MAT1071 MATHEMATICS I

7.1 CURVE SKETCHING EXAMPLES

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MAT1071 Ytu Bologna:

9	Asymptotes of Graphs, Curve Sketching, Antiderivatives, Indefinite Integrals, Integral Tables

Curve Sketching

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the intercepts
3. Identify any asymptotes that may exist
4. Find f'.

   Find the critical points of f, if any, and identify the function's behavior at each one. Find where the curve is increasing and where it is decreasing.
5. Find f''.

   Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Construct the sign table for f' and f''.
7. Plot key points, such as the intercepts and the points found in Steps 2–5, and sketch the curve together with any asymptotes that exist.

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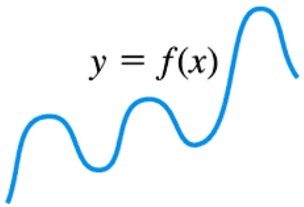
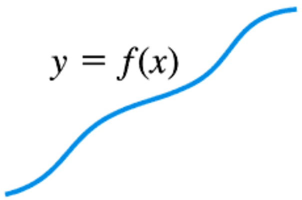
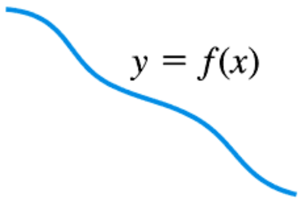
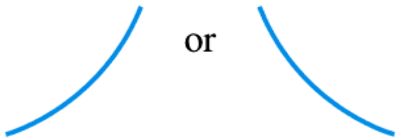
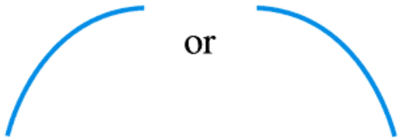
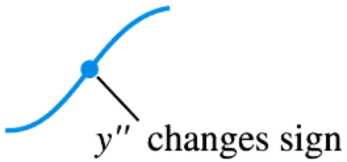
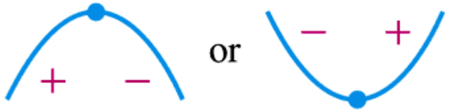
 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 Inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

EXAMPLE Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

Solution

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Solution

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EXAMPLE Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

Solution

1. Domain $= \{x \mid x+1>0\} = \{x \mid x>-1\} = (-1, \infty)$
2. The x- and y-intercepts are both 0. $\boxed{(0,0)}$
3. Symmetry: None
Since

$f(-x) \neq f(x)$

$f(-x) \neq -f(x)$

4. horizontal asy $\lim_{x\to\infty} \dfrac{x^2}{\sqrt{x+1}} = \infty \Rightarrow$ none

vertical asy $\Rightarrow \lim_{x\to -1^+} \dfrac{x^2}{\sqrt{x+1}} = \infty \Rightarrow \boxed{x=-1}$

5. $$f'(x) = \frac{2x\sqrt{x+1} - x^2 \cdot 1/(2\sqrt{x+1})}{x+1} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

critical points

$\boxed{x = -\frac{4}{3}}$ $x=0$

$\boxed{x=-1}$

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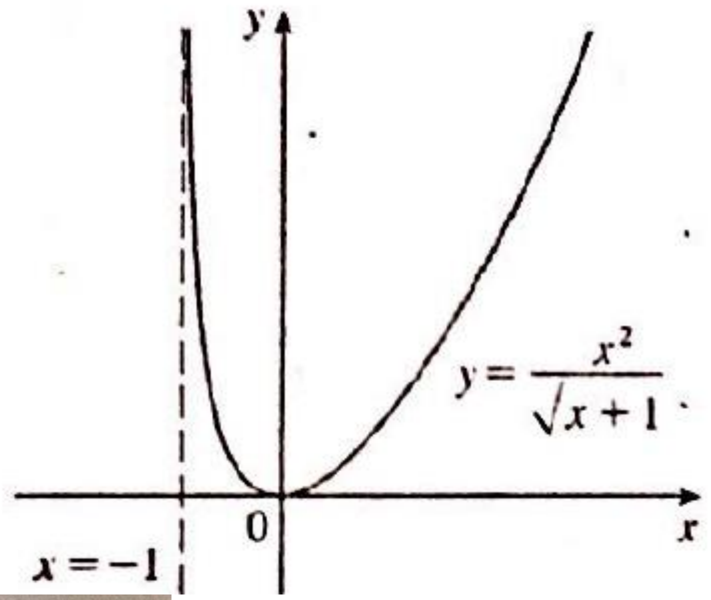
⑥ $f''(x) = \dfrac{3x^2 + 8x + 8}{4(x+1)^{5/2}} \rightarrow$ no root $\boxed{x=-1}$

x	$-\infty$		$-4/3$		-1		0		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f''(x)$		$-$		$-$	0	$+$		$+$	
					∞		0		$+\infty$

Domain of f.



f is decreasing on $(-1, 0)$
increasing on $(0, \infty)$
at $x=0$ $f(0)=0$
local min. point

f is concave up on $(-1, \infty)$
no point of inflection

EXAMPLE $f(x) = \frac{(x-2)^2}{x-1}$ ✓

Solution

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Solution

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Solution

EXAMPLE $f(x) = \dfrac{(x-2)^2}{x-1}$ ✓

Solution

① $D_f = \mathbb{R} - \{1\}$

② Intercepts $x=0 \Rightarrow y=-4$ $(0,-4)$

$y=0 \Rightarrow x=2$ $(2,0)$

③ $\lim\limits_{x \to \pm\infty} f(x) = \infty \Rightarrow$ there exists oblique asymptote

$\lim\limits_{x \to -\infty} = -\infty$

$\boxed{y = x-3}$ oblique $\left(x^2-4x+4 \,\middle|\, \begin{matrix} x-1 \\ x-3 \end{matrix}\right.$

$m = \lim\limits_{x \to \infty} \underbrace{\dfrac{(x-2)^2}{(x-1)} \cdot \dfrac{1}{x}}_{\frac{f(x)}{x}} = 1$ $\lim\limits_{x \to \infty} f(x) - mx = -3$

vertical asy $\Rightarrow \lim\limits_{x \to 1^+} \dfrac{(x-2)^2}{x-1} = \infty \Rightarrow \boxed{x=1}$ $x \to 1^- \to -\infty$

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④ $f'(x) = \dfrac{x(x-2)}{(x-1)^2} = 0 \Rightarrow \boxed{x=0}, \boxed{x=2}, \boxed{x=1} \to$ two fold

Critical points

⑤ $f''(x) = \dfrac{2}{(x-1)^3}$

⑥

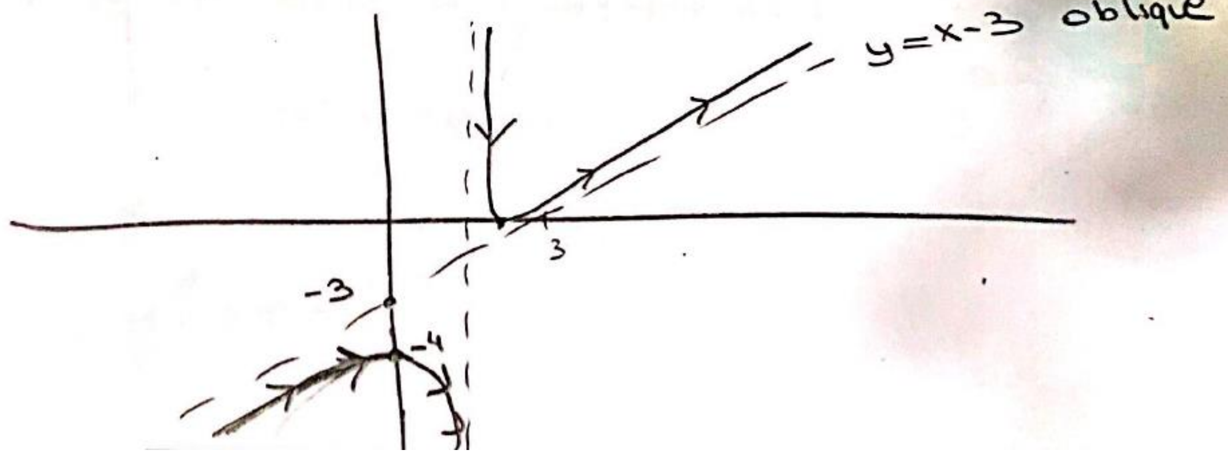
x	$-\infty$		0		1		2		$+\infty$
f'		+	0	−	0	−	0	+	
f''		−		−	0	+		+	
			-4		$-\infty \mid \infty$		0		

increasing $\to (-\infty,0)\cup(2,\infty)$

decreasing $\to (0,1)\cup(1,2)$

concave up $\to (1,\infty)$

concave down $\to (-\infty,1)$



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EXAMPLE Sketch the graph of function $f(x) = \frac{x^2}{x-2}$

Solution

EXAMPLE Sketch the graph of function $f(x)=\dfrac{x^2}{x-2}$

Solution

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Solution

EXAMPLE Exam Q

Sketch the graph of function $f(x)=\frac{x^2}{x-2}$

Solution

① $D_f = \mathbb{R}-\{2\}$

② Symmetries $f(-x)=\frac{x^2}{-x-2} \neq f(x)$ (f is not even
$\neq -f(x)$ and not odd)

③ Intercepts

$x=0 \Rightarrow f(x)=y=0$. $\boxed{(0,0)}$

$y=0 \Rightarrow x=0$

④ $f'(x) = \dfrac{x^2-4x}{(x-2)^2}$ → Critical points $\boxed{x=0}$ $\boxed{x=4}$

→ $\boxed{\begin{array}{l} x=2 \\ \text{two fold} \end{array}}$

⑤ $f''(x) = \dfrac{8}{(x-2)^3} \Rightarrow \boxed{x=2}$

⑥ Asymptotes

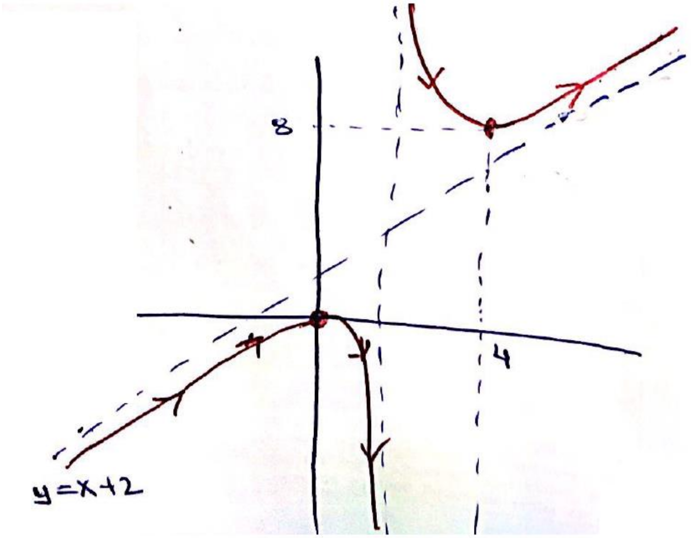
$x^2 \;\big|\; \underline{x-2}$
$\quad\;\; x+2$

$\boxed{y=x+2 \text{ oblique asym}}$

$$\boxed{\begin{array}{l} \lim\limits_{x\to 2^+} f(x) = \infty \\ \lim\limits_{x\to 2^-} f(x) = -\infty \\ x=2 \text{ vertical asy} \end{array}}$$

⑥

x	$-\infty$		0		2		4		$+\infty$
f'		+ ↗		− ↘	0	−		+	
f''		−		−	0	+		+	
y	$-\infty$		0		$-\infty$ ∞ 4		8		$+\infty$



f is increasing on $(-\infty,0)\cup(2,4)$
f is decreasing on $(0,2)\cup(2,4)$
$(0,0)$ → local max
$(4,8)$ → local min

f is concave up $(2,\infty)$
f is concave down $(-\infty,2)$

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3) f fonksiyonu $\boxed{f(x) = \frac{x^3-3x^2+1}{x^3}}$ olacak şekilde tanımlansın.

a) f fonksiyonunun tanım kümesini bulunuz

b) Eğer varsa, f fonksiyonunun tüm asimptotlarını bulunuz

$$f(x) = \frac{x^3 - 3x^2 + 1}{x^3}$$

c) f fonksiyonunun artan/azalan olduğu aralıkları belirleyiniz. Eğer varsa, yerel ekstremum değerlerini bulunuz

$$f(x)=\frac{x^3-3x^2+1}{x^3}$$

d) f fonksiyonunun konkavlığını inceleyiniz ve büküm nokta(lar)ını bulunuz

3) f fonksiyonu $\boxed{f(x) = \frac{x^3-3x^2+1}{x^3}}$ olacak şekilde tanımlansın.

a) f fonksiyonunun tanım kümesini bulunuz

$D_f = \mathbb{R} - \{0\}$

b) Eğer varsa, f fonksiyonunun tüm asimptotlarını bulunuz

vertical asymptote

$\lim\limits_{x \to 0} f(x) = \infty$ olduğundan $x=0$ Düşey asimptot

$\lim\limits_{x \to \infty} \frac{x^3-3x^2+1}{x^3} = 1$ olduğundan $y=1$ Yatay asimptot

horizontal asym.

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c) f fonksiyonunun artan/azalan olduğu aralıkları belirleyiniz. Eğer varsa, yerel ekstremum değerlerini bulunuz

$$f'(x) = \frac{3}{x^2} - \frac{3}{x^4} = 0 \Rightarrow x^4 - x^2 = 0 \Rightarrow x^2(x^2-1) = 0 \Rightarrow \begin{cases} x = 0 \notin T(f) \\ x = 1 \\ x = -1 \end{cases}$$

$x=1$, $x=-1$ kritik noktalar

	$-\infty$	-1	0	1	$+\infty$
$f'(x)$	+	−	−	+	
	↗	↘	↘	↗	

$(-\infty,-1) \cup (1,\infty) \Rightarrow f$ is increa

$(-1,0) \cup (0,1) \Rightarrow f$ is decrea

$(-1,3)$ ... ma

$(1,-1)$ yerel min

ecker
x

d) f fonksiyonunun konkavlığını inceleyiniz ve büküm nokta(lar)ını bulunuz

→ inflection point

$$f''(x) = -\frac{6}{x^3} + \frac{12}{x^5} = 0 \Rightarrow -x^5 + 2x^3 = 0 \Rightarrow x^3(2-x^2) = 0$$

→ $x = 0 \notin$ T.K

→ $x = \sqrt{2}$ } olası

→ $x = -\sqrt{2}$ } büküm n.

	$-\infty$ $-\sqrt{2}$	$-\sqrt{2}$ 0	0 $\sqrt{2}$	$\sqrt{2}$ $+\infty$
x^3	$-$	$-$	$+$	$+$
$2-x^2$	$-$	$+$	$+$	$-$
$f''(x)$	$+$	$-$	$+$	$-$

the graph of f

$(-\infty, -\sqrt{2}) \cup (0, \sqrt{2}) \Rightarrow f$ is concav

$(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty) \Rightarrow f$ is concave

$\left(-\sqrt{2}, \frac{5+2\sqrt{2}}{2\sqrt{2}}\right)$ and $\left(\sqrt{2}, \frac{-5+2\sqrt{2}}{2\sqrt{2}}\right)$

inflection points

EXAMPLE Sketch the graph of $f(x) = x \ln x$

Solution

Sketch the graph of $f(x) = x \ln x$

EXAMPLE Sketch the graph of $f(x) = x\ln x$

Solution

① Domain $x > 0$ $\boxed{D_f = (0,\infty)}$

② $y = 0 \Rightarrow x\ln x = 0 \Rightarrow \underset{(x \neq 0)}{x = 0}$ or $x = 1$ $(1,0)$

③ $\lim\limits_{x \to 0^+} \underbrace{x\ln x}_{0\cdot\infty} = \lim\limits_{x \to 0^+} \dfrac{\ln x}{\frac{1}{x}}$ $\frac{\infty}{\infty}$

tanım kümesinden bak

$= \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}} = 0$ → no vertical asym.

$\lim\limits_{x \to \infty} x\ln x = \infty$ no horizontal asym.

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④ $f'(x) = \ln x + 1 = 0 \Rightarrow \boxed{x = \frac{1}{e}}$ critical point

⑤ $f''(x) = \frac{1}{x} \Rightarrow \boxed{x = 0}$

x		0		$\frac{1}{e}$		1	
f'	$-$		$-$	0	$+$		$+$
f''	$-$	0	$+$		$+$		$+$
				$\frac{-1}{e}$		0	

Domain

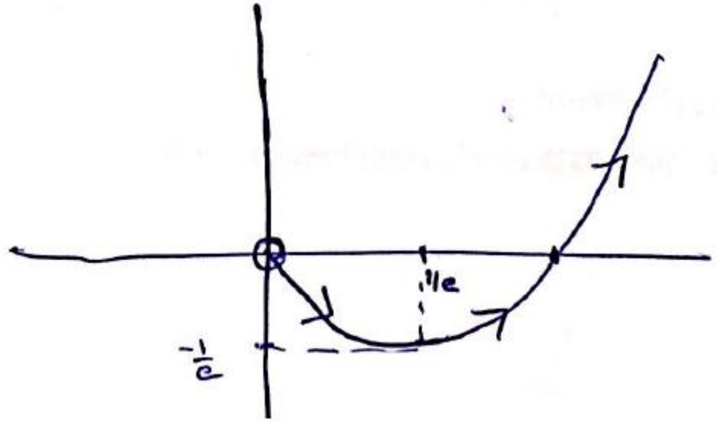
f is inc. on $(\frac{1}{e}, 1)$

decreasing $(0, \frac{1}{e})$

Concave up on $(0, \infty)$

$x = \frac{1}{e} \Rightarrow f(\frac{1}{e}) = -\frac{1}{e}$

local min



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EXAMPLE Sketch the graph of $y = \ln(4 - x^2)$.

Solution

Sketch the graph of $y = \ln(4 - x^2)$.

EXAMPLE Sketch the graph of $y = \ln(4 - x^2)$.

Solution

(1) The domain is

$D_f = (-2, 2)$

(2) The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

$(0, \ln 4)$

$(-\sqrt{3}, 0)$

$(+\sqrt{3}, 0)$

We know that $\ln 1 = \log_e 1 = 0$ (since $e^0 = 1$), so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

(3) Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

(4) We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

$$\downarrow \qquad\qquad \downarrow$$

$x = 2 \qquad x = -2$

Vertical asymptotes

→ tanım kümesine bak

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$$\boxed{f'(x) = \frac{-2x}{4 - x^2}}$$

$x = 0$ $x = 2$ $x = -2$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.

$$\boxed{f''(x) = \frac{-8 - 2x^2}{(4 - x^2)^2}}$$

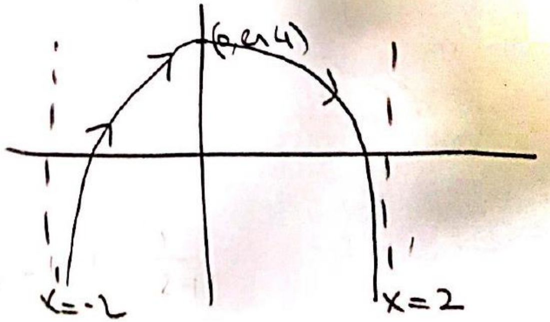
$x = -2$, $x = 2$ } two fold

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

x	$-\infty$		-2		0		2		$+\infty$
f'		$-$	$\emptyset$	$+$	0	$-$	$\emptyset$	$+$	
f''		$-$	$\emptyset$	$-$		$-$	$\emptyset$	$-$	

Df

increasing → $(-2,0)$
decreasing → $(0,2)$
concave down $(-2,2)$



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EXAMPLE Sketch the graph of $f(x) = x + \sqrt{x^2 - 1}$

Solution

EXAMPLE Sketch the graph of $f(x) = x + \sqrt{x^2-1}$

Solution ① D_f $x^2-1 \geq 0 \Rightarrow x \geq 1, x \leq -1$ $D_f = [-\infty, -1] \cup [1, \infty)$

② Asymptotes

$$\left.\begin{aligned} &\lim_{x\to -1^-} x+\sqrt{x^2-1} = -1 \\ &\lim_{x\to -1^+} x+\sqrt{x^2-1} = 1 \end{aligned}\right\} \text{no vertical. asym.}$$

$$\lim_{x\to +\infty} x+\sqrt{x^2+1} = \infty$$

$$\underset{\substack{x\to -\infty \\ \infty-\infty}}{\lim} \left(x+\sqrt{x^2+1}\right) \frac{\left(x-\sqrt{x^2-1}\right)}{\left(x-\sqrt{x^2-1}\right)} = \lim_{x\to -\infty} \frac{1}{x-|x|\left(\sqrt{1-\frac{1}{x^2}}\right)}$$

$$= \lim_{x\to \infty} \frac{1}{x\left(1+\sqrt{1-\underbrace{\frac{1}{x^2}}_{0}}\right)} = 0$$

$y=0$ is horizontal asymp.

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Oblique

$$m=\lim_{x\to\infty}\frac{x+\sqrt{x^2-1}}{x}=\lim_{x\to\infty}\frac{1+|x|\sqrt{1-\frac{1}{x^2}}}{x}=2$$

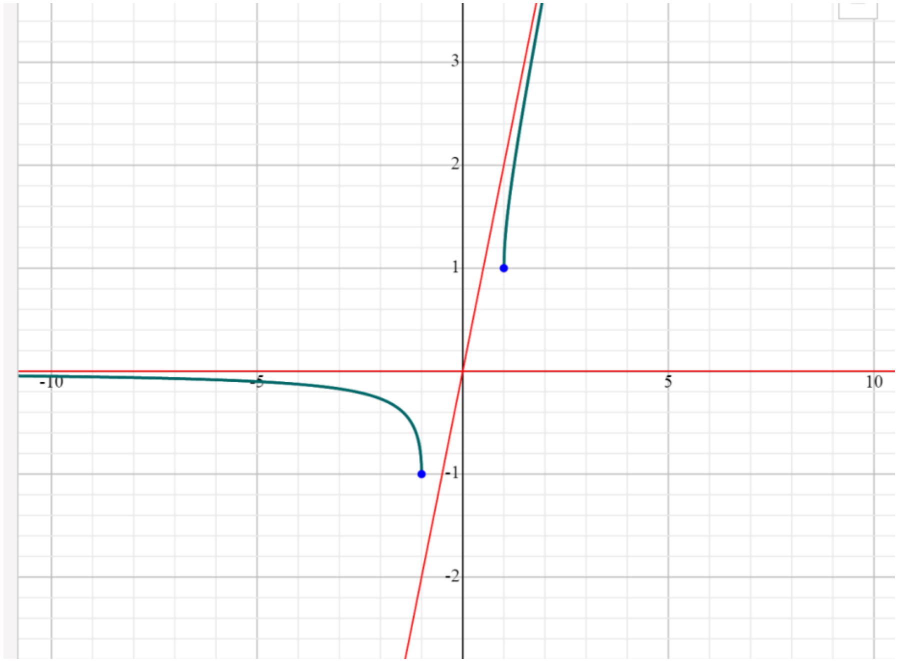
$$n=\lim_{x\to\infty} x+\sqrt{x^2-1}-2x=\lim_{x\to\infty}\frac{-1}{x-\sqrt{x^2-1}}=0$$

$$y=mx+n=2x \rightarrow \text{oblique asym.}$$

③ $f'(x)=1+\dfrac{x}{\sqrt{x^2-1}} \Rightarrow x=\pm 1$ C.P.

④ $f''(x)=\dfrac{-1}{(x-1)^{3/2}} \Rightarrow x=\pm 1$

x	$-\infty$		-1		1		$+\infty$
f'		$-$	0	$\times$	0	$+$	
f''	0	$-$	-1	$\times$	1	$-$	∞



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Graphing Equations

Use the steps of the graphing procedure to graph the equations

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

15. $y = (x - 2)^3 + 1$

Graphing Rational Functions

Graph the rational functions in Exercises 75–92.

75. $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$

76. $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

77. $y = \dfrac{x^4 + 1}{x^2}$

78. $y = \dfrac{x^2 + 4}{2x}$

79. $y = \dfrac{1}{x^2 - 1}$

80. $y = \dfrac{x^2}{x^2 - 1}$

81. $y = -\dfrac{x^2 - 2}{x^2 - 1}$

82. $y = \dfrac{x^2 - 4}{x^2 - 2}$

83. $y = \dfrac{x^2}{x + 1}$

84. $y = -\dfrac{x^2 - 4}{x + 1}$

Reference:

Thomas' Calculus, 12th Edition,
G.B Thomas, M.D.Weir, J.Hass and
F.R.Giordano, Addison-Wesley, 2012.