The Black-Scholes Model

Liuren Wu

Zicklin School of Business, Baruch College

Options Markets

(Hull chapter: 12, 13, 14)

The Black-Scholes-Merton (BSM) model

- Black and Scholes (1973) and Merton (1973) derive option prices under the following assumption on the stock price dynamics,

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad \text{(explained later)}$$

- The binomial model: Discrete states and discrete time (The number of possible stock prices and time steps are both finite).
- The BSM model: Continuous states (stock price can be anything between 0 and ∞) and continuous time (time goes continuously).
- Scholes and Merton won Nobel price. Black passed away.
- BSM proposed the model for stock option pricing. Later, the model has been extended/twisted to price currency options (Garman&Kohlhagen) and options on futures (Black).
- I treat all these variations as the same concept and call them indiscriminately the BSM model (combine chapters 13&14).

Primer on continuous time process

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- The driver of the process is W_t, a Brownian motion, or a Wiener process.
    - The sample paths of a Brownian motion are continuous over time, but nowhere differentiable.
    - It is the idealization of the trajectory of a single particle being constantly bombarded by an infinite number of infinitessimally small random forces.
    - Like a shark, a Brownian motion must always be moving, or else it dies.
    - If you sum the absolute values of price changes over a day (or any time horizon) implied by the model, you get an infinite number.
    - If you tried to accurately draw a Brownian motion sample path, your pen would run out of ink before one second had elapsed.
- The first who brought Brownian motion to finance is Bachelier in his 1900 PhD thesis: The theory of speculation.

Properties of a Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- The process W_t generates a random variable that is normally distributed with mean 0 and variance t, $\phi(0, t)$. (Also referred to as Gaussian.)

> *Everybody believes in the normal approximation, the experimenters because they believe it is a mathematical theorem, the mathematicians because they believe it is an experimental fact!*

- The process is made of independent normal increments $dW_t \sim \phi(0, dt)$.
  - "d" is the continuous time limit of the discrete time difference (Δ).
  - Δt denotes a finite time step (say, 3 months), dt denotes an extremely thin slice of time (smaller than 1 milisecond).
  - It is so thin that it is often referred to as *instantaneous*.
  - Similarly, $dW_t = W_{t+dt} - W_t$ denotes the instantaneous increment (change) of a Brownian motion.
- By extension, increments over non-overlapping time periods are independent: For $(t_1 > t_2 > t_3)$, $(W_{t_3} - W_{t_2}) \sim \phi(0, t_3 - t_2)$ is independent of $(W_{t_2} - W_{t_1}) \sim \phi(0, t_2 - t_1)$.

Properties of a normally distributed random variable

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- If $X \sim \phi(0,1)$, then $a + bX \sim \phi(a, b^2)$.
- If $y \sim \phi(m, V)$, then $a + by \sim \phi(a + bm, b^2 V)$.
- Since $dW_t \sim \phi(0, dt)$, the instantaneous price change $dS_t = \mu S_t dt + \sigma S_t dW_t \sim \phi(\mu S_t dt, \sigma^2 S_t^2 dt)$.
- The instantaneous return $\frac{dS}{S} = \mu dt + \sigma dW_t \sim \phi(\mu dt, \sigma^2 dt)$.
    - Under the BSM model, μ is the annualized mean of the instantaneous return — instantaneous mean return.
    - σ^2 is the annualized variance of the instantaneous return — instantaneous return variance.
    - σ is the annualized standard deviation of the instantaneous return — instantaneous return volatility.

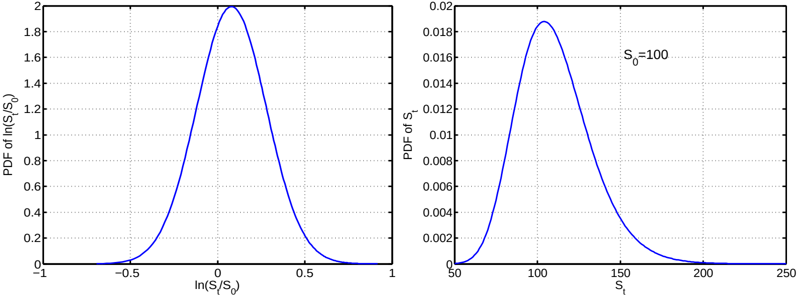
Geometric Brownian motion

$$dS_t/S_t = \mu dt + \sigma dW_t$$

- The stock price is said to follow a geometric Brownian motion.
- μ is often referred to as the drift, and σ the diffusion of the process.
- Instantaneously, the stock price change is normally distributed, $\phi(\mu S_t dt, \sigma^2 S_t^2 dt)$.
- Over longer horizons, the price change is lognormally distributed.
- The log return (continuous compounded return) is normally distributed over all horizons:
  $d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$. (By Ito's lemma)
  - $d \ln S_t \sim \phi(\mu dt - \frac{1}{2}\sigma^2 dt, \sigma^2 dt)$.
  - $\ln S_t \sim \phi(\ln S_0 + \mu t - \frac{1}{2}\sigma^2 t, \sigma^2 t)$.
  - $\ln S_T/S_t \sim \phi\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$.
- Integral form: $S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}$, $\quad \ln S_t = \ln S_0 + \mu t - \frac{1}{2}\sigma^2 t + \sigma W_t$

Normal versus lognormal distribution

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu = 10\%, \sigma = 20\%, S_0 = 100, t = 1.$$

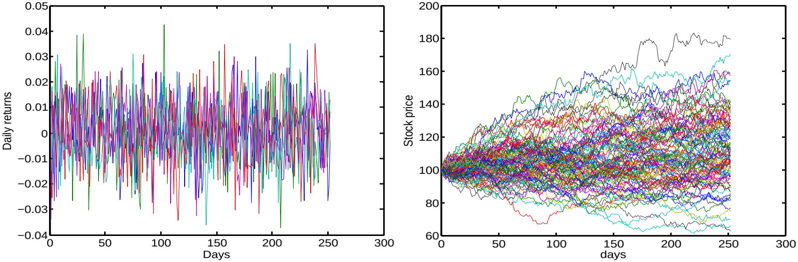


The earliest application of Brownian motion to finance is Louis Bachelier in his dissertation (1900) "Theory of Speculation." He specified the stock price as following a Brownian motion with drift:

$$dS_t = \mu dt + \sigma dW_t$$

Simulate 100 stock price sample paths

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu = 10\%, \sigma = 20\%, S_0 = 100, t = 1.$$



- Stock with the return process: $d \ln S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$.
- Discretize to daily intervals $dt \approx \Delta t = 1/252$.
- Draw standard normal random variables $\varepsilon(100 \times 252) \sim \phi(0, 1)$.
- Convert them into daily log returns: $R_d = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon$.
- Convert returns into stock price sample paths: $S_t = S_0 e^{\sum_{d=1}^{252} R_d}$.

The key idea behind BSM

- The option price and the stock price depend on the same underlying source of uncertainty.
- The Brownian motion dynamics imply that if we slice the time thin enough (dt), it behaves like a binominal tree.
- Reversely, if we cut Δt small enough and add enough time steps, the binomial tree converges to the distribution behavior of the geometric Brownian motion.
  - Under this thin slice of time interval, we can combine the option with the stock to form a riskfree portfolio.
  - Recall our hedging argument: Choose Δ such that $f - \Delta S$ is riskfree.
  - The portfolio is riskless (under this thin slice of time interval) and must earn the riskfree rate.
  - Magic: μ does not matter for this portfolio and hence does not matter for the option valuation. Only σ matters.
    - ⋆ We do not need to worry about risk and risk premium if we can hedge away the risk completely.

Partial differential equation

- The hedging argument mathematically leads to the following partial differential equation:

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

  - At nowhere do we see μ. The only free parameter is σ (as in the binominal model).

- Solving this PDE, subject to the terminal payoff condition of the derivative (e.g., $f_T = (S_T - K)^+$ for a European call option), BSM can derive analytical formulas for call and put option value.
  - Similar formula had been derived before based on distributional (normal return) argument, but μ (risk premium) was still in.
  - The realization that option valuation does not depend on μ is big. Plus, it provides a way to hedge the option position.

The BSM formulae

$$
\begin{aligned}
c_t &= S_t e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2), \\
p_t &= -S_t e^{-q(T-t)} N(-d_1) + K e^{-r(T-t)} N(-d_2),
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(S_t/K) + (r-q)(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\ln(S_t/K) + (r-q)(T-t) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.
\end{aligned}
$$

Black derived a variant of the formula for futures (which I like better):

$$
c_t = e^{-r(T-t)} \left[F_t N(d_1) - K N(d_2) \right],
$$

with $d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$.

- Recall: $F_t = S_t e^{(r-q)(T-t)}$.
- Once I know call value, I can obtain put value via put-call parity: $c_t - p_t = e^{-r(T-t)} \left[F_t - K_t \right]$.

Cumulative normal distribution

$$c_t = e^{-r(T-t)}\left[F_t N(d_1) - KN(d_2)\right], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

- $N(x)$ denotes the cumulative normal distribution, which measures the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 ($\phi(0,1)$) is less than x.
- See tables at the end of the book for its values.
- Most software packages (including excel) has efficient ways to computing this function.
- Properties of the BSM formula:
  - As S_t becomes very large or K becomes very small, $\ln(F_t/K) \uparrow \infty$, $N(d_1) = N(d_2) = 1$. $c_t = e^{-r(T-t)}\left[F_t - K\right]$.
  - Similarly, as S_t becomes very small or K becomes very large, $\ln(F_t/K) \uparrow -\infty$, $N(-d_1) = N(-d_2) = 1$. $p_t = e^{-r(T-t)}\left[-F_t + K\right]$.

Implied volatility

$$c_t = e^{-r(T-t)}\left[F_t N(d_1) - K N(d_2)\right], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

- Since F_t (or S_t) is observable from the underlying stock or futures market, (K, t, T) are specified in the contract. The only unknown (and hence free) parameter is σ.
- We can estimate σ from time series return. (standard deviation calculation).
- Alternatively, we can choose σ to match the observed option price — implied volatility (IV).
- There is a one-to-one correspondence between prices and implied volatilities.
- Traders and brokers often quote implied volatilities rather than dollar prices.
- The BSM model says that $IV = \sigma$. In reality, the implied volatility calculated from different options (across strikes, maturities, dates) are usually different.

Options on what?

Why does it matter?

- As long as we assume that the underlying security price follows a geometric Brownian motion, we can use (some versions) of the BSM formula to price European options.
- Dividends, foreign interest rates, and other types of carrying costs may complicate the pricing formula a little bit.
- A simpler approach: Assume that the underlying futures/forwards price (of the same maturity of course) process follows a geometric Brownian motion.
- Then, as long as we observe the forward price (or we can derive the forward price), we do not need to worry about dividends or foreign interest rates — They are all accounted for in the forward pricing.
- Know how to price a forward, and use the Black formula.

Risk-neutral valuation

- Recall: Under the binomial model, we derive a set of risk-neutral probabilities such that we can calculate the expected payoff from the option and discount them using the riskfree rate.
  - Risk premiums (recall CAPM) are hidden in the risk-neutral probabilities.
  - If in the real world, people are indeed risk-neutral, the risk-neutral probabilities are the same as the real-world probabilities. Otherwise, they are different.
- Under the BSM model, we can also assume that there exists such an artificial risk-neutral world, in which the expected returns on all assets earn risk-free rate.
- The stock price dynamics under the risk-neutral world becomes, $dS_t/S_t = (r - q)dt + \sigma dW_t$.
- Simply replace the actual expected return (μ) with the return from a risk-neutral world ($r - q$) [*ex-dividend return*].

The risk-neutral return on spots

$$dS_t/S_t = (r-q)dt + \sigma dW_t, \text{ under risk-neutral probabilities.}$$

- In the risk-neutral world, investing in all securities make the riskfree rate as the total return.
- If a stock pays a dividend yield of q, then the risk-neutral expected return from stock price appreciation is $(r-q)$, such as the total expected return is: dividend yield$+$ price appreciation $=r$.
- Investing in a currency earns the foreign interest rate r_f similar to dividend yield. Hence, the risk-neutral expected currency appreciation is $(r - r_f)$ so that the total expected return is still r.
- Regard q as r_f and value options as if they are the same.

The risk-neutral return on forwards/futures

- If we sign a forward contract, we do not pay anything upfront and we do not receive anything in the middle (no dividends or foreign interest rates). Any P&L at expiry is excess return.
- Under the risk-neutral world, we do not make any excess return. Hence, the forward price dynamics has zero mean (driftless) under the risk-neutral probabilities: $dF_t/F_t = \sigma dW_t$.
- The carrying costs are all hidden under the forward price, making the pricing equations simpler.

Readings behind the technical jargons: $\mathbb{P}$ v. $\mathbb{Q}$

- $\mathbb{P}$: Actual probabilities that cashflows will be high or low, estimated based on historical data and other insights about the company.
  - Valuation is all about getting the forecasts right and assigning the appropriate price for the forecasted risk — *fair wrt future cashflows and your risk preference*.
- $\mathbb{Q}$: "Risk-neutral" probabilities that we can use to aggregate expected future payoffs and discount them back with riskfree rate, regardless of how risky the cash flow is.
  - It is related to real-time scenarios, but it has nothing to do with real-time probability.
  - Since the intention is to hedge away risk under all scenarios and discount back with riskfree rate, we do not really care about the actual probability of each scenario happening.
    We just care about what all the possible scenarios are and whether our hedging works under all scenarios.
  - $\mathbb{Q}$ is not about getting close to the actual probability, but about being *fair wrt the prices of securities that you use for hedging*.

Summary

- Understand the basic properties of normally distributed random variables.
- Map a stochastic process to a random variable.
- Understand the link between BSM and the binomial model.
- Memorize the BSM formula (any version).
- Understand forward pricing and link option pricing to forward pricing.
- Can go back and forth with the put-call parity conditions, lower and upper bonds, either in forward or in spot notation.