Solutions of Application Questions I

1. $\left(\frac{1}{x-y}+\frac{x}{x^2+y^2}\right)dx+\left(\frac{1}{y-x}+\frac{y}{x^2+y^2}\right)dy=0$ H.D.E

$$\left.\begin{aligned}\frac{y}{x}=u \Rightarrow\ & y=ux\\ & dy=u\,dx+x\,du\end{aligned}\right\}$$

$$\left(\frac{1}{x-ux}+\frac{x}{x^2+u^2x^2}\right)dx+\left(\frac{1}{ux-x}+\frac{ux}{x^2+u^2x^2}\right)(u\,dx+x\,du)=0$$

$$\left(\frac{1}{x(1-u)}+\frac{1}{x(1+u^2)}+\frac{u}{x(u-1)}+\frac{u^2}{x(1+u^2)}\right)dx+\left(\frac{1}{u-1}+\frac{u}{1+u^2}\right)du=0$$

$$\left(\frac{\cancel{1-u}}{x\cancel{(1-u)}}+\frac{\cancel{1+u^2}}{x\cancel{(1+u^2)}}\right)dx+\left(\frac{1}{u-1}+\frac{u}{1+u^2}\right)du=0$$

$$\int\frac{2dx}{x}+\int\left(\frac{1}{u-1}+\frac{u}{1+u^2}\right)du=\int 0 \Rightarrow 2\ln x+\ln(u-1)+\frac{1}{2}\ln(1+u^2)=\ln c$$

$$x^2(u-1)\sqrt{1+u^2}=c \Rightarrow x^2\left(\frac{y}{x}-1\right)\sqrt{1+\frac{y^2}{x^2}}=c$$

$$\Rightarrow (y-x)\sqrt{x^2+y^2}=c$$

2) $(x-2y)=(x-2y+1)y' \Rightarrow (x-2y)dx=(x-2y+1)dy$

$$\left.\begin{aligned}x-2y&=u\\ dx-2dy&=du\end{aligned}\right\}\quad \begin{aligned}&u\,dx=(u+1)\left(\frac{dx-du}{2}\right)\\ &\left(u-\frac{u}{2}-\frac{1}{2}\right)dx=-\frac{(u+1)}{2}du \Rightarrow \frac{(u-1)}{\cancel{2}}dx=-\frac{(u+1)}{\cancel{2}}du\end{aligned}$$

$$\int dx=-\int\frac{u+1}{u-1}du \Rightarrow x+c=-\left[1+\frac{2}{u-1}du\right]$$

$$x+c=-u-2\ln(u-1) \Rightarrow x+c=-x+2y-2\ln(x-2y-1)$$

③ $\underbrace{(xy^2+y)}_{m}dx - \underbrace{(x+y)}_{N}dy = 0$

$$\frac{\partial m}{\partial y} = 2xy+1 \neq \frac{\partial N}{\partial x} = -1$$

(+) $\lambda = \lambda(x)$ not depend on x.

$$\ln\lambda = \int \frac{m_y - N_x}{N}dx = \int \frac{2xy+1+1}{-(x+y)}dx$$

(+) $\lambda = \lambda(x)$

$$\ln\lambda = \int \frac{N_x - m_y}{m}dy = \int \frac{-1-2xy-1}{y(xy+1)}dy = \int \frac{-2dy}{y} = -2\ln y \Rightarrow \lambda = \frac{1}{y^2}$$

$$\left(x+\frac{1}{y}\right)dx - \left(\frac{x}{y^2}+\frac{1}{y}\right)dy = 0$$

$$\frac{\partial u}{\partial x} = x + \frac{1}{y} \rightarrow u(x,y) = \frac{x^2}{2} + \frac{x}{y} + h(y)$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2} - \frac{1}{y}$$

$$\frac{\partial u}{\partial y} = \cancel{-\frac{x}{y^2}} + \frac{dh}{dy} = \cancel{-\frac{x}{y^2}} - \frac{1}{y} \Rightarrow \int dh = -\int\frac{dy}{y}$$

$$h(y) = -\ln y + c$$

$$u(x,y) = \frac{x^2}{2} + \frac{x}{y} - \ln y + c = k$$

④ $\sqrt{1-e^{2x}}\,y' = (1+y^2)e^x$

$$\sqrt{1-e^{2x}}\,dy = (1+y^2).e^x dx \Rightarrow \frac{dy}{1+y^2} = \frac{e^x dx}{\sqrt{1-e^{2x}}} \quad \left\{ \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right\} \int\frac{dt}{\sqrt{1-t^2}}$$

$$\arctan y = \arcsin e^x + c$$

for $y(0) = 0$

$$\arctan 0 = \arcsin 1 + c \Rightarrow 0 = \frac{\pi}{2} + c \Rightarrow \boxed{c = -\frac{\pi}{2}}$$

5 $\underbrace{(x^2+y^2+x)}_{M}\,dx + \underbrace{xy}_{N}\,dy = 0$

+ $\lambda = \lambda(x)$

$$\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = y \quad \Rightarrow \quad \ln\lambda = \int \frac{M_y - N_x}{N}\,dx = \int \frac{2y-y}{xy}\,dx = \int \frac{dx}{x}$$

$$\boxed{\lambda = x}$$

$$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0$$

$$\frac{\partial u}{\partial x} = x^3+xy^2+x^2 \quad u(x,y) = \frac{x^2y^2}{2} + h(x)$$

$$\frac{\partial u}{\partial y} = x^2y \qquad \frac{\partial u}{\partial x} = \cancel{xy^2} + \frac{dh}{dx} = x^3 + \cancel{xy^2} + x^2 \Rightarrow \int dh = \int (x^3+x^2)\,dx$$

$$h(x) = \frac{x^4}{4} + \frac{x^3}{3} + c$$

$$u(x,y) = \frac{x^2y^2}{2} + \frac{x^4}{4} + \frac{x^3}{3} + c = k$$

6 $y\left(1+e^{\frac{x}{y}}\right)dx + (y-x)\,e^{\frac{x}{y}}\,dy = 0$

$$\left.\begin{aligned} \frac{x}{y} = u \Rightarrow\ & x = uy \\ & dx = u\,dy + y\,du \end{aligned}\right\} \quad y\,(1+e^u)(u\,dy + y\,du) + (y-uy)\,e^u\,dy = 0$$

$$(uy + \cancel{uye^u} + ye^u - \cancel{uye^u})\,dy + y^2(1+e^u)\,du = 0$$

$$y\,(u+e^u)\,dy + y^2(1+e^u)\,du = 0$$

$$\int \frac{dy}{y} + \int \frac{1+e^u}{u+e^u}\,du = 0 \quad \Rightarrow \quad \ln y + \ln(u+e^u) = \ln c$$

$$y\,(u+e^u) = c$$

$$y\left(\frac{x}{y} + e^{\frac{x}{y}}\right) = c \quad \Rightarrow \quad x + y e^{x/y} = c$$

⑦ $(\cos x + y^2 \cos x)\,dx + y(1+\sin x)\,dy = 0$

$$\cos x\,(1+y^2)\,dx + y(1+\sin x)\,dy = 0$$

$$\int \frac{\cos x}{1+\sin x}\,dx + \int \frac{y\,dy}{1+y^2} = \int 0$$

$$\ln(1+\sin x) + \frac{1}{2}\ln(1+y^2) = \ln c \Rightarrow (1+\sin x)\sqrt{1+y^2} = c$$

⑧ $y' = \frac{y}{x}\left(1+\ln\left(\frac{y}{x}\right)\right) \Rightarrow dy = \frac{y}{x}\left(1+\ln\frac{y}{x}\right)dx$

$$\left.\begin{array}{l} \frac{y}{x} = u \Rightarrow y = ux \\ \qquad dy = u\,dx + x\,du \end{array}\right\} \begin{array}{l} u\,dx + x\,du = u(1+\ln u)\,dx \\ (\cancel{u} - \cancel{u} - u\ln u)\,dx + x\,du = 0 \end{array}$$

$$\int\frac{dx}{x} - \underbrace{\int\frac{du}{u\ln u}}_{} = \int 0$$

$$\left.\begin{array}{l} \ln u = t \\ \frac{du}{u} = dt \end{array}\right\} \int\frac{dt}{t} = \ln t = \ln(\ln u)$$

$$\ln x - \ln(\ln u) = \ln c \Rightarrow \frac{x}{\ln u} = c \Rightarrow x = c\ln\left(\frac{y}{x}\right)$$

For $y(1) = e$

$$1 = c\ln e \Rightarrow c = 1 \rightarrow x = \ln\left(\frac{y}{x}\right)$$

⑨ a. $$\left.\begin{array}{l} y = \sqrt{c-x} \\ y' = \frac{-1}{2\sqrt{c-x}} \end{array}\right\} y' = \frac{-1}{2y}$$

b. $$\left.\begin{array}{l} x + A\cos y = B \\ 1 - y'A\sin y = 0 \\ -y''A\sin y - (y')^2 A\cos y = 0 \end{array}\right\} \begin{array}{l} -y''\sin y - (y')^2\cos y = 0 \\ y'' + (y')^2\cot y = 0 \end{array}$$

c. $\ln y = Ae^{-x} + Be^{x} + C$

$$\frac{y'}{y} = -Ae^{-x} + Be^{x}$$

$$\frac{y''y - (y')^2}{y^2} = Ae^{-x} + Be^{x}$$

$$\frac{\left[y'''y + y''y' - 2y'y''\right]y^2 - (y''y - (y')^2)\,2yy'}{y^4} = -Ae^{-x} + Be^{x}$$

10. $y' = \dfrac{x-y-2}{x-4y+1} \Rightarrow (x-4y+1)dy - (x-y-2)dx = 0$

$$\left.\begin{array}{l} x = x_1 + h \;;\; dx = dx_1 \\ y = y_1 + k \;;\; dy = dy_1 \end{array}\right\} (x_1 - 4y_1 + h - 4k + 1)dy_1 - (x_1 - y_1 + h - k - 2)dx_1 = 0$$

$$h - 4k = -1$$
$$-\; h - k = 2$$
$$-3k = -3 \Rightarrow k = 1 \;;\; h = 3$$

$$(x_1 - 4y_1)dy_1 - (x_1 - y_1)dx_1 = 0$$

$$\left.\begin{array}{l} \dfrac{y_1}{x_1} = u \Rightarrow y_1 = ux_1 \\ \qquad dy_1 = u\,dx_1 + x_1\,du \end{array}\right\} (x_1 - 4ux_1)(u\,dx_1 + x_1\,du) - (x_1 - ux_1)dx_1 = 0$$

$$(ux_1 - 4u^2x_1 - x_1 + ux_1)dx_1 + x_1^2(1-4u)du = 0$$

$$x_1(-4u^2 + 2u - 1)dx_1 + x_1^2(1-4u)du = 0 \Rightarrow \int\frac{dx_1}{x_1} + \int\frac{(1-4u)\,du}{(-4u^2+2u-1)} = \int 0$$

$$\ln x_1 + \frac{1}{2}\ln(-4u^2 + 2u - 1) = \ln c$$

$$x_1\sqrt{-4u^2+2u-1} = c \Rightarrow x_1\sqrt{-4\frac{y_1^2}{x_1^2} + 2\frac{y_1}{x_1} - 1} = c \Rightarrow \sqrt{-4(y-1)^2 + 2(x-3)(y-1) - (x-3)^2} = c$$

(11) $(x-y)(4x+y)dx + x(5x-y)dy = 0$

$$\left.\begin{array}{l} \frac{y}{x} = u \Rightarrow y = ux \\ \qquad dy = u\,dx + x\,du \end{array}\right\} \begin{array}{l} (x-ux)(4x+ux)dx + x(5x-ux)(u\,dx + x\,du) = 0 \\ \cancel{x^2}(1-u)(4+u)dx + \cancel{x^2}(5-u)(u\,dx + x\,du) = 0 \end{array}$$

$$(4+u-4u-u^2+5u-u^2)dx + x(5-u)du = 0$$

$$(-2u^2+2u+4)dx + x(5-u)du = 0$$

$$\int \frac{dx}{x} + \int \frac{(5-u)du}{-2u^2+2u+4} = \int 0 \Rightarrow \int \frac{dx}{x} + \frac{1}{2}\int \frac{(u-5)du}{u^2-u-2} = \int 0$$

$$\frac{A}{u-2} + \frac{B}{u+1} = \frac{u-5}{u^2-u-2} \qquad \Rightarrow \begin{array}{l} A+B = 1 \\ -\ A-2B = -5 \\ \hline 3B = 6 \quad A = -1 \\ B = 2 \end{array}$$

$$\ln x + \frac{1}{2}\ln(u-2) + \ln(u+1) = \ln c \Rightarrow \frac{x(u+1)}{\sqrt{u-2}} = c \Rightarrow x(u+1) = c\sqrt{u-2}$$

$$x\left(\frac{y}{x}+1\right) = c\sqrt{\frac{y}{x}-2}$$

$$\Rightarrow y + x = c\frac{\sqrt{y-2x}}{\sqrt{x}}$$

(12) $(2\sqrt{xy} - x)dy + y\,dx = 0$

$$\left.\begin{array}{l} \frac{y}{x} = u \Rightarrow y = ux \\ \qquad dy = u\,dx + x\,du \end{array}\right\} \left[2\sqrt{x.(ux)} - x\right](u\,dx + x\,du) + ux\,dx = 0$$

$$ux(2\sqrt{u}-1)dx + x^2(2\sqrt{u}-1)du + ux\,dx = 0$$

$$2u\sqrt{u}\,x\,dx + x^2(2\sqrt{u}-1)du = 0$$

$$\int \frac{dx}{x} + \int \frac{2\sqrt{u}-1}{2u\sqrt{u}}du = \int 0$$

$$\ln x + \ln u + \frac{1}{\sqrt{u}} = \ln c \Rightarrow \ln x + \ln u - \ln c = -\frac{1}{\sqrt{u}} \Rightarrow \frac{ux}{c} = e^{-\frac{1}{\sqrt{u}}}$$

$$y = ce^{-\sqrt{\frac{x}{y}}}$$

(13) $(4x+3y^2)\,dx+2xy\,dy=0 \qquad \lambda=x^n$

$$\underbrace{(4x+3y^2)x^n\,dx}_{m}+\underbrace{2x^{n+1}y}_{N}\,dy=0$$

$$\frac{\partial m}{\partial y}=6yx^n=\frac{\partial N}{\partial x}=2(n+1)x^ny \Rightarrow 2(n+1)=6 \Rightarrow n=2 \Rightarrow \boxed{\lambda=x^2}$$

$$(4x^3+3x^2y^2)\,dx+2x^3y\,dy=0$$

$$\frac{\partial u}{\partial x}=4x^3+3x^2y^2 \qquad u(x,y)=x^3y^2+h(x)$$

$$\frac{\partial u}{\partial y}=2x^3y \qquad \frac{\partial u}{\partial x}=\cancel{3x^2y^2}+\frac{dh}{dx}=4x^3+\cancel{3x^2y^2}$$

$$dh=4x^3\,dx \Rightarrow h(x)=x^4+c$$

$$u(x,y)=x^3y^2+x^4+c=k$$

(14) $\cos y\,dx+(1+e^{-x})\sin y\,dy=0$

$$\underbrace{\int\frac{dx}{1+e^{-x}}}+\int\frac{\sin y\,dy}{\cos y}=\int 0$$

$$\int\frac{e^x\,dx}{(1+e^{-x})e^x}=\int\frac{e^x\,dx}{e^x+1}=\ln(e^x+1)$$

$$\ln(e^x+1)-\ln(\cos y)=\ln c \Rightarrow \frac{e^x+1}{\cos y}=c \Rightarrow e^x+1=c\cos y$$

(15) $y'=\frac{y}{x}\left[1-\ln\left(\frac{y}{x}\right)\right] \Rightarrow dy=\frac{y}{x}\left[1-\ln\left(\frac{y}{x}\right)\right]dx$

$$\frac{y}{x}=u \Rightarrow \begin{matrix} y=ux \\ dy=u\,dx+x\,du \end{matrix} \quad \begin{matrix} u\,dx+x\,du=u[1-\ln u]\,dx \\ (\cancel{u}-\cancel{u}+u\ln u)\,dx+x\,du=0 \end{matrix}$$

$$\int\frac{dx}{x}+\underbrace{\int\frac{du}{u\ln u}}=\int 0 \Rightarrow \ln x+\ln(\ln u)=\ln c$$

$$\left.\begin{matrix} \ln u=t \\ \frac{du}{u}=dt \end{matrix}\right\} \int\frac{dt}{t}=\ln t=\ln(\ln u)$$

$$x(\ln u)=c$$

$$x\ln\left(\frac{y}{x}\right)=c$$

(16) $(3x-2y+2)\,dy+(2y-3x)\,dx=0$

$$\left.\begin{aligned} 3x-2y&=u \\ 3\,dx-2\,dy&=du \end{aligned}\right\} \quad \begin{aligned} &(u+2)\left(\frac{3\,dx-du}{2}\right)-u\,dx=0 \\ &\left(\frac{3u}{2}+3-u\right)dx-\frac{(u+2)}{2}\,du=0 \end{aligned}$$

$$\left(\frac{u}{2}+3\right)dx-\left(\frac{u+2}{2}\right)du=0 \Rightarrow \int dx-\int\frac{u+2}{u+6}\,du=\int 0$$

$$x-\int\left(1-\frac{4}{u+6}\right)du=c$$

$$x-u+4\ln(u+6)=c \quad\Rightarrow\quad x-3x+2y+4\ln(3x-2y+6)=c$$

(17) $y'=\frac{y}{x}-\sec\left(\frac{y}{x}\right) \Rightarrow dy=\left[\frac{y}{x}-\sec\left(\frac{y}{x}\right)\right]dx$

$$\left.\begin{aligned} y&=ux \\ dy&=u\,dx+x\,du \end{aligned}\right\} \quad \begin{aligned} &u\,dx+x\,du=(u-\sec u)\,dx \\ &(\cancel{u}-\cancel{u}+\sec u)\,dx+x\,du=0 \end{aligned}$$

$$\int\frac{dx}{x}+\int\frac{du}{\sec u}=\int 0 \Rightarrow \ln x+\sin u=\ln c \Rightarrow x=ce^{-\sin u} \Rightarrow x=ce^{-\sin\frac{y}{x}}$$

For $y(1)=\frac{\pi}{2}$

$$1=ce^{-\sin\frac{\pi}{2}} \Rightarrow c=e$$

(18) $(e^{-2y}-y)\cos x\,dy=e^{y}\sin 2x\,dx$

$$\int\frac{e^{-2y}-y}{e^{y}}\,dy=\int\frac{\sin 2x}{\cos x}\,dx \Rightarrow \int\left(e^{-3y}-ye^{-y}\right)dy=\int\frac{2\sin x\cancel{\cos x}}{\cancel{\cos x}}\,dx$$

$$-\frac{1}{3}e^{-3y}+(1+y)e^{-y}=-2\cos x+c$$

For $y(0)=0$

$$-\frac{1}{3}+1=-2+c \Rightarrow c=\frac{8}{3}$$

Solutions of Application Questions II

1. $y + 2xy' = (y')^3$ Lagrange

$y' = p \Rightarrow y + 2xp = p^3$

$\underbrace{y'}_{p} + 2p + 2xp' = 3p^2 p'$

$p'(3p^2 - 2x) = 3p \Rightarrow p' = \dfrac{3p}{3p^2 - 2x} = \dfrac{dp}{dx}$

$\dfrac{dx}{dp} = \dfrac{3p^2 - 2x}{3p} = p - \dfrac{2x}{3p} \Rightarrow \dfrac{dx}{dp} + \dfrac{2x}{3p} = p$ L.D.E

$\lambda(p) = e^{\int \frac{2}{3p} dp} = e^{\frac{2}{3}\ln p} = \sqrt[3]{p^2}$

$x = \dfrac{1}{\sqrt[3]{p^2}}\left[\int \sqrt[3]{p^2} \cdot p\, dp + c\right] = \dfrac{1}{\sqrt[3]{p^2}}\left[\dfrac{3}{8} p^2 \sqrt[3]{p^2} + c\right] = \dfrac{3}{8} p^2 + c p^{-\frac{2}{3}}$

$y = p^3 - 2xp = p^3 - \dfrac{3}{4} p^3 - 2c p^{\frac{1}{3}} = \dfrac{p^3}{4} - 2c p^{\frac{1}{3}}$

2. $\left(x - y\cos\left(\dfrac{y}{x}\right)\right)dx + x\cos\left(\dfrac{y}{x}\right)dy = 0$ $f(e) = e\dfrac{\pi}{2}$

$\left.\begin{array}{l} x \to tx \\ y \to ty \end{array}\right\}$ $\left(tx - ty\cos\left(\dfrac{ty}{tx}\right)\right)dx + tx\cos\left(\dfrac{ty}{tx}\right)dy = 0$

$t\,[\text{Dif. Eq}] = 0$

it's homogeneous with degree 1.

$\dfrac{y}{x} = u \Rightarrow \left.\begin{array}{l} y = ux \\ dy = u\,dx + x\,du \end{array}\right\}$

$(x - ux\cos u)dx + x\cos u\,(u\,dx + x\,du) = 0$

$(x - \cancel{ux\cos u} + \cancel{ux\cos u})dx + x^2\cos u\,du = 0$

$x\,dx + x^2\cos u\,du = 0 \rightarrow \int \dfrac{x\,dx}{x^2} + \int \cos u\,du = \int 0$

$\ln x + \sin u = \ln c \Rightarrow \ln x - \ln c = -\sin u$

$\ln \frac{x}{c} = -\sin u \Rightarrow x = c . e^{-\sin u}$

$\boxed{x = c e^{-\sin\left(\frac{y}{x}\right)}}$ G. S

For $f(e) = e\frac{\pi}{2}$

$e = c . e^{-\sin\frac{\pi}{2}} \Rightarrow \boxed{c = e^2} \Rightarrow \boxed{x = e^{2-\sin\left(\frac{y}{x}\right)}}$

③ $y' + y^2 = \frac{y}{x} - \frac{1}{x^2}$ $\quad y_1 = \frac{1}{x}$ Riccati

$y = y_1 + \frac{1}{z}$

$y = \frac{1}{x} + \frac{1}{z}$

$y' = -\frac{1}{x^2} - \frac{z'}{z^2}$

$\cancel{-\frac{1}{x^2}} - \frac{z'}{z^2} + \cancel{\frac{1}{x^2}} + \frac{2}{xz} + \frac{1}{z^2} = \cancel{\frac{1}{x^2}} + \frac{1}{zx} - \cancel{\frac{1}{x^2}}$

$-\frac{z'}{z^2} + \frac{1}{zx} + \frac{1}{z^2} = 0 \Rightarrow -z' + \frac{z}{x} + 1 = 0$

$\Rightarrow z' - \frac{z}{x} = 1$ L.D.E

$\lambda(x) = e^{\int -\frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$

$z = x\left[\int \frac{1}{x} . 1 . dx + c\right] = x\left[\ln x + c\right]$

$y = \frac{1}{x} + \frac{1}{x[\ln x + c]} = \frac{1}{x}\left[1 + \frac{1}{c + \ln x}\right]$

④ $y = x\left[(y')^2 + 2y'\right]$

$y' = p \Rightarrow y = x(p^2 + 2p)$

$\underset{p}{y'} = p^2 + 2p + x(2pp' + 2p')$

$p'(2px + 2x) = -p^2 - p \Rightarrow p' = \frac{-p\cancel{(p+1)}}{2x\cancel{(p+1)}}$

$p' = \frac{dp}{dx} = -\frac{p}{2x} \Rightarrow \frac{2dp}{p} = -\frac{dx}{x}$ $\quad 2\ln p = -\ln x + \ln c$

$p^2 = \frac{c}{x} \Rightarrow \boxed{x = \frac{c}{p^2}}$ ★

$$y = \frac{c}{p^2}\left[p^2+2p\right] = \boxed{c+\frac{2c}{p}}$$

5) $w(f,g) = 3e^{4t} \quad f(t) = e^{2t}$

$$w(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = 3e^{4t} \Rightarrow e^{2t}g' - 2e^{2t}g = 3e^{4t}$$

$$g' - 2g = 3e^{2t} \quad \text{L.D.E}$$

$$\lambda(t) = e^{\int -2dt} = e^{-2t}$$

$$g(t) = e^{2t}\left[\int e^{-2t}\cdot 3e^{2t}\,dt + c\right] = (3t+c)e^{2t}$$

6) $(1-\sqrt{3})y' + y\sec x = y^{\sqrt{3}}\sec x$ Bernoulli

$$\left.\begin{array}{l} y^{1-\sqrt{3}} = u \\ (1-\sqrt{3})\,y^{-\sqrt{3}}\cdot y' = u' \end{array}\right\} \begin{array}{l} (1-\sqrt{3})\,y'y^{-\sqrt{3}} + y^{1-\sqrt{3}}\sec x = \sec x \\ u' + u\sec x = \sec x \quad \text{L.D.E} \end{array}$$

$$\lambda(x) = e^{\int \sec x\,dx} = e^{\ln|\sec x+\tan x|} = (\sec x+\tan x)$$

$$u(x) = \frac{1}{(\sec x+\tan x)}\left[\int(\sec x+\tan x)\sec x\,dx + c\right]$$

$$= \frac{1}{(\sec x+\tan x)}\left[\tan x+\sec x+c\right] = \boxed{1+\frac{c}{\sec x+\tan x}}$$

$$y^{1-\sqrt{3}} = u \Rightarrow y = \sqrt[1-\sqrt{3}]{u}$$

7) $\underbrace{y^2}_{m}dx + \underbrace{(3xy-e^y)}_{N}dy = 0$

$$\frac{\partial m}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = 3y$$

not depend on x.

(+) $\lambda = \lambda(x)$

$$\ln\lambda = \int \frac{m_y - N_x}{N}dx = \int \frac{2y-3y}{3xy-e^y}dx$$

7 $\lambda = \lambda(y)$

$$\ln\lambda = \int \frac{N_x - m_y}{m}\,dy = \int \frac{3y-2y}{y^2}\,dy = \int \frac{dy}{y} = \ln y \Rightarrow \lambda = y$$

$y^3\,dx + (3xy^2 - ye^y)\,dy = 0$ Exact

$$\frac{\partial u}{\partial x} = y^3 \rightarrow u(x,y) = \int y^3\,dx + h(y)$$

$$u(x,y) = xy^3 + h(y)$$

$$\frac{\partial u}{\partial y} = 3xy^2 - ye^y$$

$$\frac{\partial u}{\partial y} = \cancel{3xy^2} + \frac{dh}{dy} = \cancel{3xy^2} - ye^y$$

$$\frac{dh}{dy} = -ye^y \Rightarrow dh = -\int ye^y\,dy = -(y-1)e^y + c$$

$$u(x,y) = \boxed{xy^3 - (y-1)e^y + c = k}$$

8 $c_1(x+2) + c_2(x+2)\ln(x+2) = y$

$$c_1 + c_2\ln(x+2) + c_2\frac{\cancel{(x+2)}}{\cancel{(x+2)}} = y'$$

$$\frac{c_2}{(x+2)} = y''$$

$$c_1\cancel{(x+2)} + c_2\cancel{(x+2)}\ln(x+2) = y$$

$(x+2)$ $$\cancel{c_1} + c_2\ln(x+2) + c_2 = y'$$

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$$-(x+2)c_2 = y - (x+2)y' \Rightarrow c_2 = \frac{y-(x+2)y'}{-(x+2)}$$

$$c_2 = \frac{y-(x+2)y'}{-(x+2)} = (x+2)y''$$

$$y - (x+2)y' = -(x+2)^2y'' \Rightarrow (x+2)^2y'' - (x+2)y' + y = 0$$

9 $y(x+3)dx+(x+2)(ydx-xdy)=0$

$(2xy+5y)dx-x(x+2)dy=0$

$y(2x+5)dx-x(x+2)dy=0 \Rightarrow \frac{2x+5}{x(x+2)}dx-\frac{dy}{y}=0$

$\frac{A}{x}+\frac{B}{x+2}=\frac{2x+5}{x(x+2)} \Rightarrow \left.\begin{array}{l} A+B=2 \\ 2A=5 \end{array}\right\} \begin{array}{l} A=\frac{5}{2} \\ B=-\frac{1}{2} \end{array}$

$\frac{5}{2}\int\frac{dx}{x}-\frac{1}{2}\int\frac{dx}{x+2}-\int\frac{dy}{y}=\int 0$

$\frac{5}{2}\ln x-\frac{1}{2}\ln(x+2)-\ln y=\ln c \Rightarrow \frac{x^{5/2}}{\sqrt{x+2}\cdot y}=c$

10 $(y-xy')(x-\frac{y}{y'})=-2$

$y'=p \Rightarrow (y-xp)(x-\frac{y}{p})=-2 \Rightarrow (y-xp)(xp-y)=-2p$

$\Rightarrow (y-xp)^2=2p$

$y-xp=\mp\sqrt{2p} \Rightarrow y=xp\mp\sqrt{2p}$

$y'=\cancel{p}+xp'\mp\frac{\cancel{2}p'}{\cancel{2}\sqrt{2p}} \Rightarrow p'\left(x\mp\frac{1}{\sqrt{2p}}\right)=0$

$\cancel{p}$

$1^\circ)\ p'=0 \Rightarrow p=c \Rightarrow y=cx\mp\sqrt{2c}$ G.S

$2^\circ)\ x=\pm\frac{1}{\sqrt{2p}}$

$y=\pm\frac{\sqrt{p}}{\sqrt{2}}\mp\sqrt{2p}=$

?

(11) $y = xy' + \dfrac{1}{y'}$

$y' = p \Rightarrow y = xp + \dfrac{1}{p}$

$\cancel{y'} = \cancel{p} + xp' - \dfrac{p'}{p^2} \Rightarrow p'\left(x - \dfrac{1}{p^2}\right) = 0$

$\cancel{p}$

(+) $p' = 0 \Rightarrow p = c \Rightarrow y = cx + \dfrac{1}{c}$ G.S

(+) $x = \dfrac{1}{p^2}$

$y = \dfrac{1}{p} + \dfrac{1}{p} = \dfrac{2}{p}$

$\Bigg\}\ x = \dfrac{y^2}{4} \Rightarrow \boxed{y^2 = 4x}$

(12) $y = xy' + \sqrt{(y')^2 + 1}$

$y' = p \Rightarrow y = xp + \sqrt{p^2 + 1}$

$\cancel{y'} = \cancel{p} + xp' + \dfrac{\cancel{2}pp'}{\cancel{2}\sqrt{p^2+1}} \Rightarrow p'\left[x + \dfrac{p}{\sqrt{p^2+1}}\right] = 0$

$\cancel{p}$

(+) $p' = 0 \Rightarrow p = c \quad y = cx + \sqrt{c^2 + 1}$ G.S

$x = \dfrac{-p}{\sqrt{p^2+1}}$

$y = \dfrac{-p^2}{\sqrt{p^2+1}} + \sqrt{p^2+1} = \dfrac{1}{\sqrt{p^2+1}}$

$\Bigg\}\ \boxed{x^2 + y^2 = 1}$

(13) $y' = -\dfrac{y^2 \ln x}{x^2} + \dfrac{y}{x}$ and $y(1) = 1$

$y' - \dfrac{y}{x} = -\dfrac{y^2 \ln x}{x^2}$ Bernoulli

$y^{1-2} = y^{-1} = u$

$-y^{-2}y' = u'$

$\Bigg\}\ y'y^{-2} - \dfrac{y^{-1}}{x} = -\dfrac{\ln x}{x^2}$

$-u' - \dfrac{u}{x} = -\dfrac{\ln x}{x^2} \Rightarrow u' + \dfrac{u}{x} = \dfrac{\ln x}{x^2}$ L.D.E

$$\lambda(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$\ln x = t$, $\frac{dx}{x} = dt$ $\Big\}$ $\int t\,dt = \frac{t^2}{2} = \frac{\ln^2 x}{2}$

$$u = \frac{1}{x}\left[\int x \cdot \frac{\ln x}{x^2}\,dx + c\right] = \frac{1}{x}\left[\int \frac{\ln x\,dx}{x} + c\right]$$

$$u = \frac{1}{x}\left[\frac{\ln^2 x}{2} + c\right]$$

$$y = \frac{1}{u} = \frac{2x}{\ln^2 x + 2c}$$

$$y(1) = 1 \implies 1 = \frac{2}{0 + 2c} \Rightarrow \boxed{c = 1}$$

(14) $-y' = e^{-x}y^2 - y - e^x \qquad y_1(x) = -e^x$

$$\left.\begin{aligned} y &= -e^x + \frac{1}{z} \\ y' &= -e^x - \frac{z'}{z^2} \end{aligned}\right\} \quad \begin{aligned} &e^x + \frac{z'}{z^2} = e^{-x}\left(e^{2x} - \frac{2e^x}{z} + \frac{1}{z^2}\right) + \cancel{e^x} - \frac{1}{z} - \cancel{e^x} \\ &\cancel{e^x} + \frac{z'}{z^2} = \cancel{e^x} - \frac{2}{z} + \frac{e^{-x}}{z^2} - \frac{1}{z} \end{aligned}$$

$$z' = -3z + e^{-x} \Rightarrow z' + 3z = e^{-x}$$

$$\lambda(x) = e^{\int 3dx} = e^{3x}$$

$$z = e^{-3x}\left[\int e^{3x} \cdot e^{-x}\,dx + c\right] = e^{-3x}\left(\frac{1}{2}e^{2x} + c\right) = \frac{e^{-x}}{2} + ce^{-3x}$$

$$y = -e^x + \frac{1}{\frac{e^{-x}}{2} + ce^{-3x}}$$

(15) $\frac{dy}{dx} + xy = 6x^3y^3$

$$\left.\begin{aligned} y^{1-3} &= y^{-2} = u \\ -2y^{-3}y' &= u' \end{aligned}\right\} \quad \begin{aligned} &y'y^{-3} + xy^{-2} = 6x^3 \\ &-\frac{u'}{2} + ux = 6x^3 \Rightarrow u' - 2ux = -12x^3 \end{aligned}$$

$$\lambda(x) = e^{\int -2x\,dx} = e^{-x^2} \Rightarrow u = e^{x^2}\left[\underbrace{\int e^{-x^2}(-12x^3)\,dx}_{I} + c\right]$$

$$\left.\begin{aligned} I:\ -x^2 &= t \\ -2x\,dx &= dt \end{aligned}\right\} \int -e^t\, 6t\, dt = -6(t-1)e^t = -6(-x^2-1)e^{-x^2}$$

$$u = e^{x^2}\left[6(x^2+1)e^{-x^2} + c\right] = 6(x^2+1) + ce^{x^2}$$

$$y^{-2} = u \Rightarrow y = \frac{1}{\sqrt{u}}$$

16 $y^3 y' + \dfrac{y^4}{2x} - x = 0 \quad , y(1) = 2$

$$y' + \frac{y}{2x} = \frac{x}{y^3} \qquad \text{B.D.E}$$

$$\left.\begin{aligned} y^{1-(-3)} &= y^4 = u \\ 4y^3y' &= u' \end{aligned}\right\} \frac{u'}{4} + \frac{u}{2x} = x \Rightarrow u' + \frac{2u}{x} = 4x \qquad \text{L.D.E}$$

$$\lambda(x) = e^{\int 2\frac{dx}{x}} = e^{2\ln x} = x^2$$

$$u = \frac{1}{x^2}\left[\int x^2 \cdot 4x\, dx + c\right] = \frac{1}{x^2}\left[x^4 + c\right] = x^2 + \frac{c}{x^2}$$

$$y = \sqrt[4]{u}$$

Extra question

solve the equation $y' + \dfrac{y}{x} = \dfrac{\ln x}{x^2}$ using variation of parameters.

$$y' + \frac{y}{x} = 0 \Rightarrow \frac{dy}{dx} + \frac{y}{x} = 0$$

$$\Rightarrow \int \frac{dy}{y} + \int \frac{dx}{x} = \int 0 \Rightarrow \ln y + \ln x = \ln c \Rightarrow \boxed{y = \frac{c}{x}}$$

assume that $c = c(x)$

$$y' = \frac{c'x - c}{x^2} = \frac{c'}{x} - \frac{c}{x^2}$$

$$\frac{c'}{x} - \cancel{\frac{c}{x^2}} + \cancel{\frac{c}{x^2}} = \frac{\ln x}{x^2} \Rightarrow c' = \frac{\ln x}{x} \Rightarrow c = \frac{\ln^2 x}{2} + k$$

$$y = \frac{\ln^2 x}{2x} + \frac{k}{x}$$

Solutions of Application Questions 3

a. $y_h = c_1 e^{x} + c_2 e^{-x} + c_3 e^{\frac{2x}{3}} + c_4 e^{-\frac{2x}{3}} + c_5 e^{5x} + c_6 e^{-6x}$

b. $y_h = c_1 + c_2 x + c_3 e^{-2x} + c_4 x e^{-2x} + c_5 \cos 3x + c_6 \sin 3x$

c. $y_h = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x + c_5 \cos x + c_6 \sin x$

d. $y_h = (c_1 + c_2 x + c_3 x^2) e^{2x} + c_4 e^{-2x} + e^{2x} (c_5 \cos 3x + c_6 \sin 3x)$

e. $(c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x$

f. $y_h = e^{-x} \left((c_1 + c_2 x) \cos 3x + (c_3 + c_4 x) \sin 3x\right) + c_5 e^{2x} + c_6 e^{-2x}$

g. $y_h = e^{2x} \left[(c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x\right]$

h. $y_h = c_1 e^{\sqrt{2} x} + (c_2 + c_3 x) \cos 2x + (c_4 + c_5 x) \sin 2x + c_6 e^{\sqrt{3} x}$

i. $y_h = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x + c_5 \cos 3x + c_6 \sin 3x$

j. $y_h = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + e^{x} (c_5 \cos x + c_6 \sin x)$

k. $y_h = c_1 e^{2x} + (c_2 + c_3 x) e^{-x} + (c_4 + c_5 x + c_6 x^2) e^{-3x}$

l. $y_h = (c_1 + c_2 x + c_3 x^2) \cos \frac{3x}{2} + (c_4 + c_5 x + c_6 x^2) \sin \frac{3x}{2}$

m. $y_h = e^{-3x} \left[(c_1 + c_2 x + c_3 x^2) \cos 5x + (c_4 + c_5 x + c_6 x^2) \sin 5x\right]$

n. $y_h = e^{x} (c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x) + c_3 \cos \sqrt{2} x + c_4 \sin \sqrt{2} x + (c_5 + c_6 x) e^{-\sqrt{2} x}$

1) a. $r^3 - 3r^2 - r + 3 = 0$

$r^2 (r-3) - (r-3) = 0 \Rightarrow (r^2 - 1)(r-3) = 0 \Rightarrow \left.\begin{array}{l} r_1 = 1 \\ r_2 = -1 \\ r_3 = 3 \end{array}\right\} y_h = c_1 e^{x} + c_2 e^{-x} + c_3 e^{3x}$

b.
$$\begin{array}{l|l} r^3 + 2r^2 - 4r - 8 & r-2 \\ \hline -\;r^3 - 2r^2 & r^2 + 4r + 4 \\ \hline \quad 4r^2 - 4r - 8 & \\ -\;4r^2 - 8r & \\ \hline \quad 4r - 8 & \end{array}$$

$(r-2)(r+2)^2 = 0$

$r_1 = 2, \quad r_2 = r_3 = -2$

$y_h = c_1 e^{2x} + (c_2 + c_3 x) e^{-2x}$

c. $r^3 - 3r^2 + 3r - 1 = 0$

$$(r^3-1) - 3r(r-1) = 0 \Rightarrow (r-1)(\overbrace{r^2+r+1-3r}^{r^2-2r+1}) = 0 \Rightarrow (r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$$

$$y_h = (c_1 + c_2 x + c_3 x^2)e^x$$

d. $r^5 + 9r^3 = 0 \Rightarrow r^3(r^2+9) = 0 \Rightarrow$ $r_1 = r_2 = r_3 = 0$

$r_{4,5} = \mp 3i$

$$y_h = (c_1 + c_2 x + c_3 x^2) + c_4 \cos 3x + c_5 \sin 3x$$

e. $r^4 - 4r^3 + 29r^2 = 0$

$r^2(r^2 - 4r + 29) = 0$ $\quad r_1 = r_2 = 0$

$\Delta = 16 - 4.1.29 = -100$ $\quad r_{3,4} = \dfrac{4 \mp \sqrt{-100}}{2} = 2 \mp 5i$

$$y_h = c_1 + c_2 x + e^{2x}(c_3 \cos 5x + c_4 \sin 5x)$$

f. $r^3 - r^2 - r + 1 = 0$

$r^2(r-1) - (r-1) = 0 \Rightarrow (r^2-1)(r-1) = 0 \Rightarrow$ $r_1 = 1 = r_2 = 1$

$r_3 = -1$

$$y_h = (c_1 + c_2 x)e^x + c_3 e^{-x}$$

g. $r^3 - 5r^2 + 6r = 0 \Rightarrow r(\underset{-3\ -2}{r^2 - 5r + 6}) = 0 \Rightarrow r_1 = 0,\ r_2 = 3,\ r_3 = 2$

$$y_h = c_1 + c_2 e^{3x} + c_3 e^{2x}$$

Homework

(+) $(x+2y+1)\,dx - (2x+4y-3)\,dy = 0$

$$\left.\begin{aligned} x+2y&=u \\ dx+2dy&=du \end{aligned}\right\} \quad (u+1)\,dx-(2u-3)\left(\frac{du-dx}{2}\right)=0$$

$$\left(u+1+u-\frac{3}{2}\right)dx-\left(u-\frac{3}{2}\right)du=0$$

$$(4u-1)\,dx-(2u-3)\,du=0 \Rightarrow \int dx-\underbrace{\int\frac{2u-3}{4u-1}\,du}_{\int\left(\frac{1}{2}-\frac{5}{2}\cdot\frac{1}{4u-1}\right)du}=\int 0$$

$$x-\left(\frac{u}{2}-\frac{5}{2}\cdot\frac{1}{4}\ln(4u-1)\right)=c$$

$$x-\frac{x}{2}-y+\frac{5}{8}\ln(4x+8y-1)=c$$

(+) $y'=\dfrac{x-y-2}{x-4y+1} \Rightarrow (x-4y+1)\,dy-(x-y-2)\,dx=0$

$$\begin{vmatrix} 1 & -4 \\ -1 & 1 \end{vmatrix} \neq 0 \qquad \left.\begin{aligned} x&=x_1+h \;;\; dx=dx_1 \\ y&=y_1+k \;;\; dy=dy_1 \end{aligned}\right\}$$

$$(x_1+h-4y_1-4k+1)\,dy_1-(x_1+h-y_1-k-2)\,dx_1=0$$

$$\left.\begin{aligned} h-4k+1&=0 \\ -\ \ h-k-2&=0 \end{aligned}\right\} \qquad \begin{aligned} x&=x_1+3 \Rightarrow x_1=x-3 \\ y&=y_1+1 \Rightarrow y_1=y-1 \end{aligned}$$

$$-3k+3=0 \Rightarrow k=1$$
$$h=3$$

$$(x_1-4y_1)\,dy_1-(x_1-y_1)\,dx_1=0$$

$$\left.\begin{aligned} \frac{y_1}{x_1}=u \Rightarrow y_1&=ux_1 \\ dy_1&=u\,dx_1+x_1\,du \end{aligned}\right\} \quad \begin{aligned} &(x_1-4ux_1)(u\,dx_1+x_1\,du)- \\ &(x_1-ux_1)\,dx_1=0 \end{aligned}$$

$$(ux_1 - 4u^2x_1 - x_1 + ux_1)\,dx_1 + (x_1^2 - 4ux_1^2)\,du = 0$$

$$x_1(-4u^2+2u-1)\,dx_1 + x_1^2(1-4u)\,du = 0$$

$$\int \frac{dx_1}{x_1} + \int \frac{(1-4u)\,du}{(-4u^2+2u-1)} = \int 0$$

$$\ln x_1 + \frac{1}{2}\ln(-4u^2+2u-1) = \ln c \Rightarrow x_1\sqrt{-4u^2+2u-1} = c$$

$$x_1\sqrt{-4\frac{y_1^2}{x_1^2} + 2\frac{y_1}{x_1} - 1} = c \Longrightarrow \sqrt{-4y_1^2+2y_1x_1-x_1^2} = c$$

$$\sqrt{-4(y-1)^2+2(y-1)(x-3)-(x-3)^2} = c$$

7) $(x+2y-4)\,dx + (2x+3y-7)\,dy = 0$

$$\left.\begin{array}{l} x = x_1 + h \;;\; dx = dx_1 \\ y = y_1 + k \;;\; dy = dy_1 \end{array}\right\} (x_1+h+2y_1+2k-4)\,dx_1 + (2x_1+2h+3y_1+3k-7)\,dy_1 = 0$$

$$\left.\begin{array}{r} h+2k-4=0 \;\; /\cdot(-2) \\ 2h+3k-7=0 \end{array}\right\} \qquad \begin{array}{l} x = x_1+2 \Rightarrow \boxed{x_1 = x-2} \\ y = y_1+1 \Rightarrow \boxed{y_1 = y-1} \end{array}$$

$$-k+1=0 \Rightarrow k=1;\; h=2$$

$$(x_1+2y_1)\,dx_1 + (2x_1+3y_1)\,dy_1 = 0$$

$$\left.\begin{array}{l} \frac{y_1}{x_1} = u \Rightarrow y_1 = ux_1 \\ \qquad\qquad dy_1 = u\,dx_1 + x_1\,du \end{array}\right\}$$

$$(x_1+2ux_1)\,dx_1 + (2x_1+3ux_1)(u\,dx_1 + x_1\,du) = 0$$

$$(x_1+2ux_1+2ux_1+3u^2x_1)\,dx_1 + (2x_1^2+3ux_1^2)\,du = 0$$

$$x_1(3u^2+4u+1)\,dx_1 + x_1^2(3u+2)\,du = 0$$

$$\int \frac{dx_1}{x_1} + \int \frac{3u+2}{3u^2+4u+1}\,du = \int 0$$

$$\ln x_1 + \frac{1}{2}\ln(3u^2+4u+1) = \ln c \Rightarrow x_1\sqrt{3u^2+4u+1} = c$$

$$x_1\sqrt{3\frac{y_1^2}{x_1^2} + 4\frac{y_1}{x_1} + 1} = c \Rightarrow \sqrt{3(y-1)^2+4(y-1)(x-2)+(x-2)^2} = c$$

(+) $\underbrace{\arcsin x}_{N}\, dy + \underbrace{\dfrac{y+2\sqrt{1-x^2}\cos x}{\sqrt{1-x^2}}}_{M}\, dx = 0$

$\dfrac{\partial m}{\partial y} = \dfrac{1}{\sqrt{1-x^2}} = \dfrac{\partial N}{\partial x} = \dfrac{1}{\sqrt{1-x^2}}$ Exact D.E

$\left.\begin{aligned} \dfrac{\partial u}{\partial x} &= \dfrac{y}{\sqrt{1-x^2}} + 2\cos x \\ \dfrac{\partial u}{\partial y} &= \arcsin x \end{aligned}\right\} \rightarrow u(x,y) = y\arcsin x + h(x)$

$\dfrac{\partial u}{\partial x} = \dfrac{y}{\sqrt{1-x^2}} + \dfrac{dh}{dx} = \dfrac{y}{\sqrt{1-x^2}} + 2\cos x$

$\dfrac{dh}{dx} = 2\cos x \Rightarrow h(x) = 2\sin x + c$

$u(x,y) = y\arcsin x + 2\sin x + c = k$

(+) $y' + \dfrac{y}{x} = \dfrac{e^{2x} - \sin x}{x}$

$\left.\begin{aligned} P(x) &= \dfrac{1}{x} \\ Q(x) &= \dfrac{e^{2x} - \sin x}{x} \end{aligned}\right\}$ $\lambda(x) = e^{\int P(x)dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

$y = \dfrac{1}{\lambda}\left[\int \lambda . Q(x) dx + c\right]$

$y = \dfrac{1}{x}\left[\int x . \left(\dfrac{e^{2x} - \sin x}{x}\right) dx + c\right]$

$y = \dfrac{1}{x}\left(\dfrac{1}{2}e^{2x} + \cos x + c\right)$

(+) $(1-x^2)y' + xy = ax$

$y' + \dfrac{x}{1-x^2}y = \dfrac{ax}{1-x^2}$

$\lambda = e^{\int \frac{x dx}{1-x^2}} = e^{-\frac{1}{2}\ln(1-x^2)} \Rightarrow \lambda(x) = \dfrac{1}{\sqrt{1-x^2}}$

$y = \sqrt{1-x^2}\left[\int \dfrac{1}{\sqrt{1-x^2}} \cdot \dfrac{ax}{(1-x^2)} dx + c\right]$

$$\int \frac{ax}{(1-x^2)^{3/2}}\,dx \quad \left\}\begin{array}{l} 1-x^2 = t \\ -2x\,dx = dt \end{array}\right.$$

$$a\int \frac{-dt}{2t^{3/2}} = -\frac{a}{2}\int t^{-\frac{3}{2}}\,dt = -\frac{a}{2}\,\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} = \frac{a}{\sqrt{t}} = \frac{a}{\sqrt{1-x^2}}$$

$$y = \sqrt{1-x^2}\left[\frac{a}{\sqrt{1-x^2}} + c\right] = a + c\sqrt{1-x^2}$$

+ $y' + \frac{y}{x} = 3\cos 2x$ $(x > 0)$ variation of parameters.

$$y' + \frac{y}{x} = 0 \Rightarrow \frac{dy}{dx} + \frac{y}{x} = 0 \Rightarrow \int \frac{dy}{y} + \int \frac{dx}{x} = \int 0$$

$$\ln y + \ln x = \ln c \Rightarrow \boxed{y = \frac{c}{x}}$$

let us assume $c = c(x)$

$$y = \frac{c(x)}{x} \quad ; \quad y' = \frac{c'(x) \cdot x - c}{x^2}$$

$$\frac{c'(x)}{x} - \cancel{\frac{c(x)}{x^2}} + \cancel{\frac{c(x)}{x^2}} = 3\cos 2x$$

$$c'(x) = 3x\cos 2x \Rightarrow c(x) = 3\underbrace{\int x\cos 2x\,dx}_{I} + k$$

$$I: \quad \begin{array}{ll} x = u & dv = \cos 2x\,dx \\ dx = du & v = \frac{1}{2}\sin 2x \end{array} \left\} \begin{array}{l} I = uv - \int v\,du \\ I = \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx \end{array}\right.$$

$$I = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x$$

$$c(x) = \frac{3x}{2}\sin 2x + \frac{3}{4}\cos 2x + k$$

$$y = \frac{c(x)}{x} = \frac{3}{2}\sin 2x + \frac{3}{4x}\cos 2x + \frac{k}{x}$$

⊕ $2y'+y=3x^2$

$2y'+y=0 \Rightarrow 2\frac{dy}{dx}+y=0 \Rightarrow \int \frac{dy}{y}+\int\frac{dx}{2}=\int 0$

$\ln y+\frac{x}{2}=\ln c \Rightarrow y=ce^{-\frac{x}{2}}$

let us take $c=c(x)$

$y=c(x)e^{-\frac{x}{2}}$; $y'=c'(x)e^{-\frac{x}{2}}-\frac{c(x)}{2}e^{-\frac{x}{2}}$

$\to 2c'(x)e^{-x/2}-\cancel{c(x)e^{-x/2}}+\cancel{c(x)e^{-x/2}}=3x^2$

$c'(x)=\frac{3}{2}x^2e^{x/2} \Rightarrow c(x)=\frac{3}{2}\underbrace{\int x^2e^{\frac{x}{2}}dx}_{I}+k$

$I \to$ integration by parts

$c(x)=\frac{3}{2}e^{\frac{x}{2}}(2x^2-8x+16)+k$

$y=\frac{3}{2}(2x^2-8x+16)+ke^{-\frac{x}{2}}$

⊕ $xy'+y=y^2\ln x$

$y^{1-2}=y^{-1}=z$

$-y^{-2}y'=z'$

$xy'y^{-2}+y^{-1}=\ln x$

$-xz'+z=\ln x \Rightarrow z'-\frac{z}{x}=-\frac{\ln x}{x}$ L.D.E

$\lambda(x)=e^{-\int\frac{dx}{x}}=e^{-\ln x}=\frac{1}{x}$

$z=x\left[\int\frac{1}{x}\cdot\left(-\frac{\ln x}{x}\right)dx+c\right]=x\left[-\underbrace{\int\frac{\ln x}{x^2}dx}_{I}+c\right]$

$I:\ \ln x=u \quad \frac{dx}{x^2}=dv$

$\frac{dx}{x}=du \quad -\frac{1}{x}=v$

$I=-\frac{\ln x}{x}+\int\frac{dx}{x^2}=-\frac{\ln x}{x}-\frac{1}{x}$

$z=x\left[\frac{\ln x}{x}+\frac{1}{x}+c\right]=1+\ln x+cx$

$y^{-1}=z \Rightarrow y=\frac{1}{z}=\frac{1}{1+\ln x+cx}$

ex $x(x^2-1)y' + x^2 - (x^2-1)y - y^2 = 0$; $y_1 = x$

$$\left.\begin{aligned} y &= y_1 + \frac{1}{z} \\ y &= x + \frac{1}{z} \\ y' &= 1 - \frac{z'}{z^2} \end{aligned}\right\} \quad x(x^2-1)\left(1-\frac{z'}{z^2}\right) + x^2 - (x^2-1)\left(x+\frac{1}{z}\right) - \left(x+\frac{1}{z}\right)^2 = 0$$

$$\cancel{x^3} - \cancel{x^2} - x(x^2-1)\frac{z'}{z^2} + \cancel{x^2} - \cancel{x^3} + x - \frac{(x^2-1)}{z} - x^2 - \frac{2x}{z} - \frac{1}{z^2} = 0$$

$$-x(x^2-1)\frac{z'}{z^2} - \frac{(x^2+2x-1)}{z} - \frac{1}{z^2} + x - x^2 = 0$$

$$-x(x^2-1)z' - (x^2+2x-1)z -$$