Option Pricing by Transform Methods: Extensions, Unification, and Error Control

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Abstract

We extend and unify Fourier-analytic methods for pricing a wide class of options on any underlying state variable whose characteristic function is known. In this general setting, we bound the numerical pricing error of discretized transform computations, such as DFT/FFT. These bounds enable algorithms to select efficient quadrature parameters and to price with guaranteed numerical accuracy.

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1 Introduction

In a large and growing family of financial models, explicit formulas exist for the characteristic functions of the state variables. Given any such characteristic function, our project is to compute, efficiently and accurately, the prices of a wide class of options on those underlying state variables. Fourier-analytic solutions to various forms of this problem have appeared in the finance literature. They express option prices in terms of Fourier-inversion integrals, which are in practice evaluated numerically.

This paper extends and unifies those ideas. In a general setting, we bound the error in the numerical evaluation of these integrals as N-point sums, of the kind that may be computed as a discrete Fourier transform (DFT) by schemes including the fast Fourier transform (FFT). Then we show how these bounds lead to algorithms that make efficient choices of quadrature parameters and compute prices with guaranteed numerical accuracy.

1.1 Outline

This paper generalizes Carr-Madan (1999); unifies it with extensions of the relevant elements of Duffie-Pan-Singleton (2000), Lewis (2001), and Bakshi-Madan (2000); and develops error bounds and error minimization strategies. Carr-Madan's underlying random variable X is the logarithm of a terminal stock price, and their objective is to compute the call price, as a function of log-strike. In terms of the characteristic function of X, they calculate analytically the Fourier transform of the call price function, damped to enforce integrability. Inverting this Fourier transform by FFT and then undamping, they recover simultaneously the call prices at many strikes.

We begin by setting forth the option pricing problem and defining the options to be priced. Our scope includes not only vanilla calls on variables exponential in a single state variable, but also three other classes of payoffs. These extended payoff classes contain all of the derivative structures treated in Duffie-Pan-Singleton (2000), and in particular they allow payoffs dependent on multidimensional state variables.

Next, we derive upper bounds on option prices, intended for use at extreme strikes. These bounds will become relevant to discretization errors in transform-pricing of options at *all* strikes.

In Section 4 we extend, to all four payoff classes, Carr-Madan's analytic calculation of Fourier transforms, as well as their inversion formula recovering the option price. Also, by taking as given the Bakshi-Madan (2000) *discounted* characteristic function, we extend Carr-Madan to allow stochastic interest rates.

As Lewis (2001) observes, transform representations of option prices may be interpreted as contour integrals in the complex plane; shifts of the contours generate alternative pricing formulas. Applying this idea, we prove a unified pricing formula encompassing not just our original four but also ten complementary formulas, including as special cases some well-known transform formulas.

These formulas involve integrals over (a translate of) the real line, so approximation by an N-point sum is subject to two forms of error: sampling error because the integrand is evaluated numerically only at the grid points, and truncation error because the upper limit of numeric summation is finite. We then establish bounds for both kinds of error, in all four payoff classes.

Section 7 addresses strategic issues in error bound minimization. From an error-management perspective, we apply our bounds analysis to argue in favor of the Carr-Madan one-integral approach to call pricing, and against the traditional two-integral approach. Then we make recommendations for choosing among our five one-integral call formulas. For choosing quadrature parameters, we offer a simple algorithm as a robust alternative to the specific constant parameters suggested in Carr-Madan.

The first appendix facilitates truncation error calculations by providing bounds on the decay of characteristic functions in two prominent models. The second appendix gives sampling error bounds, for subcases deferred from the main text. The third appendix deals with specific DFT/FFT implementation issues.

1.2 Guiding Principles

Wherever possible, we observe the following principles.

First, we take as primitive the discounted characteristic function. From there, our analysis proceeds to the computation of option prices. We do not derive any characteristic functions; other papers have already taken the responsibility of finding characteristic functions given, for example, SDE or generating triplet specifications of the underlying financial dynamics; and indeed others take the characteristic function *as* the specification of the underlying dynamics. Duplication of research effort will be reduced, one hopes, by the emerging division of labor between, on one hand, those projects that specify or derive characteristic functions; and, on the other hand, projects such as this one, which derive option pricing formulas, *given* arbitrary characteristic functions.

Second, we strive to maintain generality. We do not assume that the underlying state variable is, say, a jump-diffusion or Lévy process. We do not assume that its probability distribution has a density. Time and the state space may be continuous or discrete. The state variables may be one-dimensional or multi-dimensional. Interest rates and dividends may be deterministic or stochastic. As long as the discounted characteristic function for such dynamics is known, option prices are computable. Technical restrictions do apply, which brings us to the next point.

Third, we formulate our technical conditions with the view that they should facilitate the design of provably robust pricing algorithms. So we place a premium on expressing assumptions in a complete, concise, rigorous, and readily testable way.

2 The Option Pricing Problem

Working in a filtered probability space $(\Omega, P^*, \{\mathscr{F}_t\})$, we intend to calculate numerically the time-0 price C_0 of an option paying at time T the $\mathscr{F}_T$-measurable random variable C_T.

Let r_t be the interest rate process, possibly stochastic.

Let $M_t := \exp(\int_0^t r_s ds)$ be the time-t value of a money market account.

Let B_t be the time-t value of a discount bond maturing at T.

2.1 Numeraires and Martingale Measures

Assuming that the prices (of C, M, B, and any other assets under consideration) admit no arbitrage, there must exist a risk-neutral probability measure P under which asset prices, discounted by M, are martingales. See Harrison and Kreps (1979) or Delbaen and Schachermayer (1994) for technical definitions of "admit no arbitrage" that make this statement true. Let E denote expectation with respect to P. Then the option price and bond price satisfy

$$C_0 = E[M_T^{-1} C_T]$$
$$B_0 = E[M_T^{-1}].$$

The positive price process M is an example of a *numeraire*. For *any* numeraire N there exists a probability measure P_N, said to be risk neutral with respect to N, meaning that the N_t-discounted price of any asset is a P_N-martingale; see El Karoui, Geman, and Rochet (1995). The change of measure from P to P_N is given by

$$\left.\frac{dP_N}{dP}\right|_{\mathscr{F}_T} = \frac{N_T/N_0}{M_T/M_0}.$$

When the numeraire is chosen to be the price B_t of a T-maturity discount bond, the risk-neutral measure P_B is known as the T-forward measure. Let us write $\mathbb{E}$ for expectation with respect to P_B. The option price satisfies, therefore,

$$C_0 = B_0 \mathbb{E} C_T.$$

In the case of deterministic interest rates, the forward measure is identical to the usual risk-neutral measure. In our setting, however, interest rates may be stochastic, and the measures are not necessarily identical; the forward measure has the advantage of discounting outside the expectation.

2.2 Options

Let the *state variable* X be an $\mathscr{F}_T$-measurable random variable with values in $\mathbb{R}^n$. For a *payoff* function $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, define

$$C_G(k) := B_0\mathbb{E}(G(X,k)),$$

which is the time-0 *price* of an option on X, paying $G(X,k)$ at time T. The *trigger* k is some contract variable, such as a strike, or the logarithm of a strike.

Our goal is accurate numerical computation of $C_G(k)$ for these cases of G:

$$\begin{aligned}
G_1(x,k) &:= (\exp(x) - \exp(k))^+ && b_0 := 1, b_1 := 1, x \in \mathbb{R} \\
G_2(x,k) &:= (x-k)^+ && b_0 := 1, b_1 := 0, x \in \mathbb{R} \\
G_3(x,k) &:= \exp(b_1 \cdot x)\mathbb{I}(b_0 \cdot x > k) && x \in \mathbb{R}^n \\
G_4(x,k) &:= (b_2 \cdot x)\exp(b_1 \cdot x)\mathbb{I}(b_0 \cdot x > k) && x \in \mathbb{R}^n
\end{aligned}$$

where $\mathbb{I}$ is the indicator function, so $\mathbb{I}(b_0 \cdot x > k)$ equals 1 if $b_0 \cdot x > k$, but 0 otherwise. In payoffs G_3 and G_4, the $b_0, b_1, b_2 \in \mathbb{R}^n$ are arbitrary constants. When it is clear what payoff(s) is/are under discussion, we may suppress the subscript of G or C.

We choose these four functional forms because they include a wide family of payoffs of practical interest. For example, with payoff G_1, if one chooses X to be bond yield, or the logarithm of a stock price or FX rate, then one obtains a call on a stock, bond, or currency. With payoff G_2, if one chooses X to be an interest rate, or a time-averaged interest rate, then one obtains respectively a European or an Asian option on an interest rate.

Our G_3 and G_4 are the payoff classes treated in Duffie-Pan-Singleton (2000). With payoff G_3, if one chooses b_0 and b_1 appropriately, then one can obtain asset-or-nothing, binary, equity-linked FX, and two-asset exchange/maximum options, all on the exponentials of components of X, which could be stock price logarithms or bond yields or FX-rate logarithms. With payoff G_4, if one chooses $b_1 = 0$ and b_0 and b_2 appropriately, then one can obtain basket or spread options on the components of X, which could be interest rates or their time-averages, for example.

3 Upper Bounds on Option Prices at Extreme Strikes

For practical use in bounding numerical transform-inversion errors, it is important that C_G be dominated by an expression that is easily evaluated in terms of the characteristic function of X.

For each $G = G_1, \dots, G_4$, we give two bounds; both bounds are valid for all k, but the first is intended for use with large positive k, whereas the second is intended for use with large negative k. The usual conventions about ∞ are in force, so each of Theorems 3.1-3.4 holds automatically if the expectation on the right-hand side is infinite.

The first of these four results is nearly identical to a bound obtained in Broadie-Cvitanic-Soner (1998). The differences, though minor, make it appropriate to present briefly a full proof.

Theorem 3.1. *For any $p > 0$,*

$$C_{G_1}(k) \leqslant \frac{B_0 \mathbb{E} \exp((p+1)X)}{(p+1)\exp(pk)} \left(\frac{p}{p+1}\right)^p \qquad \textit{and} \qquad C_{G_1}(k) \leqslant B_0 \mathbb{E} \exp(X).$$

Proof. For all $s \geqslant 0$ we have

$$s - e^k \leqslant \frac{s^{p+1}}{(p+1)\exp(pk)} \left(\frac{p}{p+1}\right)^p$$

because the left-hand and right-hand sides, as functions of s, have equal values and first derivatives at $s = (p+1)\exp(k)/p$, but the right-hand side has everywhere a positive second derivative. Moreover, since the right-hand side is positive, the left-hand side can be improved to $(s - \exp(k))^+$.

Now substitute $s = \exp(X)$, take expectations, and multiply by B_0 to obtain the first bound. The second bound is obvious. $\square$

Remark 3.1. Therefore, if S_T is a nonnegative random variable with $\mathbb{E} S_T^{p+1} < \infty$ for some $p > 0$, then calls on S_T must have prices that decay as $O(K^{-p})$ for strikes $K \to \infty$.

A corresponding fact for puts follows from Theorem 6.4: if $\mathbb{E} S_T^{-q} < \infty$ for some $q > 0$, then puts on on S_T must have prices that decay as $O(K^{q+1})$ for strikes $K \to 0$.

Lee (2003) uses these bounds to derive an explicit "moment formula" for the growth of implied volatility at extreme strikes.

Theorem 3.2. *For any $p > 0$,*

$$C_{G_2}(k) \leqslant \frac{B_0 \mathbb{E} \exp(pX)}{p \exp(pk+1)}.$$

For any $q > 0$,

$$C_{G_2}(k) \leqslant B_0(\mathbb{E} X - k) + \frac{B_0 \mathbb{E} \exp(-qX)}{q \exp(1 - qk)}.$$

Proof. For all $x \in \mathbb{R}$ we have

$$x - k \leqslant \frac{\exp(px)}{p \exp(pk+1)}$$

because the left-hand and right-hand sides, as functions of x, have equal values and first derivatives at $x = k + 1/p$, but the right-hand side has everywhere a positive second derivative. Substitute X for x, take expectations, and multiply by B_0 to obtain the first bound.

A similar argument shows that for all x,

$$(x-k)^+ = x-k+(k-x)^+ \leqslant x-k+\frac{\exp(-qx)}{q\exp(1-qk)},$$

which implies the second bound. □

Theorem 3.3. *For any $p > 0$,*

$$C_{G_3}(k) \leqslant \frac{B_0 \mathbb{E}\exp((pb_0+b_1)\cdot X)}{\exp(pk)} \qquad \textit{and} \qquad C_{G_3}(k) \leqslant B_0 \mathbb{E}\exp(b_1 \cdot X).$$

Proof. For all $x \in \mathbb{R}^n$ we have

$$\mathbb{I}(b_0 \cdot x > k) \leqslant \frac{\exp(pb_0 \cdot x)}{\exp(pk)},$$

which implies the first bound. The second bound is obvious. □

Theorem 3.4. *For any $p_0 > 0$ and $p_2 > 0$,*

$$C_{G_4}(k) \leqslant \frac{B_0 \mathbb{E}\exp((p_0 b_0 + p_2 b_2 + b_1)\cdot X)}{p_2 \exp(p_0 k + 1)}.$$

For any $q_0 > 0$ and $q_2 > 0$,

$$C_{G_4}(k) \leqslant B_0 \mathbb{E}((b_2 \cdot X)\exp(b_1 \cdot X)) + \frac{B_0 \mathbb{E}\exp((-q_0 b_0 - q_2 b_2 + b_1)\cdot X)}{q_2 \exp(-q_0 k + 1)}.$$

Proof. For all $x \in \mathbb{R}^n$ we have

$$(b_2 \cdot x)\mathbb{I}(b_0 \cdot x > k) \leqslant \frac{\exp(p_2 b_2 \cdot x)}{p_2 e} \frac{\exp(p_0 b_0 \cdot x)}{\exp(p_0 k)}$$

and

$$(b_2 \cdot x)\mathbb{I}(b_0 \cdot x > k) = (b_2 \cdot x)(1 - \mathbb{I}(b_0 \cdot x \leqslant k)) \leqslant b_2 \cdot x + \frac{\exp(-q_2 b_2 \cdot x)}{q_2 e} \frac{\exp(-q_0 b_0 \cdot x)}{\exp(-q_0 k)},$$

implying the two bounds. □

Remark 3.2. To bound $|C_{G_4}|$, apply Theorem 3.4 to (b_0, b_1, b_2) and $(b_0, b_1, -b_2)$, and take the *larger* of the two bounds. To bound $C_{|G_4|}$, take the *sum* of those two bounds, because $|b_2 \cdot X|$ is the sum of $(b_2 \cdot X)^+$ and $(-b_2 \cdot X)^+$.

4 From Characteristic Functions to Option Prices

Our starting point is the discounted characteristic function f of the state variable X. Unlike the usual characteristic functions of probability theory, the definition of f includes a discount factor inside the expectation, which is essential for pricing under stochastic interest rates.

We produce formulas for prices of each of the four option classes, by expressing option price transforms in terms of f, and then inverting the transforms.

4.1 The Discounted Characteristic Function

Let X be an $\mathbb{R}^n$-valued random variable. Let A_X denote the interior of the set

$$\left\{ v \in \mathbb{R}^n : \mathbb{E}e^{v \cdot X} < \infty \right\}.$$

The complex vectors whose negated imaginary parts are in A_X form a "strip" or "tube"

$$\Lambda_X := \{\zeta \in \mathbb{C}^n : -\mathrm{Im}(\zeta) \in A_X\}.$$

Adopting the terminology suggested in Bakshi-Madan (2000), define the *discounted characteristic function* of X, with respect to a discount factor $\exp(-\int_0^T r_t dt)$, to be the function $f : \Lambda_X \to \mathbb{C}$ where

$$f(\zeta) := E\left(e^{-\int_0^T r_t dt} e^{i\zeta \cdot X}\right).$$

Note that the expectation is with respect to P, but f is also related to the forward measure P_B, because

$$f(\zeta)/f(0) = \mathbb{E}e^{i\zeta \cdot X},$$

which is (for ζ restricted to $\mathbb{R}^n$) the usual characteristic function of X with respect to P_B.

Theorem 4.1. *The discounted characteristic function f is well-defined and analytic in Λ_X, which is a convex set. Partial derivatives of f may be taken through the expectation.*

Proof. This follows from Zemanian (1966), Theorems 4 and 5. □

In certain models, one can derive the discounted characteristic function from an SDE specification of state variable dynamics. For example, affine jump-diffusion specifications give rise to tractable characteristic functions, as shown in Heston (1993), Bates (1996, 2000), Bakshi-Cao-Chen (1997), Bakshi-Madan (2000), Duffie-Pan-Singleton (2000), and Chacko-Das (2002). Outside of that family, Lewis (2000), Schöbel-Zhu (1999), and Zhu (2000) obtain characteristic functions also for non-affine volatility and interest rate models.

In other models, the state variables follow Lévy processes, and one can derive the characteristic function from a specification of the generating triplet, or directly take the characteristic function to define the dynamics. Examples include the Finite Moment Log Stable model in Carr-Wu (2003), the Normal Inverse Gaussian model in Barndorff-Nielsen (1998), the Generalized Hyperbolic model in Eberlein-Prause (2002), the Variance Gamma model in Madan-Carr-Chang (1998), and the CGMY and KoBoL models in Carr-Geman-Madan-Yor (2002) and Boyarchenko-Levendorskiĭ (2002). Extensions of Lévy process models which introduce stochastic time changes also have, in certain cases, explicit solutions for characteristic functions; see Barndorff-Nielsen/Nicolato/Shephard (2002), Carr-Wu (2002), and Carr-Geman-Madan-Yor (2003).

Appendix A gives details of the characteristic functions in two models – one in the affine class, and one in the Lévy class.

Note that if discounted characteristic functions are available not just for state variables but also for path functionals of the state variables, then our pricing and error control results will apply not just to European options, but also to path-dependent options. For example, in affine models, the availability of characteristic functions for time-averages enables us to price Asian options (on the state variables, not on their exponentials). Such availability is, however, the exception rather than the rule. Transform-based pricing of exotic options is feasible even *without* a readily computable characteristic function for the path-dependent quantity, provided that the dynamics are simple enough (under geometric Brownian motion, for example, see Fu-Madan-Wang (1999) or Carr-Schröder (2003) for Asian options, and Geman-Yor (1996) or Pelsser (2000) for barrier options); but this falls outside the scope of our pricing and error control results, which assume the availability of the characteristic function.

4.2 Fourier Transform of the Damped Option Price

The usual Fourier transform of C_G itself does not exist, because $C_G(k)$ does not decay as $k \to -\infty$.

Following Carr-Madan, then, for each *damping constant* $\alpha > 0$, we define the *damped* option price function $c_{\alpha,G} : \mathbb{R} \to \mathbb{R}$ by

$$c_{\alpha,G}(k) := \exp(\alpha k) C_G(k).$$

We will show that the damped option price $c_{\alpha,G}$ *does* have a Fourier transform $\hat{c}_{\alpha,G} : \mathbb{R} \to \mathbb{C}$, well-defined by

$$\hat{c}_{\alpha,G}(u) := \int_{-\infty}^{\infty} e^{iuk} c_{\alpha,G}(k) dk,$$

provided that α is chosen appropriately.

Theorem 4.2. *Assume that G satisfies $b_1 \in A_X$. Then there exists $\alpha > 0$ with $\alpha b_0 + b_1 \in A_X$. For any such α the Fourier transform $\hat{c}_{\alpha,G}$ of $c_{\alpha,G}$ exists and*

$$\hat{c}_{\alpha,G_1}(u) = \frac{f(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u} \qquad \hat{c}_{\alpha,G_2}(u) = \frac{f(u-\alpha i)}{(\alpha+iu)^2}$$

$$\hat{c}_{\alpha,G_3}(u) = \frac{f(ub_0-(\alpha b_0+b_1)i)}{\alpha+iu} \qquad \hat{c}_{\alpha,G_4}(u) = \frac{-ib_2\cdot\nabla f(ub_0-(\alpha b_0+b_1)i)}{\alpha+iu}.$$

Proof. There exists $p > \alpha$ such that $pb_0 + b_1 \in A_X$. So Theorem 3.1, 3.2, 3.3, or 3.4 implies that $c(k)$ decays exponentially for $|k| \to \infty$. Also $c(k)$ is bounded. Therefore $c(k)$ is L^1 and has a Fourier transform; moreover, the use of Fubini in the following computation of $\hat{c}$ is justified:

$$\hat{c}_{\alpha,G}(u) := \int_{-\infty}^{\infty} e^{iuk} c_{\alpha,G}(k)dk = \int_{-\infty}^{\infty} e^{iuk} e^{\alpha k} B_0 \mathbb{E} G(X,k)dk = f(0)\mathbb{E}\int_{-\infty}^{\infty} G(X,k)e^{(\alpha+iu)k}dk.$$

Evaluating the integral,

$$\hat{c}_{\alpha,G_1}(u) = \frac{f(0)\mathbb{E}e^{(\alpha+1+iu)X}}{\alpha^2+\alpha-u^2+i(2\alpha+1)u} \qquad \hat{c}_{\alpha,G_2}(u) = \frac{f(0)\mathbb{E}e^{(\alpha+iu)X}}{(\alpha+iu)^2}$$

$$\hat{c}_{\alpha,G_3}(u) = \frac{f(0)\mathbb{E}(e^{b_1\cdot X}e^{(\alpha+iu)b_0\cdot X})}{\alpha+iu} \qquad \hat{c}_{\alpha,G_4}(u) = \frac{f(0)\mathbb{E}((b_2\cdot X)e^{b_1\cdot X}e^{(\alpha+iu)b_0\cdot X})}{\alpha+iu}.$$

The result follows because $ub_0 - (\alpha b_0 + b_1)i \in \Lambda_X$. □

4.3 Fourier Inversion

Option prices may be recovered via Fourier inversion.

Theorem 4.3. *Suppose G and α satisfy the hypotheses of Theorem 4.2.*

In cases $G = G_1, G_2$, the option price is given by

$$C_G(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\hat{c}_{\alpha,G}(u)du = \frac{e^{-\alpha k}}{\pi}\int_0^{\infty} \mathrm{Re}\left[e^{-iuk}\hat{c}_{\alpha,G}(u)\right]du. \tag{4.1}$$

In cases $G = G_3, G_4$, define the average of left and right limits $\bar{C}(k) := [C(k+0) - C(k-0)]/2$. Then

$$\bar{C}_G(k) = \frac{e^{-\alpha k}}{2\pi}\lim_{R\to\infty}\int_{-R}^{R} e^{-iuk}\hat{c}_{\alpha,G}(u)du = \frac{e^{-\alpha k}}{\pi}\int_0^{\infty} \mathrm{Re}\left[e^{-iuk}\hat{c}_{\alpha,G}(u)\right]du, \tag{4.2}$$

which can be strengthened to (4.1) *if $\hat{c}$ is L^1.*

Proof. In all cases, the damped option price $c(k)$ is L^1, as argued in Theorem 4.2.

In cases $G = G_1$ and $G = G_2$, the damped price $c(k)$ is continuous, by the dominated convergence theorem; and the transform is L^1, because $|\hat{c}(u)| \leqslant |f(-(\alpha+b_1)i)|/(u^2+\alpha^2)$. Therefore the usual Fourier inversion recovers c; see, for example, Champeney (1987) Theorem 8.2. Undamping with a factor of $e^{-\alpha k}$ yields (4.1).

In cases $G = G_3$ and $G = G_4$, the damped price $c(k)$ is locally of bounded variation, because $C(k)$ is the difference of two monotonic functions, $\exp(\alpha k)$ is monotonic, and both $C(k)$ and $\exp(\alpha k)$ are bounded on any finite interval. By, say, Champeney (1987) Theorem 8.12, we have (4.2). □

5 The Pricing Formula for General α

Transform representations of option prices can be viewed as contour integrals in the complex plane. Shifting the contour across a pole of the integrand changes the value of the integral, a technique which Lewis (2001) exploits, as will we.

Lewis differs from our approach in that he derives formulas for the transforms of option prices with respect to the *spot* variable X_0; whereas we, like Carr-Madan and Duffie-Pan-Singleton, transform with respect to the *trigger* variable k. His assumptions require that the option be written on the exponential of a variable X_T where the distribution of $X_T - X_0$ is not permitted to depend on X_0. Our formulas are not subject to this restriction and apply to a wider class of underlying state variables X, including those exhibiting mean-reversion.

One can modify the formulas of Lewis for non-independent-increments. However, the resulting formulas in that case do not allow the direct application of FFT to calibrate parameters to the prices of options at multiple strikes. For that purpose one needs transform-in-strike formulas, which we now derive.

Specifically, let $\Gamma := \Gamma_{X,G} := \{z \in \mathbb{C} : -\mathrm{Im}(z)b_0 + b_1 \in A_X\}$, and define $\hat{\mathscr{C}}_G : \Gamma_{X,G} \to \mathbb{C}$ by

$$\hat{\mathscr{C}}_{G_1}(z) := \frac{f(z-i)}{iz - z^2} \qquad \hat{\mathscr{C}}_{G_2}(z) := \frac{-f(z)}{z^2}$$

$$\hat{\mathscr{C}}_{G_3}(z) := \frac{f(b_0 z - b_1 i)}{iz} \qquad \hat{\mathscr{C}}_{G_4}(z) := \frac{-b_2 \cdot \nabla f(b_0 z - b_1 i)}{z}. \tag{5.1}$$

Theorem 4.2 proves that for *positive* α with $\alpha b_0 + b_1 \in A_X$, we have

$$\hat{c}_{\alpha,G}(u) = \hat{\mathscr{C}}_G(u - \alpha i),$$

and hence, for $z \in \Gamma$ such that $-\mathrm{Im}(z) > 0$,

$$\hat{\mathscr{C}}_G(z) = \int_{-\infty}^{\infty} e^{izk} C_G(k)\,dk. \tag{5.2}$$

Thus, in this region, $\hat{\mathscr{C}}_G(z)$ is the *complex* Fourier transform of the unmodified option price $C_G(k)$. Equivalently (modulo rotation by a factor of i), $\hat{\mathscr{C}}_G(z)$ is the bilateral Laplace transform of $C_G(k)$. Rewriting the conclusion of Theorem 4.3 shows that $\mathscr{C}_G$ may be inverted by integrating along the contour $\mathrm{Im}(z) = -\alpha$ in the complex plane:

$$\begin{aligned} C_G(k) &= \frac{1}{2\pi}\int_{-\infty-\alpha i}^{\infty-\alpha i} \hat{\mathscr{C}}_G(z)e^{-ikz}dz &&= \frac{1}{\pi}\int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}[\hat{\mathscr{C}}_G(z)e^{-ikz}]dz \qquad (G = G_1, G_2) \\ \bar{C}_G(k) &= \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R-\alpha i}^{R-\alpha i} \hat{\mathscr{C}}_G(z)e^{-ikz}dz &&= \frac{1}{\pi}\int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}[\hat{\mathscr{C}}_G(z)e^{-ikz}]dz \qquad (G = G_3, G_4), \end{aligned} \tag{5.3}$$

again assuming *positive* α with $\alpha b_0 + b_1 \in A_X$.

For *negative* α, the transform $\hat{c}_{\alpha,G}$ does not exist (for $G = G_1, \ldots, G_4$); likewise, the integral in (5.2) does not exist for $-\mathrm{Im}(z) < 0$. Nonetheless, the definitions (5.1) do make sense, and the integrals in (5.3) do exist for $\alpha < 0$, but they do not recover $\bar{C}_G$, because the integration path has shifted across the pole $z = 0$; instead they recover $\bar{C}_G$ less the contribution of the residue of $\hat{\mathscr{C}}_G$ at $z = 0$. In each case $G = G_1, \ldots G_4$, this generates one additional pricing formula. In case $G = G_1$, it generates a second additional formula, because $\hat{\mathscr{C}}_{G_1}$ has a second pole at $z = i$.

For *zero* α (and for $\alpha = -1$ in case $G = G_1$), the final integrals in (5.3) are again well-defined, but now the integration contour passes *through* a pole, and the contribution from the residue is cut in half. (The only exception is in the case $G = G_2$ which has a double pole at $z = 0$; this case calls for introducing into the integrand a term that tames the singularity, without affecting the value of the integral.) This generates two additional pricing formulas for payoff G_1, and one additional formula for the other payoffs.

Theorem 5.1 makes this discussion precise. Note that by taking $\alpha = 0$ in cases $G = G_3$ and $G = G_4$, we recover both of Duffie-Pan-Singleton's (2000, Prop 2 and Eqn 3.8) pricing formulas. Taking $\alpha > 0$ in case $G = G_1$ recovers Carr-Madan's damped-call pricing formula. Taking $\alpha = 0$ in two instances of case $G = G_3$ recovers the traditional two-integral call-pricing formulas, which we discuss further in Section 7.1. Our central pricing result is as follows.

Theorem 5.1. *Assume that $b_1 \in A_X$. Let α be any real number such that $\alpha b_0 + b_1 \in A_X$. Then in all cases except $(G = G_2; \alpha = 0)$,*

$$\bar{C}_G(k) = R_{\alpha,G} + \frac{1}{\pi}\int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}[\hat{\mathscr{C}}_G(z)e^{-ikz}]dz \tag{5.4}$$

where

$$R_{\alpha,G_1} := \begin{cases} f(-i) - e^k f(0) & \alpha < -1 \\ f(-i) - e^k f(0)/2 & \alpha = -1 \\ f(-i) & -1 < \alpha < 0 \\ f(-i)/2 & \alpha = 0 \\ 0 & \alpha > 0 \end{cases} \qquad R_{\alpha,G_2} := \begin{cases} -if'(0) - kf(0) & \alpha < 0 \\ (-if'(0) - kf(0))/2 & \alpha = 0 \\ 0 & \alpha > 0 \end{cases}$$

$$R_{\alpha,G_3} := \begin{cases} f(-b_1 i) & \alpha < 0 \\ f(-b_1 i)/2 & \alpha = 0 \\ 0 & \alpha > 0 \end{cases} \qquad R_{\alpha,G_4} := \begin{cases} -ib_2 \cdot \nabla f(-b_1 i) & \alpha < 0 \\ -ib_2 \cdot \nabla f(-b_1 i)/2 & \alpha = 0 \\ 0 & \alpha > 0 \end{cases}$$

and $\hat{\mathscr{C}}_G$ *is given in* (5.1)*. In cases* $G = G_1$ *and* $G = G_2$*, the* $\bar{C}_G$ *can be replaced by* C_G*.*

We will prove simultaneously Theorem 5.1 and the following $(G = G_2; \alpha = 0)$ theorem.

Theorem 5.2. *Assume that* $b_1 \in A_X$*. Then we have the* $(G = G_2; \alpha = 0)$ *formula*

$$C_{G_2}(k) = R_{0,G_2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\hat{\mathscr{C}}_{G_2}(z)e^{-ikz}] + \frac{1}{z^2}dz. \tag{5.5}$$

Proofs. For $\alpha > 0$, see Theorem 4.3.

For $\alpha < 0$ (except for $\alpha = -1$ in case $G = G_1$): Note that each $\hat{\mathscr{C}}_G$ is analytic in the strip $\Gamma_{X,G}$, except for a pole at $z = 0$ (and also $z = i$ in case $G = G_1$). The residue theorem applies to any rectangular path with horizontal segments on $\mathrm{Im}(z) = -\alpha_1$ and $\mathrm{Im}(z) = -\alpha_2$, and vertical segments on $\mathrm{Re}(z) = \pm R$. Since the integrals over the vertical segments approach 0 as $R \to \infty$, it follows that shifting a horizontal contour across the pole changes the value of the integral by $2\pi i$ times the residue at that pole. Residue calculation is straightforward.

For $\alpha = 0$ (including $\alpha = -1$ in case $G = G_1$): Our proof will be for $\alpha = 0$; a similar argument proves the ($G = G_1$; $\alpha = -1$) formula. Define the functions $S_{G_1}(z) := S_{G_3}(z) := S_{G_4}(z) := 0$ and $S_{G_2}(z) := 1/z^2$. On $\Gamma_{X,G} \cap -\Gamma_{X,G}$ the function

$$h(z) := S_G(z) + \frac{1}{2}\left[\hat{\mathscr{C}}_G(z)e^{-izk} + \hat{\mathscr{C}}_G(-z)e^{izk}\right]$$

(modulo a removable singularity in case G_2) is analytic. Choose $\varepsilon > 0$ such that $b_1 \pm \varepsilon b_0 \in A_X$. Applying

Cauchy's Theorem to the appropriate rectangle, and then using the relevant $\alpha \neq 0$ results, we have

$$\begin{aligned}
\frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R} h(z)dz = \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R-\varepsilon i}^{R-\varepsilon i} h(z)dz &= \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R-\varepsilon i}^{R-\varepsilon i} h(z) - S_G(z)dz \\
&= \frac{1}{4\pi}\lim_{R\to\infty}\left[\int_{-R-\varepsilon i}^{R-\varepsilon i} \hat{\mathscr{C}}_G(z)e^{-izk}dz + \int_{-R+\varepsilon i}^{R+\varepsilon i} \hat{\mathscr{C}}_G(z)e^{-izk}dz\right] \\
&= \frac{1}{2}\left[\bar{C}_G(k) + (\bar{C}_G(k) - R_{-\varepsilon,G})\right] = \bar{C}_G(k) - \frac{R_{-\varepsilon,G}}{2},
\end{aligned}$$

as claimed.

Theorem 5.1's final assertion is by continuity of C_{G_1} and C_{G_2}. □

A single piece of numerical integration code (coupled with the appropriate $R_{\alpha,G}$ adjustment) can evaluate, for example, all five formulas for payoff G_1; the only difference is the value of α passed into the procedure. Thus, without writing additional code, one gains the flexibility to choose, say, a negative or zero α if the integrand should happen to behave better there than it does along positive α. The extent to which an integrand is "well-behaved" can be quantified by the error bounds that arise from that particular choice of α. This is the subject of the next section.

Also in the next section we give alternative proofs for many of the formulas in Theorem 5.1. The contour-shift proof, given above, has the purpose of unifying the various Fourier pricing formulas; but for the purpose of deriving error bounds, it will be useful to reinterpret the results. For example, our $\alpha < 0$ bounds will exploit the equivalence between contour shifts and *parity relations*, such as put/call.

6 Bounds for Sampling and Truncation Errors

The Fourier inversion (5.4) can be approximated discretely via an N-point sum with a grid spacing of Δ in the Fourier domain. This quadrature introduces two forms of error (aside from roundoff error): truncation error because the upper limit of the numeric integration is finite, and sampling error because the integrand is evaluated numerically only at the grid points. Our bounds will account for both sources of error.

The total error is defined as the absolute difference between the true value

$$C_G(k) = R_{\alpha,G} + \frac{e^{-\alpha k}}{\pi}\int_0^\infty \mathrm{Re}\left[e^{-iuk}\hat{c}_{\alpha,G}(u)\right]du$$

and the discrete approximation given by the N-point sum

$$\Sigma^N(k) := \Sigma^{N,\Delta}_{\alpha,G}(k) := R_{\alpha,G} + e^{-\alpha k}\frac{\Delta}{\pi}\mathrm{Re}\left[\sum_{n=0}^{N-1}\hat{c}_{\alpha,G}((n+1/2)\Delta)e^{-i(n+1/2)k\Delta}\right]. \tag{6.1}$$

The total error is bounded by the sum of the *sampling error* and the *truncation error*

$$|C-\Sigma^N| \leqslant |C-\Sigma^\infty| + |\Sigma^\infty - \Sigma^N|,$$

where Σ^∞ is defined as Σ^N is, except with an infinite upper limit of summation.

Truncation errors can be bounded by a formula that applies regardless of the sign of α.

Sampling errors, however, will require treatment that depends on the sign of α. Our strategy is based on Davies (1973), but he restricts attention to the inversion of characteristic functions to recover probabilities, which is not always appropriate for us; we extend his approach to the inversion of option price transforms.

6.1 Truncation Error

Carr-Madan and Pan each suggest bounds on the tails of certain Fourier inversion integrals, but our specific need is to bound the tails of the infinite *discrete* sums that approximate our Fourier integrals.

Theorem 6.1. *Assume the hypotheses of Theorem 5.1.*

If f is such that $\hat{c}_{\alpha,G}$ decays as a power $|\hat{c}_{\alpha,G}(u)| \leqslant \Phi(u)/u^{1+\gamma}$ for all $u > u_0$, where $\gamma \equiv \gamma_{\alpha,G} > 0$ and $\Phi(u) \equiv \Phi_{\alpha,G}(u)$ is decreasing in u, then the truncation error

$$|\Sigma^\infty(k) - \Sigma^N(k)| \leqslant \frac{\Phi(N\Delta)}{\pi e^{\alpha k}\gamma(N\Delta)^\gamma}, \tag{6.2}$$

provided that $N\Delta > u_0$.

If f is such that $\hat{c}_{\alpha,G}$ decays exponentially $|\hat{c}_{\alpha,G}(u)| \leqslant \Phi(u)e^{-\gamma u}$ for all $u > u_0$, where $\gamma > 0$ and $\Phi(u)$ is decreasing in u, then the truncation error

$$|\Sigma^\infty(k) - \Sigma^N(k)| \leqslant \frac{\Delta\Phi((N+1/2)\Delta)}{2\pi e^{\alpha k+\gamma N\Delta}\sinh(\gamma\Delta/2)}, \tag{6.3}$$

provided that $N\Delta > u_0$.

Proof. In any case,

$$|\Sigma^\infty(k) - \Sigma^N(k)| \leqslant e^{-\alpha k}\frac{\Delta}{\pi}\sum_{n=N}^{\infty}\left|\hat{c}((n+1/2)\Delta)\right|.$$

In the power decay case,

$$\Delta\sum_{n=N}^{\infty}\left|\hat{c}((n+1/2)\Delta)\right| \leqslant \Delta\sum_{n=N}^{\infty}\frac{\Phi(n\Delta)}{[(n+1/2)\Delta]^{\gamma+1}} \leqslant \frac{\Phi(N\Delta)}{\Delta^\gamma}\int_N^\infty\frac{dx}{x^{\gamma+1}} = \frac{\Phi(N\Delta)}{\gamma(N\Delta)^\gamma},$$

where the middle step uses the convexity of $1/x^2$.

In the exponential decay case,

$$\sum_{n=N}^{\infty}\left|\hat{c}((n+1/2)\Delta)\right| \leqslant \Phi((N+1/2)\Delta)\sum_{n=N}^{\infty}e^{-\gamma(n+1/2)\Delta} \leqslant \frac{\Phi((N+1/2)\Delta)}{e^{\gamma(N+1/2)\Delta}-e^{\gamma(N-1/2)\Delta}},$$

as claimed. □

Remark 6.1. As observed by Carr and Madan in case $G = G_1$, and as one can verify also in case $G = G_2$, the power decay hypothesis *always* holds; specifically, let $\gamma = 1$ and $\Phi = |f(-(\alpha+b_1)i)|$. However, the resulting bound is typically poor. For practical purposes it is desirable to improve the power γ or to establish exponential decay, by factoring in the contribution from the large-u decay of $|f(u-(\alpha+b_1)i)|$. The nature of this decay presents itself in the explicit expression for f; examples appear in Appendix A.

Remark 6.2. The requirement that $N\Delta > u_0$ can be dropped, by modifying the right-hand sides of (6.2) and (6.3) as follows: first replace each $N\Delta$ by u_0 (thus bounding the $n \geqslant u_0/\Delta$ terms of truncation error). Then add a second term, to bound the $N \leqslant n < u_0/\Delta$ terms of the truncation error, by integrating, over the appropriate finite interval, a bound on $\hat{c}(u)$ valid for $u < u_0$, such as the quadratically decaying bound of Remark 6.1.

6.2 Sampling Error: Positive α

A form of the "aliasing" effect is at work here; by sampling $\hat{c}$ only at regular discrete intervals, one recovers not c but rather a periodic function equal to a combination of c and infinitely many shifted copies of c. The unwanted copies are shifted farther away as $\Delta \to 0$, so the extreme-strike bounds of Section 3 come into play.

In the main text, our sampling error analysis will focus on the payoff classes of greatest practical interest, G_1 and G_3. For sampling error in cases G_2 and G_4, see Appendix B.

Theorem 6.2. *Assume that $b_1 \in A_X$ and $\alpha b_0 + b_1 \in A_X$ with $\alpha > 0$.*

In case $G = G_1$ we have

$$|C_{G_1} - \Sigma^{\infty}_{\alpha,G_1}| \leqslant \inf_{p>\alpha:\ p+1\in A_X} \left[\frac{e^{-2\pi\alpha/\Delta} f(-i)}{1-e^{-4\pi\alpha/\Delta}} + \frac{e^{2\pi(\alpha-p)/\Delta} f(-i(p+1))}{(p+1)e^{pk}(1-e^{4\pi(\alpha-p)/\Delta})} \left(\frac{p}{p+1}\right)^p \right].$$

In case $G = G_3$, assume also that $\hat{c}(u) = O(u^{-1-\gamma})$ as $u \to \infty$, where $\gamma > 0$. Then

$$|C_{G_3} - \Sigma^{\infty}_{\alpha,G_3}| \leqslant \inf_{p>\alpha:\ pb_0+b_1\in A_X} \left[\frac{e^{-2\pi\alpha/\Delta} f(-ib_1)}{1-e^{-4\pi\alpha/\Delta}} + \frac{e^{2\pi(\alpha-p)/\Delta} f(-i(pb_0+b_1))}{e^{pk}(1-e^{4\pi(\alpha-p)/\Delta})} \right].$$

Proof. For any $\Delta > 0$ and any positive integer j,

$$\begin{aligned} c(k-2\pi j/\Delta) + c(k+2\pi j/\Delta) &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-iuk}\hat{c}(u)\cos(2\pi ju/\Delta)du \\ &= 2\int_0^{\Delta} F(u)\cos(2\pi ju/\Delta)du, \end{aligned}$$

where

$$F(u) := \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{c}(u+n\Delta)e^{-i(u+n\Delta)k}.$$

Since F is Lipschitz, the Fourier cosine series may be summed:

$$c(k)+\sum_{j=1}^{\infty}\Big[c(k-2\pi j/\Delta)+c(k+2\pi j/\Delta)\Big]\cos(2\pi ju/\Delta)=F(u)\Delta.$$

In particular, taking $u=\Delta/2$, we have

$$\big|c(k)-F(\Delta/2)\Delta\big|=\Bigg|\sum_{j=1}^{\infty}(-1)^j\Big[c(k-2\pi j/\Delta)+c(k+2\pi j/\Delta)\Big]\Bigg|.$$

Multiplying by $\exp(-\alpha k)$ to undamp the call prices,

$$\Big|C(k)-\Sigma^{\infty}(k)\Big|=\Bigg|\sum_{j=1}^{\infty}(-1)^j\Big[e^{-2\pi j\alpha/\Delta}C(k-2\pi j/\Delta)+e^{2\pi j\alpha/\Delta}C(k+2\pi j/\Delta)\Big]\Bigg|.$$

Therefore, Theorems 3.1-3.4 imply that

$$|C_{G_1}(k)-\Sigma^{\infty}(k)|\leqslant B_0\sum_{\substack{j=1\\ j\text{ odd}}}^{\infty}\left[e^{-2\pi j\alpha/\Delta}\mathbb{E}e^X+\frac{e^{2\pi j(\alpha-p)/\Delta}\mathbb{E}e^{(p+1)X}}{(p+1)e^{kp}}\left(\frac{p}{p+1}\right)^p\right] \tag{6.4}$$

and

$$|C_{G_3}(k)-\Sigma^{\infty}(k)|\leqslant B_0\sum_{\substack{j=1\\ j\text{ odd}}}^{\infty}\left[e^{-2\pi j\alpha/\Delta}\mathbb{E}e^{b_1\cdot X}+\frac{e^{2\pi j(\alpha-p)/\Delta}\mathbb{E}e^{(pb_0+b_1)\cdot X}}{e^{kp}}\right]. \tag{6.5}$$

The results follow from computing the sums. $\square$

Note that application of these bounds does not require actual computation of infimums. For example, in case $G=G_1$, *any* choice of $p>\alpha$ with $p+1\in A_X$ produces a valid upper bound, which is subject to improvement by taking more trial values of p.

6.3 Sampling Error: Negative α

The Theorem 6.2 error bounds assumed that $\alpha>0$, and must be modified for $\alpha<0$.

We have seen that shifting a Fourier inversion contour across a pole of the integrand changes the value of the integral. Sampling error bounds will now follow from the fact that the new integral value is the price of an contract related to the original option via a parity identity, such as put/call.

Specifically, for each $G=G_1,\dots,G_4$, define one "complementary" payoff G^*, and for case G_1 define a second complementary payoff G^{**} by

$$\begin{aligned}
G_1^*(x,k)&:=\min(\exp(x),\exp(k)) && b_0:=1,b_1:=1,x\in\mathbb{R}\\
G_1^{**}(x,k)&:=(\exp(k)-\exp(x))^+ && b_0:=1,b_1:=1,x\in\mathbb{R}\\
G_2^*(x,k)&:=(k-x)^+ && b_0:=1,b_1:=0,x\in\mathbb{R}\\
G_3^*(x,k)&:=\exp(b_1\cdot x)\mathbb{I}(b_0\cdot x\leqslant k) && x\in\mathbb{R}^n\\
G_4^*(x,k)&:=(b_2\cdot x)\exp(b_1\cdot x)\mathbb{I}(b_0\cdot x\leqslant k) && x\in\mathbb{R}^n.
\end{aligned}$$

These payoffs have the following time-0 values.

Theorem 6.3. *Assume that $b_1 \in A_X$ and $\alpha b_0 + b_1 \in A_X$.*

In cases $G = G_2, G_3, G_4$ with $\alpha < 0$; or in case $G = G_1$ with $-1 < \alpha < 0$, we have

$$\bar{C}_{G^*}(k) = \frac{1}{\pi}\int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}[\hat{\mathscr{C}}_G(k)e^{-izk}]dz.$$

In case $G = G_1$ with $\alpha < -1$, this holds after replacing the G^* with G^{**}.

Proof. Subtract from each original payoff function G its complementary payoff G^*; then take expectations to verify the parity relation

$$B_0\mathbb{E}G(X,k) - B_0\mathbb{E}G^*(X,k) = R_{0^-,G} \tag{6.6}$$

and similarly for G^{**}. Theorem 5.1 now implies the result.

Alternatively, without using Theorem 5.1, one may adapt Theorem 4.2 and compute directly the complex Fourier transforms of each C_{G^*}. Inverting as in Theorem 4.3 finishes the proof. Moreover, the negative-α formulas in Theorem 5.1 would then follow from (6.6). □

This equivalence between contour shifts and parity relations allows us to control the negative-α sampling error by bounding the extreme-strike values of the *complementary* payoffs. In particular, we state explicitly the complementary bounds for cases $G = G_1$ and $G = G_3$.

Theorem 6.4. *In case $G = G_1^*$ we have*

$$C_{G_1^*}(k) \leqslant B_0 e^k \quad \textit{and} \quad C_{G_1^*}(k) \leqslant B_0\mathbb{E}e^X.$$

In case $G = G_1^{**}$ we have, for any $q > 0$,

$$C_{G_1^{**}}(k) \leqslant \frac{B_0\mathbb{E}e^{-qX}}{1+q}\left(\frac{q}{1+q}\right)^q e^{(1+q)k} \quad \textit{and} \quad C_{G_1^{**}}(k) \leqslant B_0 e^k.$$

In case $G = G_3^*$ we have, for any $q > 0$,

$$C_{G_3^*}(k) \leqslant \frac{B_0\mathbb{E}e^{(-qb_0+b_1)\cdot X}}{e^{-qk}} \quad \textit{and} \quad C_{G_3^*}(k) \leqslant B_0\mathbb{E}e^{b_1\cdot X}.$$

Proof. Adapt the reasoning in Theorems 3.1 and 3.3. We omit the details. □

The sampling error bounds now follow.

Theorem 6.5. *Assume that $b_1 \in A_X$ and $\alpha b_0 + b_1 \in A_X$.*

In case $G = G_1$, for $\alpha \in (-1, 0)$:

$$|C_{G_1} - \Sigma^{\infty}_{\alpha,G_1}| \leqslant \frac{e^{k-2\pi(\alpha+1)/\Delta} f(0)}{1 - e^{-4\pi(\alpha+1)/\Delta}} + \frac{e^{2\pi\alpha/\Delta} f(-i)}{1 - e^{4\pi\alpha/\Delta}}.$$

and for $\alpha < -1$:

$$|C_{G_1} - \Sigma^{\infty}_{\alpha,G_1}| \leqslant \inf_{\substack{q > -(\alpha+1): \\ -q \in A_X}} \left[\frac{e^{k+2\pi(1+\alpha)/\Delta} f(0)}{1 - e^{4\pi(1+\alpha)/\Delta}} + \frac{e^{(1+q)k} e^{-2\pi(1+q+\alpha)/\Delta} f(iq)}{(1+q)(1 - e^{-4\pi(1+q+\alpha)/\Delta})} \left(\frac{q}{1+q} \right)^q \right].$$

In case $G = G_3$, assume also $\hat{c}(u) = O(u^{-1-\gamma})$ as $u \to \infty$, where $\gamma > 0$. Then for $\alpha < 0$,

$$|C_{G_3} - \Sigma^{\infty}_{\alpha,G_3}| \leqslant \inf_{\substack{q > -\alpha: \\ -qb_0 + b_1 \in A_X}} \left[\frac{e^{-2\pi(\alpha+q)/\Delta} f(-i(-qb_0 + b_1))}{e^{-qk}(1 - e^{-4\pi(\alpha+q)/\Delta})} + \frac{e^{2\pi\alpha/\Delta} f(-ib_1)}{1 - e^{4\pi\alpha/\Delta}} \right].$$

Proof. Adapt the reasoning in Theorem 6.2. We omit the details. □

6.4 Sampling Error: Zero α

Here we bound the sampling error along contours that pass through a pole. This means $\alpha = 0$ and, in case $G = G_1$, also $\alpha = -1$.

We present results for cases $G = G_1$ and $G = G_3$. In each case the option price function can be interpreted, after normalization, as a cumulative distribution function, so bounds from the probability literature apply directly, and we avoid reinvention of the wheel.

Our proof strategy yields, as a by-product, complete alternative proofs of 3 of the 13 formulas in Theorem 5.1, including the $(G = G_3;\ \alpha = 0)$ case, which was Duffie-Pan-Singleton's (2000, Prop 2) pricing formula; their proof influenced ours but lacks the highly convenient normalization step.

Theorem 6.6. *Assume that $b_1 \in A_X$ and $\alpha b_0 + b_1 \in A_X$.*

Then the $(\alpha, G) \in \{(0, G_1), (-1, G_1), (0, G_3)\}$ subcases of Theorem 5.1 all hold.

In case $G = G_1$, the $\alpha = -1$ and $\alpha = 0$ sampling errors are bounded by

$$\left| C_{G_1} - \Sigma^{\infty}_{-1,G_1} \right| \leqslant \max \left[\inf_{q > 0: -q \in A_X} \frac{f(iq)}{1+q} \left(\frac{q}{1+q} \right)^q \frac{e^{(1+q)k}}{e^{2\pi q/\Delta}},\ f(-i) e^{-2\pi/\Delta} \right]$$
$$\left| C_{G_1} - \Sigma^{\infty}_{0,G_1} \right| \leqslant \max \left[f(0) e^{k-2\pi/\Delta},\ \inf_{p > 0:\ p+1 \in A_X} \frac{f(-i(p+1))}{(p+1) e^{p(k+2\pi/\Delta)}} \left(\frac{p}{p+1} \right)^p \right].$$

In case $G = G_3$ assume also that $\hat{c}(u) = O(u^{-1-\gamma})$ as $u \to \infty$, where $\gamma > 0$. Then

$$\left| C_{G_3} - \Sigma^{\infty}_{0,G_3} \right| \leqslant \inf_{\substack{p > 0:\ pb_0 + b_1 \in A_X \\ q > 0: -qb_0 + b_1 \in A_X}} \max \left[\frac{f(i(qb_0 - b_1))}{e^{-q(k-2\pi/\Delta)}}, \frac{f(-i(pb_0 + b_1))}{e^{p(k+2\pi/\Delta)}} \right].$$

Proof. Case $G = G_1,\ \alpha = -1$:

On some probability space $(\Omega_1, P_1, \mathscr{F})$ there exists a real-valued random variable Y with density $\varphi(y) := e^{-y}\mathbb{E}[e^X \mathbb{I}(X > y)]$. It is easy to verify that

$$C_{G_1^{**}}(k) = f(0)e^k P_1(Y < k),$$

and that Y has P_1-characteristic function $f(u)/[f(0)(1-iu)]$. By the Gil-Pelaez (1951) formula,

$$C_{G_1^{**}}(k) = f(0)e^k\left(\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{f(u)/f(0)}{iu(1-iu)}e^{-iuk}\right]du\right) = \frac{f(0)e^k}{2} + \frac{1}{\pi}\int_{0+i}^{\infty+i}\mathrm{Re}\left[\frac{f(z-i)}{iz(iz+1)}e^{-izk}\right]dz.$$

Davies (1973) now directly implies the sampling error bound

$$|C_{G_1} - \Sigma^{\infty}_{-1,G_1}| \leqslant f(0)e^k \max\left[P_1(Y < k - 2\pi/\Delta),\ P_1(Y > k + 2\pi/\Delta)\right].$$

So for any $q > 0$,

$$|C_{G_1} - \Sigma^{\infty}_{-1,G_1}| \leqslant \max\left[\frac{B_0\mathbb{E}e^{-qX}}{1+q}\left(\frac{q}{1+q}\right)^q\frac{e^{(1+q)k}}{e^{2\pi q/\Delta}},\ e^{-2\pi/\Delta}B_0e^X\right],$$

as claimed.

Case $G = G_1,\ \alpha = 0$:

On some probability space $(\Omega_1, P_1, \mathscr{F})$ there exists a real-valued random variable Y with density $\varphi(y) := e^y P_B(X > y)f(0)/f(-i)$. It is easy to verify that

$$C_{G_1}(k) = f(-i)P_1(Y > k),$$

and that Y has P_1-characteristic function $f(u-i)/[f(-i)(iu+1)]$. By the Gil-Pelaez formula,

$$C_{G_1}(k) = f(-i)\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{f(z-i)/f(-i)}{iz(iz+1)}e^{-izk}\right]dz\right).$$

By Davies (1973),

$$|C_{G_1} - \Sigma^{\infty}_{0,G_1}| \leqslant f(-i)\max\left[P_1(Y < k - 2\pi/\Delta),\ P_1(Y > k + 2\pi/\Delta)\right].$$

So for any $p > 0$,

$$|C_{G_1} - \Sigma^{\infty}_{0,G_1}| \leqslant \max\left[f(0)e^{k-2\pi/\Delta},\ \frac{B_0\mathbb{E}e^{(p+1)X}}{(p+1)e^{p(k+2\pi/\Delta)}}\left(\frac{p}{p+1}\right)^p\right].$$

as claimed.

Case $G = G_3,\ \alpha = 0$:

Define the probability measure P_3 by $dP_3/dP := M_T^{-1}\exp(b_1\cdot X)/E(M_T^{-1}\exp(b_1\cdot X))$. Then

$$C_{G_3} = E[M_T^{-1}e^{b_1\cdot X}\mathbb{I}(b_0\cdot X>k)] = f(-b_1 i)E\left[\frac{dP_3}{dP}\mathbb{I}(b_0\cdot X>k)\right] = f(-b_1 i)P_3(b_0\cdot X>k),$$

and $f(b_0 z - b_1 i)/f(-b_1 i)$ is the characteristic function of $b_0\cdot X$ with respect to P_3. So

$$\bar{C}_{G_3} = f(-b_1 i)\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{f(b_0 z-b_1 i)/f(-b_1 i)}{iz}e^{-izk}\right]dz\right),$$

according to Gil-Pelaez. By Davies (1973),

$$|C_{G_3}-\Sigma_{0,G_3}^{\infty}| \leqslant f(-b_1 i)\max\Big[P_3(b_0\cdot X<k-2\pi/\Delta),\ P_3(b_0\cdot X>k+2\pi/\Delta)\Big]. \tag{6.7}$$

Therefore, writing E_3 for expectation with respect to P_3,

$$|C_{G_3}-\Sigma_{0,G_3}^{\infty}| \leqslant f(-b_1 i)\max\left[\frac{E_3 e^{-qb_0\cdot X}}{e^{-q(k-2\pi/\Delta)}},\ \frac{E_3 e^{pb_0\cdot X}}{e^{p(k+2\pi/\Delta)}}\right]$$

for any positive p and q, as claimed. □

Remark 6.3. In the case of $G=G_3$ with $b_0=b_1=1$ and $\alpha=0$, Pan (2002), following Davies, gives sampling error bounds.

Our $G=G_3$ proof extends Pan, because an alternative way to proceed from (6.7) is (writing Hf for the Hessian matrix of f):

$$|C_{G_3}-\Sigma_{0,G_3}^{\infty}| \leqslant f(-b_1 i)E_3\frac{(b_0\cdot X-k)^2}{(2\pi/\Delta)^2} = \frac{-b_0^\top Hf(-b_1 i)b_0 + 2ikb_0\cdot\nabla f(-b_1 i)+k^2 f(-b_1 i)}{(2\pi/\Delta)^2},$$

which improves her bound. Our other incremental contributions here include the complete explicit formulation of technical assumptions and the generality of vectors $b_0, b_1\in\mathbb{R}^n$.

This "quadratic" strategy also applies in case $G=G_1$ with $\alpha=0$ or -1. However, we prefer sampling error bounds that go to zero exponentially in $-1/\Delta$, rather than quadratically in Δ, so Theorem 6.6 reports only the exponential results.

6.5 Overall Error Bound: an Example

Consider a call on a stock, under Variance Gamma dynamics, as described in Section A.1. Of the five formulas in case $G=G_1$, we choose $\alpha>0$. The domain condition $\alpha b_0+b_1\in A_X$ entails the restriction $\alpha+1<a_+$, where a_+ is defined in (A.1). By (A.2) and Theorem 4.2,

$$|\hat{c}_{\alpha,G_1}(u)| \leqslant \frac{|f(u-(\alpha+1)i)|}{u^2} \leqslant \frac{\exp(-rT+(\alpha+1)(\log S_0+\mu T))}{(u^2\nu\sigma^2/2)^{T/\nu}u^2}.$$

By Theorem 6.1, truncation error is bounded by

$$\frac{\exp(-rT+(\alpha+1)(\log S_0+\mu T))}{\pi e^{\alpha k}(\nu\sigma^2/2)^{T/\nu}(1+2T/\nu)(N\Delta)^{1+2T/\nu}}.$$

By Theorem 6.2, sampling error is bounded by

$$\frac{e^{-2\pi\alpha/\Delta}f(-i)}{1-e^{-4\pi\alpha/\Delta}}+\frac{e^{2\pi(\alpha-p)/\Delta}f(-i(p+1))}{(p+1)e^{kp}(1-e^{4\pi(\alpha-p)/\Delta})}\left(\frac{p}{p+1}\right)^p$$

for any $p>\alpha$ such that $p+1<a_+$.

Summing the sampling and truncation bounds gives an overall error bound.

7 How To Minimize Error Bounds?

We propose some strategies for choosing α and the quadrature parameters N and Δ to obtain small error bounds, given limited computational resources.

Throughout this section, our illustrative problem is to price a vanilla call on a non-dividend-paying stock whose terminal price is $S_T=\exp(X)$. According to Theorem 5.1, we may price using any α such that $\alpha+1\in A_X$.

Assuming that $0\in A_X$ and $1\in A_X$ (mild assumptions since $\mathbb{E}e^{0\cdot X}<\infty$ and by no-arbitrage $\mathbb{E}e^{1\cdot X}<\infty$), we may write

$$A_X=(-\bar{q},\ 1+\bar{p}),$$

where $\bar{p}$ and $\bar{q}$ are positive. One determines $\bar{p}$ and $\bar{q}$ from the explicit expression for the characteristic function; see Appendix A for examples.

So α may be chosen anywhere in $(-\bar{q}-1,\ \bar{p})$. This interval comprises five subintervals, corresponding to the five G_1 formulas in Theorem 5.1.

A central question is how to choose from among these five G_1 formulas. While the *pricing* algorithm is invariant across all five α regimes, the fundamental nature of the *bounds* differs across the α regimes.

Before addressing this question, let us reject a sixth alternative.

7.1 How *Not* To Minimize Error Bounds

Instead of pricing a call as a G_1 payoff, one can price it as the difference of two G_3 payoffs. Indeed, the latter approach has dominated the literature (exceptions include Carr-Madan and Lewis).

Specifically, most authors have priced the call by decomposing it as long an asset-or-nothing call and short a binary call. Writing $K = \exp(k)$ for the strike,

$$\begin{aligned}
C(k) &= E[M_T^{-1}(S_T - K)^+] = E[M_T^{-1} S_T \mathbb{I}(S_T > K)] - KE[M_T^{-1}\mathbb{I}(S_T > K)] \\
&= S_0 E\left[\frac{S_T/S_0}{M_T}\mathbb{I}(S_T > K)\right] - KB_0 E\left[\frac{1/B_0}{M_T}\mathbb{I}(S_T > K)\right] \\
&= S_0 P_S(S_T > K) - KB_0 P_B(S_T > K).
\end{aligned}$$

They calculate both pseudo-probabilities via Gil-Pelaez inversions of the P_S-characteristic function and the P_B-characteristic function of X_T.

In other words,

$$C_{G_1}(k) = C_{G_3}^{b_0=b_1=1}(k) - e^k C_{G_3}^{b_0=1,b_1=0}(k),$$

and each C_{G_3} is evaluated according to the $\alpha = 0$ formula proved in Theorem 5.1 and again in Theorem 6.6 (the popular proof corresponds to our second proof). This G_3 approach to call-pricing has some merits, but from a computational point of view, it has significant disadvantages.

The generalized Carr-Madan approach of directly pricing G_1 has the computational advantage that we need invert only *one* Fourier transform, instead of *two* distinct characteristic functions.

Moreover, our direct G_1 error bounds have several advantages over combining two G_3 error bounds. The first is in truncation error control: $\hat{c}_{\alpha,G_1}$, unlike $\hat{c}_{\alpha,G_3}$, decays like f divided by the *square* of u, instead of u. The second is in sampling error control: the strategy of using exponential functions to dominate payoff functions produces tighter bounds when the payoff is a call than when the payoff is a binary. Third, note that summing two G_3 error bounds does not take advantage of possible error cancellation between the two components; in contrast, with only *one* integral to bound, the direct G_1 approach is not subject to this inefficiency.

7.2 Choice of Quadrature Parameters, Given an α Regime

Deferring once again the discussion of how to choose from among the five α regimes, we address here the question of how to choose quadrature parameters α, Δ, and N, *given* an α regime.

In two of the five regimes, the α interval consists of a single point, so the only question is how to choose N and Δ. For illustrative purposes, suppose we seek quadrature parameters in the $\alpha > 0$ regime, so all three quadrature parameters are in question.

The computational burden of the numeric Fourier inversion is determined by the grid point count N. First suppose that N is fixed and the goal is to find α and Δ that minimize the total error bound for a given contract k.

Theorem 6.1 gives truncation error bounds; to be concrete, let us suppose the discounted characteristic function has power decay, as defined there, for $u > 0$. Theorem 6.2 gives a sampling error bound. Combining the two, we have the optimization problem

$$\min_{\substack{(\alpha,\Delta,p):\ \Delta>0,\\ 0<\alpha<p<\bar{p}}} \left[\frac{\Phi_{\alpha,G_1}(N\Delta)}{\pi e^{\alpha k}\gamma(N\Delta)^{\gamma}} + \frac{e^{-2\pi\alpha/\Delta} f(-i)}{1-e^{-4\pi\alpha/\Delta}} + \frac{e^{2\pi(\alpha-p)/\Delta} f(-i(p+1))}{(p+1)e^{kp}(1-e^{4\pi(\alpha-p)/\Delta})} \left(\frac{p}{p+1} \right)^{p} \right]. \tag{7.1}$$

The choice of (α, Δ, p) can be automated by commonly available simplex optimization algorithms.

To modify (7.1) for models where the desired decay in $\hat{c}(u)$ is guaranteed only for $u > u_0 > 0$ (see Theorem 6.1), various options exist. The simplest is to change the Δ constraint to $\Delta > u_0/N$, but a more flexible solution is to allow also $\Delta \leqslant u_0/N$, but use the Remark 6.2 bound instead.

Instead of minimizing error bounds for a given computational budget N, an alternative goal would be to specify a desired error tolerance η, and to find the smallest N for which we can guarantee total error bounded by η. One strategy here is to choose as trial values for N successively increasing powers of 2. For each trial N, optimize, over (α, Δ, p), the total error bound in (7.1). Terminate the N loop when the error bound is smaller than the target η. Pricing can then proceed using the optimal α, Δ, and N.

7.3 Choice of Regime for α

The remaining question is how to choose from among the five regimes.

With unlimited resources, the answer is simple: compute error bounds in all five α regimes, and choose the one with the smallest error bound. Indeed, even with limited resources, this may prove to be a workable solution.

However, when the potential benefits of testing all five regimes do not justify the computing or programming effort, it is useful to have rules of thumb regarding which of the five formulas to implement. In any event, these rules also embody a comparative summary of our various bounds for pricing G_1 payoffs.

First consider sampling error. In each of the five regimes, our sampling error bound has $\Delta \to 0$ decay of order $\exp(-2\pi\rho/\Delta)$, where the "decay rate" ρ is apparent from the relevant Theorem.

Since a greater decay rate yields a better sampling error bound for Δ sufficiently small, Table 1 suggests the following sampling error guideline. The "call" regime and "put" regime are intervals with widths $\bar{p}$ and $\bar{q}$ respectively; choose α from inside the *wider* of these two intervals – unless both widths $\bar{p}$ and $\bar{q}$ are smaller than 2. In that case, choose $\alpha = -1$ or $\alpha = 0$, according to whether $\bar{q}$ or $\bar{p}$ is the larger – unless both $\bar{p}$ and $\bar{q}$ are smaller than $1/2$. In that case choose $\alpha \in (-1, 0)$.

Table 1: Decay rates of sampling error bounds

α range	Value of integral $\frac{1}{\pi}\int_{0-\alpha i}^{\infty-\alpha i}\mathrm{Re}[\hat{\mathscr{C}}_{G_1}(k)e^{-izk}]dz$	Decay rate ρ of bound on sampling error		Reference
$(-\bar{q}-1,-1)$	put	$\min(-\alpha-1,1+q+\alpha)$	$\leqslant \bar{q}/2$	Thm 6.5
-1	half-cash-secured put	$\min(q,1)$	$\leqslant \min(\bar{q},1)$	Thm 6.6
$(-1,0)$	cash-secured put = covered call	$\min(\alpha+1,-\alpha)$	$\leqslant 1/2$	Thm 6.5
0	half-covered call	$\min(p,1)$	$\leqslant \min(\bar{p},1)$	Thm 6.6
$(0,\bar{p})$	call	$\min(\alpha,p-\alpha)$	$\leqslant \bar{p}/2$	Thm 6.2

In other words, a high $\bar{p}=\sup\{p:\mathbb{E}\exp((p+1)X)<\infty\}$ indicates an X distribution with thin right-hand tail. Similarly, a high $\bar{q}$ indicates an X distribution with thin left-hand tail. To control sampling error, the guideline is to price the call if the right-hand tail is thinner in the sense that $\bar{p}>\bar{q}$, but otherwise price the corresponding put. However, if both tails are sufficiently thick, then instead price either a covered call or one of the "hybrid" payoffs induced by $\alpha=-1$ or 0.

Now consider truncation error. To develop intuition, we treat only the trivial case where X has zero variance; it should be understood that the resulting rule of thumb will lose accuracy for X with high variance. Writing $F_0=S_0/B_0$ for the T-forward price, we have

$$|\hat{c}_{\alpha,G_1}(u)| \leqslant \frac{|f(u-(\alpha+1)i)|}{u^2} = \frac{|B_0\mathbb{E}e^{i(u-(\alpha+1)i)X}|}{u^2} = \frac{B_0F_0^{\alpha+1}}{u^2} = \frac{S_0F_0^{\alpha}}{u^2}.$$

So Theorem 6.1 gives the truncation error bound

$$\frac{S_0(F_0/K)^{\alpha}}{\pi N\Delta}, \tag{7.2}$$

which is increasing in α if $K<F_0$, but decreasing in α if $K>F_0$. The rule of thumb, therefore, is that to minimize truncation error at low strikes, price the put; at high strikes, price the call. Specifically, in this zero-variance case, the rule for a given strike is to price whichever contract (put or call) is out-of-the-money-forward at that strike.

By combining these sampling error and truncation error heuristics, we will generate overall recommendations.

7.4 Recommendations

In this subsection, assume that we are to price options on equities, whose (risk-adjusted) return distributions typically exhibit significant negative skew, consistent with $\bar{p} > \bar{q}$.

For the common task of pricing a set of contracts with strikes nearly at-the-money-forward, the sampling-error heuristics (Table 1) tend to outweigh the truncation-error heuristics (7.2), which are in this case relatively insensitive to α. Therefore we recommend that the default procedure be to take $\alpha > 0$ and *price the call*.

The primary exception to this rule occurs at strikes away from the money, where truncation error tends to become more α-sensitive. At large strikes, the effect of (7.2) still favors the $\alpha > 0$ strategy of pricing the call. However, for *small strikes*, it favors the opposite strategy; indeed for strikes sufficiently small, this effect can swamp the sampling error effect, resulting in the opposite recommendation: take $\alpha < -1$ and *price the put*.

A second exception arises when the X distribution's tails are thick, in the sense that $\bar{p} < 2$ (implying that stock prices have infinite third moment). In this case the default procedure should be to price the half-covered call, by taking $\alpha = 0$, which outperforms $\alpha = -1$ on sampling error bounds. However, if the X distribution's tails are *very* thick, in the sense that $\bar{p} < 1/2$, then the default procedure should be to price price the covered call, by taking $\alpha = -1/2$, which uniquely in $(-1,0)$ attains the sampling bound decay rate of $1/2$.

A third exception could arise when one wishes to avoid optimizing α, possibly because of the computing, programming, or mathematical effort involved (where "mathematical" effort refers to the analytic determination of $\bar{p}$ and $\bar{q}$, given an unfamiliar characteristic function). Suppose one needs only a simple choice for α, that still guarantees the error bounds will go to zero for large N. Since A_X contains the interval $[0,1]$, the three choices $\alpha \in \{0, -1/2, -1\}$ are all acceptable.

Some caveats apply to the "simple" choices $\alpha \in \{0, -1/2, -1\}$. If one declines to optimize over $\alpha > 0$ or $\alpha < -1$, then one relinquishes the possibility of obtaining better error bounds – possibly *much* better, especially for N small and $\bar{p}$ large. Moreover, we put "simple" in quotation marks, because such α still require choices for Δ and N; and making those choices in an efficient way still requires optimization in some sense. Note that these caveats apply also to the traditional approach of computing a difference of two integrals, which has furthermore the error-management disadvantages of Section 7.1.

7.5 Numerical Examples and Discussion

For numerical examples we take the Variance Gamma model in Table 2 and the Heston model in Table 3. The parameters come from empirical studies of S&P 500 futures options: VG parameters from Madan-Carr-Chang (1998) which uses data from 1992–1994, and Heston parameters from Bates (2000) which uses data from 1988–1993.

For each model we generate two sub-tables: one for options at $T = 1$ month, and one for options at $T = 4$ months to expiry. Each one shows calls with strikes ranging from 80 to 120, on an underlying with value 100.

For each model and expiry, we choose the number of quadrature points N large enough to guarantee error smaller than one penny (0.01) at all listed strikes. For each strike and each of the five α regimes, we choose α and Δ to minimize our error bounds. The tables report the *a priori* error bounds and the realized errors.

Remark 7.1. These tables demonstrate that in certain examples with plausible parameters, we can *guarantee* accuracy of within one penny (which is 0.0001 times the underlying $S_0 = 100$), by sampling at a number of points N not in the thousands, but instead well under one hundred, and indeed in some cases under *ten*.

Remark 7.2. For each contract, our recommended quadrature parameters delivered a *realized* accuracy of within one-tenth of a penny (which is 0.00001 times the underlying).

Remark 7.3. The effect of increasing the time to expiry T depends on the model. To maintain the same accuracy in the VG model the computational burden decreased (from $N = 32$ to $N = 8$), whereas in the Heston model the burden increased (from $N = 8$ to $N = 16$).

Consider the following two pieces of intuition. One effect of increasing T is that return densities become smoother, which thins the tails of the characteristic function, hence *decreasing* truncation error; this effect is more significant in *VG* than Heston, because the former characteristic function decays polynomially, but the latter decays exponentially. Another effect of increasing T, however, is that return densities have fatter tails, which tends to make the characteristic function less smooth, hence *increasing* sampling error; this effect is more significant in *Heston* than VG, because the measure of tail-thinness relevant to our bounds is the number of finite moments. Under VG, the returns follow a Lévy process so the number of moments is invariant to time horizon, unlike Heston, where volatility is persistent, and hence works to decrease the number of moments and increase sampling error as T increases. To see this numerically in Tables 2 and 3, refer to A_X, which shows the number of moments to be T-dependent under Heston, but not under VG.

Remark 7.4. The optimal choice of α regime in our examples agrees with the rules of thumb proposed in Sections 7.3 and 7.4. Near-the-money and out-of-the-money, the best error bounds arise from choosing $\alpha > 0$ and pricing the call. However, for strikes sufficiently deep-in-the-money, the best error bounds arise from choosing $\alpha < -1$ and pricing the out-of-the-money put. The other choices $\alpha \in [-1, 0]$ underperformed, as we would anticipate, given the sufficiently thin tails of the return distributions; the thickest (Heston at 4 months) had $\bar{p} = 24.32$ and $\bar{q} = 9.97$, well above 2.

Remark 7.5. The numerics reflect a general viewpoint of this paper, which holds that the freedom to choose integration path, via the α parameter, plays an essential role in the accuracy, efficiency, and robustness of the transform approach.

Table 2: Realized errors and *a priori* bounds in five α regimes, under VG dynamics.

2(a). $T = 1$ month. $N = 32$ points.

Strike:	80		90		100		110		120	
Call price:	20.0057		10.0877		1.2678		0.0138		0.0004	
α	Error	Bound	Error	Bound	Error	Bound	Error	Bound	Error	Bound
< -1	**0.0000**	**0.0006**	**-0.0001**	**0.0032**	-0.0061	0.0128	-0.0008	0.0370	-0.0115	0.0829
$= -1$	0.1045	0.5271	0.0880	0.5771	0.4670	0.6258	-0.0109	0.6731	0.2862	0.7193
$\in (-1,0)$	0.5403	1.8857	0.0419	2.0033	1.4950	2.1127	0.1028	2.2151	0.2985	2.3115
$= 0$	0.0872	0.5987	0.1248	0.6157	0.4632	0.6312	-0.0271	0.6453	0.2713	0.6584
> 0	-0.0147	0.1056	-0.0019	0.0342	**0.0005**	**0.0058**	**-0.0000**	**0.0006**	**-0.0000**	**0.0001**
Optimal α	-14.98		-13.63		21.25		25.56		28.56	

2(b). $T = 4$ months. $N = 8$ points.

Strike:	80		90		100		110		120	
Call price:	20.0565		10.4903		2.8992		0.2310		0.0129	
α	Error	Bound	Error	Bound	Error	Bound	Error	Bound	Error	Bound
< -1	**-0.0000**	**0.0013**	**0.0004**	**0.0057**	-0.0001	0.0191	-0.0138	0.0505	-0.0338	0.1109
$= -1$	4.3683	7.4092	4.3888	7.7501	5.0984	8.0651	4.8697	8.3584	4.9340	8.6331
$\in (-1,0)$	21.7868	36.2630	23.1801	38.5159	25.3678	40.6206	25.9590	42.5988	26.4961	44.4674
$= 0$	4.3509	7.1282	4.3498	7.6780	5.1691	8.2023	5.1130	8.7045	5.1763	9.1873
> 0	-0.0144	0.0923	0.0017	0.0259	**-0.0001**	**0.0055**	**-0.0003**	**0.0009**	**-0.0000**	**0.0001**
Optimal α	-12.85		-11.73		17.55		20.66		23.31	

Underlying: $S_0 = 100$.

Variance Gamma model parameters: $\sigma = 0.1213$, $\nu = 0.1686$, $\theta = -0.1436$.

The interval of permissible $\alpha + 1$ values is $A_X = (-20.26,\ 39.78)$ at both time horizons.

CPU time for the N-point quadrature evaluations totaled 0.25 sec. for all of 2(a), and 0.08 sec. for all of 2(b).

Table 3: Realized errors and *a priori* bounds in five α regimes, under Heston dynamics.

3(a). $T = 1$ month. $N = 8$ points.

Strike:		80		90		100		110		120
Call price:		20.0043		10.1213		1.8314		0.0150		0.0001
α	Error	Bound	Error	Bound	Error	Bound	Error	Bound	Error	Bound
< -1	**-0.0000**	**0.0003**	**-0.0005**	**0.0034**	-0.0012	0.0225	-0.0066	0.0903	-0.0365	0.2529
$= -1$	10.1124	15.2531	10.1303	15.7479	10.8452	16.2042	10.5570	16.6283	11.1423	17.0249
$\in (-1,0)$	39.6391	66.4649	41.6391	70.5012	44.6160	74.3163	46.0391	77.9426	48.3202	81.4054
$= 0$	9.0364	13.8209	9.6049	15.0579	10.8497	16.2574	11.1304	17.4243	12.1809	18.5626
> 0	-0.0547	0.2081	-0.0021	0.0383	**0.0001**	**0.0031**	**-0.0000**	**0.0001**	**-0.0000**	**0.0000**
Optimal α		-22.20		-19.34		33.12		44.03		52.80

3(b). $T = 4$ months. $N = 16$ points.

Strike:		80		90		100		110		120
Call price:		20.3808		11.2277		3.7412		0.5343		0.0770
α	Error	Bound	Error	Bound	Error	Bound	Error	Bound	Error	Bound
< -1	**-0.0004**	**0.0078**	-0.0009	0.0157	-0.0057	0.0284	-0.0186	0.0473	-0.0170	0.0735
$= -1$	0.4526	0.9876	0.2692	1.0609	0.6069	1.1306	0.4485	1.1972	0.3552	1.2611
$\in (-1,0)$	2.4046	5.6995	2.5307	6.0546	3.5693	6.3855	3.3662	6.6960	2.7545	6.9886
$= 0$	0.5299	1.0613	0.2947	1.1131	0.5986	1.1610	0.4855	1.2058	0.3176	1.2479
> 0	-0.0017	0.0107	**-0.0010**	**0.0040**	**-0.0001**	**0.0015**	**-0.0001**	**0.0005**	**-0.0000**	**0.0002**
Optimal α		-6.11		9.84		10.96		11.98		12.91

Underlying: $S_0 = 100$.

Heston model parameters: $\kappa = 1.49$, $\theta = 0.0671$, $\sigma = 0.742$, $\rho = -0.571$. State variable: $V_0 = 0.0262$.

Interval of permissible $\alpha + 1$ values is $A_X = (-38.41,\ 89.59)$ at $T = 1$, and $A_X = (-9.97,\ 25.32)$ at $T = 4$ months.

CPU time for the N-point quadrature evaluations totaled 0.16 sec. for all of 3(a), and 0.29 sec. for all of 3(b).

A Appendix: Examples of Known Characteristic Functions

A.1 Variance Gamma

Reference: Madan-Carr-Chang (1998). The VG model has parameters σ, θ, ν.

The log price $X = \log S_T$ has discounted characteristic function

$$f(\zeta) = \frac{\exp(-rT + i\zeta[\log S_0 + \mu T])}{(1 - i\nu\theta\zeta + \nu\sigma^2\zeta^2/2)^{T/\nu}},$$

where $\mu := r + (1/\nu)\log(1 - \theta\nu - \sigma^2\nu/2)$; its domain is the strip Λ_X induced by $A_X = (a_-, a_+)$, where

$$a_\pm = -\frac{\theta}{\sigma^2} \pm \sqrt{\frac{2}{\nu\sigma^2} + \frac{\theta^2}{\sigma^4}}. \tag{A.1}$$

We have the following bound on the large-u decay of f. For $u > 0$,

$$|f(u + wi)| \leqslant \phi(w)u^{-2T/\nu}, \tag{A.2}$$

where

$$\phi(w) := \exp(-rT - w[\log S_0 + \mu T])(\nu\sigma^2/2)^{-T/\nu}. \tag{A.3}$$

Hence the VG model's discounted characteristic function satisfies the power decay condition of Theorem 6.1. So in, for example, the case $G = G_1$, one can take $\Phi_{\alpha,G_1}(u) = \phi(-(\alpha+1))$ and $1 + \gamma = 2 + 2T/\nu$.

A.2 Square-Root Stochastic Volatility

Reference: Heston (1993). The model has parameters κ, θ, σ, ρ, and a state variable V_0.

The log price $X = \log S_T$ has discounted characteristic function

$$f(\zeta) = \exp[-rT + i\zeta(\log S_0 + rT) + C(\zeta) + D(\zeta)V_0],$$

where

$$\begin{aligned}
C(\zeta) &:= \frac{\kappa\theta}{\sigma^2}\left[(\kappa - \rho\sigma\zeta i + d)T - 2\log\left(\frac{1 - ge^{dT}}{1 - g}\right)\right] \\
D(\zeta) &:= \frac{\kappa - \rho\sigma\zeta i + d}{\sigma^2}\left(\frac{1 - e^{dT}}{1 - ge^{dT}}\right) \\
g := g(\zeta) &:= \frac{\kappa - \rho\sigma\zeta i + d}{\kappa - \rho\sigma\zeta i - d} \\
d := d(\zeta) &:= \sqrt{(\rho\sigma\zeta i - \kappa)^2 + \sigma^2(\zeta i + \zeta^2)}.
\end{aligned}$$

The square root and the complex logarithm are multi-valued functions. For the square root here, either of the two values may be chosen, because f is even in d. For the logarithm, however, choosing the wrong value can lead to wildly incorrect answers. To define $f(wi)$ for real w, the correct choice of $\log(z)$ is the principal branch $\log|z| + \arg(z)$, where $-\pi < \arg(z) < \pi$. However, as pointed out by Schöbel and Zhu (1999), to define $f(\zeta)$ for general ζ, the correct choice of log is *not* necessarily the principal branch. Instead, the value of log when $\zeta = u + wi$ is determined by the analyticity of f, which implies that log must vary continuously as ζ varies from $0 + wi$ to $u + wi$.

This issue presents a challenge to the traditional approach of taking the Fourier integrals in Heston and simply passing the integrands into a numerical integration routine from a standard software library. Enforcing the required continuity of the log is tricky if the integration routine samples the integrand at an unpredictable sequence of points. On the other hand, for a method, such as ours, that samples the integrand at an increasing sequence of points with spacing Δ, enforcing continuity typically does not present any difficulty.

The domain of f is the strip Λ_X induced by $A_X = (a_-, a_+)$, where $a_- < 0$ and $a_+ > 1$ solve

$$g(-ia)\exp(d(-ia)T) = 1.$$

Specifically, if we assume $\kappa - \rho\sigma > 0$, then a_- is the largest (closest to 0) solution in $(-\infty, y_-)$, and a_+ is the smallest solution in (y_+, ∞), where

$$y_\pm := \frac{\sigma - 2\kappa\rho \pm \sqrt{\sigma^2 - 4\kappa\rho\sigma + 4\kappa^2}}{2\sigma(1-\rho^2)}.$$

For $\zeta = u + wi$ we bound the large-u decay of f, as follows. Define

$$\begin{aligned}
H_{R1}(u) &:= u^2\sigma^2(1-\rho^2) \\
H_{R2}(w) &:= w^2\sigma^2(1-\rho^2) - w(2\kappa\rho\sigma - \sigma^2) - \kappa^2 \\
H_R(u,w) &:= \mathrm{Re}(d^2) = H_{R1}(u) - H_{R2}(w) \\
H_I(u,w) &:= \mathrm{Im}(d^2) = \sigma u(2w\sigma(1-\rho^2) + \sigma - 2\kappa\rho) \\
h(u,w) &:= \sqrt{H_R(u,w)},
\end{aligned}$$

and define

$$\begin{aligned}
g^*(u,w) &:= \frac{\kappa}{\sigma\sqrt{u^2+w^2}} + \frac{|\sigma - 2\kappa\rho| + \kappa^2/(\sigma\sqrt{u^2+w^2})}{h(u,w) + \sqrt{(u^2-w^2)\sigma^2(1-\rho^2)}} \\
\underline{g}(u,w) &:= (1 - g^*(u,w))/(1 + g^*(u,w)) \\
J(u,w) &:= \left(1 + \frac{1}{\underline{g}(u)}\right)\left(1 + \frac{1}{\underline{g}(u)\exp(Th(u,w)) - 1}\right).
\end{aligned}$$

Let $u_0 > |w|$ satisfy $1 > g^*(u_0,w)$ and $h(u_0,w) > (1/T)\max(\log(1/\underline{g}(u_0,w)),1)$ and $H_{R1}(u_0) > |H_{R2}(w)|$.

Then for all $u > u_0$, we have

$$|f(u+wi)| \leqslant \phi(u,w)\exp\left(-\sqrt{1-\rho^2}\frac{V_0+\kappa\theta T}{\sigma}u\right),$$

where (suppressing the arguments (u,w) for convenience) we let

$$\begin{aligned}\phi(u,w) := J^{2\kappa\theta/\sigma^2}\exp&\left[-rT-(\log S_0+rT)w+\frac{V_0+\kappa\theta T}{\sigma^2}(\kappa+\rho\sigma w+\sqrt{\max(0,H_{R2})})\right]\\ \times\exp&\left[\frac{V_0}{\sigma^2}\frac{J}{\exp(Th)}\Big(\kappa+|\rho\sigma u|\max(1,\sqrt{H_R/H_{R1}})+|\rho\sigma w|+\sqrt{H_R+|H_I|}\Big)\right].\end{aligned}$$

Hence the square-root stochastic volatility model's discounted characteristic function satisfies the exponential decay condition of Theorem 6.1. So in, for example, the case $G = G_1$, one can take $\Phi_{\alpha,G_1}(u) = \phi(u,-(\alpha+1))/u^2$.

B Appendix: Sampling Error Bounds for Payoffs G_2 and G_4

Sections 6.2-6.4 gave sampling error bounds for $G = G_1$ and $G = G_3$, which are the payoff classes of greatest practical interest. The other cases $G = G_2$ and $G = G_4$ can be treated by similar techniques, albeit with messier results. Specifically, the G_2/G_4 version of Theorem 6.2 is as follows.

Theorem B.1. *Assume that $b_1 \in A_X$ and $\alpha b_0 + b_1 \in A_X$ with $\alpha > 0$.*

In case $G = G_2$ we have

$$\begin{aligned}|C_{G_2}-\Sigma^\infty_{\alpha,G_2}| \leqslant \inf_{\substack{p>\alpha:\ p\in A_X\\ q>0:-q\in A_X}}&\left[\frac{e^{-2\pi\alpha/\Delta}(-if'(0)-k)}{1-e^{-4\pi\alpha/\Delta}}+\frac{4\pi e^{-2\pi\alpha/\Delta}f(0)}{\Delta(1-e^{-4\pi\alpha/\Delta})^2}\right.\\ &\left.+\frac{e^{-2\pi(q+\alpha)/\Delta}f(iq)}{qe^{1-qk}(1-e^{-4\pi(q+\alpha)/\Delta})}+\frac{e^{2\pi(\alpha-p)/\Delta}f(-ip)}{pe^{pk+1}(1-e^{4\pi(\alpha-p)/\Delta})}\right].\end{aligned}$$

In case $G = G_4$, assume also that $\hat{c}(u) = O(u^{-1-\gamma})$ as $u \to \infty$, where $\gamma > 0$. Then

$$\begin{aligned}|C_{G_4}-\Sigma^\infty_{\alpha,G_4}| \leqslant \max_{\chi=\pm1}\inf_{(p_0,p_2,q_0,q_2)}&\left[\frac{e^{-2\pi\alpha/\Delta}\chi b_2\cdot\nabla f(-ib_1)}{1+e^{-2\pi\alpha/\Delta}}\right.\\ &+\frac{f(-i(-q_0b_0-q_2\chi b_2+b_1))+e^{-2\pi(\alpha+q_0)/\Delta}f(-i(-q_0b_0+q_2\chi b_2+b_1))}{q_2e^{-q_0k+1}(e^{2\pi(\alpha+q_0)/\Delta}-e^{-2\pi(\alpha+q_0)/\Delta})}\\ &\left.+\frac{f(-i(p_0b_0+p_2\chi b_2+b_1))+e^{2\pi(\alpha-p_0)/\Delta}f(-i(p_0b_0-p_2\chi b_2+b_1))}{p_2e^{p_0k+1}(e^{-2\pi(\alpha-p_0)/\Delta}-e^{2\pi(\alpha-p_0)/\Delta})}\right],\end{aligned}$$

where the inf *is over all positive p_0,p_2,q_0,q_2 such that $p_0 > \alpha$ and $p_0b_0 \pm p_2b_2 + b_1 \in A_X$ and $-q_0b_0 \mp q_2b_2 + b_1 \in A_X$.*

Proof. In the proof of Theorem 6.2, replace equations (6.4) and (6.5) with

$$|C_{G_2}(k)-\Sigma^\infty(k)| \leqslant B_0 \sum_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \left[e^{-2\pi j\alpha/\Delta}\left(\mathbb{E}X - k + 2\pi(j+1)/\Delta + \frac{\mathbb{E}e^{-qX}}{qe^{1-q(k-2\pi j/\Delta)}}\right) + \frac{e^{2\pi j(\alpha-p)/\Delta}\mathbb{E}e^{pX}}{pe^{kp+1}} \right]$$

and

$$\begin{aligned} |C_{G_4}(k)-\Sigma^\infty(k)| \leqslant B_0 \max_{\chi=\pm1} \sum_{j=1}^{\infty} \Bigg[& e^{-2\pi j\alpha/\Delta}\mathbb{E}((\chi(-1)^j b_2\cdot X)e^{b_1\cdot X}) + \frac{\mathbb{E}e^{(-q_0b_0-\chi(-1)^j q_2b_2+b_1)\cdot X}}{q_2e^{-q_0k+1}e^{2\pi j(\alpha+q_0)/\Delta}} \\ & + \frac{e^{2\pi j(\alpha-p_0)/\Delta}\mathbb{E}e^{(p_0b_0+\chi(-1)^j p_2b_2+b_1)\cdot X}}{p_2e^{p_0k+1}} \Bigg]. \end{aligned}$$

The rest of the proof holds. □

C Appendix: Application of DFT/FFT

We treat here two issues: a recipe for DFT evaluation of the quadrature scheme, and modifications to the Section 7.2 optimization problem so that the DFT output has the desired contract spacing.

First, our Davies-style discretization samples $\hat{c}$ at the midpoints $(n+1/2)\Delta$ of intervals of length Δ, whereas Carr-Madan's sampling scheme applies Simpson's rule to the endpoints of those intervals. For completeness we describe how to adapt their formulas to midpoint sampling.

Define the discrete Fourier transform (DFT) of an N-vector x to be the vector X where

$$X_m = \sum_{n=1}^{N} e^{-i(2\pi/N)(n-1)(m-1)}x_n, \qquad m=1,\ldots,N.$$

Other definitions exist; this is the one in Carr-Madan, and in a number of standard software packages, including Matlab.

Using a spacing of $\lambda = 2\pi/(N\Delta)$ between consecutive triggers, we want to use DFT to compute prices $\Sigma^N(k_m)$ at triggers

$$k_m := k_1 + \lambda(m-1), \qquad m=1,\ldots,N$$

for arbitrary k_1. Typically one would choose k_1 such that the interval $[k_1, k_1+(N-1)\lambda]$ contains all of the contracts to be priced. By (6.1),

$$\begin{aligned} \Sigma^N(k_m) &= \frac{\Delta}{\pi e^{\alpha k_m}}\mathrm{Re}\left[\sum_{n=1}^{N}\hat{c}((n-1/2)\Delta)e^{-i(n-1/2)(k_1+\lambda(m-1))\Delta}\right] \\ &= \frac{\Delta}{\pi e^{\alpha k_m}}\mathrm{Re}\left[e^{-i(m-1)\lambda\Delta/2}\sum_{n=1}^{N}\hat{c}((n-1/2)\Delta)e^{-i(n-1)(m-1)\lambda\Delta}e^{-i(n-1/2)k_1\Delta}\right] \\ &= \frac{\Delta}{\pi e^{\alpha k_m}}\mathrm{Re}\left[e^{-i\pi(m-1)/N}\sum_{n=1}^{N}e^{-i(2\pi/N)(n-1)(m-1)}\hat{c}((n-1/2)\Delta)e^{-ik_1(n-1/2)\Delta}\right], \end{aligned}$$

where the sum is computable as the m-th component of the DFT of the vector whose n-th component is $\hat{c}((n-1/2)\Delta)\exp(-ik_1(n-1/2)\Delta)$.

The second issue is the reciprocity relation $\lambda\Delta = 2\pi/N$. For N fixed, a decrease in Fourier-domain grid spacing Δ would cause the contract spacing λ to increase.

If one wishes to impose an upper bound $\bar{\lambda}$ on spacing between contracts, then the minimum in (7.1) should be taken over $\Delta > 2\pi/(N\bar{\lambda})$ instead of $\Delta > 0$. Moreover, in certain instances it is desirable to constrain Δ to be an integer times $2\pi/(N\bar{\lambda})$, because this forces λ to divide $\bar{\lambda}$, so that a set of contracts with trigger spacings of $\bar{\lambda}$ can be priced in a single DFT, without interpolation.

We deferred this material to an Appendix to emphasize that the analysis in the body of this paper does not make any assumption on *how* the sum in (6.1) is computed (aside from absence of roundoff error). One can use the DFT; or its efficient implementation the fast Fourier transform (FFT); or, perhaps even more efficiently (if few enough strikes need to be simultaneously priced), simple direct summation.

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