MAT1071 MATHEMATICS I
4. WEEK
PART 2

TRANSCENDENTAL FUNCTIONS

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Transcendental Functions

In this chapter

we investigate the calculus of important transcendental functions, including

1. the logarithmic, exponential,
2. inverse trigonometric,
3. hyperbolic functions.
4. inverse hyperbolic functions.

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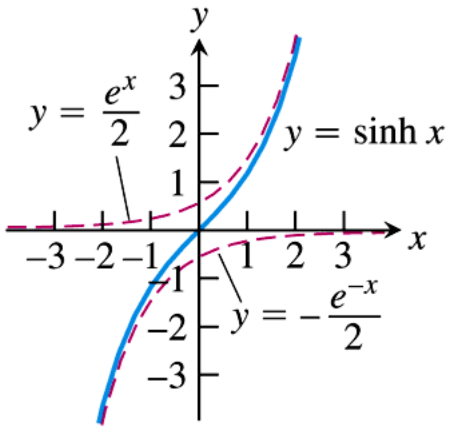
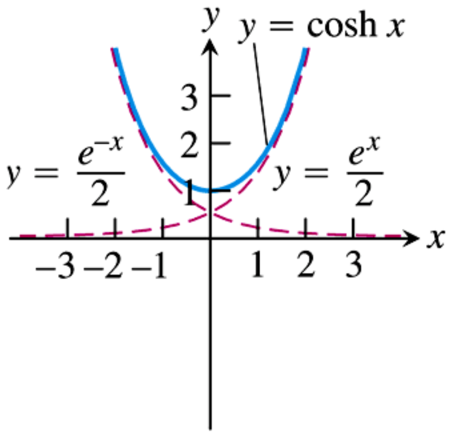
3. Hyperbolic Functions

Hyperbolic sine of x: $\sinh x = \dfrac{e^x - e^{-x}}{2}$



(a)

Hyperbolic cosine of x: $\cosh x = \dfrac{e^x + e^{-x}}{2}$



(b)

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$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

☆ Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

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$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$$

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Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

EXAMPLE

$$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$$

$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

EXAMPLE

$$* \ \frac{d}{dx}\left(6\sinh\frac{x}{3}\right) = 2\cosh\frac{x}{3}$$

$$* \ \frac{d}{dx}\left(\frac{1}{2}\sinh(2x+1)\right) = \cosh(2x+1)$$

$$* \ \frac{d}{dt}\left(2\sqrt{t}\,\tanh\sqrt{t}\right) = \frac{1}{\sqrt{t}}\cdot\tanh\sqrt{t} + \frac{2\sqrt{t}}{2\sqrt{t}}\operatorname{sech}^2\sqrt{t}$$

$$* \ \frac{d}{dz}\left(\ln(\sinh z)\right) = \frac{\cosh z}{\sinh z} = \coth z$$

4. Inverse Hyperbolic Functions

$y = \sinh^{-1} x \Rightarrow x = \sinh y$

$y = \cosh^{-1} x \Rightarrow x = \cosh y$

$y = \tanh^{-1} x \Rightarrow$

$y = \coth^{-1} x \Rightarrow$

$y = \operatorname{sech}^{-1} x \Rightarrow$

$y = \operatorname{csch}^{-1} x \Rightarrow$

Identities

$$\boxed{\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}}$$

$$\boxed{\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}}$$

$$\boxed{\coth^{-1} x = \tanh^{-1} \frac{1}{x}}$$

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$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)=\frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)}=\frac{1}{\left(\frac{1}{x}\right)}=x.$$

Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \qquad u \neq 0$$

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EXAMPLE Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

Solution

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\
&= \frac{1}{\sinh(\cosh^{-1} x)} && f'(u) = \sinh u \\
&= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} && \cosh^2 u - \sinh^2 u = 1, \\
& && \sinh u = \sqrt{\cosh^2 u - 1} \\
&= \frac{1}{\sqrt{x^2 - 1}} && \cosh(\cosh^{-1} x) = x
\end{aligned}$$

The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

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EXAMPLE

* $f(x) = x\sinh^{-1}x - \sqrt{1+x^2}$

$\Rightarrow f'(x) = \sinh^{-1}x + x \cdot \dfrac{1}{\sqrt{1+x^2}} - \dfrac{2x}{2\sqrt{1+x^2}}$

* $y = x^2\cosh^{-1}x^2 \Rightarrow y' = 2x\cosh^{-1}x^2 + x^2\left(\dfrac{2x}{\sqrt{x^4-1}}\right)$

EXAMPLE

$$* \quad y=\tanh^{-1}\sqrt{x} \Rightarrow y'=\frac{\frac{1}{2\sqrt{x}}}{1-x}$$

$$* \quad y=\cosh^{-1}(x^2+1) \Rightarrow y'=\frac{2x}{\sqrt{(x^2+1)^2-1}}$$

EXAMPLE

$$* \quad y = \sinh^{-1}(\coth x^2) \Rightarrow y' = \frac{(\coth x^2)'}{\sqrt{1+\coth^2 x^2}}$$

$$\Rightarrow y' = \frac{-2x \operatorname{cosech}^2 x}{\sqrt{1+\coth^2 x^2}}$$

$$* \quad y = \tanh^{-1}(\sin x) \Rightarrow y' = \frac{(\sin x)'}{1-\sin^2 x} = \frac{\cos x}{\cos^2 x} = \sec x$$

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HW:

Finding Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6 \sinh \frac{x}{3}$

14. $y = \frac{1}{2} \sinh (2x + 1)$

15. $y = 2\sqrt{t} \tanh \sqrt{t}$

16. $y = t^2 \tanh \frac{1}{t}$

17. $y = \ln (\sinh z)$

18. $y = \ln (\cosh z)$

19. $y = \operatorname{sech} \theta (1 - \ln \operatorname{sech} \theta)$

20. $y = \operatorname{csch} \theta (1 - \ln \operatorname{csch} \theta)$

21. $y = \ln \cosh v - \frac{1}{2} \tanh^2 v$

22. $y = \ln \sinh v - \frac{1}{2} \coth^2 v$

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EXAMPLES

EXAMPLE let $\arcsin\frac{2}{3}=\alpha \Rightarrow$ find $\cos\alpha, \tan\alpha, \cot\alpha, \sec\alpha, \operatorname{cosec}\alpha$

Solution $\Rightarrow \sin\alpha=\frac{2}{3}$

$\cos\alpha=\frac{\sqrt{5}}{3}$

$\tan\alpha=\frac{2}{\sqrt{5}}$

$\cot\alpha=\frac{\sqrt{5}}{2}$

$\sec\alpha=\frac{3}{\sqrt{5}}$

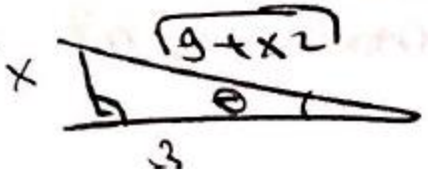
$\operatorname{cosec}\alpha=\frac{3}{2}$

EXAMPLE $\sec\left(\arctan\frac{x}{3}\right)=?$

Solution $\arctan\frac{x}{3}=\theta$

$\Rightarrow \tan\theta=\frac{x}{3}$

$\Rightarrow \sec\theta=\frac{\sqrt{x^2+9}}{3}$



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EXAMPLE Find interval of x where the given function is defined:

$$\arcsin\left(\frac{2x+1}{3}\right)$$

Solution Domain of arcsinx $\rightarrow [-1,1]$

$$-1 \leq \frac{2x+1}{3} \leq 1$$

$$-2 \leq x \leq 1$$

EXAMPLE $\cot\left(\underbrace{\sin^{-1}\left(-\frac{1}{2}\right)}_{-\pi/6} - \underbrace{\sec^{-1}(2)}_{\frac{\pi}{3}}\right) = ?$

Solution

$\Rightarrow \cot(-\pi/3) = -\cot\frac{\pi}{3} = \frac{-1}{\sqrt{3}}$ //

EXAMPLE

$y = \sinh(e^{\cosh x}) \Rightarrow y' = \cosh(e^{\cosh x}) \cdot \sinh x \, e^{\cosh x}$

$y = \operatorname{sech}(\ln(\cos x)) \Rightarrow y' = -\operatorname{sech}(\ln\cos x) \cdot \tanh(\ln\cos x)(-\tan x)$

$y = \arcsin e^x \Rightarrow y' = \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x$

$y = \arccos \frac{1}{x} \Rightarrow y' = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \left(-\frac{1}{x^2}\right)$

$y = \arccos(\ln \sin x) \Rightarrow y' = \frac{-1}{\sqrt{1-\ln^2 \sin x}} \cdot \cot x$

$y = \ln(\arctan x) \Rightarrow y' = \frac{\frac{1}{1+x^2}}{\arctan x}$

$y = \arctan(\ln x) \Rightarrow y' = \frac{1}{1+\ln^2 x} \cdot \frac{1}{x}$

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EXAMPLE $2\cosh(\ln x) = ?$

Solution $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$\Rightarrow 2\cosh(\ln x) = 2\frac{e^{\ln x} + e^{-\ln x}}{2}$$

$$= 2\,\frac{x + \frac{1}{x}}{2} = x + \frac{1}{x}$$

EXAMPLE

$$\sin\left(\arccos\frac{\sqrt{2}}{2}\right)=? \Rightarrow \sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

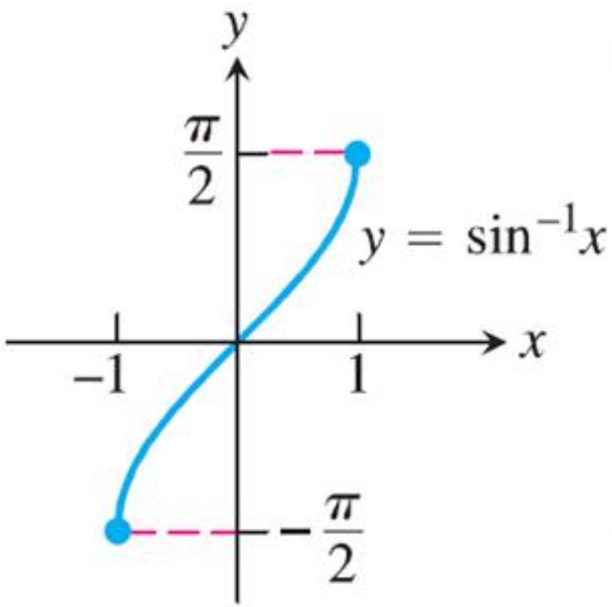
$$\sec\left(\arccos\frac{1}{2}\right)=? \Rightarrow \sec\left(\frac{\pi}{3}\right)=2$$

$$\tan\left(\arcsin\left(\frac{-1}{2}\right)\right)=? \Rightarrow \tan\left(-\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{3}$$

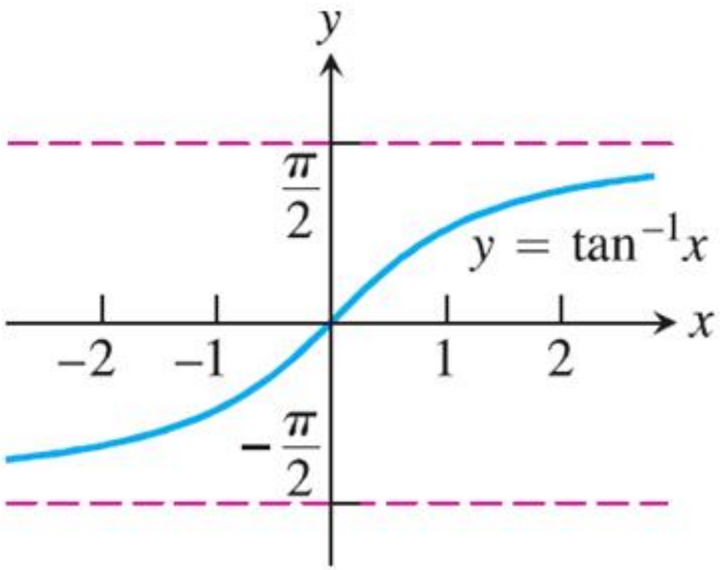
EXAMPLE

$$\lim_{x \to 1^-} \arcsin x = \frac{\pi}{2}$$

$$\lim_{x \to \infty} \arctan x = \frac{\pi}{2}$$



Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

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EXAMPLE $\lim_{x \to 0} \frac{\arcsin x}{x} = ?$

Solution

$\arcsin x = y$

$\sin y = x$

$x \to 0 \quad y \to 0$

$\Rightarrow \lim_{y \to 0} \frac{y}{\sin y} = 1$ //

EXAMPLE $\lim\limits_{x \to 0^+} \dfrac{(\arctan\sqrt{x})^2}{x\sqrt{x+1}} = ?$

Solution $\arctan\sqrt{x} = y$

$\tan y = \sqrt{x}$

$x \to 0^+ \to y \to 0^+$

$\boxed{1+\tan^2 y = \sec^2 y}$

$\Rightarrow \lim\limits_{y \to 0^+} \dfrac{y^2}{(\tan y)^2 \sqrt{1+\tan^2 y}}$

$\Rightarrow \lim\limits_{y \to 0^+} \underbrace{\left(\dfrac{y}{\tan y}\right)^2}_{1} \cdot \underbrace{\dfrac{1}{\sec y}}_{1} = 1$ //

EXAMPLE $\lim_{x \to \infty} \left(\frac{x+7}{x+3}\right)^{2x+3} = ?$

Solution $\lim_{x \to \infty} \left[\underbrace{\left(1+\frac{4}{x+3}\right)^{x+3}}_{e^4}\right]^{\underbrace{\frac{2x+3}{x+3}}_{2}} = (e^4)^2 = e^8$

EXAMPLE $\lim_{x \to \infty} \left(\frac{x}{x+2}\right)^{3x} = ?$

Solution $\lim_{x \to \infty} \left(\frac{x}{x+2}\right)^{3x} = \lim_{x \to \infty} \frac{1}{\left(\left(1+\frac{2}{x}\right)^{x}\right)^{3}} = \frac{1}{e^{6}}$

or

$$\lim_{x \to \infty} \left[\left(1+\frac{(-2)}{x+2}\right)^{x+2}\right]^{\frac{3x}{\cancel{x+2}} \nearrow 3} = \left(e^{-2}\right)^{3} = e^{-6}$$

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EXAMPLE

Show that

$$\sinh(x+y) = \sinh x \cdot \cosh y + \cosh x . \sinh y$$

Solution

$$\sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2} \quad -- (1)$$

$\sinh x \cosh y + \cosh x \sinh y$

$$= \frac{e^{x} - e^{-x}}{2} \cdot \frac{e^{y} + e^{-y}}{2} + \frac{e^{x} + e^{-x}}{2} \cdot \frac{e^{y} - e^{-y}}{2}$$

$$= \frac{e^{x+y} - e^{-x-y}}{2} \quad -- (2)$$

$(1) = (2)$

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EXAMPLE Show that $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ //

Solution $f(x) = \sin x$

$f'(x) = \cos x$

$f^{-1}(x) = \arcsin x$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{\cos(\underbrace{\sin^{-1}x}_{\alpha})} = \frac{1}{\sqrt{1-\sin^2(\sin^{-1}x)}} = \frac{1}{\sqrt{1-x^2}}$$

veya $= \dfrac{1}{\cos\alpha}$

$\arcsin x = \alpha$

$\sin\alpha = x$

$\cos\alpha = \sqrt{1-x^2}$ //

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Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.