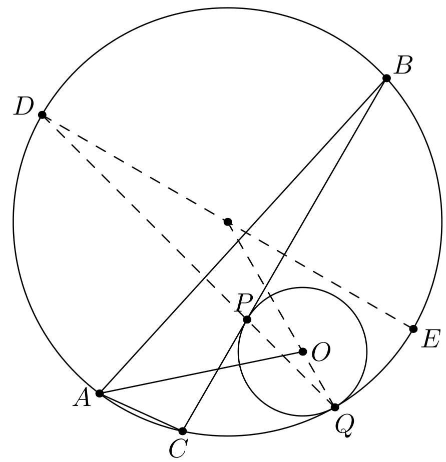
Problem 4.48 (Japanese Olympiad 2009). Triangle ABC is inscribed in circle Γ. A cirle with center O is drawn, tangent to side BC at a point P, and internally tangent to the arc BC of Γ not containing A at a point Q. Show that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.

Solution



Proof. Let D be another intersection between PQ and Γ, E be the midpoint of arc BC not containing A. There's a homothety centered at Q, sending P to D. So $OP \parallel ED$. If $\angle BAO = \angle QAO$, then A, O, and E are concurrent. Since $\angle AQP = \angle AED = \angle AOP$. So $AQOP$ is a cyclic quadrilateral. Then we know $\angle PAO = \angle PQO = \angle QDE = \angle QAO$.

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