M1103 Chapter 3

Determinants

Definition

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$

The determinant of A is the number denoted by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \in \mathbb{K}$$

Consequence

We know that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible

then, $ad - bc \neq 0$, and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus, if A is invertible, we have

$$\det(A) \neq 0, \text{ and } A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- For $n = 1$, so for $A = (a) \in M_1(\mathbb{K})$,

we set, $\det(A) = \det(a) = a$

- For $n = 2$, see before.

In fact, we can see the given definition as follows:

$$\det(A) = \det(a) \cdot \det(d) - \det(b) \cdot \det(c)$$

- Suppose now, the determinant has been defined for matrices of size $(n-1) \in \mathbb{K}$:

Definition

$A = (a_{ij}) \in M_n(\mathbb{K})$

1. The <u>minor</u> of a_{ij} is the determinant of the sub-matrix $A_{ij} \in M_{n-1}(\mathbb{K})$ obtained by deleting the i^{th} row, and the j^{th} column of A.
2. The <u>co-factor</u> of a_{ij} is the number

$$\text{cof}(a_{ij}) = (-1)^{i+j} \det(A_{ij})$$

Example 1

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{K})$

$$- \ \text{cof}(a_{11}) = (-1)^{1+1} \det(A_{11}) = \det(a_{22}) = a_{22}$$

$$- \ \text{cof}(a_{12}) = (-1)^{1+2} \det(A_{12}) = -\det(a_{21}) = -a_{21}$$

$$- \ \text{cof}(a_{21}) = (-1)^{2+1} \det(A_{21}) = -\det(a_{12}) = -a_{12}$$

$$- \ \text{cof}(a_{22}) = (-1)^{2+2} \det(A_{22}) = \det(a_{11}) = a_{11}$$

We notice:

$$a_{11}\text{cof}(a_{11}) + a_{12}\text{cof}(a_{12}) = a_{11}a_{22} - a_{12}a_{21} = \det(A)$$

$$a_{21}\text{cof}(a_{21}) + a_{22}\text{cof}(a_{22}) = -a_{21}a_{12} + a_{22}a_{11} = \det(A)$$

$$\vdots$$
$$= \det(A)$$

We will always get $\det(A)$.

Theorem

given $A \in M_n(\mathbb{K})$,

Regardless of which row or column is chosen, the number obtained by multiplying the entries of that row or column by the corresponding co-factors and adding the resulting products is always the same.

Proof

Math 401 □ (4th year)

Definition

given $A \in M_n(\mathbb{K})$,

The common number defined in the previous theorem is called the determinant of A and is denoted by $\det(A)$.

Thus, we have:

$$\det(A) = a_{i1}\mathrm{cof}(a_{i1}) + \cdots + a_{in}\mathrm{cof}(a_{in}) \to i^{th} \text{ row}$$

$$\det(A) = a_{1j}\mathrm{cof}(a_{1j}) + \cdots + a_{jn}\mathrm{cof}(a_{jn}) \to j^{th} \text{ column}$$

Example 2

Find the $\det(A)$ of $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3 \cdot (-1)^{1+1} \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} + 1 \cdot (-1)^{1+2} \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \ldots \begin{vmatrix} \ldots & \ldots \\ \ldots & \ldots \end{vmatrix}$$

$$= -1$$

Alternatively,

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3 \cdot (-1)^{1+1} \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} + (-2) \cdot (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \cdot (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) + 2(-2) + 5(3) = -1$$

Example 3

Find the $\det(A)$ of $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$

We choose the second column since it has the most zeros

$$\det(A) = 1 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot (-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2 \cdot (3) = -6$$

Theorem

$A = (a_{ij}) \in M_n(\mathbb{K})$

If A is upper triangular, (resp. lower triangular) then,

$$\det(A) = a_{11} \dots a_{nn}$$

Proof

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} \dots & \dots & \dots \\ & \vdots & \\ \dots & \dots & \dots \end{vmatrix} = a_{11}a_{22} \left| \dots \right| = \text{keep the patern going}$$

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§ 3.2

Theorem

given $A \in M_n(\mathbb{K})$,

1. if A has a zero-row or a zero-column, then
$$\det(A) = 0$$
2. We have $\det({}^tA) = \det(A)$

Proof

Immediate. □

Definition

a Permutation is a bijection,

$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$$

Example

$$\begin{aligned} \sigma : \{1, 2, 3\} &\to \{1, 2, 3\} \\ \text{where} \\ 1 &\to 3 \\ 2 &\to 1 \\ 3 &\to 2 \end{aligned}$$

Definition

A flip is a permutation $(i,j)_n : \{1,\dots,n\} \to \{1,\dots,n\}$
such that

$$\begin{aligned} & i \to j \\ & j \to 1 \\ & k \to k, \quad \forall k \notin \{i,j\} \end{aligned}$$

Theorem

Any flip is a composition of an odd number of adjacent flips.

Proof

$$(i,j)_n = (j+1,j)_n \dots (i-1,i-2)_n \dots (i,i-1)_n \dots (i-1,i-2)_n, \dots (j+1,j)_n$$

$$\forall i > j$$

Example 2

Take $n = 9$, consider the flip $(7,2)$

$$(7,2)_9 = (3,2)_9 \to (4,3)_9 \to (5,4)_9 \to (6,5)_9 \to \underbrace{(7,6)_9}_{central\ flip} \to (6,5)_9 \to \dots \to (3,2)_9$$

Thus, all of the flips will revert to their original state, remaining unchanged. **EXCEPT** for $(7,2)$.

$$\begin{aligned} 7 \to 2 \\ 2 \to 7 \end{aligned}$$

Lemma

Suppose $A, B, C \in M_n(\mathbb{K})$

Suppose that A, B, C differ only by the i^{th} row, and the i^{th} row of C is the sum of the i^{th} row of A and the i^{th} row of B. Then

$$\det(C) = \det(A) + \det(B)$$

Proof

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(B) = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{33} \\ \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(C) = \begin{vmatrix} a_{11}+b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} (a_{11}\mathrm{cof}(a_{11}) + \dots + \dots) + (b_{11}\mathrm{cof} b_{11} + \dots + \dots) &= (a_{11}+b_{11})\mathrm{cof}(a_{11}) + \dots + \dots \\ &= a_{11} + b_{11}\mathrm{cof}(a_{11}+b_{11}) + \dots + \dots \\ &= \det(C) \end{aligned}$$

Theorem

$A = (a_{ij}) \in M_n(\mathbb{K})$
$B = (b_{ij}) \in M_n(\mathbb{K})$
$C \in \mathbb{K} - \{0\}$

1. If B is obtained from A by multiplying a single row or a single column by C, then

$$\det(B) = c\det(A)$$

2. If B is obtained from A by interchanging two rows or two columns then,

$$\det(B) = -\det(A)$$

In particular, if two rows or two columns are identical, then

$$\det(A) = 0$$

3. If B is obtained from A by adding a multiple of one row or column to another row or column, then

$$\det(B) = \det(A)$$

Proof

1-

$$\det(A) = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{i1} & \dots & a_{in} \\ a_{n_1} & \dots & a_{nn} \end{vmatrix}$$

$$\det(B) = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ ca_{i_1} & \dots & ca_{in} \\ a_{n_1} & \dots & a_{nn} \end{vmatrix}$$

$$\det(B) = ca_{i_1}\overbrace{\mathrm{cof}(ca_{i_1})}^{\mathrm{cof}(a_{i_1})} + \dots + ca_{in}\overbrace{\mathrm{cof}(ca_{in})}^{\mathrm{cof}a_{in}} \qquad \square$$

2-

$$\det(A) = \begin{vmatrix} & * & \\ a_{i_1} & \dots & a_{in} \\ a_{i+1_1} & \dots & a_{i+1_n} \\ & * & \end{vmatrix}$$

We flip the two rows

$$\det(B) = \begin{vmatrix} & * & \\ a_{i+1,1} & \dots & a_{i+1,n} \\ a_{i_1} & \dots & a_{in} \\ & * & \end{vmatrix}$$

$$\det(B_{i_1}) = \det(A_{i+1,1})$$
$$\vdots$$
$$\det(B_{in}) = \det(A_{i+1,n})$$

$$\begin{aligned} \det(B) &= b_{i_1}\mathrm{cof}(b_{i_1}) + \dots \\ &= a_{i+1,1}\mathrm{cof}(b_{i_1}) + \dots \\ &= a_{i+1,1}(-1)^{i+1}\det(B_{i_1}) + \dots \\ &= a_{i+1,1}(-1)^{i+1}\det(A_{i+1,1}) + \dots \\ &= -a_{i+1,1}(-1)^{i+1+1}\det(A_{i+1,1}) + \dots \\ &= -\det(A) \qquad \square \end{aligned}$$

3-

$$\det(A) = \begin{vmatrix} & * & \\ a_{i_1} & \dots & a_{in} \\ a_{l1} & \dots & a_{ln} \\ & * & \end{vmatrix}$$

$$\det(C) = \begin{vmatrix} & * & \\ ca_{l_1} & \dots & ca_{ln} \\ ca_{l_1} & \dots & ca_{ln} \\ & * & \end{vmatrix}$$

$$\det(B) = \begin{vmatrix} & * & \\ a_{i_1} + ca_{l_1} & \dots & a_{in} + ca_{ln} \\ a_{l1} & \dots & a_{ln} \\ & * & \end{vmatrix}$$

So,

$$\det(B) = \det(A) + \det(C)$$

$$\text{but} \det(C) = 0,$$

$$so,$$
$$\det(B) = \det(A) \qquad \square$$

Corollary

let $E \in M_n(\mathbb{K})$ elementary matrix
$C \in \mathbb{K} - \{0\}$

1. If E is obtained from I_n by multiplying a single row by C, then

$$\det(E) = c$$

2. Similarly, if E is obtained from I_n by interchanging two rows, then

$$\det(E) = -1$$

3. If E is obtained from I_n by adding a multiple of one row to another row then,

$$\det(E) = 1$$

Proof

Immediate since $\det(I_n) = 1$ $\square$

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§ 3.3

Theorem :

$A \in M_n(\mathbb{K})$, $\lambda \in \mathbb{K}$

Then, $\det(\lambda A) := \lambda^n \det(A)$

Proof:

Immediate $\square$.

Remark

In general $\det(A + B) \neq \det(A) + \det(B)$

Lemma

$A \in M_n(\mathbb{K})$

$$E_1, \ldots, E_k \in M_n(\mathbb{K}) \text{ element matrices}$$

Then $\det(E_k, \ldots, E_1 A) = (\det(E_k) \ldots \det(E_1) \det(A))$

Proof

IF $E \in M_n(\mathbb{K})$ is an element matrix since EA is obtained from A by doing the row operation corresponding to E, we have $\det(EA) = \det(E)\det(A)$

The result following by induction. □

Theorem

$A \in M_n\mathbb{K}$
Then, A is invertible $\iff \det(A) \neq 0$

Proof

Let R be the RREF of A. Then $R = E_k, \ldots, E_1 A$ for some element matrices $E_1, \ldots, E_k$

By the preceding lemma: $\det(R) = \det(E_k) \ldots \det(E_1) \det(A)$, where $\det(E_k) \ldots \det(E_1 A) \neq 0$

$$\therefore \det(R) = 0 \iff \det(A)$$

- If A is invertible, then $R = I_n$,
  so $\det(R) = 1 \neq 0$, and so $\det(A) \neq 0$
- If $\det(A) \neq 0$, then $\det(R) \neq 0$ then R cant have a zero-row, so $R = I_n$ and so A is invertible.

Theorem

$A, B \in M_n(\mathbb{K})$, then $\det(AB) = \det(A)\det(B)$

Proof

- If A is not invertible, then AB is not invertible. So $\det(AB) = 0 = (0)\det(B) = \det(A)\det(B)$.

- Suppose now that A is invertible, then $A = E_1, \ldots, E_k$ for some element matrices $E_1, \ldots, E_k$

Then,

$$\begin{aligned}\det(AB) = \det(E_1, \ldots, E_k B) &= \det(E_1) \ldots \det(E_k) \det(B \\ &= \det(E_k, \ldots, E_1) \det(B) \\ &= \det(A) \det(B) \quad \square\end{aligned}$$

Consequences

$$\begin{aligned}\det : GL_n(\mathbb{K}) &\longrightarrow \mathbb{K} - \{0\} \\ A &\mapsto \det(A)\end{aligned}$$

is a group morphism. It is surjective and $ker(\det)$ is a subgroup of $GL_n(\mathbb{K})$.

$$\ker(\det) = \{A \in GL_n(\mathbb{K}) : \det(A) = 1\};$$

The group is called a special linear group and it is denoted by $SL_n(\mathbb{K})$

Corollary

$A \in GL_n(\mathbb{K})$, then $\det(A^{-1}) = \frac{1}{\det A}$

Proof

We have $1 = \det(I_n) = \det(AA^{-1}) = \det(A) \det(A^{-1}) \quad \square$

Definition

$A = (a_{ij}) \in M_n(\mathbb{K})$

The co-matrix of A is the matrix $com(A) = cof(a_{ij})$

Theorem

$A \in M_n(\mathbb{K})$

Then $A^{-1} = \frac{1}{\det(A)}\ {}^t com(A)$

Proof

${}^t com(A) = (c_{jk})$ then $c_{jk} = cof(a_{kj})$

Theorem

$A \in GL_n(\mathbb{K})$
Then $A^{-1} = \frac{1}{\det(A)} com(A)$

Proof

${}^t com(A) = (c_{jk})$. Then $C_{jk} = cof(a_{ij})$

Write: $A\ {}^t com(A) = (p_{ik})$. Then,

$$p_{ik} = \sum_{j=1}^{n} a_{ij} c_{jk}$$

$$= \sum_{j=1}^{n} a_{ij}\ com(a_{kj})$$

In particular,

$$p_{ii} = sum_{j=1}^{n} a_{ij}\ cof(a_{ij}) = \det(A)$$

- If $k \neq i$, then $p_{ik} = 0$

Indeed, let B be the matrix obtained from A by replacing the K^{th} row by $(a_{ij}, \ldots, a_{in})$. Then,

$$p_{ik} = \sum_{j=1}^{n} a_{ij}\ \mathrm{cof}(a_{kj}) = \sum_{j=1}^{n} a_{ij}\ \mathrm{cof}(a_{ij}) = \det(B) = 0$$
