M1105 Chapter 1

Set $\mathbb{R}^n$ and sequence in $\mathbb{R}^n$.

1.1.1 The space $\mathbb{R}^n$

We define the space $\underbrace{\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}}_{n\ times}$

Addition:

$x + y = (x_1 \ldots x_n) + (y_1 \ldots y_n) = (x_1 + y_1 \ldots x_n + y_n)$

For $n = 2, \mathbb{R}^2 = \{X = (x, y) : x, y \in \mathbb{R}\}$

Definition 1.2

When you multiply $x \text{ by } y$,

$$x \cdot y =< x, y >$$

Theorem

Cauchy-Schwarz inequality:

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$$

Proof:

Let us suppose that x and y are not colinear.

We then have $\sum_{i=1}^{n}(tx_i + y_i)^2 > 0$, for all $t \in \mathbb{R}$. Then
$\sum_{i=1}^{n}(tx_i + y_i)^2 = \sum_{i=1}^{n}(t^2 x_i^2 + 2tx_i y_i + y_i^2) = \left(\sum_{i=1}^{n} x_i^2\right)t^2 + 2\left(\sum_{i=1}^{n} x_i y_i\right)t + \left(\sum_{i=1}^{n} y_i^2\right) > 0$

Let $a = \sum_{i=1}^{n} x_i^2,\ b = \sum_{i=1}^{n} x_i y_i \text{ and } c = \sum_{i=1}^{n} y_i^2 \implies at^2 + 2bt + c > 0$,
as $a > 0 \implies \Delta' = b^2 - ac < 0 \implies b^2 < ac$
$\implies \left(\sum_{i=1}^{n} x_i y_i\right)^2 < \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$, hence the inequality.

If x and y are indeed collinear, then $\exists t_0 \in \mathbb{R}^*$ such that $y = t_0 x$, therefore
$\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} t_0^2 x_i^2\right)^{\frac{1}{2}} = |t_0| \sum_{i=1}^{n} x_i^2$
and $\left|\sum_{i=1}^{n} x_i y_i\right| = |t_0| \sum_{i=1}^{n} x_i^2$, hence the inequality.

1.1.2 Norms and distances

A norm on $\mathbb{R}^n$ is all mapping

$$N : \mathbb{R}^n \to [0, \infty[$$

verifying the properties:

$$\begin{cases} (N_1) & \forall x \in \mathbb{R}^n, N(x) = 0 \iff x = 0; (\text{Positivity}) \\ (N_2) & \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n, N(\alpha x) = |\alpha| N(x); (\text{Homogeneity}) \\ (N_3) & \forall x, y \in \mathbb{R}^n,\ N(x+y) \leq N(x) + N(y); (\text{Triangular inequality}) \end{cases}$$

A distance on $\mathbb{R}^n$ is all mapping:

$$d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty[,$$

verifying the properties:

$$\begin{cases} (D_1) & \forall x, y in \mathbb{R}^n, d(x,y) = 0 \iff x = y \\ (D_2) & \forall x, y \in \mathbb{R}^n, d(x,y) = d(y,x) \\ (D_3) & \forall x, y, z \in \mathbb{R}^n, d(x,z) \leq d(x,y) + d(y,z) \end{cases}$$

1.1.3 Usual norms and associated distances

- First usual norm on $\mathbb{R}^2$: Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$

The first usual norm on $\mathbb{R}^2$ is defined by:

$$||x||_1 = |x_1| + |x_2|$$

and its associated distance is given by:

$$d_1(x,y) = ||y - x||_1 = |y_1 - x_1| + |y_2 - x_2|$$

- Second usual norm on $\mathbb{R}^2$: Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$.

The second usual norm, called euclidean norm, on $\mathbb{R}^2$ is defined by

$$||x||_2 = \sqrt{x_1^2 + x_2^2}$$

and its associated distance is given by

$$d_2(x,y) = ||y - x||_2 = \sqrt{(y_1 - x_2)^2 + (y_2 - x_2)^2}$$

- Third usual norm on $\mathbb{R}^2$:

1.2.3 Equivalent norms

Definition: Two norms N_1 and N_2 on $\mathbb{R}^n$ are said to be equivalent if there exists $\alpha > 0$ and $\beta > 0$ such that:

$$\forall x \in \mathbb{R}^n, \quad \alpha N_2(x) \leq N_1(x) \leq \beta N_2(x)$$

1.4.3 Convex and Connected sets

Definition 1.25

Let $a, b \in \mathbb{R}^n$ we define the segment denoted $[a, b]$ by

$$[a, b] = \{x \in R^n : x = \alpha a + \beta b; \alpha, \beta \in \mathbb{R}^+ \text{ and } \alpha + \beta = 1\}$$

Equation of a line

$$\begin{cases} x = a + t\alpha \\ y = b + t\beta \\ z = c + t\gamma \end{cases}$$

Equation of a plane

Let $A(a, b, c)$ and $\vec{N} = \alpha\vec{i} + \beta\vec{j} + \gamma\vec{k}$

We define the plane P having $\vec{N}$ as a normal vector and A belonging to P by:

$$\vec{AM} = \begin{cases} x - a \\ y - b \\ z - c \end{cases}$$

$$\begin{aligned} \vec{AM} \cdot \vec{N} = 0 &\implies \alpha(x - a) + \beta(y - b) + \gamma(z - c) = 0 \\ &\implies \boxed{\alpha x + \beta y + \gamma z = \alpha a + \beta b + \gamma c} \end{aligned}$$

1.3 Convergence on $\mathbb{R}^n$

Definition

A vector sequence of $\mathbb{R}^n$ is all sequence $(x_k)_{k\geq 0}$ such that $x_k = (x_k^1, \ldots, x_k^n)$ with $x^{i_k} \in \mathbb{R},\ \forall i = 1, \ldots n$.

Definition 1.13

Let $(x_k)_{k\geq 0}$ be a vector sequence of $\mathbb{R}^n$, $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $||\cdot||$ be a norm on $\mathbb{R}^n$. We say that $(x_k)_{k\geq 0}$ converges to a with respect to $||\cdot||$ if one of the following properties is verified:

$$\begin{aligned} (i)\ (\forall \epsilon > 0)(\exists k_0 \in \mathbb{N})(\forall k \geq k_0, ||x_k - a|| < \epsilon) \\ (ii)\ \text{The numerical sequence } (||x_k - a||_{k\geq 0}) \text{ tends to } 0 \end{aligned}$$

In this case, we denote $x_k \overset{||\cdot||}{\to} a$ when $k \to \infty$ and we say that a is the limit of (x_k).

Example:

Show that $\lim_{n\to\infty}(\frac{n}{n+2}, 2 - \frac{1}{n^2}) = (1,2)$.

solution:

Let the vector sequence $(x_n)_{n\geq 1}$ such that $x_n = (\frac{n}{n+2}, 2 - \frac{1}{n^2})$.

$$||x_n - (1,2)||_\infty = \max(|\frac{n}{n+2} - 1|, |2 - \frac{1}{n^2} - 2|) = \max(|\frac{2}{n+2}|, |\frac{1}{n^2}|).$$

We have $\lim_{n\to\infty} |\frac{2}{n+2}| = \lim_{n\to\infty} \frac{1}{n^2} = 0 \implies \lim_{n\to\infty} ||x_n - (1,2)||_\infty = 0$.

Proposition 1.7

A vector sequence $(x_k)_k$ is convergent on $\mathbb{R}^n$ if and only if the sequences $(x_k^1)_k, \ldots, (x_k^n)_k$ are convergent in $\mathbb{R}$, and we have:

$$\lim_{k\to\infty} x_k = (\lim_{k\to\infty} x_k^1, \ldots, \lim_{k\to\infty} x_k^n)$$

Definition 1.14

A vector sequence is said to be divergent if it doesn't admit a limit.

Example

Study the convergence of $(x_n)_{n\geq 1}$ such that $x_n = (2^n, \frac{1}{n})$.

Solution:

$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(2^{n,}\frac{1}{n}) = (\lim_{n\to\infty} 2^n, \lim_{n\to\infty} \frac{1}{n}) = (\infty, 0)$, therefore the sequence is divergent.

Definition 1.15

We call sub-sequence of the sequence $(x_k)_{k\geq 0}$ of $\mathbb{R}^n$, every sequence of the form $(x_{\alpha(k)})$ where $\alpha : \mathbb{N} \to \mathbb{N}$ is a strictly increasing mapping.

Example:

Study the convergence of the sequence $(x_n)_{n\geq 1}$ such that $x_n = (\frac{(-1)^n n}{n+1}, \frac{n+(-1)^n}{n^2})$.

Solution:

We have $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty}(\frac{2n}{2n+1}, \frac{2n+1}{4n^2}) = (1,0)$ and
$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty}(\frac{-(2n+1)}{2n+2}, \frac{2n}{(2n+1)^2}) = (-1,0)$. Therefore $(x_n)_{n\geq 1}$ is divergent.

Proposition 1.10

Let x_k be a vector sequence on $\mathbb{R}^n$. If $x_k \overset{||\cdot||}{\to} a$ when $k \to \infty$, then the numerical sequence $(||x_k||_k)$ converges to $||a||$.

Definition 1.16

Let x_k be a vector sequence on $\mathbb{R}^n$ and $||\cdot||$ be a norm on $\mathbb{R}^n$. We say that $(x_k)_k$ is bounded in $\mathbb{R}^n$ if there exists $M > 0$ such that $\forall k \geq 0$, $||x_k|| \leq M$.

Proposition 1.11

Let $(x_k)_k$ be a vector sequence on $\mathbb{R}^n$. If $x_k \overset{||\cdot|}{\to} a$ when $k \to \infty$, then the sequence $(x_k)_k$ is bounded in $\mathbb{R}^n$.

Proof:

First, let us recall that $(x_k)_k$ is bounded in $\mathbb{R}^n \iff$ the sequences $(x_k^1)_k, \dots, (x_k^n)_k$ are also bounded in $\mathbb{R}$.

Then, as $x_k \overset{||\cdot||}{\to} a$, $(\forall \epsilon > 0)(\exists k_0 \in \mathbb{N})(\forall k \geq k_0, ||x_k - a|| < \epsilon)$

Now, for $\epsilon = 1$, $(\exists k_0 \in \mathbb{N})(\forall k \geq k_0, ||x_k|| < 1 + ||a||)$

Take $M = max\{||x_k||, \dots, ||x_{k_0}||, 1 + ||a||\} \implies \forall k \geq 0, ||x_k|| \leq M$

Remark

For a sequence to be divergent, we can simply show that it is not bounded. Or we can show that at least one component is divergent.

If a sequence is bounded, it does not necessarily mean it is convergent. But, if a sequence is convergent, then we can say it is bounded.

Example

Let $x_n = (\cos n, \sin n),\ \text{for } n \geq 0$
$||x_n||_1 = |\cos n| + |\sin n| \leq 2, \forall n \geq 0$, but $(x_k)_{n\geq 0}$ is not convergent.

Theorem 1.2

Let $(x_k)_k$ and $(y_k)_k$ be two vector sequence of $\mathbb{R}^n$. If $x_k \overset{||\cdot||}{\to} a$ and $y_k \overset{||\cdot||}{\to} b$ when $k \to \infty$, then the sequence $(\alpha x_k + \beta y_k)$ converges to $\alpha a + \beta b$, for $\alpha, \beta \in \mathbb{R}$.

Proof:

We have $\forall k \geq 0,\ \alpha x_k + \beta y_k - \alpha a - \beta b = \alpha(x_k - a) + \beta(y_k - b)$
$\implies \forall k \geq 0, 0 \leq ||\alpha x_k + \beta y_k - \alpha a - \beta b|| \leq |\alpha| ||x_k - a|| + |\beta| ||y_k - b||$
$\implies 0 \leq \lim_{k\to\infty} ||\alpha x_k + \beta y_k - \alpha a - \beta b|| \leq |\alpha| \lim_{k\to\infty} ||x_k - a|| + |\beta| \lim_{k\to\infty} ||y_k - b|| \leq 0$
$\implies \lim_{k\to\infty} ||\alpha x_k + \beta y_k - \alpha a - \beta b|| = 0.$

Theorem 1.3

Let $(x_k)_k$ be a vector sequence of $\mathbb{R}^n$ and $(\alpha_k)_k$ be a scalar sequence of $\mathbb{R}$. If $x_k \overset{||\cdot||}{\to} a$ and $\alpha_k \to a$ when $k \to \infty$, then the sequence $(\alpha_k x_k)$ converges to αa.

Proof:

We have $\forall k \geq 0, \alpha_k x_k - \alpha a = \alpha_k x_k - \alpha_k a + \alpha_k a - \alpha a$
$\implies \forall k \geq 0, 0 \leq ||\alpha_k x_k - \alpha a|| \leq |\alpha_k| ||x_k - a|| + |\alpha_k - a| ||a||$
$\implies 0 \leq \lim_{k\to\infty} ||\alpha_k x_k - \alpha a|| \leq \lim_{k\to\infty} |\alpha_k| ||x_k - a|| + ||a|| \lim_{k\to\infty} |\alpha_k - \alpha| \leq 0$
$\implies \lim_{k\to\infty} ||\alpha_k x_k - \alpha a|| = 0.$
