MODULE 5

9.4 BACKWARD CHAINING

The second major family of logical inference algorithms uses the **backward chaining** approach introduced in Section 7.5 for definite clauses. These algorithms work backward from the goal, chaining through rules to find known facts that support the proof. We describe the basic algorithm, and then we describe how it is used in **logic programming**, which is the most widely used form of automated reasoning. We also see that backward chaining has some disadvantages compared with forward chaining, and we look at ways to overcome them. Finally, we look at the close connection between logic programming and constraint satisfaction problems.

9.4.1 A backward-chaining algorithm

Figure 9.6 shows a backward-chaining algorithm for definite clauses. FOL-BC-ASK(KB, $goal$) will be proved if the knowledge base contains a clause of the form $lhs \Rightarrow goal$, where lhs (left-hand side) is a list of conjuncts. An atomic fact like $American(West)$ is considered as a clause whose lhs is the empty list. Now a query that contains variables might be proved in multiple ways. For example, the query $Person(x)$ could be proved with the substitution
GENERATOR $\{x/John\}$ as well as with $\{x/Richard\}$. So we implement FOL-BC-ASK as a **generator**—a function that returns multiple times, each time giving one possible result.

Backward chaining is a kind of AND/OR search—the OR part because the goal query can be proved by any rule in the knowledge base, and the AND part because all the conjuncts in the lhs of a clause must be proved. FOL-BC-OR works by fetching all clauses that might unify with the goal, standardizing the variables in the clause to be brand-new variables, and then, if the rhs of the clause does indeed unify with the goal, proving every conjunct in the lhs, using FOL-BC-AND. That function in turn works by proving each of the conjuncts in turn, keeping track of the accumulated substitution as we go. Figure 9.7 is the proof tree for deriving $Criminal(West)$ from sentences (9.3) through (9.10).

Backward chaining, as we have written it, is clearly a depth-first search algorithm. This means that its space requirements are linear in the size of the proof (neglecting, for now, the space required to accumulate the solutions). It also means that backward chaining (unlike forward chaining) suffers from problems with repeated states and incompleteness. We will discuss these problems and some potential solutions, but first we show how backward chaining is used in logic programming systems.

```
function FOL-BC-ASK(KB, query) returns a generator of substitutions
  return FOL-BC-OR(KB, query, { })
```

```
generator FOL-BC-OR(KB, goal, θ) yields a substitution
  for each rule (lhs ⇒ rhs) in FETCH-RULES-FOR-GOAL(KB, goal) do
    (lhs, rhs) ← STANDARDIZE-VARIABLES((lhs, rhs))
    for each θ′ in FOL-BC-AND(KB, lhs, UNIFY(rhs, goal, θ)) do
      yield θ′
```

```
generator FOL-BC-AND(KB, goals, θ) yields a substitution
  if θ = failure then return
  else if LENGTH(goals) = 0 then yield θ
  else do
    first,rest ← FIRST(goals), REST(goals)
    for each θ′ in FOL-BC-OR(KB, SUBST(θ, first), θ) do
      for each θ″ in FOL-BC-AND(KB, rest, θ′) do
        yield θ″
```

Figure 9.6 A simple backward-chaining algorithm for first-order knowledge bases.

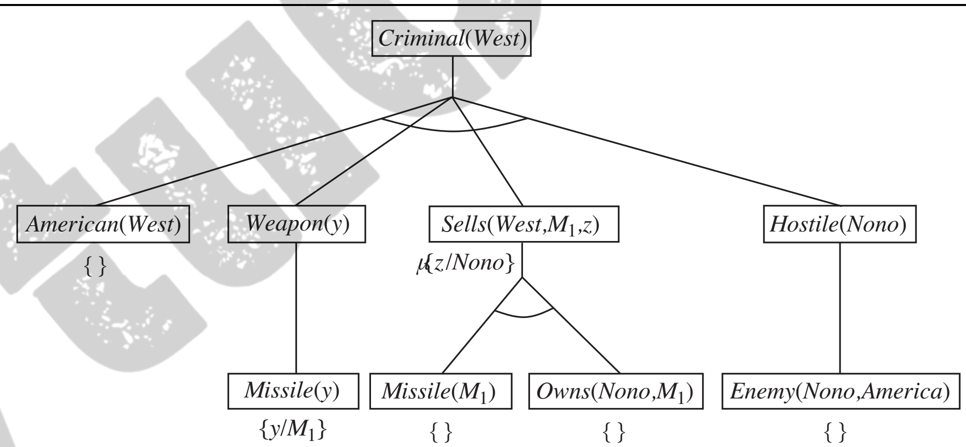

Figure 9.7 Proof tree constructed by backward chaining to prove that West is a criminal. The tree should be read depth first, left to right. To prove $Criminal(West)$, we have to prove the four conjuncts below it. Some of these are in the knowledge base, and others require further backward chaining. Bindings for each successful unification are shown next to the corresponding subgoal. Note that once one subgoal in a conjunction succeeds, its substitution is applied to subsequent subgoals. Thus, by the time FOL-BC-ASK gets to the last conjunct, originally $Hostile(z)$, z is already bound to $Nono$.

9.4.2 Logic programming

Logic programming is a technology that comes fairly close to embodying the declarative ideal described in Chapter 7: that systems should be constructed by expressing knowledge in a formal language and that problems should be solved by running inference processes on that knowledge. The ideal is summed up in Robert Kowalski's equation,

$$Algorithm = Logic + Control .$$

PROLOG

Prolog is the most widely used logic programming language. It is used primarily as a rapid-prototyping language and for symbol-manipulation tasks such as writing compilers (Van Roy, 1990) and parsing natural language (Pereira and Warren, 1980). Many expert systems have been written in Prolog for legal, medical, financial, and other domains.

Prolog programs are sets of definite clauses written in a notation somewhat different from standard first-order logic. Prolog uses uppercase letters for variables and lowercase for constants—the opposite of our convention for logic. Commas separate conjuncts in a clause, and the clause is written "backwards" from what we are used to; instead of $A \wedge B \Rightarrow C$ in Prolog we have `C :- A, B`. Here is a typical example:

```
criminal(X) :- american(X), weapon(Y), sells(X,Y,Z), hostile(Z).
```

The notation `[E|L]` denotes a list whose first element is `E` and whose rest is `L`. Here is a Prolog program for `append(X,Y,Z)`, which succeeds if list `Z` is the result of appending lists `X` and `Y`:

```
append([],Y,Y).
append([A|X],Y,[A|Z]) :- append(X,Y,Z).
```

In English, we can read these clauses as (1) appending an empty list with a list `Y` produces the same list `Y` and (2) `[A|Z]` is the result of appending `[A|X]` onto `Y`, provided that `Z` is the result of appending `X` onto `Y`. In most high-level languages we can write a similar recursive function that describes how to append two lists. The Prolog definition is actually much more powerful, however, because it describes a *relation* that holds among three arguments, rather than a *function* computed from two arguments. For example, we can ask the query `append(X,Y,[1,2])`: what two lists can be appended to give `[1,2]`? We get back the solutions

```
X=[]    Y=[1,2];
X=[1]   Y=[2];
X=[1,2] Y=[]
```

The execution of Prolog programs is done through depth-first backward chaining, where clauses are tried in the order in which they are written in the knowledge base. Some aspects of Prolog fall outside standard logical inference:

- Prolog uses the database semantics of Section 8.2.8 rather than first-order semantics, and this is apparent in its treatment of equality and negation (see Section 9.4.5).
- There is a set of built-in functions for arithmetic. Literals using these function symbols are "proved" by executing code rather than doing further inference. For example, the

goal "`X is 4+3`" succeeds with `X` bound to 7. On the other hand, the goal "`5 is X+Y`" fails, because the built-in functions do not do arbitrary equation solving.[5]

- There are built-in predicates that have side effects when executed. These include input–output predicates and the `assert/retract` predicates for modifying the knowledge base. Such predicates have no counterpart in logic and can produce confusing results—for example, if facts are asserted in a branch of the proof tree that eventually fails.
- The **occur check** is omitted from Prolog's unification algorithm. This means that some unsound inferences can be made; these are almost never a problem in practice.
- Prolog uses depth-first backward-chaining search with no checks for infinite recursion. This makes it very fast when given the right set of axioms, but incomplete when given the wrong ones.

Prolog's design represents a compromise between declarativeness and execution efficiency—inasmuch as efficiency was understood at the time Prolog was designed.

9.4.3 Efficient implementation of logic programs

The execution of a Prolog program can happen in two modes: interpreted and compiled. Interpretation essentially amounts to running the FOL-BC-ASK algorithm from Figure 9.6, with the program as the knowledge base. We say "essentially" because Prolog interpreters contain a variety of improvements designed to maximize speed. Here we consider only two.

CHOICE POINT

First, our implementation had to explicitly manage the iteration over possible results generated by each of the subfunctions. Prolog interpreters have a global data structure, a stack of **choice points**, to keep track of the multiple possibilities that we considered in FOL-BC-OR. This global stack is more efficient, and it makes debugging easier, because the debugger can move up and down the stack.

TRAIL

Second, our simple implementation of FOL-BC-ASK spends a good deal of time generating substitutions. Instead of explicitly constructing substitutions, Prolog has logic variables that remember their current binding. At any point in time, every variable in the program either is unbound or is bound to some value. Together, these variables and values implicitly define the substitution for the current branch of the proof. Extending the path can only add new variable bindings, because an attempt to add a different binding for an already bound variable results in a failure of unification. When a path in the search fails, Prolog will back up to a previous choice point, and then it might have to unbind some variables. This is done by keeping track of all the variables that have been bound in a stack called the **trail**. As each new variable is bound by UNIFY-VAR, the variable is pushed onto the trail. When a goal fails and it is time to back up to a previous choice point, each of the variables is unbound as it is removed from the trail.

Even the most efficient Prolog interpreters require several thousand machine instructions per inference step because of the cost of index lookup, unification, and building the recursive call stack. In effect, the interpreter always behaves as if it has never seen the program before; for example, it has to *find* clauses that match the goal. A compiled Prolog

[5] Note that if the Peano axioms are provided, such goals can be solved by inference within a Prolog program.

procedure APPEND($ax, y, az, continuation$)

$trail \leftarrow$ GLOBAL-TRAIL-POINTER()
if ax = [ ] and UNIFY(y, az) **then** CALL($continuation$)
RESET-TRAIL($trail$)
$a, x, z \leftarrow$ NEW-VARIABLE(), NEW-VARIABLE(), NEW-VARIABLE()
if UNIFY($ax, [a \mid x]$) and UNIFY($az, [a \mid z]$) **then** APPEND($x, y, z, continuation$)

Figure 9.8 Pseudocode representing the result of compiling the `Append` predicate. The function NEW-VARIABLE returns a new variable, distinct from all other variables used so far. The procedure CALL($continuation$) continues execution with the specified continuation.

program, on the other hand, is an inference procedure for a specific set of clauses, so it *knows* what clauses match the goal. Prolog basically generates a miniature theorem prover for each different predicate, thereby eliminating much of the overhead of interpretation. It is also possible to **open-code** the unification routine for each different call, thereby avoiding explicit analysis of term structure. (For details of open-coded unification, see Warren *et al.* (1977).)

OPEN-CODE

The instruction sets of today's computers give a poor match with Prolog's semantics, so most Prolog compilers compile into an intermediate language rather than directly into machine language. The most popular intermediate language is the Warren Abstract Machine, or WAM, named after David H. D. Warren, one of the implementers of the first Prolog compiler. The WAM is an abstract instruction set that is suitable for Prolog and can be either interpreted or translated into machine language. Other compilers translate Prolog into a high-level language such as Lisp or C and then use that language's compiler to translate to machine language. For example, the definition of the `Append` predicate can be compiled into the code shown in Figure 9.8. Several points are worth mentioning:

- Rather than having to search the knowledge base for `Append` clauses, the clauses become a procedure and the inferences are carried out simply by calling the procedure.
- As described earlier, the current variable bindings are kept on a trail. The first step of the procedure saves the current state of the trail, so that it can be restored by RESET-TRAIL if the first clause fails. This will undo any bindings generated by the first call to UNIFY.

CONTINUATION

- The trickiest part is the use of **continuations** to implement choice points. You can think of a continuation as packaging up a procedure and a list of arguments that together define what should be done next whenever the current goal succeeds. It would not do just to return from a procedure like APPEND when the goal succeeds, because it could succeed in several ways, and each of them has to be explored. The continuation argument solves this problem because it can be called each time the goal succeeds. In the APPEND code, if the first argument is empty and the second argument unifies with the third, then the APPEND predicate has succeeded. We then CALL the continuation, with the appropriate bindings on the trail, to do whatever should be done next. For example, if the call to APPEND were at the top level, the continuation would print the bindings of the variables.

Before Warren's work on the compilation of inference in Prolog, logic programming was too slow for general use. Compilers by Warren and others allowed Prolog code to achieve speeds that are competitive with C on a variety of standard benchmarks (Van Roy, 1990). Of course, the fact that one can write a planner or natural language parser in a few dozen lines of Prolog makes it somewhat more desirable than C for prototyping most small-scale AI research projects.

OR-PARALLELISM

AND-PARALLELISM

Parallelization can also provide substantial speedup. There are two principal sources of parallelism. The first, called **OR-parallelism**, comes from the possibility of a goal unifying with many different clauses in the knowledge base. Each gives rise to an independent branch in the search space that can lead to a potential solution, and all such branches can be solved in parallel. The second, called **AND-parallelism**, comes from the possibility of solving each conjunct in the body of an implication in parallel. AND-parallelism is more difficult to achieve, because solutions for the whole conjunction require consistent bindings for all the variables. Each conjunctive branch must communicate with the other branches to ensure a global solution.

9.4.4 Redundant inference and infinite loops

We now turn to the Achilles heel of Prolog: the mismatch between depth-first search and search trees that include repeated states and infinite paths. Consider the following logic program that decides if a path exists between two points on a directed graph:

```
path(X,Z) :- link(X,Z).
path(X,Z) :- path(X,Y), link(Y,Z).
```

A simple three-node graph, described by the facts `link(a,b)` and `link(b,c)`, is shown in Figure 9.9(a). With this program, the query `path(a,c)` generates the proof tree shown in Figure 9.10(a). On the other hand, if we put the two clauses in the order

```
path(X,Z) :- path(X,Y), link(Y,Z).
path(X,Z) :- link(X,Z).
```

then Prolog follows the infinite path shown in Figure 9.10(b). Prolog is therefore **incomplete** as a theorem prover for definite clauses—even for Datalog programs, as this example shows—because, for some knowledge bases, it fails to prove sentences that are entailed. Notice that forward chaining does not suffer from this problem: once `path(a,b)`, `path(b,c)`, and `path(a,c)` are inferred, forward chaining halts.

Depth-first backward chaining also has problems with redundant computations. For example, when finding a path from A_1 to J_4 in Figure 9.9(b), Prolog performs 877 inferences, most of which involve finding all possible paths to nodes from which the goal is unreachable. This is similar to the repeated-state problem discussed in Chapter 3. The total amount of inference can be exponential in the number of ground facts that are generated. If we apply forward chaining instead, at most n^2 `path(X,Y)` facts can be generated linking n nodes. For the problem in Figure 9.9(b), only 62 inferences are needed.

DYNAMIC PROGRAMMING

Forward chaining on graph search problems is an example of **dynamic programming**, in which the solutions to subproblems are constructed incrementally from those of smaller

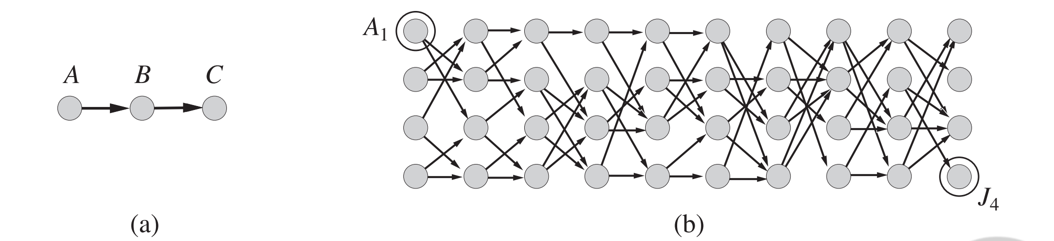

Figure 9.9 (a) Finding a path from A to C can lead Prolog into an infinite loop. (b) A graph in which each node is connected to two random successors in the next layer. Finding a path from A_1 to J_4 requires 877 inferences.

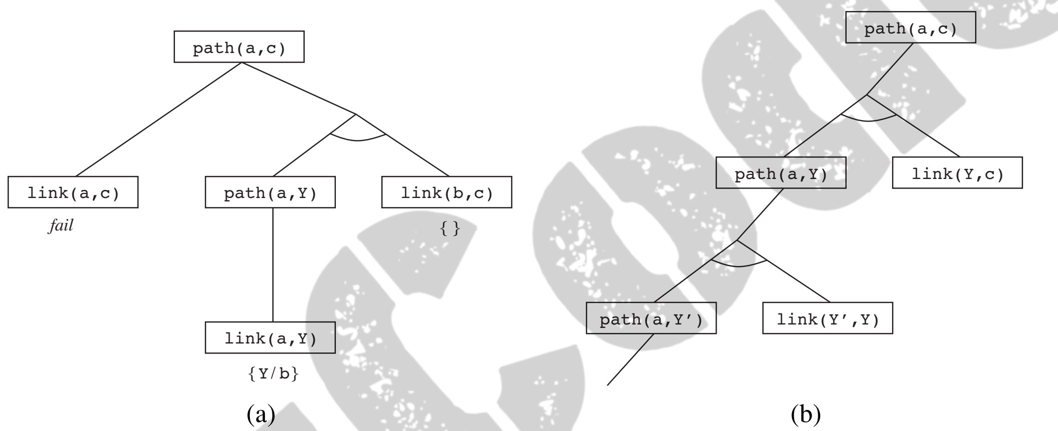

Figure 9.10 (a) Proof that a path exists from A to C. (b) Infinite proof tree generated when the clauses are in the "wrong" order.

subproblems and are cached to avoid recomputation. We can obtain a similar effect in a backward chaining system using **memoization**—that is, caching solutions to subgoals as they are found and then reusing those solutions when the subgoal recurs, rather than repeating the previous computation. This is the approach taken by **tabled logic programming** systems, which use efficient storage and retrieval mechanisms to perform memoization. Tabled logic programming combines the goal-directedness of backward chaining with the dynamic-programming efficiency of forward chaining. It is also complete for Datalog knowledge bases, which means that the programmer need worry less about infinite loops. (It is still possible to get an infinite loop with predicates like `father(X,Y)` that refer to a potentially unbounded number of objects.)

TABLED LOGIC PROGRAMMING

9.4.5 Database semantics of Prolog

Prolog uses database semantics, as discussed in Section 8.2.8. The unique names assumption says that every Prolog constant and every ground term refers to a distinct object, and the closed world assumption says that the only sentences that are true are those that are entailed by the knowledge base. There is no way to assert that a sentence is false in Prolog. This makes Prolog less expressive than first-order logic, but it is part of what makes Prolog more efficient and more concise. Consider the following Prolog assertions about some course offerings:

$$Course(CS, 101),\ Course(CS, 102),\ Course(CS, 106),\ Course(EE, 101). \quad (9.11)$$

Under the unique names assumption, *CS* and *EE* are different (as are 101, 102, and 106), so this means that there are four distinct courses. Under the closed-world assumption there are no other courses, so there are exactly four courses. But if these were assertions in FOL rather than in Prolog, then all we could say is that there are somewhere between one and infinity courses. That's because the assertions (in FOL) do not deny the possibility that other unmentioned courses are also offered, nor do they say that the courses mentioned are different from each other. If we wanted to translate Equation (9.11) into FOL, we would get this:

$$\begin{aligned} Course(d, n) \quad \Leftrightarrow \quad & (d = CS \land n = 101) \lor (d = CS \land n = 102) \\ & \lor (d = CS \land n = 106) \lor (d = EE \land n = 101)\,. \end{aligned} \quad (9.12)$$

COMPLETION

This is called the **completion** of Equation (9.11). It expresses in FOL the idea that there are at most four courses. To express in FOL the idea that there are at least four courses, we need to write the completion of the equality predicate:

$$\begin{aligned} x = y \quad \Leftrightarrow \quad & (x = CS \land y = CS) \lor (x = EE \land y = EE) \lor (x = 101 \land y = 101) \\ & \lor (x = 102 \land y = 102) \lor (x = 106 \land y = 106)\,. \end{aligned}$$

The completion is useful for understanding database semantics, but for practical purposes, if your problem can be described with database semantics, it is more efficient to reason with Prolog or some other database semantics system, rather than translating into FOL and reasoning with a full FOL theorem prover.

9.4.6 Constraint logic programming

In our discussion of forward chaining (Section 9.3), we showed how constraint satisfaction problems (CSPs) can be encoded as definite clauses. Standard Prolog solves such problems in exactly the same way as the backtracking algorithm given in Figure 6.5.

Because backtracking enumerates the domains of the variables, it works only for **finite-domain** CSPs. In Prolog terms, there must be a finite number of solutions for any goal with unbound variables. (For example, the goal `diff(Q,SA)`, which says that Queensland and South Australia must be different colors, has six solutions if three colors are allowed.) Infinite-domain CSPs—for example, with integer or real-valued variables—require quite different algorithms, such as bounds propagation or linear programming.

Consider the following example. We define `triangle(X,Y,Z)` as a predicate that holds if the three arguments are numbers that satisfy the triangle inequality:

```
triangle(X,Y,Z) :-
    X>0, Y>0, Z>0, X+Y>=Z, Y+Z>=X, X+Z>=Y.
```

If we ask Prolog the query `triangle(3,4,5)`, it succeeds. On the other hand, if we ask `triangle(3,4,Z)`, no solution will be found, because the subgoal `Z>=0` cannot be handled by Prolog; we can't compare an unbound value to 0.

CONSTRAINT LOGIC PROGRAMMING

Constraint logic programming (CLP) allows variables to be *constrained* rather than *bound*. A CLP solution is the most specific set of constraints on the query variables that can be derived from the knowledge base. For example, the solution to the `triangle(3,4,Z)` query is the constraint `7 >= Z >= 1`. Standard logic programs are just a special case of CLP in which the solution constraints must be equality constraints—that is, bindings.

CLP systems incorporate various constraint-solving algorithms for the constraints allowed in the language. For example, a system that allows linear inequalities on real-valued variables might include a linear programming algorithm for solving those constraints. CLP systems also adopt a much more flexible approach to solving standard logic programming queries. For example, instead of depth-first, left-to-right backtracking, they might use any of the more efficient algorithms discussed in Chapter 6, including heuristic conjunct ordering, backjumping, cutset conditioning, and so on. CLP systems therefore combine elements of constraint satisfaction algorithms, logic programming, and deductive databases.

METARULE

Several systems that allow the programmer more control over the search order for inference have been defined. The MRS language (Genesereth and Smith, 1981; Russell, 1985) allows the programmer to write **metarules** to determine which conjuncts are tried first. The user could write a rule saying that the goal with the fewest variables should be tried first or could write domain-specific rules for particular predicates.

9.5 RESOLUTION

The last of our three families of logical systems is based on **resolution**. We saw on page 250 that propositional resolution using refutation is a complete inference procedure for propositional logic. In this section, we describe how to extend resolution to first-order logic.

9.5.1 Conjunctive normal form for first-order logic

As in the propositional case, first-order resolution requires that sentences be in **conjunctive normal form** (CNF)—that is, a conjunction of clauses, where each clause is a disjunction of literals.[6] Literals can contain variables, which are assumed to be universally quantified. For example, the sentence

$$\forall x \;\; American(x) \wedge Weapon(y) \wedge Sells(x,y,z) \wedge Hostile(z) \;\Rightarrow\; Criminal(x)$$

becomes, in CNF,

$$\neg American(x) \vee \neg Weapon(y) \vee \neg Sells(x,y,z) \vee \neg Hostile(z) \vee Criminal(x)\,.$$

Every sentence of first-order logic can be converted into an inferentially equivalent CNF sentence. In particular, the CNF sentence will be unsatisfiable just when the original sentence is unsatisfiable, so we have a basis for doing proofs by contradiction on the CNF sentences.

[6] A clause can also be represented as an implication with a conjunction of atoms in the premise and a disjunction of atoms in the conclusion (Exercise 7.13). This is called **implicative normal form** or **Kowalski form** (especially when written with a right-to-left implication symbol (Kowalski, 1979)) and is often much easier to read.

The procedure for conversion to CNF is similar to the propositional case, which we saw on page 253. The principal difference arises from the need to eliminate existential quantifiers. We illustrate the procedure by translating the sentence "Everyone who loves all animals is loved by someone," or

$$\forall x \ [\forall y \ \ Animal(y) \Rightarrow Loves(x,y)] \Rightarrow [\exists y \ \ Loves(y,x)] \ .$$

The steps are as follows:

- **Eliminate implications**:
$$\forall x \ [\neg\forall y \ \neg Animal(y) \lor Loves(x,y)] \lor [\exists y \ \ Loves(y,x)] \ .$$
- **Move $\neg$ inwards**: In addition to the usual rules for negated connectives, we need rules for negated quantifiers. Thus, we have
$$\neg\forall x \ \ p \quad \text{becomes} \quad \exists x \ \ \neg p$$
$$\neg\exists x \ \ p \quad \text{becomes} \quad \forall x \ \ \neg p \ .$$
Our sentence goes through the following transformations:
$$\forall x \ [\exists y \ \ \neg(\neg Animal(y) \lor Loves(x,y))] \lor [\exists y \ \ Loves(y,x)] \ .$$
$$\forall x \ [\exists y \ \ \neg\neg Animal(y) \land \neg Loves(x,y)] \lor [\exists y \ \ Loves(y,x)] \ .$$
$$\forall x \ [\exists y \ \ Animal(y) \land \neg Loves(x,y)] \lor [\exists y \ \ Loves(y,x)] \ .$$
Notice how a universal quantifier ($\forall y$) in the premise of the implication has become an existential quantifier. The sentence now reads "Either there is some animal that x doesn't love, or (if this is not the case) someone loves x." Clearly, the meaning of the original sentence has been preserved.
- **Standardize variables**: For sentences like $(\exists x \, P(x)) \lor (\exists x \, Q(x))$ which use the same variable name twice, change the name of one of the variables. This avoids confusion later when we drop the quantifiers. Thus, we have
$$\forall x \ [\exists y \ \ Animal(y) \land \neg Loves(x,y)] \lor [\exists z \ \ Loves(z,x)] \ .$$

SKOLEMIZATION

- **Skolemize**: **Skolemization** is the process of removing existential quantifiers by elimination. In the simple case, it is just like the Existential Instantiation rule of Section 9.1: translate $\exists x \ P(x)$ into $P(A)$, where A is a new constant. However, we can't apply Existential Instantiation to our sentence above because it doesn't match the pattern $\exists v \ \alpha$; only parts of the sentence match the pattern. If we blindly apply the rule to the two matching parts we get
$$\forall x \ [Animal(A) \land \neg Loves(x,A)] \lor Loves(B,x) \ ,$$
which has the wrong meaning entirely: it says that everyone either fails to love a particular animal A or is loved by some particular entity B. In fact, our original sentence allows each person to fail to love a different animal or to be loved by a different person. Thus, we want the Skolem entities to depend on x and z:
$$\forall x \ [Animal(F(x)) \land \neg Loves(x,F(x))] \lor Loves(G(z),x) \ .$$

SKOLEM FUNCTION

Here F and G are **Skolem functions**. The general rule is that the arguments of the Skolem function are all the universally quantified variables in whose scope the existential quantifier appears. As with Existential Instantiation, the Skolemized sentence is satisfiable exactly when the original sentence is satisfiable.

- **Drop universal quantifiers**: At this point, all remaining variables must be universally quantified. Moreover, the sentence is equivalent to one in which all the universal quantifiers have been moved to the left. We can therefore drop the universal quantifiers:

$$[Animal(F(x)) \land \neg Loves(x, F(x))] \lor Loves(G(z), x) .$$

- **Distribute $\lor$ over $\land$**:

$$[Animal(F(x)) \lor Loves(G(z), x)] \land [\neg Loves(x, F(x)) \lor Loves(G(z), x)] .$$

This step may also require flattening out nested conjunctions and disjunctions.

The sentence is now in CNF and consists of two clauses. It is quite unreadable. (It may help to explain that the Skolem function $F(x)$ refers to the animal potentially unloved by x, whereas $G(z)$ refers to someone who might love x.) Fortunately, humans seldom need look at CNF sentences—the translation process is easily automated.

9.5.2 The resolution inference rule

The resolution rule for first-order clauses is simply a lifted version of the propositional resolution rule given on page 253. Two clauses, which are assumed to be standardized apart so that they share no variables, can be resolved if they contain complementary literals. Propositional literals are complementary if one is the negation of the other; first-order literals are complementary if one *unifies with* the negation of the other. Thus, we have

$$\frac{\ell_1 \lor \cdots \lor \ell_k, \qquad m_1 \lor \cdots \lor m_n}{\textsc{Subst}(\theta, \ell_1 \lor \cdots \lor \ell_{i-1} \lor \ell_{i+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n)}$$

where $\textsc{Unify}(\ell_i, \neg m_j) = \theta$. For example, we can resolve the two clauses

$$[Animal(F(x)) \lor Loves(G(x), x)] \quad \text{and} \quad [\neg Loves(u, v) \lor \neg Kills(u, v)]$$

by eliminating the complementary literals $Loves(G(x), x)$ and $\neg Loves(u, v)$, with unifier $\theta = \{u/G(x), v/x\}$, to produce the **resolvent** clause

$$[Animal(F(x)) \lor \neg Kills(G(x), x)] .$$

BINARY RESOLUTION

This rule is called the **binary resolution** rule because it resolves exactly two literals. The binary resolution rule by itself does not yield a complete inference procedure. The full resolution rule resolves subsets of literals in each clause that are unifiable. An alternative approach is to extend **factoring**—the removal of redundant literals—to the first-order case. Propositional factoring reduces two literals to one if they are *identical*; first-order factoring reduces two literals to one if they are *unifiable*. The unifier must be applied to the entire clause. The combination of binary resolution and factoring is complete.

9.5.3 Example proofs

Resolution proves that $KB \models \alpha$ by proving $KB \land \neg\alpha$ unsatisfiable, that is, by deriving the empty clause. The algorithmic approach is identical to the propositional case, described in

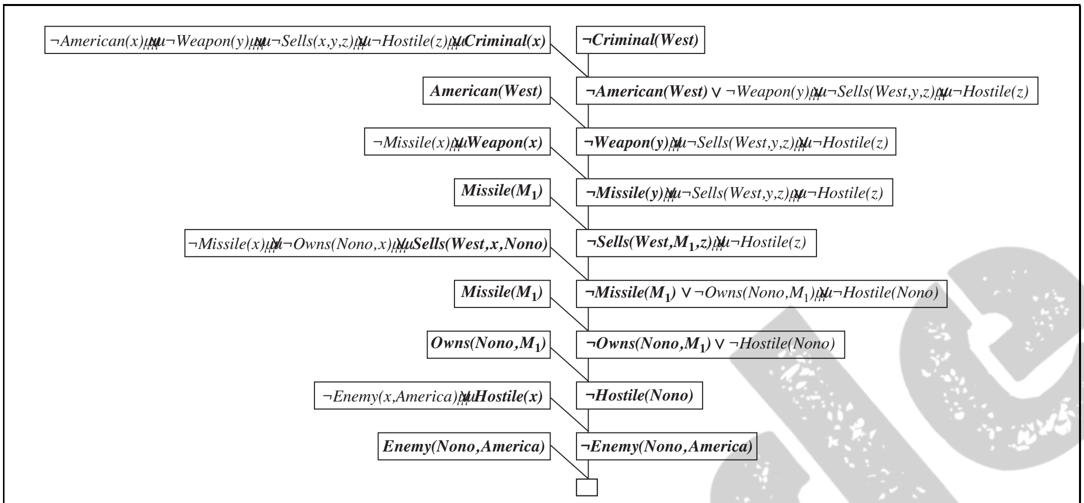

Figure 9.11 A resolution proof that West is a criminal. At each step, the literals that unify are in bold.

Figure 7.12, so we need not repeat it here. Instead, we give two example proofs. The first is the crime example from Section 9.3. The sentences in CNF are

$$\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x)$$
$$\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)$$
$$\neg Enemy(x, America) \lor Hostile(x)$$
$$\neg Missile(x) \lor Weapon(x)$$
$$Owns(Nono, M_1) \qquad Missile(M_1)$$
$$American(West) \qquad Enemy(Nono, America)\ .$$

We also include the negated goal $\neg Criminal(West)$. The resolution proof is shown in Figure 9.11. Notice the structure: single "spine" beginning with the goal clause, resolving against clauses from the knowledge base until the empty clause is generated. This is characteristic of resolution on Horn clause knowledge bases. In fact, the clauses along the main spine correspond *exactly* to the consecutive values of the *goals* variable in the backward-chaining algorithm of Figure 9.6. This is because we always choose to resolve with a clause whose positive literal unified with the leftmost literal of the "current" clause on the spine; this is exactly what happens in backward chaining. Thus, backward chaining is just a special case of resolution with a particular control strategy to decide which resolution to perform next.

Our second example makes use of Skolemization and involves clauses that are not definite clauses. This results in a somewhat more complex proof structure. In English, the problem is as follows:

Everyone who loves all animals is loved by someone.
Anyone who kills an animal is loved by no one.
Jack loves all animals.
Either Jack or Curiosity killed the cat, who is named Tuna.
Did Curiosity kill the cat?

First, we express the original sentences, some background knowledge, and the negated goal G in first-order logic:

A. $\forall x \; [\forall y \;\; Animal(y) \Rightarrow Loves(x,y)] \Rightarrow [\exists y \;\; Loves(y,x)]$

B. $\forall x \; [\exists z \;\; Animal(z) \land Kills(x,z)] \Rightarrow [\forall y \;\; \neg Loves(y,x)]$

C. $\forall x \;\; Animal(x) \Rightarrow Loves(Jack,x)$

D. $Kills(Jack, Tuna) \lor Kills(Curiosity, Tuna)$

E. $Cat(Tuna)$

F. $\forall x \;\; Cat(x) \Rightarrow Animal(x)$

¬G. $\neg Kills(Curiosity, Tuna)$

Now we apply the conversion procedure to convert each sentence to CNF:

A1. $Animal(F(x)) \lor Loves(G(x),x)$

A2. $\neg Loves(x,F(x)) \lor Loves(G(x),x)$

B. $\neg Loves(y,x) \lor \neg Animal(z) \lor \neg Kills(x,z)$

C. $\neg Animal(x) \lor Loves(Jack,x)$

D. $Kills(Jack, Tuna) \lor Kills(Curiosity, Tuna)$

E. $Cat(Tuna)$

F. $\neg Cat(x) \lor Animal(x)$

¬G. $\neg Kills(Curiosity, Tuna)$

The resolution proof that Curiosity killed the cat is given in Figure 9.12. In English, the proof could be paraphrased as follows:

> Suppose Curiosity did not kill Tuna. We know that either Jack or Curiosity did; thus Jack must have. Now, Tuna is a cat and cats are animals, so Tuna is an animal. Because anyone who kills an animal is loved by no one, we know that no one loves Jack. On the other hand, Jack loves all animals, so someone loves him; so we have a contradiction. Therefore, Curiosity killed the cat.

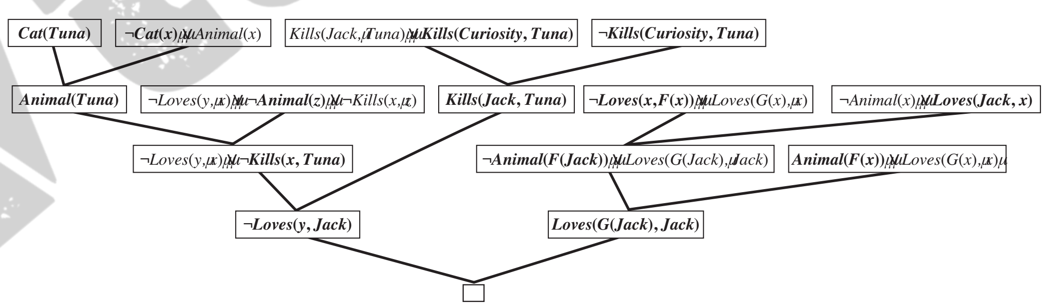

Figure 9.12 A resolution proof that Curiosity killed the cat. Notice the use of factoring in the derivation of the clause $Loves(G(Jack), Jack)$. Notice also in the upper right, the unification of $Loves(x, F(x))$ and $Loves(Jack, x)$ can only succeed after the variables have been standardized apart.

The proof answers the question "Did Curiosity kill the cat?" but often we want to pose more general questions, such as "Who killed the cat?" Resolution can do this, but it takes a little more work to obtain the answer. The goal is $\exists\, w\ \mathit{Kills}(w, \mathit{Tuna})$, which, when negated, becomes $\neg \mathit{Kills}(w, \mathit{Tuna})$ in CNF. Repeating the proof in Figure 9.12 with the new negated goal, we obtain a similar proof tree, but with the substitution $\{w/\mathit{Curiosity}\}$ in one of the steps. So, in this case, finding out who killed the cat is just a matter of keeping track of the bindings for the query variables in the proof.

NONCONSTRUCTIVE PROOF

Unfortunately, resolution can produce **nonconstructive proofs** for existential goals. For example, $\neg \mathit{Kills}(w, \mathit{Tuna})$ resolves with $\mathit{Kills}(\mathit{Jack}, \mathit{Tuna}) \lor \mathit{Kills}(\mathit{Curiosity}, \mathit{Tuna})$ to give $\mathit{Kills}(\mathit{Jack}, \mathit{Tuna})$, which resolves again with $\neg \mathit{Kills}(w, \mathit{Tuna})$ to yield the empty clause. Notice that w has two different bindings in this proof; resolution is telling us that, yes, someone killed Tuna—either Jack or Curiosity. This is no great surprise! One solution is to restrict the allowed resolution steps so that the query variables can be bound only once in a given proof; then we need to be able to backtrack over the possible bindings. Another solution is to add a special **answer literal** to the negated goal, which becomes $\neg \mathit{Kills}(w, \mathit{Tuna}) \lor \mathit{Answer}(w)$. Now, the resolution process generates an answer whenever a clause is generated containing just a *single* answer literal. For the proof in Figure 9.12, this is $\mathit{Answer}(\mathit{Curiosity})$. The nonconstructive proof would generate the clause $\mathit{Answer}(\mathit{Curiosity}) \lor \mathit{Answer}(\mathit{Jack})$, which does not constitute an answer.

ANSWER LITERAL

9.5.4 Completeness of resolution

This section gives a completeness proof of resolution. It can be safely skipped by those who are willing to take it on faith.

REFUTATION COMPLETENESS

We show that resolution is **refutation-complete**, which means that *if* a set of sentences is unsatisfiable, then resolution will always be able to derive a contradiction. Resolution cannot be used to generate all logical consequences of a set of sentences, but it can be used to establish that a given sentence is entailed by the set of sentences. Hence, it can be used to find all answers to a given question, $Q(x)$, by proving that $KB \land \neg Q(x)$ is unsatisfiable.

We take it as given that any sentence in first-order logic (without equality) can be rewritten as a set of clauses in CNF. This can be proved by induction on the form of the sentence, using atomic sentences as the base case (Davis and Putnam, 1960). Our goal therefore is to prove the following: *if S is an unsatisfiable set of clauses, then the application of a finite number of resolution steps to S will yield a contradiction.*

Our proof sketch follows Robinson's original proof with some simplifications from Genesereth and Nilsson (1987). The basic structure of the proof (Figure 9.13) is as follows:

1. First, we observe that if S is unsatisfiable, then there exists a particular set of *ground instances* of the clauses of S such that this set is also unsatisfiable (Herbrand's theorem).
2. We then appeal to the **ground resolution theorem** given in Chapter 7, which states that propositional resolution is complete for ground sentences.
3. We then use a **lifting lemma** to show that, for any propositional resolution proof using the set of ground sentences, there is a corresponding first-order resolution proof using the first-order sentences from which the ground sentences were obtained.

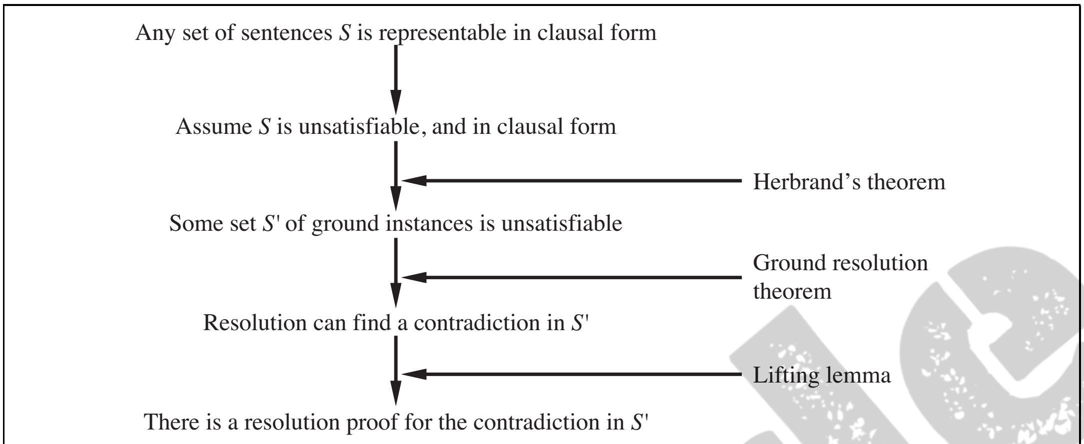

Figure 9.13 Structure of a completeness proof for resolution.

To carry out the first step, we need three new concepts:

HERBRAND UNIVERSE

- **Herbrand universe**: If S is a set of clauses, then H_S, the Herbrand universe of S, is the set of all ground terms constructable from the following:
  a. The function symbols in S, if any.
  b. The constant symbols in S, if any; if none, then the constant symbol A.

  For example, if S contains just the clause $\neg P(x, F(x, A)) \vee \neg Q(x, A) \vee R(x, B)$, then H_S is the following infinite set of ground terms:

  $$\{A, B, F(A, A), F(A, B), F(B, A), F(B, B), F(A, F(A, A)), \ldots\} .$$

SATURATION

- **Saturation**: If S is a set of clauses and P is a set of ground terms, then $P(S)$, the saturation of S with respect to P, is the set of all ground clauses obtained by applying all possible consistent substitutions of ground terms in P with variables in S.

HERBRAND BASE

- **Herbrand base**: The saturation of a set S of clauses with respect to its Herbrand universe is called the Herbrand base of S, written as $H_S(S)$. For example, if S contains solely the clause just given, then $H_S(S)$ is the infinite set of clauses

  $$\begin{aligned}&\{\neg P(A, F(A, A)) \vee \neg Q(A, A) \vee R(A, B),\\ &\;\neg P(B, F(B, A)) \vee \neg Q(B, A) \vee R(B, B),\\ &\;\neg P(F(A, A), F(F(A, A), A)) \vee \neg Q(F(A, A), A) \vee R(F(A, A), B),\\ &\;\neg P(F(A, B), F(F(A, B), A)) \vee \neg Q(F(A, B), A) \vee R(F(A, B), B), \ldots\}\end{aligned}$$

HERBRAND'S THEOREM

These definitions allow us to state a form of **Herbrand's theorem** (Herbrand, 1930):

> If a set S of clauses is unsatisfiable, then there exists a finite subset of $H_S(S)$ that is also unsatisfiable.

Let S' be this finite subset of ground sentences. Now, we can appeal to the ground resolution theorem (page 255) to show that the **resolution closure** $RC(S')$ contains the empty clause. That is, running propositional resolution to completion on S' will derive a contradiction.

Now that we have established that there is always a resolution proof involving some finite subset of the Herbrand base of S, the next step is to show that there is a resolution

GÖDEL'S INCOMPLETENESS THEOREM

By slightly extending the language of first-order logic to allow for the **mathematical induction schema** in arithmetic, Kurt Gödel was able to show, in his **incompleteness theorem**, that there are true arithmetic sentences that cannot be proved.

The proof of the incompleteness theorem is somewhat beyond the scope of this book, occupying, as it does, at least 30 pages, but we can give a hint here. We begin with the logical theory of numbers. In this theory, there is a single constant, 0, and a single function, S (the successor function). In the intended model, $S(0)$ denotes 1, $S(S(0))$ denotes 2, and so on; the language therefore has names for all the natural numbers. The vocabulary also includes the function symbols $+$, $\times$, and $Expt$ (exponentiation) and the usual set of logical connectives and quantifiers. The first step is to notice that the set of sentences that we can write in this language can be enumerated. (Imagine defining an alphabetical order on the symbols and then arranging, in alphabetical order, each of the sets of sentences of length 1, 2, and so on.) We can then number each sentence α with a unique natural number $\#\alpha$ (the **Gödel number**). This is crucial: number theory contains a name for each of its own sentences. Similarly, we can number each possible proof P with a Gödel number $G(P)$, because a proof is simply a finite sequence of sentences.

Now suppose we have a recursively enumerable set A of sentences that are true statements about the natural numbers. Recalling that A can be named by a given set of integers, we can imagine writing in our language a sentence $\alpha(j, A)$ of the following sort:

> $\forall i \;\; i$ is not the Gödel number of a proof of the sentence whose Gödel number is j, where the proof uses only premises in A.

Then let σ be the sentence $\alpha(\#\sigma, A)$, that is, a sentence that states its own unprovability from A. (That this sentence always exists is true but not entirely obvious.)

Now we make the following ingenious argument: Suppose that σ *is* provable from A; then σ is false (because σ says it cannot be proved). But then we have a false sentence that is provable from A, so A cannot consist of only true sentences—a violation of our premise. Therefore, σ is *not* provable from A. But this is exactly what σ itself claims; hence σ is a true sentence.

So, we have shown (barring $29\frac{1}{2}$ pages) that for any set of true sentences of number theory, and in particular any set of basic axioms, there are other true sentences that *cannot* be proved from those axioms. This establishes, among other things, that we can never prove all the theorems of mathematics *within any given system of axioms*. Clearly, this was an important discovery for mathematics. Its significance for AI has been widely debated, beginning with speculations by Gödel himself. We take up the debate in Chapter 26.

proof using the clauses of S itself, which are not necessarily ground clauses. We start by considering a single application of the resolution rule. Robinson stated this lemma:

> Let C_1 and C_2 be two clauses with no shared variables, and let C_1' and C_2' be ground instances of C_1 and C_2. If C' is a resolvent of C_1' and C_2', then there exists a clause C such that (1) C is a resolvent of C_1 and C_2 and (2) C' is a ground instance of C.

LIFTING LEMMA

This is called a **lifting lemma**, because it lifts a proof step from ground clauses up to general first-order clauses. In order to prove his basic lifting lemma, Robinson had to invent unification and derive all of the properties of most general unifiers. Rather than repeat the proof here, we simply illustrate the lemma:

$$\begin{aligned}
C_1 &= \neg P(x, F(x, A)) \lor \neg Q(x, A) \lor R(x, B)\\
C_2 &= \neg N(G(y), z) \lor P(H(y), z)\\
C_1' &= \neg P(H(B), F(H(B), A)) \lor \neg Q(H(B), A) \lor R(H(B), B)\\
C_2' &= \neg N(G(B), F(H(B), A)) \lor P(H(B), F(H(B), A))\\
C' &= \neg N(G(B), F(H(B), A)) \lor \neg Q(H(B), A) \lor R(H(B), B)\\
C &= \neg N(G(y), F(H(y), A)) \lor \neg Q(H(y), A) \lor R(H(y), B)\ .
\end{aligned}$$

We see that indeed C' is a ground instance of C. In general, for C_1' and C_2' to have any resolvents, they must be constructed by first applying to C_1 and C_2 the most general unifier of a pair of complementary literals in C_1 and C_2. From the lifting lemma, it is easy to derive a similar statement about any sequence of applications of the resolution rule:

> For any clause C' in the resolution closure of S' there is a clause C in the resolution closure of S such that C' is a ground instance of C and the derivation of C is the same length as the derivation of C'.

From this fact, it follows that if the empty clause appears in the resolution closure of S', it must also appear in the resolution closure of S. This is because the empty clause cannot be a ground instance of any other clause. To recap: we have shown that if S is unsatisfiable, then there is a finite derivation of the empty clause using the resolution rule.

The lifting of theorem proving from ground clauses to first-order clauses provides a vast increase in power. This increase comes from the fact that the first-order proof need instantiate variables only as far as necessary for the proof, whereas the ground-clause methods were required to examine a huge number of arbitrary instantiations.

9.5.5 Equality

None of the inference methods described so far in this chapter handle an assertion of the form $x = y$. Three distinct approaches can be taken. The first approach is to axiomatize equality—to write down sentences about the equality relation in the knowledge base. We need to say that equality is reflexive, symmetric, and transitive, and we also have to say that we can substitute equals for equals in any predicate or function. So we need three basic axioms, and then one for each predicate and function:

$$\begin{array}{l}
\forall x \;\; x = x \\
\forall x, y \;\; x = y \;\Rightarrow\; y = x \\
\forall x, y, z \;\; x = y \land y = z \;\Rightarrow\; x = z \\
\\
\forall x, y \;\; x = y \;\Rightarrow\; (P_1(x) \;\Leftrightarrow\; P_1(y)) \\
\forall x, y \;\; x = y \;\Rightarrow\; (P_2(x) \;\Leftrightarrow\; P_2(y)) \\
\qquad\vdots \\
\forall w, x, y, z \;\; w = y \land x = z \;\Rightarrow\; (F_1(w, x) = F_1(y, z)) \\
\forall w, x, y, z \;\; w = y \land x = z \;\Rightarrow\; (F_2(w, x) = F_2(y, z)) \\
\qquad\vdots
\end{array}$$

Given these sentences, a standard inference procedure such as resolution can perform tasks requiring equality reasoning, such as solving mathematical equations. However, these axioms will generate a lot of conclusions, most of them not helpful to a proof. So there has been a search for more efficient ways of handling equality. One alternative is to add inference rules rather than axioms. The simplest rule, **demodulation**, takes a unit clause $x = y$ and some clause α that contains the term x, and yields a new clause formed by substituting y for x within α. It works if the term within α unifies with x; it need not be exactly equal to x. Note that demodulation is directional; given $x = y$, the x always gets replaced with y, never vice versa. That means that demodulation can be used for simplifying expressions using demodulators such as $x + 0 = x$ or $x^1 = x$. As another example, given

$$\begin{array}{l}
Father(Father(x)) = PaternalGrandfather(x) \\
Birthdate(Father(Father(Bella)), 1926)
\end{array}$$

we can conclude by demodulation

$$Birthdate(PaternalGrandfather(Bella), 1926)\ .$$

More formally, we have

DEMODULATION

- **Demodulation**: For any terms x, y, and z, where z appears somewhere in literal m_i and where $\textsc{Unify}(x, z) = \theta$,

$$\frac{x = y, \qquad m_1 \lor \cdots \lor m_n}{\textsc{Sub}(\textsc{Subst}(\theta, x), \textsc{Subst}(\theta, y), m_1 \lor \cdots \lor m_n)}\ .$$

  where $\textsc{Subst}$ is the usual substitution of a binding list, and $\textsc{Sub}(x, y, m)$ means to replace x with y everywhere that x occurs within m.

The rule can also be extended to handle non-unit clauses in which an equality literal appears:

PARAMODULATION

- **Paramodulation**: For any terms x, y, and z, where z appears somewhere in literal m_i, and where $\textsc{Unify}(x, z) = \theta$,

$$\frac{\ell_1 \lor \cdots \lor \ell_k \lor x = y, \qquad m_1 \lor \cdots \lor m_n}{\textsc{Sub}(\textsc{Subst}(\theta, x), \textsc{Subst}(\theta, y), \textsc{Subst}(\theta, \ell_1 \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_n)}\ .$$

For example, from

$$P(F(x, B), x) \lor Q(x) \qquad \text{and} \qquad F(A, y) = y \lor R(y)$$

we have $\theta = \text{UNIFY}(F(A, y), F(x, B)) = \{x/A, y/B\}$, and we can conclude by paramodulation the sentence

$$P(B, A) \lor Q(A) \lor R(B) .$$

Paramodulation yields a complete inference procedure for first-order logic with equality.

A third approach handles equality reasoning entirely within an extended unification algorithm. That is, terms are unifiable if they are *provably* equal under some substitution, where "provably" allows for equality reasoning. For example, the terms $1 + 2$ and $2 + 1$ normally are not unifiable, but a unification algorithm that knows that $x + y = y + x$ could unify them with the empty substitution. **Equational unification** of this kind can be done with efficient algorithms designed for the particular axioms used (commutativity, associativity, and so on) rather than through explicit inference with those axioms. Theorem provers using this technique are closely related to the CLP systems described in Section 9.4.

EQUATIONAL UNIFICATION

9.5.6 Resolution strategies

We know that repeated applications of the resolution inference rule will eventually find a proof if one exists. In this subsection, we examine strategies that help find proofs *efficiently*.

UNIT PREFERENCE

Unit preference: This strategy prefers to do resolutions where one of the sentences is a single literal (also known as a **unit clause**). The idea behind the strategy is that we are trying to produce an empty clause, so it might be a good idea to prefer inferences that produce shorter clauses. Resolving a unit sentence (such as P) with any other sentence (such as $\neg P \lor \neg Q \lor R$) always yields a clause (in this case, $\neg Q \lor R$) that is shorter than the other clause. When the unit preference strategy was first tried for propositional inference in 1964, it led to a dramatic speedup, making it feasible to prove theorems that could not be handled without the preference. **Unit resolution** is a restricted form of resolution in which every resolution step must involve a unit clause. Unit resolution is incomplete in general, but complete for Horn clauses. Unit resolution proofs on Horn clauses resemble forward chaining.

The OTTER theorem prover (Organized Techniques for Theorem-proving and Effective Research, McCune, 1992), uses a form of best-first search. Its heuristic function measures the "weight" of each clause, where lighter clauses are preferred. The exact choice of heuristic is up to the user, but generally, the weight of a clause should be correlated with its size or difficulty. Unit clauses are treated as light; the search can thus be seen as a generalization of the unit preference strategy.

SET OF SUPPORT

Set of support: Preferences that try certain resolutions first are helpful, but in general it is more effective to try to eliminate some potential resolutions altogether. For example, we can insist that every resolution step involve at least one element of a special set of clauses—the *set of support*. The resolvent is then added into the set of support. If the set of support is small relative to the whole knowledge base, the search space will be reduced dramatically.

We have to be careful with this approach because a bad choice for the set of support will make the algorithm incomplete. However, if we choose the set of support S so that the remainder of the sentences are jointly satisfiable, then set-of-support resolution is complete. For example, one can use the negated query as the set of support, on the assumption that the original knowledge base is consistent. (After all, if it is not consistent, then the fact that the query follows from it is vacuous.) The set-of-support strategy has the additional advantage of generating goal-directed proof trees that are often easy for humans to understand.

INPUT RESOLUTION

Input resolution: In this strategy, every resolution combines one of the input sentences (from the KB or the query) with some other sentence. The proof in Figure 9.11 on page 348 uses only input resolutions and has the characteristic shape of a single "spine" with single sentences combining onto the spine. Clearly, the space of proof trees of this shape is smaller than the space of all proof graphs. In Horn knowledge bases, Modus Ponens is a kind of input resolution strategy, because it combines an implication from the original KB with some other sentences. Thus, it is no surprise that input resolution is complete for knowledge bases that are in Horn form, but incomplete in the general case. The **linear resolution** strategy is a slight generalization that allows P and Q to be resolved together either if P is in the original KB or if P is an ancestor of Q in the proof tree. Linear resolution is complete.

LINEAR RESOLUTION

SUBSUMPTION

Subsumption: The subsumption method eliminates all sentences that are subsumed by (that is, more specific than) an existing sentence in the KB. For example, if $P(x)$ is in the KB, then there is no sense in adding $P(A)$ and even less sense in adding $P(A) \lor Q(B)$. Subsumption helps keep the KB small and thus helps keep the search space small.

Practical uses of resolution theorem provers

SYNTHESIS

VERIFICATION

Theorem provers can be applied to the problems involved in the **synthesis** and **verification** of both hardware and software. Thus, theorem-proving research is carried out in the fields of hardware design, programming languages, and software engineering—not just in AI.

In the case of hardware, the axioms describe the interactions between signals and circuit elements. (See Section 8.4.2 on page 309 for an example.) Logical reasoners designed specially for verification have been able to verify entire CPUs, including their timing properties (Srivas and Bickford, 1990). The AURA theorem prover has been applied to design circuits that are more compact than any previous design (Wojciechowski and Wojcik, 1983).

In the case of software, reasoning about programs is quite similar to reasoning about actions, as in Chapter 7: axioms describe the preconditions and effects of each statement. The formal synthesis of algorithms was one of the first uses of theorem provers, as outlined by Cordell Green (1969a), who built on earlier ideas by Herbert Simon (1963). The idea is to constructively prove a theorem to the effect that "there exists a program p satisfying a certain specification." Although fully automated **deductive synthesis**, as it is called, has not yet become feasible for general-purpose programming, hand-guided deductive synthesis has been successful in designing several novel and sophisticated algorithms. Synthesis of special-purpose programs, such as scientific computing code, is also an active area of research.

DEDUCTIVE SYNTHESIS

Similar techniques are now being applied to software verification by systems such as the SPIN model checker (Holzmann, 1997). For example, the Remote Agent spacecraft control program was verified before and after flight (Havelund *et al.*, 2000). The RSA public key encryption algorithm and the Boyer–Moore string-matching algorithm have been verified this way (Boyer and Moore, 1984).

10 CLASSICAL PLANNING

In which we see how an agent can take advantage of the structure of a problem to construct complex plans of action.

We have defined AI as the study of rational action, which means that **planning**—devising a plan of action to achieve one's goals—is a critical part of AI. We have seen two examples of planning agents so far: the search-based problem-solving agent of Chapter 3 and the hybrid logical agent of Chapter 7. In this chapter we introduce a representation for planning problems that scales up to problems that could not be handled by those earlier approaches.

Section 10.1 develops an expressive yet carefully constrained language for representing planning problems. Section 10.2 shows how forward and backward search algorithms can take advantage of this representation, primarily through accurate heuristics that can be derived automatically from the structure of the representation. (This is analogous to the way in which effective domain-independent heuristics were constructed for constraint satisfaction problems in Chapter 6.) Section 10.3 shows how a data structure called the planning graph can make the search for a plan more efficient. We then describe a few of the other approaches to planning, and conclude by comparing the various approaches.

This chapter covers fully observable, deterministic, static environments with single agents. Chapters 11 and 17 cover partially observable, stochastic, dynamic environments with multiple agents.

10.1 DEFINITION OF CLASSICAL PLANNING

The problem-solving agent of Chapter 3 can find sequences of actions that result in a goal state. But it deals with atomic representations of states and thus needs good domain-specific heuristics to perform well. The hybrid propositional logical agent of Chapter 7 can find plans without domain-specific heuristics because it uses domain-independent heuristics based on the logical structure of the problem. But it relies on ground (variable-free) propositional inference, which means that it may be swamped when there are many actions and states. For example, in the wumpus world, the simple action of moving a step forward had to be repeated for all four agent orientations, T time steps, and n^2 current locations.

In response to this, planning researchers have settled on a **factored representation**—one in which a state of the world is represented by a collection of variables. We use a language called **PDDL**, the Planning Domain Definition Language, that allows us to express all $4Tn^2$ actions with one action schema. There have been several versions of PDDL; we select a simple version and alter its syntax to be consistent with the rest of the book.[1] We now show how PDDL describes the four things we need to define a search problem: the initial state, the actions that are available in a state, the result of applying an action, and the goal test.

PDDL

Each **state** is represented as a conjunction of fluents that are ground, functionless atoms. For example, $Poor \wedge Unknown$ might represent the state of a hapless agent, and a state in a package delivery problem might be $At(Truck_1, Melbourne) \wedge At(Truck_2, Sydney)$. **Database semantics** is used: the closed-world assumption means that any fluents that are not mentioned are false, and the unique names assumption means that $Truck_1$ and $Truck_2$ are distinct. The following fluents are *not* allowed in a state: $At(x, y)$ (because it is non-ground), $\neg Poor$ (because it is a negation), and $At(Father(Fred), Sydney)$ (because it uses a function symbol). The representation of states is carefully designed so that a state can be treated either as a conjunction of fluents, which can be manipulated by logical inference, or as a *set* of fluents, which can be manipulated with set operations. The **set semantics** is sometimes easier to deal with.

SET SEMANTICS

Actions are described by a set of action schemas that implicitly define the $\text{ACTIONS}(s)$ and $\text{RESULT}(s, a)$ functions needed to do a problem-solving search. We saw in Chapter 7 that any system for action description needs to solve the frame problem—to say what changes and what stays the same as the result of the action. Classical planning concentrates on problems where most actions leave most things unchanged. Think of a world consisting of a bunch of objects on a flat surface. The action of nudging an object causes that object to change its location by a vector Δ. A concise description of the action should mention only Δ; it shouldn't have to mention all the objects that stay in place. PDDL does that by specifying the result of an action in terms of what changes; everything that stays the same is left unmentioned.

ACTION SCHEMA

A set of ground (variable-free) actions can be represented by a single **action schema**. The schema is a **lifted** representation—it lifts the level of reasoning from propositional logic to a restricted subset of first-order logic. For example, here is an action schema for flying a plane from one location to another:

$$\begin{array}{l} Action(Fly(p, from, to), \\ \quad \text{PRECOND:} At(p, from) \wedge Plane(p) \wedge Airport(from) \wedge Airport(to) \\ \quad \text{EFFECT:} \neg At(p, from) \wedge At(p, to)) \end{array}$$

PRECONDITION

EFFECT

The schema consists of the action name, a list of all the variables used in the schema, a **precondition** and an **effect**. Although we haven't said yet how the action schema converts into logical sentences, think of the variables as being universally quantified. We are free to choose whatever values we want to instantiate the variables. For example, here is one ground

[1] PDDL was derived from the original STRIPS planning language(Fikes and Nilsson, 1971). which is slightly more restricted than PDDL: STRIPS preconditions and goals cannot contain negative literals.

action that results from substituting values for all the variables:

$$\begin{aligned}&Action(Fly(P_1, SFO, JFK),\\&\quad \text{PRECOND:} At(P_1, SFO) \land Plane(P_1) \land Airport(SFO) \land Airport(JFK)\\&\quad \text{EFFECT:} \neg At(P_1, SFO) \land At(P_1, JFK))\end{aligned}$$

The precondition and effect of an action are each conjunctions of literals (positive or negated atomic sentences). The precondition defines the states in which the action can be executed, and the effect defines the result of executing the action. An action a can be executed in state s if s entails the precondition of a. Entailment can also be expressed with the set semantics: $s \models q$ iff every positive literal in q is in s and every negated literal in q is not. In formal notation we say

$$(a \in \text{ACTIONS}(s)) \;\Leftrightarrow\; s \models \text{PRECOND}(a)\,,$$

where any variables in a are universally quantified. For example,

$$\begin{aligned}&\forall\, p, from, to \;\; (Fly(p, from, to) \in \text{ACTIONS}(s)) \;\Leftrightarrow\\&\quad s \models (At(p, from) \land Plane(p) \land Airport(from) \land Airport(to))\end{aligned}$$

APPLICABLE

We say that action a is **applicable** in state s if the preconditions are satisfied by s. When an action schema a contains variables, it may have multiple applicable instantiations. For example, with the initial state defined in Figure 10.1, the Fly action can be instantiated as $Fly(P_1, SFO, JFK)$ or as $Fly(P_2, JFK, SFO)$, both of which are applicable in the initial state. If an action a has v variables, then, in a domain with k unique names of objects, it takes $O(v^k)$ time in the worst case to find the applicable ground actions.

PROPOSITIONALIZE

Sometimes we want to **propositionalize** a PDDL problem—replace each action schema with a set of ground actions and then use a propositional solver such as SATPLAN to find a solution. However, this is impractical when v and k are large.

DELETE LIST

ADD LIST

The **result** of executing action a in state s is defined as a state s' which is represented by the set of fluents formed by starting with s, removing the fluents that appear as negative literals in the action's effects (what we call the **delete list** or $\text{DEL}(a)$), and adding the fluents that are positive literals in the action's effects (what we call the **add list** or $\text{ADD}(a)$):

$$\text{RESULT}(s, a) = (s - \text{DEL}(a)) \cup \text{ADD}(a)\,. \tag{10.1}$$

For example, with the action $Fly(P_1, SFO, JFK)$, we would remove $At(P_1, SFO)$ and add $At(P_1, JFK)$. It is a requirement of action schemas that any variable in the effect must also appear in the precondition. That way, when the precondition is matched against the state s, all the variables will be bound, and $\text{RESULT}(s, a)$ will therefore have only ground atoms. In other words, ground states are closed under the RESULT operation.

Also note that the fluents do not explicitly refer to time, as they did in Chapter 7. There we needed superscripts for time, and successor-state axioms of the form

$$F^{t+1} \;\Leftrightarrow\; ActionCausesF^t \lor (F^t \land \neg ActionCausesNotF^t)\,.$$

In PDDL the times and states are implicit in the action schemas: the precondition always refers to time t and the effect to time $t + 1$.

A set of action schemas serves as a definition of a planning *domain*. A specific *problem* within the domain is defined with the addition of an initial state and a goal. The **initial**

$$
\begin{array}{l}
\mathit{Init}(\mathit{At}(C_1,\ \mathit{SFO}) \wedge \mathit{At}(C_2,\ \mathit{JFK}) \wedge \mathit{At}(P_1,\ \mathit{SFO}) \wedge \mathit{At}(P_2,\ \mathit{JFK}) \\
\quad \wedge\ \mathit{Cargo}(C_1) \wedge \mathit{Cargo}(C_2) \wedge \mathit{Plane}(P_1) \wedge \mathit{Plane}(P_2) \\
\quad \wedge\ \mathit{Airport}(\mathit{JFK}) \wedge \mathit{Airport}(\mathit{SFO})) \\
\mathit{Goal}(\mathit{At}(C_1,\ \mathit{JFK}) \wedge \mathit{At}(C_2,\ \mathit{SFO})) \\
\mathit{Action}(\mathit{Load}(c,\ p,\ a), \\
\quad \text{PRECOND: } \mathit{At}(c,\ a) \wedge \mathit{At}(p,\ a) \wedge \mathit{Cargo}(c) \wedge \mathit{Plane}(p) \wedge \mathit{Airport}(a) \\
\quad \text{EFFECT: } \neg\, \mathit{At}(c,\ a) \wedge \mathit{In}(c,\ p)) \\
\mathit{Action}(\mathit{Unload}(c,\ p,\ a), \\
\quad \text{PRECOND: } \mathit{In}(c,\ p) \wedge \mathit{At}(p,\ a) \wedge \mathit{Cargo}(c) \wedge \mathit{Plane}(p) \wedge \mathit{Airport}(a) \\
\quad \text{EFFECT: } \mathit{At}(c,\ a) \wedge \neg\, \mathit{In}(c,\ p)) \\
\mathit{Action}(\mathit{Fly}(p,\ \mathit{from},\ \mathit{to}), \\
\quad \text{PRECOND: } \mathit{At}(p,\ \mathit{from}) \wedge \mathit{Plane}(p) \wedge \mathit{Airport}(\mathit{from}) \wedge \mathit{Airport}(\mathit{to}) \\
\quad \text{EFFECT: } \neg\, \mathit{At}(p,\ \mathit{from}) \wedge \mathit{At}(p,\ \mathit{to}))
\end{array}
$$

Figure 10.1 A PDDL description of an air cargo transportation planning problem.

INITIAL STATE

GOAL

state is a conjunction of ground atoms. (As with all states, the closed-world assumption is used, which means that any atoms that are not mentioned are false.) The **goal** is just like a precondition: a conjunction of literals (positive or negative) that may contain variables, such as $At(p, SFO) \wedge Plane(p)$. Any variables are treated as existentially quantified, so this goal is to have *any* plane at SFO. The problem is solved when we can find a sequence of actions that end in a state s that entails the goal. For example, the state $Rich \wedge Famous \wedge Miserable$ entails the goal $Rich \wedge Famous$, and the state $Plane(Plane_1) \wedge At(Plane_1, SFO)$ entails the goal $At(p, SFO) \wedge Plane(p)$.

Now we have defined planning as a search problem: we have an initial state, an ACTIONS function, a RESULT function, and a goal test. We'll look at some example problems before investigating efficient search algorithms.

10.1.1 Example: Air cargo transport

Figure 10.1 shows an air cargo transport problem involving loading and unloading cargo and flying it from place to place. The problem can be defined with three actions: $Load$, $Unload$, and Fly. The actions affect two predicates: $In(c, p)$ means that cargo c is inside plane p, and $At(x, a)$ means that object x (either plane or cargo) is at airport a. Note that some care must be taken to make sure the At predicates are maintained properly. When a plane flies from one airport to another, all the cargo inside the plane goes with it. In first-order logic it would be easy to quantify over all objects that are inside the plane. But basic PDDL does not have a universal quantifier, so we need a different solution. The approach we use is to say that a piece of cargo ceases to be At anywhere when it is In a plane; the cargo only becomes At the new airport when it is unloaded. So At really means "available for use at a given location." The following plan is a solution to the problem:

$$
\begin{array}{l}
[\mathit{Load}(C_1, P_1, \mathit{SFO}), \mathit{Fly}(P_1, \mathit{SFO}, \mathit{JFK}), \mathit{Unload}(C_1, P_1, \mathit{JFK}), \\
\ \mathit{Load}(C_2, P_2, \mathit{JFK}), \mathit{Fly}(P_2, \mathit{JFK}, \mathit{SFO}), \mathit{Unload}(C_2, P_2, \mathit{SFO})]\ .
\end{array}
$$

Finally, there is the problem of spurious actions such as $Fly(P_1, JFK, JFK)$, which should be a no-op, but which has contradictory effects (according to the definition, the effect would include $At(P_1, JFK) \land \neg At(P_1, JFK)$). It is common to ignore such problems, because they seldom cause incorrect plans to be produced. The correct approach is to add inequality preconditions saying that the *from* and *to* airports must be different; see another example of this in Figure 10.3.

10.1.2 Example: The spare tire problem

Consider the problem of changing a flat tire (Figure 10.2). The goal is to have a good spare tire properly mounted onto the car's axle, where the initial state has a flat tire on the axle and a good spare tire in the trunk. To keep it simple, our version of the problem is an abstract one, with no sticky lug nuts or other complications. There are just four actions: removing the spare from the trunk, removing the flat tire from the axle, putting the spare on the axle, and leaving the car unattended overnight. We assume that the car is parked in a particularly bad neighborhood, so that the effect of leaving it overnight is that the tires disappear. A solution to the problem is $[Remove(Flat, Axle), Remove(Spare, Trunk), PutOn(Spare, Axle)]$.

$Init(Tire(Flat) \land Tire(Spare) \land At(Flat, Axle) \land At(Spare, Trunk))$
$Goal(At(Spare, Axle))$
$Action(Remove(obj, loc),$
  PRECOND: $At(obj, loc)$
  EFFECT: $\neg At(obj, loc) \land At(obj, Ground))$
$Action(PutOn(t, Axle),$
  PRECOND: $Tire(t) \land At(t, Ground) \land \neg At(Flat, Axle)$
  EFFECT: $\neg At(t, Ground) \land At(t, Axle))$
$Action(LeaveOvernight,$
  PRECOND:
  EFFECT: $\neg At(Spare, Ground) \land \neg At(Spare, Axle) \land \neg At(Spare, Trunk)$
    $\land \neg At(Flat, Ground) \land \neg At(Flat, Axle) \land \neg At(Flat, Trunk))$

Figure 10.2 The simple spare tire problem.

10.1.3 Example: The blocks world

BLOCKS WORLD

One of the most famous planning domains is known as the **blocks world**. This domain consists of a set of cube-shaped blocks sitting on a table.[2] The blocks can be stacked, but only one block can fit directly on top of another. A robot arm can pick up a block and move it to another position, either on the table or on top of another block. The arm can pick up only one block at a time, so it cannot pick up a block that has another one on it. The goal will always be to build one or more stacks of blocks, specified in terms of what blocks are on top

[2] The blocks world used in planning research is much simpler than SHRDLU's version, shown on page 20.

$$
\begin{array}{l}
\mathit{Init}(\mathit{On}(A, \mathit{Table}) \wedge \mathit{On}(B, \mathit{Table}) \wedge \mathit{On}(C, A) \\
\quad \wedge\ \mathit{Block}(A) \wedge \mathit{Block}(B) \wedge \mathit{Block}(C) \wedge \mathit{Clear}(B) \wedge \mathit{Clear}(C)) \\
\mathit{Goal}(\mathit{On}(A, B) \wedge \mathit{On}(B, C)) \\
\mathit{Action}(\mathit{Move}(b, x, y), \\
\quad \textsc{Precond}: \mathit{On}(b, x) \wedge \mathit{Clear}(b) \wedge \mathit{Clear}(y) \wedge \mathit{Block}(b) \wedge \mathit{Block}(y) \wedge \\
\qquad\qquad (b \neq x) \wedge (b \neq y) \wedge (x \neq y), \\
\quad \textsc{Effect}: \mathit{On}(b, y) \wedge \mathit{Clear}(x) \wedge \neg \mathit{On}(b, x) \wedge \neg \mathit{Clear}(y)) \\
\mathit{Action}(\mathit{MoveToTable}(b, x), \\
\quad \textsc{Precond}: \mathit{On}(b, x) \wedge \mathit{Clear}(b) \wedge \mathit{Block}(b) \wedge (b \neq x), \\
\quad \textsc{Effect}: \mathit{On}(b, \mathit{Table}) \wedge \mathit{Clear}(x) \wedge \neg \mathit{On}(b, x))
\end{array}
$$

Figure 10.3 A planning problem in the blocks world: building a three-block tower. One solution is the sequence $[\mathit{MoveToTable}(C, A), \mathit{Move}(B, \mathit{Table}, C), \mathit{Move}(A, \mathit{Table}, B)]$.

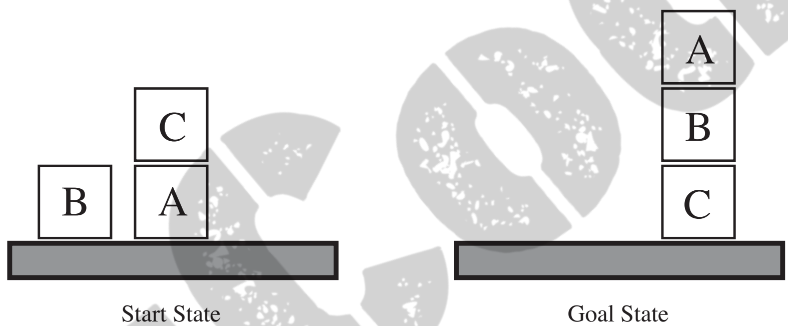

Figure 10.4 Diagram of the blocks-world problem in Figure 10.3.

of what other blocks. For example, a goal might be to get block A on B and block B on C (see Figure 10.4).

We use $\mathit{On}(b, x)$ to indicate that block b is on x, where x is either another block or the table. The action for moving block b from the top of x to the top of y will be $\mathit{Move}(b, x, y)$. Now, one of the preconditions on moving b is that no other block be on it. In first-order logic, this would be $\neg \exists\, x\ \mathit{On}(x, b)$ or, alternatively, $\forall\, x\ \neg \mathit{On}(x, b)$. Basic PDDL does not allow quantifiers, so instead we introduce a predicate $\mathit{Clear}(x)$ that is true when nothing is on x. (The complete problem description is in Figure 10.3.)

The action Move moves a block b from x to y if both b and y are clear. After the move is made, b is still clear but y is not. A first attempt at the Move schema is

$$
\begin{array}{l}
\mathit{Action}(\mathit{Move}(b, x, y), \\
\quad \textsc{Precond}: \mathit{On}(b, x) \wedge \mathit{Clear}(b) \wedge \mathit{Clear}(y), \\
\quad \textsc{Effect}: \mathit{On}(b, y) \wedge \mathit{Clear}(x) \wedge \neg \mathit{On}(b, x) \wedge \neg \mathit{Clear}(y)) \,.
\end{array}
$$

Unfortunately, this does not maintain Clear properly when x or y is the table. When x is the Table, this action has the effect $\mathit{Clear}(\mathit{Table})$, but the table should not become clear; and when $y = \mathit{Table}$, it has the precondition $\mathit{Clear}(\mathit{Table})$, but the table does not have to be clear for us to move a block onto it. To fix this, we do two things. First, we introduce another action to move a block b from x to the table:

$$\begin{aligned}&Action(MoveToTable(b, x),\\&\quad \text{PRECOND:} On(b, x) \wedge Clear(b),\\&\quad \text{EFFECT:} On(b, Table) \wedge Clear(x) \wedge \neg On(b, x))\ .\end{aligned}$$

Second, we take the interpretation of $Clear(x)$ to be "there is a clear space on x to hold a block." Under this interpretation, $Clear(Table)$ will always be true. The only problem is that nothing prevents the planner from using $Move(b, x, Table)$ instead of $MoveToTable(b, x)$. We could live with this problem—it will lead to a larger-than-necessary search space, but will not lead to incorrect answers—or we could introduce the predicate $Block$ and add $Block(b) \wedge Block(y)$ to the precondition of $Move$.

10.1.4 The complexity of classical planning

PLANSAT
BOUNDED PLANSAT

In this subsection we consider the theoretical complexity of planning and distinguish two decision problems. **PlanSAT** is the question of whether there exists any plan that solves a planning problem. **Bounded PlanSAT** asks whether there is a solution of length k or less; this can be used to find an optimal plan.

The first result is that both decision problems are decidable for classical planning. The proof follows from the fact that the number of states is finite. But if we add function symbols to the language, then the number of states becomes infinite, and PlanSAT becomes only semidecidable: an algorithm exists that will terminate with the correct answer for any solvable problem, but may not terminate on unsolvable problems. The Bounded PlanSAT problem remains decidable even in the presence of function symbols. For proofs of the assertions in this section, see Ghallab *et al.* (2004).

Both PlanSAT and Bounded PlanSAT are in the complexity class PSPACE, a class that is larger (and hence more difficult) than NP and refers to problems that can be solved by a deterministic Turing machine with a polynomial amount of space. Even if we make some rather severe restrictions, the problems remain quite difficult. For example, if we disallow negative effects, both problems are still NP-hard. However, if we also disallow negative preconditions, PlanSAT reduces to the class P.

These worst-case results may seem discouraging. We can take solace in the fact that agents are usually not asked to find plans for arbitrary worst-case problem instances, but rather are asked for plans in specific domains (such as blocks-world problems with n blocks), which can be much easier than the theoretical worst case. For many domains (including the blocks world and the air cargo world), Bounded PlanSAT is NP-complete while PlanSAT is in P; in other words, optimal planning is usually hard, but sub-optimal planning is sometimes easy. To do well on easier-than-worst-case problems, we will need good search heuristics. That's the true advantage of the classical planning formalism: it has facilitated the development of very accurate domain-independent heuristics, whereas systems based on successor-state axioms in first-order logic have had less success in coming up with good heuristics.

10.2 ALGORITHMS FOR PLANNING AS STATE-SPACE SEARCH

Now we turn our attention to planning algorithms. We saw how the description of a planning problem defines a search problem: we can search from the initial state through the space of states, looking for a goal. One of the nice advantages of the declarative representation of action schemas is that we can also search backward from the goal, looking for the initial state. Figure 10.5 compares forward and backward searches.

10.2.1 Forward (progression) state-space search

Now that we have shown how a planning problem maps into a search problem, we can solve planning problems with any of the heuristic search algorithms from Chapter 3 or a local search algorithm from Chapter 4 (provided we keep track of the actions used to reach the goal). From the earliest days of planning research (around 1961) until around 1998 it was assumed that forward state-space search was too inefficient to be practical. It is not hard to come up with reasons why.

First, forward search is prone to exploring irrelevant actions. Consider the noble task of buying a copy of *AI: A Modern Approach* from an online bookseller. Suppose there is an

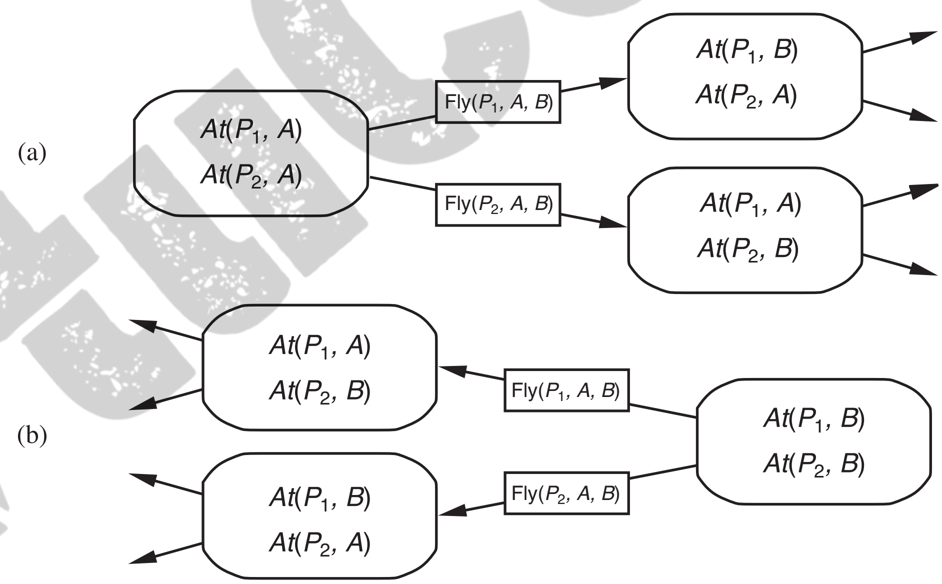

Figure 10.5 Two approaches to searching for a plan. (a) Forward (progression) search through the space of states, starting in the initial state and using the problem's actions to search forward for a member of the set of goal states. (b) Backward (regression) search through sets of relevant states, starting at the set of states representing the goal and using the inverse of the actions to search backward for the initial state.

action schema $Buy(isbn)$ with effect $Own(isbn)$. ISBNs are 10 digits, so this action schema represents 10 billion ground actions. An uninformed forward-search algorithm would have to start enumerating these 10 billion actions to find one that leads to the goal.

Second, planning problems often have large state spaces. Consider an air cargo problem with 10 airports, where each airport has 5 planes and 20 pieces of cargo. The goal is to move all the cargo at airport A to airport B. There is a simple solution to the problem: load the 20 pieces of cargo into one of the planes at A, fly the plane to B, and unload the cargo. Finding the solution can be difficult because the average branching factor is huge: each of the 50 planes can fly to 9 other airports, and each of the 200 packages can be either unloaded (if it is loaded) or loaded into any plane at its airport (if it is unloaded). So in any state there is a minimum of 450 actions (when all the packages are at airports with no planes) and a maximum of 10,450 (when all packages and planes are at the same airport). On average, let's say there are about 2000 possible actions per state, so the search graph up to the depth of the obvious solution has about 2000^{41} nodes.

Clearly, even this relatively small problem instance is hopeless without an accurate heuristic. Although many real-world applications of planning have relied on domain-specific heuristics, it turns out (as we see in Section 10.2.3) that strong domain-independent heuristics can be derived automatically; that is what makes forward search feasible.

10.2.2 Backward (regression) relevant-states search

RELEVANT-STATES

In regression search we start at the goal and apply the actions backward until we find a sequence of steps that reaches the initial state. It is called **relevant-states** search because we only consider actions that are relevant to the goal (or current state). As in belief-state search (Section 4.4), there is a *set* of relevant states to consider at each step, not just a single state.

We start with the goal, which is a conjunction of literals forming a description of a set of states—for example, the goal $\neg Poor \wedge Famous$ describes those states in which $Poor$ is false, $Famous$ is true, and any other fluent can have any value. If there are n ground fluents in a domain, then there are 2^n ground states (each fluent can be true or false), but 3^n descriptions of sets of goal states (each fluent can be positive, negative, or not mentioned).

In general, backward search works only when we know how to regress from a state description to the predecessor state description. For example, it is hard to search backwards for a solution to the n-queens problem because there is no easy way to describe the states that are one move away from the goal. Happily, the PDDL representation was designed to make it easy to regress actions—if a domain can be expressed in PDDL, then we can do regression search on it. Given a ground goal description g and a ground action a, the regression from g over a gives us a state description g' defined by

$$g' = (g - \text{ADD}(a)) \cup Precond(a) \ .$$

That is, the effects that were added by the action need not have been true before, and also the preconditions must have held before, or else the action could not have been executed. Note that $\text{DEL}(a)$ does not appear in the formula; that's because while we know the fluents in $\text{DEL}(a)$ are no longer true after the action, we don't know whether or not they were true before, so there's nothing to be said about them.

To get the full advantage of backward search, we need to deal with partially uninstantiated actions and states, not just ground ones. For example, suppose the goal is to deliver a specific piece of cargo to SFO: $At(C_2, SFO)$. That suggests the action $Unload(C_2, p', SFO)$:

$$\begin{array}{l} Action(Unload(C_2, p', SFO), \\ \quad \textsc{Precond}{:}\, In(C_2, p') \wedge At(p', SFO) \wedge Cargo(C_2) \wedge Plane(p') \wedge Airport(SFO) \\ \quad \textsc{Effect}{:}\, At(C_2, SFO) \wedge \neg In(C_2, p') \,. \end{array}$$

(Note that we have **standardized** variable names (changing p to p' in this case) so that there will be no confusion between variable names if we happen to use the same action schema twice in a plan. The same approach was used in Chapter 9 for first-order logical inference.) This represents unloading the package from an *unspecified* plane at SFO; any plane will do, but we need not say which one now. We can take advantage of the power of first-order representations: a single description summarizes the possibility of using *any* of the planes by implicitly quantifying over p'. The regressed state description is

$$g' = In(C_2, p') \wedge At(p', SFO) \wedge Cargo(C_2) \wedge Plane(p') \wedge Airport(SFO) \,.$$

RELEVANCE

The final issue is deciding which actions are candidates to regress over. In the forward direction we chose actions that were **applicable**—those actions that could be the next step in the plan. In backward search we want actions that are **relevant**—those actions that could be the *last* step in a plan leading up to the current goal state.

For an action to be relevant to a goal it obviously must contribute to the goal: at least one of the action's effects (either positive or negative) must unify with an element of the goal. What is less obvious is that the action must not have any effect (positive or negative) that negates an element of the goal. Now, if the goal is $A \wedge B \wedge C$ and an action has the effect $A \wedge B \wedge \neg C$ then there is a colloquial sense in which that action is very relevant to the goal—it gets us two-thirds of the way there. But it is not relevant in the technical sense defined here, because this action could not be the *final* step of a solution—we would always need at least one more step to achieve C.

Given the goal $At(C_2, SFO)$, several instantiations of $Unload$ are relevant: we could chose any specific plane to unload from, or we could leave the plane unspecified by using the action $Unload(C_2, p', SFO)$. We can reduce the branching factor without ruling out any solutions by always using the action formed by substituting the most general unifier into the (standardized) action schema.

As another example, consider the goal $Own(0136042597)$, given an initial state with 10 billion ISBNs, and the single action schema

$$A = Action(Buy(i), \textsc{Precond}{:}\, ISBN(i), \textsc{Effect}{:}\, Own(i)) \,.$$

As we mentioned before, forward search without a heuristic would have to start enumerating the 10 billion ground Buy actions. But with backward search, we would unify the goal $Own(0136042597)$ with the (standardized) effect $Own(i')$, yielding the substitution $\theta = \{i'/0136042597\}$. Then we would regress over the action $Subst(\theta, A')$ to yield the predecessor state description $ISBN(0136042597)$. This is part of, and thus entailed by, the initial state, so we are done.

We can make this more formal. Assume a goal description g which contains a goal literal g_i and an action schema A that is standardized to produce A'. If A' has an effect literal e'_j where $Unify(g_i, e'_j) = \theta$ and where we define $a' = \text{SUBST}(\theta, A')$ and if there is no effect in a' that is the negation of a literal in g, then a' is a relevant action towards g.

Backward search keeps the branching factor lower than forward search, for most problem domains. However, the fact that backward search uses state sets rather than individual states makes it harder to come up with good heuristics. That is the main reason why the majority of current systems favor forward search.

10.2.3 Heuristics for planning

Neither forward nor backward search is efficient without a good heuristic function. Recall from Chapter 3 that a heuristic function $h(s)$ estimates the distance from a state s to the goal and that if we can derive an **admissible** heuristic for this distance—one that does not overestimate—then we can use A* search to find optimal solutions. An admissible heuristic can be derived by defining a **relaxed problem** that is easier to solve. The exact cost of a solution to this easier problem then becomes the heuristic for the original problem.

By definition, there is no way to analyze an atomic state, and thus it it requires some ingenuity by a human analyst to define good domain-specific heuristics for search problems with atomic states. Planning uses a factored representation for states and action schemas. That makes it possible to define good domain-independent heuristics and for programs to automatically apply a good domain-independent heuristic for a given problem.

Think of a search problem as a graph where the nodes are states and the edges are actions. The problem is to find a path connecting the initial state to a goal state. There are two ways we can relax this problem to make it easier: by adding more edges to the graph, making it strictly easier to find a path, or by grouping multiple nodes together, forming an abstraction of the state space that has fewer states, and thus is easier to search.

IGNORE PRECONDITIONS HEURISTIC

We look first at heuristics that add edges to the graph. For example, the **ignore preconditions heuristic** drops all preconditions from actions. Every action becomes applicable in every state, and any single goal fluent can be achieved in one step (if there is an applicable action—if not, the problem is impossible). This almost implies that the number of steps required to solve the relaxed problem is the number of unsatisfied goals—almost but not quite, because (1) some action may achieve multiple goals and (2) some actions may undo the effects of others. For many problems an accurate heuristic is obtained by considering (1) and ignoring (2). First, we relax the actions by removing all preconditions and all effects except those that are literals in the goal. Then, we count the minimum number of actions required such that the union of those actions' effects satisfies the goal. This is an instance of the **set-cover problem**. There is one minor irritation: the set-cover problem is NP-hard. Fortunately a simple greedy algorithm is guaranteed to return a set covering whose size is within a factor of $\log n$ of the true minimum covering, where n is the number of literals in the goal. Unfortunately, the greedy algorithm loses the guarantee of admissibility.

SET-COVER PROBLEM

It is also possible to ignore only *selected* preconditions of actions. Consider the sliding-block puzzle (8-puzzle or 15-puzzle) from Section 3.2. We could encode this as a planning problem involving tiles with a single schema *Slide*:

$$
\begin{aligned}
&Action(Slide(t, s_1, s_2),\\
&\quad \text{PRECOND:}\, On(t, s_1) \wedge Tile(t) \wedge Blank(s_2) \wedge Adjacent(s_1, s_2)\\
&\quad \text{EFFECT:}\, On(t, s_2) \wedge Blank(s_1) \wedge \neg On(t, s_1) \wedge \neg Blank(s_2))
\end{aligned}
$$

As we saw in Section 3.6, if we remove the preconditions $Blank(s_2) \wedge Adjacent(s_1, s_2)$ then any tile can move in one action to any space and we get the number-of-misplaced-tiles heuristic. If we remove $Blank(s_2)$ then we get the Manhattan-distance heuristic. It is easy to see how these heuristics could be derived automatically from the action schema description. The ease of manipulating the schemas is the great advantage of the factored representation of planning problems, as compared with the atomic representation of search problems.

IGNORE DELETE LISTS

Another possibility is the **ignore delete lists** heuristic. Assume for a moment that all goals and preconditions contain only positive literals[3] We want to create a relaxed version of the original problem that will be easier to solve, and where the length of the solution will serve as a good heuristic. We can do that by removing the delete lists from all actions (i.e., removing all negative literals from effects). That makes it possible to make monotonic progress towards the goal—no action will ever undo progress made by another action. It turns out it is still NP-hard to find the optimal solution to this relaxed problem, but an approximate solution can be found in polynomial time by hill-climbing. Figure 10.6 diagrams part of the state space for two planning problems using the ignore-delete-lists heuristic. The dots represent states and the edges actions, and the height of each dot above the bottom plane represents the heuristic value. States on the bottom plane are solutions. In both these problems, there is a wide path to the goal. There are no dead ends, so no need for backtracking; a simple hillclimbing search will easily find a solution to these problems (although it may not be an optimal solution).

STATE ABSTRACTION

The relaxed problems leave us with a simplified—but still expensive—planning problem just to calculate the value of the heuristic function. Many planning problems have 10^{100} states or more, and relaxing the *actions* does nothing to reduce the number of states. Therefore, we now look at relaxations that decrease the number of states by forming a **state abstraction**—a many-to-one mapping from states in the ground representation of the problem to the abstract representation.

The easiest form of state abstraction is to ignore some fluents. For example, consider an air cargo problem with 10 airports, 50 planes, and 200 pieces of cargo. Each plane can be at one of 10 airports and each package can be either in one of the planes or unloaded at one of the airports. So there are $50^{10} \times 200^{50+10} \approx 10^{155}$ states. Now consider a particular problem in that domain in which it happens that all the packages are at just 5 of the airports, and all packages at a given airport have the same destination. Then a useful abstraction of the problem is to drop all the At fluents except for the ones involving one plane and one package at each of the 5 airports. Now there are only $5^{10} \times 5^{5+10} \approx 10^{17}$ states. A solution in this abstract state space will be shorter than a solution in the original space (and thus will be an admissible heuristic), and the abstract solution is easy to extend to a solution to the original problem (by adding additional $Load$ and $Unload$ actions).

[3] Many problems are written with this convention. For problems that aren't, replace every negative literal $\neg P$ in a goal or precondition with a new positive literal, P'.

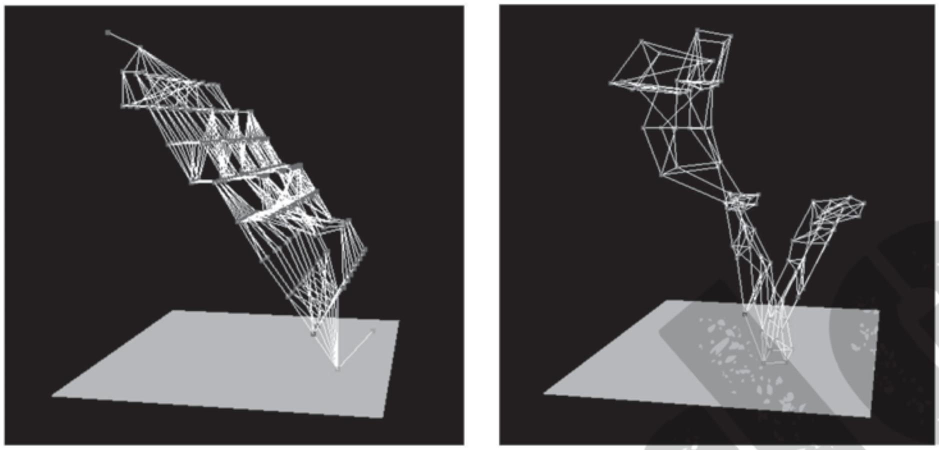

Figure 10.6 Two state spaces from planning problems with the ignore-delete-lists heuristic. The height above the bottom plane is the heuristic score of a state; states on the bottom plane are goals. There are no local minima, so search for the goal is straightforward. From Hoffmann (2005).

DECOMPOSITION

SUBGOAL INDEPENDENCE

A key idea in defining heuristics is **decomposition**: dividing a problem into parts, solving each part independently, and then combining the parts. The **subgoal independence** assumption is that the cost of solving a conjunction of subgoals is approximated by the sum of the costs of solving each subgoal *independently*. The subgoal independence assumption can be optimistic or pessimistic. It is optimistic when there are negative interactions between the subplans for each subgoal—for example, when an action in one subplan deletes a goal achieved by another subplan. It is pessimistic, and therefore inadmissible, when subplans contain redundant actions—for instance, two actions that could be replaced by a single action in the merged plan.

Suppose the goal is a set of fluents G, which we divide into disjoint subsets $G_1, \ldots, G_n$. We then find plans $P_1, \ldots, P_n$ that solve the respective subgoals. What is an estimate of the cost of the plan for achieving all of G? We can think of each $Cost(P_i)$ as a heuristic estimate, and we know that if we combine estimates by taking their maximum value, we always get an admissible heuristic. So $\max_i \text{COST}(P_i)$ is admissible, and sometimes it is exactly correct: it could be that P_1 serendipitously achieves all the G_i. But in most cases, in practice the estimate is too low. Could we sum the costs instead? For many problems that is a reasonable estimate, but it is not admissible. The best case is when we can determine that G_i and G_j are **independent**. If the effects of P_i leave all the preconditions and goals of P_j unchanged, then the estimate $\text{COST}(P_i) + \text{COST}(P_j)$ is admissible, and more accurate than the max estimate. We show in Section 10.3.1 that planning graphs can help provide better heuristic estimates.

It is clear that there is great potential for cutting down the search space by forming abstractions. The trick is choosing the right abstractions and using them in a way that makes the total cost—defining an abstraction, doing an abstract search, and mapping the abstraction back to the original problem—less than the cost of solving the original problem. The techniques of **pattern databases** from Section 3.6.3 can be useful, because the cost of creating the pattern database can be amortized over multiple problem instances.

An example of a system that makes use of effective heuristics is FF, or FASTFORWARD (Hoffmann, 2005), a forward state-space searcher that uses the ignore-delete-lists heuristic, estimating the heuristic with the help of a planning graph (see Section 10.3). FF then uses hill-climbing search (modified to keep track of the plan) with the heuristic to find a solution. When it hits a plateau or local maximum—when no action leads to a state with better heuristic score—then FF uses iterative deepening search until it finds a state that is better, or it gives up and restarts hill-climbing.

10.3 PLANNING GRAPHS

All of the heuristics we have suggested can suffer from inaccuracies. This section shows how a special data structure called a **planning graph** can be used to give better heuristic estimates. These heuristics can be applied to any of the search techniques we have seen so far. Alternatively, we can search for a solution over the space formed by the planning graph, using an algorithm called GRAPHPLAN.

PLANNING GRAPH

A planning problem asks if we can reach a goal state from the initial state. Suppose we are given a tree of all possible actions from the initial state to successor states, and their successors, and so on. If we indexed this tree appropriately, we could answer the planning question "can we reach state G from state S_0" immediately, just by looking it up. Of course, the tree is of exponential size, so this approach is impractical. A planning graph is polynomial-size approximation to this tree that can be constructed quickly. The planning graph can't answer definitively whether G is reachable from S_0, but it can *estimate* how many steps it takes to reach G. The estimate is always correct when it reports the goal is not reachable, and it never overestimates the number of steps, so it is an admissible heuristic.

LEVEL

A planning graph is a directed graph organized into **levels**: first a level S_0 for the initial state, consisting of nodes representing each fluent that holds in S_0; then a level A_0 consisting of nodes for each ground action that might be applicable in S_0; then alternating levels S_i followed by A_i; until we reach a termination condition (to be discussed later).

Roughly speaking, S_i contains all the literals that *could* hold at time i, depending on the actions executed at preceding time steps. If it is possible that either P or $\neg P$ could hold, then both will be represented in S_i. Also roughly speaking, A_i contains all the actions that *could* have their preconditions satisfied at time i. We say "roughly speaking" because the planning graph records only a restricted subset of the possible negative interactions among actions; therefore, a literal might show up at level S_j when actually it could not be true until a later level, if at all. (A literal will never show up too late.) Despite the possible error, the level j at which a literal first appears is a good estimate of how difficult it is to achieve the literal from the initial state.

Planning graphs work only for propositional planning problems—ones with no variables. As we mentioned on page 368, it is straightforward to propositionalize a set of ac-

$Init(Have(Cake))$
$Goal(Have(Cake) \land Eaten(Cake))$
$Action(Eat(Cake)$
  PRECOND: $Have(Cake)$
  EFFECT: $\neg\, Have(Cake) \land Eaten(Cake))$
$Action(Bake(Cake)$
  PRECOND: $\neg\, Have(Cake)$
  EFFECT: $Have(Cake))$

Figure 10.7 The "have cake and eat cake too" problem.

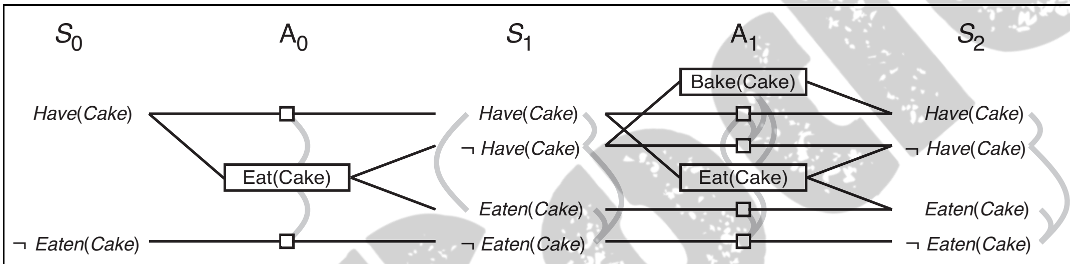

Figure 10.8 The planning graph for the "have cake and eat cake too" problem up to level S_2. Rectangles indicate actions (small squares indicate persistence actions), and straight lines indicate preconditions and effects. Mutex links are shown as curved gray lines. Not all mutex links are shown, because the graph would be too cluttered. In general, if two literals are mutex at S_i, then the persistence actions for those literals will be mutex at A_i and we need not draw that mutex link.

tion schemas. Despite the resulting increase in the size of the problem description, planning graphs have proved to be effective tools for solving hard planning problems.

Figure 10.7 shows a simple planning problem, and Figure 10.8 shows its planning graph. Each action at level A_i is connected to its preconditions at S_i and its effects at S_{i+1}. So a literal appears because an action caused it, but we also want to say that a literal can persist if no action negates it. This is represented by a **persistence action** (sometimes called a *no-op*). For every literal C, we add to the problem a persistence action with precondition C and effect C. Level A_0 in Figure 10.8 shows one "real" action, $Eat(Cake)$, along with two persistence actions drawn as small square boxes.

PERSISTENCE ACTION

Level A_0 contains all the actions that *could* occur in state S_0, but just as important it records conflicts between actions that would prevent them from occurring together. The gray lines in Figure 10.8 indicate **mutual exclusion** (or **mutex**) links. For example, $Eat(Cake)$ is mutually exclusive with the persistence of either $Have(Cake)$ or $\neg Eaten(Cake)$. We shall see shortly how mutex links are computed.

MUTUAL EXCLUSION

MUTEX

Level S_1 contains all the literals that could result from picking any subset of the actions in A_0, as well as mutex links (gray lines) indicating literals that could not appear together, regardless of the choice of actions. For example, $Have(Cake)$ and $Eaten(Cake)$ are mutex:

depending on the choice of actions in A_0, either, but not both, could be the result. In other words, S_1 represents a belief state: a set of possible states. The members of this set are all subsets of the literals such that there is no mutex link between any members of the subset.

We continue in this way, alternating between state level S_i and action level A_i until we reach a point where two consecutive levels are identical. At this point, we say that the graph has **leveled off**. The graph in Figure 10.8 levels off at S_2.

LEVELED OFF

What we end up with is a structure where every A_i level contains all the actions that are applicable in S_i, along with constraints saying that two actions cannot both be executed at the same level. Every S_i level contains all the literals that could result from any possible choice of actions in A_{i-1}, along with constraints saying which pairs of literals are not possible. It is important to note that the process of constructing the planning graph does *not* require choosing among actions, which would entail combinatorial search. Instead, it just records the impossibility of certain choices using mutex links.

We now define mutex links for both actions and literals. A mutex relation holds between two *actions* at a given level if any of the following three conditions holds:

- *Inconsistent effects:* one action negates an effect of the other. For example, $Eat(Cake)$ and the persistence of $Have(Cake)$ have inconsistent effects because they disagree on the effect $Have(Cake)$.
- *Interference:* one of the effects of one action is the negation of a precondition of the other. For example $Eat(Cake)$ interferes with the persistence of $Have(Cake)$ by negating its precondition.
- *Competing needs:* one of the preconditions of one action is mutually exclusive with a precondition of the other. For example, $Bake(Cake)$ and $Eat(Cake)$ are mutex because they compete on the value of the $Have(Cake)$ precondition.

A mutex relation holds between two *literals* at the same level if one is the negation of the other or if each possible pair of actions that could achieve the two literals is mutually exclusive. This condition is called *inconsistent support*. For example, $Have(Cake)$ and $Eaten(Cake)$ are mutex in S_1 because the only way of achieving $Have(Cake)$, the persistence action, is mutex with the only way of achieving $Eaten(Cake)$, namely $Eat(Cake)$. In S_2 the two literals are not mutex, because there are new ways of achieving them, such as $Bake(Cake)$ and the persistence of $Eaten(Cake)$, that are not mutex.

A planning graph is polynomial in the size of the planning problem. For a planning problem with l literals and a actions, each S_i has no more than l nodes and l^2 mutex links, and each A_i has no more than $a + l$ nodes (including the no-ops), $(a + l)^2$ mutex links, and $2(al + l)$ precondition and effect links. Thus, an entire graph with n levels has a size of $O(n(a + l)^2)$. The time to build the graph has the same complexity.

10.3.1 Planning graphs for heuristic estimation

A planning graph, once constructed, is a rich source of information about the problem. First, if any goal literal fails to appear in the final level of the graph, then the problem is unsolvable. Second, we can estimate the cost of achieving any goal literal g_i from state s as the level at which g_i first appears in the planning graph constructed from initial state s. We call this the

LEVEL COST

level cost of g_i. In Figure 10.8, $Have(Cake)$ has level cost 0 and $Eaten(Cake)$ has level cost 1. It is easy to show (Exercise 10.10) that these estimates are admissible for the individual goals. The estimate might not always be accurate, however, because planning graphs allow several actions at each level, whereas the heuristic counts just the level and not the number of actions. For this reason, it is common to use a **serial planning graph** for computing heuristics. A serial graph insists that only one action can actually occur at any given time step; this is done by adding mutex links between every pair of nonpersistence actions. Level costs extracted from serial graphs are often quite reasonable estimates of actual costs.

SERIAL PLANNING GRAPH

To estimate the cost of a *conjunction* of goals, there are three simple approaches. The **max-level** heuristic simply takes the maximum level cost of any of the goals; this is admissible, but not necessarily accurate.

MAX-LEVEL

LEVEL SUM

The **level sum** heuristic, following the subgoal independence assumption, returns the sum of the level costs of the goals; this can be inadmissible but works well in practice for problems that are largely decomposable. It is much more accurate than the number-of-unsatisfied-goals heuristic from Section 10.2. For our problem, the level-sum heuristic estimate for the conjunctive goal $Have(Cake) \land Eaten(Cake)$ will be $0 + 1 = 1$, whereas the correct answer is 2, achieved by the plan $[Eat(Cake), Bake(Cake)]$. That doesn't seem so bad. A more serious error is that if $Bake(Cake)$ were not in the set of actions, then the estimate would still be 1, when in fact the conjunctive goal would be impossible.

SET-LEVEL

Finally, the **set-level** heuristic finds the level at which all the literals in the conjunctive goal appear in the planning graph without any pair of them being mutually exclusive. This heuristic gives the correct values of 2 for our original problem and infinity for the problem without $Bake(Cake)$. It is admissible, it dominates the max-level heuristic, and it works extremely well on tasks in which there is a good deal of interaction among subplans. It is not perfect, of course; for example, it ignores interactions among three or more literals.

As a tool for generating accurate heuristics, we can view the planning graph as a relaxed problem that is efficiently solvable. To understand the nature of the relaxed problem, we need to understand exactly what it means for a literal g to appear at level S_i in the planning graph. Ideally, we would like it to be a guarantee that there exists a plan with i action levels that achieves g, and also that if g does not appear, there is no such plan. Unfortunately, making that guarantee is as difficult as solving the original planning problem. So the planning graph makes the second half of the guarantee (if g does not appear, there is no plan), but if g does appear, then all the planning graph promises is that there is a plan that *possibly* achieves g and has no "obvious" flaws. An obvious flaw is defined as a flaw that can be detected by considering two actions or two literals at a time—in other words, by looking at the mutex relations. There could be more subtle flaws involving three, four, or more actions, but experience has shown that it is not worth the computational effort to keep track of these possible flaws. This is similar to a lesson learned from constraint satisfaction problems—that it is often worthwhile to compute 2-consistency before searching for a solution, but less often worthwhile to compute 3-consistency or higher. (See page 211.)

One example of an unsolvable problem that cannot be recognized as such by a planning graph is the blocks-world problem where the goal is to get block A on B, B on C, and C on A. This is an impossible goal; a tower with the bottom on top of the top. But a planning graph cannot detect the impossibility, because any two of the three subgoals are achievable. There are no mutexes between any pair of literals, only between the three as a whole. To detect that this problem is impossible, we would have to search over the planning graph.

10.3.2 The GRAPHPLAN algorithm

This subsection shows how to extract a plan directly from the planning graph, rather than just using the graph to provide a heuristic. The GRAPHPLAN algorithm (Figure 10.9) repeatedly adds a level to a planning graph with EXPAND-GRAPH. Once all the goals show up as non-mutex in the graph, GRAPHPLAN calls EXTRACT-SOLUTION to search for a plan that solves the problem. If that fails, it expands another level and tries again, terminating with failure when there is no reason to go on.

```
function GRAPHPLAN(problem) returns solution or failure
  graph ← INITIAL-PLANNING-GRAPH(problem)
  goals ← CONJUNCTS(problem.GOAL)
  nogoods ← an empty hash table
  for tl = 0 to ∞ do
     if goals all non-mutex in S_t of graph then
        solution ← EXTRACT-SOLUTION(graph, goals, NUMLEVELS(graph), nogoods)
        if solution ≠ failure then return solution
     if graph and nogoods have both leveled off then return failure
     graph ← EXPAND-GRAPH(graph, problem)
```

Figure 10.9 The GRAPHPLAN algorithm. GRAPHPLAN calls EXPAND-GRAPH to add a level until either a solution is found by EXTRACT-SOLUTION, or no solution is possible.

Let us now trace the operation of GRAPHPLAN on the spare tire problem from page 370. The graph is shown in Figure 10.10. The first line of GRAPHPLAN initializes the planning graph to a one-level (S_0) graph representing the initial state. The positive fluents from the problem description's initial state are shown, as are the relevant negative fluents. Not shown are the unchanging positive literals (such as $Tire(Spare)$) and the irrelevant negative literals. The goal $At(Spare, Axle)$ is not present in S_0, so we need not call EXTRACT-SOLUTION—we are certain that there is no solution yet. Instead, EXPAND-GRAPH adds into A_0 the three actions whose preconditions exist at level S_0 (i.e., all the actions except $PutOn(Spare, Axle)$), along with persistence actions for all the literals in S_0. The effects of the actions are added at level S_1. EXPAND-GRAPH then looks for mutex relations and adds them to the graph.

$At(Spare, Axle)$ is still not present in S_1, so again we do not call EXTRACT-SOLUTION. We call EXPAND-GRAPH again, adding A_1 and S_1 and giving us the planning graph shown in Figure 10.10. Now that we have the full complement of actions, it is worthwhile to look at some of the examples of mutex relations and their causes:

- *Inconsistent effects:* $Remove(Spare, Trunk)$ is mutex with $LeaveOvernight$ because one has the effect $At(Spare, Ground)$ and the other has its negation.

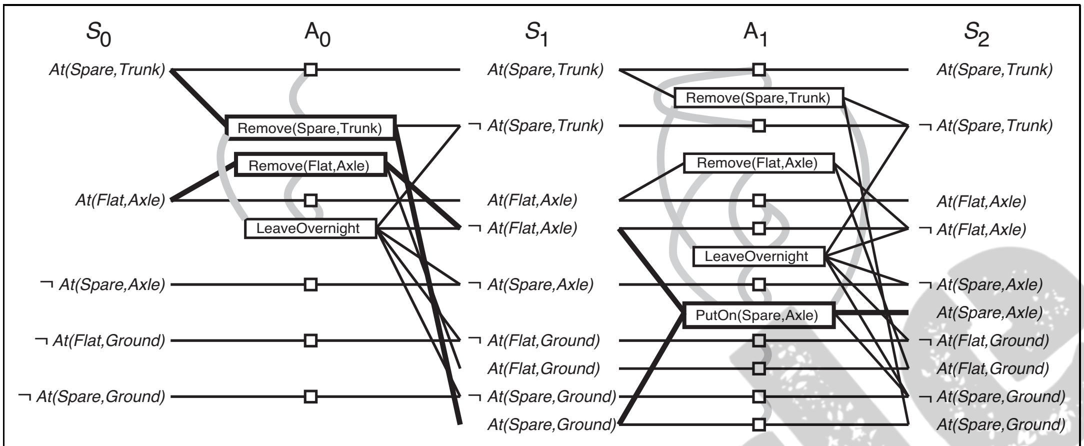

Figure 10.10 The planning graph for the spare tire problem after expansion to level S_2. Mutex links are shown as gray lines. Not all links are shown, because the graph would be too cluttered if we showed them all. The solution is indicated by bold lines and outlines.

- *Interference:* $Remove(Flat, Axle)$ is mutex with $LeaveOvernight$ because one has the precondition $At(Flat, Axle)$ and the other has its negation as an effect.
- *Competing needs:* $PutOn(Spare, Axle)$ is mutex with $Remove(Flat, Axle)$ because one has $At(Flat, Axle)$ as a precondition and the other has its negation.
- *Inconsistent support:* $At(Spare, Axle)$ is mutex with $At(Flat, Axle)$ in S_2 because the only way of achieving $At(Spare, Axle)$ is by $PutOn(Spare, Axle)$, and that is mutex with the persistence action that is the only way of achieving $At(Flat, Axle)$. Thus, the mutex relations detect the immediate conflict that arises from trying to put two objects in the same place at the same time.

This time, when we go back to the start of the loop, all the literals from the goal are present in S_2, and none of them is mutex with any other. That means that a solution might exist, and EXTRACT-SOLUTION will try to find it. We can formulate EXTRACT-SOLUTION as a Boolean constraint satisfaction problem (CSP) where the variables are the actions at each level, the values for each variable are *in* or *out* of the plan, and the constraints are the mutexes and the need to satisfy each goal and precondition.

Alternatively, we can define EXTRACT-SOLUTION as a backward search problem, where each state in the search contains a pointer to a level in the planning graph and a set of unsatisfied goals. We define this search problem as follows:

- The initial state is the last level of the planning graph, S_n, along with the set of goals from the planning problem.
- The actions available in a state at level S_i are to select any conflict-free subset of the actions in A_{i-1} whose effects cover the goals in the state. The resulting state has level S_{i-1} and has as its set of goals the preconditions for the selected set of actions. By "conflict free," we mean a set of actions such that no two of them are mutex and no two of their preconditions are mutex.

- The goal is to reach a state at level S_0 such that all the goals are satisfied.
- The cost of each action is 1.

For this particular problem, we start at S_2 with the goal $At(Spare, Axle)$. The only choice we have for achieving the goal set is $PutOn(Spare, Axle)$. That brings us to a search state at S_1 with goals $At(Spare, Ground)$ and $\neg At(Flat, Axle)$. The former can be achieved only by $Remove(Spare, Trunk)$, and the latter by either $Remove(Flat, Axle)$ or $LeaveOvernight$. But $LeaveOvernight$ is mutex with $Remove(Spare, Trunk)$, so the only solution is to choose $Remove(Spare, Trunk)$ and $Remove(Flat, Axle)$. That brings us to a search state at S_0 with the goals $At(Spare, Trunk)$ and $At(Flat, Axle)$. Both of these are present in the state, so we have a solution: the actions $Remove(Spare, Trunk)$ and $Remove(Flat, Axle)$ in level A_0, followed by $PutOn(Spare, Axle)$ in A_1.

In the case where EXTRACT-SOLUTION fails to find a solution for a set of goals at a given level, we record the $(level, goals)$ pair as a **no-good**, just as we did in constraint learning for CSPs (page 220). Whenever EXTRACT-SOLUTION is called again with the same level and goals, we can find the recorded no-good and immediately return failure rather than searching again. We see shortly that no-goods are also used in the termination test.

We know that planning is PSPACE-complete and that constructing the planning graph takes polynomial time, so it must be the case that solution extraction is intractable in the worst case. Therefore, we will need some heuristic guidance for choosing among actions during the backward search. One approach that works well in practice is a greedy algorithm based on the level cost of the literals. For any set of goals, we proceed in the following order:

1. Pick first the literal with the highest level cost.
2. To achieve that literal, prefer actions with easier preconditions. That is, choose an action such that the sum (or maximum) of the level costs of its preconditions is smallest.

10.3.3 Termination of GRAPHPLAN

So far, we have skated over the question of termination. Here we show that GRAPHPLAN will in fact terminate and return failure when there is no solution.

The first thing to understand is why we can't stop expanding the graph as soon as it has leveled off. Consider an air cargo domain with one plane and n pieces of cargo at airport A, all of which have airport B as their destination. In this version of the problem, only one piece of cargo can fit in the plane at a time. The graph will level off at level 4, reflecting the fact that for any single piece of cargo, we can load it, fly it, and unload it at the destination in three steps. But that does not mean that a solution can be extracted from the graph at level 4; in fact a solution will require $4n - 1$ steps: for each piece of cargo we load, fly, and unload, and for all but the last piece we need to fly back to airport A to get the next piece.

How long do we have to keep expanding after the graph has leveled off? If the function EXTRACT-SOLUTION fails to find a solution, then there must have been at least one set of goals that were not achievable and were marked as a no-good. So if it is possible that there might be fewer no-goods in the next level, then we should continue. As soon as the graph itself and the no-goods have both leveled off, with no solution found, we can terminate with failure because there is no possibility of a subsequent change that could add a solution.

Now all we have to do is prove that the graph and the no-goods will always level off. The key to this proof is that certain properties of planning graphs are monotonically increasing or decreasing. "X increases monotonically" means that the set of Xs at level $i + 1$ is a superset (not necessarily proper) of the set at level i. The properties are as follows:

- *Literals increase monotonically:* Once a literal appears at a given level, it will appear at all subsequent levels. This is because of the persistence actions; once a literal shows up, persistence actions cause it to stay forever.
- *Actions increase monotonically:* Once an action appears at a given level, it will appear at all subsequent levels. This is a consequence of the monotonic increase of literals; if the preconditions of an action appear at one level, they will appear at subsequent levels, and thus so will the action.
- *Mutexes decrease monotonically:* If two actions are mutex at a given level A_i, then they will also be mutex for all *previous* levels at which they both appear. The same holds for mutexes between literals. It might not always appear that way in the figures, because the figures have a simplification: they display neither literals that cannot hold at level S_i nor actions that cannot be executed at level A_i. We can see that "mutexes decrease monotonically" is true if you consider that these invisible literals and actions are mutex with everything.

  The proof can be handled by cases: if actions A and B are mutex at level A_i, it must be because of one of the three types of mutex. The first two, inconsistent effects and interference, are properties of the actions themselves, so if the actions are mutex at A_i, they will be mutex at every level. The third case, competing needs, depends on conditions at level S_i: that level must contain a precondition of A that is mutex with a precondition of B. Now, these two preconditions can be mutex if they are negations of each other (in which case they would be mutex in every level) or if all actions for achieving one are mutex with all actions for achieving the other. But we already know that the available actions are increasing monotonically, so, by induction, the mutexes must be decreasing.
- *No-goods decrease monotonically:* If a set of goals is not achievable at a given level, then they are not achievable in any *previous* level. The proof is by contradiction: if they were achievable at some previous level, then we could just add persistence actions to make them achievable at a subsequent level.

Because the actions and literals increase monotonically and because there are only a finite number of actions and literals, there must come a level that has the same number of actions and literals as the previous level. Because mutexes and no-goods decrease, and because there can never be fewer than zero mutexes or no-goods, there must come a level that has the same number of mutexes and no-goods as the previous level. Once a graph has reached this state, then if one of the goals is missing or is mutex with another goal, then we can stop the GRAPHPLAN algorithm and return failure. That concludes a sketch of the proof; for more details see Ghallab *et al.* (2004).

Year	Track	Winning Systems (approaches)
2008	Optimal	GAMER (model checking, bidirectional search)
2008	Satisficing	LAMA (fast downward search with FF heuristic)
2006	Optimal	SATPLAN, MAXPLAN (Boolean satisfiability)
2006	Satisficing	SGPLAN (forward search; partitions into independent subproblems)
2004	Optimal	SATPLAN (Boolean satisfiability)
2004	Satisficing	FAST DIAGONALLY DOWNWARD (forward search with causal graph)
2002	Automated	LPG (local search, planning graphs converted to CSPs)
2002	Hand-coded	TLPLAN (temporal action logic with control rules for forward search)
2000	Automated	FF (forward search)
2000	Hand-coded	TALPLANNER (temporal action logic with control rules for forward search)
1998	Automated	IPP (planning graphs); HSP (forward search)

Figure 10.11 Some of the top-performing systems in the International Planning Competition. Each year there are various tracks: "Optimal" means the planners must produce the shortest possible plan, while "Satisficing" means nonoptimal solutions are accepted. "Hand-coded" means domain-specific heuristics are allowed; "Automated" means they are not.