MAT1071 MATHEMATICS I
1. WEEK
PART 1

FUNCTIONS

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Functions

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

We say that "y is a function of x" and write this symbolically as

$$y = f(x) \qquad (\text{"}y \text{ equals } f \text{ of } x\text{"}).$$

In this notation, the symbol f represents the function, the letter x is the **independent variable** representing the input value of f, and y is the **dependent variable** or output value of f at x.

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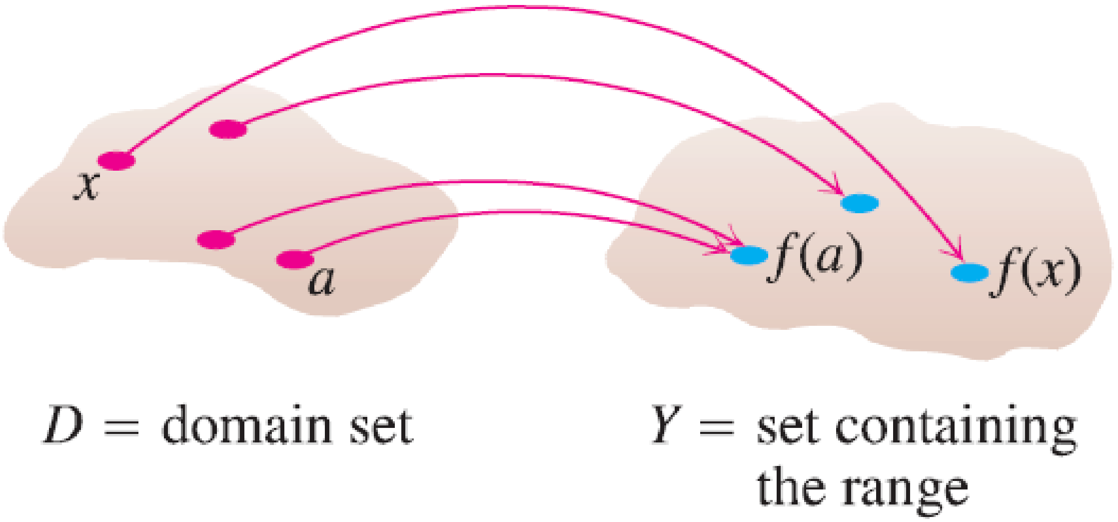
☆ The set D of all possible input values is called the **domain** of the function.

☆ The set of all values of $f(x)$ as x varies throughout D is called the **range** of the function.

☆ The range may not include every element in the set Y.

☆ The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line.

x
Input
(domain)
f
Output
(range)
$f(x)$



x
a
$f(a)$
$f(x)$
D = domain set
Y = set containing the range

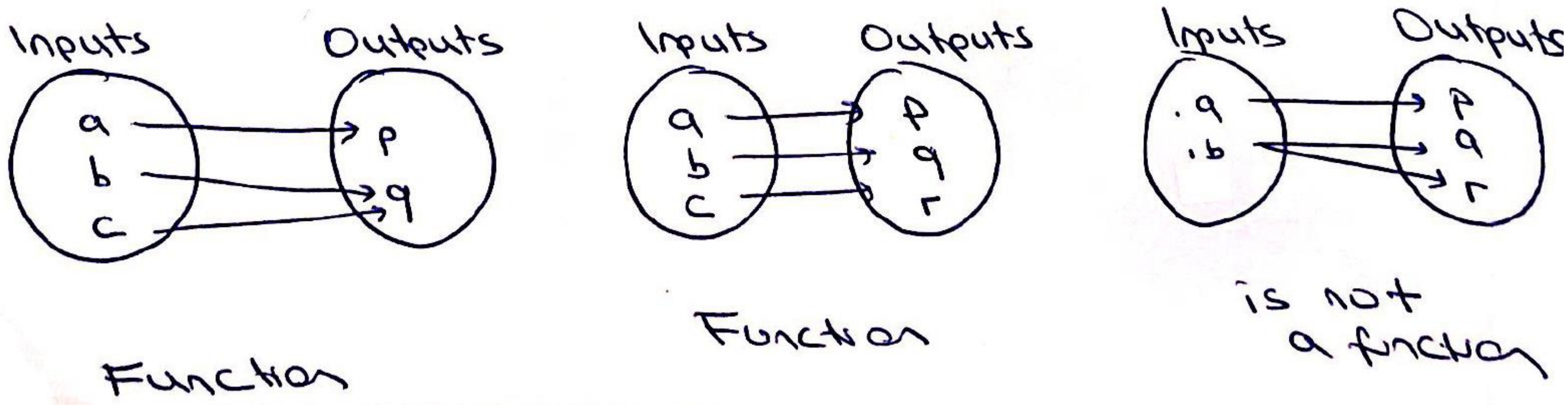
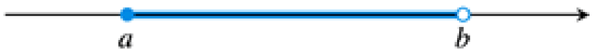
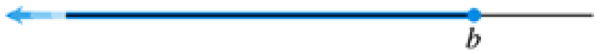

Inputs
Outputs
a
b
c
p
q
Function
Inputs
Outputs
a
b
c
p
q
r
Function
Inputs
Outputs
a
b
p
q
r
is not
a function

TABLE Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

EXAMPLE Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.

Function	Domain (x)	Range (y)
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

Function	Domain (x)	Range (y)
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

Function	Domain (x)	Range (y)
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

Function	Domain (x)	Range (y)
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

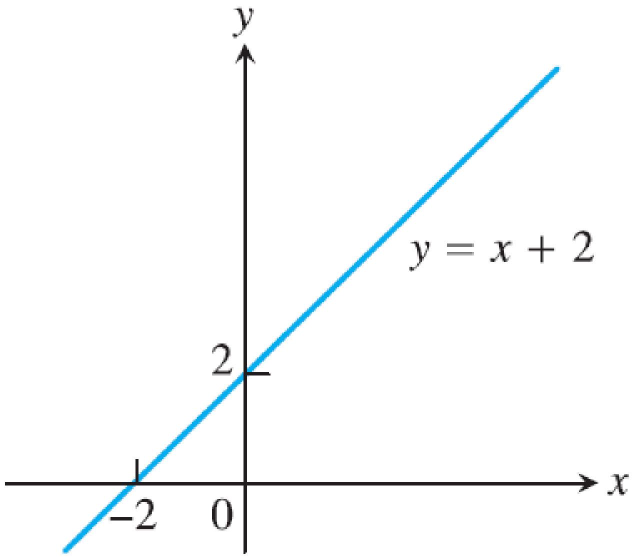
Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

Example

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$.



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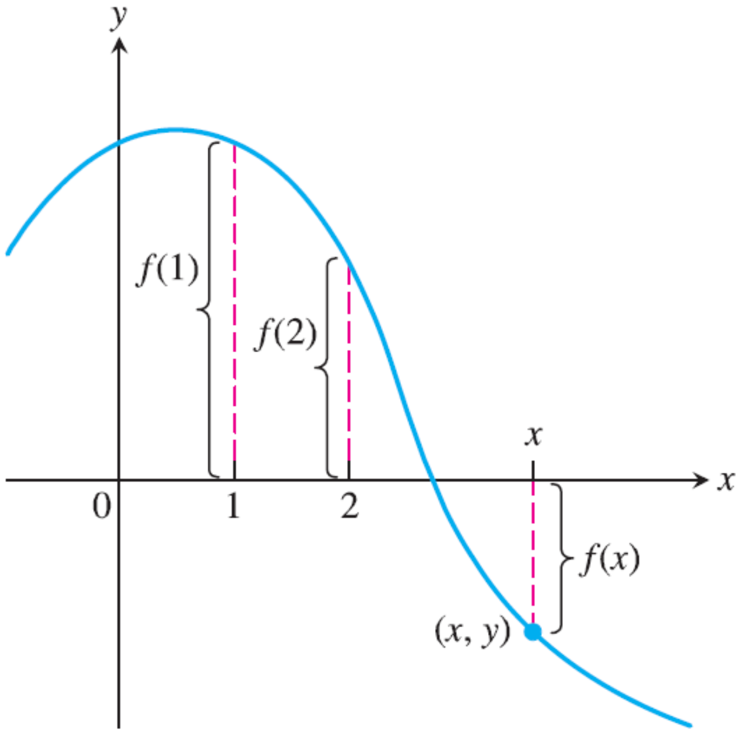
The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above the point x. The height may be positive or negative, depending on the sign of $f(x)$



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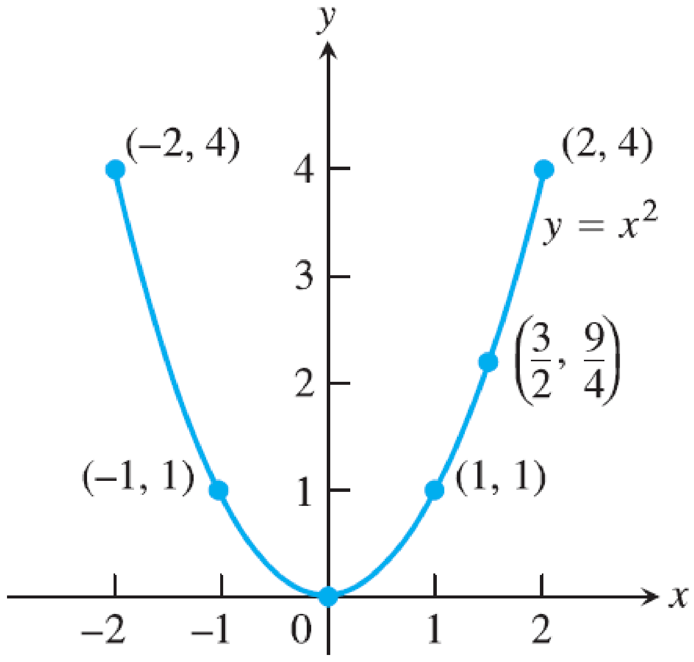
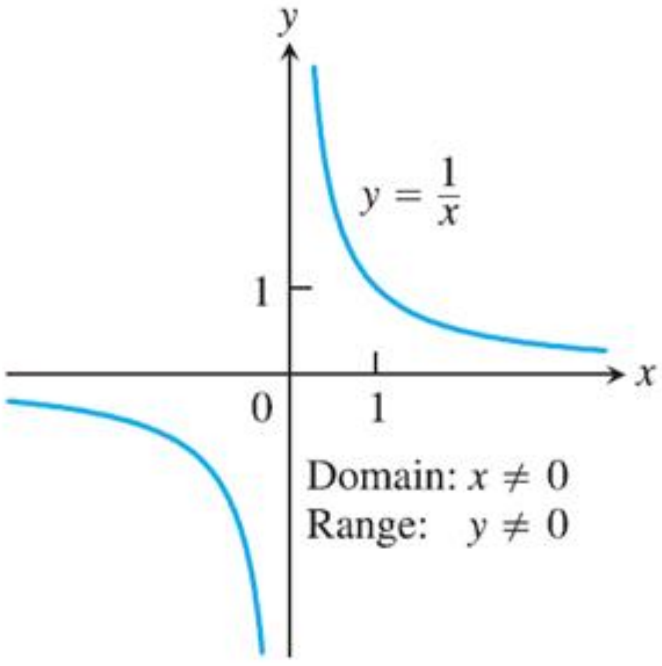
Example Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points

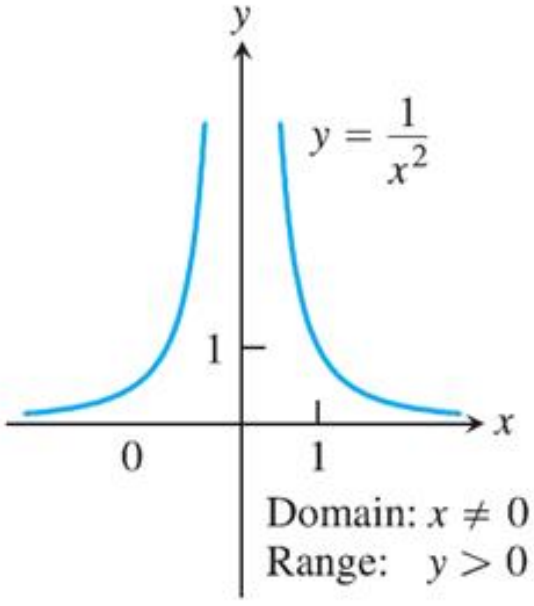
x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



Example



(a)



(b)

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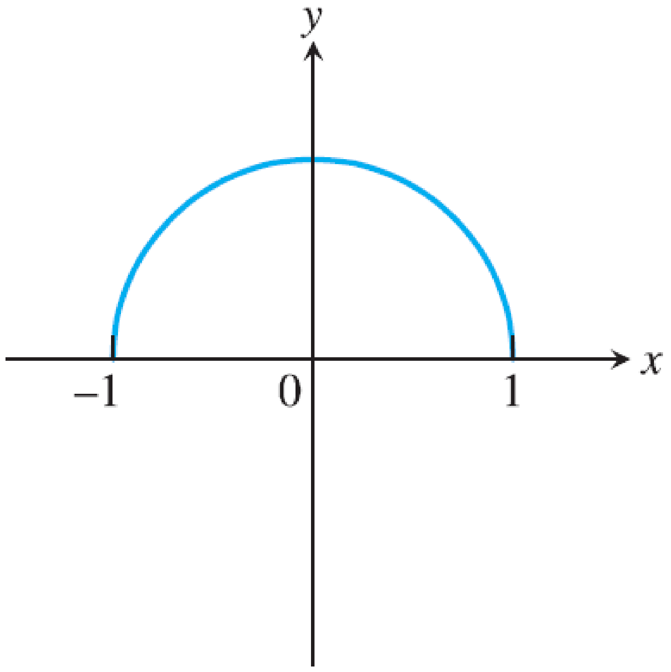
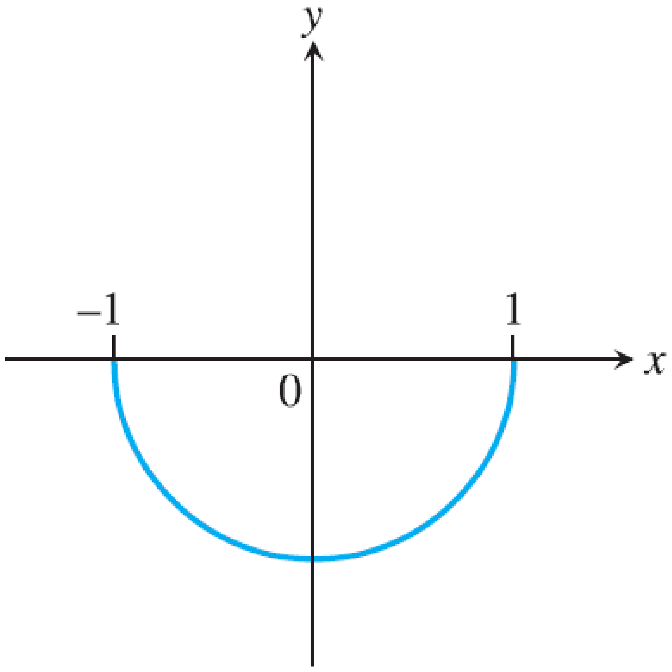
Example



(b) $y = \sqrt{1 - x^2}$



(c) $y = -\sqrt{1 - x^2}$

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical* line can intersect the graph of a function more than once. If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

Example

A circle cannot be the graph of a function since some vertical lines intersect the circle twice.

the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$

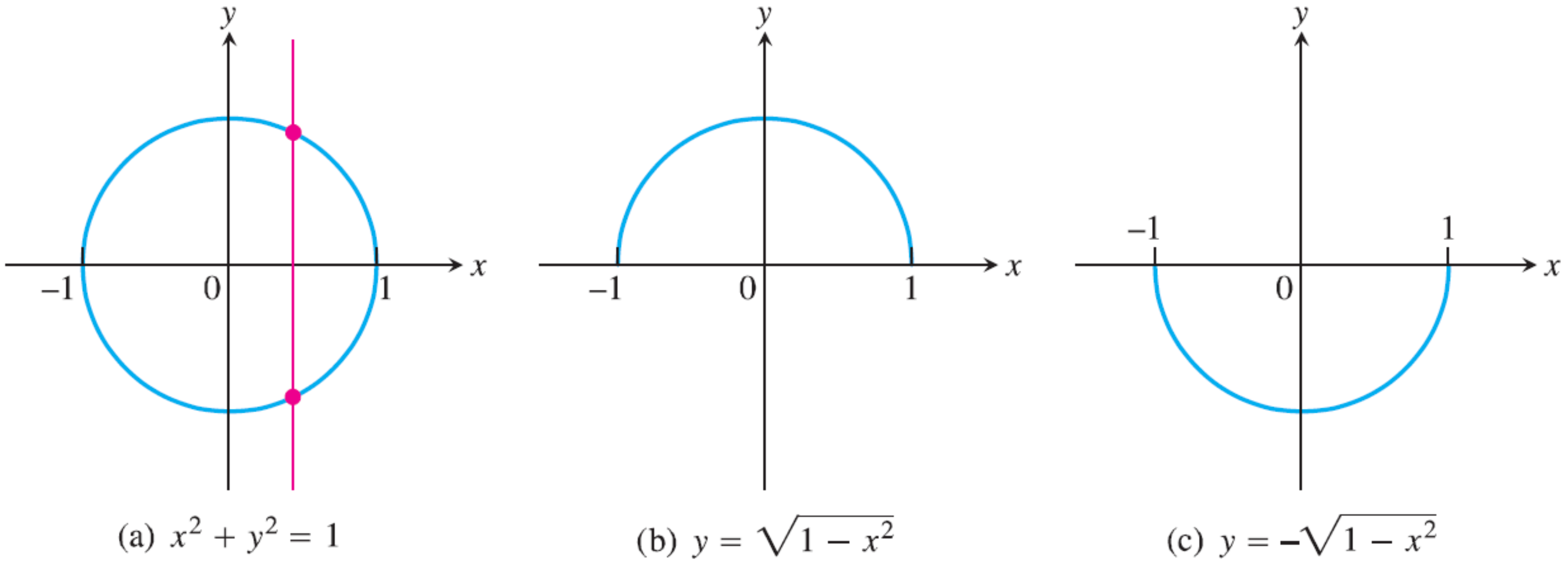
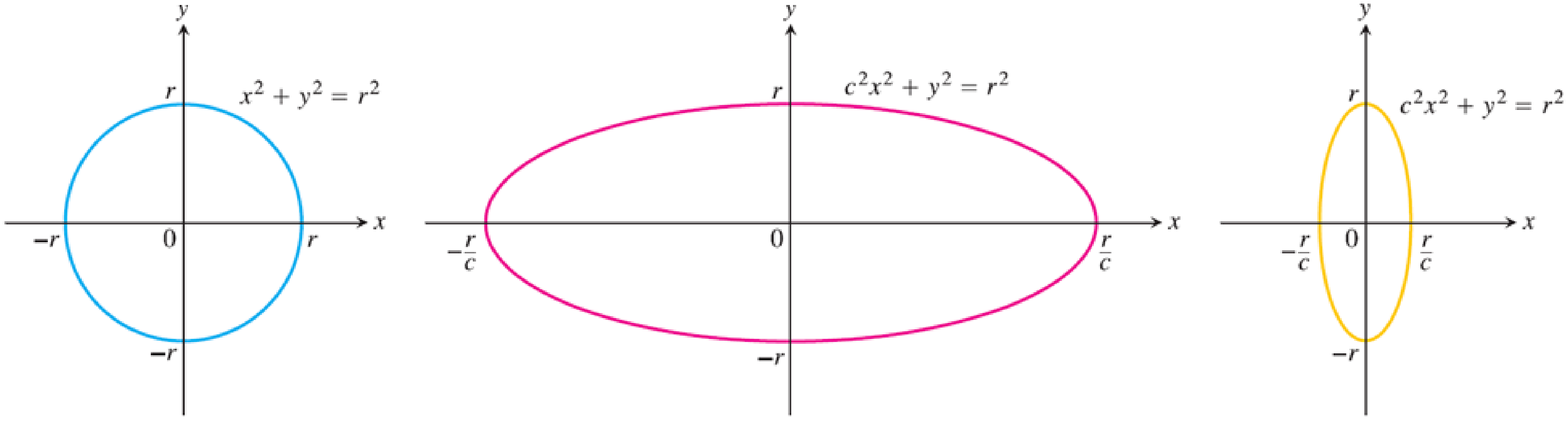


FIGURE (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

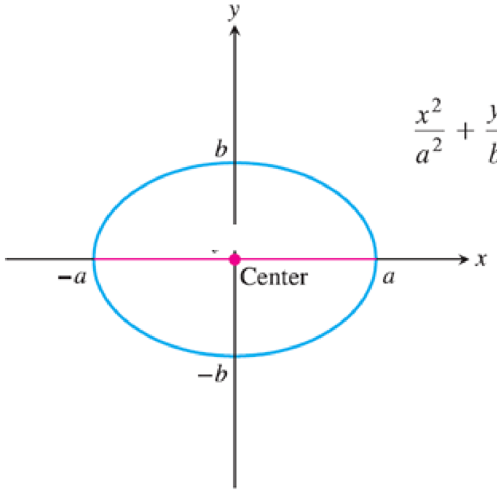
Ellipses

Although they are not the graphs of functions, circles can be stretched horizontally or vertically in the same way as the graphs of functions. The standard equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$



(a) circle (b) ellipse, $0 < c < 1$ (c) ellipse, $c > 1$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b,$$

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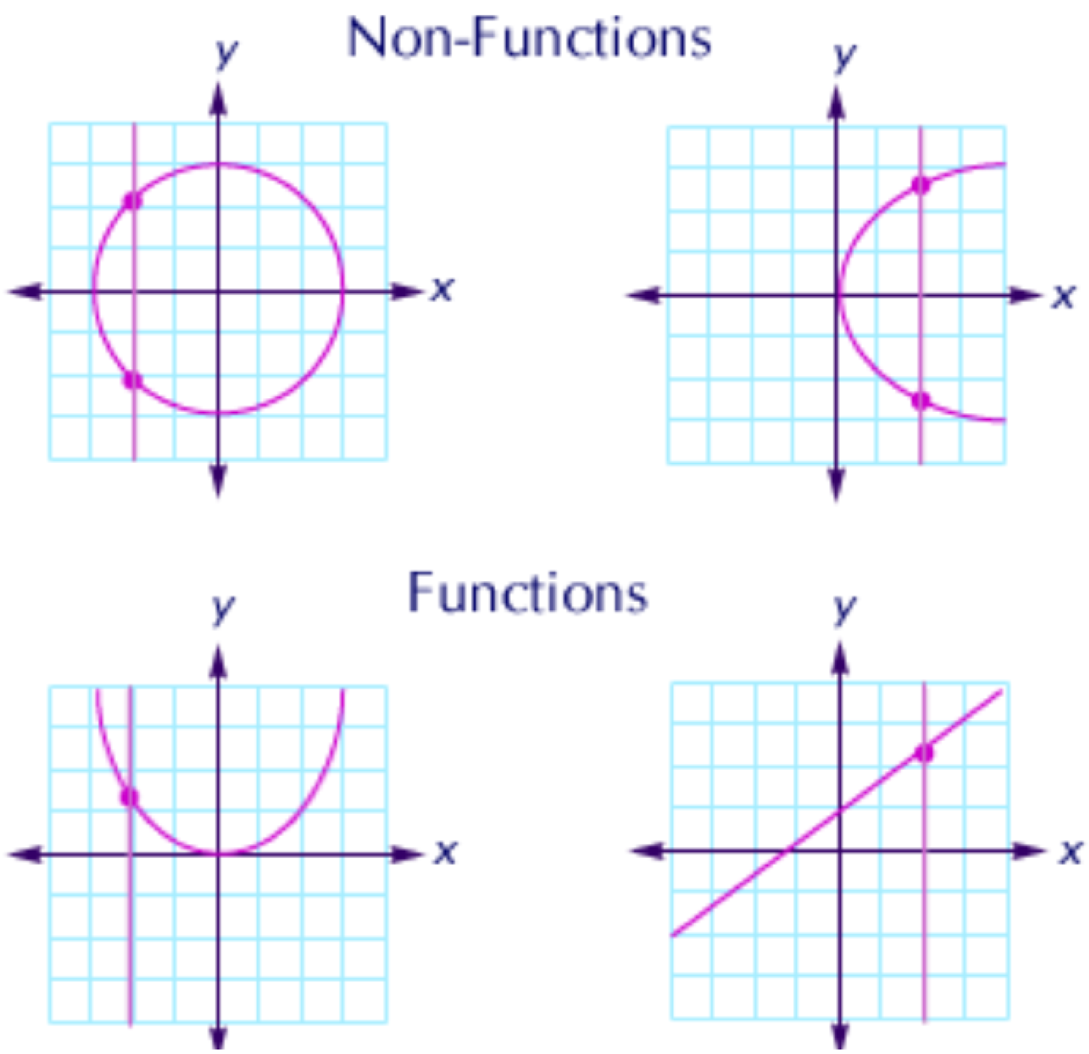
Example



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Some Special Functions

1. Constant function
2. Identity function
3. Zero function
4. One-to-one function
5. Onto function
6. Inverse function
7. Piecewise-defined functions
8. Increasing and decreasing functions
9. Even and odd functions
10. Linear functions
11. Power functions
12. Polynomials
13. Rational functions
14. Exponential functions
15. Logarithmic functions

1. Constant function $f(x) = c$ c; constant

2. Identity function $f(x) = x$

3. Zero function $f(x) = 0$

4. One-to-one function

A function f from A to B is called one-to-one (or 1-1) if whenever $f(a)=f(b)$ then $a=b$
or $a \neq b$ then $f(a) \neq f(b)$

<u>For ex:</u> $f(x)=x^2$ $2 \neq -2$ but $f(2)=f(-2)=4$
so $f(x)=x^2$ is not one-to-one

5. Onto function

A function f from A to B is called onto if for all b in B there is an a in A such that $f(a)=b$

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6. Inverse function

Let $f: A \to B$ be 1-1 and onto

then the inverse of f $\quad f^{-1}: B \to A$ where

$f(x) = y \Rightarrow f^{-1}(y) = x$

Ex: State whether $\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = 1+x^2 \end{cases}$ one-to one and onto?

Solution $f(x) = 1+x^2$

for $x=1$ $f(1) = 2$ $\quad 1 \neq -1$ but $f(1) = f(-1)$

for $x=-1$ $f(-1) = 2$ $\quad f$ is not 1-1.

If f is onto then for $\underbrace{0 \in \mathbb{R}}_{\text{range set}}$ there exists $f(x) = 0$

where $\underbrace{x \in \mathbb{R}}_{\text{domain set}}$

$1+x^2 = 0 \Rightarrow x^2 = -1$

$\Rightarrow$ no such x value in $\mathbb{R}$

$\Rightarrow f$ is not onto.

7. Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain.

EXAMPLE The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x. The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers

Example **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$.

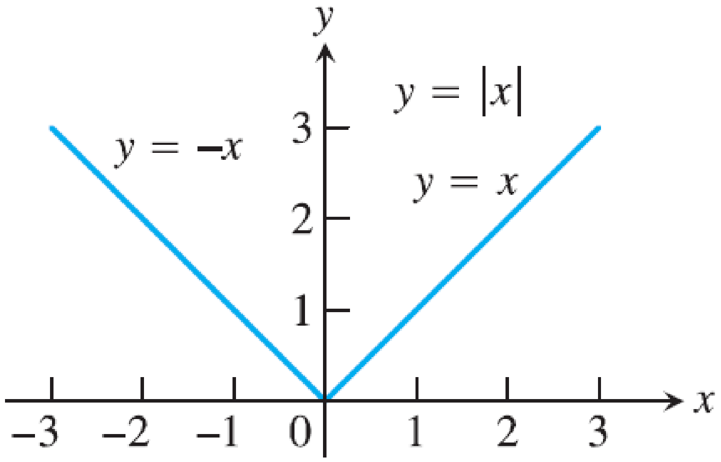

FIGURE The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

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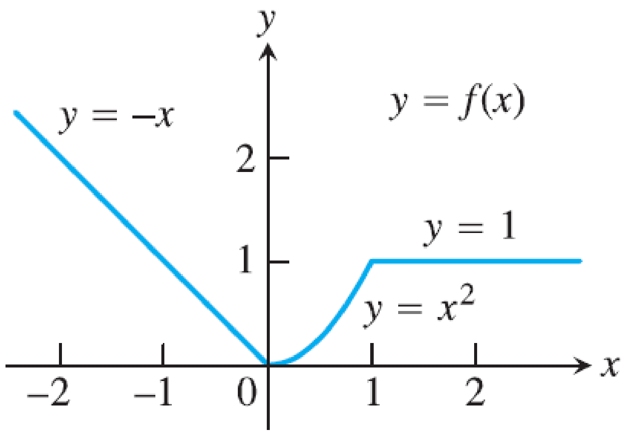
8. Increasing and Decreasing Functions

If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*. If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Example



The function is decreasing on $(-\infty, 0]$ and increasing on $[0, 1]$. The function is neither increasing nor decreasing on the interval $[1, \infty)$ because of the strict inequalities used to compare the function values in the definitions.

Example

Function	Where increasing	Where decreasing
$y = x^2$	$0 \leq x < \infty$	$-\infty < x \leq 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \leq x < \infty$	Nowhere
$y = x^{2/3}$	$0 \leq x < \infty$	$-\infty < x \leq 0$

9. Even Functions and Odd Functions:

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

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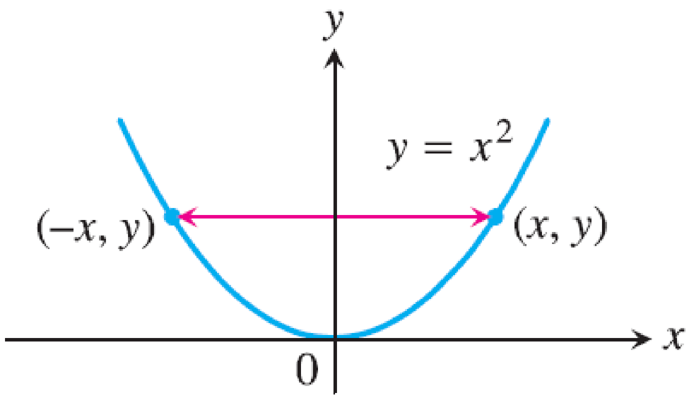
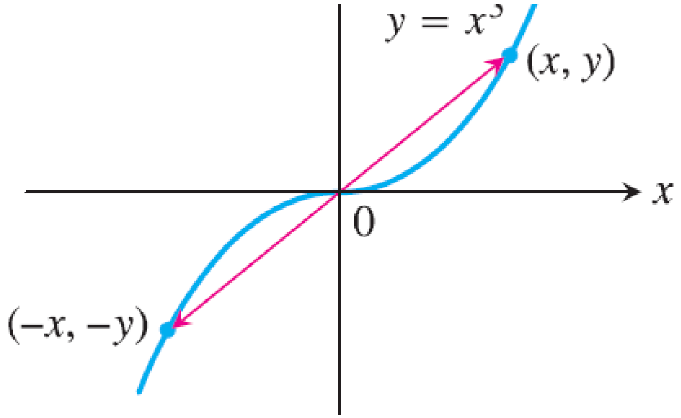
☆ The graph of an even function is **symmetric about the *y*-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph
A reflection across the y-axis leaves the graph unchanged.

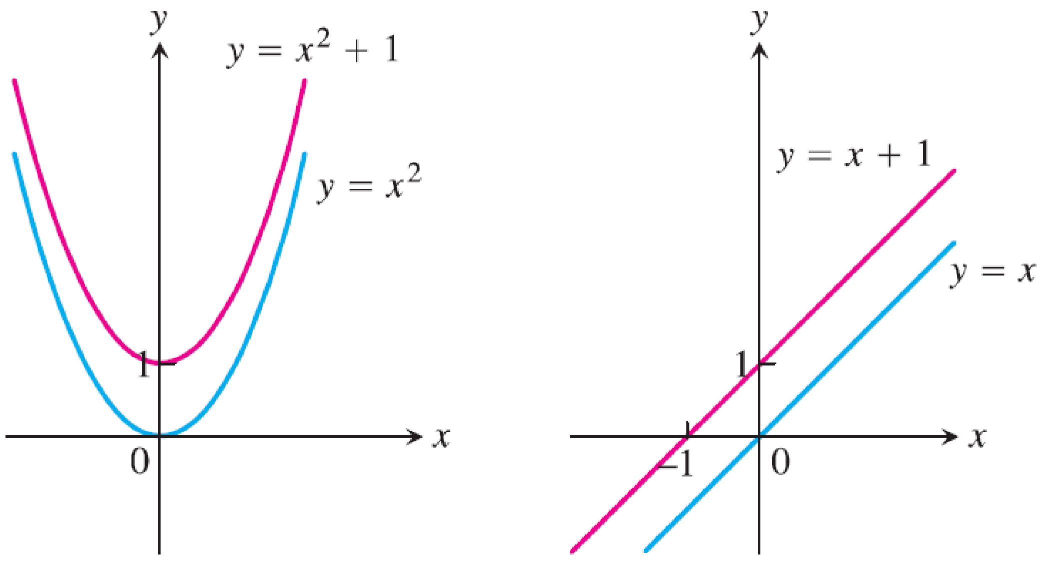


☆ The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and $-x$ must be in the domain of f.



EXAMPLE

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$



(a) (b)

10. **Linear Functions** A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**.

The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope $m = 0$

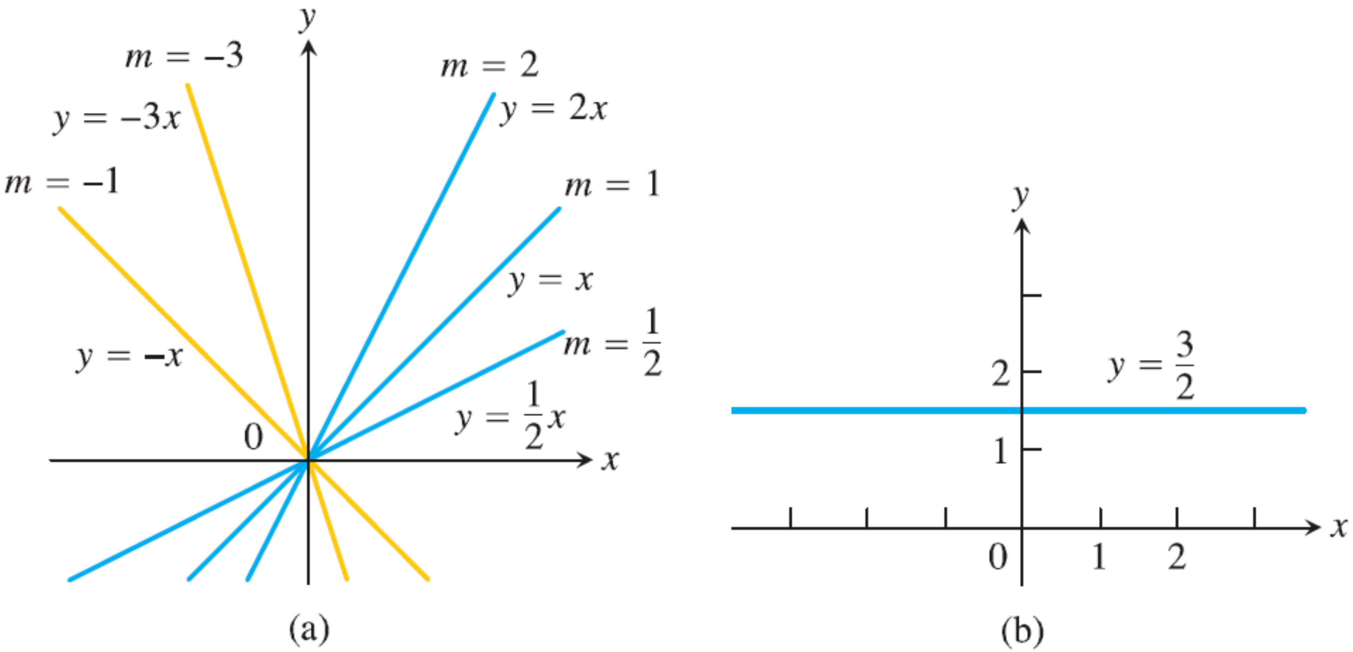


FIGURE (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

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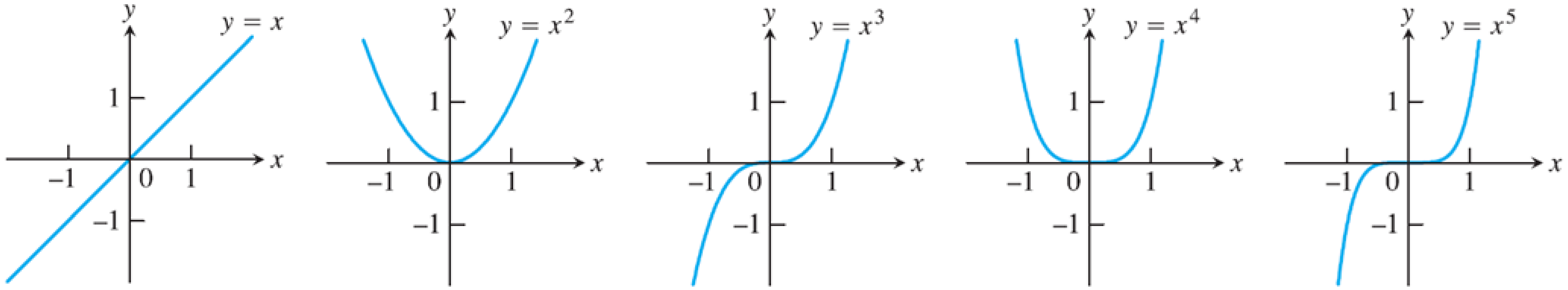
11. **Power Functions** A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $a = n$, a positive integer.



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Power Functions $f(x) = x^a$, where a is a constant,

(b) $a = -1$ or $a = -2$.

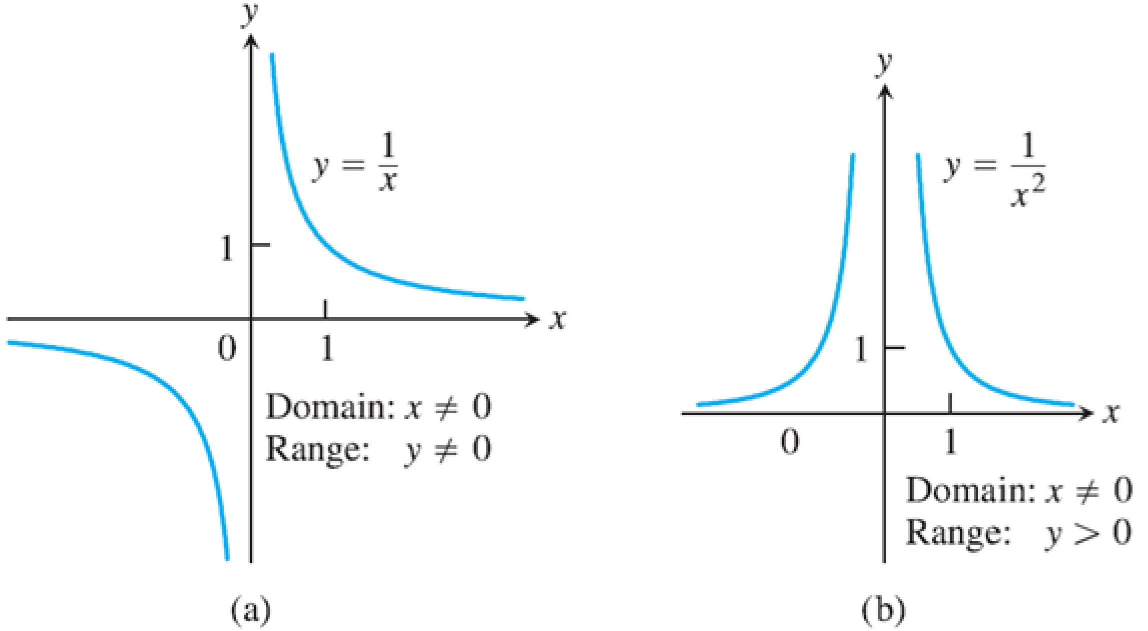

FIGURE Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

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Power Functions $f(x) = x^a$, where a is a constant,

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

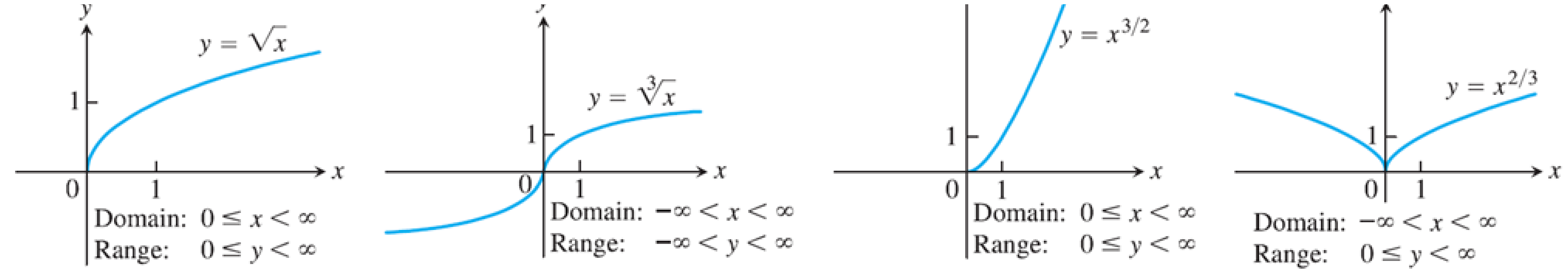

FIGURE Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

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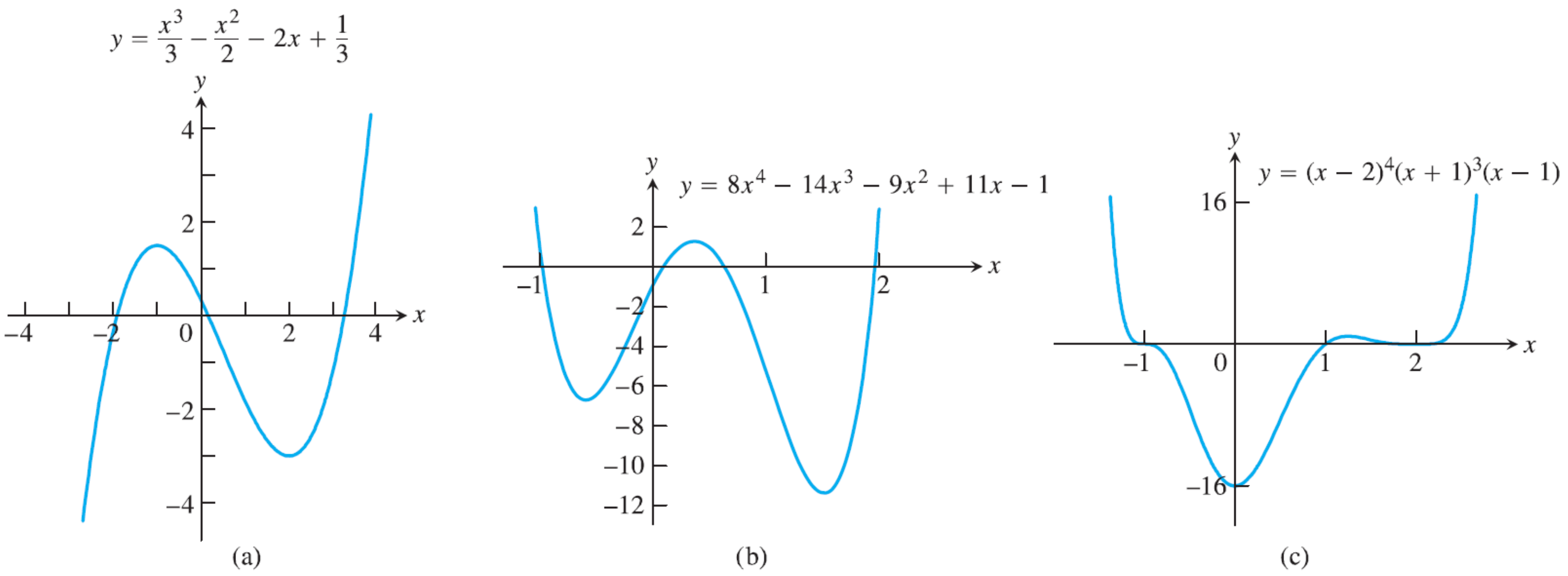
12. **Polynomials** A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure shows the graphs of three polynomials.



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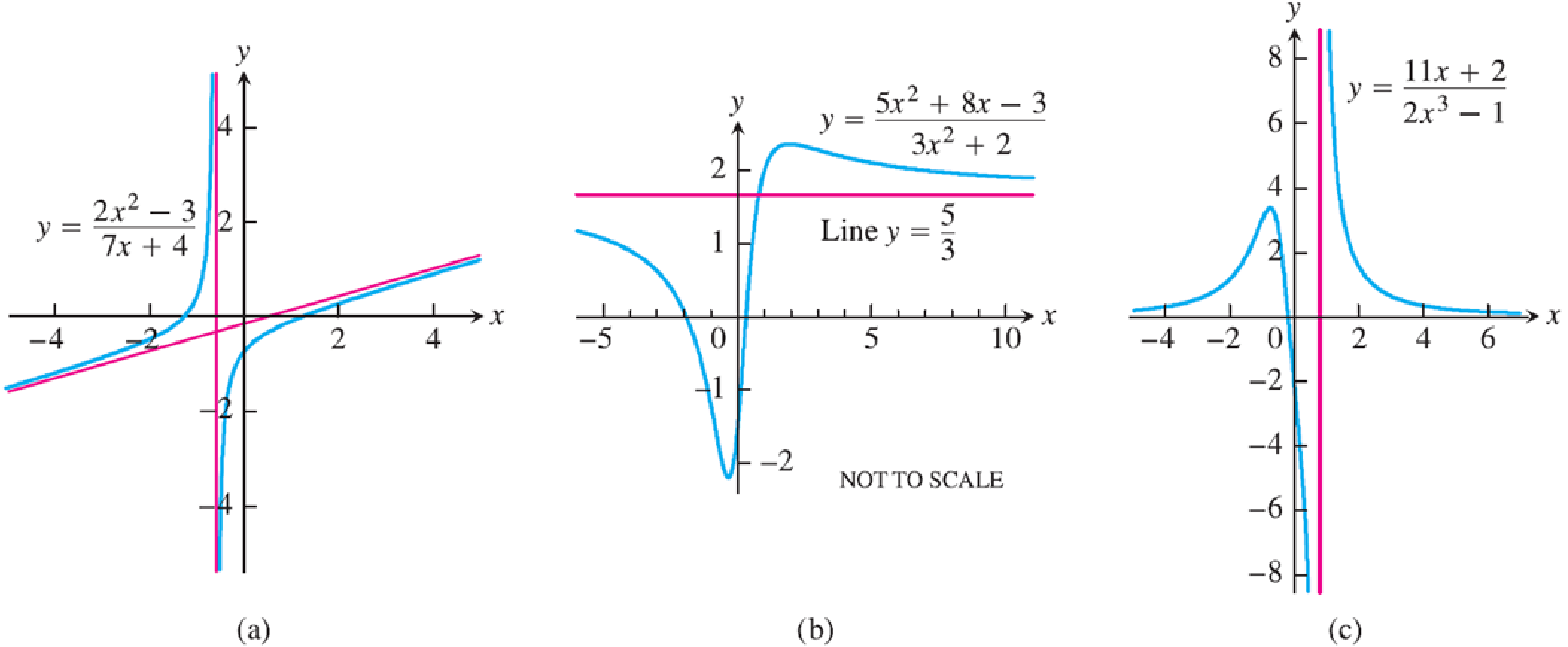
13.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure .



14. **Exponential Functions** Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0.

$$Domain : \mathbb{R}$$

$$Range : (0, \infty)$$

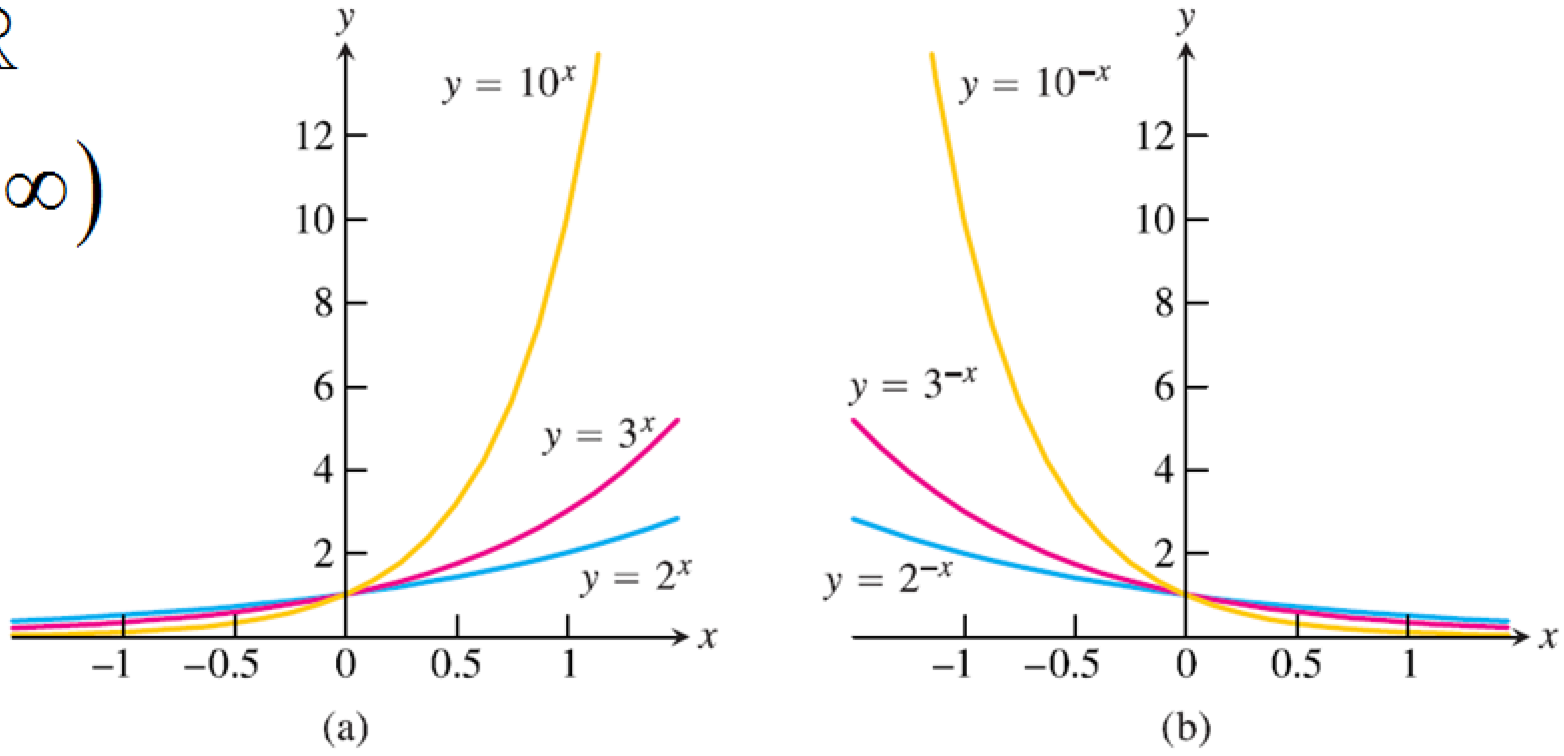


FIGURE Graphs of exponential functions.

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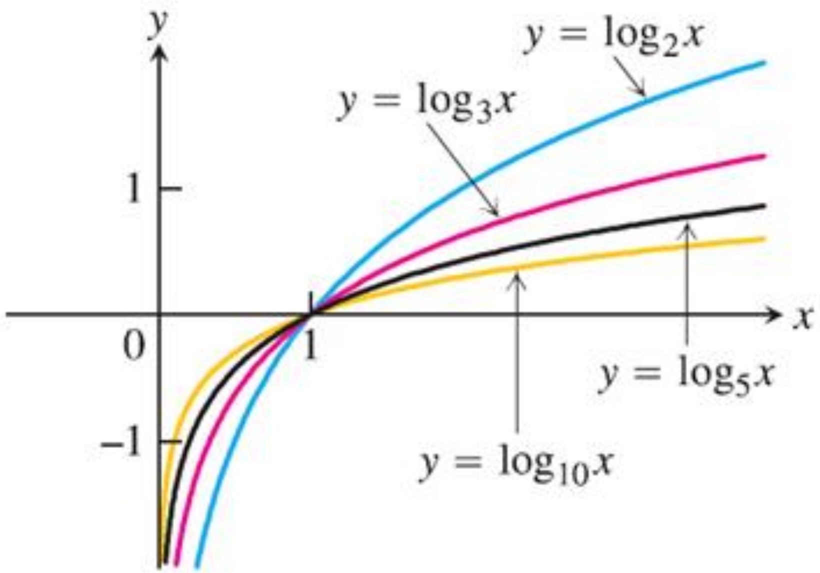
15.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions.

Figure shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.



$$Domain : (0, \infty)$$

$$Range : \mathbb{R}$$

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$(f + g)(x) = f(x) + g(x).$$
$$(f - g)(x) = f(x) - g(x).$$
$$(fg)(x) = f(x)g(x).$$

Notice that the $+$ sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the $+$ on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

$f: D_f \to \mathbb{R}$, $g: D_g \to \mathbb{R}$ $\quad c \in \mathbb{R}$

① $f \pm g : D_f \cap D_g \longrightarrow \mathbb{R}$, $(f \pm g)(x) = f(x) \pm g(x)$

② $f \cdot g : D_f \cap D_g \longrightarrow \mathbb{R}$, $(fg)(x) = f(x) \cdot g(x)$

③ $\frac{f}{g} : (D_f \cap D_g) - \{x \mid g(x) = 0\} \to \mathbb{R}$; $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

④ $cf : D_f \longrightarrow \mathbb{R}$, $(cf)(x) = cf(x)$, c is const.

EXAMPLE The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \sqrt{\dfrac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\dfrac{g}{f}(x) = \dfrac{g(x)}{f(x)} = \sqrt{\dfrac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

Composite Functions

Composition is another method for combining functions.

DEFINITION If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

* $f \circ g \neq g \circ f$
* $f \circ f^{-1} = I$

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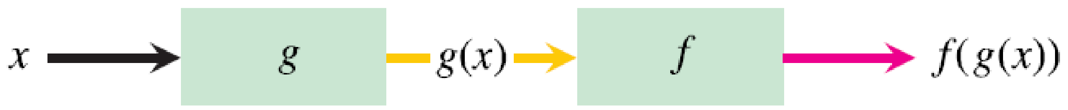

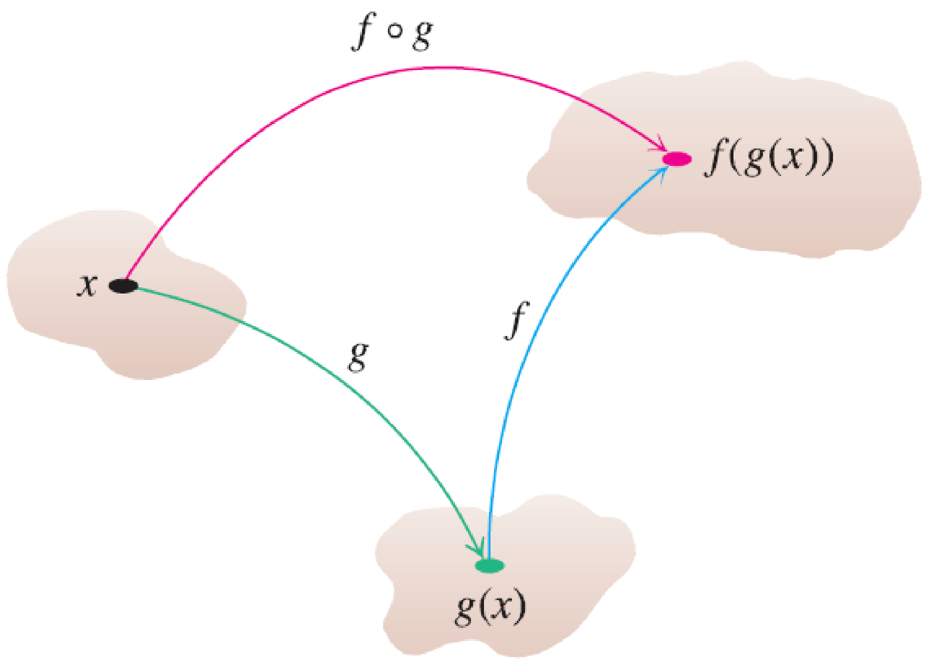

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EXAMPLE If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

	Composite	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = \left(\sqrt{x}\right)^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

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Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$ Shifts it *down* $\lvert k \rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$ Shifts it *right* $\lvert h \rvert$ units if $h < 0$

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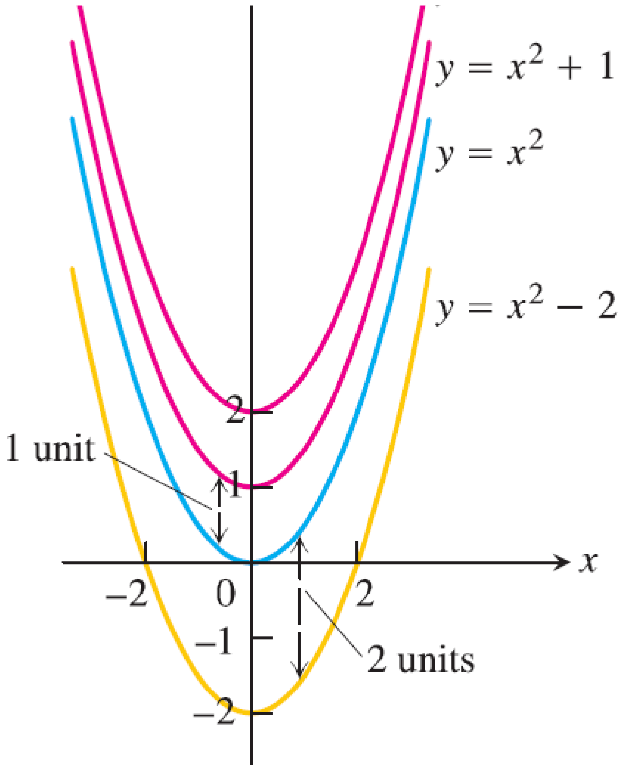
EXAMPLE

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit

(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units



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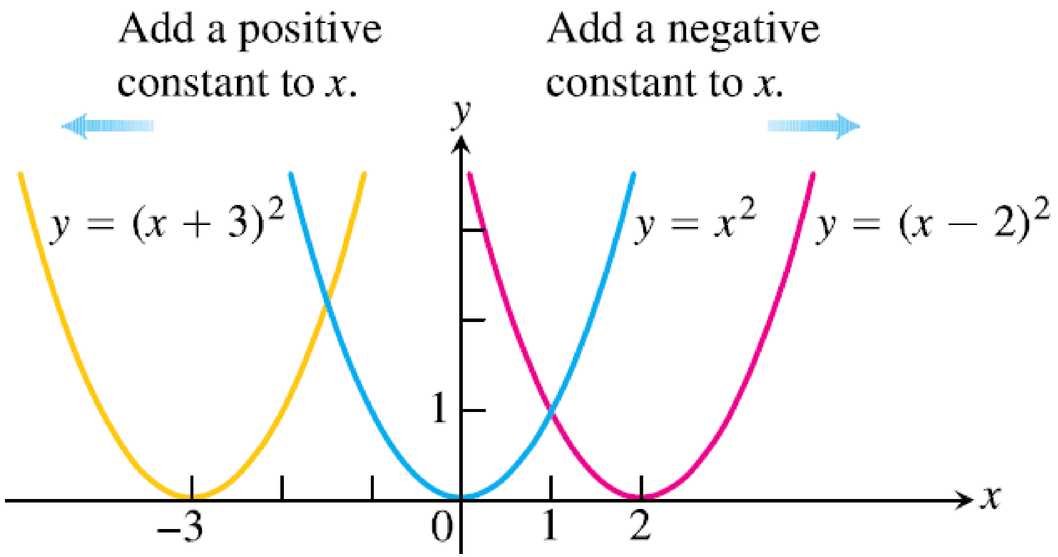
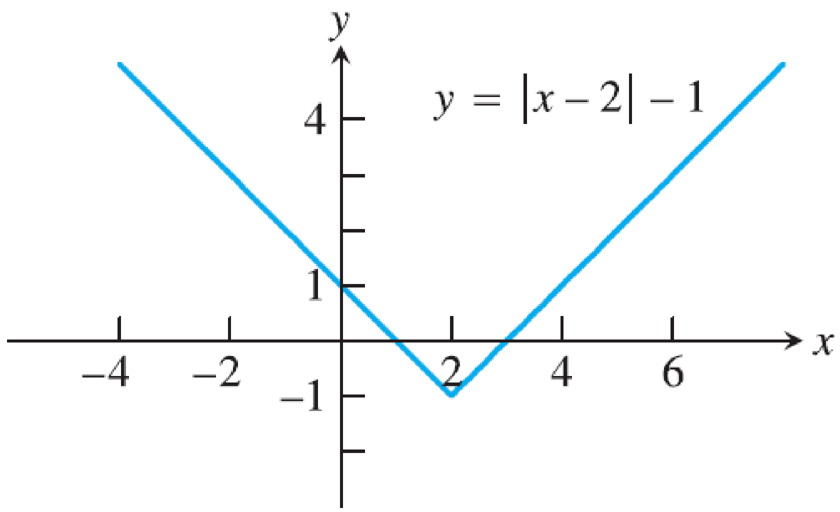
EXAMPLE

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left

(d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down





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Vertical and Horizontal Scaling Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

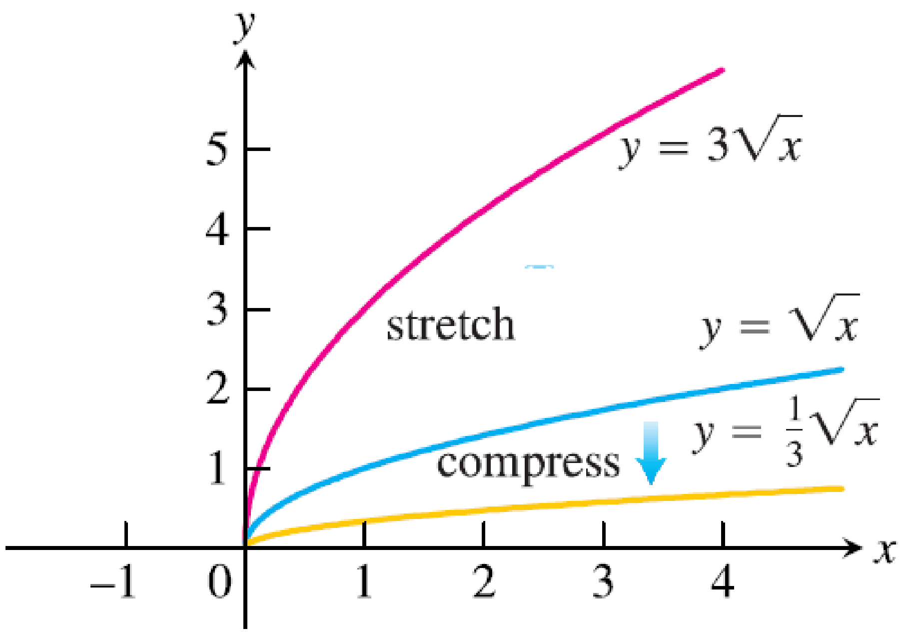
Reflecting Formulas

For $c = -1$, the graph is reflected:

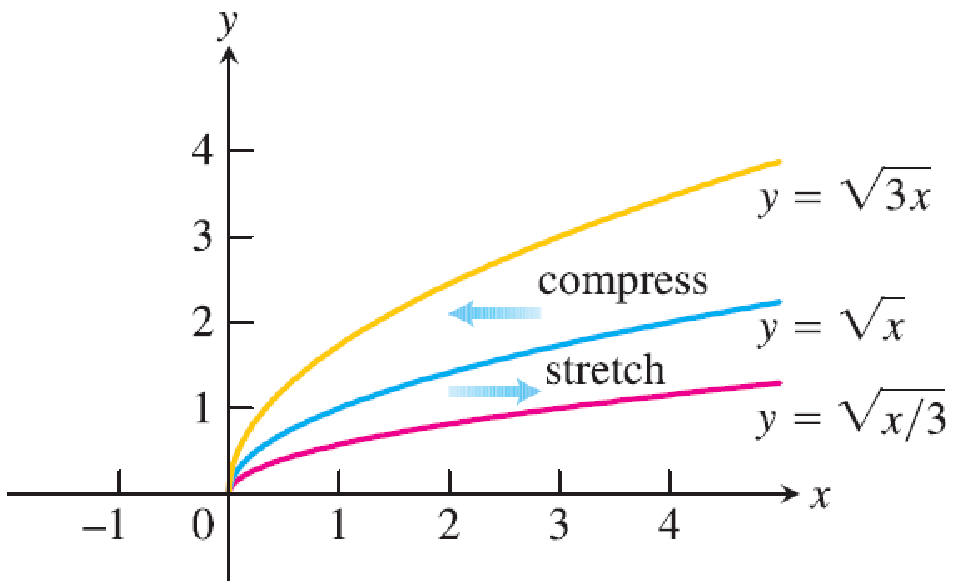
$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

EXAMPLE scale the graph of $y = \sqrt{x}$.

Vertical Scaling



Horizontal Scaling



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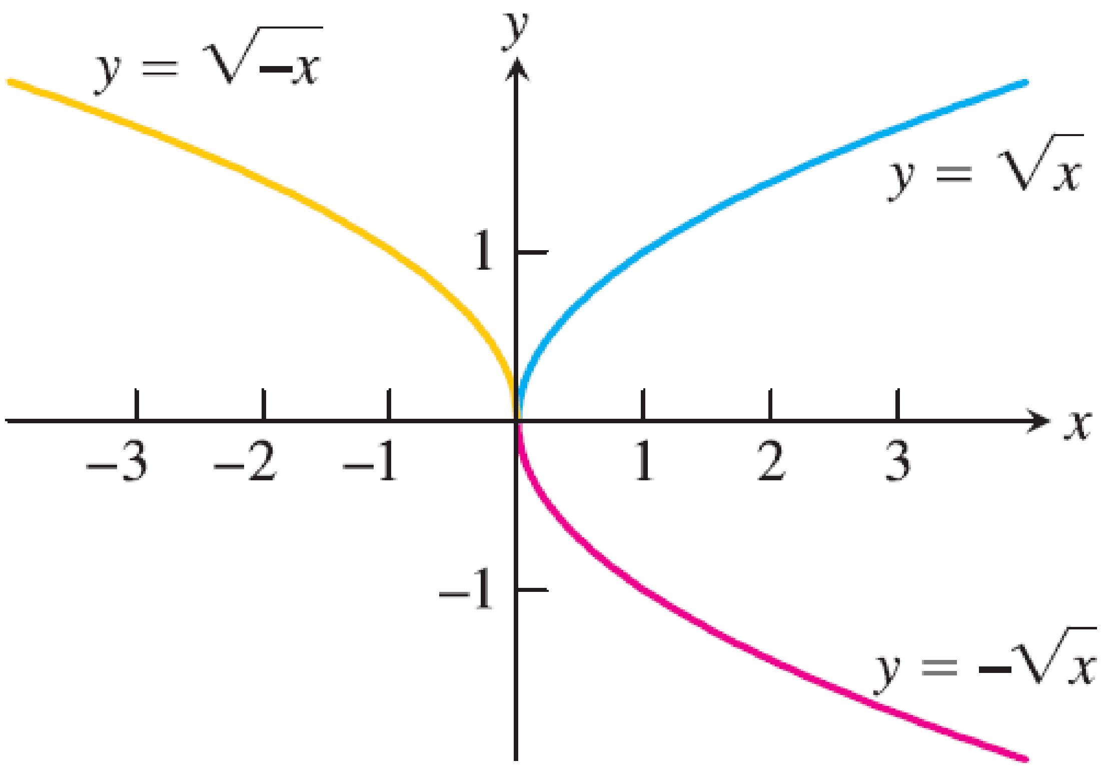
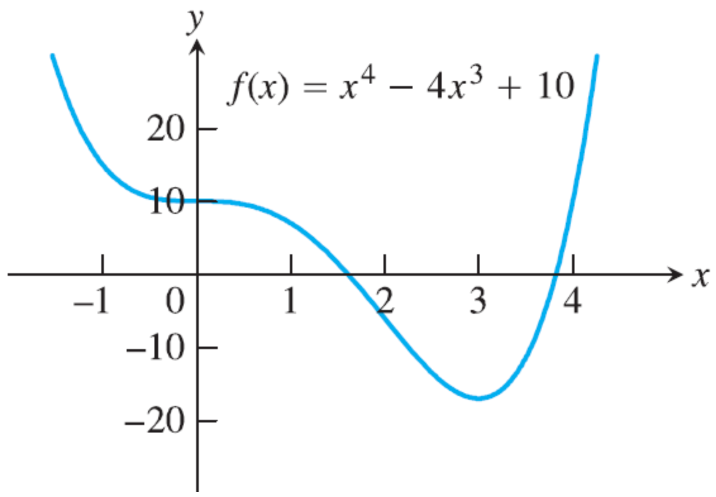
EXAMPLE reflect the graph of $y = \sqrt{x}$.



EXAMPLE Given the function $f(x) = x^4 - 4x^3 + 10$



sketch the graph of

(a) $f(-2x)$

(b) $-\frac{1}{2}f(x)$

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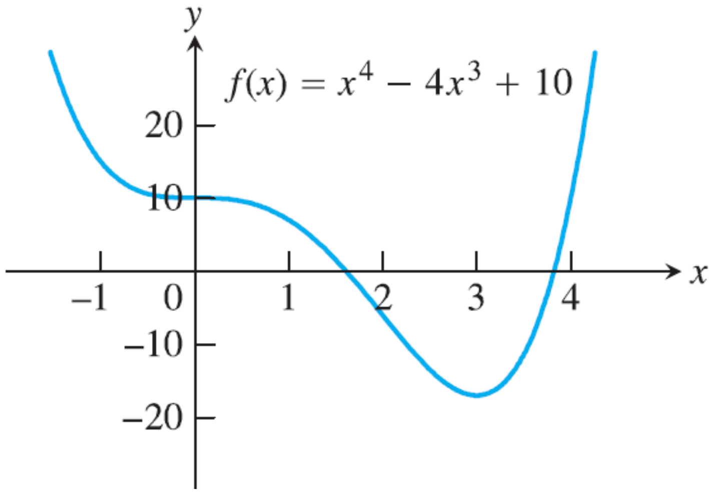
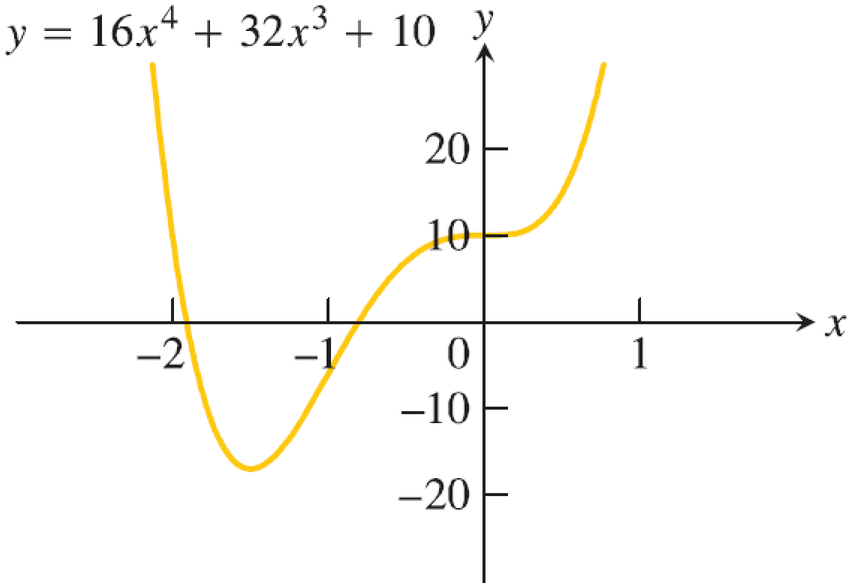
Solution

(a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y-axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f:

$$\begin{aligned} y &= f(-2x) = (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(-x)$	Reflects the graph of f across the y-axis.

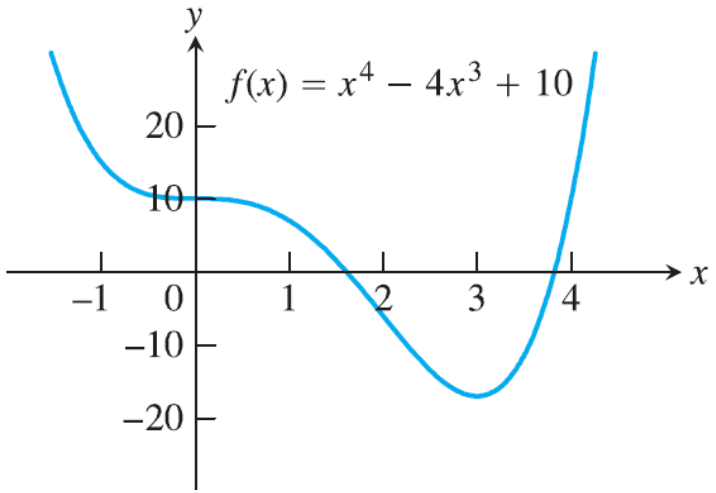


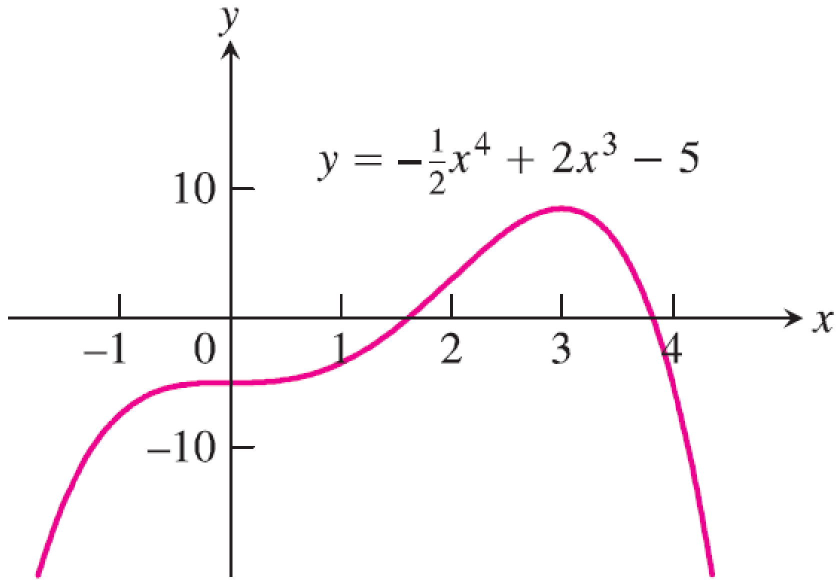


(b) The formula is

$$y = -\frac{1}{2} f(x) = -\frac{1}{2} x^4 + 2x^3 - 5.$$

$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = -f(x)$	Reflects the graph of f across the x-axis.





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HW:

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(x) = \sqrt{5x + 10}$

4. $g(x) = \sqrt{x^2 - 3x}$

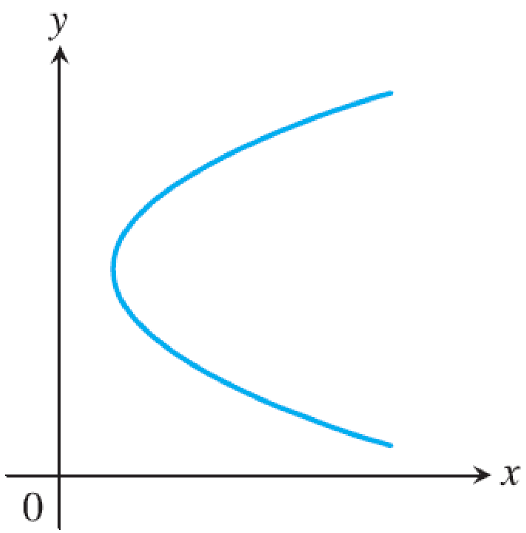
5. $f(t) = \dfrac{4}{3 - t}$

6. $G(t) = \dfrac{2}{t^2 - 16}$

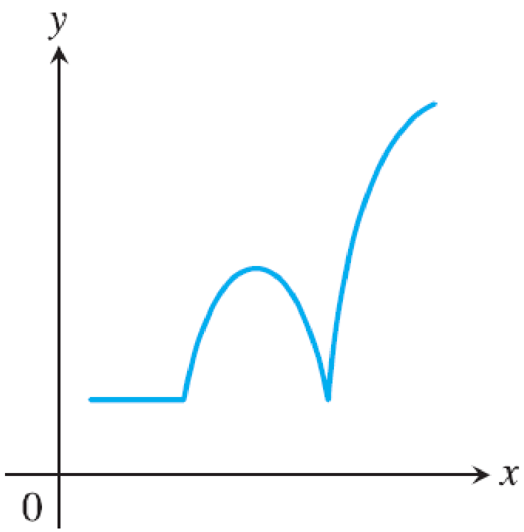
HW:

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

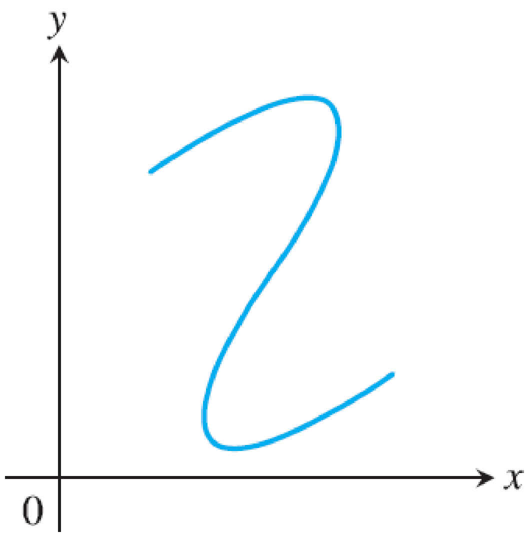
7. a.



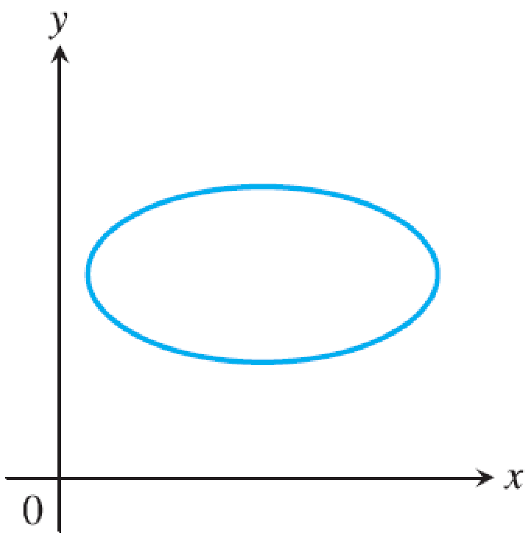
b.



8. a.



b.



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HW:

Functions and Graphs

Find the domain and graph the functions in Exercises 15–20.

15. $f(x) = 5 - 2x$

16. $f(x) = 1 - 2x - x^2$

17. $g(x) = \sqrt{|x|}$

18. $g(x) = \sqrt{-x}$

19. $F(t) = t/|t|$

20. $G(t) = 1/|t|$

Piecewise-Defined Functions

Graph the functions in Exercises 25–28.

25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

26. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

27. $F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$

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HW:

Increasing and Decreasing Functions

Graph the functions in Exercises 37–46. What symmetries, if any, dc the graphs have? Specify the intervals over which the function is in- creasing and the intervals where it is decreasing.

37. $y = -x^3$

38. $y = -\dfrac{1}{x^2}$

39. $y = -\dfrac{1}{x}$

40. $y = \dfrac{1}{|x|}$

HW:

Even and Odd Functions

In Exercises 47–58, say whether the function is even, odd, or neither. Give reasons for your answer.

47. $f(x) = 3$

48. $f(x) = x^{-5}$

49. $f(x) = x^2 + 1$

50. $f(x) = x^2 + x$

51. $g(x) = x^3 + x$

52. $g(x) = x^4 + 3x^2 - 1$

53. $g(x) = \dfrac{1}{x^2 - 1}$

54. $g(x) = \dfrac{x}{x^2 - 1}$

55. $h(t) = \dfrac{1}{t - 1}$

56. $h(t) = |t^3|$

57. $h(t) = 2t + 1$

58. $h(t) = 2|t| + 1$

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HW:

Algebraic Combinations

In Exercises 1 and 2, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

1. $f(x) = x, \quad g(x) = \sqrt{x - 1}$
2. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

HW:

Composites of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$ **b.** $g(f(0))$

c. $f(g(x))$ **d.** $g(f(x))$

e. $f(f(-5))$ **f.** $g(g(2))$

g. $f(f(x))$ **h.** $g(g(x))$

HW:

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 11 and 12 as a composite involving one or more of f, g, h, and j.

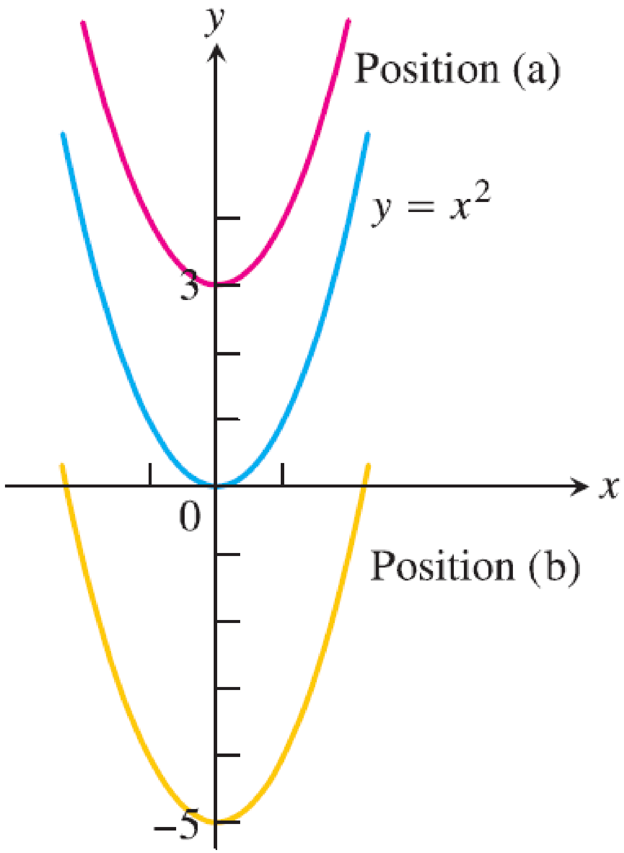
11. **a.** $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$

c. $y = x^{1/4}$ **d.** $y = 4x$

e. $y = \sqrt{(x-3)^3}$ **f.** $y = (2x - 6)^3$

HW:

22. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.

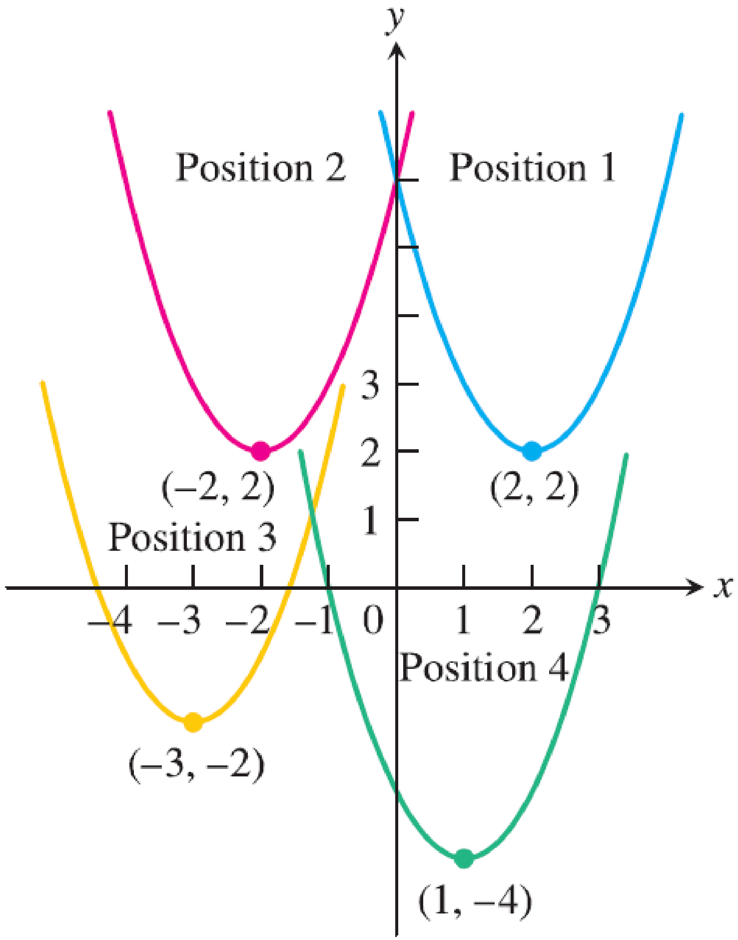


HW:

23. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$

c. $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$



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HW:

Vertical and Horizontal Scaling

Exercises 57–66 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

57. $y = x^2 - 1$, stretched vertically by a factor of 3

58. $y = x^2 - 1$, compressed horizontally by a factor of 2

59. $y = 1 + \dfrac{1}{x^2}$, compressed vertically by a factor of 2

60. $y = 1 + \dfrac{1}{x^2}$, stretched horizontally by a factor of 3

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.