Chapter 1 - Real Numbers System

0 Notation

We denote $R_+ = \{x \in \mathbb{R}; x \geq 0\}$, $\mathbb{R}_- = \{x \in \mathbb{R}; x \leq 0\}$, $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}_+^* = \mathbb{R}_+ - \{0\}$, $\mathbb{R}_-^* = \mathbb{R}_- - \{0\}$.

1 Introduction

The set $\mathbb{N}$ of natural numbers is the basis for counting.

$$\mathbb{N} = \{0, 1, 2 \ldots\}$$

Since there is no elements in $\mathbb{N}$ such that the sum with 1, or with 2, $\ldots$ gives 0, This leads to the introduction of a new set of integers.

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$$

Then, since there is no elements in $\mathbb{Z}$ such that the *product* with 2, or with $3 \ldots$ gives us 1, this leads to the construction of the set of rational numbers.

$$\mathbb{Q} = \left\{ \frac{p}{q};\ (p, q) \in \mathbb{Z} \times \mathbb{Z}^* \right\}$$

We can observe that there are no rationals with a square that leads to 2. Indeed, if there exists, $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $2 = (\frac{m}{n})^2$ and m and n are mutually prime, then $2n^2 = m^2$. This implies that 2 divides m^2 and then 2 divides m. Assume that $m = 2p$, thus $2n^2 = m^2 = 4p^2$, therefore $n^2 = 2p^2$ and 2 divides n^2 and then 2 divides n. So we find that 2 divides m and n. This contradicts the fact that m and n are mutually prime.

Numbers such as e and π are not rational.

Consequently, we need a new field, $\mathbb{R}$, larger than $\mathbb{Q}$.

2 Real Numbers

2.1 Existence and Uniqueness of $\mathbb{R}$

We admit the existence and uniqueness of the set of real numbers $\mathbb{R}$ and equipped of the two internal laws $+, .$ and a relation $\leq$. such that:

$$\begin{cases} 1) & (\mathbb{R}, +, .) \text{ is a commuatative field.} \\ 2) & \leq \text{ is a total order relation in } \mathbb{R} \\ 3) & \forall (a,b,c) \in \mathbb{R}^3, \begin{cases} a \leq b \implies a+c \leq b+c \\ a \leq b \; and \; 0 \leq c \implies ac \leq bc \end{cases} \\ 4) & \text{Any non empty upper bounded subset of } \mathbb{R} \text{ has a supremum in } \mathbb{R}. \end{cases}$$

Reminder that:
$(\mathbb{R}, +, .)$ is a commutative field, meaning that:

$$\begin{aligned} &+ \text{ is associative: } \forall (a,b,c) \in \mathbb{R}^3, (a+b)+c = a+(b+c) \\ &+ \text{ is commutative: } \forall (a,b) \in \mathbb{R}^2, a+b = b+a \\ &\mathbb{R} \text{ has a neutral element for } + \text{ denoted } 0 : \forall a \in \mathbb{R}, a+0 = 0+a = a \\ &\text{Any element } a \text{ in } \mathbb{R} \text{ has an opposite, denoted by } -a : \end{aligned}$$

$$\forall a \in \mathbb{R}, \; a + (-a) = (-a) + a = 0$$

$$\begin{aligned} &. \text{ is associative: } \forall (a,b,c) \in \mathbb{R}^3, (ab)c = a(bc) \\ &. \text{ is commutative: } \forall (a,b) \in \mathbb{R}^2, ab = ba \end{aligned}$$

$\mathbb{R}$ has an identity element for the multiplication ., denoted by $1 : \forall a \in \mathbb{R}, a1 = 1a = a$.

Any element a in $\mathbb{R} - \{0\}$ has an inverse denoted by a^{-1}:

$$\forall a \in \mathbb{R} - \{0\}, \quad aa^{-1} = a^{-1}a = 1$$

is distributive relative to the addition:

$$\forall (a,b,c) \in \mathbb{R}^3, \begin{cases} a(b+c) = ab+ac \\ (b+c)a = ba+ca \end{cases}$$

$\leq$ is a total order relation in $\mathbb{R}$, meaning that:

$$\begin{aligned} &\leq \text{ is reflexive: } \forall a \in \mathbb{R}, a \leq a \\ &\leq \text{ is antisymmetric: } \forall (a,b) \in \mathbb{R}^2, a \leq b \; and \; b \leq a \implies a = b \\ &\leq \text{ is transitive: } \forall (a,b,c) \in \mathbb{R}^3, a \leq band \; b \leq c \implies a \leq c \\ &\leq \text{ is total: } \forall (a,b) \in \mathbb{R}^2, (a \leq b \; or \; b \leq a) \end{aligned}$$

For any pair $(a,b) \in \mathbb{R}^2, a < b$ means that $a \leq b$ and $a \neq b$. We can use the notation $b \geq a$ $(resp. \; b > a)$ instead of $a \leq b$ $(resp. \; a < b)$.

The elements in $\mathbb{R}$ are called **Real Numbers**.

Definitions 1.1

Let A be a non-empty set of $\mathbb{R}$.

- A is said to be **Upper Bounded** in $\mathbb{R}$ if there exists $k \in \mathbb{R}$ such that

$$\forall a \in A, \quad a \leq k$$

k is called an **upper bound** of A in $\mathbb{R}$.

- A is said to be **Lower Bounded** in $\mathbb{R}$ if there exists $k' \in \mathbb{R}$ such that

$$\forall a \in A, \quad k' \leq a$$

k' is called a **lower bound** of A in $\mathbb{R}$

- A is said to be **bounded** if it is upper, and lower bounded.
- We name **Supremum** of A in $\mathbb{R}$ the least of all upper bounds of A in $\mathbb{R}$, if it exists; this element is then denoted $Sup_{\mathbb{R}}(A)$ or $Sup(A)$.
- We name **Infimum** of A in $\mathbb{R}$ the greatest lower bound of A in $\mathbb{R}$, if it exists; this element is the denoted $Inf_{\mathbb{R}}(A)$ or $Inf(a)$.

Remarks

1. If k is an upper bound of A in $\mathbb{R}$, then for any $l \geq k$, l is also an upper bound of A in $\mathbb{R}$. Therefore, an upper bounded set has infinitely many upper bounds.
2. If k' is a lower bound of A in $\mathbb{R}$, then for any $l' \leq k'$, l' is also a lower bound of A in $\mathbb{R}$. Therefore, a lower bounded set has infinitely many lower bounds.

Examples

1. $[0,1[$ and $]0,1]$ are bounded.
2. $\mathbb{N}$ is lower bounded, but not upper bounded.
3. $\mathbb{Z}$ is not lower bounded, nor is it upper bounded.
4. $Sup_{\mathbb{R}}([0,1[) = Sup_{\mathbb{R}}(]0,1]) = 1$ and $Inf_{\mathbb{R}}([0,1[) = Inf_{\mathbb{R}}(]0,1]) = 0$.
5. $Sup(]-1,+\infty[)$ doesn't exist and $Inf(]-1,+\infty) = -1$.

Remark

In order to establish that a real number α is the *supremum* of a non-empty set A in $\mathbb{R}$, first we show that α is an upper bound of A in $\mathbb{R}$, then show that any other upper bound of A is greater than or equal to α. That is:

$$\begin{cases} \forall x \in A, & x \leq \alpha \quad \text{that is } \alpha \text{ is an upper bound of } A \\ \forall c \in \mathbb{R}, & (c \text{ upper bound of } A \implies \alpha \leq c) \end{cases}$$

or:

$$\begin{cases} \forall x \in A, & x \leq \alpha \\ \forall \epsilon \in \mathbb{R}_+^*, & \exists x \in A, \alpha - e < x \leq \alpha \end{cases}$$

Indeed, let $\epsilon \in \mathbb{R}_+^*$, we have $\alpha - \epsilon < \alpha$, then $\alpha - \epsilon$ is not an upper bound. Then there exists $x \in A$ such that $\alpha - \epsilon < x \leq \alpha$.

Conversely, if there exists an upper bound c of A such that $c < \alpha$, then let $\epsilon = a - c$, and using the hypothesis, there exists $x \in A$ such that $\alpha - \epsilon < x$, then $c < x$ which is impossible since c is an upper bound of A.

Similarly, to establish that a real number β is the *infimum* of a non-empty subset A of $\mathbb{R}$, first we show that β is a lower bound of A in $\mathbb{R}$ and then, we show that any lower bound is less than or equal to β. That is:

$$\begin{cases} \forall x \in A, & \beta \leq x \quad \text{that is } \beta \text{ is a lower bound of } A \\ \forall c \in \mathbb{R}, & (c \text{ lower bound of } A \implies c \leq \beta) \end{cases}$$

or:

$$\begin{cases} \forall x \in A, & \beta \leq x \\ \forall \epsilon \in \mathbb{R}_+^*, & \exists x \in A, \quad \beta \leq x < \beta + \epsilon \end{cases}$$

Definitions 1.2

One defines in $\mathbb{R}$ nine types of **intervals**, for $(a, b) \in \mathbb{R}^2$ such that $a \leq b$.

$$\begin{array}{ll} [a,b] = \{x \in \mathbb{R}; a \leq x \leq b\}, & \text{is said to be closed and bounded or simply a segment} \\ [a,b[= \{x \in \mathbb{R}; a \leq x < b\} &]a,b] = \{x \in \mathbb{R}; a < x \leq b\} \\ ]a,b[= \{x \in \mathbb{R}; a < x < b\} &]-\infty,a[= \{x \in \mathbb{R}; x < a\} \\ ]-\infty,a] = \{x \in \mathbb{R}; x \leq a\} &]a,+\infty[= \{x \in \mathbb{R}; a < x\} \\ [a,+\infty[= \{x \in \mathbb{R}; a \leq x\} &]-\infty,+\infty[= \mathbb{R} \end{array}$$

The intervals $[a,b], \;]-\infty,a], \; [a,+\infty[, \; and \;]-\infty,+\infty[$ are **closed**.
The intervals $]a,b[, \;]-\infty,a[, \;]a,+\infty[, \; and \;]-\infty,+\infty[$ are **open**.
The intervals $[a,b[,]a,b]$ are **semi closed** or **semi open**.

With these notations, the real numbers a and b are named the *endpoints* of the interval

2.2 Elementary Properties of Real numbers

1) $\forall x, y, z \in \mathbb{R}, (x \leq y \implies x + z \leq y + z)$
2) $\forall x, y, u, v \in \mathbb{R}, (x \leq y \text{ and } u \leq v \implies x + u \leq y + v)$.

Indeed,

$$\begin{cases} x \leq y \implies x + u \leq y + u \\ u \leq v \implies u + y \leq v + y \end{cases} \implies x + u \leq y + v$$

Then, by induction, for any $n \in \mathbb{N}^*, x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$:

$$(\forall i \in \{1, \ldots, n\}, x_i \leq y_i) \implies \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i$$

3) $\forall x \in \mathbb{R}^*, \left(0 < x \iff 0 < \frac{1}{x}\right)$. (Generally, we use the notation $\frac{1}{x}$ instead of x^{-1}).
4) $\forall x, y \in \mathbb{R}, \forall z \in \mathbb{R}^*, (x \leq y \implies xz \leq yz)$.
5) $\forall x, y, u, v \in \mathbb{R}, (0 \leq x \leq y \text{ and } 0 \leq u \leq v \implies xu \leq yv)$.

Indeed,

$$\begin{cases} x \leq y \text{ and } 0 \leq u \implies xu \leq yu \\ u \leq v \text{ and } 0 \leq y \implies yu \leq yv \end{cases} \implies xu \leq yv$$

Also, by induction, we show that for all $n \in \mathbb{N}^*, x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$:

$$(\forall i \in \{1,\dots,n\}, 0 \leq x_i \leq y_i) \implies \prod_{i=1}^{n} x_i \leq \prod_{i=1}^{n} y_i$$

In particular, $\forall n \in \mathbb{N}, \forall (x,y) \in \mathbb{R}^2, (0 \leq x \leq y \implies x^n \leq y^n)$

6) $\forall (x,y) \in (\mathbb{R}^*)^2, \left(x < y \iff \frac{1}{y} < \frac{1}{x}\right).$
7) $\forall x,y,u,v \in \mathbb{R}, (x \leq y \text{ and } u < v \implies x+u < y+v)$

If $x \leq y$ and $u \leq v$, then $x \leq y$ and $u \leq v$. Thus by $2)$, $x+u \leq y+v$. If $x+u = y+v$, then $x-y = v-u$. But $x-y \leq 0$, and $v-u \geq 0$. This gives $x-y = v-u = 0 \implies u = v$ which is impossible since $u < v$. We conclude that $x+u < y+v$.

We deduce that for any $n \in \mathbb{N}^*, x_1,\dots,x_n, y_1,\dots,y_n \in \mathbb{R}$:

$$(\forall i \in \{1,\dots,n\}, x_i \leq y_i) \text{ and } (\exists i_0 \in \{1,\dots,n\}, x_{i_0} \leq y_{i_0}) \implies \sum_{i=1}^{n} x_i < \sum_{i=1}^{n} y_i$$

This property is generally used in the following form:

$$(\forall i \in \{1,\dots,n\}, x_i \leq y_i) \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \implies (\forall i \in \{1,\dots,n\}, x_i = y_i).$$

2.3 Absolute Value

Definition 1.3

The absolute value of a real number x is the positive (or null) real number $|x|$, defined by:

$$|x| = \begin{cases} x \quad if \quad x \geq 0 \\ -x \quad if \quad x \leq 0 \end{cases}$$

Properties of the absolute value:

1. $\forall x \in \mathbb{R},\ |x| \geq 0$
2. $|x| = 0 \iff x = 0$
3. $\forall x \in \mathbb{R},\ x \leq |x|$
4. $\forall x,y \in \mathbb{R},\ |xy| = |x||y|$
5. $\text{triangular inequality: } \forall x,y \in \mathbb{R},\ |x+y| \leq |x| + |y|$
6. $\forall x,y \in \mathbb{R},\ ||x| - |y|| \leq |x-y|$

2.4 Distance on $\mathbb{R}$

Definition 1.4

The distance between two real numbers x and y is the positive real number

$$d(x,y) = |x-y|$$

Properties of the distance:

1. $\text{Positivity: } \forall x, y \in \mathbb{R},\ d(x,y) \geq 0$
2. $\text{Symmetry: } \forall x, y \in \mathbb{R},\ d(x,y) = d(y,x)$
3. $\text{Seperation: } \forall x, y \in \mathbb{R},\ d(x,y) = 0 \iff x = y$
4. $\text{Triangular Inequality: } \forall x, y, z \in \mathbb{R}, d(x,z) \leq d(x,y) + d(y,z)$

2.5 Integer Part of a Real Number

Definition 1.5

For any real number x, there exists a unique integer $p \in \mathbb{Z}$ that satisfies

$$p \leq x < p + 1$$

This integer p is called the integer part of x and it will be denoted $E(x)$ or $\lfloor x \rfloor$.

Examples:

1. $E(3,4) = 3$
2. $E(-2,1) = -3$

Definition 1.6

The integer part of a real number x is the greatest integer less or equal to x and we have

$$\forall x \in \mathbb{R},\ E(x) \leq x < E(x) + 1$$

2.5 Neighborhood of a Real Number

Definition 1.7

Let N be a non-empty subset of $\mathbb{R}$. V is called the neighborhood of $x \in \mathbb{R}$ if N contains an open interval centered at x. More precisely:

$$\exists \alpha > 0,\ ]x - \alpha, x + \alpha[\subset V$$

Properties of the neighborhoods of a point

1. If N is a neighborhood of x, then $x \in N$.
2. If N is a neighborhood of x, and $N \subset M$, then M is a neighborhood of x.
3. If $N_1 \text{ and } N_2$ are neighborhoods of x, then $N_1 \cap N_2$ is a neighborhood of x.
4. Any open interval is a neighborhood of any of its points.
5. If $x, y \in \mathbb{R}$, such that $x \neq y$, then there exists N_1 neighborhood of x and N_2 neighborhood of y such that $N_1 \cap N_2 = \emptyset$. The set $\mathbb{R}$ is said to be Hausdorff (separated).

Neighborhood of $+\infty$

A neighborhood of $+\infty$ is of the form $]A, +\infty[$, $A \in \mathbb{R}$.
A neighborhood of $-\infty$ is of the form $]-\infty, A[$, $A \in \mathbb{R}$.

2.6 Adherent (closure) Point

Definition 1.8

A subset A of $\mathbb{R}$. We say that a real number a is an adherent point to A if any neighborhood of a intersects A. In other words, if

$$\forall \epsilon > 0,]a-\epsilon, a+\epsilon[\cap A \neq \emptyset$$

Example

let $A =]0, 1[$. The point 0 is an adherent point to A since

$$\forall \epsilon > 0,]-\epsilon, \epsilon[\cap]0, 1[\neq \emptyset$$

but the real number 2 is not an adherent point to A because there exists an open interval (for example) $]1.5, 2.5[$ centered at 2 that does not intersect A.

Theorem

Any real number x is an adherent point to $\mathbb{Q}$. Equivalently

$$\forall a, b \in \mathbb{R}, \text{ such that } a < b, \ ]a, b[\ \cap \ \mathbb{Q} \neq 0$$

Definition 1.9

The set of rational numbers $\mathbb{Q}$ is then said to be dense in $\mathbb{R}$. We have $\bar{\mathbb{Q}} = \mathbb{R}$.