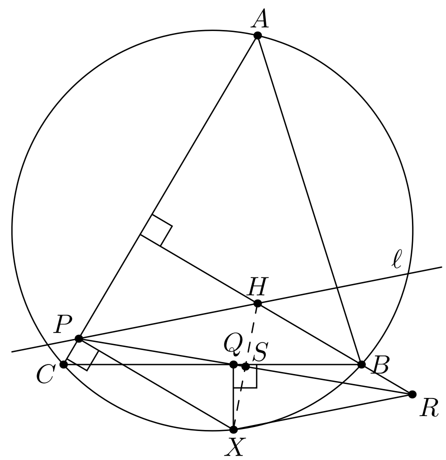
Problem 4.44 (USA TST 2014). Let ABC be an acute triangle and let X be a variable interior point on the minor arc $\widehat{BC}$. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. leet R be the intersection of line PQ and the perpendicular from B to $\overline{AC}$. Let ℓ be the line through P parallel to $\overline{XR}$. Prove that as X varies along minor arc $\widehat{BC}$, the line ℓ always passes through a fixed point.

Solution Our solution is based off the following lemma.

Lemma. Let ABC be a triangle with orthocenter H. If P is a point on (ABC) then its Simson line bisects $\overline{PH}$.



Proof. According to the lemma, PQ bisects XH. Given that $\ell \parallel XR$. Let S be the intersection point of HX and PR. We have

$$\angle HPR = \angle XRP$$
$$\angle HSP = \angle XSR$$
$$HS = SX$$

This gives us that $\triangle HPS \cong \triangle XRS$, which immediately tells $PH = XR$. So $HPXR$ is a parallelogram. ℓ passes through the orthocenter H of $\triangle ABC$. $\square$