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CHANGES OF NUMÉRAIRE, CHANGES OF PROBABILITY MEASURE AND OPTION PRICING

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Abstract

The use of the risk-neutral probability measure has proved to be very powerful for computing the prices of contingent claims in the context of complete markets, or the prices of redundant securities when the assumption of complete markets is relaxed. We show here that many other probability measures can be defined in the same way to solve different asset-pricing problems, in particular option pricing. Moreover, these probability measure changes are in fact associated with numéraire changes; this feature, besides providing a financial interpretation, permits efficient selection of the numéraire appropriate for the pricing of a given contingent claim and also permits exhibition of the hedging portfolio, which is in many respects more important than the valuation itself.

The key theorem of general numéraire change is illustrated by many examples, among which the extension to a stochastic interest rates framework of the Margrabe formula, Geske formula, etc.

PROBABILITY MEASURE CHANGES; MARTINGALES; PRICES RELATIVE TO A NUMÉRAIRE; HEDGING PORTFOLIO; FORWARD VOLATILITY

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1. Introduction

One of the most popular technical tools for computing asset prices is the so-called 'risk-adjusted probability measure'. Elaborating on an initial idea of Arrow, Ross (1978) and Harrison and Kreps (1979) have shown that the absence of arbitrage opportunities implies the existence of a probability measure Q, such that the current price of any basic security is equal to the Q-expectation of its discounted future payments. In particular, between two payment dates, the discounted price of any security is a Q-martingale. When markets are complete, i.e. when enough non-redundant securities are being traded, Q is unique.

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By using a very simple technical argument (Theorem 1, Section 2) we prove that many other probability measures can be defined in a similar way, and prove equally useful in various kinds of option pricing problems. More specifically, if $X(t)$ is the price process of a non-dividend-paying security (at least in the relevant time period), our main theorem states the existence of a probability measure Q_X such that the price of any security S relative to the numèraire X is a Q_X-martingale. A very general numéraire change formula is then provided and different applications to exchange options and options on options in a stochastic interest rates environment, options on bonds, etc. illustrate the efficiency of the right choice of numéraire. Some of the results in the paper may be found more or less explicitly in the existing literature. Our goal is to emphasize the generality and the efficiency of the numéraire change methodology.

2. The model and the crucial theorem

We consider a stochastic intertemporal economy, where uncertainty is represented by a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$. The only role of the probability $\boldsymbol{P}$ is in fact to define the negligible sets. Most of our applications will be taken in a continuous-time framework, within a bounded time interval $[0, T]$ but our basic argument is also valid for a discrete-time economy.

We will not completely specify the underlying assumptions on the economy. The flow of information accruing to all the agents in the economy is represented by a filtration $(\mathscr{F}_t)t \in [0, T]$, satisfying 'the usual hypotheses', i.e. the filtration $(\mathscr{F}_t)_{0 \leqq t \leqq \infty}$ is right continuous and $\mathscr{F}_0$ contains all the $\boldsymbol{P}$-null sets of $\mathscr{F}$.

In the following, the word 'asset' represents a general financial instrument. We distinguish two classes of assets. One class consists of the basic securities, which are traded on the markets and are the components of the portfolio defined below. The other class of assets to be considered is the class of derivative securities, also called contingent claims, for which the key issues are the valuation and hedging. All asset price processes are continuous $\mathscr{F}_t$-semimartingales. The prices $S_1(t), \cdots, S_n(t)$ of the basic securities are observed on the financial markets and almost surely strictly positive for all t; more generally, unless otherwise specified, the price of any asset is almost surely positive.

The fundamental concept in the pricing or hedging of contingent claims is the self-financing replicating portfolio, and these self-financing portfolios consequently deserve particular attention (buy and hold portfolios are the simplest example of self-financing portfolios since there is no trade). More generally, these portfolios track the target changes over time with no addition of money.

The financial value $V(t)$ of a portfolio which includes the quantities $w_1(t), \cdots, w_n(t)$ of the assets $1, 2, \cdots, n$ is given by

$$(1) \qquad V(t) = \sum_{k=1}^{n} w_k(t) S_k(t) \quad \text{and} \quad V(t) \geqq 0 \quad \text{for all } t$$

where the processes $(w_1(t))_{t \geqq 0}, \cdots, (w_n(t))_{t \geqq 0}$ are adapted, i.e. the quantities $w_1(t), \cdots, w_n(t)$ are chosen according to the information available at time t. The vector process $(w_1(t)), \cdots, (w_n(t))_{t \geqq 0}$ is called the portfolio strategy.

Definition 1. The portfolio is called self-financing if the vector stochastic integral $\int_0^T \Sigma_{k=1}^n w_k(t)dS_k(t)$ exists and

$$dV(t) = \sum_{k=1}^{n} w_k(t)dS_k(t). \tag{2}$$

Remark. To understand the intuition behind Equation (2), let us take the example of a simple strategy, i.e. a strategy rebalanced only at fixed dates $0 = t_0 < t_1 < \cdots < t_n$ and that we suppose left continuous. The self-financing equation can then be written as

$$V(t_j) - V(t_{j-1}) = \sum_{k=1}^{n} w_k(t_{j-1}^{+})[S_k(t_j) - S_k(t_{j-1})].$$

By definition,

$$V(t_j) = \sum_{k=1}^{n} w_k(t_j)S_k(t_j).$$

The self-financing condition is $V(t_j) = V(t_j^+)$ for all j or in other terms,

$$\sum_{k=1}^{n} w_k(t_j)S_k(t_j) = \sum_{k=1}^{n} w_k(t_j^+)S_k(t_j). \tag{3}$$

Using (3) at time t_{j-1} and remembering that $w_k(t_j) = w_k(t_{j-1}^+)$, the self-financing condition can also be written as

$$V(t_j) - V(t_{j-1}) = \int_{]t_{j-1}, t_j]} \sum_{k=1}^{n} w_k(u)dS_k(u).$$

More generally, the change of the portfolio value between any dates $t < t'$ is

$$V(t') - V(t) = \int_t^{t'} \sum_{k=1}^{n} w_k(u)dS_k(u).$$

For non-elementary strategies, this will be the definition of self-financing strategies. We have not emphasized so far the fact that there was an implicit numéraire behind the prices $S_1, S_2, \cdots, S_n$; it is the numéraire relevant for domestic transactions at time t and obviously plays a particular role. Our objective is to show that other quantities may be chosen as numéraires and that, for a given problem, there is a 'best' numéraire.

Definition 2. A numéraire is a price process $X(t)$ almost surely strictly positive for each $t \in [0, T]$.

Proposition 1. Self-financing portfolios remain self-financing after a numéraire change.

Proof. This property is straightforward from a financial viewpoint. Mathematically, it is also clear that Equation (3) still holds after a numéraire change. Let X be a new numéraire. From Itô's lemma, we derive

$$d\left(\frac{S_k(t)}{X(t)}\right)=S_k(t)d\left(\frac{1}{X(t)}\right)+\frac{1}{X(t)}dS_k(t)+d\langle S_k,1/X\rangle_t$$

where $d\langle S_k, 1/X\rangle$ denotes the instantaneous covariance between the semimartingales S_k and $1/X$. In the same manner

$$d\left(\frac{V(t)}{X(t)}\right)=V(t)d\left(\frac{1}{X(t)}\right)+\frac{1}{X(t)}dV(t)+d\langle V,1/X\rangle_t.$$

The self-financing condition

$$dV(t)=\sum_{k=1}^{n}w_k(t)dS_k(t)\quad\text{and}\quad V(t)=\sum_{k=1}^{n}w_k(t)S_k(t)$$

implies that

$$\begin{aligned}d\left(\frac{V(t)}{X(t)}\right)&=\sum_{k=1}^{n}w_k(t)\left\{S_k(t)d\left(\frac{1}{X(t)}\right)+\frac{1}{X(t)}dS_k(1)+d\langle S_k,1/X\rangle_t\right\}\\&=\sum_{k=1}^{n}w_k(t)d\left(\frac{S_k(t)}{X(t)}\right)\end{aligned}$$

and the portfolio expressed in the new numéraire remains self-financing.

Corollary and Definition 3

(a) A contingent claim (i.e. a random cash-flow H paid at time T) is called attainable if there exists a self-financing portfolio whose terminal value equals $H(T)$.

(b) If a contingent claim is attainable in a given numéraire, it is also attainable in any other numéraire and the replicating strategy is the same.

This property is immediately derived from Proposition 1.

The pricing methodology developed in the paper follows Harrison and Kreps (1979) and Harrison and Pliska (1981) in the no arbitrage assumption: for every self-financing portfolio V belonging to a particular class of portfolios, $V(0)=0$ and $V(T)\geqq 0$ almost surely imply $V(T)=0$.

If Ω is finite as well as the set of transaction dates, there is no restriction on the class of portfolios and the no arbitrage assumption is equivalent to the existence of a 'risk-neutral probability measure' (see Harrison and Kreps (1979), Harrison and Pliska (1981)). In our setting, as observed by Duffie and Huang (1985), this equivalence no longer holds and some requirements have to be put on the portfolios: the natural one involves square integrability conditions of the weights of the portfolio with respect to the instantaneous variance–covariance matrix of the basic assets. Delbaen and Schachermayer (1992) introduce a weaker formulation of the no arbitrage assumption, the no free lunch with vanishing risk (NFLVR), which only requires portfolios bounded below. The NFLVR condition is necessary and sufficient to exhibit a (local) martingale measure.

The former condition is clearly not invariant in a numéraire change. The latter one is if the lower bound is zero, hence the condition $V(t)\geqq 0$ for all t that we introduced earlier and which will remain valid throughout the paper unless otherwise specified. More precisely, our no arbitrage assumption will be expressed in the following manner.

Assumption 1. There exists a non-dividend-paying asset $n(t)$ and a probability π equivalent to the initial probability P such that for any basic security S_k without intermediate payments, the price of S_k relative to n, i.e. $S_k(t)/n(t)$ is a local martingale with respect to π.

By convention, we will take $n(0) = 1$.

Observations

- Portfolios themselves expressed in this numéraire will be, by definition, π-local martingales.
- Moreover, if they are positive for all t, portfolios are supermartingales, i.e.

$$\frac{V(t)}{n(t)} \geqq E_\pi\left[\frac{V(T)}{n(T)} \,\middle|\, \mathscr{F}_t\right] \quad \text{almost surely.}$$

- If the terminal value $V(T)/n(T)$ is square integrable, i.e. $E_\pi[(V(T)/n(T))^2]$ is finite, then the portfolio value is a π-martingale and

$$\frac{V(t)}{n(t)} = E_\pi\left[\frac{V(T)}{n(T)} \,\middle|\, \mathscr{F}_t\right].$$

- The consequence is that if a contingent claim H is attainable and its terminal value in the numéraire n is π-square integrable, then all replicating portfolios have the same value at any intermediary date t. This value is the price at time t of the contingent claim.
- In the general case (relaxing the assumption of square integrability), all replicating (positive) portfolios do not necessarily have the same value at any date t (see Dudley (1977), developed in Karatzas and Shreve (1988)) for the non-unicity of these portfolios but all these values are bounded below by $E_\pi[H(T)/n(T) \mid \mathscr{F}_t]$.

Moreover, if there exists one replicating portfolio whose value at any time t is equal to this expectation, this value will be called the price (or *fair price*) of the contingent claim (with respect to (n, π)) and the corresponding portfolio called the hedging portfolio. Keeping the same numéraire n, if there exists another probability π' satisfying the same replicability property, then

$$E_{\pi'}\left[\frac{H(T)}{n(T)} \,\middle|\, \mathscr{F}_t\right] = E_\pi\left[\frac{H(T)}{n(T)} \,\middle|\, \mathscr{F}_t\right].$$

Hence, the fair price does not depend on the choice of the 'risk-neutral' probability measure π (as long as this price exists); this remark is very important for the main purpose of this paper, namely the choice of the optimal numéraire when pricing and hedging a given contingent claim.

We now give an example of a situation where all contingent claims have a fair price. Suppose that the prices $S_1(t), S_2(t), \cdots, S_n(t)$ of the basic securities expressed in the numéraire n are stochastic integrals with respect to q Brownian motions $W_1, \cdots, W_q$ and that the filtration $\mathscr{F}_t$ is generated by these Brownian motions. Since $S_1, S_2, \cdots, S_n$ are local martingales, their dynamics under π are driven by the stochastic differential equations

$$\frac{dS_i}{S_i} = \sum_{j=1}^{q} \sigma_{ij}\, dW_j, \qquad i = 1, \cdots, n.$$

If we assume that the matrix $\Sigma = [\sigma_{ij}]$ is invertible (which obviously implies $q = n$), then the martingale representation allows us to express any conditional expectation as a stochastic integral with respect to the basic asset prices, hence as a portfolio.

The same property holds if the number n of basic securities is greater than the number of Brownian motions (and rank $\Sigma = q$).

Theorem 1. Let $X(t)$ be a non-dividend paying numéraire such that $X(t)$ is a π-martingale. Then there exists a probability measure Q_X defined by its Radon–Nikodym derivative with respect to π

$$\left.\frac{dQ_X}{d\pi}\right|\mathscr{F}_T = \frac{X(T)}{X(0)n(T)}$$

such that

(i) the basic securities prices are Q_X-local martingales,

(ii) if a contingent claim H has a fair price under (n, π), then it has a fair price under (X, Q_X) and the hedging portfolio is the same.

Proof

(i) If we denote by $\tilde{S} = (S(t)/X(t))$ the relative price of a security S with respect to the numéraire X, the conditional expectations formula gives

$$\boldsymbol{E}_\pi\left(\frac{dQ_X}{d\pi}\tilde{S}(T) \mid \mathscr{F}_t\right) = \boldsymbol{E}_{Q_x}[\tilde{S}(T) \mid \mathscr{F}_t]\boldsymbol{E}_\pi\left[\frac{dQ_X}{d\pi} \;\middle|\; \mathscr{F}_t\right].$$

By Assumption 1, we have

$$\frac{S(t)}{n(t)X(0)} = \boldsymbol{E}_\pi\left[\frac{dQ_X}{d\pi}\tilde{S}(T) \mid \mathscr{F}_t\right]$$

and similarly

$$\frac{X(t)}{n(t)X(0)} = \boldsymbol{E}_\pi\left[\frac{dQ_X}{d\pi} \;\middle|\; \mathscr{F}_t\right].$$

This gives the martingale property $\tilde{S}(t)$ under Q_X, and consequently for any portfolio.

(ii) If H has a fair price under (n, π), $\boldsymbol{E}_\pi[H(T)/n(T) \mid \mathscr{F}_t]$ is a self-financing portfolio. Since

$$\boldsymbol{E}_{Q_x}\left[\frac{H(T)}{X(T)} \;\middle|\; \mathscr{F}_t\right] = \boldsymbol{E}_\pi\left[\frac{H(T)}{n(T)} \;\middle|\; \mathscr{F}_t\right] \Big/ \frac{X(t)}{n(t)}$$

and we observed earlier that the property of being a self-financing portfolio is invariant through a numéraire change, $\boldsymbol{E}_{Q_x}[H(T)/n(T) \mid \mathscr{F}_t]$ is also a self-financing theorem and Theorem 1 holds.

From now on, we will concentrate on the changes of numéraire techniques.

Corollary 2. If X and Y are two arbitrary securities, the general numéraire change formula can be written at any time $t < T$ as

$$X(0)E_{Q_X}[Y(T)\Phi \mid \mathscr{F}_t] = Y(0)E_{Q_Y}[X(T)\Phi \mid \mathscr{F}_t]$$

where Φ is any random cash flow $\mathscr{F}_T$-measurable.

Proof. The formula can be immediately derived from Theorem 1, which entails

$$\left.\frac{dQ_X}{dQ_Y}\right|_{\mathscr{F}_T} = \frac{dQ_X}{d\pi}\frac{1}{dQ_Y/d\pi} = \frac{X(T)}{X(0)n(T)} \cdot \frac{1}{Y(T)/Y(0)n(T)} = \frac{X(T)/Y(T)}{X(0)/Y(0)}\,.$$

We will show later on in the paper how the choice of an appropriate numéraire permits us to simplify pricing and hedging problems.

We start by giving two examples of such numéraire changes already encountered in the literature. We want to emphasize the fact that in both cases, the important message is the financial suitability of the chosen numéraire to a given problem; the probability changes that follow are useful technically but also convey an economic interpretation.

Example 1: *The money market account as a numéraire*. It is natural to take as a first example of numéraire the riskless asset (assuming it exists). More precisely, we define $\beta(t)$ (also called the accumulation factor) as the value at date t of a fund created by investing one dollar at time 0 on the money market and continuously reinvested at the (instantaneously riskless) instantaneous interest rate $r(t)$. The interest rate process is denoted by $(r_t)_{t\geq 0}$. At this point we need a technical assumption.

Assumption 2. For almost all ω, $t \to r_t(\omega)$ is strictly positive and continuous and r_t is an $\mathscr{F}_t$-measurable process on $(\Omega, \mathscr{F}, \boldsymbol{P})$. Under this assumption, it is clear that

$$\beta(t) = \exp \int_0^t r(s)ds.$$

Then the relative price $\tilde{S}(t)$ of a security with respect to the numéraire β is simply its discounted price

$$\tilde{S}(t) = \left[\exp - \int_0^t r(s)ds\right] S(t).$$

The probability measure Q_β is the usual 'risk-neutral' probability measure Q defined by

$$\frac{dQ}{d\pi} = \frac{1}{n(T)} \exp \int_0^T r(s)ds.$$

'Historically' (see Harrison and Pliska (1981)), $Q = \pi$ was the first 'risk-neutral' probability measure (associated with the numéraire β) expressing that discounted asset prices are Q-martingales.

Example 2: *Zero-coupon bonds as numéraires*. A zero-coupon bond imposes itself as the numéraire when one looks at the price at time t of an asset giving right to a single cash-flow at a well-defined future time T. Keeping in mind the general martingale property of Theorem 1, the right numéraire to introduce, whether interest rates are stochastic or not, is the zero-coupon bond maturing at time T. Let us make explicit the corresponding probability measure change.

The price process of the bond will be denoted either by $B(t, T)$ or by $B_T(t)$,

$$B_T(t) = \boldsymbol{E}_Q\left[\exp - \int_t^T r(s)ds \mid \mathcal{F}_t\right]$$

where Q is the probability defined in Example 1.

Corollary 2 of Theorem 1 gives

$$\frac{dQ_T}{dQ} = \frac{1}{\beta(T)} \frac{\beta(0)}{B(0, T)} = \frac{1}{B(0, T)} \exp - \left[\int_0^T r(s)dt\right].$$

The relative price $S(t)/B(t, T)$ is precisely the forward price $F_S(t)$ of the security S and from Theorem 1 we get

$$F_S(t) = \boldsymbol{E}_{Q_T}\left[\frac{S(T)}{B(T, T)} \;\middle|\; \mathcal{F}_t\right].$$

In other words, the forward price, relative to time T, of a security which pays no dividend up to time T is equal to the expectation of the value at time T of this security under the 'forward neutral' probability.

The financial intuition of this result can be found in Bick (1988) and Merton (1973); the mathematical treatment was developed in the general case of stochastic interest rates by Geman (1989) and in a Gaussian interest rate framework by Jamshidian (1989). Even if this change of numéraire depends on time T and is not as universal as the 'accumulating factor' presented in Example 1, it turns out to be the right numéraire when evaluating a future random cash-flow in a stochastic interest rates environment. Besides its applications to option pricing presented in the next section, this numéraire change gives remarkable results in the pricing of floating-rate notes and of interest rate swaps, as shown in El Karoui and Geman (1991), (1994).

3. Applications to options

In this section we focus on finding interesting expressions for option prices rather than discussing their existence. Consequently, we will suppose that all options considered in the following are attainable assets. We gave earlier an example of a situation where this property holds; we must observe, though, that in many situations weaker assumptions suffice. The best example is the classical Black and Scholes framework where no assumption of completeness is necessary since it is easy to replicate the conditional expectation of the terminal pay-off by a portfolio of the riskless asset and the risky asset, hence to derive a fair price for the European call.

3.1. *A general formula.* Let us consider a call written on a security whose price dynamics $S(t)$ does not require any other specification than the fact of being a positive semimartingale.

Theorem 2. Under Assumptions 1 and 2, and denoting by T and K respectively its maturity and exercise price, the price at time 0 of the call can be written as

$$\frac{C(0)}{B(0,T)} = E_{Q_T}\left[\left(\frac{S(T)}{B(T,T)} - K\right)^+\right]$$

or

$$C(0) = S(0)Q_S(A) - KB(0,T)Q_T(A)$$

where $A = \{\omega \mid S(T,\omega) > KB(T,T)\}$.

The first expression of $C(0)$ is immediately derived from Theorem 1 used in the context of Example 2. We will prove the second one:

$$\frac{C(0)}{B(0,T)} = E_{Q_T}[(S(T)-K)^+] = E_{Q_T}\left(\frac{S(T)}{B(T,T)}\,\mathbf{1}_A\right) - KQ_T(A).$$

From the general numéraire change formula (Corollary 2 of Theorem 1), we get

$$E_{Q_T}\left(\frac{S(T)}{B(T,T)}\,\mathbf{1}_A\right) = \frac{S(0)}{B(0,T)}E_{Q_S}(\mathbf{1}_A).$$

We thus obtain the second expression of $C(0)$.

Corollary 3. In the same way, the option of exchanging asset 2 against asset 1 at time T gives right to the cash-flow $[S_1(T) - KS_2(T)]^+$ with $K = 1$ and its price $C(0)$ at time 0 is such that

$$\frac{C(0)}{S_2(0)} = E_{Q_{S_2}}\left\{\left[\frac{S_1(T)}{S_2(T)} - K\right]^+\right\}$$

or

$$C(0) = S_1(0)Q_{S_2}(A) - KS_2(0)Q_{S_1}(A)$$

where $A = \{\omega \in \Omega \mid S_1(T,\omega) \geqq KS_2(T,\omega)\}$.

This formula holds even when risky asset volatilities and interest rates are stochastic.

Corollary 4. More generally, an option which gives right to the payment at time T of the quantity $(\sum_{k=1}^n \lambda_k X_k(T))^+$, where $\lambda_1, \cdots, \lambda_n$ are any real numbers, $X_1, \cdots, X_n$ are risky assets and possibly $X_1(T) = K$ (the usual strike price of the option), has a value at time 0 which can be written as

$$C(0) = \sum_{k=1}^{n} \lambda_k X_k(0) Q_{X_k}(A).$$

Obviously, this is the situation encountered with options on bonds.

3.2. *Applications of Theorem* 2

(a) *A reexamination of the Black and Scholes' formula.* We will make the usual assumptions of the Black and Scholes' model, except that we will allow interest rates and risky asset volatility to be stochastic. Theorem 2 entails the following call price:

$$C(0) = S(0)Q_S(A) - KB(0, T)Q_T(A).$$

The asset involved in the second term is in fact the forward price of S,

$$F^S(t) = \frac{S(t)}{B(t, T)},$$

which is a positive martingale under Q_T and can therefore be written as a stochastic integral of a Brownian process:

$$\frac{dF^S(t)}{F^S(t)} = \sigma_{F_S}(t)dW_t^{F_S}$$

where $(\sigma_{F_S})^2 = (1/dt)\mathrm{Var}(dF^S/F^S)$.

Assuming that σ_{F_S} is deterministic, $Q_T(A)$ is equal to $\Pr(u \geqq 0)$, where u is a Gaussian variable with mean $\ln(S(0)/KB(0, T)) - \frac{1}{2}\sigma_{F_S}T$ and variance $\sigma_{F_S}T$. Consequently, $Q_T(A) = N(d_2)$ with

$$d_2 = \frac{1}{\sigma_{F_S}\sqrt{T}}\left\{\ln\frac{S(0)}{KB(0, T)} - \tfrac{1}{2}\sigma_{F_S}T\right\}.$$

The first term in the Theorem 2 formula involves the asset

$$\frac{B(t, T)}{S(t)} = \frac{1}{F^S(t)} = Z^T(t).$$

Whether stochastic or not, the volatility of Z^T under Q_S is the same as the volatility of F^S under Q_T (with possibly different Brownian processes). Assuming these volatilities deterministic,

$$Q_S(A) = Q_S\left(Z^T(T) \leqq \frac{1}{K}\right) = Q_S\left(Z(0)\exp\left(\sigma_{F_S}W_T - \frac{\sigma_{F_S}}{2}T\right) \leqq \frac{1}{K}\right)$$

$$Q_S(A) = Q_S\left(\sigma_{F_S}W_T - \frac{\sigma_{F_S}}{2}T \leqq \ln\frac{1}{KZ(0)}\right)$$

$$= N\left[\frac{1}{\sigma_{F_S}\sqrt{T}}\ln\frac{S(0)}{KB((0, T))} + \frac{1}{2}\sigma_{F_S}\sqrt{T}\right] = N(d_1).$$

In fact we obtain the Merton formula, and, if we define r by $B(0, T) = e^{-rT}$, it becomes the Black and Scholes' formula.

We have thus shown that these two formulae hold under the sole hypothesis of a volatility for the forward contract $S(t)/B(t, T)$, without any necessary specification on the asset price or interest rates.

Obviously, in the Black and Scholes' framework, interest rates are assumed constant and the hypothesis of a deterministic volatility of the forward contract is equivalent to the hypothesis of a deterministic volatility of the stock price.

(b) *Application to the exchange option*. In this case, there is no 'forward contract' of S_1 with respect to S_2. Consequently we need to specify the movement of the two asset prices under the risk neutral probability:

$$\frac{dX_1}{X_1} = rdt + \sigma_1 dW_1, \qquad \frac{dX_2}{X_2} = rdt + \sigma_2 dW_2$$

where σ_1 and σ_2 are not supposed deterministic and $\langle dW_1, dW_2 \rangle = rdt$.

Consequently, the volatility of X_1/X_2 is equal to $\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ and, for the same reasons as earlier, we see that Margrabe's formula (1978) holds under the sole hypothesis of a deterministic volatility for X_1/X_2, without summing non-stochastic interest rates.

We observe that the same methodology could be applied to the pricing of equity-linked foreign exchange options also called quanto options (see Reiner (1992)).

(c) *Application to hedging*. From the calculations conducted in (a), we see that in the general situation of stochastic interest rates, the right way of hedging should not be read in the Black and Scholes' formula but in the Merton formula

$$C(0) = S(0)N(d_1) - KB(0, T)N(d_2).$$

From this, we can derive very symmetrically the quantity $N(d_1)$ to invest in the risky asset and the quantity $N(d_2)$ to invest in K zero-coupon bonds maturing at time T.

Practitioners who use these weights to hedge the option with the underlying asset and money market instruments implicitly assume non-stochastic interest rates. Moreover, it is clear that if interest rates are stochastic, what is usually denominated as the 'implied volatility' of the asset is in fact the implied volatility of the forward contract.

If interest rates are stochastic and if one wants to hedge the option with the underlying asset and short-term bills, it is necessary to assume that the same Brownian motion perturbates the movement of the risky asset and the one of the zero-coupon bond maturing at time T, namely that under the risk neutral probability, we have the following dynamics:

$$\frac{dS}{S} = rdt + \sigma_1 dW, \qquad \frac{dB^T}{B^T} = rdt + \sigma_2 dW$$

from which we derive

$$\frac{dB^T}{B^T} = r\left(1 - \frac{\sigma_2}{\sigma_1}\right) dt + \frac{\sigma_2}{\sigma_1}\frac{dS}{S}.$$

Consequently, we see that the quantity to hold short in the risky asset in order to hedge the option is not $N(d_1)$ but in fact

$$\frac{\partial C}{\partial S}+\frac{\sigma_2}{\sigma_1}\frac{B(t,T)}{S(t)}\frac{\partial C}{\partial B},$$

i.e.

$$N(d_1)-K\frac{\sigma_2}{\sigma_1}\frac{B(t,T)}{S(t)}N(d_2).$$

The number of risky stocks involved in the self-financing portfolio replicating the European call is not $N(d_1)$, but the partial derivative of the Black and Scholes price with respect to the underlying stock and well-known as the delta of the call.

The classical 'Δ hedging' is correct under stochastic interest rates only if the hedging portfolio involves, besides the risky stock, the zero-coupon bond maturing at time T.

(d) *Application to compound options.* We now extend the pricing formula given by Geske (1979) for a compound option but without assuming deterministic interest rates. This involves, besides the risky stock, the zero-coupon. Let $C_1(t, S)$ be the price at date t of a European call option on the stock, with strike price K_1 and exercise data T_1; $C_2(t, S)$ be the price at date t of a European call option on C_1, with strike price K_2 and $T_2 < T_1$; $A_1=\{\omega\in\Omega \mid S(T_1,\omega)\geqq K_1\}$ be the exercise set of option C_1; and $A_2=\{\omega\in\Omega \mid S(T_2,\omega)\geqq S^*\}$ be the exercise set of option C_2, where S^* is defined implicitly by $C_1(T_2, S^*)=K_2$.

In the same spirit as earlier, we write

$$C_2(0)=B(0,T_2)E_{Q_{T_2}}\{[C_1(T_2,S(T_2))-K_2]^+\}$$

$$C_2(0)=-K_2B(0,T_2)Q_{T_2}(A_2)+B(0,T_2)E_{Q_{T_2}}[C_1(T_2,S(T_2))\mathbf{1}_{A_2}].$$

Making explicit $C_1(T_2, S(T_2))$, we get

$$\begin{aligned}&B(0,T_2)\boldsymbol{E}_{Q_{T_2}}[C_1(T_2,S(T_2)\mathbf{1}_{A_2}]\\&\quad=B(0,T_2)\boldsymbol{E}_{Q_{T_2}}[\mathbf{1}_{A_2}B(T_2,T_1)\boldsymbol{E}_{Q_{T_1}}\{(S(T_1)-K_1)^+\}\mid\mathcal{F}_{T_2}].\end{aligned}$$

Taking in Corollary 1 the asset X as the zero-coupon bond maturing at time T_1, Y as the zero-coupon bond maturing at time T_2 and $T=T_2$, we rewrite this expression as

$$B(0,T_1)\boldsymbol{E}_{Q_{T_1}}[\mathbf{1}_{A_2}E_{Q_{T_1}}\{(S(T_1)-K_1)^+\mid\mathcal{F}_{T_2}\}]$$

or

$$B(0,T_1)\boldsymbol{E}_{Q_{T_1}}[\mathbf{1}_{A_2}S(T_1)\mathbf{1}_{A_1}]-K_1B(0,T_1)Q_{T_1}(A_1\cap A_2).$$

Using again the change of numéraire formula, we obtain

$$B(0,T_1)E_{Q_{T_1}}[S(T_1)\mathbf{1}_{A_1\cap A_2}]=S(0)Q_S(A_1\cap A_2).$$

Regrouping the different terms, we write the price of the compound option as

$$C_2(0)=S(0)Q_S(A_1\cap A_2)-K_1B(0,T_1)Q_{T_1}(A_1\cap A_2)-K_2B(0,T_2)Q_{T_2}(A_2).$$

This formula does not assume interest rates and stock price volatility to be non-stochastic.

If we make the assumption of a deterministic stock price volatility, we can prove by the same arguments as in Section 2.2 that

$$Q_{T_2}(A_2) = N(\delta_2) = N\left(\frac{1}{\sigma\sqrt{T_2}} \ln \frac{S(0)}{S^*B(0, T_2)} - \frac{1}{2}\sigma\sqrt{T_2}\right),$$

$$Q_{T_1}(A_1 \cap A_2) = N(\delta_1, \delta_2)$$

where

$$\delta_1 = \frac{1}{\sigma\sqrt{T_1}} \ln \frac{S(0)}{K_1 B(0, T_1)} - \frac{1}{2}\sigma\sqrt{T_1}$$

and $N(\cdot, \cdot)$ is the cumulative function of a centred bivariate Gaussian distribution with covariance matrix

$$\begin{bmatrix} 1 & \sqrt{\dfrac{T_2}{T_1}} \\ \sqrt{\dfrac{T_2}{T_1}} & 1 \end{bmatrix},$$

$$Q_S(A_1 \cap A_2) = N(\delta_1 + \sigma\sqrt{T_1}, \delta_2 + \sigma\sqrt{T_2}).$$

Regrouping the different terms, we obtain Geske's formula for stochastic interest rates

$$C_2(0) = S(0)N(\delta_1 + \sigma\sqrt{T_1}, \delta_2 + \sigma\sqrt{T_2})$$
$$- K_1 B(0, T_1)N(\delta_1, \delta_2) - K_2 B(0, T_2)N(\delta_2).$$

4. Options on bonds

This section concerns European calls on default-free bonds.

4.1. *Options on zero-coupon bonds*. Theorem 2 entails the following formula for the price of a call maturing at date T_0 written on zero-coupon bonds maturing at date T_1

$$C(0) = B(0, T_1)Q_{T_1}(A) - KB(0, T_0)Q_{T_0}(A)$$

where K is the exercise price and A the exercise set.

(a) In the same manner as Heath et al. (1987), we will assume the following dynamics of the term structure of interest rates

$$\frac{dB(t, T)}{B(t, T)} = r(t)dt + \sigma(t, T)dW_t$$

where $\sigma(t, T)$ is decreasing in t and $\sigma(T, T) = 0$.

Assuming $\sigma(t, T)$ deterministic, the same arguments as in Example 1 of Section 3 provide a formula of the Black–Scholes type

$$C(0) = B(0, T_1)N(d_1) - KB(0, T_0)N(d_2)$$

where

$$d_2 = \frac{1}{2}\sigma\sqrt{T_0} + \frac{1}{\sigma\sqrt{T_0}}\ln\frac{B(0, T_1)}{KB(0, T_0)}$$

$$d_1 = d_2 - \sigma\sqrt{T_0}$$

and σ is the volatility of the forward price $B(t, T_1)/B(t, T_0)$ of the zero-coupon bond.

(b) Another explicit formula was obtained by Cox et al. (1985) in the context of stochastic volatilities, but with a one-state variable description of the term structure of interest rates. The (risk-adjusted) dynamics of the short rate is defined by

$$dr(t) = a(b - r(t))dt + \sigma dW_t \quad \text{where } a, b \text{ and } \sigma \text{ are positive constants.}$$

From that dynamics, it follows that the short rate is distributed under Q (respectively Q_{T_0}, Q_{T_1}) as a non-central χ^2 process with parameter of non-centrality q (respectively q_0, q_1). Cox et al. show that the call is exercised if and only if $r(T_0)$ is less than a critical level d_0; they obtain an explicit formula for a call on a zero-coupon bond, which can be written in our notation as

$$C(0) = B(0, T_1)\chi^2(d_1, n_1, q_1) - KB(0, T_0)\chi^2(d_0, n, q_0)$$

where $\chi^2(\cdot, n, q)$ is the non-centred χ^2 distribution with n degrees of freedom and parameter of non-centrality q; n, q_0, q_1, d_0 and d_1 are parameters depending on a, b, σ and the characteristics of the call. It is interesting to notice that the assumption of a deterministic volatility of interest rates is not necessary to obtain a formula à la Black and Scholes. The result holds because the spot rate driven by the dynamics

$$dr = a(b - r)dt + \sigma\sqrt{r}dW$$

follows a χ^2 distribution and that under the 'forward neutral' probability associated to any date T, this is still true (with a change in the drift of dr); consequently, the probabilities of exercise under the different probabilities are expressed in terms of non-centred χ^2 distributions.

4.2. *Options on coupon bonds*. Let us suppose that the underlying asset is a general default-free bond, characterized by the sequence $F_1, F_2, \cdots, F_n$ of fixed payments it generates at times $T_1, \cdots, T_n$. Under the assumptions and notation of Section 3, its price at date t $(t < T_1 < \cdots < T_n)$ is given by $P(t) = \Sigma_{i=1}^n F_i B(t, T_i)$, where

$$B(t, T) = E_Q\left[\exp - \int_t^T r(s)ds / F_t\right].$$

We consider a call written on the bond, with exercise price K and maturity $T_0 < T_1$.

The probability of exercise will involve the distribution of n variables, namely the prices of the n zero-coupon bonds $B(0, T_1), \cdots, B(0, T_n)$. To obtain a formula of the Black and Scholes type, it is necessary that these n prices depend on only one state-variable, for instance the spot rate.

In the Gaussian case with one source of randomness (as in Section 3, 1(a)), this is equivalent to assuming a Markovian spot rate, or in other words, a deterministic volatility $\sigma(t, T)$ which has the form

$$\sigma(t, T) = [h(T) - h(t)]g(t).$$

Jamshidian (1989) and El Karoui and Rochet (1989) obtain under these hypotheses a quasi-explicit formula for the call price

$$C(0) = \sum_{i=1}^{n} F_i B(0, T_i) N(d_i) - KB(0, T_0) N(d_0)$$

where $d_i = d_0 + \mu_i$,

$$\mu_i^2 = \int_0^{T_0} [\sigma(s, T_i) - \sigma(s, T_0)]^2 ds$$

and d_0 is defined implicitly by

$$\sum_{i=1}^{n} F_i B(0, T_i) \exp\{-\tfrac{1}{2}\mu_i^2 + d_0\mu_i\} = KB(0, T_0).$$

5. Conclusion

The paper has shown that a change of numéraire does not change the self-financing portfolios, and hence does not change the hedging or replicating portfolios either. An immediate consequence in option pricing is that, depending on whether the option under analysis is written on a stock, on a bond, is an exchange option or a compound option, the choice of the appropriate numéraire will provide the easiest calculations and the relevant hedging portfolio.

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