MAT1071 MATHEMATICS I
2. WEEK
PART 1

LIMIT

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Limit of a Function

Let examine a function's behaviour $(y=f(x))$ near a particular point a, but not a.

Suppose $f(x)$ is defined on an open interval about a, except possibly at a itself. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to a, we say that f approaches the limit L as x approaches a, and write

$$\lim_{x \to a} f(x) = L \qquad \begin{array}{l} x \to a \\ f(x) \to L. \end{array}$$

= "the limit of $f(x)$ as x approaches a is L"

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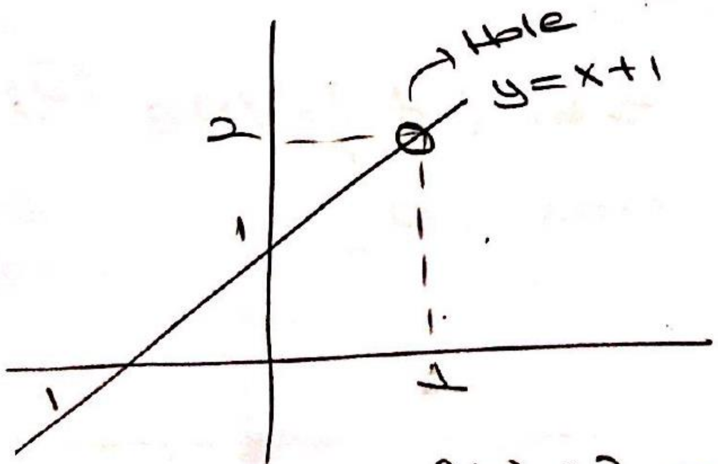
EXAMPLE $f(x)=\dfrac{x^2-1}{x-1}$ How does the given function behave near $x=1$?

Solution For $f(x)$, $D_f=\mathbb{R}-\{1\}$. For any $x\neq 1$ we can write

$$f(x)=\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}=x+1$$

Even though $f(1)$ is not defined, it's clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1.



x	$f(x)$
0.99	1.99
0.999	1.999
0.9999	1.9999
↓	↓
1	2
↑	↑
1.0001	2.0001
1.001	2.001
1.01	2.01

Then $x\to 1$ $f(x)\to 2$ (1)

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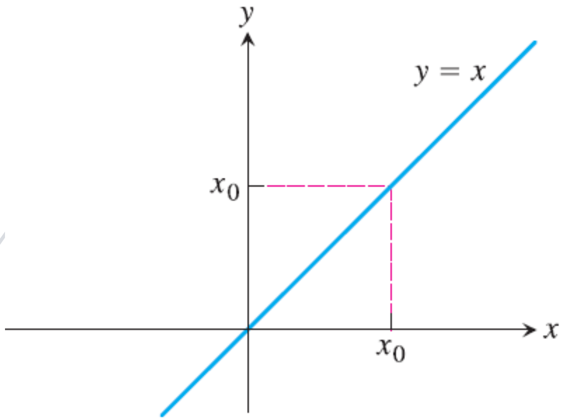
EXAMPLE How does $f(x) = x+2$ behave near $x=1$?

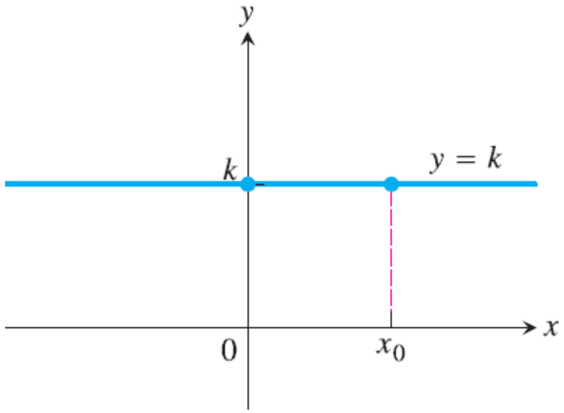
Solution

x	$x+2$
0.9	2.9
0.99	2.99
0.999	2.999
(1)	(3)
1.001	3.001
1.01	3.01
1.1	3.1

$x \to 1$, $f(x) \to 3$



(a) Identity function



(b) Constant function

EXAMPLE

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0 .$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k .$$

For instances of each of these rules we have

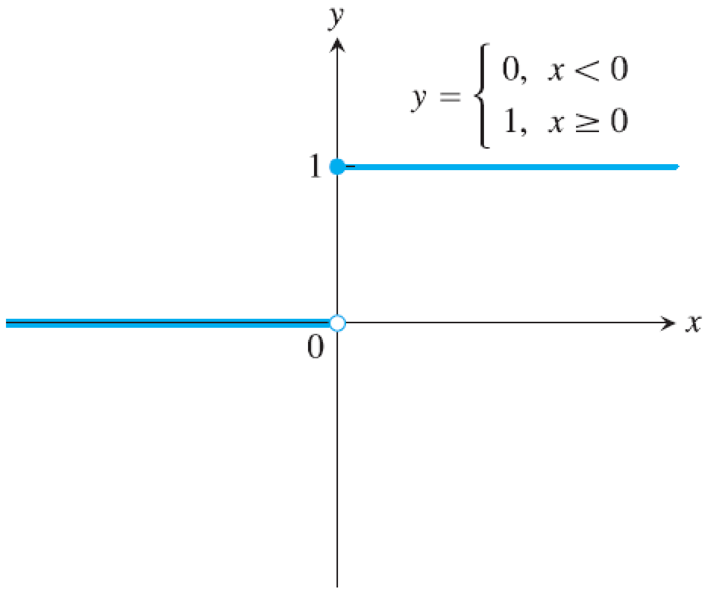
$$\lim_{x \to 3} x = 3 \qquad \text{and} \qquad \lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4 .$$

EXAMPLE Discuss the behavior of the following function as $x \to 0$.

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Solution

It *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$



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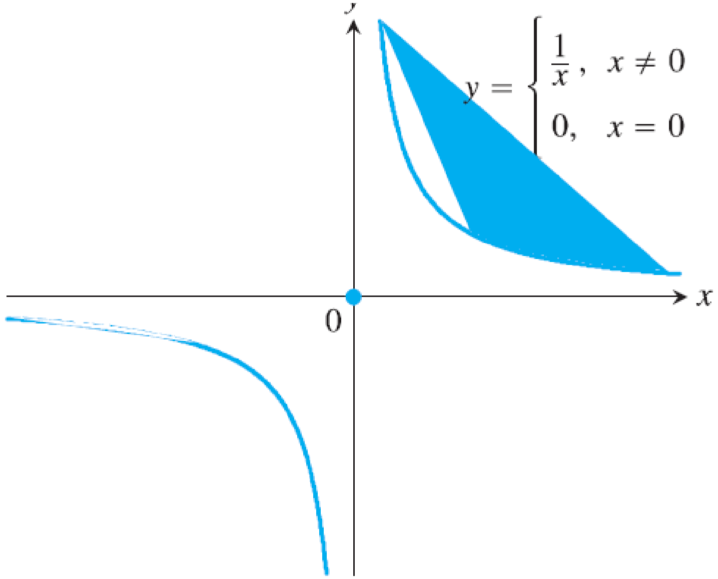
EXAMPLE Discuss the behavior of the following function as $x \to 0$.

$$g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution

It *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* fixed real number



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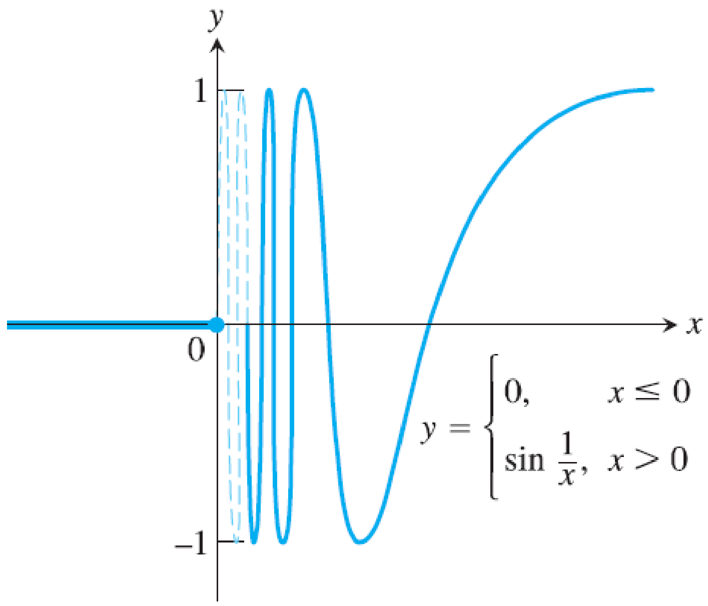
EXAMPLE Discuss the behavior of the following function as $x \to 0$.

$$f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$$

Solution

It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$



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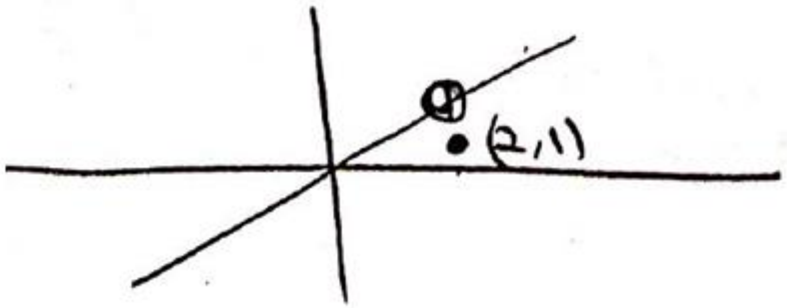
EXAMPLE $g(x) = \begin{cases} x, & x \neq 2 \\ 1, & x = 2 \end{cases}$ $\quad \lim_{x \to 2} g(x) = ?$

Solution



$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x = 2$$

(It's not necessary to be defined on "2")

The Limit Laws

THEOREM —Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

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☆ Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

☆ Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

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☆ $$\lim_{x \to a} \underbrace{k}_{\text{constant}}{}^{f(x)} = k^{\lim_{x \to a} f(x)}$$

☆ $$\lim_{x \to a} \log_k f(x) = \log_k \left(\lim_{x \to a} f(x) \right)$$

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EXAMPLE Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ and the properties of limits to find the following limits.

(a) $\lim\limits_{x\to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim\limits_{x\to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim\limits_{x\to -2} \sqrt{4x^2 - 3}$

Solution

(a) $$\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$$ Sum and Difference Rules

$$= c^3 + 4c^2 - 3$$ Power and Multiple Rules

(b) $$\lim_{x\to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim\limits_{x\to c}(x^4 + x^2 - 1)}{\lim\limits_{x\to c}(x^2 + 5)}$$ Quotient Rule

$$= \frac{\lim\limits_{x\to c} x^4 + \lim\limits_{x\to c} x^2 - \lim\limits_{x\to c} 1}{\lim\limits_{x\to c} x^2 + \lim\limits_{x\to c} 5}$$ Sum and Difference Rules

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$ Power or Product Rule

(c) $$\lim_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x\to -2}(4x^2 - 3)}$$ Root Rule with $n = 2$

$$= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3}$$ Difference Rule

$$= \sqrt{4(-2)^2 - 3}$$ Product and Multiple Rules

$$= \sqrt{16 - 3}$$

$$= \sqrt{13}$$

EXAMPLE

$$\lim_{x \to -7} 2x+5 = -9$$

$$\lim_{t \to 6} 8(t-5)(t-7) = -8$$

$$\lim_{s \to \frac{2}{3}} 3s(2s-1) = \frac{2}{3}$$

$$\lim_{x \to 2} (-x^2+5x-2) = 4$$

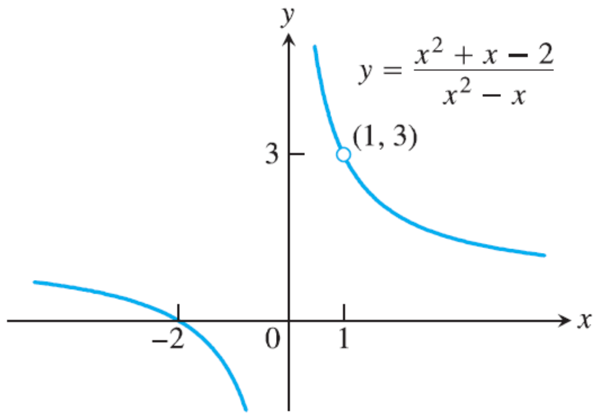
$$\lim_{y \to 2} \frac{y+2}{y^2+5y+6} = \frac{1}{5}$$

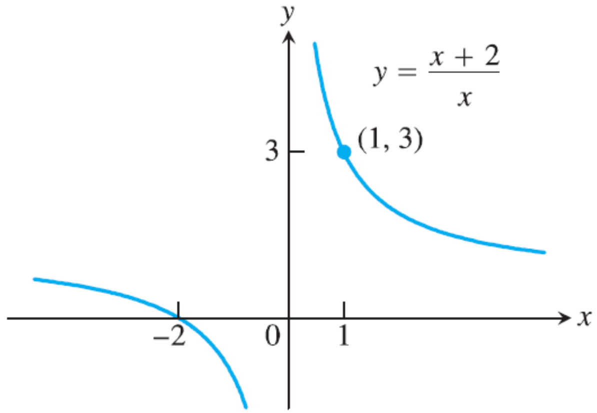
$$\lim_{x \to -2} x^3-2x^2+4x-8 = -32$$

EXAMPLE

$$\lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Eliminating Zero Denominators Algebraically





EXAMPLE Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

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EXAMPLE

a) $\lim\limits_{x \to 5} \dfrac{x-5}{x^2-25} = \lim\limits_{x \to 5} \dfrac{1}{x+5} = \dfrac{1}{10}$

b) $\lim\limits_{x \to -3} \dfrac{x+3}{x^2+4x+3} = \lim\limits_{x \to -3} \dfrac{x+3}{(x+3)(x+1)} = -\dfrac{1}{2}$

c) $\lim\limits_{x \to -5} \dfrac{x^2+3x-10}{x+5} = \lim\limits_{x \to -5} \dfrac{(x+5)(x-2)}{x+5} = -7$

d) $\lim\limits_{x \to 2} \dfrac{x^2-7x+10}{x-2} = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x-5)}{\cancel{x-2}} = -3$

EXAMPLE Evaluate

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}.$$

Solution We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned}
\frac{\sqrt{x^2+100}-10}{x^2} &= \frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} \\
&= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)} \\
&= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} && \text{Common factor } x^2 \\
&= \frac{1}{\sqrt{x^2+100}+10}. && \text{Cancel } x^2 \text{ for } x \neq 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10} \\
&= \frac{1}{\sqrt{0^2+100}+10} && \text{Denominator not 0 at } x=0\text{; substitute} \\
&= \frac{1}{20} = 0.05.
\end{aligned}$$

THEOREM —The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

EXAMPLE Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \qquad \text{for all } x \ne 0,$$

find $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x\to 0}(1 - (x^2/4)) = 1 \qquad \text{and} \qquad \lim_{x\to 0}(1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$

EXAMPLE The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta \to 0} \sin\theta = 0$ **(b)** $\lim_{\theta \to 0} \cos\theta = 1$

(c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

Solution

(a) we established that $-|\theta| \le \sin\theta \le |\theta|$ for all θ

Since $\lim_{\theta \to 0}(-|\theta|) = \lim_{\theta \to 0}|\theta| = 0$, we have

$$\lim_{\theta \to 0} \sin\theta = 0.$$

(b) $0 \le 1 - \cos\theta \le |\theta|$ for all θ

$\lim_{\theta \to 0}(1 - \cos\theta) = 0$ or

$$\lim_{\theta \to 0} \cos\theta = 1.$$

(c) Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x \to c} f(x) = 0$. ■

THEOREM If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

The Precise Definition of a Limit

DEFINITION Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

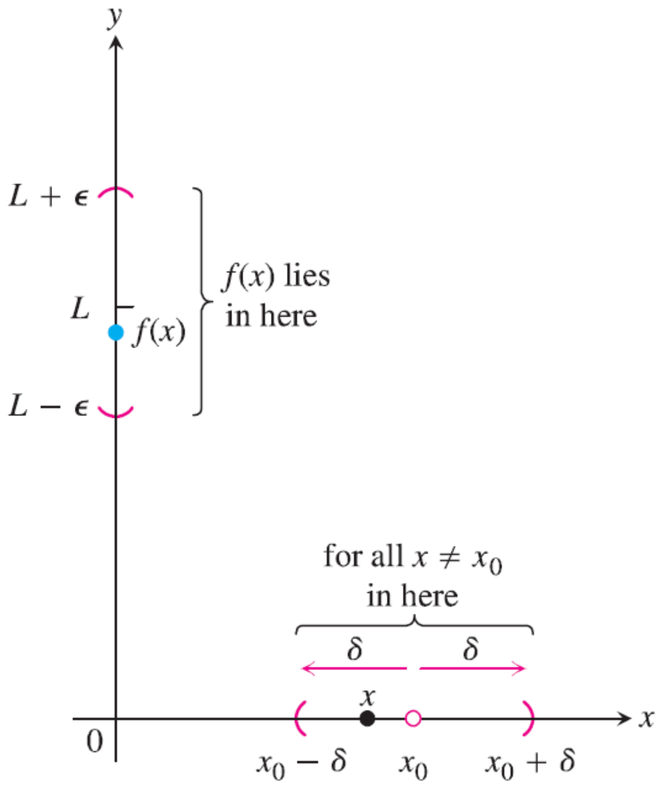

FIGURE The relation of δ and ϵ in the definition of limit.

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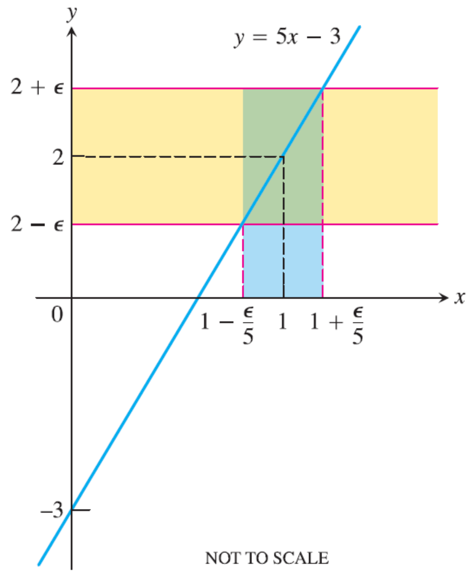
EXAMPLE Show that $\lim\limits_{x \to 1} 5x-3=2$



Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x-1| < \delta \Rightarrow |\underbrace{(5x-3)-2}_{f(x)-L}| < \varepsilon$$

$$|f(x)-2| = |5x-3-2| = |5x-5| < \varepsilon$$

$$\Rightarrow 5|x-1| < \varepsilon$$

$$\Rightarrow |x-1| < \frac{\varepsilon}{5}$$

Thus we can take $\delta = \frac{\varepsilon}{5}$.

Then $\lim\limits_{x \to 1} 5x-3 = 2$.

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

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EXAMPLE Show that $\lim_{x \to 4} x+1=5$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x-4| < \delta \Rightarrow |x+1-5| < \varepsilon$$

$$|x-4| < \varepsilon$$

By choosing $\delta = \varepsilon$

we prove that $\lim_{x \to 4} x+1=5$

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

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EXAMPLE Show that $\lim_{x \to -2} 2x - 2 = -6$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x+2| < \delta \Rightarrow |2x-2+6| < \varepsilon$$

$$|2x+4| < \varepsilon \Rightarrow 2|x+2| < \varepsilon$$

$$\Rightarrow |x+2| < \frac{\varepsilon}{2}$$

By choosing $\delta = \frac{\varepsilon}{2}$

$$\lim_{x \to -2} 2x - 2 = -6$$

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

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EXAMPLE $\lim_{x \to 3} x^2 - 5 = 4$

Solution For given number $\varepsilon > 0$, let check there exists a corresponding number $\delta > 0$ such that

$$0 < |x-3| < \delta \Rightarrow |f(x) - 4| < \varepsilon$$

$$\Rightarrow |f(x) - 4| < \varepsilon \Rightarrow |x^2 - 9| < \varepsilon$$

$$\Rightarrow |x-3|\,|x+3| < \varepsilon$$

<u>First way</u>

let choose $0 < \delta < 1$

$$|x-3| < \delta < 1$$

$$-1 < x-3 < 1$$

$$5 < x+3 < 7$$

$$|x+3| < 7$$

$$|x-3|\,|x+3| < 7\delta$$

$$\Rightarrow \boxed{7\delta = \varepsilon}$$

$$\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$$

<u>Second way</u>

$$|x-3| < \delta$$

$$-\delta + 3 < x < \delta + 3$$

$$-\delta + 6 < x + 3 < \delta + 6$$

$$|x-3|\,|x+3| < \underbrace{\delta \cdot (\delta + 6)}_{\varepsilon}$$

$$\varepsilon = \delta^2 + 6\delta$$

$$\delta^2 + 6\delta - \varepsilon = 0$$

$$\delta_1 = \frac{-6 + \sqrt{36 + 4\varepsilon}}{2} \checkmark$$

$$\delta_2 = \frac{-6 - \sqrt{36 + 4\varepsilon}}{2} < 0$$

$$\boxed{\delta = \frac{-6 + \sqrt{36 + 4\varepsilon}}{2}}$$

EXAMPLE Prove that $\lim_{x \to 2} x^2 = 4$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x-2| < \delta \Rightarrow |x^2-4| < \varepsilon$$

$$|x^2-4| < \varepsilon \Rightarrow |x-2||x+2| < \varepsilon$$

$$|x-2| < \delta \Rightarrow 2-\delta < x < 2+\delta$$

$$\Rightarrow 4-\delta < x+2 < 4+\delta$$

$$\Rightarrow |x+2| < 4+\delta$$

$$|x-2|\,|x+2| < \delta \cdot (4+\delta)$$

Let $\delta(4+\delta) = \varepsilon \Rightarrow \delta^2 + 4\delta - \varepsilon = 0$

$$\Rightarrow \delta_1 = -2 + \sqrt{4+\varepsilon}$$

$$\delta_2 = -2 - \sqrt{4+\varepsilon} < 0 \; \times$$

remember $\delta > 0$

So $\delta_1 = -2 + \sqrt{4+\varepsilon}$

$$\lim_{x \to 2} x^2 = 4$$

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EXAMPLE Show that $\lim_{x \to 1} 2x+3 \neq 4$

Solution For given $\varepsilon > 0$, let check thee exists a corresponding number $\delta(\varepsilon) > 0$ such that or not.

$$0 < |x-1| < \delta \Rightarrow |f(x)-4| < \varepsilon \quad (?)$$

$$|f(x)-4| = |2x+3-4| < \varepsilon$$
$$\Rightarrow |2x-1| < \varepsilon$$
$$\Rightarrow |2x-2+1| < \varepsilon$$

$$|2x-2+1| \leq |2x-2| + 1 < \varepsilon \qquad (|x+y| \leq |x|+|y|)$$
$$= 2|x-1| + 1 < 2\delta + 1 = \varepsilon$$

Hence $\delta = \frac{\varepsilon - 1}{2}$. For $0 < \varepsilon < 1$ we have

$$-\frac{1}{2} < \frac{\varepsilon-1}{2} < 0 \qquad \delta < 0 \text{ so,}$$

$$\lim_{x \to 1} 2x+3 \neq 4$$

Finding Deltas Algebraically for Given Epsilons

EXAMPLE For the limit $\lim_{x\to 5}\sqrt{x-1}=2$, find a $\delta > 0$ that works for $\epsilon = 1$.

Solution We organize the search into two steps, as discussed below.

That is, find a $\delta > 0$ such that for all x

$$0 < |x-5| < \delta \quad \Rightarrow \quad |\sqrt{x-1}-2| < 1.$$

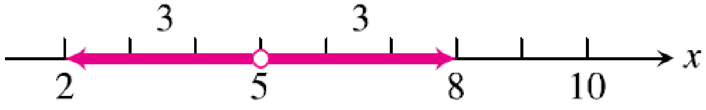
1. *Solve the inequality* $|\sqrt{x-1}-2| < 1$ *to find an interval containing* $x_0 = 5$ *on which the inequality holds for all* $x \neq x_0$.

$$\begin{gathered}
|\sqrt{x-1}-2| < 1 \\
-1 < \sqrt{x-1}-2 < 1 \\
1 < \sqrt{x-1} < 3 \\
1 < x-1 < 9 \\
2 < x < 10
\end{gathered}$$

The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well.

2. *Find a value of* $\delta > 0$ *to place the centered interval* $5 - \delta < x < 5 + \delta$ (centered at $x_0 = 5$) *inside the interval* $(2, 10)$.

The distance from 5 to the nearer endpoint of $(2, 10)$ is 3



If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x - 1} - 2| < 1$

$$0 < |x - 5| < 3 \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

EXAMPLE Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, prove that

$$\lim_{x\to c}(f(x) + g(x)) = L + M.$$

Solution Let $\epsilon > 0$ be given. We want to find a positive number δ such that for all x

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) + g(x) - (L + M)| < \epsilon.$$

Regrouping terms, we get

$$\begin{aligned}|f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M|.\end{aligned}$$

Triangle Inequality:
$|a + b| \le |a| + |b|$

Since $\lim_{x\to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x

$$0 < |x - c| < \delta_1 \quad \Rightarrow \quad |f(x) - L| < \epsilon/2.$$

Similarly, since $\lim_{x\to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x

$$0 < |x - c| < \delta_2 \quad \Rightarrow \quad |g(x) - M| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \epsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \epsilon/2$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{x\to c}(f(x) + g(x)) = L + M$.

EXAMPLE Prove the following results

(a) $\lim_{x \to x_0} x = x_0$ **(b)** $\lim_{x \to x_0} k = k$ (k constant)

Solution

(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |x - x_0| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number
This proves that $\lim_{x \to x_0} x = x_0$.

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold
This proves that $\lim_{x \to x_0} k = k$.

One-Sided Limits

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**.

One-Sided Limits

☆ if $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

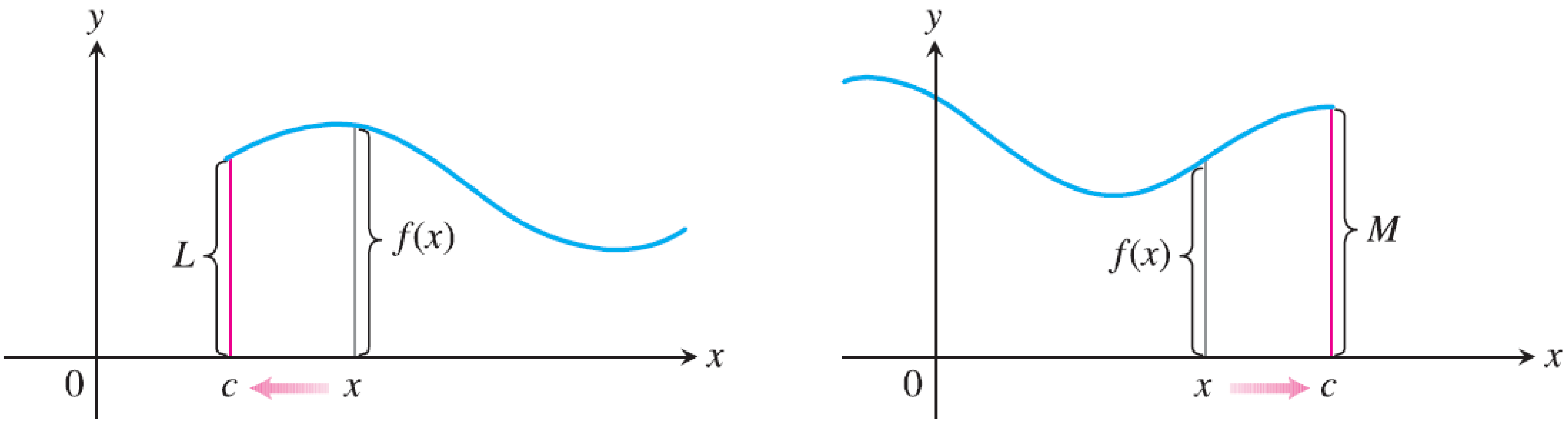
$$\lim_{x \to c^+} f(x) = L.$$

The symbol "$x \to c^+$" means that we consider only values of x greater than c.

☆ if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider only x values less than c.



(a) $\lim_{x \to c^+} f(x) = L$ (b) $\lim_{x \to c^-} f(x) = M$

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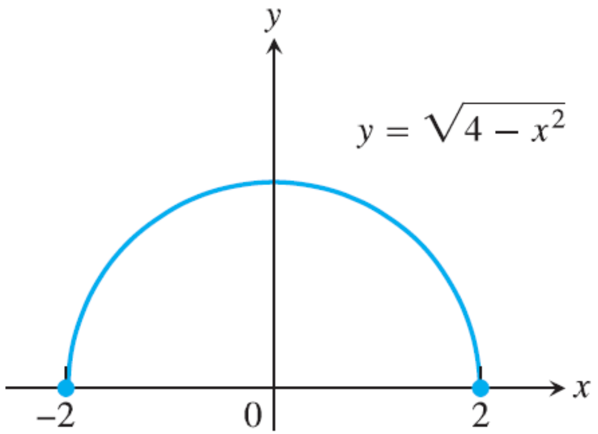
FIGURE (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

EXAMPLE The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2. ■



THEOREM A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

EXAMPLE $f(x) = \frac{x}{|x|}$

At $x = 0$:

$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = -1$$

does not exist.

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EXAMPLE For the function graphed in Figure

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$,

$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

$\lim_{x \to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

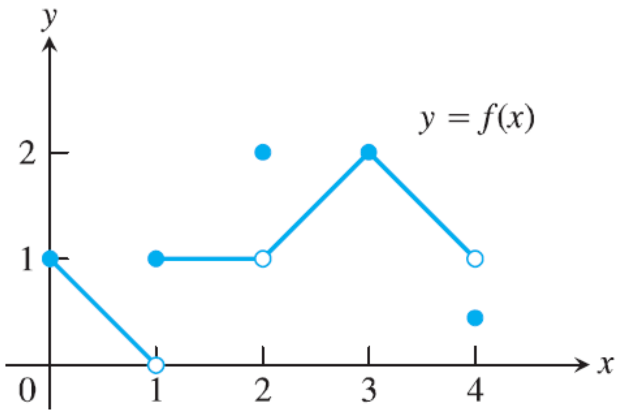

FIGURE Graph of the function

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Precise Definitions of One-Sided Limits

DEFINITIONS We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

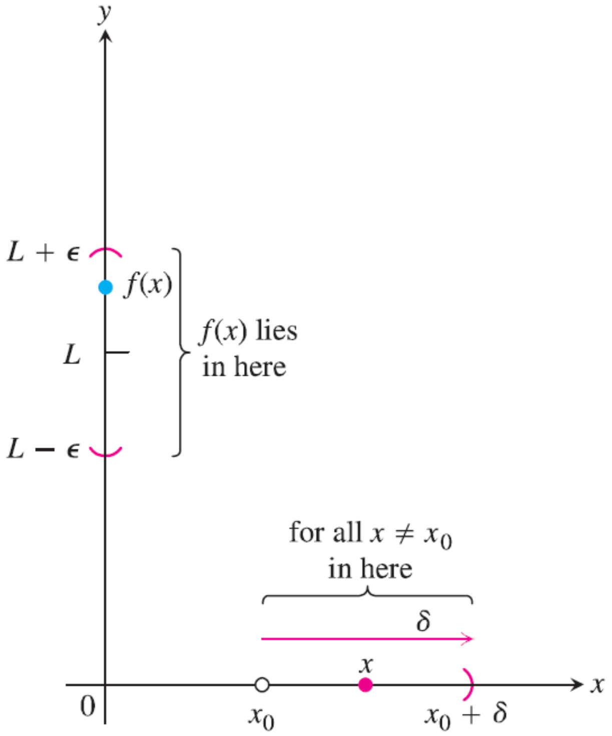

FIGURE Intervals associated with the definition of right-hand limit.

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Precise Definitions of One-Sided Limits

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

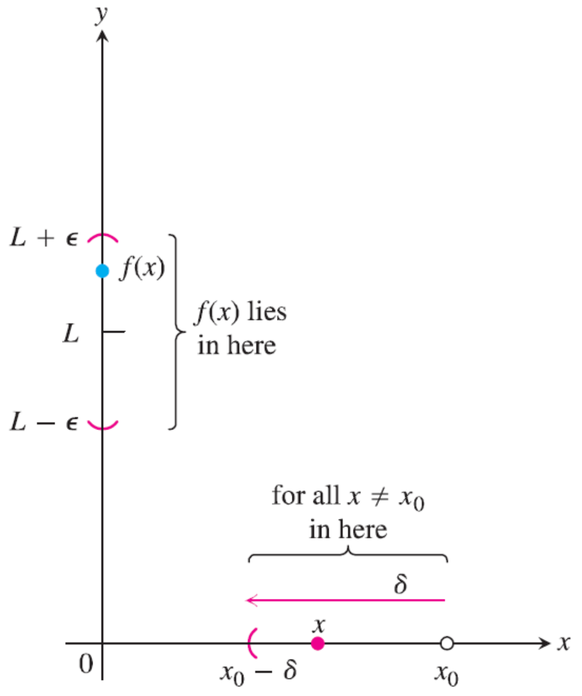

FIGURE Intervals associated with the definition of left-hand limit.

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EXAMPLE Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\epsilon > 0$ be given. Here $x_0 = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \epsilon.$$

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \epsilon^2$ we have

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

EXAMPLE

Ex a) $\lim\limits_{x \to 2^+} x+1 = 3 \qquad \lim\limits_{x \to 2^-} x+1 = 3$

b) $f(x) = \begin{cases} 2x+1, & x > 2 \\ 3, & x = 2 \\ 3x-1, & x < 2 \end{cases}$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 2x+1 = 5$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 3x-1 = 5$

c) $\lim\limits_{x \to 1^+} \frac{|x-1|}{x^2-1} = \lim\limits_{x \to 1^+} \frac{x-1}{(x-1)(x+1)} = \frac{1}{2}$

$\lim\limits_{x \to 1^-} \frac{|x-1|}{x^2-1} = \lim\limits_{x \to 1^-} \frac{-(x-1)}{(x-1)(x+1)} = -\frac{1}{2}$

EXAMPLE Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side

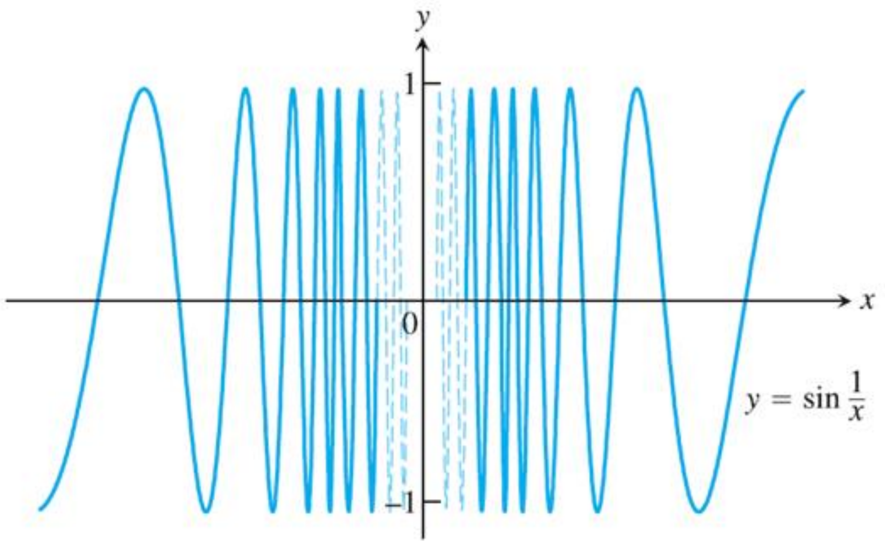

FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero
The graph here omits values very near the y-axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

Limits of Trigonometric Functions

* $\lim_{x \to 0} \sin x = 0$
* $\lim_{x \to 0} \cos x = 1$
* $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$
* $\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$
* $\lim_{x \to 0} \frac{x}{\sin x} = 1$
* $\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$
* $\lim_{x \to 0} \frac{\tan ax}{bx} = \frac{a}{b}$
* $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$
* $\lim_{x \to 0} \frac{\tan ax}{\tan bx} = \frac{a}{b}$
* $\lim_{x \to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$

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Limits Involving $(\sin \theta)/\theta$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{x \to c} \frac{\sin f(x)}{f(x)} = 1 .$$

Here are several examples.

a. $\lim_{x \to 0} \frac{\sin x^2}{x^2} = 1$

b. $\lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} \frac{\sin x^2}{x^2} \lim_{x \to 0} \frac{x^2}{x} = 1 \cdot 0 = 0$

Limits Involving $(\sin \theta)/\theta$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

c. $\lim_{x \to -1} \dfrac{\sin(x^2 - x - 2)}{x + 1}$

$$= \lim_{x \to -1} \frac{\sin(x^2 - x - 2)}{(x^2 - x - 2)} \cdot \lim_{x \to -1} \frac{(x^2 - x - 2)}{x + 1} = 1 \cdot \lim_{x \to -1} \frac{(x + 1)(x - 2)}{x + 1} = -3$$

d. $\lim_{x \to 1} \dfrac{\sin\left(1 - \sqrt{x}\right)}{x - 1}$

$$= \lim_{x \to 1} \frac{\sin\left(1 - \sqrt{x}\right)}{1 - \sqrt{x}} \frac{1 - \sqrt{x}}{x - 1} = 1 \cdot \lim_{x \to 1} \frac{\left(1 - \sqrt{x}\right)\left(1 + \sqrt{x}\right)}{(x - 1)\left(1 + \sqrt{x}\right)}$$

$$= \lim_{x \to 1} \frac{1 - x}{(x - 1)\left(1 + \sqrt{x}\right)} = -\frac{1}{2}$$

EXAMPLE

$$\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$$

$$= \lim_{x \to 0} \frac{1-(1-2\sin^2 x)}{x^2} = \lim_{x \to 0} \frac{1-1+2\sin^2 x}{x^2}$$

$$= \lim_{x \to 0} 2\,\frac{\sin x}{x} \cdot \frac{\sin x}{x} = 2$$

EXAMPLE Show that **(a)** $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and **(b)** $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h} \\
&= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \sin \theta \qquad \text{Let } \theta = h/2. \\
&= -(1)(0) = 0.
\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 2x}{5x} &= \lim_{x \to 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\
&= \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} \\
&= \frac{2}{5}(1) = \frac{2}{5}
\end{aligned}$$

EXAMPLE Find $\lim_{t \to 0} \dfrac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}\lim_{t \to 0} \frac{\tan t \sec 2t}{3t} &= \frac{1}{3} \lim_{t \to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.\end{aligned}$$

Limits Involving Infinity

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

☆ All the limit laws are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0} = \begin{cases} \frac{a_n}{b_n}, & n = m \\ \infty \text{ or } -\infty, & n > m \\ 0 & n < m \end{cases}$$

$\boxed{a>1}$ $\lim_{x\to\infty} a^x = +\infty$ / $\lim_{x\to-\infty} a^x = 0$

Ex: $\lim_{x\to\infty} 2^x = \infty$ $\lim_{x\to-\infty} 2^x = 0$

$\boxed{0<a<1}$ $\lim_{x\to\infty} a^x = 0$, $\lim_{x\to-\infty} a^x = +\infty$

Ex: $\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = 0$, $\lim_{x\to-\infty} \left(\frac{1}{2}\right)^x = \infty$

EXAMPLE Show that

(a) $\lim_{x \to \infty} \frac{1}{x} = 0$ **(b)** $\lim_{x \to -\infty} \frac{1}{x} = 0.$

Solution

(a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

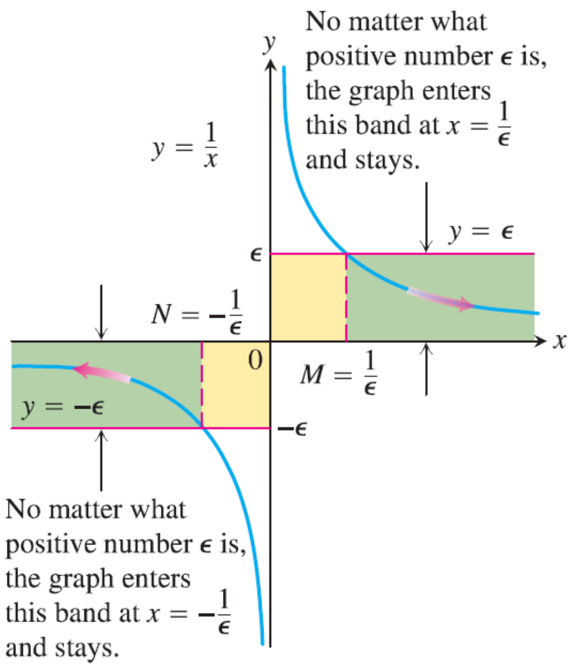
$$x > M \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number. This proves $\lim_{x\to\infty}(1/x) = 0.$

(b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$. This proves $\lim_{x\to-\infty}(1/x) = 0.$



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EXAMPLE

(a) $\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x}$ Sum Rule

$= 5 + 0 = 5$ Known limits

(b) $\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty} \pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$

$= \lim_{x\to-\infty} \pi\sqrt{3}\cdot \lim_{x\to-\infty}\frac{1}{x}\cdot \lim_{x\to-\infty}\frac{1}{x}$ Product Rule

$= \pi\sqrt{3}\cdot 0\cdot 0 = 0$ Known limits

(*) $\lim_{x\to\infty} x\,\sin\frac{3}{x} = \lim_{x\to\infty}\frac{\sin\frac{3}{x}}{\frac{1}{x}}$ $\qquad \frac{3}{x}=\theta$

for $x\to\infty$ $\theta\to 0$

$= \lim_{\theta\to 0}\frac{\sin 3\theta}{\theta} = 3$ //

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EXAMPLE These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

(a) $$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$$ Divide numerator and denominator by x^2.

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$

(b) $$\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$ Divide numerator and denominator by x^3.

$$= \frac{0 + 0}{2 - 0} = 0$$

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EXAMPLE Find $\lim_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right)$.

Solution Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \to \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic result:

$$\lim_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right) = \lim_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right) \frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \to \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x \to \infty} \frac{-16}{x + \sqrt{x^2 + 16}}.$$

As $x \to \infty$, the denominator in this last expression becomes arbitrarily large, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x \to \infty} \frac{-16}{x + \sqrt{x^2 + 16}} = \lim_{x \to \infty} \frac{-\dfrac{16}{x}}{1 + \sqrt{\dfrac{x^2}{x^2} + \dfrac{16}{x^2}}} = \frac{0}{1 + \sqrt{1 + 0}} = 0.$$

■

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Infinite Limits

EXAMPLE

f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty .$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x \to 0^+} (1/x)$ *does not exist because $1/x$ becomes arbitrarily large and positive as $x \to 0^+$.*

As $x \to 0^-$, $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty .$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative.*

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x \to 0} (1/x)$ is that it does not exist.

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Precise Definitions of Infinite Limits

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches minus infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

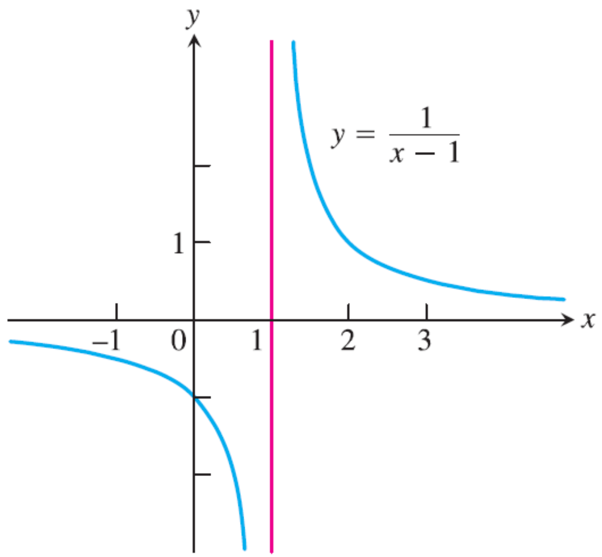
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$

EXAMPLE Find $\lim_{x\to 1^+} \dfrac{1}{x-1}$ and $\lim_{x\to 1^-} \dfrac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x-1)$ is the graph of $y = 1/x$ shifted 1 unit to the right Therefore, $y = 1/(x-1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $1/(x-1) \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $1/(x-1) \to -\infty$. ■



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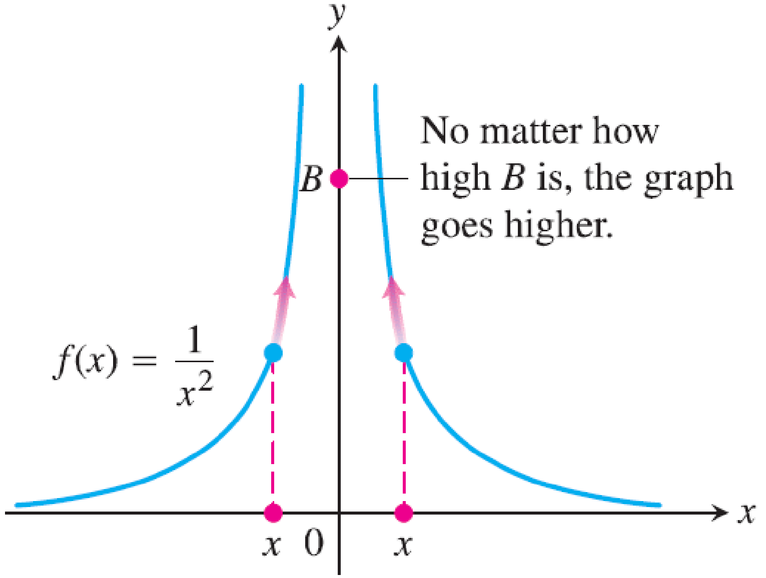
EXAMPLE Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{as} \quad x \to 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large . This means that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty.$$ ■



The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0}(1/x^2) = \infty$.

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EXAMPLE Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if } x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

EXAMPLE These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

(a) $\lim_{x \to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x \to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x-2}{x+2} = 0$

(b) $\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$

(c) $\lim_{x \to 2^+} \frac{x-3}{x^2-4} = \lim_{x \to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$ The values are negative for $x > 2$, x near 2.

(d) $\lim_{x \to 2^-} \frac{x-3}{x^2-4} = \lim_{x \to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$ The values are positive for $x < 2$, x near 2.

(e) $\lim_{x \to 2} \frac{x-3}{x^2-4} = \lim_{x \to 2} \frac{x-3}{(x-2)(x+2)}$ does not exist. See parts (c) and (d).

(f) $\lim_{x \to 2} \frac{2-x}{(x-2)^3} = \lim_{x \to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x \to 2} \frac{-1}{(x-2)^2} = -\infty$

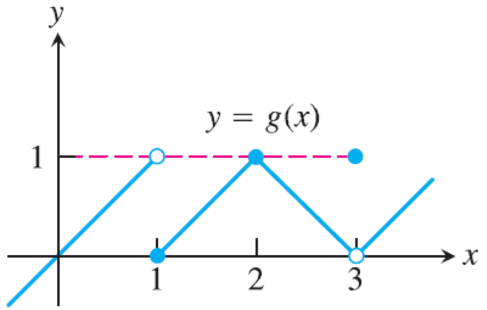
In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

HW:

Limits from Graphs

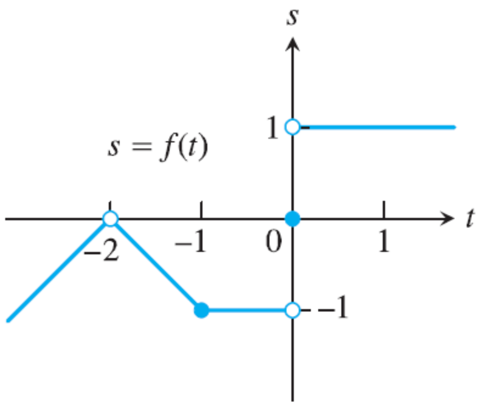
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{x\to 1} g(x)$ b. $\lim_{x\to 2} g(x)$ c. $\lim_{x\to 3} g(x)$ d. $\lim_{x\to 2.5} g(x)$



2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{t\to -2} f(t)$ b. $\lim_{t\to -1} f(t)$ c. $\lim_{t\to 0} f(t)$ d. $\lim_{t\to -0.5} f(t)$



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HW:

4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

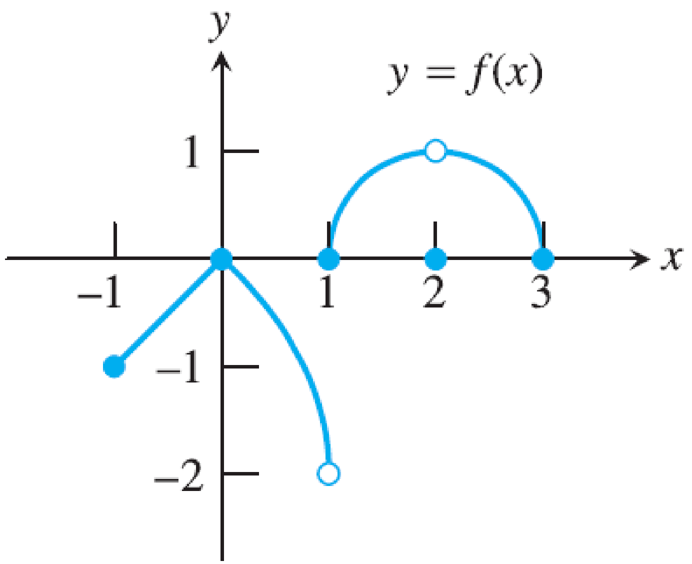
a. $\lim_{x \to 2} f(x)$ does not exist.

b. $\lim_{x \to 2} f(x) = 2$

c. $\lim_{x \to 1} f(x)$ does not exist.

d. $\lim_{x \to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$.

e. $\lim_{x \to x_0} f(x)$ exists at every point x_0 in $(1, 3)$.



HW:

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x \to -7} (2x + 5)$

12. $\lim_{x \to 2} (-x^2 + 5x - 2)$

13. $\lim_{t \to 6} 8(t - 5)(t - 7)$

14. $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x \to 2} \dfrac{x + 3}{x + 6}$

16. $\lim_{s \to 2/3} 3s(2s - 1)$

17. $\lim_{x \to -1} 3(2x - 1)^2$

18. $\lim_{y \to 2} \dfrac{y + 2}{y^2 + 5y + 6}$

19. $\lim_{y \to -3} (5 - y)^{4/3}$

20. $\lim_{z \to 0} (2z - 8)^{1/3}$

21. $\lim_{h \to 0} \dfrac{3}{\sqrt{3h + 1} + 1}$

22. $\lim_{h \to 0} \dfrac{\sqrt{5h + 4} - 2}{h}$

HW:

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x \to 5} \frac{x - 5}{x^2 - 25}$

24. $\lim_{x \to -3} \frac{x + 3}{x^2 + 4x + 3}$

25. $\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$

26. $\lim_{x \to 2} \frac{x^2 - 7x + 10}{x - 2}$

27. $\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 - 1}$

28. $\lim_{t \to -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$

HW:

Limits with trigonometric functions Find the limits in Exercises 43–50.

43. $\lim_{x \to 0} (2 \sin x - 1)$

44. $\lim_{x \to 0} \sin^2 x$

45. $\lim_{x \to 0} \sec x$

46. $\lim_{x \to 0} \tan x$

47. $\lim_{x \to 0} \dfrac{1 + x + \sin x}{3 \cos x}$

48. $\lim_{x \to 0} (x^2 - 1)(2 - \cos x)$

HW:

53. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim\limits_{x\to c} f(x)g(x)$

b. $\lim\limits_{x\to c} 2f(x)g(x)$

c. $\lim\limits_{x\to c} (f(x) + 3g(x))$

d. $\lim\limits_{x\to c} \dfrac{f(x)}{f(x) - g(x)}$

Using the Sandwich Theorem

63. If $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

64. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

HW:

Prove the limit statements in Exercises 37–50.

37. $\lim_{x \to 4} (9 - x) = 5$

38. $\lim_{x \to 3} (3x - 7) = 2$

39. $\lim_{x \to 9} \sqrt{x - 5} = 2$

40. $\lim_{x \to 0} \sqrt{4 - x} = 2$

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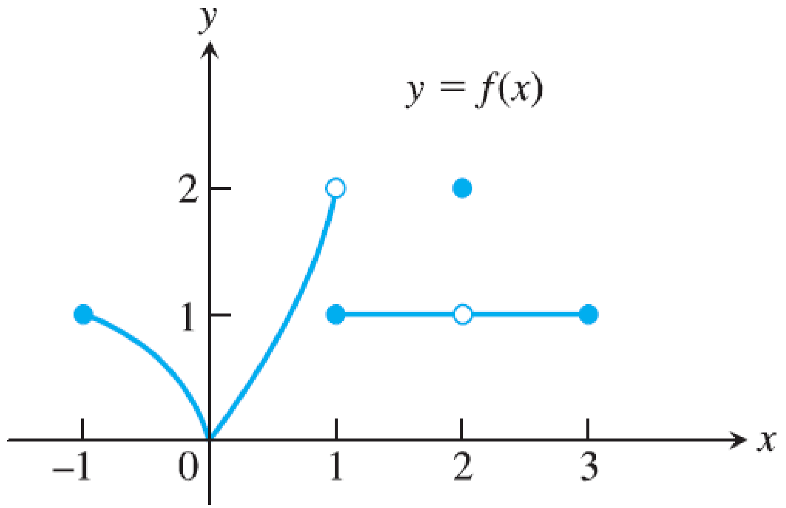
HW:

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x \to -1^+} f(x) = 1$

b. $\lim_{x \to 2} f(x)$ does not exist.

c. $\lim_{x \to 2} f(x) = 2$

d. $\lim_{x \to 1^-} f(x) = 2$

e. $\lim_{x \to 1^+} f(x) = 1$

f. $\lim_{x \to 1} f(x)$ does not exist.

g. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

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HW:

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–18.

11. $\lim_{x \to -0.5^-} \sqrt{\dfrac{x + 2}{x + 1}}$

12. $\lim_{x \to 1^+} \sqrt{\dfrac{x - 1}{x + 2}}$

13. $\lim_{x \to -2^+} \left(\dfrac{x}{x + 1}\right)\left(\dfrac{2x + 5}{x^2 + x}\right)$

14. $\lim_{x \to 1^-} \left(\dfrac{1}{x + 1}\right)\left(\dfrac{x + 6}{x}\right)\left(\dfrac{3 - x}{7}\right)$

HW:

Using $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Find the limits in Exercises 21–42.

21. $\lim_{\theta \to 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\lim_{t \to 0} \frac{\sin kt}{t}$ (k constant)

23. $\lim_{y \to 0} \frac{\sin 3y}{4y}$

24. $\lim_{h \to 0^-} \frac{h}{\sin 3h}$

25. $\lim_{x \to 0} \frac{\tan 2x}{x}$

26. $\lim_{t \to 0} \frac{2t}{\tan t}$

HW:

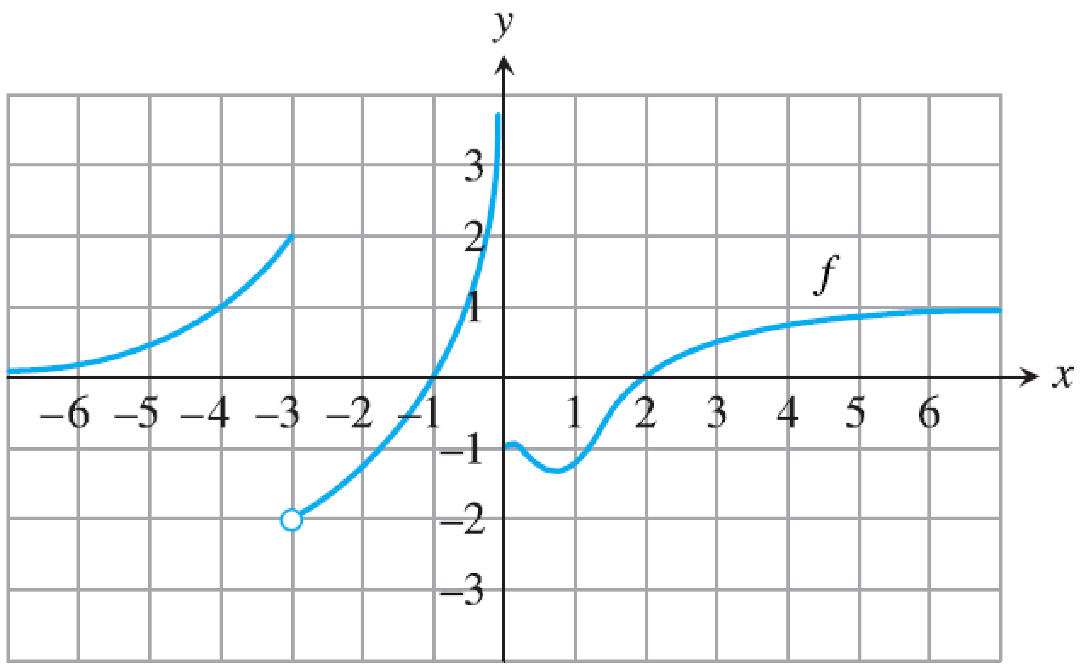
Finding Limits

1. For the function f whose graph is given, determine the following limits.

a. $\lim_{x \to 2} f(x)$ **b.** $\lim_{x \to -3^+} f(x)$ **c.** $\lim_{x \to -3^-} f(x)$

d. $\lim_{x \to -3} f(x)$ **e.** $\lim_{x \to 0^+} f(x)$ **f.** $\lim_{x \to 0^-} f(x)$

g. $\lim_{x \to 0} f(x)$ **h.** $\lim_{x \to \infty} f(x)$ **i.** $\lim_{x \to -\infty} f(x)$



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HW:

Find the limits in Exercises 9–12.

9. $\lim_{x\to\infty} \dfrac{\sin 2x}{x}$

10. $\lim_{\theta\to-\infty} \dfrac{\cos\theta}{3\theta}$

11. $\lim_{t\to-\infty} \dfrac{2 - t + \sin t}{t + \cos t}$

12. $\lim_{r\to\infty} \dfrac{r + \sin r}{2r + 7 - 5\sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

13. $f(x) = \dfrac{2x + 3}{5x + 7}$

14. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

15. $f(x) = \dfrac{x + 1}{x^2 + 3}$

16. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

HW:

Limits as $x \to \infty$ or $x \to -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\displaystyle \lim_{x\to\infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$

24. $\displaystyle \lim_{x\to-\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

25. $\displaystyle \lim_{x\to-\infty} \left(\frac{1 - x^3}{x^2 + 7x}\right)^{5}$

26. $\displaystyle \lim_{x\to\infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$

HW:

Find the limits in Exercises 37–48.

37. $\lim_{x \to 0^+} \frac{1}{3x}$

38. $\lim_{x \to 0^-} \frac{5}{2x}$

39. $\lim_{x \to 2^-} \frac{3}{x - 2}$

40. $\lim_{x \to 3^+} \frac{1}{x - 3}$

Find the limits in Exercises 53–58.

53. $\lim \frac{1}{x^2 - 4}$ as

a. $x \to 2^+$

b. $x \to 2^-$

c. $x \to -2^+$

d. $x \to -2^-$

54. $\lim \frac{x}{x^2 - 1}$ as

a. $x \to 1^+$

b. $x \to 1^-$

c. $x \to -1^+$

d. $x \to -1^-$

HW:

Finding Limits of Differences when $x \to \pm\infty$

Find the limits in Exercises 80–86.

80. $\lim_{x\to\infty} \left(\sqrt{x+9} - \sqrt{x+4}\right)$

81. $\lim_{x\to\infty} \left(\sqrt{x^2+25} - \sqrt{x^2-1}\right)$

Limits at Infinity

Find the limits in Exercises 37–46.

37. $\lim_{x\to\infty} \dfrac{2x+3}{5x+7}$

38. $\lim_{x\to-\infty} \dfrac{2x^2+3}{5x^2+7}$

39. $\lim_{x\to-\infty} \dfrac{x^2-4x+8}{3x^3}$

40. $\lim_{x\to\infty} \dfrac{1}{x^2-7x+1}$

HW:

Find the limits in Exercises 25–30.

25. $\lim_{x \to 0} \frac{\sin(1 - \cos x)}{x}$

26. $\lim_{x \to 0^+} \frac{\sin x}{\sin \sqrt{x}}$

27. $\lim_{x \to 0} \frac{\sin(\sin x)}{x}$

28. $\lim_{x \to 0} \frac{\sin(x^2 + x)}{x}$

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.