

Chapter 1 - Real Numbers System

0 Notation

We denote $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$, $\mathbb{R}_- = \{x \in \mathbb{R}; x \leq 0\}$, $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}_+^* = \mathbb{R}_+ - \{0\}$, $\mathbb{R}_-^* = \mathbb{R}_- - \{0\}$.

1 Introduction

The set \mathbb{N} of natural numbers is the basis for counting.

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

Since there is no elements in \mathbb{N} such that the sum with 1, or with 2, \dots gives 0, This leads to the introduction of a new set of integers.

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

Then, since there is no elements in \mathbb{Z} such that the *product* with 2, or with 3 \dots gives us 1, this leads to the construction of the set of rational numbers.

$$\mathbb{Q} = \left\{ \frac{p}{q}; (p, q) \in \mathbb{Z} \times \mathbb{Z}^* \right\}$$

We can observe that there are no rationals with a square that leads to 2. Indeed, if there exists, $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $2 = \left(\frac{m}{n}\right)^2$ and m and n are mutually prime, then $2n^2 = m^2$. This implies that 2 divides m^2 and then 2 divides m . Assume that $m = 2p$, thus $2n^2 = m^2 = 4p^2$, therefore $n^2 = 2p^2$ and 2 divides n^2 and then 2 divides n . So we find that 2 divides m and n . This contradicts the fact that m and n are mutually prime.

Numbers such as e and π are not rational.

Consequently, we need a new field, \mathbb{R} , larger than \mathbb{Q} .

2 Real Numbers

2.1 Existence and Uniqueness of \mathbb{R}

We admit the existence and uniqueness of the set of real numbers \mathbb{R} and equipped of the two internal laws $+$, \cdot and a relation \leq . such that:

- $$\left\{ \begin{array}{l} 1) \quad (\mathbb{R}, +, \cdot) \text{ is a commutative field.} \\ 2) \quad \leq \text{ is a total order relation in } \mathbb{R} \\ 3) \quad \forall (a, b, c) \in \mathbb{R}^3, \begin{cases} a \leq b \implies a + c \leq b + c \\ a \leq b \text{ and } 0 \leq c \implies ac \leq bc \end{cases} \\ 4) \quad \text{Any non empty upper bounded subset of } \mathbb{R} \text{ has a supremum in } \mathbb{R}. \end{array} \right.$$

Reminder that:

$(\mathbb{R}, +, \cdot)$ is a commutative field, meaning that:

$+$ is associative: $\forall (a, b, c) \in \mathbb{R}^3, (a + b) + c = a + (b + c)$

$+$ is commutative: $\forall (a, b) \in \mathbb{R}^2, a + b = b + a$

\mathbb{R} has a neutral element for $+$ denoted $0 : \forall a \in \mathbb{R}, a + 0 = 0 + a = a$

Any element a in \mathbb{R} has an opposite, denoted by $-a$:

$$\forall a \in \mathbb{R}, a + (-a) = (-a) + a = 0$$

\cdot is associative: $\forall (a, b, c) \in \mathbb{R}^3, (ab)c = a(bc)$

\cdot is commutative: $\forall (a, b) \in \mathbb{R}^2, ab = ba$

\mathbb{R} has an identity element for the multiplication \cdot , denoted by $1 : \forall a \in \mathbb{R}, a1 = 1a = a$.

Any element a in $\mathbb{R} - \{0\}$ has an inverse denoted by a^{-1} :

$$\forall a \in \mathbb{R} - \{0\}, aa^{-1} = a^{-1}a = 1$$

\leq is distributive relative to the addition:

$$\forall (a, b, c) \in \mathbb{R}^3, \begin{cases} a(b + c) = ab + ac \\ (b + c)a = ba + ca \end{cases}$$

\leq is a total order relation in \mathbb{R} , meaning that:

\leq is reflexive: $\forall a \in \mathbb{R}, a \leq a$

\leq is antisymmetric: $\forall (a, b) \in \mathbb{R}^2, a \leq b \text{ and } b \leq a \implies a = b$

\leq is transitive: $\forall (a, b, c) \in \mathbb{R}^3, a \leq b \text{ and } b \leq c \implies a \leq c$

\leq is total: $\forall (a, b) \in \mathbb{R}^2, (a \leq b \text{ or } b \leq a)$

For any pair $(a, b) \in \mathbb{R}^2, a < b$ means that $a \leq b$ and $a \neq b$. We can use the notation $b \geq a$ (resp. $b > a$) instead of $a \leq b$ (resp. $a < b$).

The elements in \mathbb{R} are called **Real Numbers**.

Definitions 1.1

Let A be a non-empty set of \mathbb{R} .

- A is said to be **Upper Bounded** in \mathbb{R} if there exists $k \in \mathbb{R}$ such that

$$\forall a \in A, a \leq k$$

k is called an **upper bound** of A in \mathbb{R} .

- A is said to be **Lower Bounded** in \mathbb{R} if there exists $k' \in \mathbb{R}$ such that

$$\forall a \in A, k' \leq a$$

k' is called a **lower bound** of A in \mathbb{R}

- A is said to be **bounded** if it is upper, and lower bounded.
- We name **Supremum** of A in \mathbb{R} the least of all upper bounds of A in \mathbb{R} , if it exists; this element is then denoted $Sup_{\mathbb{R}}(A)$ or $Sup(A)$.
- We name **Infimum** of A in \mathbb{R} the greatest lower bound of A in \mathbb{R} , if it exists; this element is then denoted $Inf_{\mathbb{R}}(A)$ or $Inf(a)$.

Remarks

- If k is an upper bound of A in \mathbb{R} , then for any $l \geq k$, l is also an upper bound of A in \mathbb{R} . Therefore, an upper bounded set has infinitely many upper bounds.
- If k' is a lower bound of A in \mathbb{R} , then for any $l' \leq k'$, l' is also a lower bound of A in \mathbb{R} . Therefore, a lower bounded set has infinitely many lower bounds.

Examples

- $[0, 1[$ and $]0, 1]$ are bounded.
- \mathbb{N} is lower bounded, but not upper bounded.
- \mathbb{Z} is not lower bounded, nor is it upper bounded.
- $Sup_{\mathbb{R}}([0, 1[) = Sup_{\mathbb{R}}(]0, 1]) = 1$ and $Inf_{\mathbb{R}}([0, 1]) = Inf_{\mathbb{R}}(]0, 1[) = 0$.
- $Sup(]-1, +\infty[)$ doesn't exist and $Inf(]-1, +\infty)) = -1$.

Remark

In order to establish that a real number α is the **supremum** of a non-empty set A in \mathbb{R} , first we show that α is an upper bound of A in \mathbb{R} , then show that any other upper bound of A is greater than or equal to α . That is:

$$\begin{cases} \forall x \in A, & x \leq \alpha \quad \text{that is } \alpha \text{ is an upper bound of } A \\ \forall c \in \mathbb{R}, & (c \text{ upper bound of } A \implies \alpha \leq c) \end{cases}$$

or:

$$\begin{cases} \forall x \in A, & x \leq \alpha \\ \forall \epsilon \in \mathbb{R}_+^*, & \exists x \in A, \alpha - \epsilon < x \leq \alpha \end{cases}$$

Indeed, let $\epsilon \in \mathbb{R}_+^*$, we have $\alpha - \epsilon < \alpha$, then $\alpha - \epsilon$ is not an upper bound. Then there exists $x \in A$ such that $\alpha - \epsilon < x \leq \alpha$.

Conversely, if there exists an upper bound c of A such that $c < \alpha$, then let $\epsilon = \alpha - c$, and using the hypothesis, there exists $x \in A$ such that $\alpha - \epsilon < x$, then $c < x$ which is impossible since c is an upper bound of A .

Similarly, to establish that a real number β is the **infimum** of a non-empty subset A of \mathbb{R} , first we show that β is a lower bound of A in \mathbb{R} and then, we show that any lower bound is less than or equal to β . That is:

$$\begin{cases} \forall x \in A, & \beta \leq x \quad \text{that is } \beta \text{ is a lower bound of } A \\ \forall c \in \mathbb{R}, & (c \text{ lower bound of } A \implies c \leq \beta) \end{cases}$$

or:

$$\begin{cases} \forall x \in A, & \beta \leq x \\ \forall \epsilon \in \mathbb{R}_+^*, & \exists x \in A, \quad \beta \leq x < \beta + \epsilon \end{cases}$$

Definitions 1.2

One defines in \mathbb{R} nine types of **intervals**, for $(a, b) \in \mathbb{R}^2$ such that $a \leq b$.

$[a, b] = \{x \in \mathbb{R}; a \leq x \leq b\}$, is said to be closed and bounded or simply a segment

$$[a, b[= \{x \in \mathbb{R}; a \leq x < b\} \qquad]a, b] = \{x \in \mathbb{R}; a < x \leq b\}$$

$$]a, b[= \{x \in \mathbb{R}; a < x < b\} \qquad]-\infty, a[= \{x \in \mathbb{R}; x < a\}$$

$$]-\infty, a] = \{x \in \mathbb{R}; x \leq a\} \qquad]a, +\infty[= \{x \in \mathbb{R}; a < x\}$$

$$[a, +\infty[= \{x \in \mathbb{R}; a \leq x\} \qquad]-\infty, +\infty[= \mathbb{R}$$

The intervals $[a, b]$, $]-\infty, a]$, $[a, +\infty[$, and $]-\infty, +\infty[$ are **closed**.

The intervals $]a, b[$, $]-\infty, a[$, $]a, +\infty[$, and $]-\infty, +\infty[$ are **open**.

The intervals $[a, b[$, $]a, b]$ are **semi closed** or **semi open**.

With these notations, the real numbers a and b are named the **endpoints** of the interval

2.2 Elementary Properties of Real numbers

$$1) \forall x, y, z \in \mathbb{R}, (x \leq y \implies x + z \leq y + z)$$

$$2) \forall x, y, u, v \in \mathbb{R}, (x \leq y \text{ and } u \leq v \implies x + u \leq y + v).$$

Indeed,

$$\begin{cases} x \leq y \implies x + u \leq y + u \\ u \leq v \implies u + y \leq v + y \end{cases} \implies x + u \leq y + v$$

Then, by induction, for any $n \in \mathbb{N}^*$, $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$:

$$(\forall i \in \{1, \dots, n\}, x_i \leq y_i) \implies \sum_{i=1}^n x_i \leq \sum_{i=1}^n y_i$$

$$3) \forall x \in \mathbb{R}^*, (0 < x \iff 0 < \frac{1}{x}). \text{ (Generally, we use the notation } \frac{1}{x} \text{ instead of } x^{-1} \text{).}$$

$$4) \forall x, y \in \mathbb{R}, \forall z \in \mathbb{R}^*, (x \leq y \implies xz \leq yz).$$

$$5) \forall x, y, u, v \in \mathbb{R}, (0 \leq x \leq y \text{ and } 0 \leq u \leq v \implies xu \leq yv).$$

Indeed,

$$\begin{cases} x \leq y \text{ and } 0 \leq u \implies xu \leq yu \\ u \leq v \text{ and } 0 \leq y \implies yu \leq yv \end{cases} \implies xu \leq yv$$

Also, by induction, we show that for all $n \in \mathbb{N}^*$, $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$:

$$(\forall i \in \{1, \dots, n\}, 0 \leq x_i \leq y_i) \implies \prod_{i=1}^n x_i \leq \prod_{i=1}^n y_i$$

In particular, $\forall n \in \mathbb{N}, \forall (x, y) \in \mathbb{R}^2, (0 \leq x \leq y \implies x^n \leq y^n)$

$$6) \forall (x, y) \in (\mathbb{R}^*)^2, \left(x < y \iff \frac{1}{y} < \frac{1}{x}\right).$$

$$7) \forall x, y, u, v \in \mathbb{R}, (x \leq y \text{ and } u < v \implies x + u < y + v)$$

If $x \leq y$ and $u \leq v$, then $x \leq y$ and $u \leq v$. Thus by 2), $x + u \leq y + v$. If $x + u = y + v$, then $x - y = v - u$. But $x - y \leq 0$, and $v - u \geq 0$. This gives $x - y = v - u = 0 \implies u = v$ which is impossible since $u < v$. We conclude that $x + u < y + v$.

We deduce that for any $n \in \mathbb{N}^*, x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$:

$$(\forall i \in \{1, \dots, n\}, x_i \leq y_i) \text{ and } (\exists i_0 \in \{1, \dots, n\}, x_{i_0} < y_{i_0}) \implies \sum_{i=1}^n x_i < \sum_{i=1}^n y_i$$

This property is generally used in the following form:

$$(\forall i \in \{1, \dots, n\}, x_i \leq y_i) \text{ and } \sum_{i=1}^n x_i = \sum_{i=1}^n y_i \implies (\forall i \in \{1, \dots, n\}, x_i = y_i).$$

2.3 Absolute Value

Definition 1.3

The absolute value of a real number x is the positive (or null) real number $|x|$, defined by:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Properties of the absolute value:

1. $\forall x \in \mathbb{R}, |x| \geq 0$
2. $|x| = 0 \iff x = 0$
3. $\forall x \in \mathbb{R}, x \leq |x|$
4. $\forall x, y \in \mathbb{R}, |xy| = |x||y|$
5. triangular inequality: $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$
6. $\forall x, y \in \mathbb{R}, ||x| - |y|| \leq |x - y|$

2.4 Distance on \mathbb{R}

Definition 1.4

The distance between two real numbers x and y is the positive real number

$$d(x, y) = |x - y|$$

Properties of the distance:

1. Positivity: $\forall x, y \in \mathbb{R}, d(x, y) \geq 0$
2. Symmetry: $\forall x, y \in \mathbb{R}, d(x, y) = d(y, x)$
3. Separation: $\forall x, y \in \mathbb{R}, d(x, y) = 0 \iff x = y$
4. Triangular Inequality: $\forall x, y, z \in \mathbb{R}, d(x, z) \leq d(x, y) + d(y, z)$

2.5 Integer Part of a Real Number

Definition 1.5

For any real number x , there exists a unique integer $p \in \mathbb{Z}$ that satisfies

$$p \leq x < p + 1$$

This integer p is called the integer part of x and it will be denoted $E(x)$ or $\lfloor x \rfloor$.

Examples:

1. $E(3,4) = 3$
2. $E(-2,1) = -3$

Definition 1.6

The integer part of a real number x is the greatest integer less or equal to x and we have

$$\forall x \in \mathbb{R}, E(x) \leq x < E(x) + 1$$

2.5 Neighborhood of a Real Number

Definition 1.7

Let N be a non-empty subset of \mathbb{R} . V is called the neighborhood of $x \in \mathbb{R}$ if N contains an open interval centered at x . More precisely:

$$\exists \alpha > 0,]x - \alpha, x + \alpha[\subset V$$

Properties of the neighborhoods of a point

1. If N is a neighborhood of x , then $x \in N$.
2. If N is a neighborhood of x , and $N \subset M$, then M is a neighborhood of x .
3. If N_1 and N_2 are neighborhoods of x , then $N_1 \cap N_2$ is a neighborhood of x .
4. Any open interval is a neighborhood of any of its points.
5. If $x, y \in \mathbb{R}$, such that $x \neq y$, then there exists N_1 neighborhood of x and N_2 neighborhood of y such that $N_1 \cap N_2 = \emptyset$. The set \mathbb{R} is said to be separated.

Neighborhood of $+\infty$

A neighborhood of $+\infty$ is of the form $]A, +\infty[$, $A \in \mathbb{R}$.

A neighborhood of $-\infty$ is of the form $] - \infty, A[$, $A \in \mathbb{R}$.

2.6 Adherent (closure) Point

Definition 1.8

A subset A of \mathbb{R} . We say that a real number a is an adherent point to A if any neighborhood of a intersects A . In other words, if

$$\forall \epsilon > 0,]a - \epsilon, a + \epsilon[\cap A \neq \emptyset$$

Example

let $A =]0, 1[$. The point 0 is an adherent point to A since

$$\forall \epsilon > 0,] - \epsilon, \epsilon[\cap]0, 1[\neq \emptyset$$

but the real number 2 is not an adherent point to A because there exists an open interval (for example) $]1.5, 2.5[$ centered at 2 that does not intersect A .

Theorem

Any real number x is an adherent point to \mathbb{Q} . Equivalently

$$\forall a, b \in \mathbb{R}, \text{ such that } a < b,]a, b[\cap \mathbb{Q} \neq \emptyset$$

Definition 1.9

The set of rational numbers \mathbb{Q} is then said to be dense in \mathbb{R} . We have $\bar{\mathbb{Q}} = \mathbb{R}$.
