M1103 Chapter 2

Matrices

Definition:

A matrix of size (m, n) is a map from $\underbrace{\{1, \ldots, m\}}_{i} \times \underbrace{\{1, \ldots, n\}}_{j} \to \mathbb{K}$.

A matrix is a family denoted (a_{ij}), $i = \{1, \ldots, m\}$ and $j = \{1, \ldots, n\}$ of scalars.

In practice, we write

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i_1} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m_1} & \ldots & a_{mj} & \ldots & a_{mn} \end{pmatrix}$$

- The set of matrices of size (m, n) on $\mathbb{K}$ is denoted by $M_{m,n}(\mathbb{K})$
- If $A = (a_{ij})$ and $B = (b_{ij})$ are two elements of $M_{m,n}(\mathbb{K})$, we have

$$A = B \iff a_{ij} = b_{ij},\ \forall (i, j)$$

Definition

- The row matrix ,respectively column matrix, on $\mathbb{K}$ is a matrix of size $(1, n)$, (respectively $(m, 1)$), on $\mathbb{K}$.

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \end{pmatrix}$$

- A square matrix of size n is a matrix size (n, n) on $\mathbb{K}$.
  Consequences:

1. Let $A = (a_{ij})$ be a square matrix of size n on $\mathbb{K}$. The main diagonal of that matrix is the family $(a_{ii})_{1 \leq i \leq n}$.

$$\begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ldots & \\ & & & a_{nn} \end{pmatrix}$$

2. We will write $M_n(\mathbb{K})$ instead of $M_{n,n}(\mathbb{K})$.

Definition

Let $A \in M_{m,n}(\mathbb{K})$:

A sub-matrix of A is the restriction of the map A, to $I \times J$, where $I \subset \{1, \dots m\}$ and $J \subset \{1, \dots, n\}$ (with $I \neq \emptyset, J \neq \emptyset$).

Example

$$\begin{pmatrix} 1 & -5 & 0 & 2 \\ 2 & 1 & 4 & 3 \\ 7 & 0 & 0 & -2 \end{pmatrix} = M_{3,4}(\mathbb{R})$$

We want to restrict this matrix to $I = \{1, 3\}$, $J = \{1, 4\}$:

$$\begin{pmatrix} 1 & 2 \\ 7 & -2 \end{pmatrix}$$

Thus, we got a square matrix from the restrictions.

Suppose we chose $I = \{1, 2\}$ and $J = \{1, 2, 3\}$, we get:

$$\begin{pmatrix} 1 & -5 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

We call this a block, corresponding to successive integers.

Section 2.2

Definition

The zero-matrix of size (m, n) on $\mathbb{K}$ is $A = (a_{ij}) \in M_{m,n}(\mathbb{K})$ such that:

$$a_{ij} = 0, \ \forall (i, j)$$

We write: $0_{m,n}$ (or 0).

Definition

$A = (a_{ij}) \in M_{m,n}(\mathbb{K})$,
$B = (b_{ij}) \in M(m, n)(\mathbb{K})$.

The sum of A and B is the matrix in $M_{m,n}(\mathbb{K})$ denoted by:

$$A + B = (c_{ij}), \text{ with } c_{ij} = a_{ij} + b_{ij}, \ \forall (i, j)$$

Definition

$A = (a_{ij}) \in M_{m,n}(\mathbb{K})$,
$\lambda \in \mathbb{K}$.

The scalar multiplication of λ and A is the matrix in $M_{m,n}\mathbb{K}$ denoted by:

$$\lambda A = (d_{ij}) \in M_{m,n}(\mathbb{K}), \text{where}$$
$$d_{ij} = \lambda a_{ij},\ \forall (i,j)$$

Theorem

Suppose $\lambda, \mu \in \mathbb{K}$,
$A, B, C \in M_{m,n}(\mathbb{K})$

Then,

1. $(A+B)+C = A+(B+C)$
2. $A+0 = 0+A = A$
3. $A+(-A) = (-A)+A = 0$
4. $A+B = B+A$
5. $\lambda(A+B) = \lambda A + \lambda B$
6. $(\lambda+\mu)A = \lambda A + \mu A$
7. $\lambda(\mu A) = (\lambda\mu)A$
8. $1A = A$

Proof

Trivial □

Note
By the previous theroem, we can say that $M_{m,n}(\mathbb{K},+)$ is an abelian group.

Definition

The identity matrix of size n on $\mathbb{K}$, is the square matrix $A = a_{ij}$ with

$$(a_{ij}) = \begin{cases} 1\ if\ i = j \\ 0\ if\ i \neq j \end{cases}$$

$$\begin{pmatrix} 1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & 1 \end{pmatrix} = I_n$$

Definition

$A = (a_{ij}) \in M_{m,n}(\mathbb{K})$,
$B = (b_{jk}) \in M(n,p)(\mathbb{K})$.

The product of A and B is the matrix $AB = (p_{ij}) \in M_{m,p}(\mathbb{K})$ where:

$$p_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk}, \ \ \forall (i,k)$$

Example

(1)

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -3 & 4 \end{pmatrix} \in M_{2,3}(\mathbb{K})$$

$$B = \begin{pmatrix} 0 & -1 & 0 & -5 \\ 0 & 2 & 1 & 2 \\ 1 & -3 & 4 & 1 \end{pmatrix} \in M_{3,4}(\mathbb{K})$$

The product is well defined, and $AB \in M_{2,4}(\mathbb{K})$.
So

$$AB = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{pmatrix}$$

$$p_{11} = \sum_{j=1}^{3} a_{1j} b_{j_1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = (1)(0) + (2)(0) + (0)(1) = 0$$

$$p_{12} = \sum_{j=1}^{3} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = (1)(-1) + (2)(2) + (0)(-3) = 3$$

We will continue doing this directly,

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & -3 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & -5 \\ 0 & 2 & 1 & 2 \\ 1 & -3 & 4 & 1 \end{pmatrix}$$

Continue applying the equation mentally to make it easier.

$$AB = \begin{pmatrix} 0 & 3 & 2 & -1 \\ 4 & -12 & 13 & 3 \end{pmatrix} : \checkmark$$

Theorem

1. $\forall A \in M_{m,n}(\mathbb{K})$, $\forall B \in M_{n,p}(\mathbb{K})$, $\forall C \in M_{p,q}(\mathbb{K})$, we have: $(AB)B = A(BC)$.
2. $\forall A \in M_{m,n}(\mathbb{K})$, we have: $\underbrace{A}_{m,n}\underbrace{I_n}_{n,n} = \underbrace{A}_{m,n} = \underbrace{I_m}_{m,m}\underbrace{A}_{m,n}$.
3. $\forall A_1, A_2 \in M_{m,n}(\mathbb{K}), \forall B \in M_{n,p}(\mathbb{K})$, we have $(A_1 + A_2)B = A_1B + A_2B$.
4. $\forall A \in M_{m,n}(\mathbb{K}), \forall B_1, B_2 \in M_{n,p}(\mathbb{K})$, then $A(B_1 + B_2) = AB_1 + AB_2$.
5. $\forall A \in M_{m,n}(\mathbb{K}), \forall B \in M_{n,p}(\mathbb{K}), \forall \lambda \in \mathbb{K}$, we have $(\lambda A)B = A(\lambda B) = \lambda(AB)$.

Proof

We shall only prove one of them:

3. $(A_1 + A_2)B \stackrel{?}{=} A_1B + A_2B$

We write

$$A_1 = (a_{ij}),\ \ A_2 = (a'_{ij}),\ \ B = (b_{jk}),$$

Then,

$$(A_1 + A_2)B = (p_{ik}),\ \ A_1B + A_2B = (q_{ik})$$

$$\begin{aligned} p_{ik} &= \sum_{j=1}^{n}(a_{ij} + a'_{ij})b_{jk} = \sum_{j=1}^{n}(a_{ij}b_{jk} + a'_{ij}b_{jk}) \\ &= \sum_{j=1}^{n} a_{ij}b_{jk} + \sum_{j=1}^{n} a'_{ij}b_{jk} = q_{ik} \quad \square \end{aligned}$$

Definition

Suppose $A = a_{ij} \in M_{m,n}(\mathbb{K})$

The transpose of A is the matrix defined by:

$$\begin{aligned} tA = (b_{ij}) \in M_{m,n}(\mathbb{K}) \\ b_{ij} = a_{ij},\ \ \forall (i,j) \end{aligned}$$

Let $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{pmatrix} \in M_{2,3}(\mathbb{R})$

Then,

$$tA = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix} \in M_{3,2}(\mathbb{R})$$

Theorem

1. $\forall A, B \in M_{m,n}(\mathbb{K}), t(A+B) = tA + tB$
2. $\forall \lambda \in \mathbb{K}, \forall A \in M_{m,n}(\mathbb{K}), t(\lambda A) = \lambda tA$
3. $\forall A \in M_{m,n}(\mathbb{K}), \forall B \in M_{m,n}(\mathbb{K}), tAB = tBtA$
4. $t(tA) = A$

Proof

1, 2 and 4 are easy

3. Write $A = (a_{ij}), B = (b_{jk}), AB = (p_{ik})$

$$\begin{aligned} tB = (\beta_{ij}),\ tA = (\alpha_{jk}) \\ tBtA = (\pi_{ik}) \text{ and } t(AB) = (\gamma_{ik}) \\ \pi_{ik} = \sum_{j=1}^{n} b_{ij}a_{kj} = \sum_{j=1}^{n} a_{jk}b_{ij} = p_{ki} = \gamma_{ik},\ \forall (i,k) \quad \square \end{aligned}$$

Section 2.3

Theorem

Consider the set $M_n(\mathbb{K})$ of square matrices of size n, equipped with addition and multiplication of matrices is a ring, with zero matrix $0_{n,n}$ and unit element I_n

Proof

by the theorem at the end of the last session, restricted to the case $m = n = p = q$.

Consequences

for $n = 1$, $M_1(\mathbb{K}) \cong \mathbb{K}$:

$$\begin{aligned} M_1(\mathbb{K}) = \{(a); a \in \mathbb{K}\} \\ \mathbb{K} \to M_1(\mathbb{K}) \\ a \to (a) \end{aligned}$$

So, $M_1(\mathbb{K})$, like $\mathbb{K}$, is a commutative field.

for $n \geq 2$, the ring $M_n(\mathbb{K})$ is NOT commutative, and it has zero divisors.

Indeed, consider $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{K})$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{K})$

$$AB = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We notice that $AB \neq BA$, ergo it is not commutative.

Newtons binomial formula

$$\begin{aligned} (x+y)^2 &= (x+y)(x+y) \\ &= x^2 + xy + yx + y^2 \\ &= x^2 + 2xy + y^2 \qquad \text{if } x \text{ and } y \text{ commute} \end{aligned}$$

Newtons binomial formula for matrices:

If

$$(A+B)^n = \sum_{k=0}^{n} C_n^k A^{n-k} B^k$$

If $\mathbb{R}$ is a ring, then x invertible $\iff \exists y \in \mathbb{R}; xy = yx = 1$, which we denote x^{-1}

$$xx^{-1} = x^{-1}x = 1$$

Also, a matrix $A \in M_n(\mathbb{K})$ is invertible if:

$$\exists B \in M_n(\mathbb{K}); AB = BA = I_n$$

then, if A is invertible, this B will be unique. We write it A^{-1}. Then we will have

$$AA^{-1} = A^{-1}A = I_n$$

Example

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We claim that if $ad - bc \neq 0$, then A is invertible, and we have:

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Indeed,

Let $B = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Then $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right]$. We recall that $A(\lambda B) = \lambda(AB)$

$$AB = \frac{1}{ad-bc}[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}]$$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ad-bc}{ad-bc} & 0 \\ 0 & \frac{ad-bc}{ad-bc} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Also, $BA = \cdots = I_2$

So A is invertible, and $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. $\square$

Note
Like in any ring, the set of invertible matrices is a group for multiplication. It is called the general linear group and is denoted by: $GL_n(\mathbb{K})$.
(Also, $\forall A, B \in GL_n(\mathbb{K}), (AB)^{-1} = B^{-1}A^{-1}$)

Theorem

$A \in M_n(\mathbb{K})$, then A invertible $\iff$ $transpose A$ invertible.

In this case, the inverse of the transpose is:

$$t(A^{-1})$$

Proof

If A is invertible, then:

$$tA \cdot tA^{-1} = t(A^{-1}A) = t(I_n) = I_n$$

Also,

$$t(A^{-1})tA = t(AA^{-1}) = t(I_n) = I_n$$

Therefore tA is invertible and $(tA)^{-1} = t(A^{-1})$. $\square$

Definition

Let $A = (a_{ij})$

1. We say that A is upper triangular (resp. lower triangular) if:

   $$i > j \implies a_{ij} = 0$$

   respectively

   $$i < j \implies a_{ij} = 0$$

What does this mean?

$$\begin{pmatrix} \ddots & & value \\ & \ddots & \\ 0 & & \ddots \end{pmatrix} = Upper\ triangular$$

$$\begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ value & & \ddots \end{pmatrix} = Lower\ triangular$$

2. We say that A is diagonal if:

   $$i \neq j \implies a_{ij=0}$$

   What does this mean?

$$\begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix} \text{(Where the dots indicate values)}$$

3. We say that A is scalar if it is diagonal and all its diagonal entries are equality

What does this mean?

$$diag(a_1, \dots, a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

$$diag(a, \dots, a) = \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = aI_n$$

Consequences

1. A upper triangular $\iff tA$ lower triangular
2. A diagonal $\iff A$ both upper and lower triangular
3. If $A = diag(\lambda, \dots, \lambda) = \lambda I_n$
4. A is invertible $\iff \lambda \neq 0$, and then

$$A^{-1} = \begin{pmatrix} \frac{1}{\lambda} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda} \end{pmatrix}$$

such matrix is invertible $\iff \forall i, a_i \neq 0$

Theorem

The set of upper triangular matrices is a sub-ring of $M_n(\mathbb{K})$. (resp. lower triangular)

Proof

1. Clearly, $\sum$ of two upper triangular matrices is upper triangular.
2. I_n is also clearly upper triangular.
3. Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are upper triangular, so that

$$a_{ij} = 0, b_{ij} = 0, \ \text{ whenever } \ i > j$$

Write $AB = (c_{ik})$;
Now,

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

Suppose $i > k$, if $\exists j; a_{ij} \neq 0 \text{ and } b_{jk} \neq 0$, then $i \leq j$, and $j \leq k$

But then we will have $i \leq k$, which is a contradiction. Thus, $\forall j, a_{ij} = 0 \text{ or } b_{jk} = 0$.
So the product $a_{ij} b_{jk}$ is necessarily 0

Thus, $c_{ik} = 0$. And finally, AB is upper triangular $\square$.

Remark

$$c_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji} = a_{ii} b_{ii}$$

Remark

Clearly, if λ a scalar and $A \in M_n(\mathbb{K})$, then if $(A \text{ upper triangular (resp. lower)})$

$$\implies \lambda A \text{ is upper/lower triangular or diagonal}$$

Definition

Let $A \in M_n(\mathbb{K})$.

We say that A is symmetric (resp. anti-symmetric/skew symmetric) if:

$$tA = A \text{ (resp. } tA = -A)$$

Consequence

Write $A = a_{ij}$, then

- A symmetric means $a_{ij} = a_{ji}, \ \forall i, j$
- A anti-symmetric means $a_{ij} = -a_{ji}, \ \forall i, j$

In particular, if A anti-symmetric, then the diagonal entries a_{ii} are all 0.

We denote by $MS_n(\mathbb{K})$ (resp. $MA_n\mathbb{K}$) the set of symmetric (resp. anti-symmetric) matrices

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Theorem

Each of $MS_n(\mathbb{K})$ and $MA_n(\mathbb{K})$ are an additive sub-group of $(M_n(\mathbb{K}), +)$.

Proof

Consider the 0 matrix, $0 \in MS_n(\mathbb{K})$ and $0 \in MA_n(\mathbb{K})$.

Let $A, B \in MS_n(\mathbb{K})$, then $t(A-B) = tA - tB = A - B$. Then $A - B \in MS_n(\mathbb{K})$.

Thus we proved that $MS_n(\mathbb{K})$ is an additive sub-group.

Take $C, D \in MA_n(\mathbb{K})$. Then $t(C-D) = tC - tD = -C - (-D) = -C + D$. Then $C - D = -(C + D)$. So, $C - D \in MA_n(\mathbb{K})$.

Thus we proved that $MA_n(\mathbb{K})$ is an additive sub-group. □

Remark

Suppose $A, B \in MS_n(\mathbb{K})$,

$$t(AB) = tBtA = BA$$

So, AB is symmetric if $AB = BA$.

Similarly, if $A, B \in MA_n(\mathbb{K})$,

$$t(AB) = tBtA = (-B) \cdot (-A)$$

So, $AB \in MA_n(\mathbb{K}) \iff AB = -BA$.

Exercise

Show that,

$$\begin{aligned} \forall \lambda \in \mathbb{K}, \forall A \in M_n(\mathbb{K}) \\ \text{we have} \\ A \in MS_n(\mathbb{K}) \implies \lambda A \in MS_n(\mathbb{K}) \\ A \in MA_n(\mathbb{K}) \implies \lambda A \in MA_n(\mathbb{K}) \end{aligned}$$

do at home

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§ 2.4

Definition

Let $A, B \in M_{m,n}(\mathbb{K})$

We say that A and B are row equivalent and we write $A \equiv B$ if B can be obtained from A by a sequence of elementary row operations.

Remark

- $\forall A \in M_{m,n}(\mathbb{K})$, we have $A \equiv A$
- $\forall A, B \in M_{m,n}(\mathbb{K})$, we have $A \equiv B \implies B \equiv A$.
- $\forall A, B, C \in M_{m,n}(\mathbb{K}), (A \equiv B \text{ and } B \equiv C) \implies A \equiv C$.

Definition

An elementary matrix of size n is a matrix $E \in M_{m,n}(\mathbb{K})$ obtained from the identity matrix by a single row operation.

examples

1. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

   This is an elementary matrix, that comes from I_3 by multiplying the third row by (-2).
2. $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

   This comes from I_4 by interchanging the second and fourth row.
3. $$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

   This comes from I_2 by doing $R_1 + 3R_2$
4. I_3

Theorem

If an elementary matrix E comes from preforming a certain row operation on I_n, the the matrix $E \cdot A$ comes from A by preforming the same row operations.

Proof

This is trivial, but lets take an example:

Let $c \neq 0$. Suppose,

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This comes from I_3 by multiplying the second row by c.

Let,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then,

$$E \cdot A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Which we got by simply multiplying the second row by c. $\square$

Let us now attempt this with other kinds of row operations:

Take,

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This comes from interchanging row 1 and 2 of I_2

Let,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then,

$$E \cdot A = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Which can be derived from interchanging the first and second row of A. $\square$

Let us take a last example:

Let,

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

This comes from I_3 by doing $R_1 + 3R_3$

Let,

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$$

Then,

$$E \cdot A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{pmatrix}$$

This matrix comes from A by doing the same row operation of $R_3 + 3R_1$. $\square$

This now tells us that elementary matrices can be derived from elementary row operations

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Theorem

Every elementary matrix is invertible, and it's inverse is an elementary matrix.

Proof

Suppose we drive E from I_n by some row operation.

Suppose we derive E' from I_n by the inverse row operations

Then, $E'E$ comes from preforming the inverse operations, and so we get back the identity I_n
$E'E = I_n$ (since the operations cancel each other out.)

Similarly, $EE' = I_n$.

So E is invertible and $E^{-1} = E'$ $\square$.

Remark

Consider the system,

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_m x_1 + \cdots + a_{mn}x_n = b_n \end{cases}$$

We can write this system as,

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_n & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

That is, $Ax = b$, where A is the matrix of coefficients, the matrix x is the coefficients, and the matrix b is the results.

Theorem

$A \in M_n(\mathbb{K})$, the following statements are equivalent:

1. A is invertible.
2. The homogeneous linear system $Ax = 0$, has only the trivial solution.
3. The RREF of A is the identity matrix I_n.
4. A is expressible as a product of elementary matrices.

Proof

1- $(1) \implies (2)$
2- $(2) \implies (3)$
3- $(3) \implies (4)$
4- $(4) \implies (1)$

1. Let x_0 be a solution of $Ax = 0$, then $Ax_0 = 0$. Then

$$A^{-1}(Ax_0) = A^{-1}0$$

$$\begin{aligned} \text{so } (A^{-1}A)x_0 &= 0 \qquad \square \\ I_n x_0 &= 0 \\ x_0 &= 0 \end{aligned}$$

2. Since $Ax = 0$ has only the trivial solution, it's RREF has to be:

$$\begin{cases} x_1 & & & = 0 \\ & x_2 & & = 0 \\ & & \ddots & \\ & & & x_n = 0 \end{cases}$$

   Which corresponds to the following augmented matrix:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

3. Since A can be reduced to I_n by the sequence of elementary row operations, there exists elementary matrices $E_1, \dots, E_k \in GL_n(\mathbb{K})$, such that $E_k \dots E_1 A = I_n$.

$$E_1^{-1} \dots E_k^{-1} \dots E_1 A = E_1^{-1} \dots E_k^{-1} I_n$$

   so,

$$A = E_1^{-1} \dots E_k^{-1}$$

   And we know that the inverse of an elementary matrix is an elementary matrix. Thus,

A can be expressed as a product of elementary matrices.

4. Every elementary matrix is invertible + the fact that the product of invertibles is invertible.

Important Remark

Let $A \in GL_n(\mathbb{K})$,

Then, there exists elementary matrices $E_1, \dots, E_k \in GL_n(\mathbb{K})$ such that

$$E_k \dots E_1 A = I_n$$
$$\text{Then,}$$

$$E_k \dots E_1 A A^{-1} = I_n A^{-1}$$
$$E_k \dots E_1 I_n = A^{-1}$$

Example

Find the inverse of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

Consider the augmented matrix $(A \mid I_3)$. That is,

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 2 & 5 & 3 & | & 0 & 1 & 0 \\ 1 & 0 & 8 & | & 0 & 0 & 1 \end{pmatrix}$$

$R_2 - 2R_1$
$R_3 - R_1$

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & -2 & 5 & | & -1 & 0 & 1 \end{pmatrix}$$

Cover first row,

$R_3 + 2R_2$

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & -5 & 2 & 1 \end{pmatrix}$$

Cover first two rows,

$R_3 \cdot (-1)$

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 5 & -2 & -1 \end{pmatrix}$$

We have reached REF. Lets proceed to RREF:

$R_2 + 3R_3$

$R_1 - 3R_3$

$$\begin{pmatrix} 1 & 2 & 0 & | & -14 & 6 & 3 \\ 0 & 1 & 0 & | & 13 & -5 & -3 \\ 0 & 0 & 1 & | & 5 & -2 & -1 \end{pmatrix}$$

$R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & 0 & | & -40 & 16 & 9 \\ 0 & 1 & 0 & | & 13 & -5 & -3 \\ 0 & 0 & 1 & | & 5 & -2 & -1 \end{pmatrix}$$

Thus, the inverse of the matrix A is:

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Remark

Often, we dont know if the matrix is invertible or not. We can still continue doing this. But by the theorem, RREF of A will not be the identity and, at some point, we will get a row of zeros.

Once we see the row of zeros, we can stop the operation and conclude that A is not invertible.

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§ 2.5

Theorem

A linear system has zero, one, or infinitely many solutions. There are no other possibilities.

Proof:

Suppose that the linear system $Ax = b$ has two distinct solutions x_1 and x_2. Let $x_0 = x_1 = x_2 \neq 0$.

Then:

$$\begin{aligned} Ax_0 &= A(x_1 - x_2) \\ &= Ax_1 - Ax_2 \\ &= b - b = 0 \end{aligned}$$

So the main solution of the homogeneous linear system is $Ax = 0$

$\forall \lambda \in \mathbb{K}$,

$$\begin{aligned} A(x_1 + \lambda x_0) &= Ax_1 + A(\lambda x_0) \\ &= Ax_1 + \lambda A x_0 \\ &= b + \lambda(0) \\ &= b \end{aligned}$$

So $\{x_1 + \lambda x_0; \lambda \in \mathbb{K}\}$ is contained in the set of solutions of $Ax = b$

and $\{x_1 + \lambda x_0; \lambda \in \mathbb{K}\}$ is infitnite (since $x_0 \neq 0$). $\square$

Theorem

If $A \in GL_n(\mathbb{K})$, then $\forall b \in M_{m,n}(\mathbb{K})$, the linear system $Ax = b$ has a unique solution, namely $x = A^{-1}b$.

Proof:

$A(A^{-1}b) = (AA^{-1})b = I_n b = b$
So $A^{-1}b$ is a solution of $Ax = b$.

Let x be a solution of $Ax = b$
then $Ax = b \implies A^{-1}(Ax) = A^{-1}b$
$\implies (A^{-1}Ax) = A^{-1}b$
$\implies I_n x = A^{-1}b$
$\implies x = A^{-1}b \,\square$.

Theorem

Let $A \in M_n(\mathbb{K})$

(1) If there exists $B \in M_n(\mathbb{K})$ such that $BA = I_n$, then A is invertible and $A^{-1} = B$

(2) If there exists $B \in M_n(\mathbb{K})$ such that $AB = I_n$, then A is invertible and $A^{-1} = B$

Proof:

(1)

$$\begin{aligned} Let\ x \in M_{n,1}(\mathbb{K}); Ax = 0 \\ then\ B(Ax) = B(0) \\ &\implies (BA)x = 0 \\ &\implies I_n x = 0 \\ &\implies x = 0 \end{aligned}$$

So, the homogeneous linear system only has the trivial solution, so A is invertible.

Then $BA = I_n$
$\implies (BA)A^{-1} = I_n A^{-1}$
$\implies B(AA^{-1}) = A^{-1}$
$\implies BI_n = A^{-1}$
$\implies B = A^{-1}$

(2)

Since $AB = I_n$, then B is invertible.
So $(AB)B^{-1} = I_n B^{-1}$
$\implies A(BB^{-1}) = B^{-1}$
$\implies AI_n = B^{-1}$
$\implies A = B^{-1}$
$\implies B = A^{-1}$ $\square$

Theorem

(continuation of the previous theorem)

5. The linear system $Ax = b$ has a unique solution
6. The linear system $Ax = b$ is consistent $\forall b \in M_{n,1}(\mathbb{K})$.

Proof:

$(5) \to (6)$: immediate

$(6) \to (1)$:

There exists $x_1, x_2, \ldots, x_n; Ax_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, Ax_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, Ax_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Let $B = (x_2, \ldots, x_n)$

Then, $AB = (Ax_1, \ldots, Ax_n) = I_n$. So A is invertible. $\square$
