Exercise Sheet 1 - Real Numbers System

Ex. 1:

Exercice 1

1. Prove that : $\forall x \in \mathbb{R},\ E(x)+E(-x) = \begin{cases} -1 & \text{if} \quad x \in \mathbf{R} \setminus \mathbb{Z} \\ 0 & \text{if} \quad x \in \mathbb{Z} \end{cases}$
2. Prove that : $\forall x, y \in \mathbb{R},\ E(x+y) - E(x) - E(y) \in \{0, 1\}$.
3. Prove that : $\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R},\ E\left(\frac{E(nx)}{n}\right) = E(x)$.

Proof:

1. Take $x = m + r, m \in \mathbb{Z}, r \in [0, 1[$.

If $r = 0$, then $x = m$. Thus $\lfloor x \rfloor = m$ and $\lfloor -x \rfloor = -m$ hence:

$$\lfloor x \rfloor + \lfloor -x \rfloor = m + (-m) = 0$$

If $r > 0$ (meaning if $x \notin \mathbb{Z}$), then $m \leq m + r < m + 1$. Multiply by -1 to get:

$$-m - 1 < -m - r \leq -m$$

So, we get:

$$\lfloor -x \rfloor = \lfloor -m - r \rfloor = -m - 1$$

And since $\lfloor x \rfloor = m$, we get:

$$\lfloor x \rfloor + \lfloor -x \rfloor = m + (-m - 1) = m - m - 1 = -1 \quad \square$$

2.

Let $x = m + \alpha, y = n + \beta$ where $m, n \in \mathbb{Z}, \text{ and } \alpha, \beta \in [0, 1)$. Therefore, $E(x) = m,\ E(y) = n$

Now,

$$x + y = (m + n) + (\alpha + \beta)$$

So,

$$E(x+y) = \lfloor x+y \rfloor = m + n + \lfloor \alpha + \beta \rfloor$$

Now, since $\alpha, \beta \in [0,1)$, we have $\alpha + \beta \in [0,2)$. Hence, $\lfloor \alpha + \beta \rfloor$ has to be either 0, or 1. Ergo:

$$E(x+y) - E(x) - E(y) = (m + n + \lfloor \alpha + \beta \rfloor) - m - n = \lfloor \alpha + \beta \rfloor \in [0,1) \ \square.$$

3.

Let $x = m + \alpha$, where $m = \lfloor x \rfloor \in \mathbb{Z}$ and $\alpha \in [0,1)$. Multiplying by some $n \in \mathbb{N}^*$, we get:

$$nx = nm + n\alpha$$

taking the floors, we get:

$$\lfloor nx \rfloor = nm + \lfloor n\alpha \rfloor$$

Dividing by n:

$$\frac{\lfloor nx \rfloor}{n} = m + \frac{\lfloor n\alpha \rfloor}{n}$$

Now, since $0 \leq \alpha < 1$, we have $0 \leq n\alpha < n$. So $\lfloor n\alpha \rfloor$ is an integer in $\{0, 1, \ldots, n-1\}$. Therefore:

$$\lfloor \frac{\lfloor nx \rfloor}{n} \rfloor = \lfloor m + \frac{\lfloor n\alpha \rfloor}{n} \rfloor = m = \lfloor x \rfloor \ \square.$$
