MAT1071 MATHEMATICS I
3. WEEK
PART 2

- IMPLICIT DIFFERENTIATION
- MOTION ALONG A LINE
- LINEARIZATION AND DIFFERENTIALS
- INCREASING AND DECREASING FUNCTIONS

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Implicit Differentiation

$y = f(x)$ $\quad y = x^2 + 2x$ that expresses

$y = \ln x$ $\quad$ y explicitly

$y = 2^{x^2+2x-1}$ ... in terms of the variable x.

Sometimes we encounter equations like

$x^3 + y^3 - 9xy = 0$

$y^2 - x = 0$

$y^3 x + y\cos(xy) = 0$

These equations define an implicit relation between the variable x and y.

$F(x,y) = 0$

$y = y(x)$

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Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

EXAMPLE Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$y' = \frac{x^2}{y}, \qquad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

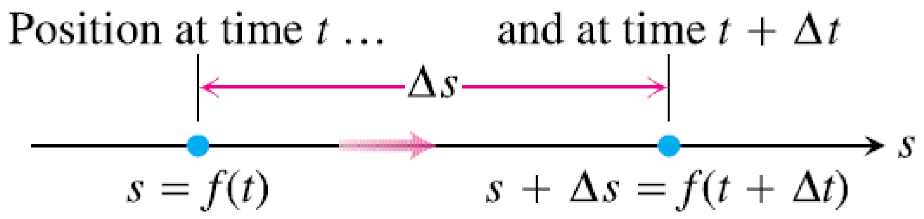
Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad \text{when } y \neq 0$$

Motion Along a Line

Suppose that an object is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t:

$$s = f(t).$$



- The **displacement** of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t),$$

- and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t.

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DEFINITION **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

EXAMPLE Let a particle moves along a line with $s = t^3$. Find the velocity of this particle at t:

Solution

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(t+\Delta t)^3 - t^3}{\Delta t} = 3t^2$$

$$\Rightarrow v(t) = 3t^2 \;//$$

Linearization and Differentials

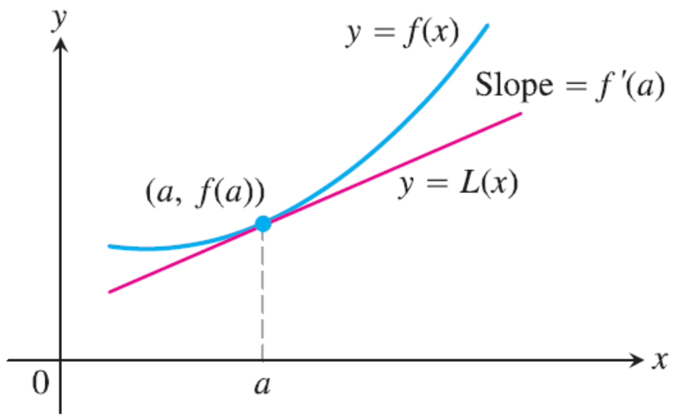

FIGURE The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

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EXAMPLE Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$.

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}(1 + x)^{-1/2}\bigg|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x - 3) = \frac{5}{4} + \frac{x}{4}.$$

EXAMPLE $f(x) = \sqrt{x}$

Find: $\sqrt{0.9}$, $\sqrt{1.2}$ = ?

Solution Linear approximation at $a = 1$

$f'(x) = \dfrac{1}{2\sqrt{x}}$ $\qquad f(1) = 1 \qquad f'(1) = \frac{1}{2}$

$$L(x) = f(a) + f'(a)(x-a) = 1 + \frac{1}{2}(x-1) = \frac{x}{2} + \frac{1}{2}$$

$$L(x) = \frac{x}{2} + \frac{1}{2}$$

$$f(x) \approx L(x) \Rightarrow f(0.9) \approx L(0.9) = \frac{0.9}{2} + \frac{1}{2} \Rightarrow \sqrt{0.9} \approx 0{,}95$$

$$f(x) \approx L(x) \Rightarrow f(1.2) \approx L(1.2) = \frac{1.2}{2} + \frac{1}{2} \Rightarrow \sqrt{1.2} \approx 1{,}1$$

EXAMPLE Find the app. value of

$$(1.001)^5 - 3(1.001)^{3/2} + 2$$

Solution Linearization at $a=1$ (known value)

$$L(x) = f(a) + f'(a)(x-a) \qquad f(x) = x^5 - 3x^{3/2} + 2$$

$$= f(1) + f'(1)(x-1) \qquad f'(x) = 5x^4 - \frac{9}{2}x^{1/2}$$

$$= 0 + \left(5 - \frac{9}{2}\right)(x-1) = \frac{1}{2}(x-1)$$

$$f(x) \approx L(x) \Rightarrow f(1.001) \cong \frac{1}{2}(1.001-1) \approx 0.0005$$

EXAMPLE Find the linearization of $f(x) = \cos x$ at $x = \pi/2$

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

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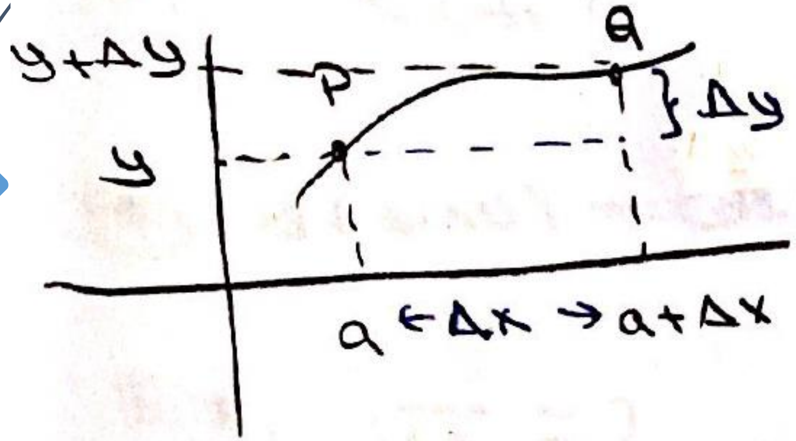
Differentials

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential *dx*** is an independent variable. The **differential *dy*** is

$$dy = f'(x)\, dx.$$



$$m_{PQ} = \frac{\Delta y}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

Differentials

- ☆ $dC = 0$ $\quad C$ is constant
- ☆ $d(u+v) = du + dv$
- ☆ $d(u-v) = du - dv$
- ☆ $d(Cu) = C\,du$ $\quad C$ is constant
- ☆ $d(uv) = v\,du + u\,dv$
- ☆ $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

EXAMPLE $y = x^2 \Rightarrow dy = 2x\,dx \Rightarrow \dfrac{dy}{dx} = 2x$

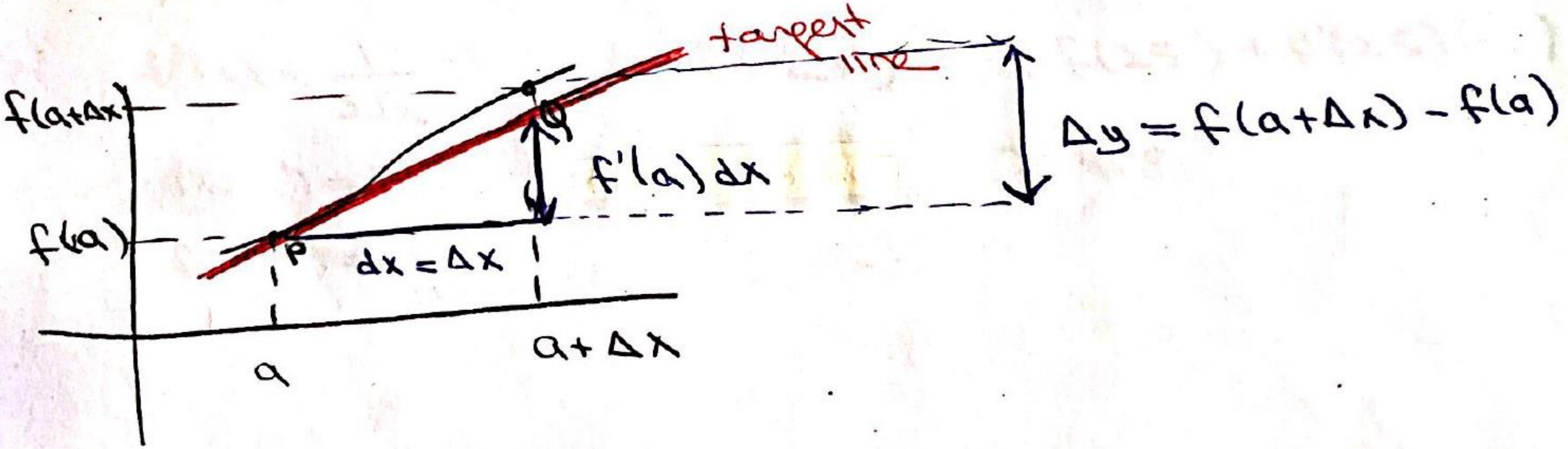
$$y = x^5 + 37x \Rightarrow dy = (5x^4 + 37)\,dx$$

$$\circledast \quad d(\tan 2x) = \sec^2 2x \; d(2x) = 2\sec^2 2x\,dx$$

$$\circledast \quad d\left(\frac{x}{x+1}\right) = \frac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \frac{dx}{(x+1)^2}$$

$$\circledast \quad d(\sin u) = \cos u\,du.$$

The Geom. Meaning of Differentials



$$\Delta f = \Delta y = f(a+\Delta x) - f(a)$$

$$\boxed{\Delta x \to 0 \qquad \Delta f = \Delta y \to df = dy = f'(x)\,dx}$$

$$f(a+dx) - f(a) \approx f'(a)\,dx$$

$$\Rightarrow f(a+dx) \approx \underbrace{f(a) + f'(a)\,dx\,(x-a)}_{L(x) \text{ at } x=a}$$

$$\Rightarrow f(a+dx) \approx L(x) \quad //$$

EXAMPLE $\sqrt{23} \cong ?$

Solution Diff. approximation at $a=25$ (known value)

$a=25$
$x=23$

$dx = \Delta x = x - a = -2$

$\underbrace{f(a+dx)}_{L(x)} \approx f(a) + f'(a)\underbrace{dx}_{\Delta x}$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f(25) = 5$

$f'(25) = \frac{1}{10}$

$f(23) \approx f(25) + f'(25)\cdot(-2)$

≈ 4.8

EXAMPLE $e^{0.2} \cong ?$

Solution $f(a+dx) \simeq f(a) + f'(a)dx$ Diff. approximation

at $a=0 \rightarrow$ known value

$f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \Rightarrow f(0) = 1$

$x = 0.2$

$dx = \Delta x = x - a = 0.2$

$\Rightarrow f(0.2) \approx f(0) + f'(0) \cdot dx = 1.2$

EXAMPLE- Find $\sqrt[4]{85}$ by using linear approximation.

Solution $L(x) = f(a) + f'(a)(x-a)$ → Δx

$f(a)$ → known value

So let apply linear approximation at $a=81$

$f(x) = \sqrt[4]{x} \Rightarrow f(81) = 3$

$f'(x) = \frac{1}{4} x^{-3/4} \Rightarrow f'(81) = \frac{1}{4}(3^4)^{-\frac{3}{4}} = \frac{1}{108}$

$L(x) = f(81) + f'(81)(x-81)$

$L(x) = 3 + \frac{1}{108}(x-81)$

$f(x) \approx L(x)$ for $x=85 \Rightarrow f(85) \approx L(85) = 3 + \frac{1}{108} \cdot 4$

≈ 3.037 //

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EXAMPLE Find the approximate value $(25)^{1/3}$

Solution $f(x)=\sqrt[3]{x}$ Use the linearization at $a=27$ known value

$$L(x)=f(a)+f'(a)(\underbrace{x-a}_{\Delta x})$$

$$f(x)=\sqrt[3]{x} \Rightarrow f(27)=3$$

$$f'(x)=\frac{1}{3}x^{-2/3} \Rightarrow f'(27)=\frac{1}{3}(3^3)^{-\frac{2}{3}}=\frac{1}{27}$$

$$L(x)=f(27)+f'(27)(x-27)$$

$$L(x)=3+\frac{1}{27}(x-27)$$

$$f(x)\approx L(x) \text{ for } x=25 \Rightarrow f(25)\approx 3+\frac{1}{27}(25-27)$$

$$\Rightarrow f(27)\approx 2.926$$

EXAMPLE $\cos 91^\circ = ?$

$$f(a+\overset{dx}{\Delta x}) \approx f(a) + f'(a)\,\overset{dx}{\Delta x}$$

Solution

$f(x) = \cos x$

$f'(x) = -\sin x$

$a \Rightarrow 90^\circ$ (known value)

$x = 91^\circ$

$\Delta x = x - a = 91^\circ - 90^\circ = 1^\circ$

Diff. approximation

$$f(91^\circ) \approx f(90^\circ) + f'(90^\circ)\cdot 1^\circ$$

$$\approx 0 + (-1)\,1^\circ \approx -1^\circ \approx -\frac{\pi}{180} \,//$$

By using linear approximation

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = f(90^\circ) + f'(90^\circ)(x-90^\circ)$$

$$L(x) = 90^\circ - x$$

$$f(x) \cong L(x) \Rightarrow f(91) \cong L(91) = 90^\circ - 91^\circ \approx -1^\circ \approx -\frac{\pi}{180} \,//$$

Increasing Functions and Decreasing Functions

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

Increasing Functions and Decreasing Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

☆ Critical points → $f' = 0$
↘ f' is undefined

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EXAMPLE

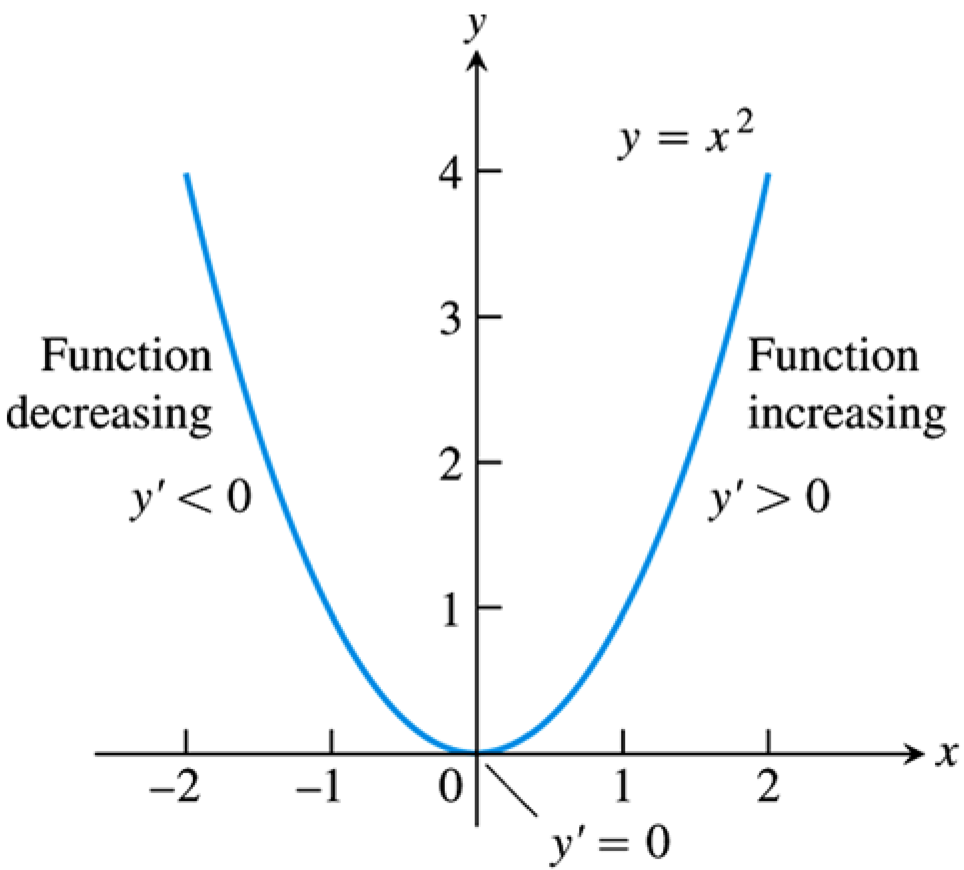
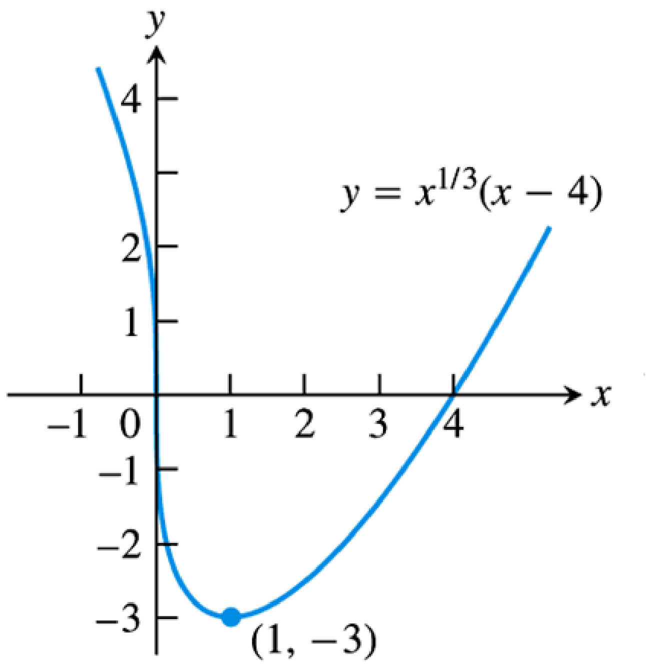


FIGURE The function $f(x) = x^2$ is monotonic on the intervals $(-\infty, 0]$ and $[0, \infty)$, but it is not monotonic on $(-\infty, \infty)$.

EXAMPLE

$$f(x) = x^{\frac{1}{3}}(x-4)$$

identify the intervals on which f is increasing and decreasing.



Solution

$D_f = \mathbb{R}$

$$f'(x) = \frac{4(x-1)}{3\sqrt[3]{x^2}}$$

Critical points
- $f' = 0 \Rightarrow \boxed{x=1}$
- f' is undef. $\Rightarrow \boxed{x=0 \text{ two fold}}$

x	$-\infty$		0		1		$+\infty$
f'		$-$	0	$-$	0	$+$	
f		↘		↘		↗	

f is increasing on $(1,\infty)$

f is decreasing on $(-\infty,0) \cup (0,1)$

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EXAMPLE Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

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EXAMPLE $f(x)=5x^2-x^4+7$ find the intervals for which $f(x)$ is inc. and decreasing.

Solution $f'(x)=10x-4x^3=0$

$$x(10-4x^2)=0 \Rightarrow x=0 \qquad x^2=\frac{\sqrt{10}}{2}$$

$$x=\pm\frac{\sqrt{10}}{2}$$

x	$-\infty$		$-\frac{\sqrt{10}}{2}$		0		$\frac{\sqrt{10}}{2}$	
$f'(x)$		$+$	0	$-$	0	$+$	0	$-$
$f(x)$		$\nearrow$		$\searrow$		$\nearrow$		$\searrow$

increasing on

$\left(-\infty,-\frac{\sqrt{10}}{2}\right)\cup\left(0,\frac{\sqrt{10}}{2}\right)$

decreasing on

$\left(-\frac{\sqrt{10}}{2},0\right)\cup\left(\frac{\sqrt{10}}{2},\infty\right)$

EXAMPLE $f(x) = 3x^5 - 15x^4 + 15x^3$

Solution

$x = 0$

$x = 3$

$x = 1$

$(-\infty, 1) \cup (3, \infty) \rightarrow$ increasing

$(1, 3) \longrightarrow$ decreasing

EXAMPLE $f(x) = \dfrac{x^2+100}{x^2-25}$ find the intervals on which f is increasing and decreasing.

Solution

$D_f = \mathbb{R} - \{-5, 5\}$

$f'(x) = \dfrac{-250x}{(x^2-25)^2}$

Critical points → $f' = 0 \Rightarrow x = 0$

↘ f' is undefined $\Rightarrow$ $\boxed{\begin{array}{l} x=5 \text{ two fold} \\ x=-5 \text{ two fold} \end{array}}$

x	$-\infty$		-5 (two fold)		0		5 (two fold)		$+\infty$
f'		$+$	‖	$+$	0	$-$	‖	$-$	
f		↗	‖	↗		↘	‖	↘	

$(-\infty, -5) \cup (-5, 0) \rightarrow f$ is increasing

$(0, 5) \cup (5, \infty) \rightarrow f$ is decreasing

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EXAMPLE Find the intervals on which $f(x) = x^{8/3} - 4x^{2/3}$ is increasing and decreasing. $D_f = \mathbb{R}$

Solution $f'(x) = \frac{8}{3}x^{-\frac{1}{3}}(x-1)(x+1).$

$$f'(x) = \frac{8}{3}\,\frac{(x-1)(x+1)}{\sqrt[3]{x}}$$

Roots of f'

$f' = 0$: $\boxed{x=1}$ $\boxed{x=-1}$

f' is und.: $\boxed{x=0}$

Critical points

x	$-\infty$		-1		0		1		$+\infty$
f'		$-$	0	$+$	0	$-$	0	$+$	
f		$\searrow$		$\nearrow$		$\searrow$		$\nearrow$	

$(-1,0) \cup (1,\infty) \rightarrow f$ is increasing

$(-\infty,-1) \cup (0,1) \rightarrow f$ is decreasing

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HW:

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–14.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $2xy + y^2 = x + y$
4. $x^3 - xy + y^3 = 1$
5. $x^2(x - y)^2 = x^2 - y^2$
6. $(3xy + 7)^2 = 6y$

Second Derivatives

In Exercises 19–26, use implicit differentiation to find dy/dx and then d^2y/dx^2.

19. $x^2 + y^2 = 1$
20. $x^{2/3} + y^{2/3} = 1$
21. $y^2 = x^2 + 2x$
22. $y^2 - 2x = 1 - 2y$
23. $2\sqrt{y} = x - y$
24. $xy + y^2 = 1$

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HW:

Slopes, Tangents, and Normals

In Exercises 29–38, verify that the given point is on the curve and find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

29. $x^2 + xy - y^2 = 1, \quad (2, 3)$

30. $x^2 + y^2 = 25, \quad (3, -4)$

31. $x^2y^2 = 9, \quad (-1, 3)$

32. $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$

HW:

Motion Along a Coordinate Line

Exercises 1–6 give the positions $s = f(t)$ of a body moving on a coordinate line, with s in meters and t in seconds.

a. Find the body's displacement and average velocity for the given time interval.

b. Find the body's speed and acceleration at the endpoints of the interval.

1. $s = t^2 - 3t + 2, \quad 0 \le t \le 2$

2. $s = 6t - t^2, \quad 0 \le t \le 6$

3. $s = -t^3 + 3t^2 - 3t, \quad 0 \le t \le 3$

4. $s = (t^4/4) - t^3 + t^2, \quad 0 \le t \le 3$

5. $s = \dfrac{25}{t^2} - \dfrac{5}{t}, \quad 1 \le t \le 5$

6. $s = \dfrac{25}{t + 5}, \quad -4 \le t \le 0$

HW: Finding Linearizations

In Exercises 1–5, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$
2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$
3. $f(x) = x + \dfrac{1}{x}, \quad a = 1$

In Exercises 7–12, find a linearization at a suitably chosen integer near x_0 at which the given function and its derivative are easy to evaluate.

7. $f(x) = x^2 + 2x, \quad x_0 = 0.1$
8. $f(x) = x^{-1}, \quad x_0 = 0.9$
9. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -0.9$

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HW:

Answer the following questions about the functions whose derivative are given in Exercises 1–14:

a. What are the critical points of f?

b. On what intervals is f increasing or decreasing?

1. $f'(x) = x(x - 1)$

2. $f'(x) = (x - 1)(x + 2)$

3. $f'(x) = (x - 1)^2(x + 2)$

4. $f'(x) = (x - 1)^2(x + 2)^2$

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.