M1103 Chapter 4

Vector Spaces

Definition

A <u>vector space</u> over the field $\mathbb{K}$ is a set E with two operations:

1. $$\begin{aligned} + : E \times E &\longrightarrow E \\ (x, y) &\longmapsto x + y \end{aligned}$$

2. $$\begin{aligned} \mathbb{K} \times E &\longrightarrow E \\ (\alpha, x) &\longmapsto \alpha x \end{aligned}$$

   Satisfying the following axioms:

A1: $(x+y)+z = x+(y+z), \ \ \forall x, y, z \in E$

A2: There is an element $0_E \in E$ such that $x + 0_E = 0_E + x = x, \ \ \forall x \in E$

A3: $\forall x \in E$, there is an element $-x \in E$ such that $x + (-x) = (-x) + x = 0_E$

A4: $x + y = y + x, \ \ \forall x, y \in E$

A5: $\alpha(x+y) = \alpha x + \alpha y, \forall \alpha \in \mathbb{K}, \forall x, y \in E$

A6: $(\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in \mathbb{K}, \forall x \in E$

A7: $\alpha(\beta x) = (\alpha\beta)x, \forall \alpha, \beta \in \mathbb{K}, \forall x \in E$

A8: $1x = x, \forall x \in E$

The elements of a vector space E are called vectors. The element 0_E is called the zero-vector.

Examples

1. Consider a singleton $\{0\}$.

$$\begin{aligned} \{0\} \times \{0\} &\longrightarrow \{0\} \\ (0, 0) &\longmapsto 0 \end{aligned}$$

$$0 + 0 := 0$$

$$\mathbb{K} \times \{0\} \longrightarrow \{0\}$$
$$(\alpha, 0) \longmapsto 0$$

$$\alpha 0 = 0$$

The 8 axioms are clearly satisfied

$$\therefore \{0\} \text{ is a vector space over } \mathbb{K}$$

called a zero-vector space.

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2. $\mathbb{K}^n = \mathbb{K} \underbrace{\times \cdots \times}_{n-times} \mathbb{K}$

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$
$$\alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$$

$\mathbb{K}^n$ becomes a vector space for these operations.

Zero-vector in $\mathbb{K}^n$: $(0, \ldots, 0)$

Exercise:

Check that the 8 axioms are satisfied for the above example:

Consider a vector space $\mathbb{K}^n$.
Let us take $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n),\ z = (z_1, \ldots, z_n),\ \forall x, y, z \in \mathbb{K}^n$. Let α, β scalars in $\mathbb{K}$.

A1: Associativity of addition -

$$(x + y) + z = ((x_1 + y_1) + z_1, \ldots) = (x_1 + (y_1 + z_1), \ldots) = x_1 + (y_1 + z_1)$$

$$\therefore \text{ Associativity holds.}$$

A2: Existence of an element 0 -

$$x + 0 = (x_1 + 0, \ldots, x_n + 0) = x$$
$$Where,\ 0 = (0, \ldots, 0)$$

A3: Existence of an additive inverse -

Take $-x = (-x_1, \ldots, -x_n) \in \mathbb{K}^n$

$$x + (-x) = (x_1 - x_1, \ldots, x_n - x_n) = (0, \ldots, 0) = 0$$

Which holds, since $0 = (0, \ldots, 0) \in \mathbb{K}^n$

A4: Commutativity of additions -

$$x + y = (x_1 + y_1, \ldots, x_n + y_n) = (y_1 + x_1, \ldots, y_n + x_n) = y + x$$

A5: Scalar distributivity over addition -

$$\alpha(x + y) = \alpha(x_1 + y_1, \ldots, x_n + y_n) = (\alpha x_1 + \alpha y_1, \ldots, \alpha x_n + \alpha y_n) = \alpha x + \alpha y$$

A6: Scalar addition over scalar multiplication -

$$(\alpha + \beta)x = ((\alpha + \beta)x_1, \ldots, (\alpha + \beta)x_n) = (\alpha x_1 + \beta x_1, \ldots, \alpha x_n + \beta x_n) = \alpha x + \beta x$$

A7: Compatibility of Scalar multiplication -

$$\alpha(\beta x) = \alpha(\beta x_1, \ldots, \beta x_n) = (\alpha\beta x_1, \ldots, \alpha\beta x_n) = (\alpha\beta)x$$

A8: Identity element in scalar multiplication -

$$1 \cdot x = (1 \cdot x_1, \ldots, 1 \cdot x_n) = x \quad \square.$$

Example 2

Consider $\mathbb{K}^{\mathbb{N}}$. It is the set of all maps

$$\begin{aligned}\{f : \mathbb{N} &\longrightarrow \mathbb{K}\} \\ n &\mapsto f(n)\end{aligned}$$

So, $\mathbb{K}^{\mathbb{N}}$ is the set of all sequences of elements in $\mathbb{K}$.

$$(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$$

$$\lambda(a_n)_{n\in\mathbb{N}} = (\lambda a_n)_{n\in\mathbb{N}}$$

Conclusion:

The set of sequences becomes a vector space for these operations.

Zero-vector $= (0, \ldots, 0)$
opposite: $(-a_n)_{n\in\mathbb{N}}$

Exercise

Check that the 8 axioms hold for the previous example

Let $\mathbb{K}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{K}\}$. We denote a sequence by $(a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$.

We denote $u = (u_n), v = (v_n), w = (w_n) \in \mathbb{K}^{\mathbb{N}}$. Let $\alpha, \beta \in \mathbb{K}$

A1: Associativity of addition -

$$(u+v)+w = ((u_n+v_n)+w_n) = (u_n+(v_n+w_n)) = u+(v+w), \ \ \forall n \in \mathbb{K}$$

A2: Existence of an element 0 -

$$u+0 = (u_n+0) = (u_n) = u$$

$$Where, \ 0 = (0,0,0\ldots,0)$$

A3: Existence of an additive inverse -

Let $-u = (-u_n)$:

$$u+(-u) = (u_n - u_n) = 0$$

A4: Commutativity of additions -

$$u+v = (u_n+v_n) = (v_n+u_n) = v+u$$

Because addition in $\mathbb{K}$ is commutative.

A5: Scalar distributivity over addition -

$$\alpha(u+v) = \alpha(u_n+v_n) = \alpha u_n + \alpha v_n = \alpha u + \alpha v$$

A6: Scalar addition over scalar multiplication -

$$(\alpha+\beta)u = ((\alpha+\beta)u_n)(\alpha u_n + \beta u_n) = \alpha u + \beta u$$

A7: Compatibility of scalar multiplication -

$$\alpha(\beta u) = \alpha(\beta u_n) = (\alpha\beta u_n) = (\alpha\beta)u$$

A8: Identity element for scalar multiplication -

$$1 \cdot u = 1 \cdot u_n = u_n = u \qquad \square.$$

Example 4

Consider $(M_{m,n}(\mathbb{K}), +, scalar\ multiplication)$

This is a vector space over $\mathbb{K}$.

Example 5

Consider X a non-empty set. $F(X,\mathbb{K}) = \{f : X \longrightarrow \mathbb{K}\ function\} = \mathbb{K}^X$

1. Given $f, g \in F(X,\mathbb{K})$, and given $\alpha \in \mathbb{K}$,

$$\begin{aligned} f+g: X &\longrightarrow \mathbb{K} \\ x &\longmapsto (f+g)(x) = f(x)+g(x) \end{aligned}$$

$$\begin{aligned} \alpha f: X &\longrightarrow \mathbb{K} \\ x &\longmapsto (\alpha f)(x) = \alpha f(x) \end{aligned}$$

Thus, $f+g \in F(X,\mathbb{K})$, and $\alpha f \in F(X,\mathbb{K})$

$$\begin{aligned} 0: X &\longrightarrow \mathbb{K} \\ x &\longmapsto 0(x) = 0 \end{aligned}$$

$$\begin{aligned} -f: X &\longrightarrow \mathbb{K} \\ x &\longmapsto (-f)(x) = -f(x) \end{aligned}$$

Then $F(X,\mathbb{K})$ is a vector space over $\mathbb{K}$.

Exercise

Show that the 8 axioms hold
at home

Example 6

Consider $\mathbb{K}[X]$ the set of polynomials with coefficients in $\mathbb{K}$.

Given $P, Q \in \mathbb{K}[X]$,

$$P+Q := \sum_{n\in\mathbb{N}} (a_n+b_n)X^n \in \mathbb{K}X$$

Given $\lambda \in \mathbb{K}$

$$\lambda P := \sum_{n\in\mathbb{N}} (\lambda a_n)X^n \in \mathbb{K}[X]$$

Thus, $\mathbb{K}X$ is a vector space over $\mathbb{K}$

Example 7

Suppose E_1, E_2 vector spaces over $\mathbb{K}$,

$$E_1 \times E_2 = \{(x_1,x_2); x_1 \in E_1, x_2 \in E_2\}$$

Given $(x_1,x_2),(y_1,y_2) \in E_1 \times E_2$, given $\alpha \in \mathbb{K}$

$$(x_1, x_2) + (y_1 + y_2) := (x_1 + y_1, x_2 + y_2)$$

$$\alpha(x_1, x_2) := (\alpha x_1, \alpha x_2)$$

Thus, $E_1 \times E_2$ is also a vector space over $\mathbb{K}$.

$0_{E_1 \times E_2} = (0_{E_1}, 0_{E_2})$

Theorem

E vector space over $\mathbb{K}$, $x, y \in E$, $\alpha, \beta \in \mathbb{K}$.

Then,

1. $\alpha(x - y) = \alpha x - \alpha y$
2. $(\alpha - \beta)x = \alpha x - \beta x$
3. $0x = 0_E$
4. $\alpha 0_E = 0_E$
5. $(-1)x = -x$
6. $\alpha x = 0_E \implies (\alpha = 0 \ or \ x = 0_E)$

Proof

1. $\alpha(x - y) = \alpha x - \alpha y$

By A5:

$$= \alpha[x - y + y]$$

By A3:

$$= \alpha(x + 0_E)$$

By A2:

$$= \alpha x$$

Now,

$$\begin{aligned} \alpha(x-y) + \alpha y &= \alpha x \\ \implies \alpha(x-y) + \alpha y + (-\alpha y) &= \alpha x + (-\alpha y) \\ By \ A3 \ : \implies & \alpha x - \alpha y \end{aligned}$$

2. Similar to 1 do as exercise at home
3. $0x = (0 - 0)x = 0x - 0x = 0_E$
4. $\alpha 0_E = \alpha(0_E - 0_E) = \alpha 0_E - \alpha 0_E = 0_E$
5. $x + (-1)x = 1x + (-1)x = [1 + (-1)]x = 0x = 0_E$

6. Suppose $ax = 0$ and $\alpha \neq 0$
   then $\alpha^{-1}\alpha x = \alpha^{-1}0_E \implies (\alpha^{-1}\alpha)x = 0_E \implies 1x = 0_E \implies x = 0_E$

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§ 4.2

Definition

E vector space over $\mathbb{K}$.

A linear sub-space of E is a subset F of E closed under addition and scalar multiplication such that F is itself a vector space over $\mathbb{K}$ for the operations induced on F.

Theorem

E vector space over $\mathbb{K}$, $F \subset E$

$$F \text{ linear subspace of } E \iff \begin{cases} 1. \quad 0_E \in F. \\ \\ 2. \quad \forall x, y \in F, (x+y) \in F. \\ \\ 3. \quad \forall \alpha \in \mathbb{K}, \forall x \in F, \alpha x \in F \end{cases}$$

Proof

2 and 3 are trivial.

<u>PROOF OF 1:</u>

1. Let 0_F be the zero-vector of F.
   we have

$$\forall x \in E, x + 0_E = x$$

In particular,

$$0_F + 0_E = 0_F$$

On the other hand,

$$\forall y \in F, y + 0_F = y$$

In particular,

$$0_F + 0_F = 0_F$$

From the previous two statements, combining them together yields us:

$$\implies 0_F + 0_E = 0_F + 0_F$$

$$\implies 0_E = 0_F$$

But 0_F belongs to F

$$\therefore 0_E \in F$$

2. Let us now prove the converse.

F is closed under addition and scalar multiplication.

$$\forall x, y \in F, \underbrace{(-y)}_{-1y} \in F$$

so $x - y \in F$. Thus, since $F \neq \emptyset$, F is an additive subgroup of E, so $(F, +)$ is an abelian group

Then, it is easy to check that $A5$ to $A8$ are satisfied. $\square$

Example 1

Let E be a vector space over $\mathbb{K}$

Consider the singleton $\{0_E\} \subset E$

1. $0_E \in \{0_E\}$
2. $0_E + 0_E = 0_E \in \{0_E\}$
3. $\forall \lambda \in \mathbb{K}, \lambda 0_E = 0_E \in \{0_E\}$

So, $\{0_E\}$ is a linear subspace of E.

Also, $E \subset E$ and E is a linear subspace of E.

Example 2

Consider the vector space $\mathbb{K}^n$.

Let $a \in \mathbb{K}^n - \{0\}$

$$\mathbb{K}a := \{\lambda a, \lambda \in \mathbb{K}\}.$$

1. $0 = 0a$, so $0 \in \mathbb{K}a$
2. Let $x, y \in \mathbb{K}a$. Thus, $x = \lambda a, y = \lambda' a, \forall \lambda \in \mathbb{K}$ then, $x + y = \lambda a + \lambda' a = a(\lambda + \lambda') \in \mathbb{K}a$
3. Let $x \in \mathbb{K}a, \alpha \in \mathbb{K}$, then $x = \lambda a$, for some $\lambda \in \mathbb{K}$. Then $\alpha x = \alpha(\lambda a) = (\alpha\lambda)a = (\lambda\alpha)a \in \mathbb{K}a$.

Thus $\mathbb{K}a$ is a linear subspace of $\mathbb{K}^n$.

Example 3

Consider the vector space $\mathbb{K}^n$, take $a \in \mathbb{K}^n$ and $b \in \mathbb{K}^n$ such that $b \notin \mathbb{K}a$

$$F := \{\lambda a + \mu b, \lambda, \mu \in \mathbb{K}\}$$

$$x \in F \iff x = \lambda a + \mu b$$

1. $0 = 0a + 0b \in F$
2. Let $x, y \in F$, $x = \lambda a + \mu b$, $y = \lambda' a + \mu' b$.
   $x + y = \lambda a + \mu b + \lambda' a + \mu' b = a(\lambda + \lambda') + b(\mu + \mu') \in F$
3. $\forall \alpha \in \mathbb{K}, \forall x \in F$, $x = \lambda a + \mu b$, so $\alpha x = \alpha\lambda a + \alpha\mu b = (\alpha\lambda)a + (\alpha\mu)b \in F$.

Thus F is a linear subspace of $\mathbb{K}^n$.

Example 4

let $\mathbb{R}^2$ linear subspace

$$F := \{(x_1, x_2) \in \mathbb{R}^2, x_1, x_2 \geq 0\}$$

F is NOT a linear subspace of $\mathbb{R}^2$

Example 5

Consider the vector space $\mathbb{K}^{\mathbb{N}}$, let $F = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}\ such\ that\ a_n\ converges\}$

Then, F is a linear subspace of $\mathbb{K}^{\mathbb{N}}$.

Consider the vector space $\mathbb{K}^{\mathbb{N}}$, let $G = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}\ such\ that\ a_n\ is\ bounded\}$

Then, G is a linear subspace of $\mathbb{K}^{\mathbb{N}}$

Example 6

Consider the vector space $M_n(\mathbb{K})$. Let
$F = \{A \in M_n(\mathbb{K}) : F\ is\ upper/lower\ triangular/diagonal\}$

Thus, F is a linear subspace of $M_n(\mathbb{K})$

Example 7

In $M_n(\mathbb{K})$, each of $MS_n(\mathbb{K})$ and $MA_n(\mathbb{K})$ is a linear subspace of $M_n(\mathbb{K})$.

Example 8

In $M_n(\mathbb{K})$, $GL_n(\mathbb{K})$ is not a linear subspace of $M_n(\mathbb{K})$ because a 0 matrix does not belong to $GL_n(\mathbb{K})$

Example 9

Suppose I an interval of $\mathbb{R}$. In $F(I,\mathbb{K})(=\mathbb{K}^I)$, consider $C(I,\mathbb{K}) = \{I \mapsto \mathbb{K} : f\ is\ continuous\}$

Then $C(I,\mathbb{K})$ is a linear subspace of $F(I,\mathbb{K})$.

Example 10

In $\mathbb{K}[X]$, let $\mathbb{K}_n[X] = \{P \in \mathbb{K}[X] : \deg P \leq n\}$

- $\deg 0 = -\infty \leq n$
- closed under addition
- closed under scalar multiplication

So $\mathbb{K}_n[X]$ is a linear subspace of $\mathbb{K}[X]$.

Theorem

Suppose E is a vector space over $\mathbb{K}$, suppose $F_1, \ldots, F_n$ are linear subspaces of E. Then,

$$F_1 \cap \cdots \cap F_n$$

is a linear subspace of E.

Proof

- $0_E \in F_i, \forall i \implies 0_E \in F_1 \cap \cdots \cap F_n$.

Let $x, y \in F_1 \cap \cdots \cap F_n$

- $\forall i, x, y \in F_i \implies x + y \in F_i$
  Therefore, $x + y \in F_1 \cap \cdots \cap F_n$

Let $\alpha \in \mathbb{K}, x \in F_1 \cap \cdots \cap F_n$

- $\forall i, \alpha \in \mathbb{K}, x \in F_i \implies \alpha x \in F_i$
  Therefore, $\alpha x \in F_1 \cap \cdots \cap F_n$. $\square$

Remark

Still true for an arbitrary family $(F_i)_{i \in I}$ linear subspace of E. (Then $\bigcap F_i$ is also a linear subspace).

Remark

In general, the union of two linear sub-spaces is not a linear sub-space.

Definition

E vector space over $\mathbb{K}$. $S \subset E$

The linear subspace spanned by S is the smallest linear subspace of E that contains S. It is denoted by

$$\mathrm{span}(S).$$

Definition

let E vector space over $\mathbb{K}$. F, G linear subspace of E

The sum of F and G is the linear subspace $F + G = \mathrm{span}(\text{F U G})$.

Example 11

In $\mathbb{R}^3$, let $F = \mathbb{R}(1,0,0)$ and $G = \mathbb{R}(0,1,0)$

Then, $F+G=\{\lambda(1,0,0)+\mu(0,1,0), \forall\lambda,\mu\in\mathbb{R}\}=\{(\lambda,\mu,0),\lambda,\mu\in\mathbb{R}\}$.

Theorem

E vector space over $\mathbb{K}$. F, G linear subspaces of E.
Then, $F+G=\{x+y; x\in F\ and\ y\in G\}$.

Proof

Let us denote $H=\{x+y; x\in F\ and\ y\in G\}$.

- $F\subset H\ (\forall x\in F, x=x+0_E, so\ x\in H)$.
- $G\subset H\ (\forall y\in G, y=0_E+y, so\ y\in G)$.

Thus, $(F\ U\ G)\in H$. □

1. $0_E=0_E+0_E,\ so\ 0_E\in H$
2. Let $z,z'\in H$
   Then, $z=x+y$ for some $x\in F\ and\ y\in G$
   Also, $z'=x'+y'$ for some $x'\in F\ and\ y'\in G$

Then,

$$z+z'=(x+y)+(x'+y')$$

So, $z,z'\in H$

3. Let $\alpha\in\mathbb{K}, z\in H$
   Then, $z=x+y$ for some $x\in F, y\in G$

Thus,

$$\alpha z=\alpha(x+y)=\alpha x+\alpha y$$

Ergo, H is a linear subspace of E.

Let H' be a linear subspace of E that contains $F\ U\ G$.

Let $z\in H$. Then $z=x+y;\ x\in F, y\in G$.

So $x\in H'$, also $y\in H'$.

Therefore, $x+y\in H'$ (since H' is a linear subspace)

Then $z\in H'$.

We conclude that $H\subset H'$. □.

Definition

E vector space over $\mathbb{K}$, F, G linear subspaces of E.

We say that F and G are in direct sum if $F \cap G = \{0_E\}$. In this case, we write

$$F \oplus G$$

Example 12

In $\mathbb{R}^3$, let $F = \mathbb{R}(1,0,0)$ and $G = \mathbb{R}(0,1,0)$

Then $F \cap G = \{(0,0,0)\}$ thus F and G are in direct sum.

Then $F + G = \{(\lambda, \mu, 0); \lambda, \mu \in \mathbb{R}\}$

Example 13

In $\mathbb{R}^3$, let $F = \mathbb{R}(1,0,0) \oplus \mathbb{R}(0,1,0)$, and $G = \mathbb{R}(1,0,0) \oplus \mathbb{R}(0,0,1)$

Then $F \cap G = \mathbb{R}(1,0,0) \neq \{(0,0,0)\}$

Thus, F and G are NOT in direct sum.

In fact, $F + G = \mathbb{R}^3$.

Indeed, $\forall x, y, z \in \mathbb{R}^3$,

$$(x, y, z) = \underbrace{(49x, y, 0)}_{\in F} + \underbrace{(-48x, 0, z)}_{\in G}$$

Definition

E vector space over $\mathbb{K}$. F, G linear subspace of E.

We say that F and G are <u>complementary</u> in E if

$$E = F \oplus G$$

Example 14

In $\mathbb{R}^3$, let $F=\mathbb{R}(1,0,0)\oplus\mathbb{R}(0,1,0)$, and $G=\mathbb{R}(0,0,1)$. Then $F\cap G=\{(0,0,0)\}$

Also, we have $F+G=\mathbb{R}^3$.

Indeed. $\forall x,y,z\in\mathbb{R}^3$, we can write

$$(x,y,z)=\underbrace{(x,y,0)}_{\in F}+\underbrace{(0,0,z)}_{\in G}$$

Thus, we conclude that $\mathbb{R}^3=F\oplus G$

In other words, F and G are complimentary in $\mathbb{R}^3$.

Theorem

E vector space over $\mathbb{K}$. F and G linear subspaces over E.

Then the following statements are equivalent

1. F and G are complimentary in E
2. $\forall z\in E,\exists!(x,y)\in F\times G; z=x+y$

Proof

- $1 \implies 2$:

We have existence since $E=F+G$.

Let $z\in E$, suppose $\exists x,y\in F\times G; z=x+y$. $\exists z=x'+y'$

Indeed,

$$x+y=x'+y'$$

So, $\underbrace{x-x'}_{\in F}=\underbrace{y'-y}_{\in G}$.

Thus, $x-x'\in F\cap G=\{0_E\}$.

Then, $x=x'$, and so $y=y'$.

- $2 \implies 1$:

Existence gives us $E=F+G$

Indeed, let $z\in F\cap G$. We can write $z=\underbrace{z}_{\in F}+\underbrace{0_E}_{\in G}$ z=\underbrace{0*{E}}*{\in F}+\underbrace{z}*{\in G}\implies z=0*{E}$ by the uniqueness of the decomposition.

Thus, $F \cap G = \{0_E\}$. $\square$

Remark

$F_1 + \cdots + F_n = \operatorname{span}(F_1 \cup \cdots \cup F_n) = \{x_1 + \ldots x_n ; x_i \in F_i, \forall i\}.$

$F_1 + \ldots F_n$ is direct $\iff F_i \cap (F_1 + \cdots + F_{i-1}) = \{0_E\},\ \forall i$

Then we write $F_1 \oplus \cdots \oplus F_n$

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§ 4.3

Definition

An almost-zero family of scalars is a family $(\lambda_i)_{i\in I}$ of elements of $\mathbb{K}$ such that

$$\{i \in I : \lambda_i \neq 0\} \text{ is finite}$$

the set of almost-zero families of scalars is denoted by $\mathbb{K}^I$

Definition

Let E be a vector space over $\mathbb{K}$

$(a_i)_{i\in I}$ be a family of vectors in E, $x \in E$

We say that x is a linear combination of $(a_i)_{i\in I}$ if

$$\exists (\lambda_i)_{i\in I} \in \mathbb{K}^I : x = \sum_{i\in I} \lambda_i a_i$$

Example

Show that $x = (9, 2, 7)$ is a linear combination of $a = (1, 2, -1)$ and $b = (6, 4, 2)$

We need to show that $x = \lambda a + \mu b$ for some $\lambda, \mu \in \mathbb{R}$

$$(9,2,7) = \lambda(1,2,-1) + \mu(6,4,2) = (\lambda + 6\mu, 2\lambda + 4\mu. -\lambda + 2\mu)$$

$$\implies \begin{cases} \lambda + 6\mu = 9 \\ 2\lambda + 4\mu = 2 \\ -\lambda + 2\mu = 7 \end{cases}$$

Solving this linear system yields $\lambda = -3$ and $\mu = -2$

So $x = -3a + 2b$

Theorem

Let E be a vector space over $\mathbb{K}$.

$(a_i)_{i \in I}$ be a family of vectors in E

the set of all linear combinations of $(a_i)_{i \in I}$ is $\text{span}\{a_i, i \in I\}$

Proof

Let F be the set of all linear combinations of $(a_i)_{i \in I}$

- $\forall k \in I, a_k \in F$?

we need to show that $a_k = \sum_{i \in I} \lambda_i a_i$ for some $(\lambda_i)_{i \in I} \in \mathbb{K}^I$

it suffices to take

$$\lambda_i = \begin{cases} 1 & if \ \ i = k \\ \\ 0 & if \ \ i \neq k \end{cases}$$

Then, $\sum_{i \in K} \lambda_i a_i + \lambda_k a_k = a_k$

So, $a_k \in F, \forall k \in I$

Definition

Let E be a vector space over $\mathbb{K}$
$(a_i)_{i \in I}$ be a family of vectors of E

We say that $(a_i)_{i \in I}$ is a generating family for E if

$$\text{span}\{a_i, i \in I\} = E$$

$\iff \forall x \in E, x = \sum_{i \in I} \lambda_i a_i$ for some $(\lambda_i)_{i \in I} \in \mathbb{K}^I$

Example

In $\mathbb{R}^3$ let $A = (1,1,2), B = (1,0,1), C = (2,1,3)$ is (a,b,c) a generating family of $\mathbb{R}^3$?

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

We need to show that $x = \lambda_1 a + \lambda_2 b + \lambda_3 c$ for some $\lambda_1, \lambda_2, \lambda_3$

Then, $(x_1, x_2, x_3) = (\lambda_1 + \lambda_2 + 2\lambda_3, \ldots)$

$$\implies \begin{cases} \lambda_1 + \lambda_2 + 2\lambda_3 = x_1 \\ \\ \lambda_1 + \lambda_3 = x_2 \\ \\ 2\lambda_1 + \lambda_2 + 3\lambda_3 = x_3 \end{cases}$$

Example 3

Find $\operatorname{span}\{a, b, c\}$

$$\begin{cases} \alpha + \beta + 2\gamma = x_1 \\ \alpha + \gamma = x_2 \\ 2\alpha + \beta + 3\gamma = x_3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix} \implies \det(A) = 0$$

Since its non consistent.

Augmented matrix is:

$$\begin{pmatrix} 1 & 1 & 2 & | & x_1 \\ 1 & 0 & 1 & | & x_2 \\ 2 & 1 & 3 & | & x_3 \end{pmatrix}$$

By gaussian elimination,

$$\begin{pmatrix} 1 & 1 & 2 & | & x_1 \\ 0 & 1 & 1 & | & x_1 - x_2 \\ 0 & 0 & 0 & | & x_3 - x_1 - x_2 \end{pmatrix}$$

The system is consistent $\iff x_3 - x_1 - x_2 = 0$

Thus, the $\operatorname{span}\{x_1, x_2, x_3\}$ that satisfies this is $\operatorname{span}\{x_3 - x_1 - x_2 = 0\}$

Example 4

Determine whether the polynomials
$Q_1 = X + X^2, \quad Q_2 = X - X^2, \quad Q_3 = 1 + X, \quad Q_4 = 1 - X$
Generate $\mathbb{R}_2[X]$.

We need to see if $\forall P = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$, we have
$P = \alpha Q_1 + \beta Q_2 + \gamma Q_3 + \delta Q_4$

i.e. $a_0 + a_1x + a_2x^2 = (\alpha + \beta + \gamma - \delta)X + (\alpha - \beta)X^2$

i.e.

$$\begin{cases} \gamma + \delta = a_0 \\ \alpha + \beta + \gamma - \delta = a_1 \\ \alpha - \beta = a_2 \end{cases}$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & | & a_0 \\ 1 & 1 & 1 & -1 & | & a_1 \\ 1 & -1 & 0 & 0 & | & a_2 \end{pmatrix}$$

By gaussian elimination, we get

$$\begin{pmatrix} 1 & 0 & 0 & -1 & | & -\frac{a_0+a_1+a_2}{2} \\ 0 & 1 & 0 & -1 & | & -\frac{a_0+a_1-a_2}{2} \\ 0 & 0 & 1 & 1 & | & a_0 \end{pmatrix}$$

nb of variables: 4
rank=3 (leading variables)

This system is consistent $\forall a_0, a_1, a_2 \in \mathbb{R}$

i.e. $\{Q_1, Q_2, Q_3, Q_4\}$ is a generating family for $\mathbb{R}_2[X]$

---

§ 4.4

Definition

Let E vector space over $\mathbb{K}$
$(a_i)_{i\in I}$ family in E

We say that $(a_i)_{i\in I}$ is a free family or linearly independant family in E. $\forall (\lambda_i)_{i\in I} \in \mathbb{K}^{(I)}$

Otherwise, we say that $(a_i)_{i\in I}$ is linearly dependant

---

Remark

Suppose that for some $K \in I$, we have $a_k = 0_E$

Then $(a_i)_{i\in I}$ is linearly dependant

$$(12)a_k + \sum 0_{a_i} = 0_E$$

Example 1

Let $a \in E - \{0_E\}$
then (a) is free

Indeed, $\forall \lambda \in \mathbb{K}$

\begin{align} \lambda a=0_{E}&\implies \lambda=0 \ or \ a=0_{E} \\ &\implies \lambda=0 \ (since \ a \neq

\begin{align}
\exists (\lambda*{i})*{i \in I-{k}} \in \mathbb{K}^{(I)}; \
ak=\sum \lambda*{i}a*{i} \
Then, \
\sum \lambda*{i}a*{i}+(-1)ak=0_{E}
\end{align}

So, $(a_{i})_{i \in I}$ is linearly dependent. ___ ##### Theorem E vector space over $\mathbb{K}$. $(a

\begin{align}
\text{span}{a*{i}; \ i \in I}=\text{span}{a*{i}; i \in I -{k}}
\end{align}

In particular, if $(a_{i})_{i \in I}$ generates E, then $(a_{i})_{i \in I-\{k\}}$ generates E. 2. Suppose $

\begin{align}
x=\sum*{i \in I} \lambda*{i}a*{i}=\sum \lambda*{i}a*{i}+\lambda*{k}a_{k}
\end{align}

$$Then, since\$a_n = \sum_{i\neq k} \mu_i a_i\$forsome\$(\mu_i), i \in I - \{k\} \in \mathbb{K}\$, Weget:$$

x=\sum*{i \neq k} \lambda*{i}a*{i}+\lambda*{k}\left(\sum*{i \neq k}mu*{i}a*{i} \right)=\sum*{i \neq k}(\lambda*{i}+\lambda k\mu*{i})

$$So, \$x \in \operatorname{span}\{a_{i,i\in I-\{k\}}\}\$. 2. Let\$(\lambda_i)_{i\in I} \in \mathbb{K}^{(i)}\$and\$\lambda \in \mathbb{K}\$besuchthat:$$

\begin{align}
\sum*{i \in I} \lambda*{i}a_{i}+\lambda b=0
\end{align}

$$Ifwehave\$\lambda \neq 0\$, then\$b = \sum_{i\in I}\left(\frac{-\lambda_i}{\lambda}\right) a_i\$Whichcontradictsthefactthat\$b \notin \mathrm{span}\{a_i; i \in I\}\$Con.$$

\begin{align}
\sum{i \in I} \lambda{i}a_{i}=0
\end{align}

Since, $(a_{i})_{i \in I}$ is free, we get $\lambda_{i}=0, \forall i \in I$ $\square$. ___ # $\textsection 4.6$

\begin{align}
dim E=dim F + dim G
\end{align}

Proof: If $F=\{0_{E}\}$ or $F=E$, then the result is true since $E=\{0_{E}\} \oplus E$. Suppose n

\text{then, }x=x{F}+x{G}, \ \text{with}
\begin{cases}
x{F} \in F \
\
x{G} \in G
\end{cases}

\text{then, }x=x{F \cap G}+x{F{1}}, \ \text{with}
\begin{cases}
x{F \cap G} \in F \cap G \
\
x{F{1}} \in F_{1}
\end{cases}

$$Thus,$$

\begin{align}
x&=x{F \cap G}+x{F{1}}+x{G} \
&=\underbrace{x{F{1}}}{\in F{1}}+\underbrace{x{F \cap G}+x{G}}_{\in G} \
\end{align}

$$Wededucethat\$F+G=F_1+G\$andtheclaimisestablished. Consequently,$$

\begin{align}
\dim(F+G)&=\dim(F{1} \oplus G) \
&=\dim F{1}+ \dim G \
&=\dim F - \dim(F \cap G)+ \dim G \quad \square.
\end{align}

___ ##### Corollary E finite-dimensional vector space over $\mathbb{K}$. F,G linear subspace of $E

\begin{align}
L(E,F) \subset F(E,F)

\end{align}
$$ $(0 : E \longrightarrow E \text{ is a linear map, so } 0 \in L(E,F))$

Also, by the above theorem:

$$u, v \in L(E,F) \implies \begin{cases} u + v \in L(E,F) \\ \\ \lambda u \in L(E,F) \end{cases}$$

Thus, $L(E,F)$ is a linear subspace of $F(E,F)$

Remark

When $E = F$, we write

$L(E)$ instead of $L(E,E)$.

Theorem

E, F, G vector spaces over $\mathbb{K}$.

$u : E \longrightarrow F$. $v : F \longrightarrow G$ linear maps.

Then, $v \circ u : E \longrightarrow G$
$x \mapsto (v \circ u)(x) = v(u(x))$ is a linear map.

Proof

Let $x_1, x_2 \in E$, and let $\lambda \in \mathbb{K}$. Then:

$$\begin{aligned} (v \circ u)(x_1 + \lambda x_2) &= v(u(x_1 + \lambda x_2)) \\ &= v(u(x_1) + \lambda u(x_2)) \quad (\text{since } u \text{ is linear}) \\ &= v(u(x_1) + \lambda v(u(x_2))) \quad (\text{since } v \text{ is linear}) \\ &= (v \circ u)(x_1) + \lambda (v \circ u)(x_2) \end{aligned}$$

Since $(v \circ u)(x_1 + \lambda x_2) = (v \circ u)(x_1) + \lambda (v \circ u)(x_2)$, then $v \circ u$ is a linear map

$$\boxed{v \circ u \in \mathscr{L}(E,G)}$$

Theorem

E, F, G, H vector spaces over $\mathbb{K}$.

Then,

1. $\forall u \in L(E,F), \forall v \in L(F,G), \forall w \in L(G,H)$
   $w \circ (v \circ u) = (w \circ v) \circ u$
2. $\forall u \in L(E,F)$
   $u \circ Id_E = u = Id_F \circ u.$
3. $\forall u_1, u_2 \in \mathscr{L}(E,F), \forall v \in \mathscr{L}(F,G)$
   $v \circ (u_1 + u_2) = v \circ u_1 + v \circ u_2$
4. $\forall u \in \mathscr{L}(E,F). \forall v_1, v_2 \in \mathscr{L}(F,G)$
   $(v_1 + v_2) \circ u = v_1 \circ u + v_2 \circ u$
5. $\forall u \in \mathscr{L}(E,F), \forall v \in \mathscr{L}(F,G), \forall \lambda \in \mathbb{K}$
   $v \circ (\lambda u) = (\lambda v) \circ u = \lambda(v \circ u)$

Proof

1. Associativity of $\circ$ in general
2. Done.

Remark

Suppose $u : E \longrightarrow F$ is an isomorphism.

$u^{-1} : F \longrightarrow E$ isomorphism.

$u^{-1} \circ u = Id_E.$
$u \circ u^{-1} = Id_F.$

Theorem

$(\mathscr{L}(E), +, \circ)$ is a ring, with the zero element:
$0 : E \longrightarrow E$
$x \mapsto 0_E$

and the unit element:
$Id_E : E \longrightarrow E$
$x \mapsto x$

Proof

Now, $E = F = G$, so $\circ$ becomes an internal operation, and we apply the previous theorem. $\square$
.

Consequences

- In general, the ring $\mathscr{L}(E)$ is not commutative. (just like the ring of matrices).
- We have,

$(u+v)^n = \sum_{k=0}^{n} C_n^k u^{n-k} \circ v^k$
provided $u \circ v = v \circ u$

- u invertible $\Leftrightarrow$ $\exists v \in \mathscr{L}(E); u \circ v := v \circ u = Id_E \Leftrightarrow u$ is an automorphism of E and $v = u^{-1} : E \longrightarrow E$
- The automorphisms of E form a group (the group of invertible elements of the ring $\mathscr{L}(E)$.) called the general linear group of E, denoted by $GL(E)$.
- $(u \circ v)^{-1} = v^{-1} \circ u^{-1}, \forall u, v \in GL(E)$

---

§ 5.2

Definition

E vector space of dimension n over $\mathbb{K}$

$a_1, \ldots, a_p$ vectors of E. $(e_i)_{1 \leq i \leq n}$ basis of E

$\forall j \in \{1, \ldots, p\}, a_j = \sum_{i=1}^{n} \alpha_{ij} e_i$

Then the matrix $A = (\alpha_{ij}) \in M_{n,p}(\mathbb{K})$ is called the <u>representative matrix</u> of the family $a_1 \ldots a_p$ in the basis $(e_i)_{1 \leq i \leq n}$, denoted by $A = M[(a_i, \ldots, a_p); (e_i)]$

$$A = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1p} \\ \vdots & & \vdots \\ \alpha_{n1} & \ldots & \alpha_{np} \end{pmatrix}$$

Remark

If $p = 1$,

given a vector $x = \sum_{i=1}^{n} x_i$, we write $X = M[x_i; (e_i)]$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_i \end{pmatrix} \in M_n(\mathbb{K})$$

Definition

Suppose E vector space of dimension n over $\mathbb{K}$.
F vector space of dimension m over $\mathbb{K}$.
$u \in \mathscr{L}(E, F)$
$(e_j)_{1 \leq j \leq n}$ basis of E.
$(e'_i)_{1 \leq i \leq m}$ basis of F.

$\forall j \in \{1, \ldots, n\}$,
$u(e_j) = \sum_{i=1}^{m} \alpha_{ij} e'_i$

Then the matrix $A = (\alpha_{ij}) \in M_{m,n}(\mathbb{K})$ is called the representative matrix of the linear map u in the two basis $(e_j)_{1 \leq j \leq n}$ of E and $(e'_i)_{1 \leq i \leq m}$ of F, and is denoted by: $A = M[u; (e_j), (e'_i)]$.

$$A = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m_1} & \ldots & \alpha_{mn} \end{pmatrix}$$

Consequence

Matrix interpretation of a linear map:

Let,

$x = \sum_{j=1}^{n} x_j e_j \in E$, $X = M[x; (e_j)] = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$u(x) = \sum_{i=1}^{m} y_i e'_i \in F$, $Y = M[u(x); (e'_i)] = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

We have

$$\sum_{i=1}^{m} y_i e'_i = u(x) = u\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j u(e_j) = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} \alpha_{ij} e'_i$$
$$= \sum^{[} m]_{i=1} \left(\sum_{j=1}^{n} \alpha_{ij} x_j\right) e'_i$$

$$So,$$
$$y = \sum_{j=1}^{n} \alpha_{ij} x_j$$

We get:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \dots & \alpha_m \\ \vdots & & \vdots \\ \alpha_{m_1} & \dots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

So,

$$Y = AX$$

$u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$(x_1, \dots, x_n) \mapsto u(x_1, \dots, x_n) = (y_1, \dots, y_m)$

$$u(x_1, \dots, x_n) = \sum_{j=1}^{n} \alpha_{nj} x_j, \dots, \sum_{j=1}^{n} \alpha_{mj} x_j$$

special case

$u : \mathbb{R} \longrightarrow \mathbb{R}$
$x \mapsto u(x) = ax$

Example

Let

$$u : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x_1, x_2) \mapsto u(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2, -5x_1)$$

u is a linear map since each component function is a homogeneous polynomial of degree 1 in all variables.

The matrix of u:

$u(e_1) = u(1, 0) = (1, 2, -5)$
$u(e_i') = u(0, 1) = (1, -1, 0)$

$$A = M[u, (e_j), (e_i')] = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -5 & 0 \end{pmatrix} \in M_{3,2}(\mathbb{R})$$

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M[x; (e_j), (e_i')]$

$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = M[u(x); (e_1, e_2, e_3)]$
$\Leftrightarrow Y = AX$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_1 - x_2 \\ -5x_1 \end{pmatrix}$$

Theorem

E vector space of dimension n in $\mathbb{K}$.
F vector space of dimension m in $\mathbb{K}$

take the basis $(e_j)_{1 \leq j \leq n}$ basis of E
take the basis $(e'_i)_{1 \leq i \leq m}$ basis of F

Then, the map

$$\begin{aligned} \phi : \mathscr{L}(E, F) &\longrightarrow M_{m,n}(\mathbb{K}) \\ u &\mapsto \phi(u) = M[u; (e_j), (e'_i)] \end{aligned}$$

is an isomorphism from $\mathscr{L}(E, F)$ to $M_{m,n}(\mathbb{K})$.

Proof:

Let $u, v \in \mathscr{L}(E, F)$,
Let $A = \phi(u) = M[u; (e_j), (e'_i)] = (a_{ij})$
Let $B = \phi(v) = M[v; (e_j), (e'_i)] = (\beta_{ij})$

$$\phi(u + v) = M[u + v; (e_j), (e'_i)]$$

$$\begin{aligned} (u + v)(e_j) = u(e_j) + v(e_j) &= \sum_{i=1}^{m} \alpha_{ij} e'_i + \sum_{j=1}^{n} \beta_{ij} e'_i \\ &= \sum_{i=1}^{n} (\alpha_{ij} + \beta_{ij}) e'_i \end{aligned}$$

So, $\phi(u + v) = (\alpha_{ij} + \beta_{ij}) = A + B = \phi(u) + \phi(v)$.

Exercise:

Show that $\phi(\lambda(u)) = \lambda \phi(u), \lambda \in \mathbb{K}$.

- Injectivity:

Let $u \in \ker \phi$, so that $\phi(u) = 0$.

This means: $M[u; (e_j), (e'_i)] = 0$

Then, $u(e_j) = 0, \forall j \in \{1, \ldots, n\}$

And so, $u(x) = 0, \forall x \in E$

That is, $u = 0$

Thus, the $\ker \phi = 0$, and so ϕ is injective.

- surjectivity:

Let $A = M_{m,n}(\mathbb{K})$. Then $A = (\alpha_{ij})$

For each $j \in \{1, \dots, n\}$, $c_j = \sum_{i=1}^{n} \alpha_{ij} e'_i$.

let

$$\begin{aligned} u : E &\longrightarrow F \\ x &\mapsto u(x) = \sum_{j=1}^{n} x_j c_j \end{aligned}$$

==note ==
$e_k = \sum_{j=1}^{n} \delta_{jk} e_j$

Then $u \in \mathscr{L}(E, F)$ and $u(e_k) = \sum_{k=1}^{n} \delta_{jk} c_j = c_k$

It follows that $\phi(u) = M[u; (e_j), (e'_i)] = A \square$.

Theorem

E vector space of dimension p over $\mathbb{K}$
F vector space of dimension n over $\mathbb{K}$
G vector space of dimension m over $\mathbb{K}$

$u \in \mathscr{L}(E, F)$
$v \in \mathscr{L}(F, G)$

Let $(e_k)_{1 \leq k \leq p}$ basis of E
Let $(e'_j)_{1 \leq j \leq n}$ basis of F
Let $(e''_i)_{1 \leq i \leq m}$ basis of G

$B = (\beta_{ij}) = M[v; (e'_j), (e''_i)] \in M_{m,n}(\mathbb{K})$
$A = (\alpha_{jk}) = M[u; (e_k), (e'_j)] = M_{n,p}(\mathbb{K})$
$C = (\gamma_{ik}) = M[v \circ u; (e_k), (e''_i)] = M_{m,p}(\mathbb{K})$

Then, $C = BA$

Proof

$$\sum_{i=1}^{m} \gamma_{ik} e'_i = (v \circ u)(e_k) = v(u(e_k)) = v\left(\sum_{k=1}^{n} \alpha_{jk} e'_j\right) = \sum_{k=1}^{n} \alpha_{jk} v(e'_j) = \sum_{k=1}^{n} \alpha_{jk} \sum_{i=1}^{m} \beta_{ij} e''_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \beta_{ij} \alpha_{jk}\right) e''_i$$

$$\implies \gamma_{ik} = \sum_{j=1}^{n} \beta_{ij} \alpha_{jk} \quad \square.$$

Notation:

E vector space of dimension n over $\mathbb{K}$.
$u \in \mathscr{L}(E)$
$(e_i)_{1 \leq i \leq n}$ basis of E

Then, $M[u; (e_i), (e_i)] = M[u; (e_i)] \in M_n(\mathbb{K})$

Example

$\lambda \in \mathbb{K}$
$u = \lambda Id_E$

$$\begin{aligned} u : E &\longrightarrow E \\ x &\mapsto u(x) = (\lambda Id_E)(x) = (\lambda x) \end{aligned}$$

let (e_i) basis of E
