Program Verification & Logic

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Security experts hack into moving car and seize control

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REUTERS

In a controlled test, they turned on the Jeep Cherokee's radio and activated other inessential features before rewriting code embedded in the entertainment system hardware to issue commands through the internal network to steering, brakes and the engine.

Program Correctness

- Software is everywhere …
  - Pacemakers, cars, airplanes, satellites, …
  - self-driving cars, drones, robot surgeons, …

- Software bugs => significant consequences

- Program verification
  - Important … though not a panacea!

- What’s a bug?

Program Correctness

- Is the following function correct?

```
method F (x: int, y: int) {
   var z: int;
   if (x > y)
       z := x;
   else
       z := y;
   return z;
}
```

Program Correctness

- Is the following function correct?

```
// returns the minimum
// of the two arguments
method F (x: int, y: int) {
    var z: int;
    if (x > y)
        z := x;
    else
        z := y;
    return z;
}
```

Informal Specification

Formal Specification?

```
// returns the maximum
// of the two arguments
method F (x: int, y: int) {
    var z: int;
    if (x > y)
        z := x;
    else
        z := y;
    return z;
}
```

Specification Constructs:

Assertions

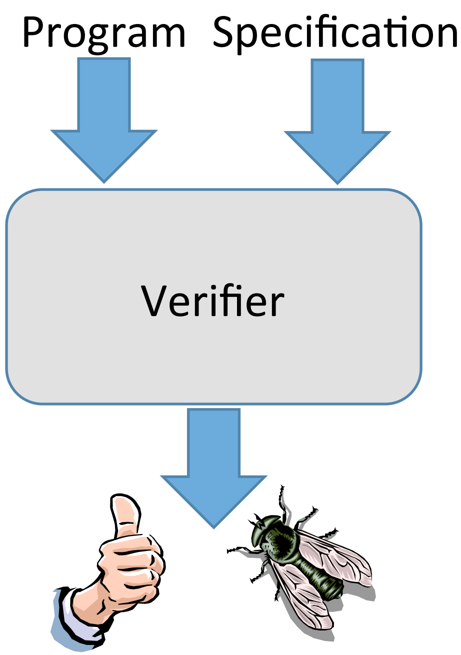
```
method F (x: int, y: int) {
    var z: int;
    if (x > y)
        z := x;
    else
        z := y;
    assert z >= x;
    return z;
}
```

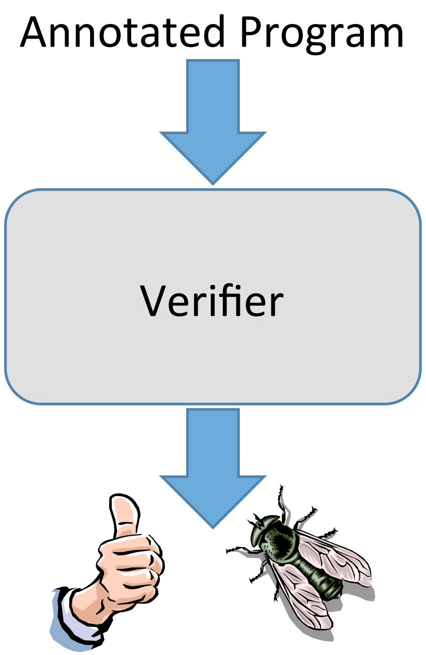
Formal Specifications As Language Constructs

```
method F (x: int, y: int) {
    var z: int;
    if (x > y)
        z := x;
    else
        z := y;
    assert z >= x;
    return z;
}
```

- As *unambiguous* documentation
- For dynamic checking
- For static verification

Specifications & Verification





Specification: Completeness

```
method F (x: int, y: int) {
    var z: int;
    if (x > y)
        z := x;
    else
        z := y;
    assert z >= x;
    assert z >= y;
    assert (z == x) || (z == y);
    return z;
}
```

- Incomplete vs. complete specifications
- Most specifications are incomplete …
    - limits the value of program-verification

Demo: Dafny

- http://rise4fun.com/dafny

Specification Constructs:

Post-condition

```
method F (x: int, y: int)
   returns (z : int)
   ensures z >= x;
   ensures z >= y;
   ensures (z == x) || (z == y);
{
   if (x > y)
      z := x;
   else
      z := y;
}
```

Is this program correct?

```
method Sum (N: int)
   returns (sum : int)
   ensures sum == N*(N+1)/2;
{
   var i := 0; sum := 0;
   while (i < N) {
      i := i + 1;
      sum := sum + i;
   }
}
```

Specification Constructs:

Pre-condition

```
method Sum (N: int)
   returns (sum : int)
   requires N > 0;
   ensures sum == N*(N+1)/2;
{
   var i := 0; sum := 0;
   while (i < N) {
      i := i + 1;
      sum := sum + i;
   }
}
```

Specification Constructs:

Assume statement

```
method Sum (N: int)
   returns (sum : int)
{
   assume N > 0;
   var i := 0; sum := 0;
   while (i < N) {
      i := i + 1;
      sum := sum + i;
   }
   assert sum == N*(N+1)/2;
}
```

Demo

```
method Sum (N: int)
    returns (sum : int)
    requires N > 0;
    ensures sum == N*(N+1)/2;
{
    var i := 0; sum := 0;
    while (i < N) {
        i := i + 1;
        sum := sum + i;
    }
}
```

100 %

Error List

1 Error | 0 Warnings | 1 Message

	Description
1	Error: A postcondition might not hold on this return path.
2	Related location: This is the postcondition that might not hold.

Mathematical Proofs

- Prove:
  - If $n>0$, then $\sum_{i=1}^{n} i = n(n+1)/2$
  - Let P(n) denote the above proposition
- Proof by induction
  - Prove: P(1).
  - Assume P(k) and prove P(k+1).
  - Inductive hypothesis: P(k)

Specification Constructs:
Loop Invariant

```
method Sum (N: int)
   returns (sum : int)
   requires N > 0;
   ensures sum == N*(N+1)/2;
{
   var i := 0;
   while (i < N)
      invariant sum == i*(i+1)/2
  {
      i := i + 1;
      sum := sum + i;
   }
}
```

- A loop invariant serves as an inductive hypothesis (for a proof-by-induction)

Demo

```
method Sum (N: int)
    returns (sum : int)
    requires N > 0;
    ensures sum == N*(N+1)/2;
{
    var i := 0;  sum := 0;
    while  (i < N)
       invariant (sum == i*(i+1)/2)
    {
        i := i + 1;
        sum := sum + i;
    }
}
```

Demo

```
method Sum (N: int)
    returns (sum : int)
    requires N > 0;
    ensures sum == N*(N+1)/2;
{
    var i := 0; sum := 0;
    while (i < N)
       invariant (sum == i*(i+1)/2)
          && (i >= 0) && (i <= N)
    {
       i := i + 1;
       sum := sum + i;
    }
}
```

Recursion

```
method Sum (N: int)
   returns (sum : int)
   requires N > 0;
   ensures sum == N*(N+1)/2;
{
   if (N <= 1)
      sum := 1;
   else
      sum := Sum(N-1) + N;
}
```

- The pre-condition/post-condition of a recursive procedure serves as an inductive hypothesis (for a proof-by-induction)

The Problem:

How to (dis)prove it?

```
method Eg1 (x, y, z: bool)
{
  var result : bool;
  if (x)
    result := y;
  else
    result := z;
  assert result;
}
```

(x,y,z) is a counterexample iff
$(\neg x \vee \neg y) \wedge (x \vee \neg z)$

- Counterexamples to assertion can be found using a Boolean satisfiability (SAT) solver
- The original NP-complete problem

The Problem:

Arithmetic satisfiability

```
method Eg2 (x, y : int)
  returns (z : int)
{
   assume x < y;
   z := (x+y)/2;
   assert x < z;
}
```

Valid iff for all x, y, z:
$((x<y) \wedge (z=x+y/2\ ) \Rightarrow (x<z)\ )$

(x,y,z) is a counterexample iff
$(x<y) \wedge (z=x+y/2\ )) \wedge (x \geq z)$

- Counterexamples to assertion can be found using an arithmetic satisfiability solver
- Related to early 20th Century work in logic, mathematics, and foundations of computing

Towards Automated Verification

Annotated Program

Verification Condition Generation

Verification Condition

Theorem Prover/ SAT/SMT Solver

```
method Eg2 (x, y : int)
  returns (z : int)
{
   assume x < y;
   z := (x+y)/2;
   assert x < z;
}
```

Valid iff for all x, y, z:
$((x<y) \wedge (z=x+y/2) \Rightarrow (x<z))$

Mathematical Logic:
An Introduction

- Barber’s paradox & Russell’s paradox
- Correctness & proofs
- Informal proofs vs. formal proofs
- Why “formal” proofs?
    - Systematic approach
    - … easier to check (for correctness)
    - ... helps avoid mistakes/paradoxes
    - … can automate checking proofs
    - … helps find proofs easier
    - … can automate proof generation

Key Ingredients

- Axiomatic reasoning

<u>Theorem</u>: $(a+b)^2 = a^2 + 2ab + b^2$
<u>Proof</u>:
$(a+b)(a+b)$
$= a(a+b) + b(a+b)$
$= aa + ab + ba + bb$
$= a^2 + 2ab + b^2$

<u>Axioms</u>:
- Distributivity: $x(y+z) = xy + xz$
- Distributivity: $(x+y)z = xz + yz$
- Commutativity: $xy = yx$
- Congruence: $(x = y) \Rightarrow (x+z) = (y+z)$
- …

Key Ingredients

- Separation of syntax and semantics

<u>Claim</u>: $(a+b)\uparrow 2 = a\uparrow 2 + 2ab + b\uparrow 2$

What are a and b?

What is $+$?

What does ab denote?

Interpretations

True for *<u>integer arithmetic</u>*
a, b: integers
$+$: integer-addition
ab: integer-multiplication

Interpretation-1

False for *<u>matrix arithmetic</u>*
a, b: $n \times n$ matrices
$+$: matrix-addition
ab: matrix-multiplication

Interpretation-2

Key Ingredients

- Syntactic approach to proofs
  - symbolic manipulation

<u>Theorem</u>: $(a+b)\uparrow 2 = a\uparrow 2 + 2ab + b\uparrow 2$
<u>Proof</u>:
$(a+b)(a+b)$
$= a(a+b) + b(a+b)$
$= aa + ab + ba + bb$
$= a\uparrow 2 + 2ab + b\uparrow 2$

<u>Axioms</u>:
- Distributivity: $x(y+z) = xy + xz$
- Distributivity: $(x+y)z = xz + yz$
- Commutativity: $xy = yx$
- Congruence: $(x=y) \Rightarrow (x+z) = (y+z)$
- …

- Similar to algebraic approaches to solving word problems

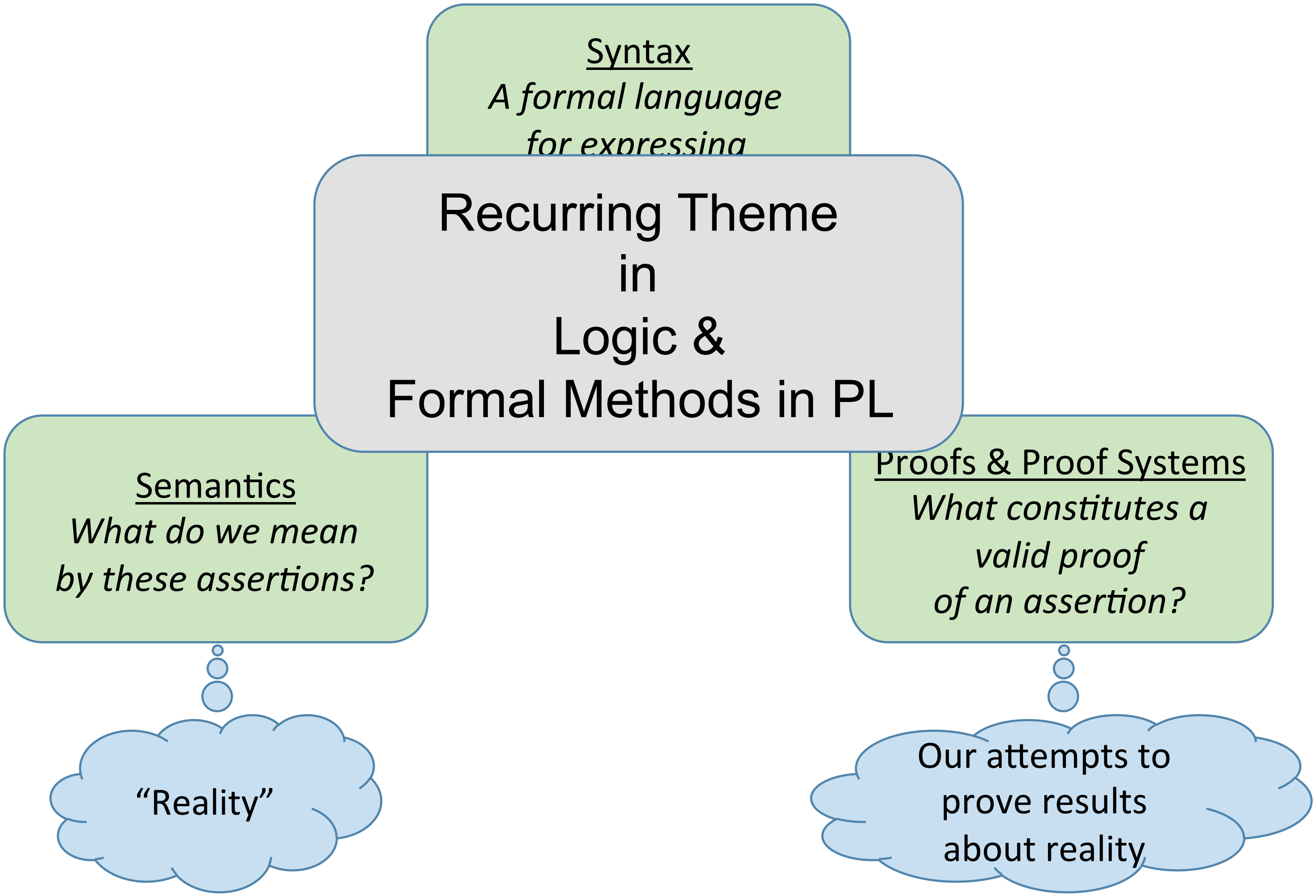
- If Alice is thrice as old as Bob and in another five years Alice will be twice as old as Bob, how old are Alice and Bob?

Symbolic manipulation

$$
\begin{aligned}
&x=3y\\
&x+5=2(y+5)\\
&\Rightarrow 3y+5=2(y+5)\\
&\Rightarrow 3y+5=2y+10\\
&\Rightarrow y=5\\
&\Rightarrow x=15
\end{aligned}
$$

Separation of syntax & semantics

Axioms? E.g., solving matrix equations


Syntax
A formal language
for expressing
Recurring Theme
in
Logic &
Formal Methods in PL
Semantics
What do we mean
by these assertions?
Proofs & Proof Systems
What constitutes a
valid proof
of an assertion?
“Reality”
Our attempts to
prove results
about reality

Propositional Logic

- A language for (pure) Boolean-expressions
- Boolean variables: p, q, r, ...
- Boolean operators:
  - And: $p \wedge q$
  - Or: $p \vee q$
  - Not: $\neg p$
  - …
- Evaluation of Boolean expressions

Propositional Logic: Syntax

- $P ::=$ a set of propositional variables

- The set of formulas over P is defined by

$$\phi \; ::= P \;\; / \;\; \phi_1 \vee \phi_2 \;\; / \;\; \phi_1 \wedge \phi_2 \;\; / \;\; \neg \phi \;\; /$$

Inductive Definitions

- Let $\Sigma = \mathcal{P} \cup \{ \wedge, \vee, \neg \}$
- Let Σ^* denote the set of all sequences of symbols from Σ

- The set of formulas is the smallest subset S of Σ^* that satisfies:
  - If $x \in \mathcal{P}$, then $x \in \mathcal{S}$
  - If $\phi_1 \in \mathcal{S}$ then $\neg \phi_1 \in \mathcal{S}$
  - If $\phi_1 \in \mathcal{S}$ and $\phi_2 \in \mathcal{S}$ then $\phi_1 \vee \phi_2 \in \mathcal{S}$
  - If $\phi_1 \in \mathcal{S}$ and $\phi_2 \in \mathcal{S}$ then $\phi_1 \wedge \phi_2 \in \mathcal{S}$

Propositional Logic: Syntax

- Other operators
- Define $\phi_1 \Rightarrow \phi_2$ to be shorthand for $(\neg \phi_1) \vee \phi_2$)
- Alternatively: take $\Rightarrow$ and $\neg$ as primitive operations
  - Exercise: Define $\wedge$ and $\vee$ in terms of $\Rightarrow$ and $\neg$
- For theoretical (formal) development, it is convenient to restrict attention to a small core language

Propositional Logic: Semantics

- Let T and F denote the values true/false
- Given $M: P \rightarrow \{T, F\}$
- We can recursively define (evaluate) the value $M(\phi)$ of any formula ϕ

$M(\phi_1)$	$M(\phi_2)$	$M(\neg\phi_1)$	$M(\phi_1 \vee \phi_2)$	$M(\phi_1 \wedge \phi_2)$
T	T	F	T	T
T	F	F	T	F
F	T	T	T	F
F	F	T	F	F

Propositional Logic: Semantics

- We say $M: P \rightarrow \{T, F\}$ is an interpretation (or truth-assignment)
- We write $M \vDash \phi$ iff $M(\phi) = T$.
  - M is said to be a model for ϕ iff $M \vDash \phi$
- ϕ is said to be satisfiable if it has a model
- ϕ is said to be unsatisfiable if it has no model
- ϕ is said to be valid (or a tautology) if every interpretation M is a model for ϕ
- We write $\vDash \phi$ iff ϕ is a tautology

Exercises

- Which of the following are satisfiable? Which are tautologies?

- $p \Rightarrow (p \vee q)$
- $p \Rightarrow (p \wedge q)$
- $p \wedge (\neg p)$

- Verify the following theorem:
  ϕ is a tautology iff $\neg \phi$ is unsatisfiable