16. $\lim_{h\to 0^-} \dfrac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$

17. a. $\lim_{x\to -2^+} (x + 3)\dfrac{|x + 2|}{x + 2}$ b. $\lim_{x\to -2^-} (x + 3)\dfrac{|x + 2|}{x + 2}$

18. a. $\lim_{x\to 1^+} \dfrac{\sqrt{2x}\,(x - 1)}{|x - 1|}$ b. $\lim_{x\to 1^-} \dfrac{\sqrt{2x}\,(x - 1)}{|x - 1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 19 and 20.

19. a. $\lim_{\theta\to 3^+} \dfrac{\lfloor \theta \rfloor}{\theta}$ b. $\lim_{\theta\to 3^-} \dfrac{\lfloor \theta \rfloor}{\theta}$

20. a. $\lim_{t\to 4^+} (t - \lfloor t \rfloor)$ b. $\lim_{t\to 4^-} (t - \lfloor t \rfloor)$

Using $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

Find the limits in Exercises 21–42.

21. $\lim_{\theta\to 0} \dfrac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\lim_{t\to 0} \dfrac{\sin kt}{t}$ (k constant)

23. $\lim_{y\to 0} \dfrac{\sin 3y}{4y}$

24. $\lim_{h\to 0^-} \dfrac{h}{\sin 3h}$

25. $\lim_{x\to 0} \dfrac{\tan 2x}{x}$

26. $\lim_{t\to 0} \dfrac{2t}{\tan t}$

27. $\lim_{x\to 0} \dfrac{x \csc 2x}{\cos 5x}$

28. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

29. $\lim_{x\to 0} \dfrac{x + x\cos x}{\sin x \cos x}$

30. $\lim_{x\to 0} \dfrac{x^2 - x + \sin x}{2x}$

31. $\lim_{\theta\to 0} \dfrac{1 - \cos\theta}{\sin 2\theta}$

32. $\lim_{x\to 0} \dfrac{x - x\cos x}{\sin^2 3x}$

33. $\lim_{t\to 0} \dfrac{\sin(1 - \cos t)}{1 - \cos t}$

34. $\lim_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

35. $\lim_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

36. $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 4x}$

37. $\lim_{\theta\to 0} \theta\cos\theta$

38. $\lim_{\theta\to 0} \sin\theta \cot 2\theta$

39. $\lim_{x\to 0} \dfrac{\tan 3x}{\sin 8x}$

40. $\lim_{y\to 0} \dfrac{\sin 3y \cot 5y}{y \cot 4y}$

41. $\lim_{\theta\to 0} \dfrac{\tan\theta}{\theta^2 \cot 3\theta}$

42. $\lim_{\theta\to 0} \dfrac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta}$

Theory and Examples

43. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

44. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

45. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

46. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

47. Given $\epsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x - 5} < \epsilon$. What limit is being verified and what is its value?

48. Given $\epsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4 - x} < \epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 49 and 50.

49. $\lim_{x\to 0^-} \dfrac{x}{|x|} = -1$

50. $\lim_{x\to 2^+} \dfrac{x - 2}{|x - 2|} = 1$

51. **Greatest integer function** Find **(a)** $\lim_{x\to 400^+} \lfloor x \rfloor$ and **(b)** $\lim_{x\to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 400} \lfloor x \rfloor$? Give reasons for your answer.

52. **One-sided limits** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find **(a)** $\lim_{x\to 0^+} f(x)$ and **(b)** $\lim_{x\to 0^-} f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 0} f(x)$? Give reasons for your answer.

2.5 Continuity

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure (Figure 2.34). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. The limit of a continuous function as x approaches c can be found simply by calculating the value of the function at c. (We found this to be true for polynomials in Theorem 2.)

Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one continuous motion without lifting the pencil is an example of a continuous function. In this section we investigate more precisely what it means for a function to be continuous.

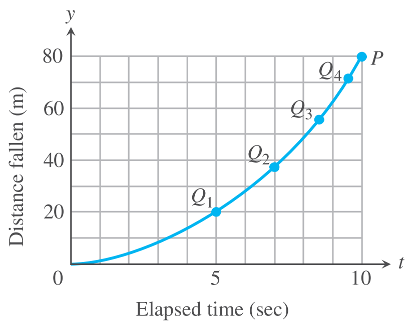

FIGURE 2.34 Connecting plotted points by an unbroken curve from experimental data $Q_1, Q_2, Q_3, \dots$ for a falling object.

We also study the properties of continuous functions, and see that many of the function types presented in Section 1.1 are continuous.

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.

EXAMPLE 1 Find the points at which the function f in Figure 2.35 is continuous and the points at which f is not continuous.

Solution The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1, x = 2$, and $x = 4$. At these points, there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$,	$\lim_{x \to 0^+} f(x) = f(0)$.
At $x = 3$,	$\lim_{x \to 3} f(x) = f(3)$.
At $0 < c < 4, c \neq 1, 2$,	$\lim_{x \to c} f(x) = f(c)$.

Points at which f is not continuous:

At $x = 1$,	$\lim_{x \to 1} f(x)$ does not exist.
At $x = 2$,	$\lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.
At $x = 4$,	$\lim_{x \to 4^-} f(x) = 1$, but $1 \neq f(4)$.
At $c < 0, c > 4$,	these points are not in the domain of f. ■

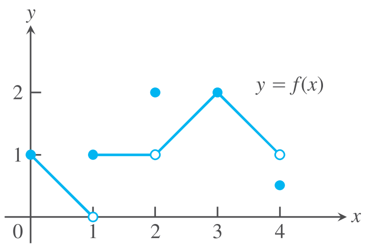

FIGURE 2.35 The function is continuous on [0, 4] except at $x = 1, x = 2$, and $x = 4$ (Example 1).

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit) (Figure 2.36).

> **DEFINITION**
> *Interior point*: A function $y = f(x)$ is **continuous at an interior point c** of its domain if
> $$\lim_{x \to c} f(x) = f(c).$$
> *Endpoint*: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if
> $$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

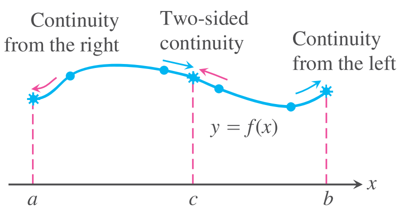

FIGURE 2.36 Continuity at points a, b, and c.

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and that c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

A function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x \to c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b. A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.36).

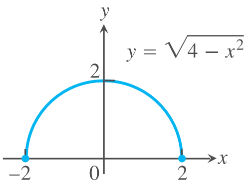

FIGURE 2.37 A function that is continuous at every domain point (Example 2).

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain $[-2, 2]$ (Figure 2.37), including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous. ■

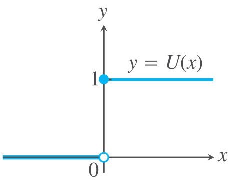

FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.38, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

We summarize continuity at a point in the form of a test.

> **Continuity Test**
> A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.
>
> **1.** $f(c)$ exists (c lies in the domain of f).
> **2.** $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
> **3.** $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

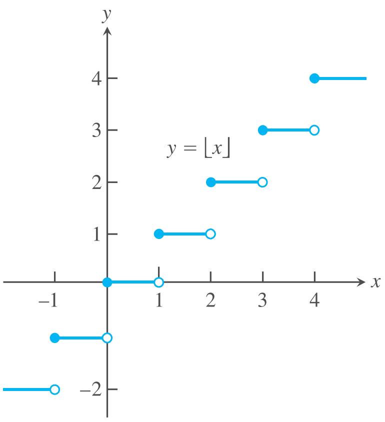

FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.39. It is discontinuous at every integer because the left-hand and right-hand limits are not equal as $x \to n$:

$$\lim_{x\to n^-} \lfloor x \rfloor = n - 1 \qquad \text{and} \qquad \lim_{x\to n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x\to 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor .$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x\to c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor . \qquad ■$$

Figure 2.40 displays several common types of discontinuities. The function in Figure 2.40a is continuous at $x = 0$. The function in Figure 2.40b would be continuous if it had $f(0) = 1$. The function in Figure 2.40c would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in Figure 2.40b and c are **removable**. Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.40d through f are more serious: $\lim_{x\to 0} f(x)$ does not exist, and there is no way to improve the situation by changing f at 0. The step function in Figure 2.40d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.40e has an **infinite discontinuity**. The function in Figure 2.40f has an **oscillating discontinuity**: It oscillates too much to have a limit as $x \to 0$.

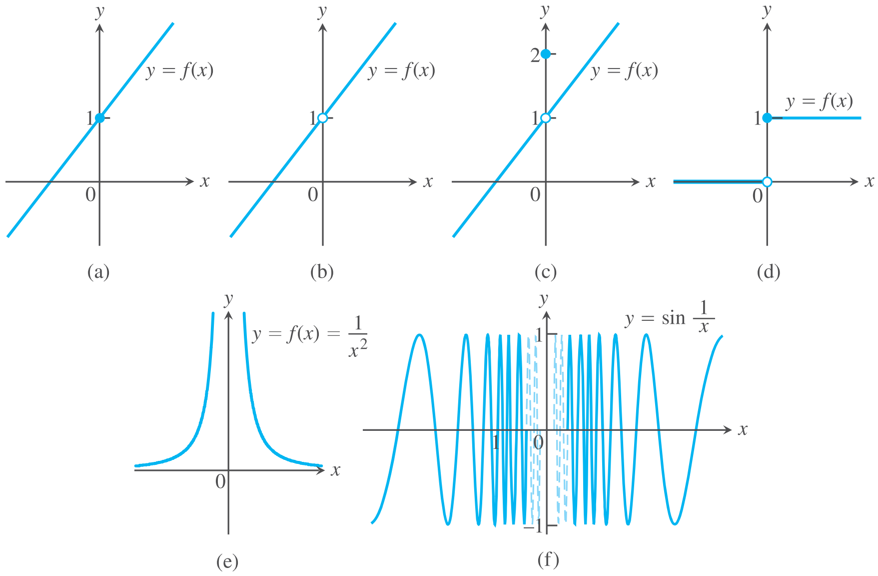

FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. For example, the semicircle function graphed in Figure 2.37 is continuous on the interval $[-2, 2]$, which is its domain. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

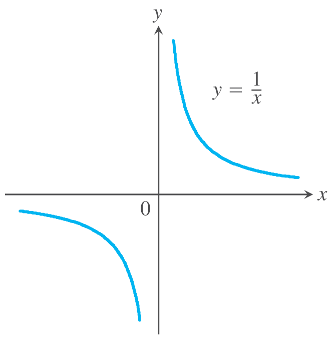

FIGURE 2.41 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

EXAMPLE 5

(a) The function $y = 1/x$ (Figure 2.41) is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there; that is, it is discontinuous on any interval containing $x = 0$.

(b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

Algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Most of the results in Theorem 8 follow from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned}
\lim_{x\to c}(f+g)(x) &= \lim_{x\to c}(f(x)+g(x)) \\
&= \lim_{x\to c} f(x) + \lim_{x\to c} g(x), && \text{Sum Rule, Theorem 1} \\
&= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\
&= (f+g)(c).
\end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6

(a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x\to c} P(x) = P(c)$ by Theorem 2, Section 2.2.

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined $(Q(c) \neq 0)$ by Theorem 3, Section 2.2. ■

EXAMPLE 7 The function $f(x) = |x|$ is continuous at every value of x. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x\to 0}|x| = 0 = |0|$. ■

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 11 of Section 2.2. Both functions are, in fact, continuous everywhere (see Exercise 70). It follows from Theorem 8 that all six trigonometric functions are then continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

Inverse Functions and Continuity

The inverse function of any function continuous on an interval is continuous over its domain. This result is suggested from the observation that the graph of f^{-1}, being the reflection of the graph of f across the line $y = x$, cannot have any breaks in it when the graph of f has no breaks. A rigorous proof that f^{-1} is continuous whenever f is continuous on an interval is given in more advanced texts. It follows that the inverse trigonometric functions are all continuous over their domains.

We defined the exponential function $y = a^x$ in Section 1.5 informally by its graph. Recall that the graph was obtained from the graph of $y = a^x$ for x a rational number by filling in the holes at the irrational points x, so the function $y = a^x$ was defined to be continuous over the entire real line. The inverse function $y = \log_a x$ is also continuous. In particular, the natural exponential function $y = e^x$ and the natural logarithm function $y = \ln x$ are both continuous over their domains.

Composites

All composites of continuous functions are continuous. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.42). In this case, the limit as $x \to c$ is $g(f(c))$.

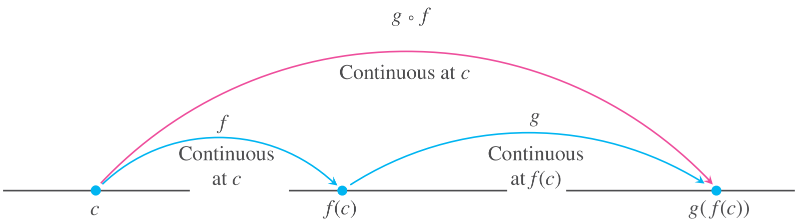

FIGURE 2.42 Composites of continuous functions are continuous.