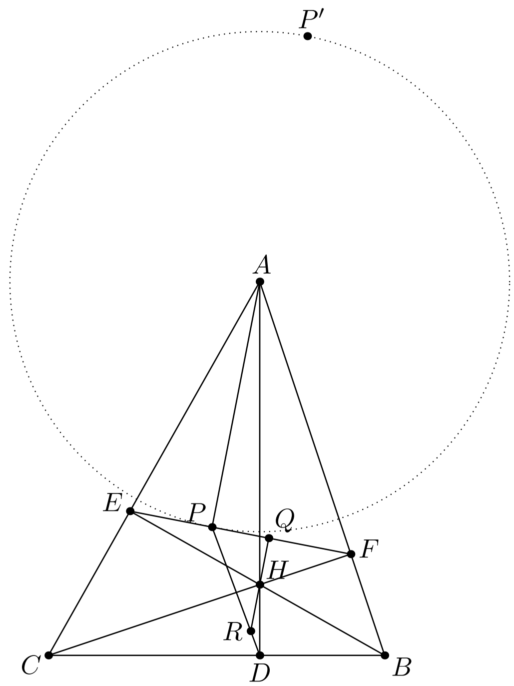
Problem 4.45 (USA TST 2011/1). In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{CA}$, $\overline{CF} \perp \overline{AB}$. Altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ meet at orthocenter H. Points P and Q lie on segment $\overline{EF}$ such that $\overline{AP} \perp \overline{EF}$ and $\overline{HQ} \perp \overline{EF}$. Lines DP and QH intersect at point R. Compute HQ/HR.

Solution Our solution is based off the following lemma.

Lemma 1. If I_A, I_B, I_C are the excenters of $\triangle ABC$, then triangle ABC is the orthic triangle of $\triangle I_A I_B I_C$, and the orthocenter is I.

Lemma 2. Let (ABC) be a triangle whose incircle is tangent to $\overline{BC}$ at D. If $\overline{DE}$ is a diameter of incircle and ray AE meets $\overline{BC}$ at X, then X is the tangency point of the A-excircle to $\overline{BC}$. Suppose XY is the diameter of the A-excircle, then D lies on $\overline{AY}$.



Proof. From the first lemma we claim that H is the incenter of triangle DEF. Q is the tangency point of the incircle of $\triangle DEF$ to $\overline{EF}$.

Let P' be a point on the D-excircle of $\triangle DEF$ so that PP' is the diameter. By applying the second lemma we have Q lies on $\overline{DP'}$. Since $RQ \parallel PP'$ and R lies on PD, we claim that there's a homothety sending $\overline{RQ}$ to $\overline{PP'}$, sending the incircle of $\triangle DEF$ to D-excircle of $\triangle DEF$. So QR is the diameter of the incircle of $\triangle DEF$. This finally tells us $HQ/HR = 1$. $\square$