MAT1320-Linear Algebra
Lecture Notes

Echelon Form of a Matrix

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Table of contents

1. Echelon Form of a Matrix

2. Elementary Row (Column) Operations

3. Finding an Inverse using Elementary Row Operations

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c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.

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d) If a column contains a leading one, then all other entries in that column are zero.

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- An,$m \times n$ matrix satisfying properties $a)$, $b)$, and $c)$ is said to be in row echelon form (REF). There may be no zero rows.
- A similar definition can be formulated in the obvious manner for reduced column echelon form and column echelon form.

Echelon Form of a Matrix

Example

The following are matrices in reduced row echelon form, since they satisfy properties $a)$, $b)$, and $d)$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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and

$$C = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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The following are matrices in row echelon form.

$$H = \begin{pmatrix} 1 & 5 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

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Note: We shall now show that every matrix can be put into row (column) echelon form, or into reduced row (column) echelon form, by means of certain row (column) operations.

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- Replace row (column) j by k times row (column) $i+$ row (column) j, Type III: $k\mathbf{r}_i + \mathbf{r}_j \to \mathbf{r}_j$ $(k\mathbf{c}_i + \mathbf{c}_j \to \mathbf{c}_j)$

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Interchanging rows 1 and 3 of A, we obtain

$$B = A_{r_1 \leftrightarrow r_3} = \begin{pmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

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Multiplying the third row of A by $\frac{1}{3}$, we obtain

$$C = A_{\frac{1}{3} r_3 \to r_3} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{pmatrix}$$

Adding (-2) times row 2 of A to row 3 of A, we obtain

$$D = A_{-2r_2+r_3 \to r_3} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{pmatrix}$$

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Note: An $m \times n$ matrix B is said to be row (column) equivalent to an $m \times n$ matrix A if B can be produced by applying a finite sequence of elementary row (column) operations to A.

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so C is row equivalent to B. It then follows that C is row equivalent to A, since we obtained C by applying two successive elementary row operations to A.

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Every nonzero $m \times n$ *matrix* $\mathbf{A} = [a_{ij}]$ *is row (column) equivalent to a matrix in row (column) echelon form.*

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Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1 + r_2 \rightarrow r_2} = \underbrace{\begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}}_{\text{REF}}$$

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Example (cont.)

Adding (-2) times row 3 of D to row 1 of D and 4 times row 3 of D to row 2 of D, we obtain

$$E = D_{\substack{4r_3+r_2\to r_2 \\ -2r_3+r_1\to r_1}} = \begin{pmatrix} 1 & 1 & 0 & -7 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

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Adding (-1) times row 2 of E to row 1 of E, we obtain

$$F = E_{-r_2+r_1\to r_1} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & -19 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 2 \end{pmatrix}}_{\text{RREF}}$$

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Note: The number of nonzero rows in the REF or RREF of a matrix $\mathbf{A}$ is called the rank of $\mathbf{A}$ and denoted by $rank\,(\mathbf{A})$.

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- Suppose the required elementary row operations are (in order) $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n$, then

$$\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}$$

which means that $\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 = \mathbf{A}^{-1}$.

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  which means that $\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 = \mathbf{A}^{-1}$.
- Furthermore, because

$$\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 \mathbf{I} = \mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1$$

  we can use the following technique:

Finding an Inverse using Elementary Row Operations

- Write **A** and **I** side-by-side.

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- Write **A** and **I** side-by-side.
- Then apply the same row operations to both **A** and **I** until **A** is reduced to the identity matrix.

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- Write $\mathbf{A}$ and $\mathbf{I}$ side-by-side.
- Then apply the same row operations to both $\mathbf{A}$ and $\mathbf{I}$ until $\mathbf{A}$ is reduced to the identity matrix.
- At the same time, the identity matrix will be "reduced" to the inverse matrix.

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Here is the fully worked out example:

Example

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$$B = [A|I_3] = \left(\begin{array}{ccc|ccc} 3 & 3 & 6 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

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$$B = [A|I_3] = \left(\begin{array}{ccc|ccc} 3 & 3 & 6 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Multiplying the first row of B by $\frac{1}{3}$, we obtain

$$C = B_{\frac{1}{3}r_1 \to r_1} = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example (cont.)

Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1+r_2\to r_2}\left(\begin{array}{rrr|rrr} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -4 & \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example (cont.)

Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1+r_2\to r_2} \left(\begin{array}{rrr|rrr} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -4 & \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-2) times row 3 of D to row 1 of D and 4 times row 3 of D to row 2 of D, we obtain

$$E = D_{\substack{4r_3+r_2\to r_2 \\ -2r_3+r_1\to r_1}} = \left(\begin{array}{rrr|rrr} 1 & 1 & 0 & \frac{1}{3} & 0 & -2 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example (cont.)

Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1+r_2\to r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -4 & \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-2) times row 3 of D to row 1 of D and 4 times row 3 of D to row 2 of D, we obtain

$$E = D_{\substack{4r_3+r_2\to r_2 \\ -2r_3+r_1\to r_1}} = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{3} & 0 & -2 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-1) times row 2 of E to row 1 of E, we obtain

$$\underset{-r_2+r_1\to r_1}{E} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -6 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example (cont.)

Adding (-2) times row 1 of C to row 2 of C, we obtain

$$D = C_{-2r_1+r_2\to r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -4 & \frac{-2}{3} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-2) times row 3 of D to row 1 of D and 4 times row 3 of D to row 2 of D, we obtain

$$E = D_{\substack{4r_3+r_2\to r_2 \\ -2r_3+r_1\to r_1}} = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{3} & 0 & -2 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Adding (-1) times row 2 of E to row 1 of E, we obtain

$$\underset{-r_2+r_1\to r_1}{E} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -6 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right), \ A^{-1} = \begin{pmatrix} 1 & -1 & -6 \\ \frac{-2}{3} & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

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