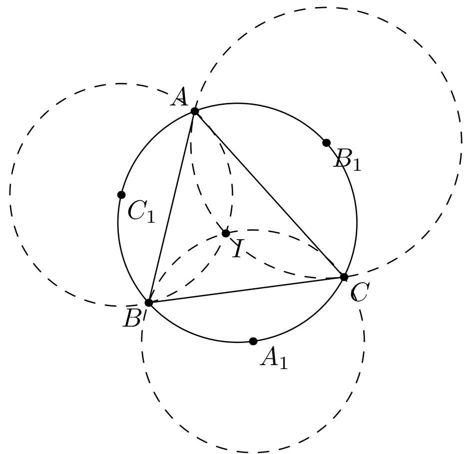
Problem 4.42 (USAMO 1988/4). Suppose $\triangle ABC$ is a triangle with incenter I. Show that the circumcenters of $\triangle IAB$, $\triangle IBC$, and $\triangle ICA$ lie on a circle whose center is the circumcenter of $\triangle ABC$.

Solution Mark the circumcenters of $\triangle IBC$, $\triangle ICA$, $\triangle IAB$ as A_1, B_1, and C_1.



Proof. If we prove C_1 lies on (ABC), we can prove A_1, B_1 lie on (ABC) similarly. We prove this by proving $\angle AC_1B + \angle ACB = 180°$.

$$\angle AIB = 180° - \tfrac{1}{2}\angle BAC - \tfrac{1}{2}\angle CBA = 180° - \tfrac{1}{2}(180° - \angle ACB) = 90° + \tfrac{1}{2}\angle ACB$$
$$\angle AC_1B = 2(180° - \angle AIB) = 180° - \angle ACB$$

Similarly A_1, B_1 lie on (ABC). Hence C_1 lies on (ABC). □