2.4 Linear First Order Equations

A first-order linear equation ($n = 1$) looks like

$$y' + P(x)y = Q(x).$$

An integrating factor can always be found by the following method. Consider

$$\frac{dy + P(x)y}{dx} = Q(x)dx,$$

$$(P(x)y - Q(x))M(x, y)dx + N(x, y)dy = 0.$$

We use the definition for the integrating factor $I(x, y)$. The equation $IMdx + INdy$ is exact if

$$IxN + INx = IyM + IMy.$$

In our case,

$$Ix + 0 = Iy(P(x)y - Q(x)) + IP(x). \quad (*)$$

We need only one solution, so we look for one of the form $I(x)$, i.e., with $I_y = 0$. Then $(*)$ becomes

$$\frac{dI}{dx} = IP(x).$$

This is separable. So

$$\frac{dI}{I} = P(x)dx,$$

$$\ln(|I|) = \int P(x)dx + C,$$

$$|I| = e^{\int P(x)dx}, \quad e^x > 0.$$

$$I = e^{\int P(x)dx}.$$

We conclude that $e^{\int P(x)dx}$ is an integrating factor for $y' + P(x)y = Q(x)$.