Chapter 1 - Real Number System

Whenever we write a set X to the power of "$*$", it means that its the non zero set.

1. Real Number System $\mathbb{R}$

1.1 Rational Numbers:

We take $\mathbb{N}$ as the set of natural numbers i.e $\mathbb{N} = \{0, 1, 2 \ldots\}$

We take $\mathbb{Z}$ as the set of integers i.e $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$

We take $\mathbb{Q}$ as the set of rational numbers i.e $\mathbb{Q} = \{\frac{p}{q}, (p, q) \in \mathbb{Z} \times \mathbb{N}^*\}$

Now,

1. Any rational number $r \in \mathbb{Q}$ is the ratio of the number p and a non zero rational q: $r = \frac{p}{q} \in \mathbb{Q}$.
2. Given $r = \frac{p}{q}$ and $r' = \frac{p'}{q'}$ two ration numbers, we have:

$$r = r' \Leftrightarrow \frac{p}{q} = \frac{p'}{q'} \Leftrightarrow pq' = qp'$$

3. The following inclusions are trivial

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

Definition 1.1.1 -

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$$

Where $\mathbb{R} - \mathbb{Q}$ is the set of irrational numbers. We have:

$$0 \in \mathbb{Q}, \quad 0 \notin (\mathbb{R} - \mathbb{Q}), \quad \sqrt{2} \in (\mathbb{R} - \mathbb{Q}), \quad \sqrt{2} \notin \mathbb{Q}, \quad \sqrt{2} \in \mathbb{R}$$

and the inclusions:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Note that $\mathbb{Q} \cap (\mathbb{R} - \mathbb{Q}) = \emptyset$.

1.2 Properties of the addition and the multiplication

- $+$ is associative

$$\forall a, b, c \in \mathbb{R}, \qquad (a+b)+c = a+(b+c)$$

- $+$ is commutative

$$\forall a, b \in \mathbb{R}, \qquad a+b = b+a$$

- The real number 0 is a neutral element for $+$

$$\forall x \in \mathbb{R}, \qquad x+0 = 0+x = x$$

- Any real number x has an opposite element denoted $(-x)$

$$\forall x \in \mathbb{R}, \qquad x+(-x) = (-x)+x = 0$$

- $\times$ is associative

$$\forall a, b, c \in \mathbb{R}, \qquad (a \times b) \times c = a \times (b \times c)$$

- $\times$ is commutative

$$\forall a, b \in \mathbb{R}, \qquad a \times b = b \times a$$

- The real number 1 is a neutral element for $\times$

$$\forall a \in \mathbb{R}, \qquad 1 \times a = a \times 1 = a$$

- Any real non zero x has an inverse element $(\frac{1}{x})$ or x^{-1}

$$\forall x \in \mathbb{R}^*, \qquad x \times \frac{1}{x} = \frac{1}{x} \times x = 1$$

- $\times$ is distributive with respect to $(+) : \forall a, b, c \in \mathbb{R}$,

$$a \times (b+c) = (a \times b) + (a \times c)$$

  and

$$(b+c) \times a = (b \times a) + (c \times a)$$

1.3 Order of the set of real numbers $\mathbb{R}$

1. $\leq$ is **reflexive**

$$\forall a \in \mathbb{R}, \qquad a \leq a$$

2. $\leq$ is **antisymmetric**

$$\forall a, b \in \mathbb{R}, \qquad \text{if } (a \leq b \text{ and } b \leq c) \text{ then } a = b$$

3. $\leq$ is **transitive**

$$\forall a, b, c \in \mathbb{R}, \qquad \text{if } (a \leq b \text{ and } b \leq c) \text{ then } a \leq c$$

4. $\leq$ is **total** i.e. $\forall a, b \in \mathbb{R}$, we have:

$$a \leq b \qquad \text{or} \qquad b \leq a$$

NOTATIONS: denote

- $a < b$ if and only if $a \leq b$ and $a \neq b$
- $b \geq a$ if and only if $a \leq b$
- $b > a$ if and only if $b \geq a$ and $b \neq a$

Compatibility of the order with the two binary operations

The following properties are admitted

1. $\forall a, b, c, \in \mathbb{R}$ we have : $\qquad a \leq b \implies a + c \leq b + c.$
2. $\forall a, b, x, y \in \mathbb{R}$ we have : $\qquad x \leq y \text{ and } a \leq b \implies x + a \leq y + b.$

3. $\forall a, b, c \in \mathbb{R}$ we have : $\quad a \leq b$ and $c > 0 \implies ac \leq bc$.

1.4 Extended real number line $\bar{\mathbb{R}}$

Add to $\mathbb{R}$ two distinct elements, not belonging to $\mathbb{R}$ denoted $-\infty$ and $+\infty$ and let

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

Define them extension of the binary operations and the order defined on $\mathbb{R}$. We obtain the following:

1. $\forall x \in \mathbb{R}, x + (+\infty) = +\infty$ and $x + (-\infty) = -\infty$
2. $\forall x > 0, x(+\infty) = +\infty$ and $x(-\infty) = -\infty$
3. $\forall x < 0, x(+\infty) = -\infty$ and $x(-\infty) = -\infty$
4. $(+\infty) + (+\infty) = +\infty$, and $(-\infty) + (-\infty) = -\infty$
5. $(+\infty)(+\infty)$ and $(-\infty)(+\infty) = -\infty$ and $(-\infty)(-\infty) = +\infty$
6. $\forall x \in \mathbb{R}, -\infty < x < +\infty$
7. $-\infty \leq -\infty$ and $+\infty \leq +\infty$

1.5 Intervals of $\mathbb{R}$

Definition 1.5.1 (segment)

Let a and b be two real numbers such that $a \leq b$. The segment $[a, b]$ is defined as:

$$[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}$$

This can be rewritten as:

$$z \in [a, b] \Leftrightarrow z = (1 - t)x + ty, \qquad 0 \leq t \leq 1.$$

Definition 1.5.2 (interval)

Let I be a subset of $\mathbb{R}$. I is called an interval of $\mathbb{R}$ if and only if

$$\forall a, b \in I, \text{ such that } a \leq b, \text{ the segment } [a, b] \subset I$$

1.6 Abs Value of a real number

Definition 1.6.1

The absolute value of a real number x is the positive (or null) real number $|x|$ defined by:

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x \leq 0 \end{cases}$$

Properties of the abs value

1. $\forall x \in \mathbb{R}, |x| \geq 0$
2. $|x| = 0 \Leftrightarrow x = 0$
3. $\forall x \in \mathbb{R}, x \leq |x|$
4. $\forall x, y \in \mathbb{R}, |xy| = |x||y|$
5. $\text{triangular inequality} : \forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$
6. $\forall x, y \in \mathbb{R}, ||x| - |y|| \leq |x - y|$

1.7 Distance on $\mathbb{R}$

Definition on 1.7.1

The distance between two real numbers x and y is the positive real number

$$d(x, y) = |x - y|$$

Properties of the distance:

1. Positivity: $\forall x, y \in \mathbb{R}, d(x, y) \geq 0$.

2. Symmetry: $\forall x, y \in \mathbb{R}, d(x, y) = d(y, x)$.
3. Seperation: $\forall x, y \in \mathbb{R}, d(x, y) = 0 \Leftrightarrow x = y$.
4. Triangular inequality: $\forall x, y, z \in \mathbb{R}, d(x, z) \leq d(x, y) + d(y, z)$.

1.8 Upper bound and Lower bound of a subset of $\mathbb{R}$

1. A real number M is an **upper bound** of the subset A if:

$$\forall x \in A, x \leq M$$

2. A real number m is a **lower bound** of the subset A if:

$$\forall x \in A, m \leq x$$

We say that a subset is upper bounded when it has an upper bound.

We say that a subset is lower bounded when it has a lower bounded.

We say that a subset is bounded when it has both an upper and a lower bound.

1.8.3 Greatest element, least element:

Let $A \subset \mathbb{R}, A \neq \emptyset$.

1. An upper bound M of A is the greatest element of A if $M \in A$
2. A lower bound m of A is the least element of A if $m \in A$

We can denote the greatest element by $\max \mathrm{A}$ and the least element by $\min \mathrm{A}$

1.8.4 Supremum, infimum:

1. The supremum of a subset A is the least upper bound of A.
2. The infimum of a subset A is the greatest lower bound of A.

Example:

Given the subset $A = [0, 1[$. We have:

$$\inf A = \min A = 0 \qquad \text{and} \qquad \sup A = 1 \notin A$$

ANY NON EMPTY UPPER BOUNDED SET HAS A SUPREMUM IN R
ANY NON EMPTY LOWER BOUNDED SET HAS AN INFIMUM IN R

Characterizations:

$$\sup A = M \Leftrightarrow \begin{cases} 1. \forall x \in A, x \leq M \\ \\ 2. \forall \epsilon > 0, \exists a \in A, M - \epsilon < a \leq M \end{cases}$$

$$\inf A = m \Leftrightarrow \begin{cases} 1. \forall x \in A, m \leq x \\ \\ 2. \forall \epsilon > 0, \exists a \in A, m \leq a < m + \epsilon \end{cases}$$

1.10 Neighborhood of a real number:

V is called a neighborgood of $x \in \mathbb{R}$ if N contains an open interval centered at x:

$$\exists \alpha > 0,]x-\alpha, x+\alpha[\subset V$$

1. If N neighborhood of x, then $x \in N$
2. If N is a neighborhood of x and $N \subset M$ then M is a neighborhood of x.
3. If N_1 and N_2 are neighborhoods of x then $N_1 \cap N_2$ is a neighborhood of x.
4. Any open interval is a neighborhood of any of its points.
5. If $x, y \in \mathbb{R}$ such that $x \neq y$, then there exists N_1 neighborhood of x and N_2 neighborhood of y such that $N_1 \cap N_2 = \emptyset$. The set $\mathbb{R}$ is said to be seperated.

1.11 Adherent (closure) point:

We say that a real number a is an adherant point to A if any neighborhood of a intersects A:

$$\forall \epsilon > 0,]a-\epsilon, a+\epsilon[\cap A \neq \emptyset$$

Example: Let $A =]0,1[$. The point 0 is adherent point to A since:

$$\forall \epsilon > 0,]-\epsilon, \epsilon[\cap]0,1[\neq \emptyset$$

but the real number 2 is not an adherent point to A because there exist an open interval $]1.5, 2.5[$ centered at 2 that does not intersect A.
