Chapter 3 - Functions

Let E and F be two sets.

A function (map or mapping) f from E to F is defined as the relation that associates to every element of E one *unique* element in F denoted $f(x)$. A function from E to F is denoted by a small letter. E is the set of inputs (domain of the function) and F is the set of outputs (range/codomain of the function). We write:

$$f : E \longrightarrow F$$

The element $f(x)$ is the image of x by the function f and we write $y = f(x)$. In the context of real numbers, we say that y is the value of f of x. So, a function f from a set E to a set F is denoted as:

$$f : E \longrightarrow F, \quad x \longrightarrow y = f(x)$$

Remark

Some people define mappings and functions differently. Sometimes, it is defined as the relation between E and F such that an element of E has no image or only one image in F. A mapping is then a relation such that every element of R has one and only one image in F.

3.1.1 Definition - Real Function

We call a **real function** of the **real variable** or a numerical function, any mapping from $\mathbb{R}$ or a subset of $\mathbb{R}$ to $\mathbb{R}$.

Imagine two boxes, each respectively named E and F.

Box E has 5 objects inside, and box F has 6. We can map every object in E to one object in F. A **mapping** (or **function**) assigns each object in E to **exactly one** object in F. Different objects in E can be assigned to the same object in F, but no object in E can be assigned to more than one object in F.

3.1.2 Definition - Domain of definition, image, graph

The domain of definition of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is denoted D_f and is defined by:

$$D_f = \{x \in \mathbb{R},\ f(x) \in \mathbb{R}\}$$

For example, the function $x \longrightarrow f(x) = \frac{1}{x}$. It's domain of definition is $\mathbb{R}^*$.

The set of images of a numerical function f is denoted $\mathrm{Im}(f)$, and is defined by:

$$\mathrm{Im}(f) = \{f(x),\ x \in D_f\}$$

For example, the function $f(x) = x^2$ defined on $\mathbb{R}$. We have $\mathrm{Im}(f) = \mathbb{R}^+$.

The graph of a mapping $f : E \longrightarrow F$ is the subset G_f of $E \times F$ defined by:

$$G_f = \{(x, f(x)),\ x \in D_f\}$$

3.1.3 Definition - Injection, surjection, bijection

Let f be a mapping $f : D_f \longrightarrow \mathbb{R}$

1. f is **injective** if:

$$\forall x, y \in D_f,\ x \neq y \implies f(x) \neq f(y)$$

   or, we can say

$$\forall x, y \in D_f,\ f(x) = f(x) \implies x = y$$

2. f is **surjective** if

$$\forall y \in \mathbb{R},\ \exists x \in D_f,\ y = f(x)$$

3. A mapping $f : E \longrightarrow F$ is **bijective** if it is injective and surjective. In this case, we define its inverse function, denoted by f^{-1}. We have:

$$f^{-1} : F \longrightarrow E, \quad y \longrightarrow x = f^{-1}(y)$$

3.2 The set $\mathbb{R}^I$

Let I be any interval of $\mathbb{R}$. Denote $\mathbb{R}^I$ the set of mappings f from I to $\mathbb{R}$,

$$\mathbb{R}^I = \{f : I \longrightarrow \mathbb{R},\ \text{mapping}\ \}$$

We define on $\mathbb{R}^I$ two operations; the addition, denoted $+$ and the multiplication denoted $\times$ as follows:

$\forall f, g \in \mathbb{R}^I,\ \forall x \in I$

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (f \times g)(x) = f(x)g(x)$$

For the multiplication in $f \in \mathbb{R}^I$, instead of writing $f \times g$, it will be denoted fg or $f \cdot g$.

Define now a third external operation of $f \in \mathbb{R}^I$ as follows: for all $f \in \mathbb{R}^I$ and for all $\lambda \in \mathbb{R}$,

$$\lambda f : I \longrightarrow \mathbb{R}, \quad x \longrightarrow (\lambda f)(x) = \lambda f(x).$$

Finally, we say that the elements f and $g \in \mathbb{R}^I$ are equal when the images of the f coincide with those of g:

$$f = g \iff f(x) = g(x),\ \ \forall x \in I$$

Proposition 3.2.1

let $f, g, h : I \longrightarrow \mathbb{R}$ be three mappings, and $\lambda.\ \mu$ two real numbers. We have

1. $(f+g)+h=f+(g+h)$
2. $(f+g)=(g+f)$
3. Denoting $O: I \longrightarrow \mathbb{R}$, $O(x)=0$ the identically null mapping. We have:

$$f+O=O+f=f$$

4. Denoting $1: I \longrightarrow \mathbb{R}$, $1(x)=1$ the constant mapping equal to 1 on I. We have:

$$1 \cdot f=f \cdot 1=f$$

5. $(\lambda f) g=\lambda(f g)$
6. $(\lambda+\mu) f=(\lambda f)+(\mu f)$
7. $\lambda(f+g)=\lambda f+\lambda g$

3.2.1 Definition

Let $f, g \in \mathbb{R}^I$

1. Suppose that $\forall x \in I, g(x) \neq 0$. We denote:

$$\frac{1}{g}: I \longrightarrow \mathbb{R}, \quad x \longrightarrow \frac{1}{g}(x)=\frac{1}{g(x)}$$

2. Suppose that $\forall x \in I, g(x) \neq 0$. We denote:

$$\frac{f}{g}: I \longrightarrow \mathbb{R}, \quad x \longrightarrow \frac{f}{g}(x)=\frac{f(x)}{g(x)}$$

3. Denote as well:

$$|f|: I \longrightarrow \mathbb{R}, \quad x \longrightarrow|f|(x)=|f(x)|$$

3.3 Order in $\mathbb{R}^I$

Define on $\mathbb{R}^I$ the relation denoted $\leq$ by:

$$\forall f, g \in \mathbb{R}^I, \quad f \leq g \iff \forall x \in I, f(x) \leq g(x)$$

This relation $\leq$ defines an order on $\mathbb{R}^I$. In fact it is:

1. **reflexive**: $\forall f \in \mathbb{R}^I, f \leq f$
2. **anti-symmetric**: $\forall f, g \in \mathbb{R}^I,\ (f \leq g \text{ and } g \leq f) \implies f=g$
3. **transitive**: $\forall f, g, h \in \mathbb{R}^I,\ (f \leq g \text{ and } g \leq h) \implies f \leq h$

The order relation $\leq$ satisfies as well the compatibility properties with the addiction and the multiplication

The order relation $\leq$ defined in $\mathbb{R}^I$ is not total, i.e., there exists functions in $\mathbb{R}^I$ that are not comparable.

Example: Suppose that I contains at least two elements a and b such that $a \neq b$. Consider the two functions f and g defined by:

$$f : I \longrightarrow \mathbb{R}, \quad f(a) = 1 \text{ and } f(x) = 0, \quad \text{if} \quad x \neq a$$
$$g : I \longrightarrow \mathbb{R}, \quad g(b) = 1 \text{ and } g(x) = 0, \quad \text{if} \quad x \neq b$$

Then we do not have $f \leq g$ since $f(a) > g(a)$ neither $g \leq f$ since $g(b) > f(b)$.

3.4 Monotonic mappings

3.4.1 Definition

Let $f \in \mathbb{R}^I$

1. We say f is **increasing** if

$$\forall x, y \in I, \quad x \leq y \implies f(x) \leq f(y)$$

2. We say that f is **decreasing** if

$$\forall x, y \in I, \quad x \leq y \implies f(x) \geq f(y)$$

3. We say that f is **monotonic** if it is only increasing or only decreasing.
4. When the inequalities in $1.$ (respectively $2.$) are strict, we say that f is strictly increasing (respectively strictly decreasing.)

3.5 Parity, Periodicity

3.5.1 Definition

Given I an interval of $\mathbb{R}$ such that

$$\forall x \in I, \quad -x \in I$$

let $f \in \mathbb{R}^I$

1. We say that f is even if

$$\forall x \in I, \ f(-x) = f(x)$$

2. We say that f is odd if

$$\forall x \in I, \ f(-x) = -f(x)$$

3.5.2 Definition

Let I be an interval of $\mathbb{R}$ and $T \in \mathbb{R}^*$. Suppose that

$$\forall x \in I, \quad x+T \in I$$

Let $f \in \mathbb{R}^I$. We say that f is periodic of period T if

$$\forall x \in I, \quad f(x+T) = f(x)$$

We say as well that the function f is $T-$periodic.

3.6 Bounded above, bounded blow and bounded functions

Let $f \in \mathbb{R}^I$. Define $f(I)$ by the set of images of the function f. $f(I)$ is a subset of $\mathbb{R}$. We have

$$f(I) = \{f(x),\ x \in I\}$$

Denote as well $f(I)$ by $\mathrm{Im}(f)$. We define then, the following:

3.6.1 Definition

1. We say that f is **bounded above** if $f(I)$ is bounded above in $\mathbb{R}$. i.e.,

$$\exists M \in \mathbb{R}, \forall x \in I,\ f(x) \leq M$$

2. We say that f is **bounded below** if $f(I)$ is bounded below in $\mathbb{R}$. i.e.,

$$\exists m \in \mathbb{R}, \forall x \in I, m \leq f$$

3. We say that f is **bounded** if it is bounded above and bounded below.