{2s^2+1}{s^2+1} \qquad \ldots (3)$$

and
$$\bar{x}-s\bar{y}=-\frac{s^2+2}{s^2+1} \qquad \ldots (4)$$

Adding (3) and (4), we get

$$(s^2+1)\bar{x}=\frac{2s^2+1}{s^2+1}-\frac{s^2+2}{s^2+1}=\frac{s^2-1}{s^2+1} \quad \text{or} \quad \bar{x}=\frac{s^2-1}{(s^2+1)^2} \qquad \ldots (5)$$

Using (5) in (4), we get

$$s\bar{y}=\frac{s^2-1}{(s^2+1)^2}+\frac{s^2+2}{(s^2+1)}=\frac{s^2+1-2}{(s^2+1)^2}+\frac{s^2+1+1}{(s^2+1)}$$

$$=\frac{1}{(s^2+1)}=\frac{2}{(s^2+1)^2}+\frac{1}{(s^2+1)}$$

$$=\frac{2}{(s^2+1)}-\frac{2}{(s^2+1)^2}+1=\frac{2s^2}{(s^2+1)}+1$$

or
$$\bar{y}=\frac{1}{s}+\frac{2s}{(s^2+1)^2} \qquad \ldots (6)$$

Taking inverse of (5) and (6),

$$x=t\cos t \quad \text{and} \quad y=(1+t\sin t).$$

ASSIGNMENT 4

Solve the following simultaneous equations

1. $\dfrac{dx}{dt}+y=\sin t, \quad \dfrac{dy}{dx}+x=\cos t$ given $x=2,\ y=0$ when $t=0$.

[UPTech, 2004; Kerala, 2005]

2. $3\dfrac{dx}{dt}+\dfrac{dy}{dt}+2x=1, \quad \dfrac{dx}{dt}+4\dfrac{dy}{dt}+3y=0;$ given $x=3,\ y=0$ when $t=0$.

3. $\dfrac{dx}{dt}+5x-2y=t, \quad \dfrac{dy}{dt}+2x+y=0;$ being given that $x=y=0$ at $t=0$

4. The currents i_1 and i_2 in a mesh are given by the differential equations:

$\frac{di_1}{dt} - wi_2 = a\cos pt, \quad \frac{di_2}{dt} + wi_1 = a\sin pt$. Find the currents i_1 and i_2 by Laplace transforms if $i_1 = i_2 = 0$ at $t = 0$.

5. Small Oscillations of a certain system with two degrees of freedom are given by the equations $\left.\begin{array}{l} D^2x + 3x - 2y = 0 \\ D^2x + D^2y - 3x + 5y = 0 \end{array}\right\}$ where $x = y = 0$ and $\left.\begin{array}{l} Dx = 3 \\ Dy = 2 \end{array}\right\}$ when $t = 0$.

Find x and y when $t = \frac{1}{2}$.

12.11 LAPLACE TRANSFORMS OF SOME SPECIAL FUNCTIONS

Sr.No.	Function Name	$f(t)$	$\bar{f}(s)$
1.	Heaviside's Unit Step Function	$u(t-a) = \begin{cases} 0, & t < a \\ 1 & t \geq a \end{cases}$	$\frac{e^{-as}}{s}, \quad s > 0$
2.	Dirac Delta function	$\delta_\in = \begin{cases} \frac{1}{\in}, & 0 \leq t \leq \in \\ 0 & t > \in \end{cases}$	$\frac{1}{s\in}\left(1 - e^{-s\in}\right)$
3.	Periodic Function	$f(t) = f(t+T)$	$\frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$
4.	Sine Integral	$S_i(t) = \int_0^t \frac{\sin u}{u} du$	$\frac{1}{s}\tan^{-1}\frac{1}{s}$
5.	Cosine Integral	$C_i(t) = \int_t^\infty \frac{\cos u}{u} du$	$\frac{1}{2s}\log\left(s^2+1\right)$
6.	Error Function	$erf\left(\sqrt{t}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2} du$	$\frac{1}{s\sqrt{s+1}}$
7.	Complementary Error Fun.	$erf_c\left(\sqrt{t}\right) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2} du$	$\frac{1}{\left\{\left(\sqrt{s+1}\right)+1\right\}\sqrt{s+1}}$
8.	Exponential Fun. of Order Zero	$e_L(t) = \int_t^\infty \frac{e^{-u}}{u} du$	$\frac{1}{s}\log(s+1)$
9.	Bessel Function of Order Zero	$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} \ldots$	$\frac{1}{\sqrt{1+s^2}}$

A few of the above functions are taken up in the subsequent discussions.

1. **Unit Step Function (*Heaviside's Unit Function*):** In engineering, many times we come across such fractions of which inverse laplace is either very difficult or can not be obtained by the formulae discussed so far. To overcome such problem, Unit step function (Heaviside's Unit Function) has been introduced.

   The unit step function is defined as follows:

$$U_a(t) = u(t-a) = \begin{cases} 0 & for \quad t < a \\ 1 & for \quad t \geq a, \end{cases}$$

where a is always positive.

From the graph, it is apparent that function is zero for all values of t up to 'a' and after which it is unity. In physical problems, t is a time variable.

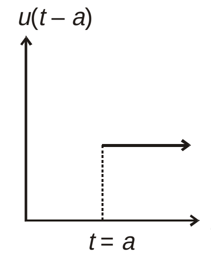

Fig. 12.5

As a particular case,
$$u(t) = 0, \quad t < 0$$
$$= 1, \quad t \geq 0$$

Observations: Generally the unit step function in mechanical engineering comes into picture as a force suddenly applied to a machine or a machine component, where as in electrical engineering it manifests as an electromotive force of a battery in circuit.

Transform of Unit Step Function:

$$L\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_0^a e^{-st}0\,dt + \int_a^\infty e^{-st}\,1\,dt = \left.\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s}$$

and in particular, when $a = 0$, $\quad L\{u(t-a)\} = \dfrac{1}{s}$

Second Shifting Theorem (t-shifting):

On the basis of the $u(t-a)$, we had the functions,
$$f(t)\,u(t-a) = 0, \quad \text{when} \quad t < a$$
$$= f(t), \quad \text{when} \quad t \geq a,$$

and the function $f(t-a)$ represents the graph of $f(t)$ displaced through a distance 'a' to the right i.e. $u_a(t)$ has a jump type of discontinuity at $t = a$.

If $\quad L(f(t)) = \bar{f}(s)$, then $\quad L\{f(t-a)u(t-a)\} = e^{-as}\bar{f}(s)$

$$L\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st}\{f(t-a)u(t-a)\}dt$$

$$= \int_0^\infty e^{-st}o\,dt + \int_0^\infty e^{-st}f(t-a)\cdot 1\,dt$$

$$= \int_0^\infty e^{-s(a+x)}f(x)\,dx \quad (\text{on taking } t-a = x)$$

$$= e^{-as}\int_0^\infty e^{-sx}f(x)\,dx = e^{-as}\bar{f}(s)$$

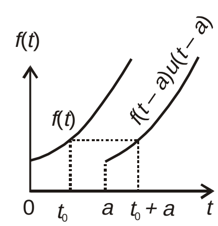

Fig. 12.6

The second shift theorem is also known as ***t–shifting*** i.e. t is replaced by $t-a$ in $f(t)$.

Example 41: Find Laplace Transform of

(*i*) $f(t) = (t-1)^2 u(t-1)$ **(*ii*)** $f(t) = e^{-t}\{1 - u(t-2)\}$

(*iii*) $f(t) = \sin t \cdot u(t-a)$ **(*iv*)** $f(t) = \begin{cases} t-1 & \text{for} \quad 1 < t < 2 \\ 3-t & \text{for} \quad 2 < t < 3 \end{cases}$

Solution:

(*i*) Given $f(t) = (t-1)^2 u(t-1)$ … (1)

On comparing the given function with $f(t-a)u(t-a)$,

We see that in this case $a = 1$, $f(t) = t^2$ and $\bar{f}(s) = L\big(f(t)\big) = L\big(t^2\big) = \frac{2}{s^3}$ … (2)

Consequently by second shift theorem,

$$L\left\{f(t-a)u(t-a)\right\} = e^{-as}\bar{f}(s) = e^{-s}\frac{2}{s^3} = \frac{2e^{-s}}{s^3}$$

(*ii*) Here in this case taking $f(t) = e^{-t}$, $\bar{f}(s) = \frac{1}{s+1}$ and then by second shift theorem

$$L\left(e^{-t}\left\{1 - u(t-2)\right\}\right) = L\left(e^{-t} - e^{-t}u(t-2)\right)$$

$$= L\left[e^{-t}\right] - L\left[\frac{e^{-(t-2)}}{e^2}u(t-2)\right]$$

$$= \frac{1}{s+1} - \frac{1}{e^2}e^{-2s}\frac{1}{s+1}$$

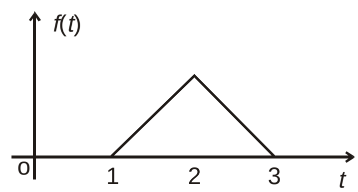

Fig. 12.7

Example 42: Find the inverse laplace transforms of the following:

(*i*)* $\frac{se^{-as}}{s^2 - w^2}$, $a > 0$ **(*ii*)**** $\frac{se^{-s/2} + \pi e^{-s}}{s^2 + \pi^2}$ **(*iii*)***** $\frac{e^{-as}}{s^2(s+b)}$, $a > 0$

(*iv*) $\frac{e^{-s}}{(s-1)(s-2)}$ **(*v*)** $\frac{e^{-s}}{(s+1)^3}$

[*KUK, 2002; **NIT Kurukshetra, ***KUK 2005; VTU 2000, 2003]

Solution:

(*i*) Let $\bar{f}(s) = \frac{s}{s^2 - w^2}$, then $f(t) = \cosh wt$

Now by second shifting theorem, $L^{-1}\left\{e^{-as}\bar{f}(s)\right\} = f(t-a)u(t-a)$

$$\Rightarrow \quad L^{-1}\left\{e^{-as}\frac{s}{s^2 - w^2}\right\} = \cosh w(t-a)u(t-a)$$

(*ii*) For $\bar{f}(s) = \frac{s}{s^2 + \pi^2}$, $f(t) = \cos \pi$

$$\bar{f}(s) = \frac{\pi}{s^2+\pi^2}, \quad f(t) = \sin \pi t$$

Also by second shifting theorem $L^{-1}\left\{e^{-as}\bar{f}(s)\right\} = f(t-a)u(t-a)$,

$$L^{-1}\left\{\frac{se^{-\frac{s}{2}}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-\frac{s}{2}}\frac{s}{s^2+\pi^2}\right\} + L^{-1}\left\{e^{-s}\frac{\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi\left(t-\frac{1}{2}\right)u\left(t-\frac{1}{2}\right) + \sin\pi(t-1)u(t-1)$$

$$= \cos\pi\left(t-\frac{1}{2}\right)u\left(t-\frac{1}{2}\right) + \sin\left(-\pi+\pi t\right)u(t-1)$$

$$= \cos\pi\left(t-\frac{1}{2}\right)u\left(t-\frac{1}{2}\right) - (\sin\pi t)u(t-1)$$

(*iii*) Given $\bar{F}(s) = \dfrac{e^{-as}}{s^2\left(s+b\right)} = e^{-as}\left[\dfrac{1}{b^2}\dfrac{1}{s+b} - \dfrac{1}{b^2}\dfrac{1}{s} + \dfrac{1}{b}\cdot\dfrac{1}{s^2}\right]$, (By partial fractions)

$$= \left[\frac{1}{b^2}e^{-as}\frac{1}{s+b} - \frac{1}{b^2}e^{-as}\frac{1}{s} + \frac{1}{b} \quad e^{-as}\frac{1}{s^2}\right]$$

Further $L^{-1}\left(\dfrac{1}{s+b}\right) = e^{-bt}, \quad L^{-1}\left(\dfrac{1}{s}\right) = 1, \quad L^{-1}\left(\dfrac{1}{s^2}\right) = t,$

Also by second shifting theorem, $L^{-1}\left(e^{-as}\bar{f}(s)\right) = f(t-a)u(t-a),$

$$L^{-1}\left(\frac{e^{-as}}{s^2\left(s+b\right)}\right) = L^{-1}\left(\frac{1}{b^2}e^{-as}\frac{1}{s+b}\right) + L^{-1}\left(-\frac{1}{b^2}e^{-as}\frac{1}{s}\right) + L^{-1}\left(\frac{1}{b}e^{-as}\frac{1}{s^2}\right)$$

$$= \frac{1}{b^2}e^{-b(t-a)}u(t-a) - \frac{1}{b^2}1\cdot u(t-a) + \frac{1}{b}(t-a)u(t-a)$$

$$= \frac{1}{b^2}\left\{e^{-b(t-a)} + b(t-a) - 1\right\}u(t-a)$$

(*iv*) $\bar{F}(s) = \dfrac{e^{-s}}{(s-1)(s-2)} = e^{-s}\left(\dfrac{1}{s-2} - \dfrac{1}{s-1}\right) = e^{-s}\dfrac{1}{s-2} - e^{-s}\dfrac{1}{s-1}$

$$\Rightarrow \qquad L^{-1}\left(\frac{e^{-s}}{(s+1)(s-2)}\right) = L^{-1}\left(e^{-s}\frac{1}{s-2}\right) - L^{-1}\left(e^{-s}\frac{1}{s-1}\right)$$

[By second shifting theorem]

$$= e^{2(t-1)}u(t-1) - e^{(t-1)}u(t-1)$$

$$= \left[e^{2(t-1)} - e^{(t-1)}\right] \cdot u(t-1)$$

(v) Here $\bar{F}(s) = \dfrac{e^{-s}}{(s+1)^3} = e^{-as}\bar{f}(s)$, where $\bar{f}(s) = \dfrac{1}{(s+1)^3}$

$$L^{-1}\left(\frac{e^{-s}}{(s+1)^3}\right) = \frac{1}{2}e^{-(t-1)}(t-1)^2\, u(t-1)$$

Example 43: Find the inverse laplace transform of $\dfrac{1}{s\left(1-e^{-s}\right)}$.

Solution: $L^{-1}\left[\dfrac{1}{s\left(1-e^{-s}\right)}\right] = L^{-1}\left[\dfrac{1}{s}\left(1-e^{-s}\right)^{-1}\right] = L^{-1}\left[\dfrac{1}{s}\left(1+e^{-s}+e^{-2s}+e^{-3s}+\ldots\right)\right]$

$$= L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(e^{-s}\frac{1}{s}\right) + L^{-1}\left(e^{-2s}\frac{1}{s}\right) + L^{-1}\left(e^{-3s}\frac{1}{s}\right) + \ldots$$

Comparable to $L^{-1}\left(e^{-as}\bar{f}(s)\right)$, where $\bar{f}(s) = \dfrac{1}{s}$ in each case.

$\Rightarrow$ $L^{-1}\left[\dfrac{1}{s}\left(1-e^{-s}\right)^{-1}\right] = 1 + u(t-1) + u(t-2) + u(t-3) + \ldots$

Thus, the function can be written as

$$f(t) = \begin{cases} 1, & 0 < t \le 1 \\ 2, & 1 < t \le 2 \\ 3, & 2 < t \le 3 \\ \ldots & \ldots\ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots\ldots \end{cases}$$

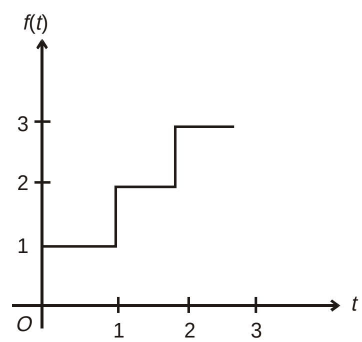

Fig. 12.8

Sometimes it is called as staircase function.

Example 44: In an electrical circuit with e.m.f. $E(t)$, resistance R and inductance L, the current i builds up at the rate given by $L\dfrac{di}{dt} + Ri = E(t)$. If the switch is connected at $t = 0$ and disconnected at $t = a$, find the current i at any instant.

Solution: Given $i = 0$ at $t = 0$ and $E(t) = \begin{cases} E, & \text{for } 0 < t < a \\ o, & \text{for } t > a \end{cases}$

On taking laplace of each term of the given equation, we get

$$L\left(s\,\bar{i}(s) - i(0)\right) + R\,\bar{i}(s) = \int_0^\infty e^{-st}E(t)\,dt \quad (L \text{ and } R \text{ are constants}) \qquad \ldots (1)$$

$$(Ls+R)\,\bar{i}(s)=\int_0^a e^{-st}E\,dt+\int_a^\infty e^{-st}0\,dt$$

$$(Ls+R)\,\bar{i}(s)=E\left.\frac{e^{-s\,t}}{-s}\right|_o^a=\frac{E}{s}\left(1-e^{-as}\right)$$

$$\Rightarrow \qquad \bar{i}(s)=E\left(\frac{1}{s(Ls+R)}-\frac{e^{-as}}{s(Ls+R)}\right) \qquad \dots (2)$$

Now taking inverse laplace on both sides, we get

$$i(t)=EL^{-1}\left(\frac{1}{s(Ls+R)}\right)-EL^{-1}\left(e^{-as}\frac{1}{s(Ls+R)}\right) \qquad \dots (3)$$

By partial Fractions, $\dfrac{1}{s(Ls+R)}=\left(\dfrac{1}{R}\dfrac{1}{s}-\dfrac{L}{R}\dfrac{1}{Ls+R}\right)$ … (4)

$\therefore$ Equation (3) becomes,

$$i(t)=EL^{-1}\left\{\frac{1}{R}\frac{1}{s}-\frac{L}{R}\frac{1}{Ls+R}\right\}-EL^{-1}\left\{e^{-as}\left(\frac{1}{R}\frac{1}{s}-\frac{L}{R}\frac{1}{Ls+R}\right)\right\}$$

$$i(t)=\frac{E}{R}\left[L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{1}{s+\frac{R}{L}}\right)\right]-\frac{E}{R}\left[L^{-1}\left(\frac{e^{-as}}{s}\right)-L^{-1}\left(e^{-as}\frac{1}{s+\frac{R}{L}}\right)\right]$$

$$i(t)=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)-\frac{E}{R}L^{-1}\left(\bar{F}(s)\right) \qquad \dots (5)$$

Now $L^{-1}\bar{F}(s)=L^{-1}\left(e^{-as}\bar{f}(s)\right)=f(t-a)\,u(t-a)$, (By second shifting property)

$$\therefore \quad L^{-1}\left[e^{-as}\frac{1}{s}\right]-L^{-1}\left(e^{-as}\frac{1}{s+\frac{R}{L}}\right)=1.u(t-a)-e^{-\frac{R}{L}(t-a)}u(t-a)=\left(1-e^{-\frac{R}{L}(t-a)}\right)u(t-a) \qquad \dots (6)$$

On putting (6) into (5), we get the value of the current

$$i(t)=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)-\frac{E}{R}\left(1-e^{-\frac{R}{L}(t-a)}\right)u(t-a) \qquad \dots (7)$$

Now, for $0<t<a$, $\quad i(t)=\dfrac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$ $\qquad \left[\because u(t-a)=0,\ 0<t<a;\ =1,\ t\ge a\right]$

for $t>a$, $\quad i(t)=\dfrac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)-\dfrac{E}{R}\left(1-e^{-\frac{R}{L}(t-a)}\right)=\dfrac{E}{R}e^{-\frac{R}{L}t}\left(e^{\frac{R}{L}a}-1\right)$

II Unit Impulse Function (Dirac Delta Funtion)

Impulse is considered as a force of very high magnitude applied for just an instant and the function representing the impulse is called Dirac–Delta Function.

Mathematically, it is the limiting form of the function

$$\delta_\in(t-a) = \begin{cases} \frac{1}{\in}, & a \le t \le a + \in \\ 0, & \text{otherwise,} \quad \text{as} \quad \in \to 0 \end{cases}$$

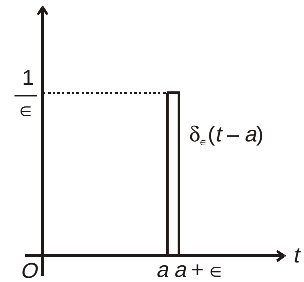

Fig. 12.9

It is apparent from the figure 12.9, that as $\in \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that the area of the strip is always unity.

The above fact can notionally be stated as $\left.\begin{matrix}\delta(t-a) = \infty, & \text{for} \quad t = a \\ o, & \text{for} \quad t \ne a\end{matrix}\right\}$ such that

$$\int_0^\infty \delta(t-a)\, dt = 1 \quad \text{for} \quad a \ge 0$$

Observation: In mechanics, the impulse of a force $f(t)$ over a time interval, say $a \le t \le (a + \in)$ is defined to be the integral of $f(t)$ from a to $(a + \in)$. The analog of an electric circuit is the integral of e.m.f. applied to the circuit, integrated from a to $(a + \in)$. Of particular practical interest is the case of very short ε with limit $\varepsilon \to 0$ i.e. impulse of a force acting for an instant.

Transform Of Unit Impulse Function:

$$L\left[\delta(t-a)\right] = \int_0^\infty e^{-st}\delta(t-a)\,dt = \int_0^a e^{-st} o\, dt + \int_a^{a+\in} e^{-st}\frac{1}{\in}dt + \int_{a+\in}^\infty e^{-st} o\, dt$$

$$= \frac{1}{\in}\int_a^{a+\in} e^{-st}dt = \frac{1}{\in}\left[\frac{e^{-st}}{-s}\right]_a^{a+\in} = \frac{1}{s\in}\left[-e^{-s(a+\in)} + e^{-as}\right] = \frac{e^{-as}}{s}\left[\frac{1-e^{-s\in}}{\in}\right]$$

Taking Limit as $\in \to o$,

$$L\left[\delta(t-a)\right] = \underset{\in \to o}{Lt}\, \frac{e^{-as}}{s}\left(\frac{1-e^{-s\in}}{\in}\right) = e^{-as}\, \underset{\in \to 0}{Lt} \cdot \frac{1-e^{-s\in}}{s\in} = e^{-as}$$

In particular, if $a = o$, $L[\delta(t)] = 1$.

Filtering of unit Impulse Function:

If $f(t)$ is continuous and integrable for all $t \ge 0$, then $\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$

Relation Bewteen $u'(t-a)$ and $\delta(t-a)$

$$L\left[u'(t-a)\right] = sL\left[u(t-a)\right] - u(-a) = s\frac{e^{-as}}{s} - o = e^{-as} = L\left[\delta(t-a)\right]$$

Example 45: The differential equation governing the flow of a current $i(t)$ in an LR circuit is given by

$$Ri + L\frac{di}{dt} = E\,\delta(t)$$

where $E\,\delta(t)$ is the impulse voltage and $i(0) = 0$. Find the current $i(t)$, $t > 0$.

Solution: On taking laplace of each individual term in the given equation,

$$R\,\bar{i}(s) + L\left[s\,\bar{i}(s) - i(o)\right] = E.1$$

implying $(R + Ls)\bar{i}(s) = E$ or $\bar{i}(s) = \dfrac{\frac{E}{L}}{s + \frac{R}{L}}$

On taking inverse, laplace, $i(t) = \dfrac{E}{L} e^{-\frac{R}{L}t}$ is the required expression for the current i.

Example 46: An impulsion voltage $E\delta(t)$ is applied to an $L\ C\ R$ circuit with zero initial condition. If $i(t)$ be the current at any subsequent time t, find the limit of $i(t)$ as $t \to 0$.

Solution: The governing equation for an $L\ C\ R$ circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_o^t i\,dt = E\,\delta(t)\ , \quad \text{where } i(t) = 0 \text{ when } t = 0$$

Taking Laplace of each individual term on both sides of the equation,

$$\left[s\,\bar{i}(s) - i(0)\right] + \frac{R}{L}\bar{i}(s) + \frac{1}{CL}\frac{\bar{i}(s)}{s} = \frac{E}{L}, \quad \text{where } i(0) = 0$$

Implying, $\left[s + \dfrac{R}{L} + \dfrac{1}{CL}\dfrac{1}{s}\right]\bar{i}(s) = \dfrac{E}{L}$ or $\left(s^2 + \dfrac{R}{L}s + \dfrac{1}{CL}\right)\bar{i}(s) = \dfrac{E}{L}s$

$$\bar{i}(s) = \frac{E}{L}\frac{s}{s^2 + \frac{R}{L}s + \frac{1}{CL}} = \frac{E}{L}\frac{s}{s^2 + 2as + \left(a^2 + b^2\right)} \quad \text{where } \frac{R}{L} = 2a \text{ and } \frac{1}{CL} = a^2 + b^2$$

$$\bar{i}\,(s) = \frac{E}{L}\left(\frac{(s + a) - a}{(s + a)^2 + b^2}\right)\frac{E}{L}\left[\frac{(s + a)}{(s + a)^2 + b^2} - a\frac{1}{(s + a)^2 + b^2}\right],$$

On inversion, $i(t) = \dfrac{E}{L}\left[e^{-at}\cos bt - \dfrac{a}{b}e^{-at}\sin bt\right]$

Taking limits as $t \to 0$, $i \to \dfrac{E}{L}$. Though, initially current is applied, yet a large current will develop instantaneously due to impulse applied at $t = 0$ and it is $\dfrac{E}{L}$.

ASSIGNMENT 5

1. Sketch the graph of the following function and express them is terms of unit step function. Hence find their laplace transforms:

(*i*) $f(t)\begin{cases} 2t, & \text{for } 0 < t < \pi \\ 1, & \text{for } t > \pi \end{cases}$

(*ii*) $f(t)=\begin{cases}0 & \text{for} \quad 0<t<a\\ t^2, & \text{for} \quad t>a\end{cases}$ [Hint: $f(t) = t^2 u_a(t)$]

2. Find the laplace transform of the followings:

(*i*) $f(t)=\begin{cases}0, & t>T\\ \cos(\omega t+\phi), & 0<t<T\end{cases}$

(*ii*) $f(t)=\begin{cases}k, & \text{for} \quad a<t<b, \text{ where } a>0\\ 0 & \text{for} \quad o<t<a \text{ and } t>b\end{cases}$

(*iii*) $f(t)=\begin{cases}\dfrac{1}{h}, & 0<t<h\\ 0, & t>h\end{cases}$ $\left[\text{Hint: Expessible as } f(t) = \dfrac{1}{h}\,[1 - u_h(t)]\right]$

(*iv*) $f(t)=\begin{cases}e^{3t}, & 0<t<1\\ 0 & t>1\end{cases}$ [Hint: $f(t) = [1 - u_1(t)]e^{3t}$]

(*v*) $f(t)=\begin{cases}0, & 0<t<\pi\\ \sin t, & t>\pi\end{cases}$ [Hint: $f(t) = \sin t\; u_\pi(t)$]

(*vi*) $f(t)=\begin{cases}\sin t\ , & 0<t\le\pi\\ 0, & t>\pi\end{cases}$ $\left[\text{Hint: } f(t)=\sin t\left[1-u_\pi(t)\right]+t\,u_\pi(t)=\sin t+\left[t-\sin t\right]u_\pi(t)\right]$

3. Find the inverse transform of

(*i*)* $\dfrac{e^{-\pi s}}{s^2+1}$ (*ii*) $\dfrac{e^{-as}}{s^2}$ (*iii*) $\dfrac{se^{-as}}{s^2+b^2}$ (*iv*) $\dfrac{1-e^{-\pi s}}{s^2+4}$ *[KUK, 2004]

4. Solve $y'' + 4y' + 3y = f(t)$, where $\left.\begin{aligned} f(t) &= 1, \quad 0<t<a\\ &= 0, \quad t>a\end{aligned}\right\}$; given that $y(0) = y'(0) = 0$

$\left[\text{Hint: } f(t)=1-u_a(t),\ \bar{y}(s)=\dfrac{1}{s(s+1)(s+3)}\left(1-e^{-as}\right),\ y(t)=f(t-a)\,u_a(t)\right]$

5. A beam has its ends clamped at $x = 0$ and $x = l$. A concentrated load W acts vertically downwards at the point $x=\dfrac{\ell}{3}$. Find the resulting defection.

$\left[\text{Hint: The differential equation and boundary conditions are } \dfrac{d^4y}{dx^4}=\dfrac{\omega}{EI}\,\delta\left(x-\dfrac{\ell}{3}\right) \text{ and } y(0)=y'(0)=y(l)=y'(l)=0\right]$

6. A cantilever beam is clamped at the end $x = 0$ and is free at the end $x = l$. It carries a uniform load per unit length from $x = 0$ to $x = l/2$. Calculate the deflection y at any point.

$\left[\text{Hint: The differential equation and boundary condition } \dfrac{d^4y}{dx^4}=\dfrac{\omega(x)}{EI},\ (0<x<\ell);\ \ \omega(x)=\begin{cases}w_{0,} & 0<x<\dfrac{\ell}{2}\\ 0, & x>\ell/2\end{cases} \text{ and } y(0)=y'(0)=0=y''(0)=y'''(0)\right]$

III Laplace Transform of Periodic Functions

If $f(t)$ be a periodic function of period, $T(>0)$ i.e. $f(t+T)=f(t)$, then

$$Lf(t)=\frac{\int_0^T e^{-st}f(t)\,dt}{\left(1-e^{-sT}\right)}$$

Proof: By definition,

$$Lf(t)=\int_0^\infty e^{-st}f(t)\,dt=\int_0^t e^{-st}f(t)\,dt+\int_t^{2t} e^{-st}f(t)\,dt+\int_{2t}^{3t} e^{-st}f(t)\,dt+\ldots\ldots$$

Substituting $t=u$, $t=u+T$, $t=u+2T$, in the successive integrals, then

$$Lf(t)=\int_0^T e^{-su}f(u)\,du+\int_0^T e^{-s(u+T)}f(u+T)\,du+\int_0^T e^{-s(u+2T)}f(u+2T)\,du+\ldots\ldots$$

Since $\qquad f(u)=f(u+T)=f(u+2T)=\ldots,$

$$\therefore \qquad L[f(t)=\int_0^T e^{-su}f(u)\,du+e^{-sT}\int_0^T e^{-su}f(u)\,du+e^{-2sT}\int_0^T e^{-su}f(u)\,du+\ldots\ldots$$

$$=\left(1+e^{-sT}+e^{-2sT}+\ldots\right)\int_0^T e^{-su}f(u)\,du$$

$$=\frac{1}{\left(1-e^{-sT}\right)}\int_0^T e^{-su}f(u)\,du \quad \text{(On changing the dummy variable } u \text{ to } t\text{)}$$

$$L\left(f(t)\right)=\frac{\int_0^T e^{-st}f(t)\,dt}{\left(1-e^{-sT}\right)}$$

Observation: Hence the laplace transform of a periodic function $f(t)$ with period T is expressible in the form of integral $\int_0^T e^{-st}f(t)\,dt$ over one period of $f(t)$ instead of over the semi-infinite interval $0 \le t \le \infty$.

Example 47: Find the laplace transform of the triangular wave of period 2a given by

$$\left.\begin{aligned} \boldsymbol{f(t)} &= \boldsymbol{t}, & \boldsymbol{0<t<a}\\ &= \boldsymbol{2a-t}, & \boldsymbol{a<t<2a}\end{aligned}\right\}$$

[Madras, 2000; UP Tech, 2002; VTU, 2003; KUK, 2004; NIT Jalandhar, 2006]

Or

Find the laplace transform of the triangular wave function $f(t)$ whose graph is shown below.

Solution: Clearly the graph of $f(t)$ is a straight line from the origin with slop 1 in the interval $o \le t \le a$

$$\therefore \qquad f(t)=t, \quad o\le t\le a$$

However in the interval $a \le t \le 2a$, the graph is a straight line whose slope is -1.

$\therefore \qquad f(t) = (-1)(t-2a) = 2a - t, \qquad a \le t \le 2a$

Thus, the function is a periodic one, with period $2a$

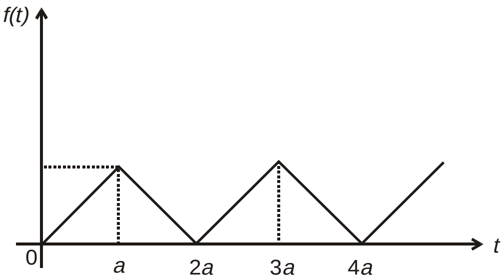

Fig. 12.10

$$\therefore \qquad L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_o^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left\{ t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right\}_o^a + \left\{ (2a-t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right\}_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left\{ a\frac{e^{-as}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} \right\} + \left\{ \frac{e^{-2as}}{s^2} + a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right\} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{s^2}\left(e^{-2as} - 2e^{-as} + 1\right) \right]$$

$$= \frac{1}{s^2\left(1-e^{-2as}\right)}\left(1-e^{-as}\right)^2$$

$$= \frac{1}{s^2 e^{-as}\left(e^{as}-e^{-as}\right)}\left[e^{-as/2}\left(e^{-as/2}-e^{-as/2}\right)\right]^2$$

$$= \frac{1}{s^2\left(e^{as/2}-e^{-as/2}\right)\left(e^{as/2}+e^{-as/2}\right)}\left(e^{as/2}-e^{-as/2}\right)^2$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

Example 48: Find the laplace transform of the full wave rectifier $f(t) = E\sin\omega t$, $o < t < \pi/\omega$; having period π/ω.

Solution: $Lf(t) = \dfrac{1}{1-e^{\frac{-s\pi}{\omega}}} \displaystyle\int_0^{\pi/w} e^{-st} E\sin\omega t\,dt$

$$= \frac{E}{1-e^{\frac{-s\pi}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t-\omega\cos\omega t\right)\right]_0^{\pi/w}$$

$$= \frac{E}{1-e^{\frac{-s\pi}{\omega}}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\left(-\sin\pi-\omega\cos\omega\frac{\pi}{\omega}\right)-\frac{1}{s^2+\omega^2}(0-\omega)\right]$$

$$= \frac{E}{1-e^{\frac{-s\pi}{\omega}}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\omega+\frac{\omega}{s^2+\omega^2}\right]$$

$$= \frac{E\cdot\omega}{\left(s^2+\omega^2\right)\left(1-e^{-s\pi/\omega}\right)}\left(e^{-s\pi/\omega}+1\right)$$

$$= \frac{E\omega}{(s^2+\omega^2)}\frac{1+e^{-s\pi/\omega}}{1-e^{-s\pi/\omega}}$$

$$= \frac{E\omega}{(s^2+\omega^2)}\coth\frac{s\pi}{2\omega} \qquad \left(\because \quad \coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{1+e^{-2x}}{1-e^{-2x}}\right)$$

The graph for the function $f(t)$ which is a full-sine wave rectifier can be drawn as below:

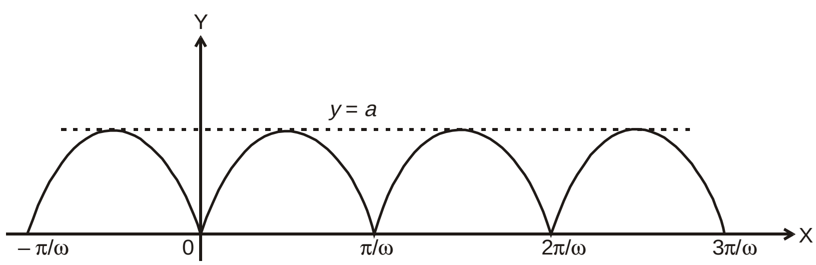

Fig. 12.11

Since $\left|\sin w\left(t+\frac{\pi}{\omega}\right)\right| = |\sin\omega t|$ for all t.

Example 49: Find the laplace transform of the square wave (*or meander*) function of a period a defined as: $\left.\begin{aligned} f(t) &= 1 \quad \text{when} \quad 0 < t < a/2 \\ &= -1, \quad \text{when} \quad a/2 < t < a \end{aligned}\right\}$ **[UP Tech, 2004]**

Solution: $L[f(t)] = \frac{1}{1-e^{-as}}\int_o^a e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-as}}\left[\int_o^{a/2} 1\cdot e^{-st}dt + \int_{a/2}^{a}(-1)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left|\frac{e^{-st}}{-s}\right|_{o}^{a/2} - \left|\frac{e^{-st}}{-s}\right|_{a/2}^{a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1}{-s}\left[e^{-sa/2}-1\right]+\frac{1}{s}\left[e^{-as}-e^{-as/2}\right]\right]$$

$$= \frac{1}{s\left(1-e^{-as}\right)}\left[1-2\,e^{-sa/2}+e^{-as}\right]$$

$$= \frac{1}{s\left(1-e^{-as}\right)}\left(1-e^{-sa/2}\right)^2$$

$$= \frac{1}{s}\frac{\left(1-e^{-sa/2}\right)}{\left(1+e^{-sa/2}\right)} = \frac{1}{s}\frac{\left(e^{sa/4}-e^{-sa/4}\right)}{\left(e^{sa/4}+e^{-sa/4}\right)} = \frac{1}{s}\tanh\frac{as}{4}$$

Note: *It may be defined like, f(t) = k, when $0 < t < a/2$*
$= -k$, when $a/2 < t < a$

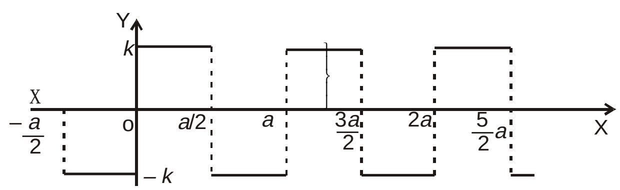

Fig. 12.12

Example 50: Find the Laplace Transform of the periodic function (*saw tooth wave*),

$$f(t+T) = f(t); \qquad f(t) = k\frac{t}{T} \quad \text{for} \quad 0 < t < T$$

Solution: $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}k\frac{t}{T}\,dt$

$$= \frac{1}{1-e^{-sT}}\frac{k}{T}\int_0^T e^{-st}t\,dt$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[t\frac{e^{-st}}{-s} - \int 1\cdot\frac{e^{-st}}{-s}\,dt\right]_o^T \quad \text{(By parts)}$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{t\,e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_o^T$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-sT}}{-s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-sT}}{-s} + \frac{1}{s^2}\left(1-e^{-sT}\right)\right]$$

$$= \frac{k}{s^2T} - \frac{ke^{-sT}}{s\left(1-e^{-sT}\right)}$$

Note: *It may result to a particular case: when k = 1,* $f(T) = \frac{t}{T}$

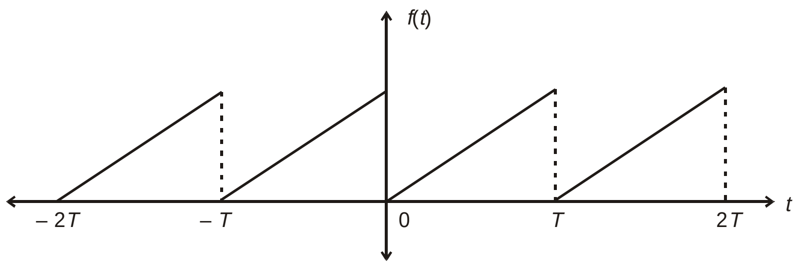

Fig. 12.13

Example 51: Find the Laplace transform of the function (*Half wave rectifier*)

$$\boldsymbol{f(t) = \begin{cases} \sin \omega t, & \text{for } \quad 0 < t < \dfrac{\pi}{\omega} \\ 0, & \text{for } \quad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}}$$ **[Madras, 2003; KUK, 2005]**

Solution: $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\pi/\omega} e^{-st} f(t)\,dt \qquad \left[\because \quad f(t) \text{ has a period, } T = \frac{2\pi}{\omega}\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\times o\times dt\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_o^{\pi/\omega} e^{-st}\sin\omega t\,dt,$$

Using $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$,

$$= \frac{1}{\left(1 - e^{\frac{-2\pi s}{\omega}}\right)} \left[\frac{e^{-st}\left(-s \sin \omega t \quad - \omega \cos \omega t\right)}{s^2 + \omega^2} \right]_0^{\pi/\omega}$$

$$= \frac{1}{\left(1 - e^{\frac{-2\pi s}{\omega}}\right)} \left[\frac{\omega e^{\frac{-\pi}{\omega} s} + \omega}{s^2 + \omega^2} \right]$$

$$= \frac{\omega \left[1 + e^{\frac{\pi s}{w}} + \omega \right]}{\left(s^2 + \omega^2\right) \left[1 - e^{\frac{-2\pi s}{\omega}} \right]}$$

$$= \frac{\omega}{\left(s^2 + \omega^2\right) \left[1 - e^{\frac{-2\pi s}{\omega}} \right]}$$

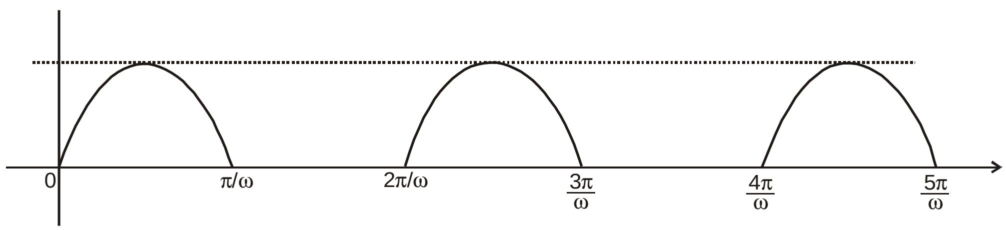

Fig. 12.14

Note: *Clearly the function f(t) = sinωt represents the half sine wave rectifier with period* $\frac{2\pi}{\omega}$. *The graph of f(t) in the internal* $0 < t < \frac{\pi}{\omega}$ *is sine function which varnishes at t = 0 and* $t = \frac{\pi}{\omega}$ *and f(t) = 0 for the rest half from* $\frac{\pi}{\omega}$ to $\frac{2\pi}{\omega}$.

Example 52: A periodic square wave function $f(t)$, in terms of unit step functions, is written as $f(t) = k[u_0(t) - 2u_a(t) + 2u_{2a}(t) - 2u_{3a}(t) + \ldots]$. Show that the Laplace transform of $f(t)$ is given by $L[f(t)] = \frac{k}{s} \tanh\left(\frac{as}{2}\right)$.

Solution: $f(t) = k[u_0(t) - 2u_a(t) + 2u_{2a}(t) - 2u_{3a}(t) + \ldots\ldots]$

$\therefore \qquad Lf(t) = k_0[L u_0(t) - 2L u_a(t) + 2L u_{2a}(t) - 2L u_{3a}(t) + \ldots\ldots]$

$$= k\left[\frac{1}{s} - 2\frac{e^{-as}}{s} + 2\frac{e^{-2as}}{s} - 2\frac{e^{-3as}}{s} + \ldots\ldots \right]$$

$$= \frac{k}{s}\left[1 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \ldots\ldots \right]$$

$$= \frac{k}{s}\left[1 - 2\left(e^{-as} - e^{-2as} + e^{-3as} - \ldots\ldots \right) \right]$$

$$= \frac{k}{s}\left[1 - 2\frac{e^{-as}}{1 + e^{-as}} \right]$$

$$= \frac{k}{s}\left[\frac{1+e^{-as}-2e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{k}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right] = \frac{k}{s}\left[\frac{e^{\frac{as}{2}}-e^{\frac{-as}{2}}}{e^{\frac{as}{2}}+e^{\frac{-as}{2}}}\right]$$

$$= \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

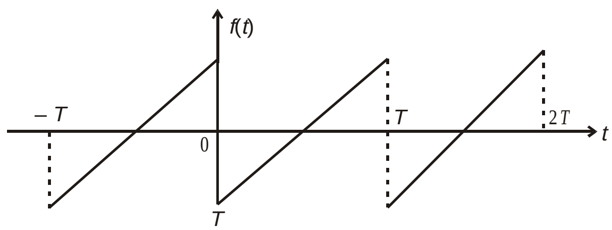

Fig. 12.15

ASSIGNMENT 6

Find the Laplace transform of the following periodic functions:

1. $f(t) = |\sin \omega t|, \ t \geq 0$ when $f(t) = f(t+T); \ T = \dfrac{\pi}{\omega}$

   [Hint: $f(t)$ represents full sine wave rectifier. Ex. 48]

2. Saw tooth wave function of period T given by $f(t) = -a + \dfrac{2at}{T}, \quad 0 \leq t < T$

3. Non–Symmetric square wave of period T, given by $f(t) = \begin{cases} a, & 0 \leq t < b \\ -a, & b \leq t < T \end{cases}$

4. Staircase function defined by $f(t) = kn$, $nT < t < (n+1)T$; where $n = 0, 1, 2, \ldots$

ANSWERS

Assignment 1

1. (*i*) $\dfrac{s^3}{s^4+4a^4}$, (*ii*) $\dfrac{a\left(s^2+2a^2\right)}{s^4+4a^4}$, (*iii*) $\dfrac{a\left(s^2-2a^2\right)}{s^4+4a^4}$,

(*iv*) $\dfrac{2a^2s}{s^4+4a^4}$, (*v*) $\dfrac{2as^2}{s^4+4a^4}$, (*vi*) $\dfrac{4a^3}{s^4+4a^4}$,

2. (*i*) $\dfrac{a}{s^2+4a^2}$, (*ii*) $\dfrac{1}{s^2\left(s^2+a^2\right)}$

3. (*i*) $\dfrac{2}{s^3}-\dfrac{e^{-2s}}{s^3}\left(2+3s+3s^2\right)+\dfrac{e^{-3s}}{s^2}(5s-1)$ (*ii*) $\dfrac{e^{-\frac{2\pi s}{3}}}{s^2+1}$, $s>0$

4. (*i*) $\dfrac{2as}{\left(s^2+a^2\right)^2}$, (*ii*) $\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$, (*iii*) $\dfrac{2as}{\left(s^2-a^2\right)^2}$, $s>|a|$ (*iv*) $\dfrac{s^2+a^2}{\left(s^2-a^2\right)^2}$,

5. (*i*) $\dfrac{2as^2}{\left(s^2+a^2\right)^2}$, (*ii*) $\dfrac{2a^3}{\left(s^2+a^2\right)^2}$,

(*iii*) $\dfrac{2as^2}{\left(s^2-a^2\right)^2}$, $s>|a|$ (*iv*) $\dfrac{-2a^3}{\left(s^2-a^2\right)^2}$, $s>|a|$

6. (*i*) $\dfrac{s^3}{\left(s^2+a^2\right)^2}$, (*ii*) $\dfrac{s^3}{\left(s^2-a^2\right)^2}$,

7. (*i*) $\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)^{1/2}$ (*ii*) $\dfrac{1}{2}\log\left(\dfrac{s^2+4}{s^2}\right)$

(*iii*) $\cot^{-1}\left(\dfrac{s}{a}\right)$ (*iv*) $\cot^{-1}(s+1)$

8. $\dfrac{a}{s^2+a^2}$, $\dfrac{s}{s^2+a^2}$ 9. $\dfrac{a}{s^2-a^2}$, $\dfrac{s}{s^2-a^2}$

10. (*i*) $\dfrac{1}{s}\left(\dfrac{s+1}{s^2+2s+2}\right)$ (*ii*) $\dfrac{1}{2s}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$

Assignment 2

1. (*i*) $\dfrac{t\sinh t}{2}$, (*ii*) $\dfrac{1}{13}\left(3e^{3t}-3\cos 2t+2\sin 2t\right)$

(*iii*) $\frac{a^2}{3}\left[e^{at} - e^{at/2}\left\{\cos\frac{\sqrt{3}}{2}at + \sqrt{3}\sin\frac{\sqrt{3}}{2}at\right\}\right]$,

(*iv*) $\frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$ (*v*) $\frac{1}{2}\sin t - \frac{1}{2}te^{-t}$

2. $\frac{e^{-at}}{2b^3C}\left[(aA - B)bt\cdot\cos bt + (B - aA + Ab^2t)\sin bt\right]$

3. (*i*) $\frac{1}{3}e^{-2t}(6\cos 3t - 7\sin 3t)$, (*ii*) $4e^{3t} - e^{-t}$, (*iii*) $e^{2t} + 2e^{-4t}$

4. (*i*) $\frac{1}{2}te^{-2t}\sin t$, (*ii*) $3te^{-3t}\sin t$, (*iii*) $\frac{1}{2}t\sinh t$

5. (*i*) $\frac{1 - 2\cos t + e^{-t}}{t}$ (*ii*) $\frac{\sin 2t}{t}$, (*iii*) $\frac{2\sinh t\,\sin t}{t}$,

(*iv*) $\frac{e^{-bt} - e^{-at}}{t}$, (*v*) $\frac{e^{-3t} + e^{-2t} - e^{-t}}{t}$, (*vi*) $\frac{2(1 - \cosh at)}{t}$,

(*vii*) $\frac{2(e^t - \cos t)}{t}$, (*viii*) $\frac{e^{-t}\sin t}{t}$, (*ix*) $\frac{\cos at - \cos bt}{t}$

6. (*i*) $\frac{1}{a^2}\cos\left(\frac{b}{a}t\right)$ (*ii*) $(1 - at)e^{-at}$, (*iii*) $\frac{1}{2}(a^2t^2 - 4at + 2)e^{-at}$

7. (*i*) $\frac{1}{8} - \frac{1}{4}\left(t^2 + t + \frac{1}{2}\right)e^{-2t}$ (*ii*) $\frac{1}{4}(e^{-2t} + 2t - 1)$ (*iii*) $\frac{1}{a^3}(at - \sin at)$

9. (*i*) $\frac{e^{-bt} - e^{-at}}{a - b}$, (*ii*) $\frac{1}{2a^3}(\sin at - at\cos at)$, (*iii*) $t(e^{-t} + 1) + 2(e^{-t} - 1)$

(*iv*) $\frac{1}{a^3}(at - \sin at)$, (*v*) $\frac{e^{-t}}{64}\left[1 - e^{-8t}(1 + 8t)\right]$, (*vi*) $\frac{1}{3}(\cot t - \cos 2t)$

(*vii*) $\frac{(a\sin at - b\sin bt)}{(a^2 - b^2)}$

Assignment 3

1. $x = \left(2 + \frac{\omega}{\omega^2 + 1}\right)e^{-t} + \frac{\sin\omega t - \omega t\cos\omega t}{\omega^2 + 1}$ 2. $y = A\cos\omega x + \frac{B}{\omega}\sin\omega x$

3. $y = (2t + 3) + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}$ 4. $x = \frac{a}{2n^2}[\sin nt\cos\alpha - nt\cos(nt + \alpha)]$

5. $y = e^t - 3e^{-t} + 2e^{-2t}$

6. $y = \left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right)e^t$

7. $x = \left(2 - 3t + \frac{1}{2}t^2\right)e^t$

8. $y = \frac{1}{8}[(3 - t^2)\sin t - 3t\cos t]$

9. $y = \frac{11}{3}(\sin t + \sin 2t)e^{-t}$

10. $y = \frac{1}{2}\left(1 + \frac{2}{t}\right)\sin t$

11. $y = e^{2t}$

12. $f(t) = \cos t + \sin t$

Assignment 4

1. $x = e^t + e^{-t}$
$y = e^t - e^{-t} + \sin t$

2. $x = \frac{1}{10}\left(5 - 2e^{-t} - 3e^{\frac{-6}{11}t}\right)$
$y = \frac{1}{5}\left(e^{-t} - e^{\frac{-6}{11}t}\right)$

3. $x = -\frac{1}{27}(1 + 6t)e^{-3t} + \frac{1}{27}(1 + 3t),$
$y = -\frac{2}{27}(2 + 3t)e^{-3t} + \frac{2}{27}(2 - 3t)$

4. $i_1 = \frac{a}{p + \omega}(\sin \omega t + \sin pt)$
$i_2 = \frac{a}{p + \omega}(\cos \omega t - \cos pt)$

5. $x = \frac{1}{4}\left(11\sin t + \frac{1}{3}\sin 3t\right), \qquad x\left(\frac{1}{2}\right) = 1.4015$
$y = \frac{1}{4}\left(11\sin t - \frac{1}{3}\sin 3t\right), \qquad y\left(\frac{1}{2}\right) = 1.069$

Assignment 5

1 (*i*) $(1 - 2t)u(t - \pi) + 2t\,u(t), \frac{2}{s^2} + \left(\frac{1 - 2\pi}{s} - \frac{2}{s^2}\right)e^{-as}$ (*ii*) $t^2 u_a(t),\ e^{-as}\left(\frac{a}{s} + \frac{2a}{s^2} + \frac{2}{s^3}\right)$

2. (*i*) $[u(t) - u(t - T)\cos(\omega t + \phi)]$ (*ii*) $\frac{k}{s}\left(e^{-as} - e^{-bs}\right)$ (*iii*) $\frac{1}{sh}[1 - e^{-sh}]$

(*iv*) $\frac{1}{(3 - s)}\left[e^{(3-s)} - 1\right]$ (*v*) $\frac{-e^{-\pi s}}{s^2 + 1},$ (*vi*) $\frac{1}{s^2 + 1} + \left(\frac{\pi}{s} + \frac{1}{s^2} + \frac{1}{s^2 + 1}\right)e^{-\pi s}$

3. (*i*) $-\sin t\, u_\pi(t)$ (*ii*) $(t - a)\,u_a(t)$ (*iii*) $\cos b(t - a)\,u_a(t)$ (*iv*) $\frac{1}{2}\sin 2t\left(1 - u_\pi(t)\right)$

4. $y(t) = f(t) - f(t - a)u_a(t)$, where $f(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$

5. $$y(x) = \begin{cases} \dfrac{2\omega x^2(3\ell - 5x)}{81 \quad EI}, & 0 < x < \dfrac{\ell}{3} \\ \dfrac{2\omega x^2(3\ell - 5x)}{81 \quad EI} + \dfrac{\omega}{6EI}\left(x - \dfrac{\ell}{3}\right)^3, & \dfrac{\ell}{3} < x < l \end{cases}$$

6. $y(x) = \frac{\omega\ell^2}{16EI}x^2 - \frac{\omega\ell}{12EI}x^3 + \frac{\omega}{24EI}x^4 - \frac{\omega}{24EI}\left(x - \ell/2\right)^4 u\left(x - \ell/2\right)$

Assignmen 6

1. $\frac{E\omega}{s^2 + w^2}\coth \frac{s\pi}{2\omega}$

2. $\frac{a}{s}\left[\frac{1}{\left(\frac{sT}{2}\right)} - \coth\left(\frac{sT}{2}\right)\right]$

3. $\frac{a\left(e^{-sT} - 2e^{-sb} + 1\right)}{s\left(1 - e^{-sT}\right)}$

4. $\frac{k\,e^{-sT}}{s\left(1 - e^{-sT}\right)}$