Chapter 2 - Sequences

2.1 Definition

An infinite numerical sequence is a mapping from $\mathbb{N} \to \mathbb{R}$

$$\begin{aligned} U : \mathbb{N} &\to \mathbb{R} \\ n &\to U(n) \end{aligned}$$

Example:

$$U_n = \frac{1}{n+1}$$

$$\begin{aligned} U : \mathbb{N} &\to \mathbb{R} \\ n &\to \frac{1}{n+1} \end{aligned}$$

- The sequence is denoted $(U_n)_{n\in\mathbb{N}}$ of general term U_n
- The elements of the sequence are $U_0, U_1, \ldots, U_n$

2.1.1 Bounded above, below, bounded

A sequence $(U_n)_n$ is bounded from above by M if $\forall n \in \mathbb{N}, U_n \leq M$

Example:

$$U_n = \frac{1}{n} : \\ U_n \text{ is bounded from above by } 1$$

A sequence $(U_n)_n$ is bounded from below by m if $\forall n \in \mathbb{N}, U_n \geq m$

Example:

$$U_n = \frac{1}{n} : \\ U_n \text{ is bounded from below by } 0$$

A sequence $(U_n)_n$ is bounded if it is bounded from above and from below $\iff \exists k \geq 0 \ / \ |U_n| \leq k, \forall n \in \mathbb{N}$

2.1.2 Arithmetic & Geometric Sequences

An arithmetic sequence is a sequence of numbers such that the difference between two consecutive numbers is constant.

$$\forall n, U_n + 1 - U_n = d$$

Remark

1. $U_p = U_q + (p - q)d$
2. $U_n = U_0 + nd$
3. $S_n = U_0 + U_1 + \cdots + U_n = (n+1) \times \frac{U_0 + U_n}{2} = (\text{nb of terms}) \times (\frac{\text{1st+last}}{2})$

A geometric sequence is a sequence where the ratio of two consecutive terms is constant.

$$\forall n, \frac{U_n + 1}{U_n} = r \iff U_n + 1 = U_n \times r$$

Remark

1. $U_n = U_0 \times r^n$
2. $U_p = U_q \times r^{(p-q)}$
3. $S = U_0 + \cdots + U_n = U_0 \times \frac{1 - r^{r+1}}{1-r} = \text{1st term} \times (\frac{1-r}{1+r})^{(\text{nb of terms})}$

$$\begin{gathered} U_n \longmapsto l \\ n \to \infty \end{gathered}$$

$$\iff \forall \epsilon > 0, \exists N \in \mathbb{N}; n \geq N \implies |U_n - l| \leq \epsilon$$
$$\iff \forall \epsilon > 0, \exists N \in \mathbb{N}; n \geq N \implies -\epsilon \leq U_n - l \leq \epsilon$$
$$\iff \forall \epsilon > 0, \exists N \in \mathbb{N}; n \geq \implies l - \epsilon \leq U_n \leq l + \epsilon$$
$$\iff \forall \epsilon > 0, \exists N \in \mathbb{N}; n \geq N \implies U_n \in [l - \epsilon, l + \epsilon]$$
$\iff$ For every neighborhood of l there exists a rank such that starting this ranks, all U_n belongs to the neighborhood.

2.2 Limit of a sequence

Definition 2.2.1

Let l be a finite number. We say that the sequence $(u_n)_n$ tends to l as n tends to $+\infty$ if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, n \in \mathbb{N} / (n \geq N \implies |u_n - l| \leq \epsilon)$$

What this means is; there exists an n beyond which, any interval of the form $]l - \epsilon, l + \epsilon[$ (with $\epsilon > 0$) contain all the terms of the sequence $(u_n)_n$

Definition 2.2.2

A sequence $(u_n)_n$ is said to be convergent if u_n tends to a finite limit l as n tends to $\pm\infty$. Else, the sequence is divergent.

Definition 2.2.3

The sequence $(u_n)_n$ has a limit equal to $+\infty$ if and only if

$$\forall A > 0, \exists N \in \mathbb{N}, (n \geq N \implies u_n \geq A)$$

The sequence $(u_n)_n$ has a limit equal to $-\infty$ if and only if

$$\forall A > 0, \exists N \in \mathbb{N}, (n \geq N \implies u_n \leq -A)$$

Remark

The nature of the sequence remains unchanged if a finite number of the terms of the sequence is removed.

Theorem 2.2.1

When a sequence $(u_n)_n$ has a finite limit l, then this limit is unique.

Proof:

Proof by contradiction. Suppose that the sequence $(u_n)_n$ has two limits l_1 and l_2 with $l_1 \neq l_2$.

Let $\epsilon = \frac{|l_1 - l_2|}{3}$. Thus:

$$\exists N_1 \in \mathbb{N}, (n \geq N_1 \implies |u_n - l_1| \leq \frac{|l_1 - l_2|}{3})$$

and

$$\exists N_2 \in \mathbb{N}, (n \geq N_2 \implies |u_n - l_2| \leq \frac{|l_1 - l_2|}{3})$$

Let $N = \max\{N_1, N_2\}$. Then, $\forall n \geq N$, we have

$$\begin{aligned} |l_1 - l_2| &= |l_1 - u_n - l_2 + u_n| \\ &\leq \frac{|l_1 - l_2|}{3} + \frac{|l_1 - l_2|}{3} \\ &\leq \frac{2}{3}|l_1 - l_2| \end{aligned}$$

which is impossible $\square$.

Remark

When a sequence $(u_n)_n$ tends to a finite limit $l \in \mathbb{R}$ as n tends to $+\infty$, we write

$$u_n \longrightarrow l \quad or \quad \lim_{n \to +\infty} = l$$

2.2.1 Eventually constant sequence

A sequence $(u_n)_{n\in\mathbb{N}}$ is called *eventually constant* if it is constant beyond a given n index, i.e.

$$\exists p \in \mathbb{N}, u_p = u_{p+1} = \cdots = u_{p+n} = \dots$$

2.2.2 Operations and limits with sequences

Theorem 2.2.2

Let $(u_n)_n$ and $(v_n)_n$ be two real sequences. l and l' two real numbers. We have the following properties

1. $u_n \longrightarrow_{n\to+\infty} l \implies |u_n| \longrightarrow_{n\to+\infty} |l|$
2. If $u_n \longrightarrow_{n\to+\infty} l$ and $v_n \longrightarrow_{n\to+\infty} l'$ then $u_n + v_n \longrightarrow_{n\to+\infty} l + l'$
3. $u_n \longrightarrow_{n\to+\infty} l \implies \lambda u_n \longrightarrow_{n+\infty} \lambda l, \quad \forall\lambda \in \mathbb{R}$
4. If $u_n \longrightarrow_{n\to+\infty} 0$ and $(v_n)_n$ is bounded then $u_n v_n \longrightarrow_{n\to+\infty} 0$
5. If $u_n \longrightarrow_{n\to+\infty} l$ and $v_n \longrightarrow_{n\to+\infty} l'$ then $u_n v_n \longrightarrow_{n\to+\infty} ll'$
6. If $u_n \longrightarrow_{n\to+\infty} l$ and $l \neq 0$ then $\dfrac{1}{u_n} \longrightarrow_{n\to+\infty} \dfrac{1}{l}$
7. If $u_n \longrightarrow_{n\to+\infty} l$ and $v_n \longrightarrow_{n\to+\infty} l' \neq 0$ then $\dfrac{u_n}{v_n} \longrightarrow_{n\to+\infty} \dfrac{l}{l'}$

2.2.3 Sign of the limit

Theorem 2.2.3

Let $(u_n)_n$ be a sequence of real numbers convergent to $l \in \mathbb{R}$. If $\forall n \in \mathbb{N}, u_n > 0$, then $l \geq 0$.

Proof

Proof by contradiction. Suppose that $l < 0$. For $\epsilon = -l > 0$, we have

$$\exists N \in \mathbb{N}; n \geq N \implies |u_n - l| \leq -l$$

or

$$|u_n - l| \leq -l \implies l \leq u_n - l \leq -l \implies 2l \leq u_n \leq 0$$

which is a contradiction with the fact that all the terms of the sequence are strictly positive.

Proposition 2.2.4

Let $(u_n)_n$ and $(v_n)_n$ be two real sequences satisfying $u_n \leq v_n$ beyond a given index. Then,

$$\lim u_n \leq \lim v_n$$

Note that if the sequence $(u_n)_n$ tends to $+\infty$ then the sequence $(v_n)_n$ tends to $+\infty$ as well. But, if one of the sequences converge, we can conclude nothing regarding the convergence of the other sequence.

Proposition 2.2.5 - Sandwich Theorem

Let $(u_n)_n$, $(v_n)_n$ and $(w_n)_n$ be three, real sequences such that

$$\forall n \in \mathbb{N}, \quad u_n \leq v_n \leq w_n$$

If $(u_n)_n$ and $(w_n)_n$ are convergent to the same limit l, then $(v_n)_n$ is convergent, and its limit is also l.

2.3 Monotonic sequence

A sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ is **increasing** (respectively **strictly increasing**) if

$$\forall n \in \mathbb{N}, \;\; u_{n+1} - u_n \geq 0 \text{ (respectively } u_{n+1} - u_n > 0)$$

A sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ is **decreasing** (respectively **strictly decreasing**) if

$$\forall n \in \mathbb{N}, \;\; u_{n+1} - u_n \leq 0 \text{ (respectively } u_{n+1} - u_n < 0)$$

A sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ is **monotonic** if its only increasing, or only decreasing. Non monotonic sequences exist in fact, for example: $u_n = (-1)^n, \; n \in \mathbb{N}$.

Theorem 2.3.1

If a sequence is increasing and bounded from above (respectively decreasing and bounded from below) then it is said to be convergent.

Remark

1. If $(u_n)_n$ is increasing and bounded above then $\lim u_n = \sup\{u_n, \; n \in \mathbb{N}\}$
2.
    1. If $(u_n)_n$ is decreasing and bounded below then $\lim u_n = \inf\{u_n, \; n \in \mathbb{N}\}$

2.4 Adjacent sequences

Definition 2.4.1

Two real sequences $(u_n)_n$ and $(v_n)_n$ are called adjacent if and only if the following three conditions are satisfied:

1. $(u_n)_n$ is increasing.
2. $(v_n)_n$ is decreasing.
3. $v_n - u_n \longrightarrow_{n \to +\infty} 0$

Proposition 2.4.1

If the two real sequences $(u_n)_n$ and $(v_n)_n$ are adjacent then they are convergent and they have the same limit.

Moreover, denoting l their common limit, we have

$$\forall n \in \mathbb{N}, u_0 \leq u_1 \leq \cdots \leq u_n \leq l \leq v_n \leq \cdots \leq v_1 \leq v_0$$

2.5 Subsequences

Definition 2.5.1

Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing mapping.

Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. A subsequence of $(u_n)_{n \in \mathbb{N}}$ is defined as the sequence $(u_{\sigma(n)})_{n \in \mathbb{N}}$. In other words, a subsequence is an infinite selection of elements of the original subsequence, with increasing indices.

Examples

$(u_{2n})_n$ and $(u_{2n+1})_n$ are two subsequences of $(u_n)_{n \in \mathbb{N}}$, since the mappings

$$\sigma_1(n) = 2n, \qquad \text{and} \qquad \sigma_2(n) = 2n + 1$$

are strictly increasing from $\mathbb{N}$ to $\mathbb{N}$.

Proposition 2.5.1

Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing mapping. Then

$$\forall n \in \mathbb{N}, \ \sigma(n) \geq n$$

Proposition 2.5.2

If a sequence $(u_n)_n$ converges to $l \in \mathbb{R}$, then any subsequence of $(u_n)_n$ also converges to l.

Theorem 2.5.3

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. This theorem is known as the *Bolzano-Weierstrass theorem*.