Exercise Sheet 1 - Real Numbers System

Exercice 1

1. Prove that : $\forall x \in \mathbb{R},\ E(x)+E(-x) = \begin{cases} -1 & \text{if} \quad x \in \mathbf{R} \setminus \mathbb{Z} \\ 0 & \text{if} \quad x \in \mathbb{Z} \end{cases}$
2. Prove that : $\forall x, y \in \mathbb{R},\ E(x+y)-E(x)-E(y) \in \{0,1\}$.
3. Prove that : $\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R},\ E\left(\frac{E(nx)}{n}\right) = E(x)$.

1. Take $x = m + r, m \in \mathbb{Z}, r \in [0,1[$.

If $r = 0$, then $x = m$. Thus $\lfloor x \rfloor = m$ and $\lfloor -x \rfloor = -m$ hence:

$$\lfloor x \rfloor + \lfloor -x \rfloor = m + (-m) = 0$$

If $r > 0$ (meaning if $x \notin \mathbb{Z}$), then $m \leq m + r < m + 1$. Multiply by -1 to get:

$$-m - 1 < -m - r \leq -m$$

So, we get:

$$\lfloor -x \rfloor = \lfloor -m - r \rfloor = -m - 1$$

And since $\lfloor x \rfloor = m$, we get:

$$\lfloor x \rfloor + \lfloor -x \rfloor = m + (-m - 1) = m - m - 1 = -1 \quad \square$$

2.

Let $x = m + \alpha, y = n + \beta$ where $m, n \in \mathbb{Z},$ and $\alpha, \beta \in [0, 1)$. Therefore, $E(x) = m,\ E(y) = n$

Now,

$$x + y = (m + n) + (\alpha + \beta)$$

So,

$$E(x + y) = \lfloor x + y \rfloor = m + n + \lfloor \alpha + \beta \rfloor$$

Now, since $\alpha, \beta \in [0, 1)$, we have $\alpha + \beta \in [0, 2)$. Hence, $\lfloor \alpha + \beta \rfloor$ has to be either 0, or 1. Ergo:

$$E(x+y)-E(x)-E(y)=(m+n+\lfloor\alpha+\beta\rfloor)-m-n=\lfloor\alpha+\beta\rfloor\in[0,1)\ \square.$$

3.

Let $x=m+\alpha$, where $m=\lfloor x\rfloor\in\mathbb{Z}$ and $\alpha\in[0,1)$. Multiplying by some $n\in\mathbb{N}^*$, we get:

$$nx=nm+n\alpha$$

taking the floors, we get:

$$\lfloor nx\rfloor=nm+\lfloor n\alpha\rfloor$$

Dividing by n:

$$\frac{\lfloor nx\rfloor}{n}=m+\frac{\lfloor n\alpha\rfloor}{n}$$

Now, since $0\le\alpha<1$, we have $0\le n\alpha<n$. So $\lfloor n\alpha\rfloor$ is an integer in $\{0,1,\ldots,n-1\}$. Therefore:

$$\lfloor\frac{\lfloor nx\rfloor}{n}\rfloor=\lfloor m+\frac{\lfloor n\alpha\rfloor}{n}\rfloor=m=\lfloor x\rfloor\ \square.$$

Ex. 2

Exercice 2

1. Prove that for all $a,\ b\in\mathbf{R}$, $\Big(a<b+\varepsilon,\ \forall\varepsilon>0\Big)\Rightarrow\Big(a\le b\Big)$.
2. Deduce that : If $a,\ b\in\mathbf{R},\ a<b+\frac{1}{n},\ \forall n\in\mathbb{N}^*$ then $a\le b$.
3. Prove that : $\forall a,b\in\mathbb{R},\ \left[x\in\mathbf{R},\ a\le x\le b\Leftrightarrow\forall n\in\mathbb{N}^*,\ a-\frac{1}{n}<x<b+\frac{1}{n}\right]$.

1. Lets start by assuming the opposite.

Let $a>b$. Let $\delta=a-b$. This means that $\delta>0$.

We take $\epsilon=\frac{\delta}{2}$, then:

$$b+\epsilon=b+\frac{\delta}{2}=\frac{b+a}{2}$$

Comparing this result to a, we get:

$$a-\frac{b+a}{2}=\frac{2a-(b+a)}{2}=\frac{a-b}{2}=\frac{\delta}{2}>0$$

this means that:

$$a > \frac{b+a}{2} = b + \epsilon$$

This is a contradiction, meaning

$$(\forall \epsilon > 0, a < b + \epsilon) \implies a \leq b \ \square.$$

2. Let $\epsilon > 0$. By the Archimedian property: $\frac{1}{n} < \epsilon$.

Then

$$a < b + \frac{1}{n}$$

and since $\frac{1}{n} < \epsilon$:

$$a < b + \frac{1}{n} < b + \epsilon$$

Therefore, $a < b + \epsilon \implies a \leq b \, \square$.

3. Left hand side:

Suppose $x \in [a, b]$, then $a \leq x \leq b$. $\forall n \in \mathbb{N}^*$ we have:

$$a - \frac{1}{n} < a \leq x \leq b < b + \frac{1}{n}$$

meaning,

$$a - \frac{1}{n} \leq x \leq b + \frac{1}{n}$$

Right hand side:

Suppose $x > b$. Lets take $\delta = x - b > 0$. Let $n \in \mathbb{N}^*$. By the archimedian property; $\frac{1}{n} < \delta$, meaning:

$$b + \frac{1}{n} < b + \delta = x$$

This is a contradiction. So $x \leq b$

Now, suppose $x < a$. Lets take $\beta = a - x > 0$. By the archimedian property; $\frac{1}{n} < \beta$, meaning:

$$a - \frac{1}{n} > a - \beta = x$$

This is a contradiction. So $x \geq a$.

Combining these two statements, we get

$$a \leq x \leq b \iff a - \frac{1}{n} < x < b + \frac{1}{n}, \ \ \forall n \in \mathbb{N}^* \quad \square.$$

Ex. 3

Exercice 3

Let A and B be two non empty bounded subsets of $\mathbb{R}$. Prove that

1. $A \subset B \Rightarrow \sup A \leq \sup B$.
2. $\inf(A \cup B) = \inf(\inf A,\ \inf B)$

1. Let $x \in A$. Since $A \subset B$, then $x \in B \implies x \leq \text{Sup } B$. Therefore, $\text{Sup } B$ is an upper bound for A. $\implies \text{Sup } A \leq \text{Sup } B$ since $\text{Sup } A$ is the smallest upper bound for A.
2. Lets write $a = \inf A$ and $b = \inf B$. Since A, B are non-empty and bounded, then a, b exist.

Lets take $\delta = \inf(A \cup B)$. We will show that $\delta = \min(a, b)$

Let $m = \min(a, b)$. If $x \in A \cup B$, then $x \in A$ OR $x \in B$.
If $x \in A$, then $a \leq x$, so $m \leq x$.
If $x \in B$, then $b \leq x$, so $m \leq x$.

In both cases, $m \leq x$, meaning m is a lower bound of $A \cup B$.

Lets take γ, a random lower bound of $A \cup B$. Then γ is a lower bound of A and of B. Then, by the property of the infimum:

$$\gamma \leq a \ \ and \ \ \gamma \leq b$$

Meaning,

$$\gamma \leq \min(A, B) = m$$

This tells us that no lower bound of $A \cup B$ is bigger than m.

From this, we can conclude that m is the greatest lower bound of $A \cup B$, so

$$\inf(A \cup B) = \min(a, b) = \inf(\inf A, \inf B) \quad \square.$$

Ex. 4

Exercice 4

Let A be a non empty bounded subset of $\mathbb{R}$. Let : $B = \{x - y,\ x \in A \text{ and } y \in A\}$.

1. Justify the existence of $\sup A$ et $\inf A$.
2. Prove that B is upper bounded by $\sup A - \inf A$.
3. Let M be an upper bound of B. Let $y \in A$ be a given number. Prove that $\sup A \leq M + y$.
4. Deduce that $\sup A - \inf A \leq M$. What can we conclude ?

1.

Since A is bounded and non-empty, then $\text{Sup}A$ and $\inf A$ exist.

2.

Let $x - y \in B$.

$$\begin{aligned} x \in A \implies & \inf A \leq x \leq \text{Sup}A \\ y \in A \implies & \inf A \leq y \leq \text{Sup}A \\ \implies & -\text{Sup}A \leq y \leq -\inf A \end{aligned}$$

$$\inf A - \text{Sup}A \leq x - y \leq \text{Sup}A - \inf A$$

Then, $\text{Sup}A - \inf A$ is an upper bound of B.

3.

M upper bound for B then $\forall x, y \in A$, $x - y \leq M$

$\forall x \in A, x - y \leq M \implies \forall x \in A, x \leq M + y \implies M + y$ is an upper bound for $A \implies \sup A \leq M + y$

4.

$$\begin{cases} SupA - infA & \text{upper bound for } B \\ \forall M \text{ upper bound} & supA \leq M + y, \quad \forall y \in A \end{cases} \implies supA - M \leq y$$

$$\begin{aligned} &\implies supA - M \quad \text{upper bound for } A \\ &\implies supA - M \leq infA \\ &\implies supA - infA \leq M \end{aligned}$$

From 2), we have $supA - infA$ upper bound for B. From 1), we have every M upper bound for B is greater or equal than $supA - infA$. Then

$$supB = supA - infA$$

Ex. 5

Exercice 5

Let A and B be two non empty bounded subsets of $\mathbb{R}$. Define the subset $A+B$ by $A+B = \{a+b,\ a \in A \text{ and } b \in B\}$ and the subset $A-B = \{a-b,\ a \in A \text{ and } b \in B\}$.

1. Prove that $\sup(A+B) = \sup A + \sup B$.
2. Let $\lambda < 0$. Prove that $\sup(\lambda A) = \lambda \inf A$.
3. Deduce that $\sup(A-B) = \sup A - \inf B$.

1.

Let $x \in A + B$. Then, $x = a + b$, for some $a \in A$ and $b \in B$. By definition of the supremum, $a \leq \sup A$, $b \leq \sup B$. So:

$$x = a + b \leq \sup A + \sup B$$

This holds for any $x \in A + B$, meaning $\sup A + \sup B$ is an upper bound of $A + B$. Which gives us:

$$\sup(A + B) \leq \sup A + \sup B$$

Now, let $\epsilon > 0$. Since $\sup A$ is the least upper bound of A, then $\exists a \in A$ such that

$$a > \sup A - \frac{\epsilon}{2}$$

Same goes for b

$$b > \sup B - \frac{\epsilon}{2}$$

adding them together, we get:

$$a + b > \sup A + \sup B - \epsilon$$

But $a + b \in A + B$. So $\forall \epsilon > 0$, there exists an element in $A + B > \sup A + \sup B - \epsilon$. Meaning, $\sup(A + B) \geq \sup A + \sup B$.

From the two inequalities we got, we conclude that:

$$\sup(A + B) = \sup A + \sup B \quad \square.$$

2.

Take $a \in A$. Then $\inf A \leq a$. If we multiply by $\lambda < 0$, we get:

$$\lambda \inf A \geq \lambda a$$

meaning $\lambda \inf A$ is an upper bound of λA.

Let $\epsilon > 0$. Take $\delta = \frac{\epsilon}{|\lambda|}$. By the property of the infimum:

$$a < \inf A + \delta$$

Multiplying by lambda, we get:

$$\lambda a > \lambda \inf A - \epsilon$$

which tells us that λA is the least upper bound of λA.

Since we have: $\sup(\lambda A) \leq \lambda \inf A$, and $\sup(\lambda A) \geq \lambda \inf A$, then:

$$sup(\lambda A) = \lambda \inf A \quad \square.$$

3.

Lets rewrite $A - B$ as $A - (-1) \cdot B$

Then, from Question 2:

$$\sup(\lambda B) = \lambda \inf B$$

Here, we can take $\lambda = -1 < 0$. So:

$$\sup(-B) = -\inf B$$

From Question 1, we can write:

$$\sup(A - B) = \sup(A + (-B)) = \sup A + \sup(-B)$$

Substituting $\sup(-B)$, we get:

$$\sup(A - B) = \sup A - \inf B \quad \square.$$

Ex. 6

Exercice 6

Consider the following subsets of $\mathbb{R}$:

$$A = \{\frac{(-1)^n}{n},\ n \in \mathbb{N}^*\} \qquad \text{and} \qquad B = \{\frac{1}{n} + \frac{1}{m},\ m \text{ and } n \in \mathbb{N}^*\}.$$

Are these parts upper bounded ? lower bounded ? Do they have a maximum ? a minimum ? an infimum ? a supremum ?

Starting with A:

Let n be odd, then $n = 2k + 1$. So:

$$A = \frac{-1}{2k+1}, \quad \forall k \in \mathbb{N}$$

Now,

$$\begin{aligned}
0 &\le k < +\infty \\
0 &\le 2k < \infty \\
1 &\le 2k+1 < \infty \\
0 &\le \frac{1}{2k+1} \le 1 \\
-1 &\le \frac{-1}{2k+1} < 0
\end{aligned}$$

Now, let n be even. Then $n = 2k$. So:

$$\begin{aligned}
1 &\le k < \infty \\
2 &\le 2k < \infty \\
0 &< \frac{1}{2k} \le \frac{1}{2}
\end{aligned}$$

So, $\forall n \in \mathbb{N}^*$,

$$-1 \leq \frac{(-1)^n}{n} \leq \frac{1}{2}$$

We have:

- -1 lower bound for A, with $-1 \in A$. Then $\min A = \inf A = -1$
- $\frac{1}{2}$ is an upper bound for A with $\frac{1}{2} \in A$. Then $\max A = \sup A = \frac{1}{2}$.

Lets move on to B:

$n \in \mathbb{N}^*$

$$\implies 1 \leq n < \infty$$
$$\implies 0 < \frac{1}{n} < 1$$

$m \in \mathbb{N}^*$

$$\implies 1 \leq m < \infty$$
$$\implies 0 < \frac{1}{m} < 1$$

so $0 < \frac{1}{n} + \frac{1}{m} \leq 2$.

- 2 is an upper bound for B. Then $\max B = \sup B = 2$
- 0 is a lower bound for B. But $0 \notin B$. So $\min B \nexists$

Let $\epsilon > 0 \implies \frac{\epsilon}{3} > 0 \text{ and } \frac{2\epsilon}{3} > 0$

Extra Exercises:

Ex. 1

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1. (a) Show, by mathematical induction, the following formulas:

   i. $S_1^n = 1 + 2 + \ldots + (n-1) + n = \dfrac{n(n+1)}{2}$, $\forall n \in \mathbb{N}^*$

   ii. $S_2^n = 1^2 + 2^2 + \ldots + (n-1)^2 + n^2 = \dfrac{n(n+1)(2n+1)}{6}$, $\forall n \in \mathbb{N}^*$

   iii. $1.2 + 2.3 + \ldots + n.(n+1) = \dfrac{n(n+1)(n+2)}{3}$, $\forall n \in \mathbb{N}^*$

   (b) Deduce the following sum :

$$S_3^n = 1^3 + 2^3 + \ldots + (n-1)^3 + n^3 \ , \qquad n \in \mathbb{N}^*$$

a) i.

$S_1^n = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}, \quad \forall n \in \mathbb{N}^*$

Let $n = 1$, then:

$$S_1^1 = 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

So this property is true for $n = 1$.

Lets assume that this property is true for any $n \geq 1$. We will show that it is also true for $n + 1$

$S_1^{n+1} = 1 + 2 + \cdots + n + (n+1) = \frac{(n)(n+1)}{2} + (n+1)$

$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Meaning this property is true for $n + 1$. So we conclude that the formula $S_1^n = \frac{n(n+1)}{2}$ holds $\forall n \in \mathbb{N}^* \square$.

Remark

$$S_1^n = 1 + 2 + 3 + \cdots + (n-1) + n$$
$$S_1^n = n + (n-1) + (n-2) + \cdots + 2 + 1$$

$$\Longrightarrow 2S_1^n = (n+1) + (n+1) + (n+1) + \cdots + (n+1) \ \text{ n times.}$$

$$\Longrightarrow 2S_1^n = n(n+1)$$

$$\Longrightarrow S_1^n = \frac{n(n+1)}{2}$$

ii)

$S_2^n = 1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \in \mathbb{N}^*$

Let $n = 1$, then:

$$S_2^1 = 1 = \frac{1(1+1)(2+1)}{6} = \frac{2 \cdot 3}{6} = \frac{6}{6} = 1$$

So this property holds for $n = 1$. Lets assume its true for $n \geq 1$. We will show that it holds for $n + 1$:

$S_2^{n+1} = 1^2 + 2^2 + \cdots + n^2 + (n+1)^2$

$$
\begin{aligned}
= S_2^n + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}
\end{aligned}
$$

So the formula is also true for $n+1$, ergo it is true for any $n \in \mathbb{N}^*$.

iii) To be continued later.