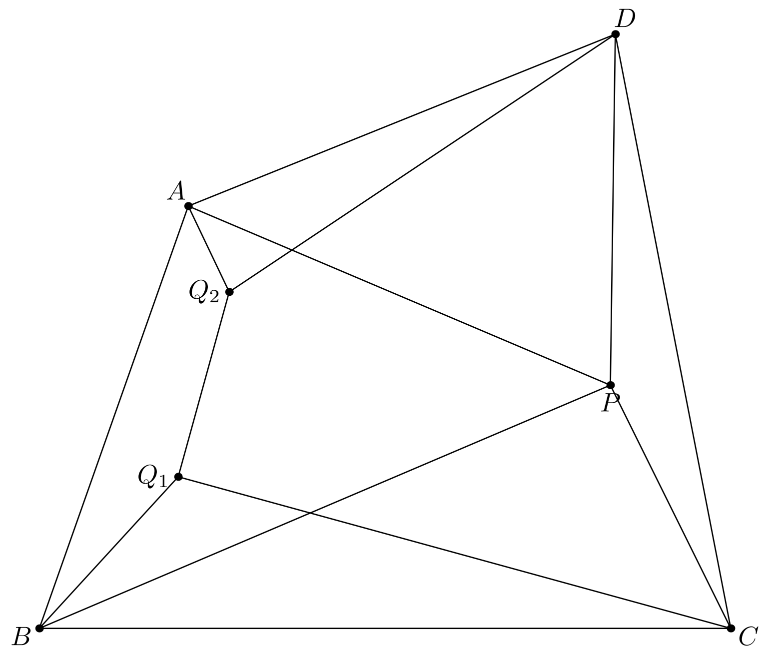
Problem 4.47 (USAMO 2011/5). Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \angle Q_1CB = \angle DCP,$$
$$\angle Q_2AD = \angle BAP, \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

Solution



Proof. To prove $AB \parallel Q_1Q_2 \iff AB \parallel CD$, we can prove the proportionality of height from Q_1, Q_2 to AB, AC are equal. let

$$\begin{aligned}
s &= \frac{AQ_2 \sin\angle BAQ_2 \cdot CQ_1 \sin\angle DCQ_1}{BQ_1 \sin\angle ABQ_1 \cdot DQ_2 \sin\angle CDQ_2} = \frac{AQ_2 \sin\angle DAP \cdot CQ_1 \sin\angle BCP}{BQ_1 \sin\angle CBP \cdot DQ_2 \sin\angle ADP} \\
&= \frac{AQ_2 \cdot DP \cdot CQ_1 \cdot BP}{BQ_1 \cdot CP \cdot DQ_2 \cdot AP} \\
&= \frac{DP \cdot BP}{CP \cdot AP} \cdot \frac{\sin\angle ADQ_2 \cdot \sin\angle CBQ_1}{\sin\angle DAQ_2 \cdot \sin\angle BCQ_1} = \frac{DP \cdot BP}{CP \cdot AP} \frac{\sin\angle CDP \cdot \sin\angle ABP}{\sin\angle BAP \cdot \sin\angle DCP} \\
&= \frac{DP \cdot BP \cdot CP \cdot AP}{CP \cdot AP \cdot BP \cdot DP} = 1
\end{aligned}$$

and we are done □

Second solution

Proof. If $AB \parallel CD$, we've proved the problem. Suppose AB and CD intersect at R. We know Q_2, P are isogonal conjugates with respect to $\triangle RAD$, Q_1, P are isogonal conjugates with respect to $\triangle RBC$. So Q_1Q_2 and RP are isogonals with respect to $\angle BRC$. R is on Q_1Q_2. If AB is parallel to Q_1Q_2 and CD is not parallel to Q_1Q_2, since AB intersects CD at R, R is on Q_1Q_2, so Q_1Q_2 intersects AB at R, which leads to a contradiction. Similarly we can prove the if AB is not parallel to Q_1Q_2 and CD is parallel to Q_1Q_2 leads to a contradiction. So

$$Q_1Q_2 \parallel AB \iff Q_1Q_2 \parallel CD$$

□

Remark. It's important to recognize the isogonal conjugate.