ERROR ANALYSIS IN FOURIER METHODS FOR OPTION PRICING

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Abstract. We provide a bound for the error committed when using a Fourier method to price European options when the underlying follows an exponential Lévy dynamic. The price of the option is described by a partial integro-differential equation (PIDE). Applying a Fourier transformation to the PIDE yields an ordinary differential equation that can be solved analytically in terms of the characteristic exponent of the Lévy process. Then, a numerical inverse Fourier transform allows us to obtain the option price. We present a novel bound for the error and use this bound to set the parameters for the numerical method. We analyse the properties of the bound for a dissipative and pure-jump example. The bound presented is independent of the asymptotic behaviour of option prices at extreme asset prices. The error bound can be decomposed into a product of terms resulting from the dynamics and the option payoff, respectively. The analysis is supplemented by numerical examples that demonstrate results comparable to and superior to the existing literature.

Keywords: Fourier transform, European option pricing, error bound, exponential Lévy process, partial integro-differential equations

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1. Introduction

Lévy processes form a rich field within mathematical finance. They allow modelling of asset prices with possibly discontinuous dynamics. An early and probably the best known model involving Lévy processes is the Merton (1976) model, which generalises the Black and Scholes (1973) model. More recently, we have seen more complex models allowing for more general dynamics of the asset price. Examples of such models include the Kou (2002) model (see also Dotsis et al. (2007)), the normal-inverse Gaussian model (Barndorff-Nielsen (1997); Rydberg (1997)), the variance gamma model (Madan and Seneta (1990); Madan et al. (1998)), and the Carr-Geman-Madan-Yor (CGMY) model (Carr et al. (2002, 2003)). For a good exposition on jump processes in finance we refer to Cont and Tankov (2004) (also see Raible (2000) and Eberlein (2001)).

Prices of European options whose underlying asset is driven by the Lévy process are solutions to partial integro-differential equations (PIDEs) (Nualart et al. (2001); Briani et al. (2004); Almendral and Oosterlee (2005); Kiessling and Tempone (2011)) that generalise the Black-Scholes equation by incorporating a non-local integral term to account for the discontinuities in the asset price. This approach has also been extended to cases where the option price features path dependence, for instance in dHalluin et al. (2004) and Lord et al. (2008).

The Lévy-Khintchine formula provides an explicit representation of the characteristic function of a Lévy process (cf, Tankov (2004)). As a consequence, one can derive an exact expression for the Fourier transform of the solution of the relevant PIDE. Using the inverse fast Fourier transform (iFFT) method, one may efficiently compute the option price for a range of asset prices simultaneously. Furthermore, in the case of European call options, one may use the duality property presented by Dupire (1997) and iFFT to efficiently compute option prices for a wide range of strike prices.

Despite the popularity of Fourier methods for option pricing, not many works can be found on the error analysis of these methods. A bound for the error not only provides an interval for the precise value of the option, but also suggests a method to select the parameters of the numerical method. An important work in this direction is the one by Lee (2004) in which several payoff functions are considered for a rather general set of models, whose characteristic function is assumed to be known.

In this work, we present a methodology for studying the error when using FFT methods to compute option prices. We also provide a systematic way of choosing the parameters of the numerical method. We focus on exponential Lévy processes that may be of either diffusive or pure-jump. Our contribution is to derive a strict error bound for a Fourier transform method when pricing options under risk-neutral Lévy dynamics. We derive a simplified bound that separates the contributions of the payoff and of the process in an easily processed and extensible product form that is independent of the asymptotic behaviour of the option price at extreme prices and at strike parameters. We specify regularity conditions on Lévy processes for which we prove the existence of a separable L_∞ error bound and establish a methodology for error control for other Lévy processes. We also provide a proof for the existence of optimal parameters of the numerical computation that minimise the presented error bound. When comparing our work with Lee's work we find that Lee's work is more general than ours in that he studies a wider range of processes, on the other hand, our results applies to a larger class of payoffs. We also find that our bound was tighter than Lee's for the test cases chosen by him.

The paper is organised in the following sections: In Section 2 we introduce the PIDE setting in the context of risk-neutral asset pricing; we show the Fourier representation of the relevant PIDE for asset pricing with Lévy processes and use that representation for derivative pricing. In Section 3 we derive a representation for the numerical error and divide it into quadrature and cutoff contributions. We also describe the methodology for choosing numerical parameters to obtain minimal error bounds. The derivation is supported by numerical examples using relevant test cases with both diffusive and pure-jump Lévy processes in Section 4. Numerics are followed by conclusions in Section 5.

2. Fourier method for option pricing

Consider an asset whose price at time t is modelled by the stochastic process $S = (S_t)$ defined by $S_t = S_0 e^{X_t}$, where $X = (X_t) \in \mathbb{R}$ is assumed to be a Lévy process whose jump measure ν satisfies

$$\int_{\mathbb{R}\setminus\{0\}} \min\{y^2, 1\}\nu\,(dy) < \infty \tag{1}$$

Assuming the risk-neutral dynamic for S_t, the price at time $t = T - \tau$ of a European option with payoff G and maturity time T is given by

$$\Pi(\tau, s) = e^{-r\tau}\mathrm{E}\left(G\left(S_T\right)|S_{T-\tau} = s\right)$$

where r is the short rate that we assume to be constant and $\tau\colon 0 \leq \tau \leq T$ is the time to maturity. Extensions to non-constant deterministic short rates are straightforward.

The infinitesimal generator of a Lévy process X is given by (see Applebaum, 2004)

$$\begin{aligned}\mathcal{L}^X f(x) &\equiv \lim_{h\to 0} \frac{\mathrm{E}\left(f(X_{t+h})|X_t = x\right) - f(x)}{h} \\ &= \gamma f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}\setminus\{0\}} \left(f(x+y) - f(x) - y1_{|y|\leq 1} f'(x)\right)\nu(dy) \qquad (2)\end{aligned}$$

where (γ, σ^2, ν) is the characteristic triple of the Lévy process. The risk-neutral assumption on (S_t) implies

$$\int_{|y|>1} e^y \nu(dy) < \infty \tag{3}$$

and fixes the drift term (see Kiessling and Tempone (2011)) γ of the Lévy process to

$$\gamma = r - \frac{1}{2}\sigma^2 - \int_{\mathbb{R}\setminus\{0\}} \left(e^y - 1 - y1_{|y|\leq 1}\right)\nu(dy) \tag{4}$$

Thus, the infinitesimal generator of X may be written under the risk-neutral assumption as

(5)

$$\mathcal{L}^X f(x) = \left(r - \frac{\sigma^2}{2}\right) f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}\setminus\{0\}} \left(f(x+y) - f(x) - (e^y - 1) f'(x)\right)\nu(dy)$$

Consider g as the reward function in log prices (ie, defined by $g(x) = G(S_0 e^x)$). Now, take f to be defined as

$$f(\tau, x) \equiv \mathrm{E}\left(g(X_T) \,|X_{T-\tau} = x\right)$$

Then f solves the following PIDE:

$$\begin{cases} \partial_\tau f(\tau, x) & = \mathcal{L}^X f(\tau, x) \\ f(0, x) & = g(x), \qquad\qquad (\tau, x) \in [0, T] \times \mathbb{R} \end{cases}$$

Observe that f and Π are related by

$$\Pi(\tau, S_0 e^x) = e^{-r\tau} f(\tau, x) \tag{6}$$

Consider a damped version of f defined by $f_\alpha(\tau, x) = e^{-\alpha x} f(\tau, x)$; we see that $\partial_\tau f_\alpha = e^{-\alpha x}\mathcal{L}^X f(\tau, x)$.

There are different conventions for the Fourier transform. Here we consider the operator $\mathcal{F}$ such that

$$\mathcal{F}[f](\omega) = \int_{\mathbb{R}} e^{i\omega x} f(x) dx$$

defined for functions f for which the previous integral is convergent. We also use $\hat{f}(\omega)$ as a shorthand notation of $\mathcal{F}[f](\omega)$. To recover the original function f, we define the inverse Fourier transform as

$$\mathcal{F}^{-1}[f](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega x} f(\omega) d\omega$$

We have that $\mathcal{F}^{-1}\left[\hat{f}\right](x) = f(x)$.

Applying $\mathcal{F}$ to f_α we get $\hat{f_\alpha}(\omega) = \hat{f}(\omega + i\alpha)$. Observe also that the Fourier transform applied to $\mathcal{L}^X f(\tau, x)$ gives $\Psi(-i\omega)\hat{f}(\tau, \omega)$, where $\Psi(\cdot)$ is the characteristic exponent of the process X, which satisfies $\mathrm{E}\left(e^{zX_t}\right) = e^{t\Psi(z)}$. The explicit expression for $\Psi(\cdot)$ is

$$\Psi(z) = \left(r - \frac{\sigma^2}{2}\right) z + \frac{\sigma^2}{2} z^2 + \int_{\mathbb{R}} \left(e^{zy} - 1 - (e^y - 1)z\right) \nu(dy) \tag{7}$$

From the previous considerations it can be concluded that

$$\partial_\tau \hat{f_\alpha} = \Psi(\alpha - i\omega)\hat{f}(\omega - i\alpha) \tag{8}$$

Now $\hat{f}(\omega - i\alpha) = \hat{f_\alpha}(\omega)$ so $\hat{f_\alpha}$ satisfies the following ODE

$$\begin{cases} \frac{\partial_\tau \hat{f_\alpha}(\tau,\omega)}{\hat{f_\alpha}(\tau,\omega)} & = \Psi(\alpha - i\omega) \\ \hat{f_\alpha}(0, \omega) & = \hat{g_\alpha}(\omega) \end{cases} \tag{9}$$

Solving the previous ODE explicitly, we obtain

$$\hat{f_\alpha}(\tau, \omega) = e^{\tau\Psi(\alpha - i\omega)}\hat{g_\alpha}(\omega) \tag{10}$$

Observe that the first factor in the right-hand side in the above equation is $\mathrm{E}\left(e^{(\alpha - i\omega)X_\tau}\right)$, (ie, $\varphi_1(-i\alpha - \omega)$), where $\varphi_\tau(\cdot)$ denotes the characteristic function of the random variable X_τ.

Now, to obtain the value function we employ the inverse Fourier transformation and obtain

$$f_\alpha(\tau,x) = \mathcal{F}^{-1}\left[\hat{f}_\alpha\right](\tau,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega x}\hat{f}_\alpha(\tau,\omega)d\omega \tag{11}$$

or

$$f_\alpha(\tau,x) = \frac{1}{\pi}\int_0^{+\infty} \mathrm{Re}\left[e^{-i\omega x}\hat{f}_\alpha(\tau,\omega)\right]d\omega \tag{12}$$

As it is typically not possible to compute the inverse Fourier transform analytically, we approximate it by truncating and discretising the integration domain using trapezoidal quadrature (11). Consider the following approximation:

$$f_{\alpha,\Delta\omega,n}(\tau,x) = \frac{\Delta\omega}{2\pi}\sum_{k=-n}^{n-1} e^{-i\left(k+\frac{1}{2}\right)\Delta\omega x}\hat{f}_\alpha\left(\tau,\left(k+\frac{1}{2}\right)\Delta\omega\right) \tag{13}$$

$$= \frac{\Delta\omega}{\pi}\sum_{k=0}^{n-1}\mathrm{Re}\left[e^{-i\left(k+\frac{1}{2}\right)\Delta\omega x}\hat{f}_\alpha\left(\tau,\left(k+\frac{1}{2}\right)\Delta\omega\right)\right] \tag{14}$$

Estimating the error in the approximation of $f(\tau,x)$ by

$$f_{\Delta\omega,n}(\tau,x) \equiv e^{\alpha x}f_{\alpha,\Delta\omega,n}(\tau,x)$$

is the main issue of this paper and will be addressed in the following section.

Remark 2.1. The fast Fourier transform algorithm provides an efficient way of computing the previous bound for several values of x at the same time. Examples of works that consider this widely extended tool are Lord et al. (2008); Jackson et al. (2008); Hurd and Zhou (2010) and Schmelzle (2010).

Remark 2.2. Although we are mainly concerned with option pricing when the payoff function can be damped in order to guarantee regularity in the L_1 sense, we note here that our main results are naturally extendable to include the Greeks of the option. Indeed, we have by (10) that

$$f(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{(\alpha-i\omega)x}\hat{f}_\alpha(\tau,\omega)\,d\omega \tag{15}$$

so the delta and gamma of the option equal

$$\Delta(t,x) \equiv \frac{\partial f(t,x)}{\partial x} = \frac{1}{2\pi}\int_{\mathbb{R}}(\alpha-i\omega)\,e^{(\alpha-i\omega)x}\hat{f}_\alpha(\tau,\omega)\,d\omega \tag{16}$$

$$\Gamma(t,x) \equiv \frac{\partial^2 f(t,x)}{\partial x^2} = \frac{1}{2\pi}\int_{\mathbb{R}}(\alpha-i\omega)^2\,e^{(\alpha-i\omega)x}\hat{f}_\alpha(\tau,\omega)\,d\omega \tag{17}$$

Because the expressions involve partial derivatives with respect to only x, the results in this work are applicable for the computation of Δ and Γ through a modification of the payoff function:

$$\hat{g}_{\alpha,\Delta}(\omega) = \hat{g}_\alpha(\omega)(\alpha-i\omega) \tag{18}$$

$$\hat{g}_{\alpha,\Gamma}(\omega) = \hat{g}_\alpha(\omega)(\alpha-i\omega)^2 \tag{19}$$

When the frequency space payoff function manifests exponential decay, the introduction of a coefficient that is polynomial in ω does not change the regularity of $\hat{g}$ in a way that would significantly change the following analysis. Last, we note that since we do our analysis for PIDEs on a mesh of xs, one may also compute the option values in one go and obtain the Greeks with little additional effort using a finite difference approach for the derivatives.

3. Error bound

The aim of this section is to compute a bound of the error when approximating the option price $f(\tau, x)$ by $f_{\alpha,\Delta\omega,n}(\tau, x)$, defined in (13). Considering

$$f_{\alpha,\Delta\omega}(\tau, x) = \frac{\Delta\omega}{2\pi} \sum_{k\in\mathbf{Z}} e^{-i\left(k+\frac{1}{2}\right)\Delta\omega x} \hat{f}_\alpha\left(\tau, \left(k + \frac{1}{2}\right)\Delta\omega\right) \tag{20}$$

the total error $\mathcal{E}$ can be split into a sum of two terms: the quadrature and truncation errors. The former is the error from the approximation of the integral in (11) by the infinite sum in (20), while the latter is due to the truncation of the infinite sum. We have

$$\mathcal{E} := |f(\tau, x) - f_{\Delta\omega,n}(\tau, x)| \leq \mathcal{E}_{\mathcal{Q}} + \mathcal{E}_{\mathcal{F}} \tag{21}$$

with

$$\begin{aligned}\mathcal{E}_{\mathcal{Q}} &= e^{\alpha x} \left| f_\alpha(\tau, x) - f_{\alpha,\Delta\omega}(\tau, x) \right| \\ \mathcal{E}_{\mathcal{F}} &= e^{\alpha x} \left| f_{\alpha,\Delta\omega}(\tau, x) - f_{\alpha,\Delta\omega,n}(\tau, x) \right|\end{aligned}$$

Observe that each $\mathcal{E}$, $\mathcal{E}_{\mathcal{Q}}$ and $\mathcal{E}_{\mathcal{F}}$ depend on three kinds of parameters:

- Parameters underlying the model and payoff such as volatility and strike price. We call these *physical parameters.*
- Parameters relating to the numerical scheme such as α and n.
- *Auxiliary parameters* that will be introduced in the process of deriving the error bound. These parameters do not enter the computation of the option price, but they need to be chosen optimally to have as tight a bound as possible.

We start by analysing the quadrature error.

3.1. **Quadrature error.**

Denote by A_a, with $a > 0$, the strip of width $2a$ around the real line:

$$A_a \equiv \{z \in \mathbb{C} \colon\ |\mathrm{Im}\,[z]| < a\}$$

The following theorem presents conditions under which the quadrature error goes to zero at a spectral rate as $\Delta\omega$ goes to zero. Later in this section, we discuss simpler conditions to verify the hypotheses and analyse in more detail the case when the process X is a diffusive process or there are "enough small jumps."

Theorem 3.1. *Assume that for $a \geq 0$*

H1. *the characteristic function of the random variable X_1 has an analytic extension to the set*

$$A_a - \alpha i \equiv \{z \in \mathbb{C} \colon\ |\mathrm{Im}\,[z] + \alpha| < a\}$$

H2. *the Fourier transform of $g_\alpha(x)$ is analytic in the strip A_a and*

H3. $\lim_{\omega\to+\infty}\sup_{\beta\in(-a,a)}\left|\hat{f}_\alpha(\tau,\omega+i\beta)\right|=0.$

Consider $M_{\alpha,a}(\tau,x)$ *defined by*

$$M_{\alpha,a}(\tau,x)=\sup_{\beta\in(-a,a)}\int_{\mathbb{R}}\left|e^{-i(\omega+i\beta)x}\hat{f}_\alpha(\tau,\omega+i\beta)\right|d\omega \tag{22}$$

and assume $M_{\alpha,a}(\tau,x)<\infty$*. Then the quadrature error is bounded by*

$$\mathcal{E}_{\mathcal{Q}}\le e^{\alpha x}\frac{M_{\alpha,a}(\tau,x)}{\pi\left(e^{2\pi a/\Delta\omega}-1\right)}$$

The proof of Theorem 3.1 is an application of Theorem 6.1 in Trefethen and Weideman (2013) that we include for ease of reading.

Lemma 3.2 (Theorem 6.1 in Trefethen and Weideman (2013))**.** *Suppose* w *is analytic in the strip* $|\mathrm{Im}\,[x]|<a$ *for some* $a>0$*. Suppose further that* $w(x)\to 0$ *uniformly as* $|x|\to\infty$ *in the strip, and for some* M *it satisfies*

$$\int_{-\infty}^{\infty}|w(x+ib)|\,dx\le M$$

for all $b\in(-a,a)$*. Then, for any* $h>0$*,* I_h *defined by*

$$I_h=h\sum_{k=-\infty}^{\infty}w(kh)$$

exists and satisfies

$$\left|I_h-\int_{-\infty}^{\infty}w(x)dx\right|\le\frac{2M}{e^{2\pi a/h}-1}$$

and the constant $2M$ *is as small as possible.*

Proof of Theorem 3.1. Call $w(z)=e^{-i\left(z+\frac{1}{2}\Delta\omega\right)x}\hat{f}_\alpha\left(\tau,z+\frac{1}{2}\Delta\omega\right)$. By H1 and H2 we know that $z\to e^{-izx}\hat{f}_\alpha(\tau,z)$ is analytic in the strip A_a. By H3 we know that

$$e^{-izx}\hat{f}_\alpha(\tau,z)\to 0$$

uniformly when $|z|\to\infty$ in A. Observe that the previous assertions remain valid for $z\mapsto w(z)$, so we can apply Lemma 3.2 to $w(z)$, with $h=\Delta\omega$ and $M=M_{\alpha,a}(\tau,x)$. Observe also that

$$\int_{\mathbb{R}}e^{-izx}\hat{f}_\alpha(\tau,z)dz=\int_{\mathbb{R}}w(z)dz$$

which allows us to express the quadrature error as

$$\frac{e^{\alpha x}}{2\pi}\left|\Delta\omega\sum_{k\in\mathbf{Z}}w(k\Delta\omega)-\int_{\mathbb{R}}w(z)dz\right| \tag{23}$$

which is $\frac{e^{\alpha x}}{2\pi}\left|I_{\Delta\omega}-I\right|$, in the notation of Lemma 3.2. The proof is completed by substituting the bound provided by Lemma 3.2 in the previous expression. ☐

Regarding the hypotheses of Theorem 3.1, a simpler condition that implies H1 is given in the following proposition, while a practical way of verifying H2 is to check that for all $b < a$, the function $x \mapsto e^{b|x|} g_\alpha(x)$ is in $L^2(\mathbb{R})$ (see Theorem IX.13 in Reed and Simon (1975)).

Proposition 3.3. *If α, a and ν are such that*

$$\int_{y>1} e^{(\alpha+a)y}\nu(dy) < \infty \quad \text{and} \quad \int_{y<-1} e^{(\alpha-a)y}\nu(dy) < \infty \tag{24}$$

then H1 in Theorem 3.1 is fulfilled.

Proof. Denoting by $\varphi_1(\cdot)$ the characteristic function of X_1, we want to prove that $z \mapsto \varphi_1(z+\alpha i)$ is analytic in A_a. Considering that $\varphi_1(z+\alpha i) = e^{\Psi(iz-\alpha)}$, the only non-trivial part of the proof is to verify that

$$z \mapsto \int p(z,y)\nu(dy) \tag{25}$$

is analytic in A_a, where $p\colon A_a \times \mathbb{R} \to \mathbb{C}$ is given by

$$p(z,y) = e^{y(iz-\alpha)} - 1 - (e^y - 1)(iz - \alpha)$$

To prove this, we show that we can apply the main result and the only theorem in Mattner (2001), which, given a measure space $(\Omega, \mathcal{A}, \mu)$ and an open subset $G \subseteq \mathbb{C}$, ensures the analyticity of $\int f(\cdot,\omega)d\mu(\omega)$, provided that $f\colon G\times\Omega \to \mathbb{C}$ satisfies: $f(z,\cdot)$ is $\mathcal{A}$-measurable for all $z \in G$; $f(\cdot,\omega)$ is holomorfic for all $\omega \in \Omega$; and $\int |f(\cdot,\omega)|\, d\mu(\omega)$ is locally bounded. In our case we consider the measure space to be $\mathbb{R}$ with the Borel σ-algebra and the Lebesgue measure, $G = A_a$ and $f = p$. It is clear that $p(x,\cdot)$ is Borel measurable and $p(\cdot,y)$ is holomorphic. It remains to verify that

$$z \mapsto \int_{\mathbb{R}^*} |p(z,y)|\, \nu(dy)$$

is locally bounded. To this end, we assume that $\mathrm{Re}\,[z] < b$ (and, since $z \in A_a$, $\mathrm{Im}\,[z] < a$) and split the integration domain in $|y| > 1$ and $0 < |y| \le 1$ to prove that both integrals are uniformly bounded.

Regarding the integral in $|y| > 1$, we observe that

$$|p(z,y)| \le e^{y(\alpha+\mathrm{Im}[z])} + 1 + (e^y + 1)(\alpha + a + b) \tag{26}$$

for $y < -1$ we have $e^{y(\alpha+\mathrm{Im}[z])} < e^{y(\alpha-a)}$ while for $y > 1$ we have $e^{y(\alpha+\mathrm{Im}[z])} < e^{y(\alpha+a)}$. Using the previous bounds and the hypotheses together with (1) and (3), we obtain the needed bound.

For the integral in $0 < |y| \le 1$, observe that, denoting $f(z,y) = |p(z,y)|$, we have, $f(z,0) = 0$ for every z, $\partial_y f(z,0) = 0$ for every z, and $|\partial_{yy} f(z,y)| < c$ for $z \in A_a, \mathrm{Re}\,[z] < b, |y| < 1$. From these observations we get that the McLaurin polynomial of degree one of $y \mapsto f(z,y)$ is null for every z, and we can bound $f(z,y)$ by the remainder term, which, in our region of interest, is bounded by $\frac{c}{2}y^2$, obtaining

$$\int_{0<|y|\le 1} |p(z,y)|\, \nu(dy) \le \frac{c}{2}\int_{0<|y|\le 1} y^2 \nu(dy) \tag{27}$$

which is finite by hypothesis on ν, which finishes the proof. $\square$

We now turn our attention to a more restricted class of Levy processes. Namely, processes such that either $\sigma^2 > 0$ or there exists $\lambda \in (0,2)$ such that $C(\lambda)$ defined in (28) is strictly positive. For this class of processes, we can state our main result explicitly in terms of the characteristic triplet.

Given $\lambda \in (0,2)$, define $C(\lambda)$ as

$$C(\lambda) = \inf_{\kappa>1} \left\{ \kappa^\lambda \int_{0<|y|<\frac{1}{\kappa}} y^2 \nu(dy) \right\} \tag{28}$$

Observe that $C(\lambda) \geq 0$ and, by our assumptions on the jump measure ν, $C(\lambda)$ is finite. Furthermore, if $\lambda \in (0,2)$ is such that

$$\liminf_{\epsilon \downarrow 0} \frac{1}{\epsilon^\lambda} \int_{0<|y|<\epsilon} y^2 \nu(\mathrm{d}y) > 0 \tag{29}$$

then $C(\lambda) > 0$. To see this, observe that (29) implies the existence of ϵ_0 such that

$$\inf_{\epsilon \leq \epsilon_0} \left\{ \frac{1}{\epsilon^\lambda} \int_{0<|y|<\epsilon} y^2 \nu(\mathrm{d}y) \right\} > 0$$

If $\epsilon_0 < 1$ observe that

$$\inf_{\epsilon_0 \leq \epsilon \leq 1} \left\{ \frac{1}{\epsilon^\lambda} \int_{0<|y|<\epsilon} y^2 \nu(\mathrm{d}y) \right\} \geq \int_{0<|y|<\epsilon_0} y^2 \nu(\mathrm{d}y) > 0$$

where for the first inequality it was taken into account that $\frac{1}{\epsilon^\lambda} \geq 1$ and that the integral is increasing with ϵ. Combining the two previous infima and considering $|\kappa| = \frac{1}{\epsilon}$ we get that $C(\lambda) > 0$.

Furthermore, we note that for a Lévy model with finite jump intensity, such as the Black-Scholes and Merton models that satisfy the first of our assumption, $C(\lambda) = 0$ identically.

Proposition 3.4. *Assuming that α and a are such that* (24) *holds and $\hat{g_\alpha} \in L^\infty_{A_a}$, if either $\sigma^2 > 0$ or $C(\lambda) > 0$ for some $\lambda \in (0,2)$, then $M_{\alpha,a}(\tau,x)$ is finite for every $\tau > 0$ and for every x. Furthermore, we have*

$$M_{\alpha,a}(\tau,x) \leq e^{a|x|} e^{\tau \sup_{|\beta-\alpha|<a}\{\Psi(\beta)\}} \|\hat{g_\alpha}\|_{L^\infty_{A_a}} \int_{\mathbb{R}} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2 + \frac{|\omega|^{2-\lambda}}{4} C(\lambda) \mathbf{1}_{|\omega|>1}\right)} d\omega \tag{30}$$

Proof. Considering $h_{\alpha,a}(\tau,x,\omega)$ defined by

$$h_{\alpha,a}(\tau,x,\omega) = \sup_{\beta \in (-a,a)} \left| e^{-i(\omega+i\beta)x} \hat{f}_\alpha(\tau, \omega + i\beta) \right| \tag{31}$$

we have that

$$M_{\alpha,a}(\tau,x) \leq \int_{\mathbb{R}} h_{\alpha,a}(\tau,x,\omega)\, d\omega$$

On the other hand

$$\left| e^{-i(\omega+i\beta)x} \hat{f}_\alpha(\tau, \omega + i\beta) \right| = e^{\beta x} \left| \hat{f}_\alpha(\tau, \omega + i\beta) \right| = e^{\beta x} \left| e^{\tau \Psi(\alpha+\beta-i\omega)} \right| \left| \hat{g_\alpha}(\omega + i\beta) \right| \tag{32}$$

Considering $\beta \in (-a, a)$, the first factor in the right-hand side can be bounded by $e^{a|x|}$, while the third one is bounded by assumption. For the second factor we have

$$\left| e^{\tau \Psi(\alpha+\beta-i\omega)} \right| = e^{\tau \mathrm{Re}[\Psi(\alpha+\beta-i\omega)]} \tag{33}$$

Now, observe that

$$\begin{aligned}\mathrm{Re}\left[\Psi\left(\alpha+\beta-i\omega\right)\right] &= \left(\alpha+\beta\right)\left(r-\frac{\sigma^2}{2}\right)+\frac{\sigma^2}{2}\left(\left(\alpha+\beta\right)^2-\omega^2\right) \\ &+ \int_{\mathbb{R}\setminus\{0\}}\left(e^{(\alpha+\beta)y}\cos(-y\omega)-1-\left(\alpha+\beta\right)\left(e^y-1\right)\right)\nu(dy)\end{aligned} \tag{34}$$

If $|\omega| \le 1$ we bound $\cos(-y\omega)$ by 1, getting

$$\begin{aligned}\mathrm{Re}\left[\Psi\left(\alpha+\beta-i\omega\right)\right] &\le \left(\alpha+\beta\right)\left(r-\frac{\sigma^2}{2}\right)+\frac{\sigma^2}{2}\left(\left(\alpha+\beta\right)^2-\omega^2\right) \\ &\quad + \int_{\mathbb{R}\setminus\{0\}}\left(e^{(\alpha+\beta)y}-1-\left(\alpha+\beta\right)\left(e^y-1\right)\right)\nu(dy) \\ &= \Psi\left(\alpha+\beta\right)-\frac{\sigma^2}{2}\omega^2\end{aligned} \tag{35}$$

Assume $|\omega| > 1$. Using that for $|x| < 1$ it holds that $\cos(x) < 1 - x^2/4$, we can bound the first term of the integral in the following manner:

$$\begin{aligned}\int_{\mathbb{R}\setminus\{0\}} e^{(\alpha+\beta)y}\cos(y\omega)\nu(dy) &\le \int_{0<|y|<1/|\omega|} e^{(\alpha+\beta)y}\left(1-\omega^2y^2/4\right)\nu(dy) \\ &\quad + \int_{|y|\ge 1/|\omega|} e^{(\alpha+\beta)y}\nu(dy) \\ &\le \int_{\mathbb{R}\setminus\{0\}} e^{(\alpha+\beta)y}\nu(dy) - \frac{|\omega|^{2-\lambda}}{4}|\omega|^{\lambda}\int_{0<|y|<1/|\omega|} y^2\nu(dy) \\ &\le \int_{\mathbb{R}\setminus\{0\}} e^{(\alpha+\beta)y}\nu(dy) - \frac{|\omega|^{2-\lambda}}{4}C\left(\lambda\right)\end{aligned} \tag{36}$$

Inserting (36) back into (34) we get

$$\begin{aligned}\mathrm{Re}\left[\Psi\left(\alpha+\beta-i\omega\right)\right] &\le \left(\alpha+\beta\right)\left(r-\frac{\sigma^2}{2}\right)+\frac{\sigma^2}{2}\left(\left(\alpha+\beta\right)^2-\omega^2\right) \\ &\quad + \int_{\mathbb{R}\setminus\{0\}}\left(e^{(\alpha+\beta)y}-1-\left(\alpha+\beta\right)\left(e^y-1\right)\right)\nu(dy) \\ &\quad - \frac{|\omega|^{2-\lambda}}{4}C\left(\lambda\right) \\ &= \Psi\left(\alpha+\beta\right)-\frac{\sigma^2}{2}\omega^2-\frac{|\omega|^{2-\lambda}}{4}C\left(\lambda\right)\end{aligned}$$

Taking the previous considerations into account we get

$$h_{\alpha,a}(\tau,x,\omega) \le e^{a|x|}e^{\tau\left(\sup_{|\beta-\alpha|<a}\{\Psi(\beta)\}-\frac{\sigma^2}{2}\omega^2-\frac{|\omega|^{2-\lambda}}{4}C(\lambda)\mathbf{1}_{|\omega|>1}\right)}\|\hat{g_\alpha}\|_{L^\infty_{A_a}}$$

Finally, integrating in $\mathbb{R}$ with respect to ω, we obtain (30). It just remains to observe that if either $\sigma^2 > 0$ or $C(\lambda) > 0$, then the integral in (30) is convergent, so $M_{\alpha,a}(\tau, x)$ is finite. □

The following corollary provides a bound when $\sigma > 0$. It should be noted that if $C(\lambda) > 0$, then the bound in Proposition 3.4 is better than the one in Corollary 3.5.

Corollary 3.5. *Under the same conditions as in the previous proposition, if $\sigma > 0$ then*

$$M_{\alpha,a}(\tau, x) \leq e^{a|x|} e^{\tau \sup_{|\beta-\alpha|<a}\{\Psi(\beta)\}} \|\hat{g_\alpha}\|_{L^\infty_{A_a}} \frac{\sqrt{2\pi}}{\sigma\sqrt{\tau}} \tag{37}$$

Proof. We can apply Proposition 3.4 for any $\lambda \in (0, 2)$. Using the fact that $C(\lambda) \geq 0$, and bounding it by 0, we conclude that the bound obtained in (30) is less than or equal to

$$e^{a|x|} e^{\tau \sup_{|\beta-\alpha|<a}\{\Psi(\beta)\}} \|\hat{g_\alpha}\|_{L^\infty_{A_a}} \int_{\mathbb{R}} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2\right)} d\omega$$

The proof is completed by evaluating the integral. □

Remark 3.6. In the case of call options, hypothesis H2 implies a dependence between the strip-width parameter a and damping parameter α. We have that the damped payoff of the call option is in $L_1(\mathbb{R})$ if and only if $\alpha > 1$ and hence the appropriate choice of strip-width parameter is given by $0 < a < \alpha - 1$. A similar argument holds for the case of put options, for which $\alpha < 0$. In such case, $a < -\alpha$.

The case of binary options whose payoff has finite support $(G(x) = \mathbf{1}_{[a,b]}(x))$ we can set any $a \in \mathbb{R}$ (ie, no damping is needed at all and even if such damping is chosen, it has no effect on the appropriate choice of a).

Remark 3.7. The bound we provide for the quadrature error is naturally positive and increasing in $\Delta\omega$. It decays to zero at a spectral rate as $\Delta\omega$ decreases to 0.

3.2. Frequency truncation error.

The frequency truncation error is given by

$$\mathcal{E}_{\mathcal{F}} = \frac{e^{\alpha x}\Delta\omega}{\pi} \left| \sum_{k=n}^{\infty} \operatorname{Re}\left[e^{-i\left(k+\frac{1}{2}\right)\Delta\omega x} \hat{f}_\alpha\left(\tau, \left(k+\frac{1}{2}\right)\Delta\omega\right)\right]\right|$$

If a function $c\colon (\omega_0, \infty) \to (0, \infty)$ satisfies

$$\left|\operatorname{Re}\left[e^{-i\left(k+\frac{1}{2}\right)\Delta\omega x} \hat{f}_\alpha\left(\tau, \left(k+\frac{1}{2}\right)\Delta\omega\right)\right]\right| \leq c\left(\left(k+\frac{1}{2}\right)\Delta\omega\right) \tag{38}$$

for every natural number k, then we have that

$$\begin{aligned}\mathcal{E}_{\mathcal{F}} &\leq \frac{e^{\alpha x}\Delta\omega}{\pi} \sum_{k=n}^{\infty} \left|\operatorname{Re}\left[e^{-i\left(k+\frac{1}{2}\right)\Delta\omega x} \hat{f}_\alpha\left(\tau, \left(k+\frac{1}{2}\right)\Delta\omega\right)\right]\right| \\ &\leq \frac{e^{\alpha x}\Delta\omega}{\pi} \sum_{k=n}^{\infty} c\left(\left(k+\frac{1}{2}\right)\Delta\omega\right)\end{aligned}$$

Furthermore, if c is a non-increasing concave integrable function, we get

$$\mathcal{E}_{\mathcal{F}} \leq \frac{e^{\alpha x}}{\pi} \int_{n\Delta\omega}^{\infty} c(\omega) \mathrm{d}\omega \tag{39}$$

When $\hat{g_\alpha} \in L^\infty_{[\omega_0,\infty)}$ and either $\sigma^2 > 0$ or $C(\lambda) > 0$, then the function c in (38) can be chosen as

$$c(\omega) = \|\hat{g_\alpha}\|_{L^\infty_{[\omega_0,\infty)}} e^{\tau\Psi(\alpha)} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2 + \frac{|\omega|^{2-\lambda}}{4}C(\lambda)\mathbf{1}_{|\omega|>1}\right)} \tag{40}$$

To prove that this function satisfies (38) we can use the same bound we found in the proof of Proposition 3.4, with $\beta = 0$ to obtain

$$\mathrm{Re}\left[\Psi(\alpha - i\omega)\right] \le \Psi(\alpha) - \frac{\sigma^2}{2}\omega^2 - \frac{|\omega|^{2-\lambda}}{4} C(\lambda)\,\mathbf{1}_{|\omega|>1}$$

from where the result is straightforward.

3.3. **Bound for the full error.** In this section we summarize the bounds obtained for the error under different assumptions and analyse their central properties.

In general the bound provided in this paper are of the form

$$\overline{\mathcal{E}} = \frac{e^{\alpha x}}{\pi}\left(\frac{\bar{M}}{e^{2\pi a/\Delta\omega} - 1} + \int_{n\Delta\omega}^\infty c(\omega)\mathrm{d}\omega\right) \tag{41}$$

where $\bar{M}$ is an upper bound of $M_{\alpha,a}(\tau,x)$ defined in (22) and c is non-increasing, integrable and satisfies (38). Both $\bar{M}$ and c may depend on the parameters of the model and the artificial parameters, but they are independent of $\Delta\omega$ and n. Typically one can remove the dependence of some of the parameters, simplifying the expressions but obtaining less tight bounds.

When analysing the behaviour of the bound one can observe that the term correspondent with the quadrature error decreases to zero spectrally when $\Delta\omega$ goes to 0. The second term goes to zero if $n\Delta\omega$ diverges, but we are unable to determine the rate of convergence without further assumptions.

Once an expression for the error bound is obtained, the problem of how to choose the parameters of the numerical method to minimise the bound arises, assuming a constraint on the computational effort one is willing to use. The computational effort of the numerical method depends on n. For this reason we aim at finding the parameters that minimise the bound for a fixed n. The following result shows that the bound obtained, as a function of $\Delta\omega$, has a unique local minimum, which is the global minimum.

Proposition 3.8. *Fix α, a, n, and λ and consider the bound $\overline{\mathcal{E}}$ as a function of $\Delta\omega$. There exists an optimal $\Delta\omega^* \in [\frac{\omega_0}{n},\infty)$ such that $\overline{\mathcal{E}}$ is decreasing in $(\frac{\omega_0}{n},\Delta\omega^*)$ and increasing in $(\Delta\omega^*,\infty)$; thus, a global minimum of $\overline{\mathcal{E}}$ is attained at $\Delta\omega^*$.*

Furthermore, the optimal $\Delta\omega$ is either the only point in which $\Delta\omega \mapsto p(n\Delta\omega, b) - c(n\Delta\omega)$, with p defined in (42), changes sign, or $\Delta\omega = \frac{\omega_0}{n}$ if $p(\omega_0, b) - c(\omega_0) > 0$.

Proof. Let us simplify the notation by calling $y = n\Delta\omega$, $b = 2\pi a n$ and $\tilde{\mathcal{E}} = \pi e^{-\alpha x}\overline{\mathcal{E}}$. We want to prove the existence of $y^*\colon y^* \ge \omega_0$ such that $\tilde{\mathcal{E}}(y)$ is decreasing for $\omega_0 < y < y^*$ and increasing for $y > y^*$. We have

$$\tilde{\mathcal{E}}(y) = \frac{\bar{M}}{e^{b/y} - 1} + \int_y^\infty c(\omega)\mathrm{d}\omega.$$

The first term is differentiable with respect to y and goes to 0 if $y \to 0^+$. This allows us to express it as an integral of its derivative. We can then express $\tilde{\mathcal{E}}(y)$ as

$$\tilde{\mathcal{E}}(y) = \tilde{\mathcal{E}}(\omega_0) + \int_{\omega_0}^{y} \left(\frac{b\bar{M}e^{b/\omega}}{\left(e^{b/\omega}-1\right)^2 \omega^2} - c(\omega) \right) d(\omega)$$

The first term on the right-hand side of the previous equation is constant. Now we move on to proving that the integrand is increasing with y and it is positive if y is large enough. Denote by

$$p(y,b) = \frac{b\bar{M}e^{b/y}}{\left(e^{b/y}-1\right)^2 y^2} \tag{42}$$

Taking into account that c is integrable, we can compute the limit of the integrand in ∞, obtaining

$$\lim_{y\to+\infty} p(y,b) - c(y) = \frac{\bar{M}}{b} > 0$$

Let us prove that $p(y,b)$ is increasing with y for all $b > 0$, which renders $p(y,b) - c(y)$ also increasing with y. The derivative of p with respect to y is given by

$$\partial_y p(y,b) = \frac{b\bar{M}e^{b/y}\left((b/y)e^{b/y} - 2e^{b/y} + b/y + 2\right)}{y^3\left(e^{b/y}-1\right)^3}$$

in which the denominator and the first factor in the numerator are clearly positive. To prove that the remainder factor is also positive, observe that $xe^x - 2e^x + x + 2 > 0$ if $x > 0$.

□

3.4. **Explicit error bounds.** In the particular case when either $\sigma^2 > 0$ or $C(\lambda) > 0$ for some $\lambda \in (0,2)$ we can give an explicit version of (41). Substituting M by the bound obtained in Proposition 3.4 and c by the function given in (40) we obtain

$$\overline{\mathcal{E}} = \overline{\mathcal{E}}_Q + \overline{\mathcal{E}}_F \tag{43}$$

where

$$\overline{\mathcal{E}}_Q = e^{\alpha x}\frac{e^{a|x|}e^{\tau \sup_{|\beta-\alpha|<a}\{\Psi(\beta)\}}}{\pi\left(e^{2\pi a/\Delta\omega}-1\right)} \|\hat{g_\alpha}\|_{L^\infty_{A_a}} \int_{\mathbb{R}} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2 + \frac{|\omega|^{2-\lambda}}{4}C(\lambda)\mathbf{1}_{|\omega|>1}\right)} d\omega \tag{44}$$

$$\overline{\mathcal{E}}_F = \frac{e^{\alpha x}}{\pi}\|\hat{g_\alpha}\|_{L^\infty_{\mathbb{R}}} e^{\tau\Psi(\alpha)} \int_{n\Delta\omega}^{\infty} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2 + \frac{|\omega|^{2-\lambda}}{4}C(\lambda)\mathbf{1}_{|\omega|>1}\right)} d\omega \tag{45}$$

Remark 3.9. Observe that the bound of both the quadrature and cutoff error is given by a product of one factor that depends exclusively on the payoff and another factor that depends on the asset dynamic. This property makes it easy to evaluate the bound for a specific option under different dynamics of the asset price. In Subsection 4.3 we analyse the terms that depend on the payoff function for the particular case of call options.

Remark 3.10. The bound of the cutoff error $\overline{\mathcal{E}}_F$ can be improved by substituting $\|\hat{g_\alpha}\|_{L^\infty_{\mathbb{R}}}$ by $\|\hat{g_\alpha}\|_{L^\infty_{[n\Delta\omega,\infty)}}$.

The integrals in (44) and (45) can, in some cases, be computed analytically, or bounded from above by a closed form expression Consider for instance dissipative models with finite jump intensity. These models are characterised by $\sigma^2 > 0$ and $C(\lambda) = 0$. Thus the integrals can be expressed in terms of the cumulative normal distribution Φ:

$$\int_{\mathbb{R}} e^{-\tau \frac{\sigma^2 \omega^2}{2}} d\omega = \sqrt{\frac{2\pi}{\tau\sigma^2}}, \tag{46}$$

$$\int_{\varsigma}^{\infty} e^{-\tau \frac{\sigma^2 \omega^2}{2}} d\omega = \sqrt{\frac{2\pi}{\tau\sigma^2}} \left(1 - \Phi\left(\varsigma\sqrt{\tau\sigma^2}\right)\right) \tag{47}$$

Now we consider the case of pure-jump processes (ie, $\sigma^2 = 0$) that satisfy the condition $C(\lambda) > 0$ for some $\lambda \in (0, 2)$. In this case the integrals are expressible in terms of the incomplete gamma function γ. First, let us define the auxiliary integral:

$$I(a, b) \equiv e^{-a} + a^{-\frac{1}{b}} \gamma\left(\frac{1}{b}, a\right)$$

for $a, b > 0$. Using this, the integrals become

$$\int_{\mathbb{R}} e^{-\tau \frac{|\omega|^{2-\lambda}}{4} C(\lambda) \mathbf{1}_{|\omega|>1}} = 2\left(1 + I\left(\frac{\tau C(\lambda)}{4}, 2 - \lambda\right)\right) \tag{48}$$

$$\int_{\varsigma}^{\infty} e^{-\tau \frac{|\omega|^{2-\lambda}}{4} C(\lambda) \mathbf{1}_{|\omega|>1}} = \begin{cases} I\left(\frac{\tau C(\lambda)}{4}, 2 - \lambda\right) + 1 - \varsigma & \varsigma < 1 \\ \varsigma I\left(\frac{\tau \varsigma^{2-\lambda} C(\lambda)}{4}, 2 - \lambda\right) & \varsigma \geq 1 \end{cases} \tag{49}$$

An example of a process for which the previous analysis works is the CGMY model presented in Carr et al. (2002, 2003), for the regime $Y > 0$.

Lastly, when both $C(\lambda)$ and σ^2 are positive, the integrals in (44) and (45) can be bounded by a simpler expression. Consider the two following auxiliary bounds for the same integral, in which $\varsigma \geq 1$:

$$\begin{aligned} \int_{\varsigma}^{\infty} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2 + \frac{|\omega|^{2-\lambda}}{4} C(\lambda)\right)} d\omega &\leq e^{-\tau \frac{\sigma^2}{2} \varsigma^2} \int_{\varsigma}^{\infty} e^{-\tau \frac{|\omega|^{2-\lambda}}{4} C(\lambda)} d\omega \\ &= \varsigma e^{-\tau \frac{\sigma^2}{2} \varsigma^2} I\left(\frac{\tau \varsigma^{2-\lambda} C(\lambda)}{4}, 2 - \lambda\right) \end{aligned} \tag{50}$$

$$\begin{aligned} \int_{\varsigma}^{\infty} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2 + \frac{|\omega|^{2-\lambda}}{4} C(\lambda)\right)} d\omega &\leq e^{-\tau \frac{\varsigma^{2-\lambda}}{4} C(\lambda)} \int_{\varsigma}^{\infty} e^{-\tau \frac{\sigma^2}{2} \omega^2} d\omega \\ &= \sqrt{\frac{2\pi}{\tau\sigma^2}} e^{-\tau \frac{\varsigma^{2-\lambda}}{4} C(\lambda)} \left(1 - \Phi(\varsigma\sqrt{\tau\sigma^2})\right) \end{aligned} \tag{51}$$

We have that $b(\varsigma)$, defined as the minimum of the right-hand sides of the two previous equations,

$$b(\varsigma) = \min\left\{\varsigma e^{-\tau \frac{\sigma^2}{2} \varsigma^2} I\left(\frac{\tau \varsigma^{2-\lambda} C(\lambda)}{4}, 2 - \lambda\right), \sqrt{\frac{2\pi}{\tau\sigma^2}} e^{-\tau \frac{\varsigma^{2-\lambda}}{4} C(\lambda)} \left(1 - \Phi\left(\varsigma\sqrt{\tau\sigma^2}\right)\right)\right\}$$

		$\arg\min \overline{\mathcal{E}}$			$\Pi(\tau, 0)$	$\min \overline{\mathcal{E}}$	Lee's bound
12τ	n	α	a	$\Delta\omega$			
4	8	20.4	18.0	10.4	2.8992	0.0031	0.0055
1	32	21.8	17.9	10.9	1.2678	0.0058	0.0058

Table 1. The error bound for at-the-money options for the VG model and the parameters relevant to evaluating the bound compared to the results by Lee.

is a bound for the integral. Bearing this in mind we have

$$\int_{\mathbb{R}} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2+\frac{|\omega|^{2-\lambda}}{4}C(\lambda)\mathbf{1}_{|\omega|>1}\right)} d\omega \leq 2\Phi\left(\sqrt{\tau\sigma^2}\right)-1+2b(1) \tag{52}$$

and

$$\int_{\varsigma}^{\infty} e^{-\tau\left(\frac{\sigma^2}{2}\omega^2+\frac{|\omega|^{2-\lambda}}{4}C(\lambda)\mathbf{1}_{|\omega|>1}\right)} d\omega \leq b(\varsigma) \tag{53}$$

provided that $\varsigma \geq 1$.

4. Computation and minimization of the bound

In this section, we present numerical examples on the bound presented in the previous section using practical models known from the literature. We gauge the tightness of the bound compared to the true error using both dissipative and pure-jump processes. We also demonstrate the feasibility of using the bound expression as a tool for setting numerical parameters for the Fourier inversion.

4.1. **Call option in variance gamma model.** The variance gamma model provides a test case to evaluate the bound in the pure-jump setting. We note that of the two numerical examples presented, it is the less regular one in the sense that $\sigma^2 = 0$ and $C(\lambda) = 0$ for $0 < \lambda < 2$, indicating that Proposition 3.4 in particular is not applicable.

The Lévy measure of the VG model is given by:

$$\nu_{\mathrm{VG}}(dy) = dy\left(\mathbf{1}_{y>0}\frac{Ke^{\eta_+}}{x} - \mathbf{1}_{y<0}\frac{Ke^{\eta_-}}{x}\right) \tag{54}$$

By Proposition 3.3 we get that

$$a < \min\{\eta_- - \alpha, \eta_+ + \alpha\} \tag{55}$$

which, combined with the requirement that $g_\alpha \in L_1(\mathbb{R})$ (cf, Remark 3.6), implies that

$$\begin{aligned} a &< \min\{\eta_+ - \alpha, \eta_- + \alpha, \alpha - 1\} \\ a &< \min\{\eta_+ - \alpha, \eta_- + \alpha, -\alpha\} \end{aligned} \tag{56}$$

for calls and puts, respectively. We note that evaluation of the integral in (11) is also possible for $\alpha \in (0, 1)$, but the result is the option price plus a constant given by residue calculus, which is not presented here.

In Lee (2004) and in our calculations the parameters equal $\eta_+ = 39.7840$, $\eta_- = 20.2648$ and $K = 5.9311$.

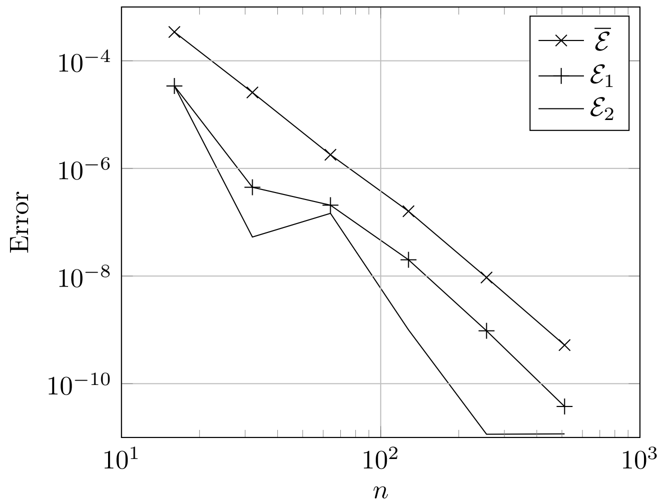

FIGURE 1. The true error and the error bound for evaluating at the money options for the VG model test case.

Table 1 presents the specific parameters and compares the bound for the VG model with the results obtained by Lee (2004). Based on the table, we note that for the VG model presented in Madan et al. (1998) we can achieve comparable or better error bounds when compared to the study by Lee.

To evaluate the bound, we perform the integration of (22) and (39) by compactifying the integration domain using a change of variables and evaluating the definite integral with Gauss quadrature. To supplement Table 1 for a wide range of n, we present the magnitude of the bound compared to the true error in Figure 1.

In Figure 1, we see that the choice of numerical parameters for the Fourier inversion has a strong influence on the error of the numerical method. One does not in general have access to the true solution, thus the parameters need to be optimised with respect to the bound. Recall that $\mathcal{E} = \mathcal{E}(\alpha, \Delta\omega, a, n)$ and $\overline{\mathcal{E}} = \overline{\mathcal{E}}(\alpha, \Delta\omega, n)$ denote the true and estimated errors, respectively. Keeping the number of quadrature points n fixed, we let $(\alpha_1, \Delta\omega_1, a_1)$ and $(\alpha_2, \Delta\omega_2)$ denote the minimisers of the estimated and true errors, respectively

$$(\alpha_1, \Delta\omega_1, a_1) = \arg\inf \overline{\mathcal{E}} \tag{57}$$

$$(\alpha_2, \Delta\omega_2) = \arg\inf \mathcal{E} \tag{58}$$

We further let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the true error as a function of the parameters minimising the estimated and the true error, respectively

$$\mathcal{E}_1 = \mathcal{E}\left(\alpha_1, \Delta\omega_1\right) \tag{59}$$

$$\mathcal{E}_2 = \mathcal{E}\left(\alpha_2, \Delta\omega_2\right) \tag{60}$$

In Figure 1 we see that the true error increases by approximately an order of magnitude when optimising to the bound instead of to the true error, translating into a two-fold difference in the number of quadrature points needed for a given tolerance. The difference

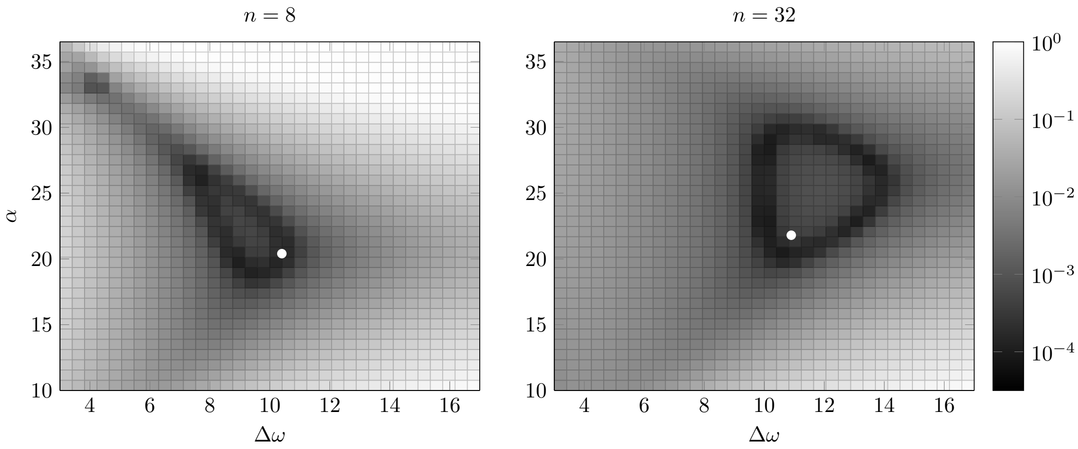

FIGURE 2. The true error $\mathcal{E}$ for the two VG test cases presented in Table 1 and the bound-minimising configurations (white circle) $(\alpha_2, \Delta\omega_2)$ for the examples.

between $\mathcal{E}_1$ and the bound is approximately another order of magnitude and necessitates another two-fold number of quadrature points compared to the theoretical minimum.

In Figure 2, we present the true error [1] for the Fourier method for the two test cases in Table 1. We note that while minimising error bounds will produce sub-optimal results, the numerical parameters that minimise the bound are a good approximation of the true optimal parameters. This, of course, is a consequence of the error bound having qualitatively similar behaviour as the true error, especially as one gets further away from the true optimal parameters.

4.2. **The binary option in the Merton model.** For dissipative models, we may employ a fast semi-closed form evaluation of the relevant integrals instead of resorting to quadrature methods. We choose the Merton model as an example of bounding the error of the numerical method for such a model. The parameters are adopted from the estimated parameters for S&P 500 Index from Andersen and Andreasen (2000).

In Figure 3, we present the bound and true error for the Merton model to demonstrate the bound on a dissipative model. The option presented is a binary option with finite support on $[95, 105]$; no damping was needed or used. We note that like in the case of the pure-jump module presented above, our bound reproduces the qualitative behaviour of the true error. The configuration resulting from optimising the bound is a good approximation of the true error. Such behaviour is consistent through the range of n of the most practical relevance.

4.3. **The call option.** In Subsection 3.4 explicit expressions to bound $\mathcal{E}$ are provided. To evaluate these bounds it is necessary to compute $\|\hat{g_\alpha}\|_{L^\infty_{\mathbb{R}}}$ and $\|\hat{g_\alpha}\|_{L^\infty_{A_a}}$. According to

[1] The reference value to compute the true error was obtained by the numerical methods with n and $\Delta\omega$ such that the level of accuracy is of the order 10^{-10}.

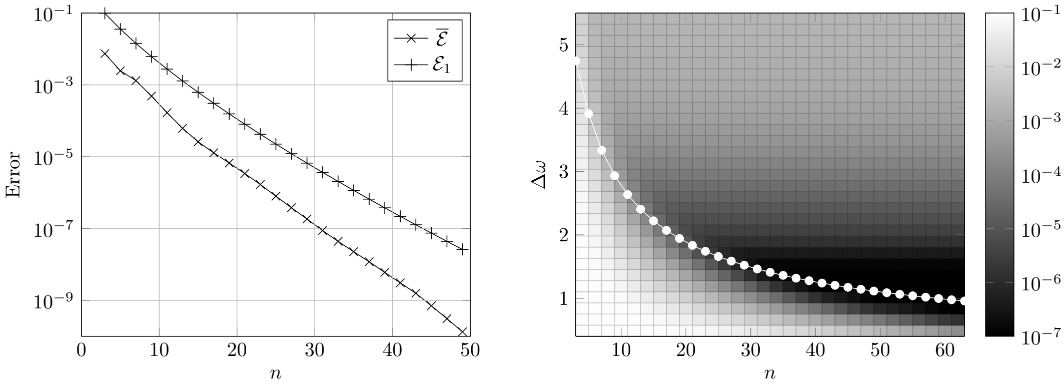

FIGURE 3. The true error $\mathcal{E}_1$ and the bound $\overline{\mathcal{E}}$ for the dissipative Merton test case for a range of number of quadrature points n, along with the bound-minimising configurations contrasted to the true error.

Remark 3.9, once we compute these values we could use them for any model, provided that they satisfy the conditions considered there.

The payoff of perhaps the most practical relevance is that of a call option. Consider g defined by:

$$g(x) = (S_0 e^x - K)^+ = S_0 \left(e^x - e^k\right)^+$$

for which a selection of a damping parameter $\alpha > 0$ is necessary to have the damped payoff in $L_1(\mathbb{R})$ and to ensure the existence of a Fourier transformation. In this case we have

$$\hat{g_\alpha}(\omega) = S_0 \int_{\mathbb{R}} \exp\left((1-\alpha+i\omega)x\right) - \exp\left(k + (i\omega - \alpha)x\right) dx \tag{61}$$

$$= \frac{S_0 \exp\left((1-\alpha+i\omega)k\right)}{(1+i\omega-\alpha)(i\omega-\alpha)} \tag{62}$$

and

$$|\hat{g_\alpha}(\omega)|^2 = \frac{S_0^2 e^{2(1-\alpha)k}}{(\alpha^2+\omega^2)\left((1-\alpha)^2+\omega^2\right)} \tag{63}$$

It is easy to see that the previous expression decreases as $|\omega|$ increase. This yields

$$\|\hat{g_\alpha}\|_{L^\infty_{\mathbb{R}}} = |\hat{g_\alpha}(0)| = \frac{S_0 e^{(1-\alpha)k}}{\alpha^2 - \alpha} \tag{64}$$

and

$$\|\hat{g_\alpha}\|_{L^\infty_{[\varsigma,\infty)}} = |\hat{g_\alpha}(\varsigma)| \tag{65}$$

The maximisation of $|\hat{g_\alpha}|$ in the strip A_a of the complex plane is more subtle. Denoting $\hat{g_\alpha}(\eta,\rho) = \hat{g_\alpha}(\eta + i\rho)$, we look for critical points that satisfy $\partial_\eta |\hat{g_\alpha}| = 0$. This gives

$$4\eta^3 + 2\eta\left(4\rho\alpha + 2\alpha^2 - 2\rho - 2\alpha + \rho^2 + 1\right) = 0. \tag{66}$$

For ρ fixed, $|\hat{g_\alpha}|$ has a vanishing derivative with respect to η at a maximum of three points. Of the three roots of the derivative, only the one characterised by $\eta = 0$ is a local maximum, giving us that for call options

$$\|\hat{g_\alpha}\|_{L^\infty_{A_a}} = \sup_{\rho\in[-a,a]} |\hat{g_\alpha}(0,\rho)| \tag{67}$$

Now, observe that $|\hat{g_\alpha}(0,\rho)|$ is a differentiable real function of ρ, whose derivative is given by the following polynomial of second degree:

$$p(\rho) \equiv k\left(\rho + \alpha - 2\rho\alpha - \alpha^2 - \rho^2\right) - 2\alpha - 2\rho + 1 \tag{68}$$

We conclude that

$$\|\hat{g_\alpha}\|_{L^\infty_{A_a}} = \max_{\rho\in B} \{|\hat{g_\alpha}(0,\rho)|\} \tag{69}$$

where B is the set of no more than four elements consisting of a; $-a$; and the real roots of p that fall in $(-a,a)$.

5. Conclusion

We have presented a decomposition of the error committed in the numerical evaluation of the inverse Fourier transform needed in asset pricing for exponential Lévy models into truncation and quadrature errors. For a wide class of Lévy models, we have presented an L_∞-bound for the error.

The error bound differs from the earlier work presented in Lee (2004) in the sense that it does not rely on the asymptotic behaviour of the option payoff at extreme strikes or option prices. The structure of the bound allows for a modular implementation that decomposes the error components arising from the dynamics of the system and the payoff into a product form for a large class of models, including all dissipative models. On select examples, we also demonstrate the performance that is comparable or superior to the earlier bounds.

We have shown that the bound works as a proxy for objective function when minimising the numerical error, and that the numerical parameters found by minimising the error bound approximate well the parameters that minimise the true numerical error.

The bound can be used in the primitive setting of establishing a strict error bound for the numerical estimation of option prices for a given set of physical and numerical parameters or as a part of a numerical scheme, whereby the end user wishes to estimate an option price either on a single point or in a domain up to a predetermined error tolerance.

In the future, the error bounds presented can be used in efforts requiring multiple evaluations of Fourier transformations. Examples of such applications include multidimensional Fourier transformations, possibly in sparse tensor grids, as well as time-stepping algorithms for American and Bermudan options. Such applications are sensitive towards the error bound being used, as any numerical scheme will be required to run multiple times, either in high dimension or for multiple time steps (or both).

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7. Declarations of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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