$P(B|A)$ = probability of B given that A has occured

Conditional Probability

Let A and B be two events such that P(A) > 0.

Denote by $\boxed{P(B|A)}$ the probability of B given that A has occurred.

Since A is known to have occurred, it becomes the new sample space replacing the original S.

If A and B are two any events in a sample space S and P(A) > 0, the conditional probability of B given A is

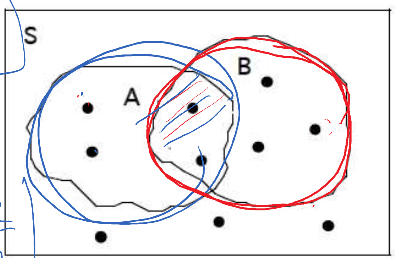
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Example 2.5. The following diagram shows two events A and B in the sample space S. Are the events A and B independent?

$P(B|A) = ?\ \frac{2}{4}$

$P(A|B) = ?\ \frac{2}{5}$



$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{2}{10}}{\frac{4}{10}} = \frac{2}{4}$

$P(A) = \frac{4}{10}$

$P(B) = \frac{5}{10}$

$P(A \cap B) = \frac{2}{10}$

Answer: There are 10 black dots in S and event A contains 4 of these dots. So the probability of A, is $P(A) = \frac{4}{10}$. Similarly, event B contains 5 black dots. Hence $P(B) = \frac{5}{10}$. The conditional probability of A given B is

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{5}.$$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{10}}{\frac{5}{10}}$

$P(A|B) = \frac{2}{5}$

Ex Find the probability that a single toss of a die will result in a number less than 4 if
(a) no other information is given and A
(b) it is $\boxed{\text{given that}}$ the toss resulted in an odd number.

$S: \{1, 2, 3, 4, 5, 6\}$

$P(A) = \frac{3}{6}$

$S_{new} = \{1, 3, 5\}$

$\boxed{P(B) = \frac{2}{3}}$

Solution. **(a)** Let B denote the event $\{\text{less than } 4\} = \{1, 2, 3\}$. Since B is the union of the events 1, 2, or 3 turning up, we see that

$$P(B) = P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

assuming equal probabilities for the sample points.
(b) Letting A be the event $\{\text{odd number}\} = \{1, 3, 5\}$, we see that $P(A) = \frac{3}{6}$. Also $P(A \cap B) = \frac{2}{6}$. Then

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}.$$

Hence, the added knowledge that the toss results in an odd number raises the probability from 1/2 to 2/3.

Ex. The probability of a flight departing on time is P(D) = 0.83.
The probability of a flight arriving on time is P(A) = 0.82.
We also know that the probability that a flight both departs and arrives on time is P(D ∩ A) = 0.78.
(a) What is the probability of a flight arriving on time if we know it departed on time?
(b) What is the probability of a flight departed on time if we know it arrived on time?

$P(D) = 0{,}83$
$P(A) = 0{,}82$
$P(A \cap D) = 0{,}78.$

$P(A|D) = \frac{P(A \cap B)}{P(D)} = \frac{0{,}78}{0{,}83} = 0{,}94$

$P(D|A) = \frac{P(A \cap B)}{P(A)} = \frac{0{,}78}{0{,}82} = 0{,}95$

Solution. **(a)** $P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94.$
(b) $P(D|A) = \frac{P(A \cap D)}{P(A)} = \frac{0.78}{0.82} = 0.95.$

Example 1.6. We toss a fair coin three successive times. We wish to find the conditional probability $\mathbf{P}(A \mid B)$ when A and B are the events

$$A = \{\text{more heads than tails come up}\}, \qquad B = \{\text{1st toss is a head}\}.$$

The sample space consists of eight sequences,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

which we assume to be equally likely. The event B consists of the four elements HHH, HHT, HTH, HTT, so its probability is

$$\mathbf{P}(B) = \frac{4}{8}.$$

The event $A \cap B$ consists of the three elements outcomes HHH, HHT, HTH, so its probability is

$$\mathbf{P}(A \cap B) = \frac{3}{8}.$$

Thus, the conditional probability $\mathbf{P}(A \mid B)$ is

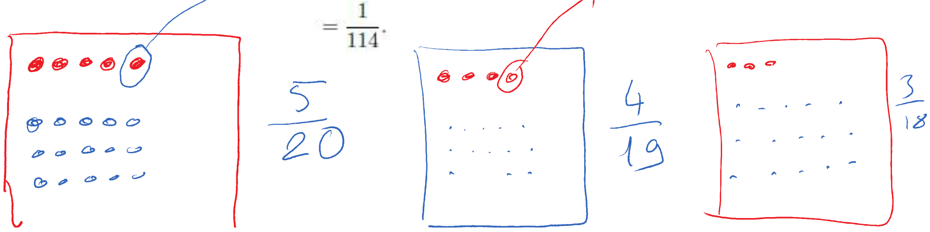
$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

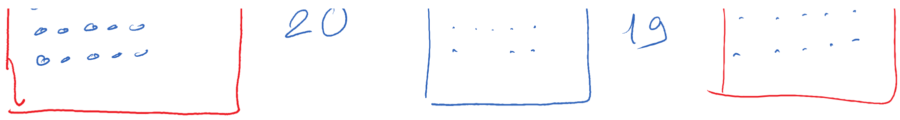
Because all possible outcomes are equally likely here, we can also compute $\mathbf{P}(A \mid B)$ using a shortcut. We can bypass the calculation of $\mathbf{P}(B)$ and $\mathbf{P}(A \cap B)$, and simply divide the number of elements shared by A and B (which is 3) with the number of elements of B (which is 4), to obtain the same result 3/4.

Example 2.4. A box of fuses contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random and removed from the box in succession without replacement, what is the probability that all three fuses are defective?

Answer: Let A be the event that the first fuse selected is defective. Let B be the event that the second fuse selected is defective. Let C be the event that the third fuse selected is defective. The probability that all three fuses selected are defective is $P(A \cap B \cap C)$. Hence

$$\begin{aligned} P(A \cap B \cap C) &= P(A)\,P(B/A)\,P(C/A \cap B) \\ &= \left(\frac{5}{20}\right)\left(\frac{4}{19}\right)\left(\frac{3}{18}\right) \\ &= \frac{1}{114}. \end{aligned}$$





Example 2.3. If we randomly pick two television sets in succession from a shipment of 240 television sets of which 15 are defective, what is the probability that they will be both defective?

Answer: Let A denote the event that the first television picked was defective. Let B denote the event that the second television picked was defective. Then $A \cap B$ will denote the event that both televisions picked were defective. Using the conditional probability, we can calculate

$$
\begin{aligned}
P(A \cap B) &= P(A)\, P(B/A) \\
&= \left(\frac{15}{240}\right)\left(\frac{14}{239}\right) \\
&= \frac{7}{1912}.
\end{aligned}
$$

In Example 2.3, we assume that we are sampling without replacement.

Multiplication Rule

If A and B are any two events in a sample space S and P(B) /= 0, then
P(A ∩ B) = P(B) · P(A|B).

$$\boxed{P(A \cap B) = P(B) \cdot P(A|B)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Ex. If we randomly pick 2 television tubes in succession from a shipment of 240 television tubes of which 15 are defective, what is the probability that they will both be defective?

$$\frac{15}{240} \cdot \frac{14}{239} = \frac{7}{1912}$$

Solution. Let $A = \{\text{the first tube is defective}\}$ and $B = \{\text{the second tube is defective}\}$. Then

$$P(A) = \frac{15}{240} \text{ and } P(B|A) = \frac{14}{239}.$$

Thus, the probability that both tubes will be defective is

$$P(A) \cdot P(B|A) = \frac{15}{240} \cdot \frac{14}{239} = \frac{7}{1,912}.$$

This assumes that we are sampling without replacement; that is, the first tube is not replaced before the second tube is selected.

Ex. Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards is we sample

4 aces

(a) without replacement

(b) with replacement.

(c) It is given that the first card was an ace.

What is the probability that the second card (without replacement) is also an ace?

$$\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$$

$$\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$$

$$P(\quad) = \frac{\frac{4}{52}\cdot\frac{3}{51}}{\frac{4}{52}} = \frac{3}{51}$$

Solution. **(a)** $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$.

(b) $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$.

(c) $\frac{\frac{4}{52}\cdot\frac{3}{51}}{\frac{4}{52}} = \frac{3}{51}$.

Example 1.10. Three cards are drawn from an ordinary 52-card deck without replacement (drawn cards are not placed back in the deck). We wish to find the probability that none of the three cards is a heart. We assume that at each step, each one of the remaining cards is equally likely to be picked. By symmetry, this implies that every triplet of cards is equally likely to be drawn. A cumbersome approach, that we will not use, is to count the number of all card triplets that do not include a heart, and divide it with the number of all possible card triplets. Instead, we use a sequential description of the sample space in conjunction with the multiplication rule (cf. Fig. 1.10).

Define the events

$$A_i = \{\text{the } i\text{th card is not a heart}\}, \qquad i = 1, 2, 3.$$

We will calculate $\mathbf{P}(A_1 \cap A_2 \cap A_3)$, the probability that none of the three cards is a heart, using the multiplication rule,

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2).$$

We have

$$\mathbf{P}(A_1) = \frac{39}{52},$$

since there are 39 cards that are not hearts in the 52-card deck. Given that the first card is not a heart, we are left with 51 cards, 38 of which are not hearts, and

$$\mathbf{P}(A_2 \mid A_1) = \frac{38}{51}.$$

Finally, given that the first two cards drawn are not hearts, there are 37 cards which are not hearts in the remaining 50-card deck, and

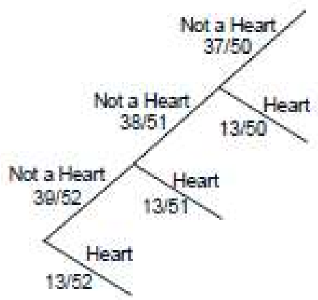
$$\mathbf{P}(A_3 \mid A_1 \cap A_2) = \frac{37}{50}.$$

These probabilities are recorded along the corresponding branches of the tree describing the sample space, as shown in Fig. 1.10. The desired probability is now obtained by multiplying the probabilities recorded along the corresponding path of the tree:

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50}.$$

Note that once the probabilities are recorded along the tree, the probability of several other events can be similarly calculated. For example,

$$\mathbf{P}(\text{1st is not a heart and 2nd is a heart}) = \frac{39}{52} \cdot \frac{13}{51},$$
$$\mathbf{P}(\text{1st two are not hearts and 3rd is a heart}) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{13}{50}.$$



Example 1.11. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into 4 groups of 4. What is the probability that each group includes a graduate student? We interpret randomly to mean that given the assignment of some students to certain slots, any of the remaining students is equally likely to be assigned to any of the remaining slots. We then calculate the desired probability using the multiplication rule, based on the sequential description shown in Fig. 1.11. Let us denote the four graduate students by 1, 2, 3, 4, and consider the events

$$A_1 = \{\text{students 1 and 2 are in different groups}\},$$
$$A_2 = \{\text{students 1, 2, and 3 are in different groups}\},$$
$$A_3 = \{\text{students 1, 2, 3, and 4 are in different groups}\}.$$

We will calculate $\mathbf{P}(A_3)$ using the multiplication rule:

$$\mathbf{P}(A_3) = \mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2).$$

We have

$$\mathbf{P}(A_1) = \frac{12}{15},$$

since there are 12 student slots in groups other than the one of student 1, and there are 15 student slots overall, excluding student 1. Similarly,

$$\mathbf{P}(A_2 \mid A_1) = \frac{8}{14},$$

since there are 8 student slots in groups other than the one of students 1 and 2, and there are 14 student slots, excluding students 1 and 2. Also,

$$\mathbf{P}(A_3 \mid A_1 \cap A_2) = \frac{4}{13},$$

since there are 4 student slots in groups other than the one of students 1, 2, and 3, and there are 13 student slots, excluding students 1, 2, and 3. Thus, the desired probability is

$$\frac{12}{15} \cdot \frac{8}{14} \cdot \frac{4}{13},$$

and is obtained by multiplying the conditional probabilities along the corresponding path of the tree of Fig. 1.11.

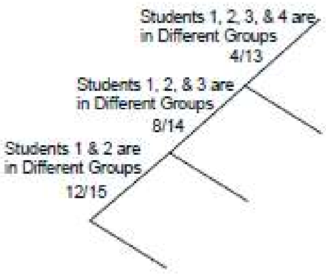


Figure 1.11: Sequential description of the sample space of the student problem in Example 1.11.

Independent Events

If $P(B|A) = P(B)$, i.e., the probability of B occurring is not affected by the occurrence or non-occurrence of A, then we say that A and B are independent events.

This is equivalent to

$$\boxed{P(A \cap B) = P(A) \cdot P(B|A)}$$

Two events A and B are *independent* if and only if

$$\boxed{P(A \cap B) = P(A) \cdot P(B).}$$

Example

A coin is tosses three times and the eight possible outcomes, HHH,HHT,HTH,THH,HTT,THT, TTH and TTT, are assumed to be equally likely.

If A is the event that a head occurs on each of the first two tosses,
B is the event that a tail occurs on the third toss, and
C is the event that exactly two tails occur in the three tosses, show that
(a) events A and B are independent?
(b) events B and C are dependent.

$A = \{HHH, HHT\}$
$B = \{HHT, HTT, THT, TTT\}$
$C = \{HTT, THT, TTH\}$

$P(B \cap C) = \{HTT, THT\}$

$P(A \cap B) = \{HHT\}$

independent

$P(A) = \frac{2}{8}$ $\quad$ $P(B) = \frac{4}{8}$ $\quad$ $P(A \cap B) = \frac{1}{8}$

$\boxed{P(A) \cdot P(B) = P(A \cap B)}$

$\frac{2}{8} \cdot \frac{4}{8} \stackrel{?}{=} \frac{1}{8}$

$\frac{1}{8} = \frac{1}{8}$

$P(B) = \frac{4}{8}$ $\quad$ $P(C) = \frac{3}{8}$ $\quad$ $P(B \cap C) = \frac{2}{8}$

$\boxed{P(B) \cdot P(C) = P(B \cap C)}$

$\frac{4}{8} \cdot \frac{3}{8} \stackrel{?}{=} \frac{2}{8}$

$\frac{3}{16} \neq \frac{1}{4}$

Solution. Since

$$A = \{HHH, HHT\}, \; B = \{HHT, HTT, THT, TTT\},$$

$C = \{HTT, THT, TTH\}, \; A \cap B = \{HHT\}, B \cap C = \{HTT, THT\}$

the assumption that the eight possible outcomes are all equiprobable yields

$$P(A) = \frac{2}{8}, \; P(B) = \frac{4}{8}, P(C) = \frac{3}{8}, \; P(A \cap B) = \frac{1}{8}, P(B \cap C) = \frac{2}{8}.$$

$$P(A) = \frac{2}{8}, \; P(B) = \frac{4}{8}, P(C) = \frac{3}{8}, \; P(A \cap B) = \frac{1}{8}, P(B \cap C) = \frac{2}{8}.$$

(a) Since

$$P(A) \cdot P(B) = \frac{2}{8} \cdot \frac{4}{8} = \frac{1}{8} = P(A \cap B),$$

the events A and B are independent.
(b) Since

$$P(B) \cdot P(C) = \frac{4}{8} \cdot \frac{3}{8} = \frac{3}{16} \neq \frac{2}{8} = P(B \cap C),$$

the events B and C are dependent.

Example 1.13. We roll a fair four-sided die. If the result is 1 or 2, we roll once more but otherwise, we stop. What is the probability that the sum total of our rolls is at least 4?

START

10.35

Let A_i be the event that the result of first roll is i, and note that $\mathbf{P}(A_i) = 1/4$ for each i. Let B be the event that the sum total is at least 4. Given the event A_1, the sum total will be at least 4 if the second roll results in 3 or 4, which happens with probability 1/2. Similarly, given the event A_2, the sum total will be at least 4 if the second roll results in 2, 3, or 4, which happens with probability 3/4. Also, given the event A_3, we stop and the sum total remains below 4. Therefore,

$$\mathbf{P}(B \mid A_1) = \frac{1}{2}, \qquad \mathbf{P}(B \mid A_2) = \frac{3}{4}, \qquad \mathbf{P}(B \mid A_3) = 0, \qquad \mathbf{P}(B \mid A_4) = 1.$$

By the total probability theorem,

$$\mathbf{P}(B) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{9}{16}.$$

The total probability theorem can be applied repeatedly to calculate probabilities in experiments that have a sequential character, as shown in the following example.

Let A_i be the event that the result of first roll is i, and note that $\mathbf{P}(A_i) = 1/4$ for each i. Let B be the event that the sum total is at least 4. Given the event A_1, the sum total will be at least 4 if the second roll results in 3 or 4, which happens with probability $1/2$. Similarly, given the event A_2, the sum total will be at least 4 if the second roll results in 2, 3, or 4, which happens with probability $3/4$. Also, given the event A_3, we stop and the sum total remains below 4. Therefore,

10^{35}

$$\mathbf{P}(B \,|\, A_1) = \frac{1}{2}, \qquad \mathbf{P}(B \,|\, A_2) = \frac{3}{4}, \qquad \mathbf{P}(B \,|\, A_3) = 0, \qquad \mathbf{P}(B \,|\, A_4) = 1.$$

By the total probability theorem,

$$\mathbf{P}(B) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{9}{16}.$$

The total probability theorem can be applied repeatedly to calculate probabilities in experiments that have a sequential character, as shown in the following example.