# Pure Jump Lévy Processes and Self-decomposability in Financial Modelling

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# Outline

- Introduction
- Levy Processes
- Self-decomposable Laws
- Ornstein-Uhlenbeck Processes
- Normal Inverse Gaussian Distribution
- Modelling of Returns
- Application of Real Data
- Conclusion

# 1. Introduction

- The basic models of modern finance are based on the assumption of normality of asset returns. But a lot of empirical studies are shown that the assumption which observations are normally distributed is a poor approximation in the real world.

- Because the returns have features such as jumps, semi-heavy tails and asymmetry. In traditional diffusion models, price movements are very small in short period of time. But in real markets, prices may be show  big jumps in short time periods.

# Introduction

- When price process model include the jumps, the perfect hedging is imposable. In this case, market participants can not be hedge risks by using only underlying assets.

- For these reasons, diffusion models used in finance is not a sufficient model. A good model must be allow for discontinuities and jumps in price process.

- Lévy processes is a valuable tool in financial modeling, they provide a good fit with reel data.

# Introduction

- In probability theory, a *Lévy process*, named after the French mathematician Paul Levy, is any continuous-time stochastic process that starts at 0, admits cadlag modification and has "stationary independent increments‟.

- The most well-known examples are the Wiener process and the Poisson process.
- Lévy process is a simple Markov process with jumps that allow us to capture asset returns without the necessity of introducing extreme parameter values

# Introduction

- Lévy models are not adequately fitting implied volatility surfaces of equity options across both strike and maturity.

- The increments of additive process present a flexible model. These processes may be obtained with self-decomposable distributions.

# Introduction

- A law in class of self-decomposable laws can be decomposable into the sum of a scaled down version themselves and an independent term.

- The class of self-decomposable distributions is obtained as a limit laws of a sequence that independent and suitably normalized

# Introduction

- The properties of the return distributions depend on length of return interval. Log returns is taken monthly are reasonably represented by a normal distribution.

- If one be dealing with tick data, then return distributions may have heavy tails.

- The aim of these paper, to review to pure jump Lévy process with self-decomposable distributions in financial modeling and testing for the presence of a Brownian motion component and discriminating between finite or infinite variation

# 2. LÉVY PROCESSES

***Definition 2.1*** A cadlag stochastic process $X = (X_t)_{t\geq 0}$ is defined on a filtered probability spaces $(\Omega, \Im, (\Im_t)_{t\geq 0}, P)$ is called a *Lévy processes* if the following conditions holds

*i)* $X_0 = 0$

*ii)* $X$ has independent increments of the past i.e. $X_t - X_s$ independent of $\{X_u ; u \leq s\}$ for $0 \leq s \leq t$

*iii)* $X$ has stationary increments i.e $X_t - X_s$ has the same distribution with $X_{t-s}$

# Lévy Processes

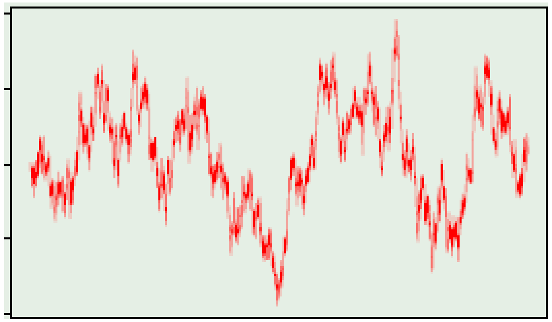
*iv)* $X$ is stochastically continuous,

$$\lim_{k \to \infty} P\left(\left|X_{t+k} - X_t\right| \geq \varepsilon\right) = 0 \qquad \forall \varepsilon > 0$$

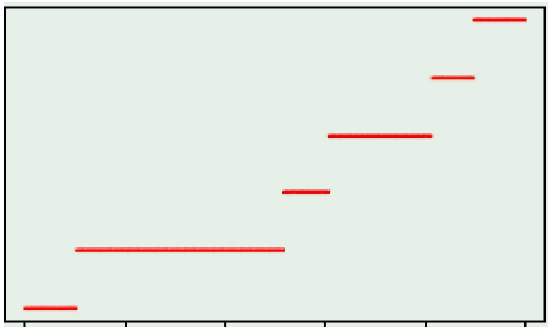
Process has cadlag paths, i.e,

- $t \to X_t$ is a.s. right-continuous with left limits.

# Basic Examples



Brownian motion $B_t \sim N\left(\mu t, \sigma^2 t\right)$



Poisson Process $N_t \sim Pois(\lambda t) \quad \lambda \in (0, \infty)$

# Independent and Stationary Increments

- A continuous-time stochastic process assigns a random variable $X_t$ to each point $t \geq 0$ in time. It is a random function of $t$. The **increments** of such a process are the differences $X_s - X_t$ between its values at different times $t < s$.

- The increments of a process is called independent if increments $X_s - X_t$ and $X_u - X_v$ are independent random variables ,whenever the two time intervals do not overlap.

- The increments is called stationary , if the probability distribution of any increment $X_s - X_t$ depends only on the length $s - t$ of the time interval; in the other words , increments with equally long time intervals are identically distributed.

# Independent and Stationary Increments

- Lévy processes can be viewed as continuous time random walks. Lévy process form building blocks for both Markov processes and semi-martingales

- There is one to one relationship between Lévy process and infinitely divisible distributions.

# Independent and Stationary Increments

- ***Theorem 2.1*** Let $X = (X_t)_{t\geq 0}$ be a Lévy process. Then $X_t$ has an infinitely divisible distribution $F$ for every t.

- Conversely if is an infinitely divisible distribution, $F$ then there exist a $X = (X_t)_{t\geq 0}$ Lévy process , such that distribution of $X_1$ is given $F$

- The distribution of a Lévy process $X = (X_t)_{t\geq 0}$ is completely determined by any of its marginal distributions

# Independent and Stationary Increments

- The probability distributions of the increments of any Lévy process are infinitely divisible
- There is a Lévy process for each infinitely divisible probability distribution.
- A continuous process with independent increments is Gaussian.
- A linear combination of independent Levy processes is also a Levy process.

# 2.1 Infinitely Divisibility

- Let $\varphi(u)$ be a characteristic function of a $X$ random variable. We say that the law of $X$ random variable is infinitely divisible, if we can always find another characteristics function. $\varphi_n(u)$ such that,

$$\varphi(u) = [\varphi_n(u)]^n$$

- If $X \sim \varphi(u)$ is infinitely divisible, then for all $n \in \mathrm{N}$ there exist i.i.d random variables

$$X_1^{(1/n)}, \ldots, X_n^{(1/n)}$$

# Infinitely Divisibility

- All distributed as $\varphi_n(u)$ and such that ,

$$X \overset{d}{=} X_1^{(1/n)} + \ldots + X_n^{(1/n)}$$

- In the other words, $X$ random variable always is decomposable into the sum of an arbitrary finite number of i.i.d. random variables

# 2.2 Jump Diffusion Model

- These models have jumps and random evolution between the jump times,

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

- where W standard Brownian motion, $\sum_{i=1}^{N_t} Y_i$ compound Poisson process.

$$E\left[e^{iuX_t}\right] = \exp\left\{t\left(iu\mu - \frac{u^2\sigma^2}{2} + \lambda\int_{-\infty}^{\infty}\left(e^{iux}-1\right)F(dx)\right)\right\}$$

# 2.3 The Probabilistic Properties of Lévy Processes

- Let $X_t$ be a Lévy process, we consider following characteristic function,

$$\varphi_{X_t}(u) = \exp(iuX_t)$$

- For time interval $[0,t]$ $\Delta t = t_i - t_{i-1} = \frac{t}{n}$ by the assumption of independent and stationary increments;

$$X_t = \left(X_{t_1} - X_{t_0}\right) + \ldots + \left(X_{t_n} - X_{t_{n-1}}\right) \overset{d}{=} n\, X_{\frac{t}{n}}$$

# The Probabilistic Properties of Lévy Processes

$$\varphi_{X_t}(u) = E\exp\left(iuX_t\right) = E\exp\left(iunX_{t/n}\right) = \left(E\exp\left(iuX_{t/n}\right)\right)^n$$

for $n = t,$

$$\varphi_{X_t}(u) = \left(E\exp\left(iuX_1\right)\right)^t = e^{t\,\psi(u)}$$

$$\psi(u) = \log E\exp\left(iuX_1\right)$$

# The Probabilistic Properties of Lévy Processes

- The every Lévy process can be represented in the following form,

$$X_t = \mu t + \sigma W_t + Z_t$$

where $Z_t$ is a jump process with infinitely many jumps.

# Lévy-Khintchine Formula

- The characteristic function of a Lévy process $X = (X_t)_{t \geq o}$ is given by the, Lévy- Khintchine formula,

$$E\, e^{iuX_t} = \exp\left\{ \left( iub - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} \left( e^{iux} - 1 - iux 1_{\{|x| \leq 1\}} \right) \nu(dx) \right) \right\}$$

where, $(b, \sigma^2, \nu)$ called generating triplet, $\nu$ does not have mass on 0

# Lévy-Khintchine Formula

$\nu(\{0\}) = 0$ and satisfies the following integrability condition;

$$\int_{\Re} \min\left(x^2, 1\right)\nu\left(dx\right) < \infty$$

# Lévy-Ito representation

- *Lévy-Ito representation* describes the path structure of a Lévy process. A Lévy process can be represented in the following way,

$$X_t = \mu t + \sigma W_t + \sum_{s \le t} \Delta X_s 1_{\{|\Delta X_s| \ge 1\}} + \lim_{\varepsilon \to 0} \left( \sum_{s \le t} \Delta X_s 1_{\{\varepsilon \le |\Delta X_s| \le 1\}} - t \int x 1_{\{\varepsilon \le |x| \le 1\}} \nu(dx) \right)$$

$b \ge 0$ , $\sigma \ge 0$ , $(W_t)_{t \ge 0}$ is a standard Brownian motion.

# Lévy measure

$\Delta X_s = X_s - X_{s^-}$ : denotes the jumps at times.

- $\nu$ is called *Lévy measure* of $\{X_t\}$
- Thus, $\nu(dx)$ is the intensity of jumps of size x .
- We assumed that the paths of Lévy process is defined over a finite intervals $[0,t]$ .
- As a consequence the sum of the jumps in time interval $[0,t]$ with absolute jump size bigger than 1 is a finite sum for each path.

# Lévy measure

$$\nu(A) = E\left(card\left\{s \in [0,1] : \Delta X_s \neq 0 \,,\, \Delta X_s \in A\right\}\right) \qquad A \in \mathrm{B}(\Re)$$

- In the other words, $\nu(A)$ *is the average number of jumps* of process
- $X$ in time interval $[0,1]$ whose sizes fall in $A$
- *In general in a Lévy process, the frequency of the big jumps determines existence of moments of process. The fine structure of the paths of the process can be read of the frequency the small jumps.*

## 2.4 Finite Activity

- A stochastic process has *finite activity* if all paths of process have a finite number of jumps on every finite interval i.e.,

$$\nu(R) = \int_R \nu(dx) < \infty$$

- If all paths of process has an infinite number of jumps on every finite interval i.e.,
- $\nu(R) = \infty$ Lévy process has *infinite activity*. i.e. An infinite mass accumulates around the origin. In a such case, small jumps of X occur infinitely.

# 2.5 Finite Variation

- Let $X = (X_t)_{t \geq 0}$ be a Lévy process.

*i)* If $\sigma^2 = 0$ and $\int 1_{\{|x| \leq 1\}} |x| \, \nu(dx) < \infty$ and $X$ process have *finite variation* i.e.,

$\sum_{s \leq t} |\Delta X_s| < \infty$ i f and only if $\int_{\Re} \min(|x|, 1) \, \nu(dx) < \infty$

*ii)* If $\sigma^2 \neq 0$ or $\int 1_{\{|x| \leq 1\}} |x| \, \nu(dx) = \infty$

then process $X$ have *infinite variation*.

# Subordinators

- A Lévy process *Z* is called a *subordinator,* if it has a.s. non-decreasing paths.
  - such a process cannot have a diffusion component.

- Given a Lévy process *X* and an independent subordinator *Z*, the process

$$Y_t = X_{Z_t}$$

- is called a *subordinated* Levy process.

# 3. SELF-DECOMPOSABLE LAWS

- There is a very close connection the laws of class L and self-decomposable laws. For these reason, first time , we describe self-decomposable laws and related.

- 3.1 Laws of Class L

- The name of class L first time used by Khintchine, any random variable in class L is infinitely divisible ([30],Theorem 9.3), [26]. The infinitely divisible laws are the limit laws of triangle arrays, where arrays of independent random variables which individually negligible. Infinite divisibility is preserved under affine transformations.

# 3.1 Laws of Class L

- Let $(Y_n\ ; n = 1,2,\ldots)$ is a sequence of independent random variables and

$$S_n = \sum_{i=1}^{n} Y_i$$

- denotes their sum.
- Suppose that, there exist centering constants $a_n \in \Re$ and scaling constants $b_n > 0$ such that the distribution of
- $b_n S_n + a_n$ converges to the distribution of some random
- variable $X$

# 3.1 Laws of Class L

- Then we say that, random variable $X$ is a member of class L .

- As explained above, we shortly can say that, if a random $X$ variable has same $x$ distribution as the limit of some sequence of normalized sums of independent random variables, random variable $X$ has a distribution of class L.

# Self-decomposability

***Definition3.1*** *(Self-decomposability)* We suppose that $\varphi(u)$ is the characteristic function of a law. We say that this law is *self-decomposable* and write $\varphi \in SD$ if for $c \in (0,1)$ *and* $u \in \Re$

$$\varphi(u) = \varphi(cu)\,\varphi_c(u)$$

- where $\varphi_c(u)$ is a characteristic function. $\varphi_c(u)$ is uniquely determined.
- For example normal case,

# Self-decomposability

- For example normal case,

$$\varphi_c(u) = \exp\left[-\frac{1}{2}(1-c)u\right]$$

- We can restate this definition for random variables as follows,

- The distribution of a random variable $X$ is self-decomposable, for any constant $c \in (0,1)$
- we can a find independent random variable $X_c$ such that,

# Self-decomposability

$$X \stackrel{d}{=} cX + X_c$$

- where $\stackrel{d}{=}$ denotes equality in distribution
- According to ([30], theorem 15.3) a random variable $X$ has a distribution of *class L* if and only if the law of the $X$ is self-decomposable.
- The class of self-decomposable distributions is a proper subset of infinitely divisible distributions.
- A random variable X or its probability $\mu_X$ or probability density $f_X$ is self-decomposable,if the corresponding characteritic function is in the class SD.

# Self-decomposability

- Self-decomposable laws arise as marginal laws in autoregressive time series models

$$X_t = c\,X_{t-1} + \varepsilon_t$$

- The Lévy measure of the self-decomposable laws is absolutely continuous with following density form,

$$\nu(dx) = \frac{k(x)}{x}\,dx$$

# Self-decomposability

- SD is closed under affine mappings, i.e., for all reals a and b if $\varphi \in SD$ then,

$$e^{ibu}\varphi(a\,u) \in SD$$

- If $X \in SD$ there exists a unique Lévy process Z such that

$$X \overset{d}{=} \int_0^\infty e^{-s}\,dZ_s$$

# Self-decomposability

- Let $\varphi \ and \ \psi$ denote the characteristic functions of $X \ and \ Z_1$ respectively, then

$$\log \varphi(t) = \int_o^t \log \psi(v) \frac{dv}{v}$$

$$\psi(t) = \exp\left[ t \left( \log \varphi(t) \right)' \right] \qquad , t \neq 0 \ , \ \psi(0) = 1$$

# Self-decomposability

- where, $k(x)$ increasing for $(-\infty, x)$ and decreasing for $(x, \infty)$.
- The density of self-decomposable distributions is unimodal.
- Let $X = (X_t)_{t \geq 0}$ be a Lévy process.
- The process $(X_1)$ self-decomposable if and only if $(X_t)$ selfdecomposable for every $t > 0$ .

# Self-decomposability

- The characteristic function of self-decomposable laws have following form,

$$E\left[e^{iuX}\right] = \exp\left\{ iu\mu - \frac{u^2\sigma^2}{2} + \int_{\Re} \left( e^{iux} - 1 - iux 1_{\{|x|<1\}} \right) \frac{k(x)}{x} dx \right\}$$

- Where, $\mu \in R$ , $k(x) \geq 0$ and $\int_{\Re} \min\left(1, |x|^2\right) \frac{k(x)}{x} dx < \infty$

# Self-decomposability

- A self-decomposable random variable $X$ is the value at unit time of some pure jump Lévy processes which sample paths have bounded variation.

- When the levy density integrates $|x|$ in the region $|x|<1$, for

$$\mu = \int_{|x|<1} x \frac{k(x)}{x} dx$$

# Self-decomposability

- The characteristic function of $X$ ,

$$E\left[e^{iuX}\right] = \exp\left[\int_{-\infty}^{\infty} \left(e^{iux_1}\right) \frac{k(x)}{x} dx\right]$$

- We can consider that, the returns are the sum of a suitable number of approximately independent random variables.
- Furthermore the return distribution is a limit distribution .
- Self-decomposable distributions can be consider as candidate for the unit period distribution of asset returns

# Self-decomposability

- For example, the Laplace (double exponential) random variable has the probability density,

$$f(x) = \frac{1}{2} e^{-|x|} \qquad , x \in R$$

- Its characteristic function is equal to,

$$\varphi(u) = \exp\left[ \int_{-\infty}^{\infty} \left(e^{iux} - 1\right) \frac{e^{-|x|}}{|x|} dx \right]$$

$$= \exp \int_0^u \left[ \int_{-\infty}^{\infty} \left(e^{-ivx} - 1\right) e^{-|x|} \, dx \right] \frac{dv}{v}$$

$\varphi \in SD$

## 3.2 Self-Decomposable Laws and Additive Processes

- Additive processes are obtained from Lévy processes by relaxing the condition of stationary of increments
- A levy process is a additive process with requirement that the increments must be stationary.
- Additive processes have markovian property hence we can obtain complete law of process, if we know the laws of increments
- The trajectories of the additive process can make jumps.
- A law is self-decomposable if and only if it is the law of unit time of an additive processes.

## 3.3 Self-similarity and Self-decomposability

- A stochastic process $X = (X_t)_{t \geq 0}$ is called self-similar for any given , $a \geq 0$ if we can find $b \in \Re$ such that,

$$(X_{at})_{t \geq 0} \stackrel{d}{=} (b\, X_t)_{t \geq 0}$$

- Where, b can be expressed as $b = a^H \quad , H \geq 0$

- In the other words, we say that *one stochastic process is self-similar such that, the change an in time scale can be compensated by a corresponding change in the spaces scale*.

## Self-similarity and Self-decomposability

- The H is called *Hurst exponent*. The connection between self-decomposable laws and self-similar additive process is given by [31].

- A law is self-decomposable if and only if it is the law at unit time of a self-similar additive process.

- Let $\varphi(u)$ be a characteristic function of a law.

- then it can take a characteristic function of Lévy process as follow,

# Self-similarity and Self-decomposability

$$\varphi_t(u) = [\varphi(u)]^{t/N}$$

- Where, N is time scale , if $\varphi(u)$ is a infinite divisible,
- $\varphi_t(u)$ is a characteristic function .
- Now we describe a new function as follows,

$$\psi_{k,H}(\theta) = \varphi\left[\left(\frac{k}{N}\right)^H \theta\right] \varphi\left[\left(\frac{h}{N}\right)^H \theta\right]^{-1}$$

## Self-similarity and Self-decomposability

- It is a characteristic function if and only if  is self-decomposable ([30],p.99), ([27],p.1884).

- The stationary process and the self-similar process are related by  using first Lamperti representation

## Self-similarity and Self-decomposability

***Proposition 3.1:*** Let $X = (X_t)_{t \in R}$ be a stationary process, the a new process defined by

$$Z_t = t^H X_{\log t} \quad , t > 0, Z_0 = 0 \, and H > 0$$

is a self-similar process with index H>0 .

- Conversely if the process Z is self-similar with H>0 , then process defined by

$$(X_t)_{t \in R} = \left(e^{-tH} Z_{e^t}\right)_{t \in R}$$

is a stationary process

## Self-similarity and Self-decomposability

- In general a Lévy process $X = (X_t)_{t \geq 0}$ may be constructed from the a H-self-similar process $Z_t$ in accordance with

$$X_t = \int_t^{e^t} \frac{1}{s^H} dZ_s$$

# 4. ORNSTEIN-UHLENBECK PROCESSES

- $X = (X_t)_{t \geq 0}$ Ornstein-Uhlenbeck process (OU) satisfies following differential equation ,

$$dX_t = -\lambda X_t dt + dZ_t$$

- where $\lambda > 0$ and $Z = (Z_t)_{t \geq 0}$ is a Lévy process. Homogenous Lévy process $Z_t$ has that property

$$E\left[\log\left(1 + |Z_1|\right)\right] < \infty$$

# Ornstein-Uhlenbeck Processes

- If $Z_t$ is non-Gaussian Lévy process, above differential equation has a unique solution. It has following form,

$$X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)} dZ_s$$

where $\lambda \in R$ and $X_0$ is initial state.

- The Ornstein-Uhlenbeck process is a Markov process and possesses a stationary regime under a mild log-moment condition.

# Ornstein-Uhlenbeck Processes

- $\nu_0$ Let be Lévy measure such that satisfies log-integrability conditition ,

$$\int_{|x|\geq 1} \log|x|\,\nu_0(dx) < \infty$$

- If a Lévy process satisfies above condition, then it has a self-decomposable distribution.

# Ornstein-Uhlenbeck Processes

- O–U process has the following, moving average representation:
- $$X(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} dZ_s$$
- O–U process has following correlation function;

$$Corr\left[X_s, X_t\right] = e^{-\lambda|s-t|}$$

# Ornstein-Uhlenbeck Processes

- O–U process can be defined as a Lamperti representation of the Brownian motion $Z_t$
-

$$X(t) = e^{-\lambda t} Z\left(e^{2\lambda t}\right)$$

# Ornstein-Uhlenbeck Processes

- If an $X = (X_t)_{t \geq 0}$ OU process is stationary, it's characteristic function has following form,

$$\varphi(u) = \varphi\left(u\, e^{-\lambda t}\right)\varphi_c(u)$$

- This denotes that, the marginal distribution of $(X_t)$ is self-decomposable.

$$\varphi_c(u) = \exp\left(\kappa(u) - \kappa\left(u\, e^{-\lambda t}\right)\right)$$

# Ornstein-Uhlenbeck Processes

- Where, $\kappa(u)$ is cumulant of $(X_t)$ and denoted by as

$$\kappa(u) = \log \varphi(u)$$

- Let Z be the background driving Lévy process of X OU processes and

$$\int_{\{|x|>2\}} \log|x|\,\nu(dx) < \infty$$

- Where v Lévy measure of Z process.

# Ornstein-Uhlenbeck Processes

- Then the law of $X_t$ converge towards $\zeta$ a self-decomposable law as $u \to \infty$

- Characteristic function of this law is given by,

$$\zeta(u) = \exp\left\{ \int_0^\infty \phi\left(u\, e^{-\lambda s}\right) ds \right\}$$

- $\phi$ is the characteristic exponent of $X_1$ process

# Ornstein-Uhlenbeck Processes

- We can say that the limit distribution of an $\zeta$ OU process is self-decomposable .

- The distribution of a random variable $X_1$ is self-decomposable if

$$X_1 \stackrel{d}{=} \int_0^\infty e^{-s} dZ_s$$

# Stochastic Volatility

- In Black-Scholes model,volatility is standart deviation of the return.Volatility is unobserved and has to be estimated from past prices.
- Volatility is not constant, it can be take as stochastic.
- For example, in BNS model,return process,

$$dX_t = \left[\mu + \beta \sigma_t^2\right]dt + \sigma_t \, dW_t$$

- Volatility process may be as an OU process,

$$d\sigma_t^2 = -\lambda \sigma_t^2 \, dt + dZ_{\lambda t}$$

- The process $\sigma^2$ is defined as a stationary markov process

# Stochastic Volatility

- Where Z is an increasing Levy process and W an independent Brownian motion
- The price process $(X_t)_{t \geq 0}$ has continuous sample paths.
- The volatility process $(\sigma_t)_{t \geq 0}$ is a non-Gaussian Ornstein-Uhlenbeck process
- The volatility process $(\sigma_t)_{t \geq 0}$ is not directly observable

- Finally ,stochastic volatility (SV) models are used to capture the impact of time-varying volatility on financial markets.

# 5. NORMAL INVERSE GAUSSIAN DISTRIBUTION

- Normal Inverse Gaussian Distribution was introduced by [5].
- This distribution is used in [9], [28], [29] to model equity returns.
- NIG distribution has semi-heavy tails and a special case with $\lambda = -1/2$ parameter in the class of Generalized Hyperbolic Distributions (GH).
- GH class are infinitely divisible and self-decomposable.

# Normal Inverse Gaussian Distribution

- The probability density function of the $NIG(\alpha, \beta, \mu, \delta)$ is defined by as follows,

$$f_{NIG}(x, \alpha, \beta, \mu, \delta) = \frac{\alpha\,\delta}{\pi} \exp\left[\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right] . \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}}$$

with parameters, $\delta > 0$ , $\mu \in (-\infty, \infty)$ *and* $0 < |\beta| < \infty$

## Normal Inverse Gaussian Distribution

- The function $K_\nu$ is modified Bessel function of third kind with index $\nu$ .

$$K_\nu(x) = \frac{1}{2}\int_0^\infty u^{\nu-1} \exp\left\{-\frac{1}{2}x\left(u+\frac{1}{u}\right)\right\} du \qquad (x>0)$$

$$K_{\mp\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}.e^{-x} \qquad\qquad K_\nu(x) = K_{-\nu}(x)$$

# Normal Inverse Gaussian Distribution

- The characteristic function of NIG distribution,

$$\varphi_{NIG}(u) = \exp\left\{\delta\sqrt{\alpha^2 - \beta^2} - \delta\sqrt{\alpha^2 - (\beta + iu)^2} + iu\mu\right\}$$

- The normal Inverse Gaussian distribution has semi-heavy tails,i.e.

$$f_{NIG}(x) \sim const\,|x|^{-3/2}\, e^{(-\alpha\,|x| + \beta x)} \qquad as \quad x \to \mp\infty$$

## Normal Inverse Gaussian Distribution

- The central moments of a random variable are

$$E[X_1] = \mu + \delta \frac{\beta}{\sqrt{\alpha^2 - \beta^2}}$$

$$Var[X_1] = \frac{\delta}{\sqrt{\alpha^2 - \beta^2}} + \frac{\delta \beta^2}{\left(\sqrt{\alpha^2 - \beta^2}\right)^3} = \delta \frac{\alpha^2}{\gamma^3}$$

## Normal Inverse Gaussian Distribution

$$Skew\left[X_1\right] = 3\frac{\beta}{\alpha\sqrt{\delta\sqrt{\alpha^2-\beta^2}}}$$

$$Kurtosis\left[X_1\right] = 3 + 3\left[1 + 4\left(\frac{\beta}{\alpha}\right)^2\right]\frac{1}{\delta\sqrt{\alpha^2-\beta^2}}$$

# Normal Inverse Gaussian Distribution

- The Lévy measure of NIG has following form,

$$\nu_{NIG}(dx) = \frac{\alpha}{\pi} \cdot \frac{\delta}{|x|} \exp\{\beta x\} K_1(\alpha |x|) dx$$

- The probability density function of NIG distribution,

$$f_{NIG}(x) \sim \sqrt{\frac{\alpha}{2\pi}} \exp\left\{\delta \sqrt{\alpha^2 - \beta^2}\right\} \frac{\delta}{|x|^{3/2}} e^{-(\alpha+\beta)|x|} \qquad as \quad x \to \pm\infty$$

# 5.1 Parameter Estimation of NIG Distribution

- ***Moment Estimators***
- These estimates can be used as parameter estimates which obtained in real data. Let $m_i \ , i=1,2,3,4$ be sample moment of real data.

$$\hat{\rho} = \frac{3}{m_2 \sqrt{3 m_4 - 5 m_3^2}}$$

$$\hat{\mu} = m_1 - \frac{\hat{\beta}\,\hat{\delta}}{\hat{\rho}} \qquad \hat{\beta} = \frac{1}{3}\left(m_3 \ m_2 \ \hat{\rho}^{\,2}\right)$$

# Parameter Estimation of NIG Distribution

$$\hat{\delta} = \frac{m_2^2 \ \hat{\rho}^3}{\hat{\beta}^2 + \hat{\rho}^2}$$

$$\hat{\alpha} = \sqrt{\hat{\rho}^2 + \hat{\beta}^2}$$

# Parameter Estimation of NIG Distribution

$$E(X) = \mu + \frac{\delta\left(\dfrac{\beta}{\alpha}\right)}{\sqrt{1-\left(\dfrac{\beta}{\alpha}\right)^2}}$$

$$Var(X) = \frac{\delta^2 / \alpha}{\sqrt{\left(1-\left(\dfrac{\beta}{\alpha}\right)^2\right)^3}}$$

$$Skew(X) = 3\alpha^{-1/4} \frac{\dfrac{\beta}{\alpha}}{\left[1-\left(\dfrac{\beta}{\alpha}\right)^2\right]^{1/4}}$$

$$Kurt(X) = 3\alpha^{-1/2} \frac{1+4\left(\dfrac{\beta}{\alpha}\right)^2}{\left[1-\left(\dfrac{\beta}{\alpha}\right)^2\right]^{1/2}}$$

# Parameter Estimation of NIG Distribution

$$\frac{\left(Skew\left(X\right)\right)^2}{Kurt\left(X\right)} = \frac{3\left(\dfrac{\beta}{\alpha}\right)^2}{1+4\left(\dfrac{\beta}{\alpha}\right)^2}$$

$$\left|Skew\left(X\right)\right| \leq \frac{3}{5} Kut\left(X\right)$$

# 5.2 The Simulation of Normal Inverse Gaussian Distributions

- The algorithm can be state as following [9], [29], [10] .
  - Generate from Y from N(0,1)
  - Generate from Z from $IG\left(\delta^2, \alpha^2 - \beta^2\right)$
  - Set $X = \mu + \beta Z + \sqrt{Z}\, Y$
- *The simulation of Inverse Gaussian distribution*
  - Sample Y from N(0,1) and set $V = Y^2$
  - Set $K = \eta + \dfrac{\eta^2 V}{2\delta^2} - \dfrac{1}{2\delta^2}\sqrt{4\eta\delta^2 V + \eta^2 V^2}$

# Parameter Estimation of NIG Distribution

- with $$\eta = \frac{\delta}{\sqrt{\alpha^2 - \beta^2}}$$

- Set $$Z = K.1_{\left\{U_1 < \frac{\eta}{\eta + K}\right\}} + \frac{\eta^2}{K}.1_{\left\{U_1 \geq \frac{\eta}{\eta + K}\right\}}$$

# Parameter Estimation of NIG Distribution

- *The simulation of V*
  - Generate $U_1$ *and* $U_2$ uniform random variables

$$N_1 = \left[-2\ln U_1\right]^{1/2} \cos\left(2\pi U_2\right)$$

$$N_2 = \left[-2\ln U_1\right]^{1/2} \sin\left(2\pi U_2\right)$$

- Both of $N_1$ *and* $N_2$ have standard normal distribution are independent.

# 6. THE MODELLING OF RETURNS

- We can consider a general exponential Levy process model for stock prices,

$$S_t = S_0 \exp\left(X_t\right)$$

- with a Levy process $\left(X_t : t \geq 0\right)$

- The distribution of a Levy process is uniquely determined by any of its one dimensional marginal distribution
- So we may use the distribution of $X_1$
- The distribution of $X_1$ is infinitely divisible and its characteristic function is given by Levy-Khintchine formula.

# 6. THE MODELLING OF RETURNS

- An infinitely divisible distribution can be estimated from the real data which available for the particular asset.
- The log returns of the model have independent and stationary increments along time intervals of length 1.
- The path properties of the levy process X carry over to S.
- If X is a pure jump levy process, then S is alsa a pure jump process.

# 6. THE MODELLING OF RETURNS

- We consider log price changes,

$$X_t = \ln S_{t+\Delta} - \ln S_t$$

- $X_t$ reflects the multiplicative character of price changes.

- When we use daily price data, $\Delta$ typically will have the value 1.

- In generally, the properties of return distributions depends on the length of the return interval $\Delta$.

# The Modelling of Returns

- For long $\Delta$ as a result of central limit theorem, the returns can be described with a Gaussian distribution
- but for tick data, the normal distribution can not be suitable
- In this case, as a model Generalized hyperbolic distributions and it's subset can be used
- These distributions are infinitely divisible and self-decomposable
- Levy markets give good fit to return distributions.

# The Modelling of Returns

- A non-parametric threshold technique is proposed to test the integrated variance which based on discrete observed prices by [13].

- We can determine whether jump type process is suitable or not as a model for price process, using with this technique.

## The Modelling of Returns

- $\left(X_t\right)_{t\geq o}$ is the Lévy process generated by the normal inverse Gaussian distribution that was fitted to the real data.
- The increments of price process a long time interval 1,
- $X_{t+1} - X_t$ are distributed according to the NIG distribution.
- The security price $S_t$ denotes the present price of security
- The risk one day a head is calculated as follow,

## The Modelling of Returns

$$Var(S_{t+1} - S_t) = S_t^2 \, Var\left[\exp\left(X_{t+1} - X_t\right)\right]$$

$$= S_t^2 \, Var\left[\exp\left(X_1\right)\right]$$

- The distribution of $X_1$ is the fitted NIG

# The Modelling of Returns

- Levy processes are semimartingales. As a consequence, the following stochastic integrals,

$$\int_0^t f(s)dX \qquad and \qquad \int_o^t A(s)\,dX_s$$

- are well definite for $L^2$

# The Modelling of Returns

- In classic Black-Scholes model, price process is solution of below stochastic differential equation

$$dS_t = S_t \mu \, dt + S_t \sigma \, dW_t \quad , S_0 > 0$$

- The solution,

$$S_t = S_0 \exp\left\{ \left( \mu - \frac{1}{2}\sigma^2 \right) dt + \sigma W_t \right\}$$

# 6.1 Test Statistics

- We want to make inference on to unobserved volatility process from the observed price process,
- Blumenthal-Getoor, index is described as follows,

$$\alpha = \inf \left\{ p \geq 0 \, , \int_{\{|x| \leq 1\}} x^p \ \nu(dx) < \infty \right\} \qquad\qquad \alpha \in [0,2]$$

- This index measures activities of small jumps of Lévy process .
- For jump type Lévy process, $\alpha = 1$
- For example, Normal inverse Gaussian motion has infinitely variation and

# Test Statistics

- An infinite activity process with Blumenthal-Getoor, index
- $\alpha < 1$ has paths with finite variation while $\alpha > 1$ then the sample paths have infinite variation
- An estimator for the integrated variance given by

$$\int_0^t \sigma_s^2 \, ds$$

- This estimator is consistent in the presence of jumps and characterize its asymptotic behavior via a central limit therem.
- V denotes threshold estimator for integrated variance
- Under the very general assumptions, realised m-power variation converge to quadratic variation.

# The estimation of quadratic variation(QV)

- Let $\delta$ denotes a time periods between observations
- QV process is defined as following,

$$\left(X_{\delta}\right)_t = \sum_{i=1}^{[t/\delta]} \left(X_{\delta i} - X_{\delta(i-1)}\right)^2$$

$$\lim_{\delta \downarrow 0} \left(X_{\delta}\right)_t \xrightarrow{p} \int_0^t \sigma_s^2 \, ds$$

- We used daily returns to measure increments of the quadratic variation

# 6.1.1 Test for the Presence of a Continuous Martingale Component of a Levy Model

- Test procedure as follows;

- Choice a coefficient $\beta \in [0.5,1]$
- We choose a threshold $r(k) = k^{\beta}$
- Set $\Delta_i Y = \Delta_i X + \sigma \sqrt{k}\, Z_i$
- Where $Z_i \sim N(0,1)$ as $n = T/k$ , $k \to 0$
- T is measured by as an annually, $\sigma$ is known standard deviation , $\Delta_i X$ denotes log-returns.

# Test for the Presence of a Continuous Martingale Component of a Lévy Model

- *Null Hypothesis:* $H_0 : \sigma \equiv 0$

- *Test statistics* is calculated as following

$$\hat{V}_k = \sum_{i=1}^{n} (\Delta_i Y)^2 1_{\{(\Delta_i Y)^2 \le r(k)\}} \qquad \hat{Q}_k = \frac{1}{3k} \sum_{i=1}^{n} (\Delta_i Y)^4 1_{\{(\Delta_i Y)^2 \le r(k)\}}$$

$$T_k = \frac{\hat{V}_k - \sigma^2 n.k}{\sqrt{2k\,\hat{Q}_k}} \to N(0,1)$$

# Test for the Presence of a Continuous Martingale Component of a Lévy Model

- If $P\left\{\left|T_k^{\alpha}\right| > 1.96\right\}$ is near 0.05, we accept null hypothesis.

- Under the alternative hypothesis $H_1 : \sigma \neq 0$ , above test statistic diverges to +∞ .

- If $P\left\{\left|T_k^{\alpha}\right| > 1.96\right\} >> 0.05$ , we reject the null hypothesis

# 6.1.2 The test whether the jump component has finite variation

- *Null Hypothesis* : $H_0 : \alpha < 1$

$$\Delta_i \hat{M} = \Delta_i X 1_{\{(\Delta_i X)^2 > r(k)\}} + \sigma\sqrt{k}\, Z_i \qquad , Z_i \sim N(0,1)$$

$$\hat{V}_k = \sum_{i=1}^{n} \left(\Delta_i \hat{M}\right)^2 1_{\{(\Delta_i M)^2 \le r(k)\}} \qquad \hat{Q}_k = \frac{1}{3k} \sum_{i=1}^{n} \left(\Delta_i \hat{M}\right)^4 1_{\{(\Delta_i M)^2 \le r(k)\}}$$

# The test whether the jump component has finite variation

$$T_k^\alpha = \frac{\hat{V}_k - \alpha^2 n.k}{\sqrt{2k\hat{Q}_k}} \rightarrow N(0,1)$$

- If $P\left\{ \left| T_k^\alpha \right| > 1.96 \right\}$ is near 0.05, we accept null.

## 6.1.3 The test of Presence Jumps

- The variance of a Lévy processes is estimated as following form,

$$S_X = \sum_{i=0}^{n-1} \left( X_{t_{i+1}} - X_{t_i} \right)^2$$

- Where each $t_i$ is a division of interval $[0, t]$ for each $n \in \mathrm{N}$ . $\left| t_i - t_{t_{i-1}} \right| = \frac{t}{n} = \Delta t_i$ and $t = n.\Delta_{t_i}$

## The Test of Presence Jumps

$$S_X^2 = \sum_{i=0}^{n-1} \left( X_{t_{i+1}} - X_{t_i} \right)\left( X_{t_i} - X_{t_{i-1}} \right)$$

$$S_X^4 = \left[ \frac{n}{t} \right] \sum_{i=4}^{n} \left| \Delta_n X_{t_{i-1}} \right| \ldots \left| \Delta_n X_{t_{i-4}} \right|$$

# The Test of Presence Jumps

- Null hypothesis: $H_0 : X_t$ is a continuous Lévy process

$$Z_n = \frac{\left( S_X - \frac{S_X^2}{K^2} \right)}{\sqrt{\frac{t}{n}} \sqrt{\frac{S_X^4}{K^4}}} \qquad K = (2/\pi)^{0.5}$$

- If $Z_n > z_\alpha$ , we reject the null hypothesis
- where $z_\alpha$ is a chosen quantile.

# 7. Application to Real data

- We consider the time series of NIKKEI 225 in Japan returns from 04.01.1982 to 30.09.2005 and ISE Compound 100 index in Turkey returns from 06.02.2002 to 15.02.2007 .

- We use daily data and $k = 1/252 = 0{,}003968$ and $r(k) = k^{0{,}75}$

- We divided return series such that each group have 500 daily return for NIKKEI 225 index
- Similarly, ISE Compound 100 index returns divided sub groups which include 300 daily return.

# Application to Real data

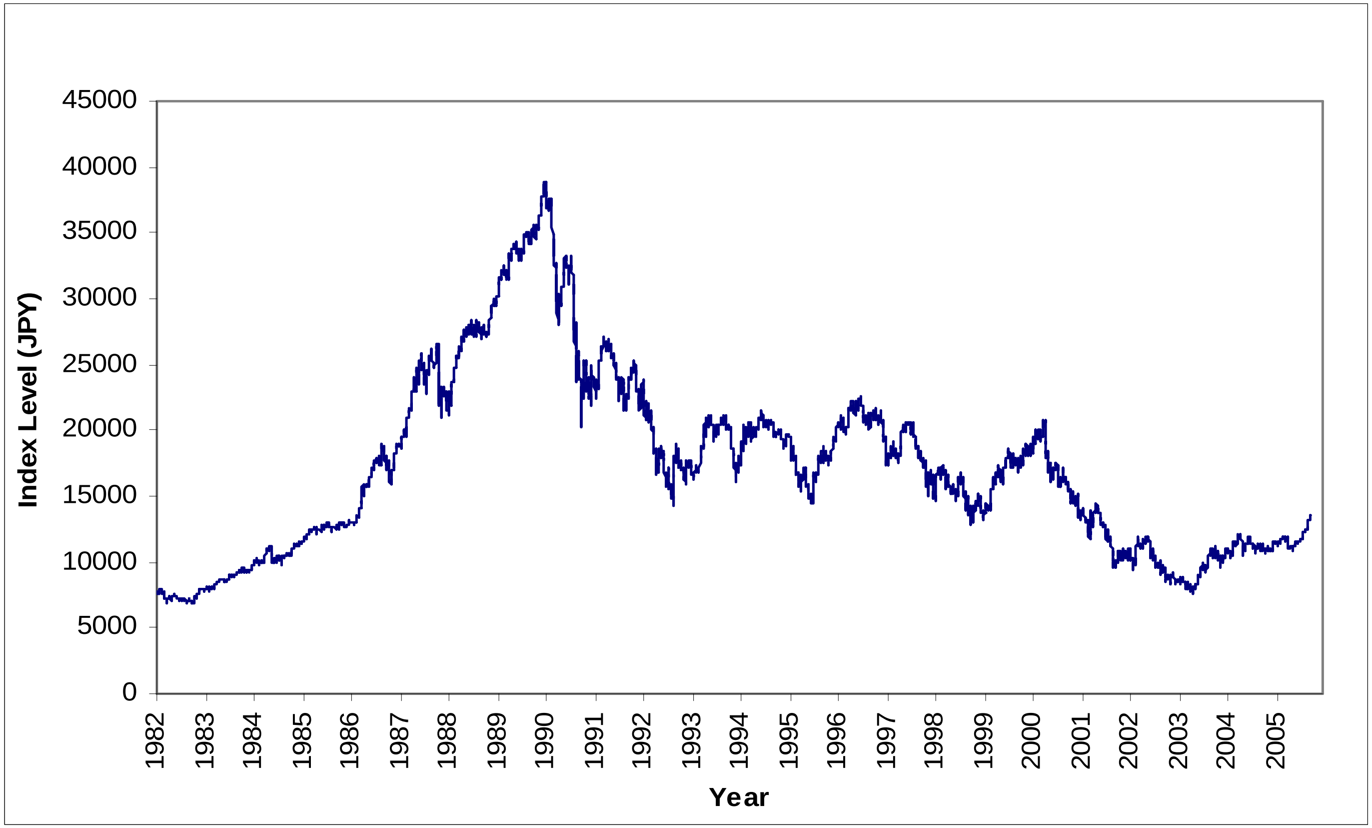



**FIGURE 1.** History of NIKKEI 225 Index in Japan

# Application to Real data

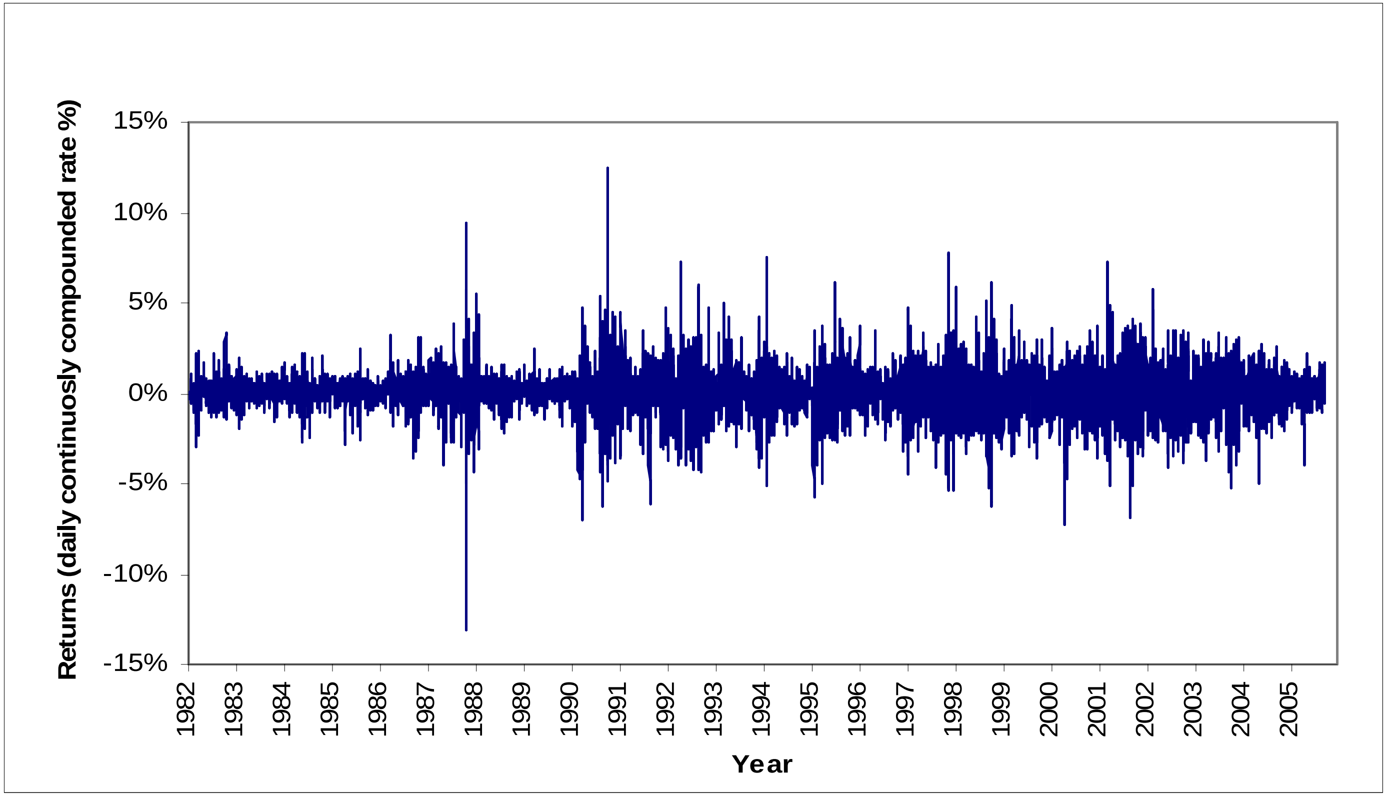



**FIGURE 2.** History of NIKKEI 225 Index returns in Japan

# Application to Real data

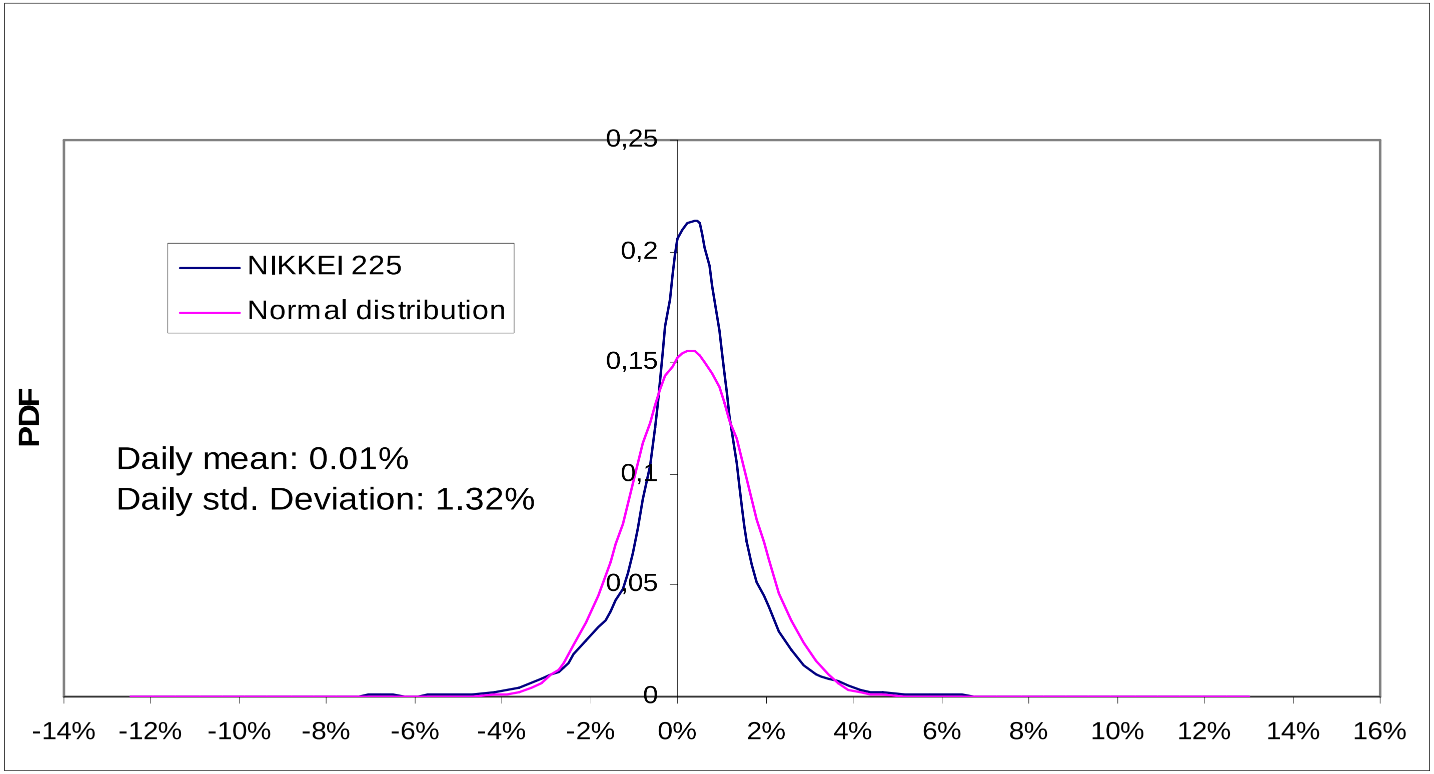



**FIGURE 3.** PDF of Daily Returns of Nikkei 225 from 1982/01/04 to 2005/09/30

# Application to Real data

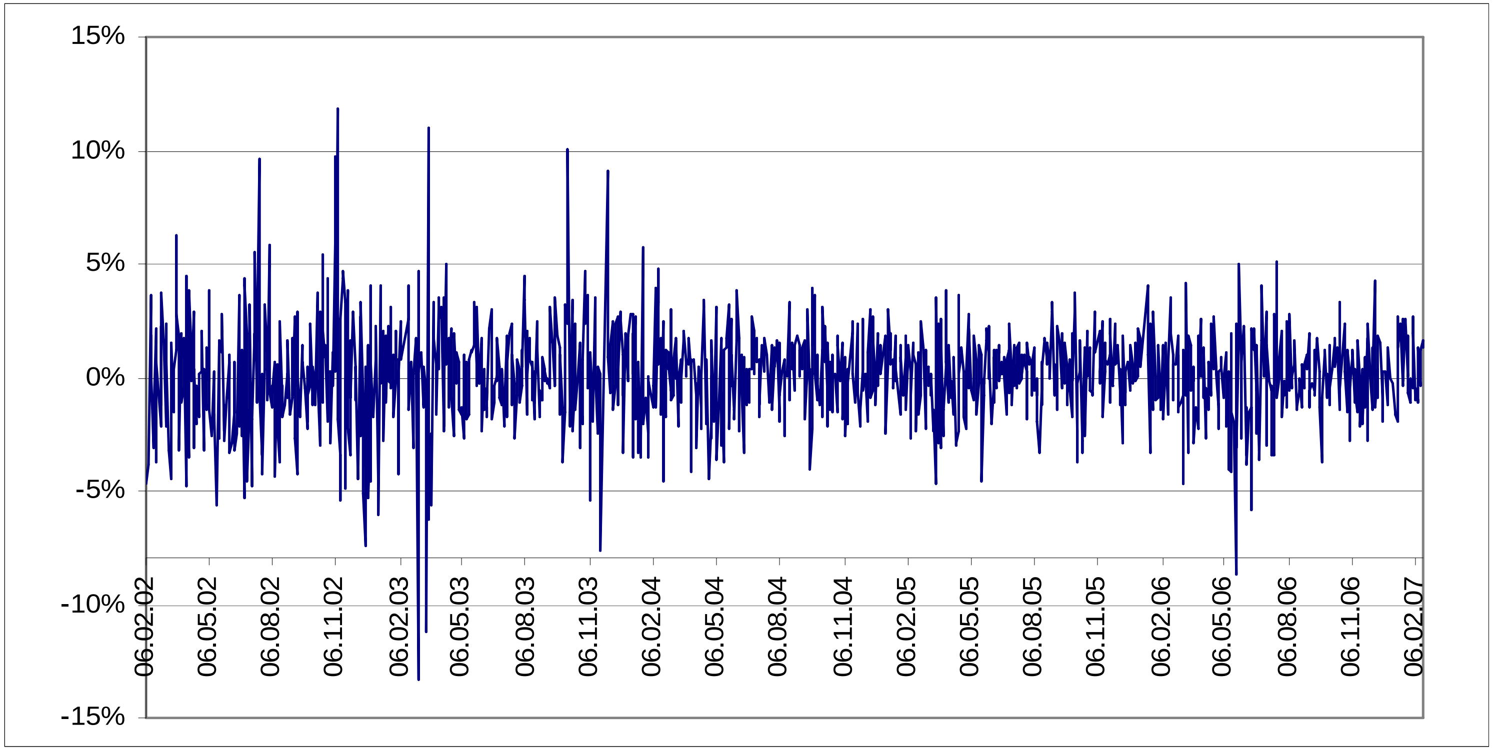



**FIGURE 4.** History of ISE Composite 100 Index return in Turkey

# Application to Real data

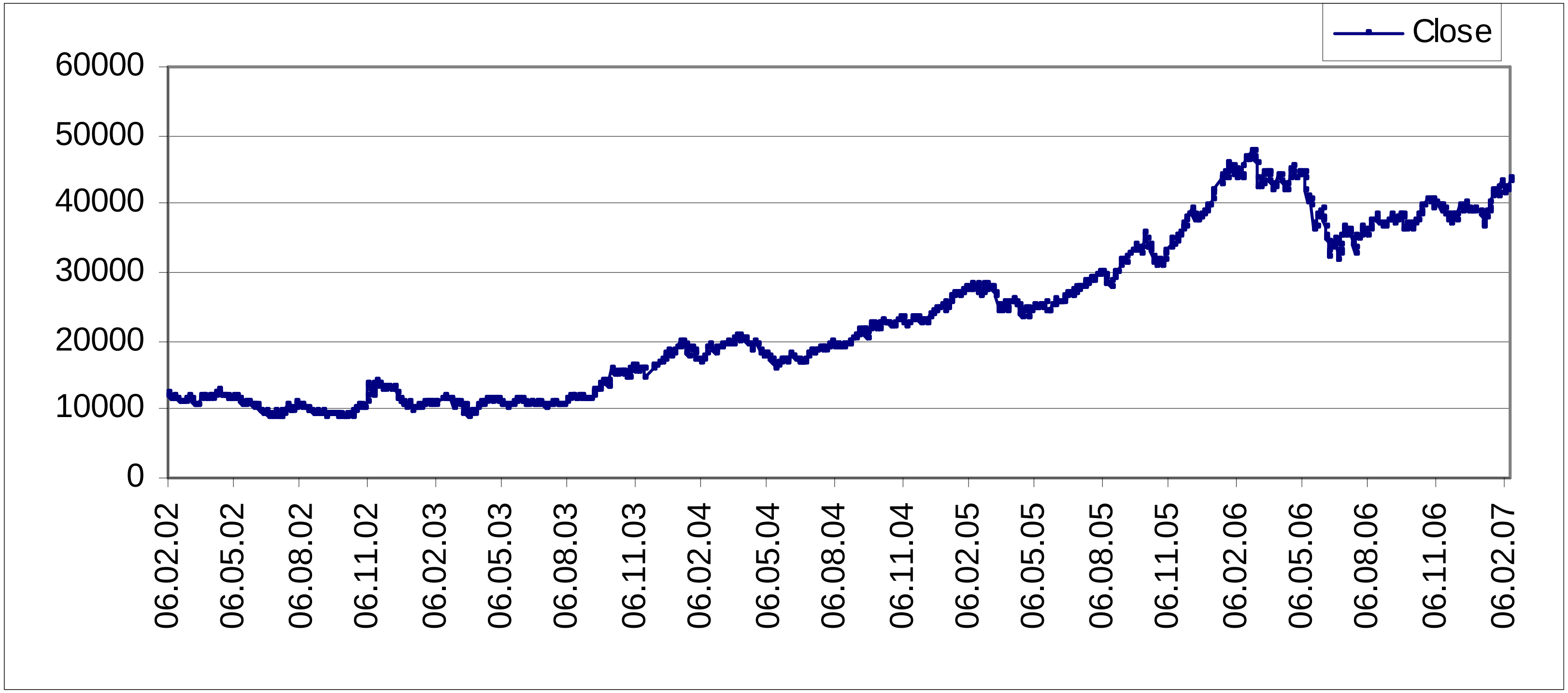



**FIGURE 5.** History of ISE Composite 100 Index in Turkey

# Application to Real data

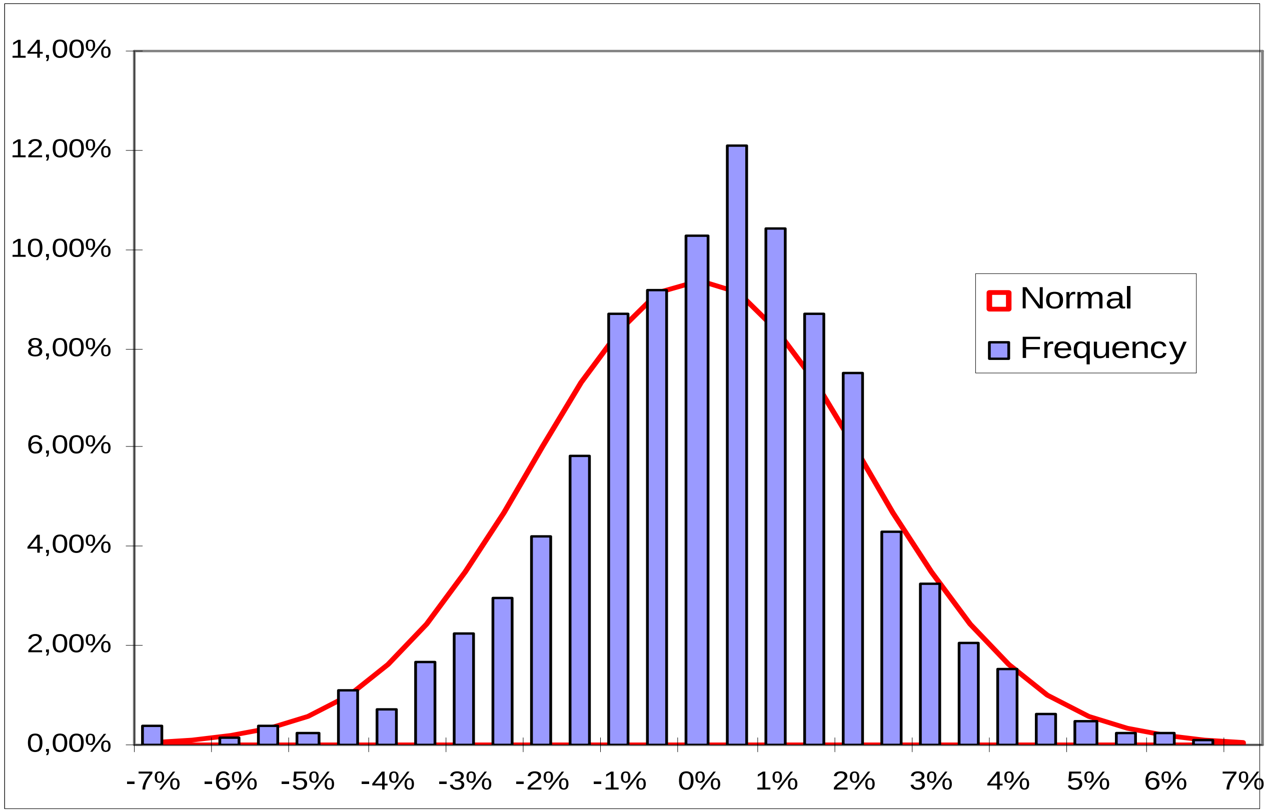



**FIGURE 6.** Frequency of ISE Composite 1000 Index in Turkey

# Application to Real data

| ISE | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\hat{V}_k$ | 0,029252 | 0,190011 | 0,130958 | 0,166943 | 0,027818 |
| $\hat{Q}_k$ | 0,586629 | 0,12095 | 0,015813 | 0,040404 | 0,003087 |
| $T_{k(j)}$ | 0,413469 | 6,116505 | 11,56244 | 9,301056 | 5,570338 |

**Table 1.** The testing for the presence of a Brownian component for ISE Composite 100 index price process

# Application to Real data

| ISE | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\hat{V}_k$ | 0,280954 | 0,067261 | 0,001438 | 0,023769 |
| $\hat{Q}_k$ | 0,525418 | 0,093732 | 1,89E-06 | 0,018 |
| $T_{k(j)}$ | 4,334696 | 2,447545 | 0,073143 | 1,956218 |

**Table 2.** The testing finite variation of jump component of ISE Composite 100 index price process

# Application to Real data

| NIKKEI 225 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\hat{V}_k$ | 0,08143 | 0,223476 | 0,559763 | 0,378513 | 0,525433 | 0,27848 |
| $\hat{Q}_k$ | 0,002493 | 0,207762 | 0,183874 | 0,064663 | 0,1103662 | 0,025918 |
| $T_{k(j)}$ | 18,26515 | 5,491945 | 14,62282 | 16,67741 | 17,72089 | 19,37838 |

**Table 3.** The testing for the presence of a Brownian component for NIKKEI 225 Index returns

# Application to Real data

| NIKKEI 225 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\hat{V}_k$ | 0,000185 | 0,059225 | 0,098538 | 0,0498 | 0,067396 | 0,000549 |
| $\hat{Q}_k$ | 9,09E-09 | 0,187877 | 0,119161 | 0,030012 | 0,0472258 | 9,39E-08 |
| $T_{k(j)}$ | 0,336161 | 1,521648 | 3,166666 | 3,181129 | 3,427891 | -0,01413 |

**Table 4.** The testing finite variation of jump component of NIKKEI 225 Index returns

# Application to Real data

| INDEX | $Z_n$ |
|---|---|
| NIKKIE 225 | 179,0877 |
| ISE Composite 100 | 113,6736 |

**Table 5 .** The testing the presence of jumps under the hypothesis that price process is continuous Lévy process.

# Application to Real data

| Index | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\mu}$ | $\hat{\delta}$ |
|---|---|---|---|---|
| **NIKKEI 225** | 0,261649 | 0,000218 | 0,000096 | 0,0000456 |
| **ISE Composite 100** | 0,525443 | 0,007822 | 0,001017 | 0,000238 |

**Table 6 .** Estimated parameters of the normal inverse Gaussian distribution

# 8. CONCLUSION

- The laws of self-decomposable distributions class are be constituted as a limit laws of Lévy – driven Ornstein-Uhlenbeck process

- These laws can be related with general additive process.

- The Lévy processes has a flexible structure to model in real phenomena in finance such as heavy-tails, jumps and volatility smile

- In this study, we focus on Levy process and it’s increments laws.

- Self-decomposable laws are sub class of infinitely divisible laws

# CONCLUSION

- The family of self-decomposable laws always has both of self-similarity and stationary of increments.
- We tested whether the pure jump Levy process is a suitable model or not for financial return series
- For this test, we used a threshold estimator of quadratic variation by proposed by Cont and Mancini
- We applied this test statistics to two international index return series.

# CONCLUSION

- Our empirical analysis show that,
- Nikkei 225 index model must include Brownian motion component
- The jumps of Nikkei 225 return index have infinite variation
- ISE 100 return index model must include Brownian motion component
- The jumps of ISE 100 return index have infinite variation

# CONCLUSION

- A self-decomposable random variable has the same distribution of as a scaled version of itself and an independent residual random variable.
- Self-decomposable processes are Levy processes with jump arrival rates that are decreasing in the jump size
- A self-decomposable random variable also has a distribution of class L which means it can motivated as a limit law with more general scaling than the Gaussian limit law.

# CONCLUSION

- This means that self-decomposable processes can be motivated as limit laws where the independent infleunces being summed are of different orders of magnitude
- 
- Thus they are appropriate building blocks for stochastic processes used to model financial markets.
- Finally, we can say that, a good fit model for real return data have to include both of a Brownian component and a infinite variation component.