Example of Hierarchical Modeling: Estimation of Population and Cosmological Parameters of Binary Neutron Stars

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The posterior of $\vec{\Omega}_P$ and $\vec{\Omega}_C$ for ith NS-NS event [Thrane & Talbot],

$$P(\vec{\Omega}_P, \vec{\Omega}_C|D_i) \propto P(\vec{\Omega}_P, \vec{\Omega}_C)L_i(D_i|\vec{\Omega}_P, \vec{\Omega}_C)$$

Quasi Likelihood L_i is given by,

$$L_i(D_i|\vec{\Omega}_P, \vec{\Omega}_C) = \int_{M_{cz},D_L} P(D_i|M_{cz}, D_L)P(M_{cz}, D_L|\vec{\Omega}_P, \vec{\Omega}_C)dM_{cz}dD_L$$

which is,

$$L_i(D_i|\vec{\Omega}_P, \vec{\Omega}_C) = \int_{M_{cz},D_L} P(D_i|M_{cz}, D_L)P(D_L)P(M_{cz}|D_L, \vec{\Omega}_P, \vec{\Omega}_C)dM_{cz}dD_L$$

which involves **Assumption 1:** D_L and $(\vec{\Omega}_P, \vec{\Omega}_C)$ are independent. Our model will come through $z = z(D_L, \vec{\Omega}_C)$, not by $D_L = D_L(z, \vec{\Omega}_C)$.

Now **Assumption 2** is,

$$P(M_{cz}|D_L, \vec{\Omega}_P, \vec{\Omega}_C) = \int_{M_c,z} P(M_{cz}|M_c, z)P(M_c|\vec{\Omega}_P)P(z|D_L, \vec{\Omega}_C)dM_cdz$$

Our models are,

$$P(M_{cz}|z, M_c) = \delta(M_{cz} - M_c(1 + z)) \tag{1}$$

$$P(M_c|\vec{\Omega}_P) = N_{M_c}(\vec{\Omega}_P) \tag{2}$$

$$P(z|D_L, \vec{\Omega}_C) = \delta(z - z(D_L, \vec{\Omega}_C)) \tag{3}$$

with those prior distributions, $P(M_{cz}|D_L, \vec{\Omega}_P, \vec{\Omega}_C)$ becomes,

$$P(M_{cz}|D_L, \vec{\Omega}_P, \vec{\Omega}_C) = \frac{1}{1 + z(D_L, \vec{\Omega}_C)} N_{\left(\frac{M_{cz}}{1+z(D_L,\vec{\Omega}_C)}\right)}(\vec{\Omega}_P)$$

Hence our Quasi-Likelihood becomes,

$$L_i(D_i|\vec{\Omega}_P, \vec{\Omega}_C) = \int_{M_{cz},D_L} P(D_i|M_{cz}, D_L)\ P(D_L)\ \frac{1}{(1 + z(D_L, \vec{\Omega}_C))}\ N_{\left(\frac{M_{cz}}{1+z(D_L,\vec{\Omega}_C)}\right)}(\vec{\Omega}_P)\ dM_{cz}dD_L$$

The posterior distribution of $(\vec{\Omega}_P, \vec{\Omega}_C)$ is given by,

$$P(\vec{\Omega}_P, \vec{\Omega}_C|\bigcap_i D_i) = P(\vec{\Omega}_P, \vec{\Omega}_C)\ L_i(\bigcap_i D_i|\vec{\Omega}_P, \vec{\Omega}_C) = P(\vec{\Omega}_P, \vec{\Omega}_C)\ \prod_i L_i(D_i|\vec{\Omega}_P, \vec{\Omega}_C)$$

Hence for N events the posterior distribution of $(\vec{\Omega}_P, \vec{\Omega}_C)$ is given by,

$$P(\vec{\Omega}_P, \vec{\Omega}_C|\bigcap_i D_i) = P(\vec{\Omega}_P, \vec{\Omega}_C)\ \prod_i \int_{M_{cz},D_L} P(D_i|M_{cz}, D_L)\ P(D_L)\ \frac{1}{(1 + z(D_L, \vec{\Omega}_C))}\ N_{\left(\frac{M_{cz}}{1+z(D_L,\vec{\Omega}_C)}\right)}(\vec{\Omega}_P)\ dM_{cz}dD_L \tag{4}$$