Exercise Sheet 1 : Real numbers system

Exercice 1

1. Prove that : $\forall x \in \mathbb{R},\ E(x)+E(-x) = \begin{cases} -1 & \text{if} \quad x \in \mathbf{R}\setminus\mathbb{Z} \\ 0 & \text{if} \quad x \in \mathbb{Z} \end{cases}$
2. Prove that : $\forall x, y \in \mathbb{R},\ E(x+y)-E(x)-E(y) \in \{0,1\}$.
3. Prove that : $\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R},\ E\Big(\dfrac{E(nx)}{n}\Big) = E(x)$.

Exercice 2

1. Prove that for all $a,\ b \in \mathbf{R}$, $\Big(a < b+\varepsilon,\ \forall \varepsilon > 0\Big) \Rightarrow \Big(a \leq b\Big)$.
2. Deduce that : If $a,\ b \in \mathbf{R},\ a < b+\frac{1}{n},\ \forall n \in \mathbb{N}^*$ then $a \leq b$.
3. Prove that : $\forall a, b \in \mathbb{R},\ \big[x \in \mathbf{R},\ a \leq x \leq b \Leftrightarrow \forall n \in \mathbb{N}^*,\ a-\frac{1}{n} < x < b+\frac{1}{n}\big]$.

Exercice 3

Let A and B be two non empty bounded subsets of $\mathbb{R}$. Prove that

1. $A \subset B \Rightarrow \sup A \leq \sup B$.
2. $\inf(A \cup B) = \inf(\inf A,\ \inf B)$

Exercice 4

Let A be a non empty bounded subset of $\mathbb{R}$. Let : $B = \{x-y,\ x \in A \text{ and } y \in A\}$.

1. Justify the existence of $\sup A$ et $\inf A$.
2. Prove that B is upper bounded by $\sup A - \inf A$.
3. Let M be an upper bound of B. Let $y \in A$ be a given number. Prove that $\sup A \leq M + y$.
4. Deduce that $\sup A - \inf A \leq M$. What can we conclude ?

Exercice 5

Let A and B be two non empty bounded subsets of $\mathbb{R}$. Define the subset $A+B$ by $A+B = \{a+b,\ a \in A \text{ and } b \in B\}$ and the subset $A-B = \{a-b,\ a \in A \text{ and } b \in B\}$.

1. Prove that $\sup(A+B) = \sup A + \sup B$.
2. Let $\lambda < 0$. Prove that $\sup(\lambda A) = \lambda \inf A$.
3. Deduce that $\sup(A-B) = \sup A - \inf B$.

Exercice 6

Consider the following subsets of $\mathbb{R}$:

$$A = \{\frac{(-1)^n}{n},\ n \in \mathbb{N}^*\} \qquad \text{and} \qquad B = \{\frac{1}{n}+\frac{1}{m},\ m \text{ and } n \in \mathbb{N}^*\}.$$

Are these parts upper bounded ? lower bounded ? Do they have a maximum ? a minimum ? an infimum ? a supremum ?