MAT1071 MATHEMATICS I
EXAMPLES_1

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EXAMPLE $f(x) = x - 1 + \dfrac{1}{x+1}$ Determine the intervals on which f is increasing and decreasing

Solution $D_f = \mathbb{R} - \{-1\}$

$$f'(x) = \frac{x(x+2)}{(x+1)^2}$$

Critical points $\rightarrow f' = 0 \Rightarrow \boxed{x = 0}$, $\boxed{x = -2}$

$\searrow$ f' is und $\Rightarrow$ $x = -1$ two fol

x	$-\infty$		-2		-1		0		$+\infty$
f'		$+$	0	$-$	$\parallel$	$-$	0	$+$	
f		$\nearrow$		$\searrow$	$\parallel$	$\searrow$		$\nearrow$	

f is increasing on $(-\infty, -2) \cup (0, \infty)$

f is decreasing on $(-2, -1) \cup (-1, 0)$

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EXAMPLE $f(x)=x\cdot\sqrt{a^2-x^2}+a^2\cdot\arcsin\left(\frac{x}{a}\right) \Rightarrow f'(x)=?$ $\quad a\in\mathbb{R}$

Solution $f'(x)=1\cdot\sqrt{a^2-x^2}+x\left(\sqrt{a^2-x^2}\right)'+a^2\cdot(\arcsin u)'$, $\quad u=\frac{x}{a}$

$$(\sqrt{v})=v^{1/2}$$

$$(\sqrt{u})'=\frac{1}{2}\cdot v'\cdot\frac{1}{\sqrt{v}}$$

$$=\frac{1}{2}\cdot(-2x)\cdot\frac{1}{\sqrt{a^2-x^2}}=-\frac{x}{\sqrt{a^2-x^2}}$$

$$(\arcsin u)'=\frac{u'}{\sqrt{1-u^2}}$$

$$=\frac{1/a}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$=\frac{1}{a\sqrt{\frac{a^2-x^2}{a^2}}}$$

$$=\frac{1}{\sqrt{a^2-x^2}}$$

$$f'(x)=\sqrt{a^2-x^2}-\frac{x^2}{\sqrt{a^2-x^2}}+\frac{a^2}{\sqrt{a^2-x^2}}$$

$$=\frac{a^2-x^2-x^2+a^2}{\sqrt{a^2-x^2}}=\frac{2(a^2-x^2)}{\sqrt{a^2-x^2}}=2\sqrt{a^2-x^2}$$

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EXAMPLE $f(x) = 2^{\cos(5x)} \Rightarrow f'(0) = ?$

Solution $u(x) = \cos(5x)$ o.ü $y = e^{u(x)}$

$$y' = \left[2^{u(x)}\right]' = u'(x) \cdot 2^{u(x)} \cdot \ln 2$$

$$u'(x) = 5.(-\sin(5x)) \Rightarrow y' = -5\sin 5x \, 2^{\cos 5x} \ln 2$$

$$y'(0) = f'(0) = -5\sin 0 \; 2^{\cos 0} . \ln 2$$

$$= 0$$

EXAMPLE $3y^2 = \frac{2}{3}x^3 + \cos y$ için $\frac{dy}{dx} = ?$

Solution

$$\frac{d}{dx}(3y^2) = \frac{d}{dx}\left(\frac{2}{3}x^3 + \cos y\right)$$

$$\Rightarrow 6y\frac{dy}{dx} = 2x^2 - \sin y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx}(6y + \sin y) = 2x^2$$

$$\Rightarrow \frac{dy}{dx} = y' = \frac{2x^2}{6y + \sin y}$$

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EXAMPLE $f(x)=(\cos x)^{\sin x}$ fonk. türevini hesaplayınız.

Solution $\ln f(x)=\ln(\cos x)^{\sin x}=\sin x.\ln\cos x$

$$[\ln f(x)]'=\frac{f'(x)}{f(x)} \Rightarrow [\ln f(x)]'=\cos x.\ln\cos x+\sin x.\frac{-\sin x}{\cos x}$$

$$[\ln f(x)]'=\cos x.\ln\cos x-\frac{\sin^2 x}{\cos x}$$

$$\Rightarrow f'(x)=\cos x^{\sin x}\left[\cos x.\ln\cos x-\frac{\sin^2 x}{\cos x}\right]$$

EXAMPLE

$f(x)=\dfrac{x}{x-1}$ fonksiyonun türevini "türevin limiti" tanımı kullanarak bulunuz.

Solution

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

$$=\lim_{h\to 0}\frac{\dfrac{x+h}{x+h-1}-\dfrac{x}{x-1}}{h}=\lim_{h\to 0}\frac{(x-1).(x+h)-x\,(x+h-1)}{h.\,(x-1).\,(x+h-1)}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+\cancel{hx}-\cancel{x}-h-\cancel{x^2}-\cancel{xh}+\cancel{x}}{h(x-1)\,(x+h-1)}=\lim_{h\to 0}-\frac{\cancel{h}}{\cancel{h}(x-1)(x+h-1)}$$

$\left(\frac{0}{0}\right)$

$$=\lim_{h\to 0}-\frac{1}{(x-1)\,(x+h-1)}=-\frac{1}{(x-1)^2}$$

EXAMPLE Türev tanımını kullanarak $f(x)=2x^2-16x+3$ fonk. türevini hesaplayınız.

Solution

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{2(x+h)^2-16(x+h)+3-2x^2+16x-3}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{2x^2}+4xh+2h^2\cancel{-16x}-16h\cancel{+3}\cancel{-2x^2}\cancel{+16x}\cancel{-3}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(4x+2h-16)}{\cancel{h}}=4x-16$$

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Exam Q.

b) $\boxed{0 < x < 1}$ olmak üzere $f(x) = \arcsin x - \arccos\sqrt{1-x^2}$ ile tanımlı f fonksiyonunun türevini bulup, ortaya çıkan durumu yorumlayınız. (10p)

find derivative of f

write comments related to the situation

$$f'(x) = \underbrace{\frac{1}{\sqrt{1-x^2}}}_{2} + \underbrace{\frac{\frac{-2x}{2\sqrt{1-x^2}}}{\sqrt{1-(1-x^2)}}}_{4} \qquad (6)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{x}{x\sqrt{1-x^2}} = 0 \qquad (2)$$

(2) $\forall\ x \in (0,1)$, $f'(x) = 0 \Rightarrow f(x)$ is constant.

(or the curve $f(x)$ has a horizontal tangent on this interval.)

Exam Q.

Examine the continuity. Classify the type of discont.

4.a. $f(x)=\frac{3|x-2|}{x^2(4-x^2)}$ fonksiyonunun sürekliliğini inceleyiniz. Eğer varsa süreksizlik noktalarını sınıflandırınız.

Fonksiyon $x^2(4-x^2)=0 \rightarrow x=0,-2,2$ de incelenmelidir.

Diğer her noktada süreklidir.

$x=0$ için

$$\lim_{x\to 0^-}\frac{3\cancel{(2-x)}}{x^2\cancel{(2-x)}(2+x)}=\lim_{x\to 0^-}\frac{3}{x^2(2+x)}=\infty=\lim_{x\to 0^+}\frac{3\cancel{(2-x)}}{x^2\cancel{(2-x)}(2+x)}=\infty$$

olduğundan, $x=0$ da $\boxed{\text{sonsuz (esas) süreksiz}}$ infinite disc.

$x=-2$ için

$$\lim_{x\to -2^-}\frac{3\cancel{(2-x)}}{x^2\cancel{(2-x)}(2+x)}=-\infty \qquad \lim_{x\to -2^+}\frac{3\cancel{(2-x)}}{x^2\cancel{(2-x)}(2+x)}=+\infty$$

olduğundan $x=-2$ da $\boxed{\text{sonsuz (esas) süreksiz}}$ infinite disc.

$x=2$ için

$$\lim_{x\to 2^-}\frac{3\cancel{(2-x)}}{x^2\cancel{(2-x)}(2+x)}=\frac{3}{16}\neq\lim_{x\to 2^+}\frac{3\cancel{(x-2)}}{x^2\cancel{(2-x)}(2+x)}=\frac{-3}{16}$$

olduğundan $x=2$ de $\boxed{\text{sıçramalı süreksiz}}$ jump disc.

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Exam Q.

4.b. $f(x)$ ve $g(x)$ türevlenebilir iki fonksiyon olsun. $f(g(x)) = x$ ve $f'(x) = 1 + (f(x))^2$ olduğunu kabul edelim. O zaman $g'(x) = \frac{1}{1+x^2}$ olduğunu ispatlayınız.

Suppose that

$\rightarrow$ Prove

$$[f(g(x))]' = 1 \Rightarrow f'(g(x)) \cdot g'(x) = 1$$

$$f'(g(x)) = 1 + [f(g(x))]^2 = 1 + x^2 \quad \text{olduğundan}$$

$$(1+x^2)\, g'(x) = 1 \Rightarrow \boxed{g'(x) = \frac{1}{1+x^2}}$$

Exam Q.

b) Assuming y is a function of x, find the equation of the tangent line to the curve $x^2y^2+\tan(x+y)-1=0$ at the point $P\left(\frac{\pi}{4},0\right)$. (13 pts)

[Do not use the formula $y'=-F_x/F_y$ derived from the equation $F(x,y)=0$].

Implicit differentiation

$$2xy^2+2x^2yy'+\sec^2(x+y)\cdot(1+y')=0$$

$$2xy^2+2x^2yy'+\sec^2(x+y)+\sec^2(x+y)y'=0$$

$$y'=-\frac{2xy^2+\sec^2(x+y)}{2x^2y+\sec^2(x+y)}$$

at $P(\frac{\pi}{4},0) \Rightarrow y'\Big|_P=\frac{-\sec^2\pi/4}{\sec^2\pi/4}=-1$

equation of tangent line

$$y-0=(-1)\left(x-\frac{\pi}{4}\right)$$

$$y=-x+\frac{\pi}{4}$$

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Exam Q.

Exam Q: Let f be a function given by

$f(x) = \frac{1}{\sqrt{x}}$ such that $x > 0$. Find the derivative of the function using the definition of derivative

Solution: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\,\sqrt{x}} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\,\sqrt{x+h}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{2x^{3/2}} \,//$$

Exam Q.

Exam Q: $f:(-\infty,1] \to \mathbb{R}$

$$f(x)=\frac{1+x}{\sqrt{1+x^2}}$$

Show that f has an inverse and calculate $(\underbrace{f^{-1})'(0}_{y_0})$.

Solution: $f'(x)=\dfrac{1-x}{(1+x^2)^{3/2}}$

$\forall x\in(-\infty,1)$, $f'(x)>0$. Hence f is 1-1 on its domain and has an inverse.

$$f(x_0)=\underbrace{y_0}_{0} \Rightarrow \frac{1+x_0}{\sqrt{1+x_0^2}}=0 \Rightarrow x_0=-1$$

$$(f^{-1})'(\underbrace{0}_{y_0})=\frac{1}{f'(\underbrace{-1}_{x_0})}=\frac{1}{\left.\dfrac{1-x}{(1+x^2)^{3/2}}\right|_{x_0=-1}}=\sqrt{2}$$

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Exam Q.

2-a) Find the equation of the tangent line at P(0,1) to the curve $y=f(x)$ which is implicitly defined by $y-2\cos(\pi y-x)=2x+3$.(Do NOT use the formula $y'=-\frac{F_x}{F_y}$) (13 Points)

$$y'+2(\pi y'-1)\sin(\pi y-x)=2$$

$$y'|_p+2(\pi . y'|_p-1)\sin(\pi-0)=2$$

$$y'|_p=2$$

Teget Dogrusu : $y-1=2.(x-0)$

$$\boxed{y=2x+1}$$

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Exam Q.

Find the tangent line of the curve

1.a. $y\sin\left(\frac{1}{y}\right) + x\cos\left(\frac{1}{y}\right) = -2x$ eğrisinin $P\left(0, \frac{1}{\pi}\right)$ noktasındaki teğet doğrusunun denklemini bulunuz ($y' = -\frac{F_x}{F_y}$ formülünü kullanmayınız).

Do not use the formula $y' = -\frac{F_x}{F_y}$.

$$y'\sin\left(\frac{1}{y}\right) + y\left(\frac{-y'}{y^2}\right)\cos\left(\frac{1}{y}\right) + \cos\left(\frac{1}{y}\right) + x\left(-\frac{y'}{y^2}\right)\left(-\sin\left(\frac{1}{y}\right)\right) = -2$$

$$y' = \frac{-2-\cos\left(\frac{1}{y}\right)}{\sin\left(\frac{1}{y}\right) - \frac{1}{y}\cos\left(\frac{1}{y}\right) + \frac{x}{y^2}\sin\frac{1}{y}}$$

$$y'\Big|_P = \frac{-2-\cos\pi}{\sin\pi - \pi\cos(\pi) + 0} = \frac{-1}{\pi}$$

Teğet doğrusu : $\boxed{\left(y - \frac{1}{\pi}\right) = -\frac{1}{\pi}(x-0)}$

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Exam Q.

linear approxima...

1.b. $g:\mathbb{R}\to\mathbb{R}$ türevlenebilen bir fonksiyon ve $g(2)=-4$, $g'(x)=\sqrt{x^2+5}$ olmak üzere, lineer yaklaşımı kullanarak $g(2{,}05)$'in yaklaşık değerini bulunuz.

$g(x)$ fonksiyonunun $a=2$ deki lineerleştirmesi:

$$L(x)=g(2)+g'(2)(x-2)$$

$$L(x)=-4+3(x-2)$$

$$g(2.05)\approx L(2.05)=-4+3(2.05-2)=-4+0{,}15=\boxed{-3.85}$$

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Exam Q.

prove that f has an inverse.

b. $f: R \to R^+$, $f(x) = e^{\arctan x}$ fonksiyonunun tersinin mevcut olduğunu gösteriniz ve $(f^{-1})'(e^{\frac{\pi}{3}})$ değerini hesaplayınız.

Calculate

increasing

$f'(x) = \frac{1}{1+x^2} e^{\arctan x} > 0$ olduğundan fonksiyon artandır ve tersi mevcuttur.

f has an inverse

$f(a) = e^{\arctan a} = e^{\frac{\pi}{3}} \Rightarrow \arctan a = \frac{\pi}{3} \Rightarrow a = \sqrt{3}$

$(f^{-1})'\left(e^{\frac{\pi}{3}}\right) = \frac{1}{f'(\sqrt{3})}$

$(f^{-1})'\left(e^{\frac{\pi}{3}}\right) = \frac{1}{\frac{1}{1+3} e^{\arctan\sqrt{3}}} = \frac{4}{e^{\frac{\pi}{3}}} = 4e^{-\pi/3}$

Exam Q.

1.a) $g:\mathbb{R}\to\mathbb{R}$ fonksiyonu $g(1)=g'(1)=4$ şartlarını sağlayan türevlenebilen bir fonksiyon olsun ve $f:\mathbb{R}\to\mathbb{R}$ fonksiyonu da $\boxed{f(x)=\frac{g(x^2)}{1+x^2}}$ ile tanımlı olsun. Lineer yaklaşım veya diferansiyel hesap kullanarak $f(1.25)$ in yaklaşık değerini bulunuz. *(13 Puan)*

linear app. or diff. app.

$$L(x)=f(1)+f'(1)(x-1) \quad 03, \quad f(1)=\frac{g(1)}{1+1}=\frac{4}{2}=2 \quad 02$$

$$f'(x)=\frac{2x.g'(x^2)(1+x^2)-g(x^2).2x}{(1+x^2)^2} \quad 02$$

$$f'(1)=\frac{2.4.2-4.2}{2^2}=2 \quad 01$$

$$f(x)\approx L(x)=2+2(x-1) \quad 02$$

$$f(1.25)\approx L(1.25)=2+2(1.25-1)=2.5 \quad 03$$

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Exam Q.

4.a: $\lim_{x\to 4}\left[\frac{4}{\pi}arctan\left(\frac{x}{4}\right)\right]^{tan\left(\frac{\pi x}{8}\right)}$ limitini hesaplayınız.

$$\lim_{x\to 4}\left[\frac{4}{\pi}arctan\left(\frac{x}{4}\right)\right]^{tan\left(\frac{\pi x}{8}\right)} = 1^{\infty}$$

$$y=\left[\frac{4}{\pi}arctan\left(\frac{x}{4}\right)\right]^{tan\left(\frac{\pi x}{8}\right)} \Rightarrow Iny = tan\left(\frac{\pi x}{8}\right) In\left[\frac{4}{\pi}arctan\left(\frac{x}{4}\right)\right]$$

$$\lim_{x\to 4} Iny = \lim_{x\to 4} tan\left(\frac{\pi x}{8}\right) In\left[\frac{4}{\pi}arctan\left(\frac{x}{4}\right)\right] = \infty.0$$

$$\lim_{x\to 4} Iny = \lim_{x\to 4}\frac{In\left[\frac{4}{\pi}arctan\left(\frac{x}{4}\right)\right]}{cot\left(\frac{\pi x}{8}\right)} = \frac{0}{0}$$

$$\lim_{x\to 4} Iny = \lim_{x\to 4}\frac{\dfrac{\frac{4}{\pi}\cdot\dfrac{\frac{1}{4}}{1+\frac{x^2}{16}}}{\frac{4}{\pi}arctan\left(\frac{x}{4}\right)}}{-\frac{\pi}{8}cosec^2\left(\frac{\pi x}{8}\right)} = -\frac{4}{\pi^2}$$

$$\lim_{x\to 4} Iny = -\frac{4}{\pi^2} \Rightarrow \lim_{x\to 4} y = e^{-\frac{4}{\pi^2}}$$

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