

- b. Find the first four terms of a series for $\sec x \tan x$. For what values of x should the series converge?
- c. Check your result in part (b) by multiplying the series for $\sec x$ by the series given for $\tan x$ in Exercise 53.
- 55. Uniqueness of convergent power series**
- a. Show that if two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are convergent and equal for all values of x in an open interval $(-c, c)$, then $a_n = b_n$ for every n . (*Hint:* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. Differentiate term by term to show that a_n and b_n both equal $f^{(n)}(0)/(n!)$.)
- b. Show that if $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x in an open interval $(-c, c)$, then $a_n = 0$ for every n .
- 56. The sum of the series $\sum_{n=0}^{\infty} (n^2/2^n)$** To find the sum of this series, express $1/(1 - x)$ as a geometric series, differentiate both sides of the resulting equation with respect to x , multiply both sides of the result by x , differentiate again, multiply by x again, and set x equal to $1/2$. What do you get?

10.8 | Taylor and Maclaurin Series

This section shows how functions that are infinitely differentiable generate power series called Taylor series. In many cases, these series can provide useful polynomial approximations of the generating functions. Because they are used routinely by mathematicians and scientists, Taylor series are considered one of the most important topics of this chapter.

Series Representations

We know from Theorem 21 that within its interval of convergence the sum of a power series is a continuous function with derivatives of all orders. But what about the other way around? If a function $f(x)$ has derivatives of all orders on an interval I , can it be expressed as a power series on I ? And if it can, what will its coefficients be?

We can answer the last question readily if we assume that $f(x)$ is the sum of a power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - a)^n \\ &= a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots \end{aligned}$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I , we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x - a) + 3a_3(x - a)^2 + \cdots + na_n(x - a)^{n-1} + \cdots, \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x - a) + 3 \cdot 4a_4(x - a)^2 + \cdots, \\ f'''(x) &= 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x - a) + 3 \cdot 4 \cdot 5a_5(x - a)^2 + \cdots, \end{aligned}$$

with the n th derivative, for all n , being

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x - a) \text{ as a factor.}$$

Since these equations all hold at $x = a$, we have

$$f'(a) = a_1, \quad f''(a) = 1 \cdot 2a_2, \quad f'''(a) = 1 \cdot 2 \cdot 3a_3,$$

and, in general,

$$f^{(n)}(a) = n!a_n.$$

These formulas reveal a pattern in the coefficients of any power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ that converges to the values of f on I (“represents f on I ”). If there is such a series (still an open question), then there is only one such series, and its n th coefficient is

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

If f has a series representation, then the series must be

$$\begin{aligned} f(x) &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 \\ &\quad + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots. \end{aligned} \quad (1)$$

But if we start with an arbitrary function f that is infinitely differentiable on an interval I centered at $x = a$ and use it to generate the series in Equation (1), will the series then converge to $f(x)$ at each x in the interior of I ? The answer is maybe—for some functions it will but for other functions it will not, as we will see.

Taylor and Maclaurin Series

The series on the right-hand side of Equation (1) is the most important and useful series we will study in this chapter.

HISTORICAL BIOGRAPHIES

Brook Taylor
(1685–1731)

Colin Maclaurin
(1698–1746)

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 \\ &\quad + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots. \end{aligned}$$

The **Maclaurin series generated by f** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

The Maclaurin series generated by f is often just called the Taylor series of f .

EXAMPLE 1 Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \dots$. Taking derivatives we get

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \quad \dots, \quad f^{(n)}(x) = (-1)^n n!x^{-(n+1)},$$

so that

$$f(2) = 2^{-1} = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}, \quad \dots, \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$\begin{aligned} f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x - 2)^n + \cdots \\ = \frac{1}{2} - \frac{(x - 2)}{2^2} + \frac{(x - 2)^2}{2^3} - \cdots + (-1)^n \frac{(x - 2)^n}{2^{n+1}} + \cdots. \end{aligned}$$

This is a geometric series with first term $1/2$ and ratio $r = -(x - 2)/2$. It converges absolutely for $|x - 2| < 2$ and its sum is

$$\frac{1/2}{1 + (x - 2)/2} = \frac{1}{2 + (x - 2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x - 2| < 2$ or $0 < x < 4$. ■

Taylor Polynomials

The linearization of a differentiable function f at a point a is the polynomial of degree one given by

$$P_1(x) = f(a) + f'(a)(x - a).$$

In Section 3.11 we used this linearization to approximate $f(x)$ at values of x near a . If f has derivatives of higher order at a , then it has higher-order polynomial approximations as well, one for each available derivative. These polynomials are called the Taylor polynomials of f .

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N , the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x - a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

We speak of a Taylor polynomial of *order n* rather than *degree n* because $f^{(n)}(a)$ may be zero. The first two Taylor polynomials of $f(x) = \cos x$ at $x = 0$, for example, are $P_0(x) = 1$ and $P_1(x) = 1$. The first-order Taylor polynomial has degree zero, not one.

Just as the linearization of f at $x = a$ provides the best linear approximation of f in the neighborhood of a , the higher-order Taylor polynomials provide the “best” polynomial approximations of their respective degrees. (See Exercise 40.)

EXAMPLE 2 Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \dots$, the Taylor series generated by f at $x = 0$ (see Figure 10.17) is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \\ = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots \\ = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

This is also the Maclaurin series for e^x . In the next section we will see that the series converges to e^x at every x .

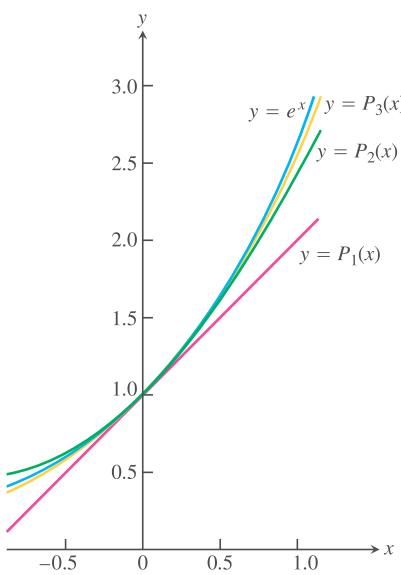


FIGURE 10.17 The graph of $f(x) = e^x$ and its Taylor polynomials

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + (x^2/2!)$$

$$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!).$$

Notice the very close agreement near the center $x = 0$ (Example 2).

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

EXAMPLE 3 Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots &&\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \quad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

This is also the Maclaurin series for $\cos x$. Notice that only even powers of x occur in the Taylor series generated by the cosine function, which is consistent with the fact that it is an even function. In Section 10.9, we will see that the series converges to $\cos x$ at every x .

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n + 1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 10.18 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y -axis.

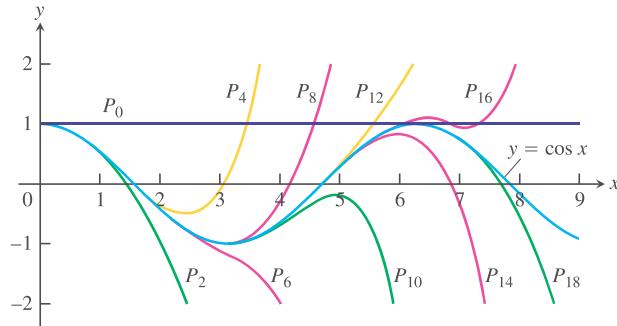


FIGURE 10.18 The polynomials

$$P_{2n}(x) = \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to $\cos x$ as $n \rightarrow \infty$. We can deduce the behavior of $\cos x$ arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

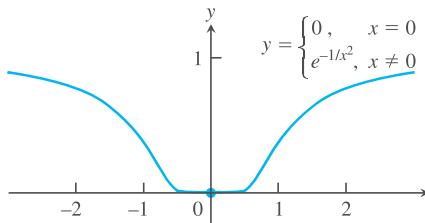


FIGURE 10.19 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4). Therefore its Taylor series is not the function itself.

EXAMPLE 4 It can be shown (though not easily) that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

(Figure 10.19) has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n . This means that the Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots \\ = 0 + 0 + \cdots + 0 + \cdots. \end{aligned}$$

The series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$. That is, the Taylor series generated by $f(x)$ in this example is *not* equal to the function $f(x)$ itself. ■

Two questions still remain.

- For what values of x can we normally expect a Taylor series to converge to its generating function?
- How accurately do a function's Taylor polynomials approximate the function on a given interval?

The answers are provided by a theorem of Taylor in the next section.

Exercises 10.8

Finding Taylor Polynomials

In Exercises 1–10, find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a .

- | | |
|-------------------------------|--------------------------------|
| 1. $f(x) = e^{2x}, a = 0$ | 2. $f(x) = \sin x, a = 0$ |
| 3. $f(x) = \ln x, a = 1$ | 4. $f(x) = \ln(1+x), a = 0$ |
| 5. $f(x) = 1/x, a = 2$ | 6. $f(x) = 1/(x+2), a = 0$ |
| 7. $f(x) = \sin x, a = \pi/4$ | 8. $f(x) = \tan x, a = \pi/4$ |
| 9. $f(x) = \sqrt{x}, a = 4$ | 10. $f(x) = \sqrt{1-x}, a = 0$ |

Finding Taylor Series at $x = 0$ (Maclaurin Series)

Find the Maclaurin series for the functions in Exercises 11–22.

- | | |
|----------------------------------------|----------------------------------------|
| 11. e^{-x} | 12. xe^x |
| 13. $\frac{1}{1+x}$ | 14. $\frac{2+x}{1-x}$ |
| 15. $\sin 3x$ | 16. $\sin \frac{x}{2}$ |
| 17. $7 \cos(-x)$ | 18. $5 \cos \pi x$ |
| 19. $\cosh x = \frac{e^x + e^{-x}}{2}$ | 20. $\sinh x = \frac{e^x - e^{-x}}{2}$ |
| 21. $x^4 - 2x^3 - 5x + 4$ | 22. $\frac{x^2}{x+1}$ |

Finding Taylor and Maclaurin Series

In Exercises 23–32, find the Taylor series generated by f at $x = a$.

23. $f(x) = x^3 - 2x + 4, a = 2$
24. $f(x) = 2x^3 + x^2 + 3x - 8, a = 1$

25. $f(x) = x^4 + x^2 + 1, a = -2$
26. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, a = -1$
27. $f(x) = 1/x^2, a = 1$
28. $f(x) = 1/(1-x)^3, a = 0$
29. $f(x) = e^x, a = 2$
30. $f(x) = 2^x, a = 1$
31. $f(x) = \cos(2x + (\pi/2)), a = \pi/4$
32. $f(x) = \sqrt{x+1}, a = 0$

In Exercises 33–36, find the first three nonzero terms of the Maclaurin series for each function and the values of x for which the series converges absolutely.

33. $f(x) = \cos x - (2/(1-x))$
34. $f(x) = (1-x+x^2)e^x$
35. $f(x) = (\sin x) \ln(1+x)$
36. $f(x) = x \sin^2 x$

Theory and Examples

37. Use the Taylor series generated by e^x at $x = a$ to show that

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots \right].$$

38. (Continuation of Exercise 37.) Find the Taylor series generated by e^x at $x = 1$. Compare your answer with the formula in Exercise 37.
39. Let $f(x)$ have derivatives through order n at $x = a$. Show that the Taylor polynomial of order n and its first n derivatives have the same values that f and its first n derivatives have at $x = a$.

- 40. Approximation properties of Taylor polynomials** Suppose that $f(x)$ is differentiable on an interval centered at $x = a$ and that $g(x) = b_0 + b_1(x - a) + \cdots + b_n(x - a)^n$ is a polynomial of degree n with constant coefficients b_0, \dots, b_n . Let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions

- i) $E(a) = 0$ The approximation error is zero at $x = a$.
ii) $\lim_{x \rightarrow a} \frac{E(x)}{(x - a)^n} = 0$, The error is negligible when compared to $(x - a)^n$.

then

$$\begin{aligned} g(x) &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots \\ &\quad + \frac{f^{(n)}(a)}{n!}(x - a)^n. \end{aligned}$$

Thus, the Taylor polynomial $P_n(x)$ is the only polynomial of degree less than or equal to n whose error is both zero at $x = a$ and negligible when compared with $(x - a)^n$.

Quadratic Approximations The Taylor polynomial of order 2 generated by a twice-differentiable function $f(x)$ at $x = a$ is called the *quadratic approximation* of f at $x = a$. In Exercises 41–46, find the (a) linearization (Taylor polynomial of order 1) and (b) quadratic approximation of f at $x = 0$.

41. $f(x) = \ln(\cos x)$ 42. $f(x) = e^{\sin x}$
43. $f(x) = 1/\sqrt{1 - x^2}$ 44. $f(x) = \cosh x$
45. $f(x) = \sin x$ 46. $f(x) = \tan x$

10.9 | Convergence of Taylor Series

In the last section we asked when a Taylor series for a function can be expected to converge to that (generating) function. We answer the question in this section with the following theorem.

THEOREM 23—Taylor's Theorem If f and its first n derivatives $f', f'', \dots, f^{(n)}$ are continuous on the closed interval between a and b , and $f^{(n)}$ is differentiable on the open interval between a and b , then there exists a number c between a and b such that

$$\begin{aligned} f(b) &= f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots \\ &\quad + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b - a)^{n+1}. \end{aligned}$$

Taylor's Theorem is a generalization of the Mean Value Theorem (Exercise 45). There is a proof of Taylor's Theorem at the end of this section.

When we apply Taylor's Theorem, we usually want to hold a fixed and treat b as an independent variable. Taylor's formula is easier to use in circumstances like these if we change b to x . Here is a version of the theorem with this change.

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a , then for each positive integer n and for each x in I ,

$$\begin{aligned} f(x) &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots \\ &\quad + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \end{aligned} \tag{1}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \tag{2}$$

When we state Taylor's theorem this way, it says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x).$$

The function $R_n(x)$ is determined by the value of the $(n + 1)$ st derivative $f^{(n+1)}$ at a point c that depends on both a and x , and that lies somewhere between them. For any value of n we want, the equation gives both a polynomial approximation of f of that order and a formula for the error involved in using that approximation over the interval I .

Equation (1) is called **Taylor's formula**. The function $R_n(x)$ is called the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I .

If $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I , and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

Often we can estimate R_n without knowing the value of c , as the following example illustrates.

EXAMPLE 1 Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x .

Solution The function has derivatives of all orders throughout the interval $I = (-\infty, \infty)$. Equations (1) and (2) with $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x) \quad \begin{array}{l} \text{Polynomial from} \\ \text{Section 10.8, Example 2} \end{array}$$

and

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

Since e^x is an increasing function of x , e^c lies between $e^0 = 1$ and e^x . When x is negative, so is c , and $e^c < 1$. When x is zero, $e^x = 1$ and $R_n(x) = 0$. When x is positive, so is c , and $e^c < e^x$. Thus, for $R_n(x)$ given as above,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \leq 0, \quad e^c < 1$$

and

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \quad \text{when } x > 0. \quad e^c < e^x$$

Finally, because

$$\lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} = 0 \quad \text{for every } x, \quad \begin{array}{l} \text{Section 10.1, Theorem 5} \end{array}$$

$\lim_{n \rightarrow \infty} R_n(x) = 0$, and the series converges to e^x for every x . Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots. \quad (3)$$

■

The Number e as a Series

We can use the result of Example 1 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

where for some c between 0 and 1,

$$R_n(1) = e^c \frac{1}{(n+1)!} < \frac{3}{(n+1)!}. \quad e^c < e^1 < 3$$

Estimating the Remainder

It is often possible to estimate $R_n(x)$ as we did in Example 1. This method of estimation is so convenient that we state it as a theorem for future reference.

THEOREM 24—The Remainder Estimation Theorem If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a , inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f , then the series converges to $f(x)$.

The next two examples use Theorem 24 to show that the Taylor series generated by the sine and cosine functions do in fact converge to the functions themselves.

EXAMPLE 2 Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x .

Solution The function and its derivatives are

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x, \\ &\vdots && \vdots \\ f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k+1)}(x) &= (-1)^k \cos x, \end{aligned}$$

so

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

The series has only odd-powered terms and, for $n = 2k + 1$, Taylor's Theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

All the derivatives of $\sin x$ have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M = 1$ to obtain

$$|R_{2k+1}(x)| \leq 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}.$$

From Theorem 5, Rule 6, we have $(|x|^{2k+2}/(2k+2)!) \rightarrow 0$ as $k \rightarrow \infty$, whatever the value of x , so $R_{2k+1}(x) \rightarrow 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x . Thus,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \quad (4)$$

EXAMPLE 3 Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x .

Solution We add the remainder term to the Taylor polynomial for $\cos x$ (Section 10.8, Example 3) to obtain Taylor's formula for $\cos x$ with $n = 2k$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \leq 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x , $R_{2k}(x) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, the series converges to $\cos x$ for every value of x . Thus,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \quad (5)$$

■

Using Taylor Series

Since every Taylor series is a power series, the operations of adding, subtracting, and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

EXAMPLE 4 Using known series, find the first few terms of the Taylor series for the given function using power series operations.

$$(a) \frac{1}{3}(2x + x \cos x) \quad (b) e^x \cos x$$

Solution

$$\begin{aligned}
 (a) \frac{1}{3}(2x + x \cos x) &= \frac{2}{3}x + \frac{1}{3}x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots\right) \\
 &= \frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{3!} + \frac{x^5}{3 \cdot 4!} - \cdots = x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots
 \end{aligned}$$

$$\begin{aligned}
 (b) e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \quad \text{Multiply the first} \\
 &\quad \text{series by each term} \\
 &\quad \text{of the second series.} \\
 &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!2!} + \frac{x^5}{2!3!} + \cdots\right) \\
 &\quad + \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!4!} + \cdots\right) + \cdots \\
 &= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots
 \end{aligned}$$

■

By Theorem 20, we can use the Taylor series of the function f to find the Taylor series of $f(u(x))$ where $u(x)$ is any continuous function. The Taylor series resulting from this substitution will converge for all x such that $u(x)$ lies within the interval of convergence of the Taylor

series of f . For instance, we can find the Taylor series for $\cos 2x$ by substituting $2x$ for x in the Taylor series for $\cos x$:

$$\begin{aligned}\cos 2x &= \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots && \text{Eq. (5) with } 2x \text{ for } x \\ &= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \dots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} x^{2k}}{(2k)!}.\end{aligned}$$

EXAMPLE 5 For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ with an error of magnitude no greater than 3×10^{-4} ?

Solution Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x . According to the Alternating Series Estimation Theorem (Section 10.6), the error in truncating

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

after $(x^3/3!)$ is no greater than

$$\left| \frac{x^5}{5!} \right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \text{or} \quad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514. \quad \text{Rounded down, to be safe}$$

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate $x - (x^3/3!)$ for $\sin x$ is an underestimate when x is positive, because then $x^5/120$ is positive.

Figure 10.20 shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials. The graph of $P_3(x) = x - (x^3/3!)$ is almost indistinguishable from the sine curve when $0 \leq x \leq 1$. ■

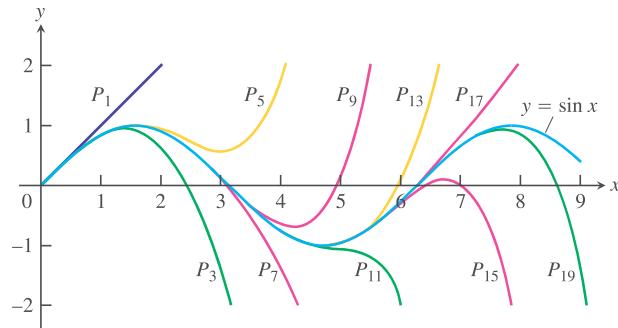


FIGURE 10.20 The polynomials

$$P_{2n+1}(x) = \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

converge to $\sin x$ as $n \rightarrow \infty$. Notice how closely $P_3(x)$ approximates the sine curve for $x \leq 1$ (Example 5).