M1103 Chapter 1

Linear Systems:

In this course, $\mathbb{K}$ will denote a commutative field (usually $\mathbb{R}$ or $\mathbb{C}$)

Definition:

1- Suppose we have $a_1, \ldots, a_n, b \in \mathbb{K}$

A linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

We call $a_1, \ldots, a_n$ the coefficients of the equation.
We call $x_1, \ldots, x_n$ the unknowns of the equation.
We call b the right hand side of the equation.

2- A solution for the equation is any n-tuple $(s_1, \ldots, s_n)$ of numbers in $\mathbb{K}$, such that if we substitute x_1 by s_1, x_n by s_n in the equation, we get a true statement.

Example:

$$x + 3y = 7 : \checkmark$$

is a linear equation.

$$\frac{1}{2}x - y + 3z = -1 : \checkmark$$

is a linear equation.

$$x_1 - 2x_2 - 3x_3 + x_4 = 0 : \checkmark$$

is a linear equation.

$$x + 3y^2 = 4$$

is NOT a linear equation, because its a polynomial of degree 2.

$$\sin x + y - 0$$

is NOT a linear equation, because we have a $\sin$ function.

$$3x + 2y - xy = 5$$

is NOT a linear equation, because its of degree 2.

$$\sqrt{x_1} + 2x_2 + x_3 = 1$$

is NOT a linear equation, because its of degree $\frac{1}{2}$. (square root.)

Definition:

A system of linear equations, or a linear system, is a collection of linear equations:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The solution of a linear system is any n-tuple, $s_1, \ldots, s_n$ that satisfies all the equations simultaneously.

Case of two unknowns:

$$\begin{cases} ax + by = c \\ \\ dx + ey = f \end{cases}$$

Definition:

Given a linear system as above, we define the corresponding augmented matrix as:

$$mat$$

Example:

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{cases}$$

Augmented matrix:

$$\begin{pmatrix} 1 & 1 & 2 & | & 9 \\ 2 & 4 & -3 & | & 1 \\ 3 & 6 & -5 & | & 0 \end{pmatrix}$$

Definition:

A zero row, is a row that consists entirely of zeros. If there's at least one number in the row that is not zero, then it is a non-zero row.

Definition:

A matrix is said to be in row echelon form (REF), if these 3 conditions are satisfied:

1. In any non-zero row, the first non-zero number is 1, called a leading 1.
2. If there are zero rows, they are grouped together at the bottom of the matrix.
3. In any two-successive non-zero rows, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

Then, we say that the matrix is in reduced row echelon form (RREF) if furthermore:

4. Each column containing the leading 1 has zeros everywhere else in that column.

Example:

1-

$$\begin{pmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & -1 \end{pmatrix} = RREF$$

Condition 1 is satisfied in this case. The second condition is trivially satisfied as well. So is the 3rd condition. Thus, it is in row echelon form (REF). We notice as well that condition 4 is satisfied. Ergo, it is in reduced row echelon form. (RREF).

2-

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = RREF$$

This is also in reduced row echelon form (RREF).

3-

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = RREF$$

This is also in reduced row echelon form (RREF).

4-

Now,

$$\begin{pmatrix} 1 & 4 & -3 & | & 7 \\ 0 & 1 & 6 & | & 2 \\ 0 & 0 & 1 & | & 5 \end{pmatrix} = REF$$

This is in row echelon form (REF), but NOT in reduced row echelon form (RREF).

5-

Same for,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = REF$$

NOTE:

Reducing matrices to RREF, makes it easier for us to solve the system.

Example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 5 \end{pmatrix}$$

$$\text{Linear system of the above matrix} = \begin{cases} x_1 & & & & = 3 \\ & x_2 & & & = -1 \\ & & x_3 & & = 0 \\ & & & x_4 & = 5 \end{cases}$$

Which has $(3, -1, 0, 5)$ as unique solutions.

Example 2:

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{pmatrix} = RREF$$

$$\begin{cases} x_1 = 0 \\ x_2 + 2x_3 = 0 \\ 0 = 1 \end{cases}$$

This equation is inconsistent, thus it has no solutions.

Example 3:

$$\begin{pmatrix} 1 & 0 & 3 & | & -1 \\ 0 & 1 & -4 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} = RREF$$

$$\begin{cases} x_1 + \quad + 3x_3 = -1 \\ \quad + x_2 - 4x_3 = 2 \end{cases}$$

x_1, x_2 are leading variables, while x_3 is a free variable:

Solving for the leading variables;

$$\implies \begin{cases} x_1 = -1 - 3x_3 \\ x_2 = 2 + 4x_3 \end{cases}$$

Call $x_3 : t$

Then, we can write:

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \\ t \in \mathbb{R} \end{cases}$$

Thus, we write:

$$\text{The set of solutions is: } \{(-1 - 3t,\ 2 + 4t,\ t); t \in \mathbb{R}\}$$

Example 4:

$$\left(\begin{array}{ccc|c} \boxed{1} & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = RREF$$

$$\{ x_1 - 5x_2 + x_3 = 4$$

Solving this;

$$\{ x_1 = 4 + 5x_2 - x_3$$

Call $x_2 : r,\ x_3 : s$

Then:

$$\begin{cases} x_1 = 4 + 5r - s \\ x_2 = r \\ x_3 = s \\ r, s \in \mathbb{R} \end{cases}$$

Thus,

$$\text{The set of solutions is: } \{(4 + 5r - s,\ r,\ s); r, s \in \mathbb{R}\}.$$

Here, we have 2 degrees of freedom (r, s) and thus, it is geometrically a plane.

Reducing Matrices into REF and RREF:

Consider the following operations:

1. Multiply an equation by a non-zero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Now, consider the same operations, but on matrices:

1. Multiply a row by a non-zero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These operations will allow us to reduce matrices into REF and RREF.

Application:

Let us apply these operations to a matrix:

Consider:

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & | & 12 \\ 2 & 4 & -10 & 6 & 12 & | & 28 \\ 2 & 4 & -5 & 6 & -5 & | & -1 \end{pmatrix}$$

- Step 1 - Locate the left-most non-zero column.
- Step 2 - Interchange row 1 and row 2:

$$\begin{pmatrix} 2 & 4 & -10 & 6 & 12 & | & 28 \\ 0 & 0 & -2 & 0 & 7 & | & 12 \\ 2 & 4 & -5 & 6 & -5 & | & -1 \end{pmatrix}$$

- Step 3 - Multiply the first row by $\frac{1}{2}$:

$$\begin{pmatrix} \boxed{1} & 2 & -5 & 3 & 6 & | & 14 \\ 0 & 0 & -2 & 0 & 7 & | & 12 \\ 2 & 4 & -5 & 6 & -5 & | & -1 \end{pmatrix} = \text{Row } 1 \times \frac{1}{2}$$

- Step 4 - Do Row 3 - 2Row 1:

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & | & 14 \\ 0 & 0 & -2 & 0 & 7 & | & 12 \\ 0 & 0 & 5 & 0 & -17 & | & -29 \end{pmatrix} = \text{Row } 3 - 2\text{Row } 1$$

  We can now cover and forget about row 1, since we achieved what we need. Move on to row 2.

- Step 5 - Locate the left-most non-zero column.
- Step 6 - Multiply Row 2 by $-\frac{1}{2}$:

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & | & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & | & -6 \\ 0 & 0 & 5 & 0 & -17 & | & -29 \end{pmatrix} = \text{Row } 2 \times \frac{1}{2}$$

- Step 7 - Do Row 3 - 5Row 2:

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & | & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & | & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & | & 1 \end{pmatrix} = \text{Row } 3 - 5 \times \text{Row } 2$$

- Step 8 - Multiply by 2:

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & | & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & | & -6 \\ 0 & 0 & 0 & 0 & 1 & | & 2 \end{pmatrix}$$

  Now, this matrix is in Reduced Echelon Form (REF). Let's reduce it to RREF:

- Step 9 - Do Row 2 + $\frac{7}{2}$ Row 3, and Row 1 - 6Row 3:

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 0 & | & 2 \\ 0 & 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & 2 \end{pmatrix}$$

- Step 10 - Row 1 + 5 Row 2:

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & | & 7 \\ 0 & 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & 2 \end{pmatrix}$$

We have achieved Reduced Row Echelon Form (RREF). ✅

Note:

The process to transform an augmented matrix to REF is called Gaussian elimination

The process to transform an augmented matrix to RREF is called Gauss-Jordan elimination

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Solving Example

Solve the linear system

$$\begin{cases} -2x_3 + 7x_5 = 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 = 28 \\ 2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 = -1 \end{cases}$$

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 7 & | & 12 \\ 2 & 4 & -10 & 6 & 12 & | & 28 \\ 2 & 4 & -5 & 6 & -5 & | & -1 \end{pmatrix}$$

We have previously solved this matrix.

The RREF of this augmented matrix is:

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 & | & 7 \\ 0 & 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & 2 \end{pmatrix}$$

So the new linear system of this matrix is:

$$\begin{cases} x_1 + 2x_2 + 3x_4 = 7 \\ x_3 = 1 \\ x_5 = 2 \end{cases}$$

Get the leading variable in terms of the free variables:

$$\begin{aligned} &x_1 = 7 - 2x_2 + 3x_4 \\ &x_3 = 1 \\ &x_5 = 2 \end{aligned}$$

Let $x_2 = r$ and $x_4 = s$
So,

$$\begin{cases} x_1 = 7 - 2r + 3s \\ x_2 = r \\ x_3 = 1 \\ x_4 = s \\ x_5 = 2 \\ r, s \in \mathbb{R} \end{cases}$$

Thus, the set of solutions is:

$$\boxed{\{(7 - 2r - 3s, r, 1, s, 2)\}; r, s \in \mathbb{R}}$$

Note: rank is the number of all leading variables of the system

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Take another augmented matrix:

$$\begin{pmatrix} 1 & 2 & -5 & 3 & 6 & | & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & | & -6 \\ 0 & 0 & 0 & 0 & 1 & | & 2 \end{pmatrix}$$

This is a different method to solve the matrix, which is called "back substitution":

$$\begin{cases} x_1 + 2x_2 - 5x_3 + 3x_4 + 6x_5 = 14 \\ \qquad\qquad\quad x_3 \qquad\quad - \frac{7}{2}x_5 = -6 \\ \qquad\qquad\qquad\qquad\qquad\quad x_5 = 2 \end{cases}$$

$$\begin{aligned} x_5 = 2 &\implies x_3 + \left(-\frac{7}{2}\right)(2) = -6 \\ &\implies x_3 = -6 + 7 = 1 \end{aligned}$$

$$
\begin{aligned}
&x_1 2x_2 + (-5)(1) + 3x_4 + 6(2) = 14 \\
&x_1 + 2x_2 + 3x_4 = 14 - 12 + 5 \\
&x_1 + 2x_2 + 3x_4 = 7 \\
&x_1 = 7 - 2x_2 - 3x_4 \\
&x_2 = r \\
&x_4 = s \\
&\implies x_1 = 7 - 2r - 3s
\end{aligned}
$$

The set of solutions is:

$$\{(7 - 3r - 3s, r, s); r, s \in \mathbb{R}\}$$

Definition

A linear system is said to be homogeneous if its RHS (right hand side) is zero

Theorem

A homogeneous linear system is always consisent

Proof

$$
(*) \begin{cases}
a_{11}x_1 + \cdots + a_{1n}x_n = & 0 \\
\vdots & \vdots \quad \vdots \\
a_{m_1}x_1 + \cdots + a_{mn}x_n = 0
\end{cases}
$$

So, $(0, \ldots, 0)$ is always a solution for $(*)$.

Example

$$
\begin{cases}
-2x_3 + 7x_5 = 0 \\
2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 \\
2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 = 0
\end{cases}
$$

is a homogeneous linear system.

$$
\begin{pmatrix}
0 & 0 & -2 & 0 & 7 & | & 0 \\
2 & 4 & -10 & 6 & 12 & | & 0 \\
2 & 4 & -5 & 6 & -5 & | & 0
\end{pmatrix}
$$

Clearly, the RREF of this augmented matrix is:

$$
\begin{pmatrix}
1 & 2 & 0 & 3 & 0 & | & 0 \\
0 & 0 & 1 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 0 & 1 & | & 0
\end{pmatrix}
$$

The linear system of the RREF is:

$$\begin{cases} x_1 + 2x_2 + 3x_4 = 0 \implies x_1 = -2x_2 - 3x_4 \\ x_3 = 0 \\ x_5 = 0 \\ x_2 = r, x_4 = s \implies x_1 = -2r - 3s \end{cases}$$

So, the set of solutions is:

$$\{(-2r - 3s, r, s, 0, 0); r, s \in \mathbb{R}\}$$

Note

1. If it has 1 degree of freedom then its geometric representation is a line.
2. If it has 2 degrees of freedom then its geometric representation is a plane.
3. In the case of a homogeneous linear equation, the line/plane passes through the origin (true for all dimensions.)

Theorem

If a homogeneous linear system has n unknowns, and if the RREF of its augmented matrix has r non-zero rows, then the system has $n - r$ free variables.

Proof

Each non-zero row in the RREF has a leading 1, so there are r leading 1s, and so r leading variables. So we have $n - r$ free variables. $\square$

Corollary

A homogeneous linear system with more unknowns than the equations has infinitely many solutions.

Proof

Suppose we have m equations with n unknowns and we assume that $m < n$. Let r be the number of non-zero rows in RREF, we have $r \leq m$ but $m \leq n \implies r < n$. Then,

$$\begin{aligned} n - r &> 0 \\ \text{and} \\ n - r &\geq 1 \end{aligned}$$

But $n-r$ is the number of free variables. So we can conclude that the set of solutions must be infitnite. $\square$