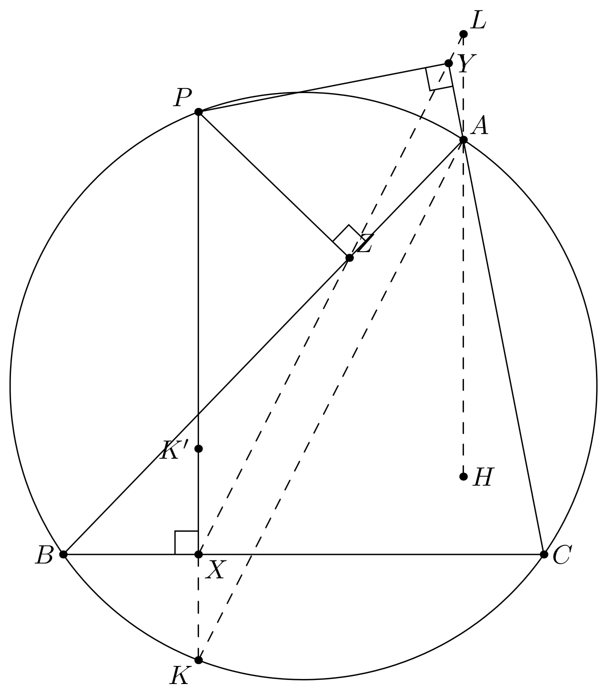
Lemma 4.4 (Simson Line Bisection). *Let ABC be a triangle with orthocenter H. If P is a point on (ABC) then its Simson line bisects $\overline{PH}$.*

Solution Our solution is based off the following lemma.

Lemma. *In cyclic quadriateral $ABCD$, points X, Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Then $AXYD$ is a parallelogram.*

Let line PX meet (ABC) again at a point K, and let line AH intersect the Simson line at the point L.



Proof. We first prove AK is parallel to the simson line, then we prove that K' is the orthocenter of $\triangle PBC$, then we prove that $LHXP$ is a parallelogram.

We claim that AK is parallel to the simson line because

$$\angle YZA = \angle YPA = 90^\circ - \angle PAY = 90^\circ - (180^\circ - \angle PBC) = 90^\circ - \angle PBC = \angle BPK = \angle BAK$$

We have $PX \perp BC$ and

$$\angle K'BX = \angle KBX = \angle KPC = 90^\circ - \angle PCB$$

so $BK' \perp PC$. K' is the orthocenter of $\triangle PBC$. According to the lemma, we claim that $AHK'P$ is a parallelogram. Since $AH \parallel PK$, $LAKX$ is a parallelogram. So we have

$$LH = LA + AH = KX + PK' = K'X + PK' = XP$$

So $LHXP$ is a parallelogram. The simson line LX bisects HP.

□