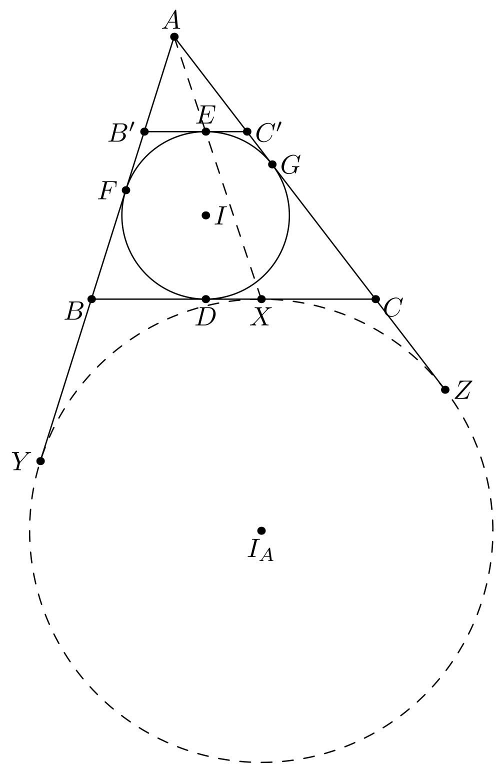
Lemma 4.9 (The Diameter of the Incircle). *Let (ABC) be a triangle whose incircle is tangent to $\overline{BC}$ at D. If $\overline{DE}$ is a diameter of incircle and ray AE meets $\overline{BC}$ at X, then $BD = CX$ and X is the tangency point of the A-excircle to $\overline{BC}$.*

Solution



Proof. Let $B'C'$ be the line parallel to BC that passes through E. The incircle of triangle ABC is the A-excircle of triangle $AB'C'$ with tangency point E. There's a homothety centered at A sending E to X. So we claim that X is the tangency point of the A-excircle to $\overline{BC}$.

Let F, G be the tangency point of the incircle of triangle ABC to $\overline{AB}$ and $\overline{AC}$, Y, Z be the tangency point of A-excircle of triangle ABC to AB and AC. We have

$$AG = AF, BF = BD, CD = CG$$

$$BD + DC = a, CG + GA = b, AF + FB = c$$

$$AY + AZ = a + b + c$$

$$AY = AZ$$

which gives us

$$BD = CX = s - b$$

where

$$s = \frac{1}{2}(a + b + c)$$

□