In this assignment, we will denote:

$$y'' + p(x)y' + q(x)y = r(x), \quad x \in I \tag{*}$$
$$y'' + p(x)y' + q(x)y = 0, \quad x \in I \tag{**}$$

where $I \subset \mathbb{R}$ is an interval and $p(x), q(x), r(x)$ are continuous functions on I.

1. Let y_1 be the solution of the IVP

$$y'' + (2x-1)y' + \sin(e^x)\, y = 0, \quad y(0) = 1, y'(0) = -1;$$

and y_2 be the solution of the IVP

$$y'' + (2x-1)y' + \sin(e^x)\, y = 0, \quad y(0) = 2, y'(0) = -1.$$

Find the Wronskian of y_1, y_2. What is the general solution of $y'' + (2x-1)y' + \sin(e^x)\, y = 0$?

Solution: We know that if y_1, y_2 are solutions of (**), then the Wronskian is give by

$$W(y_1, y_2)(x) = W(x) = c\exp\left(-\int p(x)dx\right) = ce^{-x^2+x}.$$

From the given initial conditions, we have $W(0) = 1$. So $c = 1$. Hence $W(x) = e^{-x^2+x}$.

Since $W(0) \neq 0$, we deduce that y_1, y_2 are independent solutions. Therefore, the general solution is given by $c_1y_1 + c_2y_2$.

2. Show that the set of solutions of the linear homogeneous equation (**) is a real vector space. Also show that the set of solutions of the linear non-homogeneous equation (*) is not a real vector space. If $y_1(x), y_2(x)$ are any two solutions of (*), obtain conditions on the constants a and b so that $ay_1 + by_2$ is also its solution.

Solution: Let S be the set of solutions of the linear homogeneous ODE (**). Clearly S is a subset of set of twice differentiable functions on I which is a real vector space. Thus it is sufficient to show that S is subspace of the above vector space of twice differentiable function. Now $\mathbf{0}(x) = 0$ satisfies $(^{**})$ and hence $\mathbf{0} \in S$. Thus S is nonempty. Also if u, v both satisfies (**), then $\alpha u(x) + v(x)$ is also a solution of (**). This implies $\alpha u + v \in S$. Hence S is a subspace, i.e. a vector space.

Now $\mathbf{0}(x) = 0$ is not a solution of $(^*)$, thus zero element does not exist. Hence, the set of solution of $(^*)$ is not a real vector space.

Let $y_1(x), y_2(x)$ are any two solutions of (*). Then

$$y_1'' + p(x)y_1' + q(x)y_1 = r(x) \tag{1}$$
$$y_2'' + p(x)y_2' + q(x)y_2 = r(x). \tag{2}$$

Multiplying (1) by a and (2) by b and adding, we find

$$(ay_1 + by_2)'' + p(x)\left(ay_1 + by_2\right)' + q(x)\left(ay_1 + by_2\right) = (a+b)r(x).$$

If $ay_1 + by_2$ is also a solution, then the LHS is $r(x)$ and hence $a + b = 1$.

3. Decide if the statements are true or false. If the statement is true, prove it, if it is false, give a counter example showing it is false.
(i) If $f(x)$ and $g(x)$ are linearly independent functions on an interval I, then they are linearly independent on any larger interval containing I.

If $f(x)$ and $g(x)$ are linearly independent functions on an interval I, then they are linearly independent on any smaller interval contained in I.
(ii) If $f(x)$ and $g(x)$ are linearly dependent functions on an interval I, then they are linearly dependent on any subinterval of I.

If $y_1(x)$ and $y_2(x)$ are linearly dependent functions on an interval I, then they are linearly dependent on any larger interval containing I.
(iii) If $y_1(x)$ and $y_2(x)$ are linearly independent solution of (**) on an interval I, they are linearly independent on any interval contained in I.
(iv) If $y_1(x)$ and $y_2(x)$ are linearly dependent solutions of (**) on an interval I , they are linearly dependent on any interval contained in I.

Solution:

(i) True, follows from the definition of linear independence.
False: take $f(x) = x^2$ and $g(x) = x|x|$. Then f, g linearly independent over $[-1, 1]$ but dependent over $[0, 1]$.

(ii) True, follows from definition.
False, the previous examples would work.

(iii) True, follows from the fact that, in this case y_1, y_2 is linearly independent on I iff $W(y_1, y_2) \neq 0$ on all I.

(iv) True, follows from the fact that, in this case y_1, y_2 is linearly dependent on I iff $W(y_1, y_2) = 0$ on all I.

4. Can x^3 be a solution of (**) on $I = [-1, 1]$? Find two 2nd order linear homogeneous ODE with x^3 as a solution.

Solution: No. Putting $y = x^3$ in the given equation, we get

$$6x + p(x)3x^2 + q(x)x^3 = 0, \quad \forall x \in [-1, 1]\,.$$

Canceling x, we get

$$6 + p(x)3x + q(x)x^2 = 0, \quad \forall x \in [-1, 1] \setminus \{0\}.$$

That is

$$3xp(x) + q(x)x^2 = -6 \quad \forall x \in [-1, 1] \setminus \{0\}.$$

Note that the function $f(x) = 3xp(x)+q(x)x^2$ is continuous on $[-1, 1]$ and so $\lim_{x\to 0} f(x) = f(0) = 0$. But also $f(x) = -6$ for $x \in [-1, 1] \setminus \{0\}$, which is a contradiction. Hence, x^3 cannot be a solution of (**) on I.

Two ODEs with x^3 as solution are: $xy'' = 2y'$ and $x^2y'' = 6y$. Note that here p, q are not continuous at 0 .

5. Can $x \sin x$ be a solution of a second order linear homogeneous equation with constant coefficients?

Solution: No, putting $x \sin x$ in $y'' + py' + qy = 0$, we get

$$((q - 1)x + p)\sin x + (px + 2)\cos x = 0, \quad \forall\, x \in \mathbb{R}.$$

For $x = 0$, we get $2 = 0$, which is absurd. So, this cannot possible.

6. Find the largest interval on which a unique solution is guaranteed to exist of the IVP. $(x+2)y'' + xy' + \cot(x)y = x^2 + 1, \quad y(2) = 11, \quad y'(2) = -2$.

Solution: Comparing with (*), we have

$$p(x) = \frac{x}{x+2}, q(x) = \frac{\cot(x)}{(x+2)\sin x}, \quad r(x) = \frac{x^2+1}{x+2}.$$

The discontinuities of p, q, r are $x = -2, 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$. The largest interval that contains $x_0 = 2$ but none of the discontinuities is, therefore, $(0, \pi)$.

7. Without solving determine the largest interval in which the solution is guaranteed to uniquely exist of the IVP $ty'' - y' = t^2 + t, \quad y(1) = 1, y'(1) = 5$. Verify your answer by solving it explicitly.

Solution: Since p, r are not continuous at 0 , the maximum interval of existence and uniqueness of solution of the given IVP is $(0, \infty)$.

Here dependent variable y is missing. Solving it, $y(t)=t^{3}/3+7t^{2}/4+t^{2}(\ln t)/2-13/12$ for which the max interval of validity is $(0, \infty)$.

8. Find the differential equation satisfied by each of the following two-parameter families of plane curves:
(i) $y=\cos(ax+b)$
(ii) $y=ax+\frac{b}{x}$
(iii) $y=ae^{x}+bxe^{x}$

Solution: For two arbitrary constants, the order of the ODE will be two. Eliminate constants a and b by differentiating twice.
(i) $y=\cos(ax+b) \Longrightarrow y'=-a\sin(ax+b), y''=-a^{2}\cos(ax+b)=-a^{2}y$. From this we find

$$\frac{y'^{2}}{a^{2}}+y^{2}=1 \Longrightarrow \left(1-y^{2}\right)a^{2}=y'^{2} \Longrightarrow -\left(1-y^{2}\right)\frac{y''}{y}=y'^{2} \Longrightarrow \left(1-y^{2}\right)y''+yy'^{2}=0$$

(ii) $y=ax+b/x \Longrightarrow xy=ax^{2}+b \Longrightarrow xy'+y=2ax \Longrightarrow y'+y/x=2a$ which on differentiating again gives $y''+y'/x-y/x^{2}=0 \Longrightarrow x^{2}y''+xy'-y=0$.
(iii) $y=ae^{x}+bxe^{x} \Longrightarrow e^{-x}y=a+bx \Longrightarrow e^{-x}y'-e^{-x}y=b \Longrightarrow e^{-x}y''-2e^{-x}y'+e^{-x}y=0$ which on simplification gives $y''-2y'+y=0$

9. Find general solution of the following differential equations given a known solution y_1 :
(i) $x(1-x)y''+2(1-2x)y'-2y=0 \quad y_1=1/x$
(ii) $\left(1-x^{2}\right)y''-2xy'+2y=0 \quad y_1=x$

Solution: (i) Here $y_1=1/x$. Substitute $y=u(x)/x$ to get $(1-x)u''-2u'=0$. Thus, $u'=1/(1-x)^{2}$ and $u=1/(1-x)$. Hence, $y_2=1/(x(1-x))$ and the general solution is $y=a/x+b/(x(1-x))$.
(ii) Here $y_1=x$. Substitute $y=xu(x)$ to get $x\left(1-x^{2}\right)u''=2\left(2x^{2}-1\right)u'$. Thus,

$$\frac{u''}{u'}=\frac{2\left(2x^{2}-1\right)}{x\left(1-x^{2}\right)}=-\frac{2}{x}-\frac{1}{1+x}+\frac{1}{1-x} \Longrightarrow u'=\frac{1}{x^{2}\left(1-x^{2}\right)}$$

Thus,

$$u'=\frac{1}{x^{2}}+\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right) \Longrightarrow u=-\frac{1}{x}+\frac{1}{2}\ln\left(\left(\frac{1+x}{1-x}\right)\right.$$

Hence,

$$y_2 = -1 + \frac{x}{2} \ln\left(\left(\frac{1+x}{1-x}\right)\right)$$

and the general solution is

$$y = ax + b\left\{-1 + \frac{x}{2} \ln\left(\left(\frac{1+x}{1-x}\right)\right)\right\}.$$

10. Verify that $\sin x/\sqrt{x}$ is a solution of $x^2y'' + xy' + \left(x^2 - 1/4\right)y = 0$ over any interval on the positive x-axis and hence find its general solution.

Solution: Verification is straightforward.
Substitute $y = u(x)\sin x/\sqrt{x}$ to get

$$y' = \frac{\sin x}{\sqrt{x}}u' + \left(\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}}\right)u$$
$$y'' = \frac{\sin x}{\sqrt{x}}u'' + 2\left(\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}}\right)u' + \left(-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x^{3/2}} + \frac{3}{4}\frac{\sin x}{x^{5/2}}\right)u$$

This leads to

$$\sin x u'' + 2\cos x u' = 0 \implies u' = \operatorname{cosec}^2 x \implies u = -\cot x$$

Hence, $y_2 = -\cos x/\sqrt{x}$ and the general solution is $y = (a\sin x + b\cos x)/\sqrt{x}$.

11. Solve the following differential equations:
(i) $y'' - 4y' + 3y = 0$
(ii) $y'' + 2y' + \left(\omega^2 + 1\right)y = 0, \quad \omega$ is real.

Solution: (i) Characteristic (or auxiliary) equation: $m^2 - 4m + 3 = 0 \implies m = 1, 3$. General sol: $y = Ae^x + Be^{3x}$
(ii) Characteristic equation: $m^2 + 2m + \left(1 + \omega^2\right) = 0 \implies m = -1 \pm \omega i$.

Case 1: $\omega = 0 \implies$ equal roots $m = -1, -1$ and general sol: $y = (A + Bx)e^{-x}$
Case 2: $\omega \neq 0 \implies$ complex conjugate roots $m = -1 \pm \omega i$ and general sol: $y = e^{-x}(A\sin\omega x + B\cos\omega x)$

12. Solve the following initial value problems:
(i) $y'' + 4y' + 4y = 0 \quad y(0) = 1, y'(0) = -1$
(ii) $y'' - 2y' - 3y = 0 \quad y(0) = 1, y'(0) = 3$

Solution: (i) Assume $y = e^{mx}$ is a solution. Putting in the given equation, we get the characteristic equation: $m^2+4m+4 = 0 \Longrightarrow m = -2, -2$. General sol: $y = e^{-2x}(A+Bx)$. Using initial conditions:

$$A = 1, B - 2A = -1 \Longrightarrow B = 1 \Longrightarrow y = (x + 1)e^{-2x}$$

(ii) Characteristic equation: $m^2 - 2m - 3 = 0 \Longrightarrow m = -1, 3$. General sol: $y = \left(Ae^{3x} + Be^{-x}\right)$. Using initial conditions:

$$A + B = 1, 3A - B = 3 \Longrightarrow A = 1, B = 0 \Longrightarrow y = e^{3x}$$

13. Reduce the following second order differential equation to first order differential equation and hence solve.
(i) $xy'' + y' = y'^2$
(ii) $yy'' + y'^2 + 1 = 0$
(iii) $y'' - 2y' \coth x = 0$

Solution: (i) Dependent variable y absent. Substitute $y' = p \Longrightarrow y'' = dp/dx$. Thus $xp' + p = p^2$. Solving $p = 1/(1 - ax)$ which on integrating again gives $y = b - \ln(|1 - ax|)/a$, where a and b are arbitrary constants.
(ii) Independent variable x is absent in $yy'' + y'^2 + 1 = 0$. Substitute $y' = p \Longrightarrow y'' = pdp/dy$. Thus

$$py\frac{dp}{dy} + p^2 = 1 \Longrightarrow \frac{p}{1 + p^2}\frac{dp}{dy} + \frac{1}{y} = 0 \Longrightarrow \ln\sqrt{1 + p^2}y = \ln a \Longrightarrow 1 + p^2 = \frac{a^2}{y^2}$$

From $p^2 = a^2/y^2 - 1$, we find

$$\frac{y}{\sqrt{a^2 - y^2}}\frac{dy}{dx} = \pm 1 \Longrightarrow -\sqrt{a^2 - y^2} = \pm x + b$$

Both the solutions can be written as $(x + b)^2 + y^2 = a^2$ where a and b are arbitrary constants..
(iii) $y'' - 2y' \coth x = 0$. Substitute $y' = p \Longrightarrow y'' = dp/dx$. Thus $dp/dx = 2p \coth x$.

Solving $p = a \sinh^2 x$, which on integrating again gives $y = a(\sinh 2x - 2x)/4 + b$ where a and b are arbitrary constants.

14. Find the curve $y = y(x)$ which satisfies the ODE$y'' = y'$ and the line $y = x$ is tangent at the origin.

Solution: The given conditions lead to the following problem:
Solve $y'' - y' = 0$ with $y(0) = 0, y'(0) = 1$. Integrating once gives $y' - y = a$ which on another integration gives $y + a = be^x.y(0) = 0$ gives $a = b.y'(0) = 1$ gives $b = 1$ and hence solution is $y = e^x - 1$.

15. Are the following functions linearly dependent on the given intervals?
(i) $\sin 4x, \cos 4x \quad (-\infty, \infty)$
(ii) $\ln x, \ln x^3 \quad (0, \infty)$
(iii) $\cos 2x, \sin^2 x \quad (0, \infty)$
(iv) $\quad x^3, x^2|x| \quad [-1, 1]$

Solution: (i) $a \sin 4x + b \cos 4x = 0$. For $x = 0$ we find $b = 0$ and for $x = \pi/8$ we get $a = 0$. Hence they are NOT linearly dependent.
(ii) $\ln x^3 - 3 \ln x = 0$ for $x \in (0, \infty)$. Hence linearly dependent.
(iii) $a \cos 2x + b \sin^2 x = 0$. For $x = 0$ we find $a = 0$ and for $x = \pi/2$ we get $b = 0$. Hence they are NOT linearly dependent.
(iv) $ax^3 + bx^2|x| = 0$. For $x = -1$ we find $a - b = 0$ and for $x = 1$ we get $a + b = 0$. Hence $a = b = 0$ and thus they are NOT linearly dependent.

16. (a) Show that a solution to (**) with x-axis as tangent at any point in I must be identically zero on I.
(b) Let $y_1(x), y_2(x)$ be two solutions of (**) with a common zero at any point in I. Show that y_1, y_2 are linearly dependent on I.
(c) Show that $y = x$ and $y = \sin x$ are not a pair solutions of equation (**), where $p(x), q(x)$ are continuous functions on $I = (-\infty, \infty)$.

Solution: (a) Let $\xi(x)$ be the solution. Since x axis is a tangent, at $x = x_0$, say, then $\xi(x_0) = \xi'(x_0) = 0$. Clearly $y(x) \equiv 0$ satisfies (**) and the initial conditions $y(x_0) = y'(x_0) = 0$. Since the solution is unique, $\xi(x) \equiv 0$ in $\mathcal{I}$.
(b) If $y_1(x), y_2(x)$ have a common zero at $x = x_0$, say, then $y_1(x_0) = y_2(x_0) = 0$. Hence, $W(y_1, y_2) = 0$ at $x = x_0$ and thus y_1, y_2 are linearly dependent.
(c) $y_1 = x$ and $y_2 = \sin x$ are LI on I. So if they were solution of (**), the wronskian $W(y_1, y_2)$ must never be zero. But $W(y_1, y_2) = 0$ at $x = 0$, a contradiction.

17. (a) Let $y_1(x), y_2(x)$ be two twice continuously differentiable functions on an interval I.
(i) Show that the Wronskian $W(y_1, y_2)$ does not vanish anywhere in I if and only if there exists continuous $p(x), q(x)$ on I such that (**) has y_1, y_2 as independent solutions.
(ii) Is it true that if y_1, y_2 are independent on I then there exists continuous $p(x), q(x)$ on I such that (**) has y_1, y_2 as independent solutions?
(b) Construct equations of the form (**) from the following pairs of solutions: e^{-x}, xe^{-x}.

Solution: (a)(i) Suppose that $W(y_1, y_2)$ does not vanish anywhere in I. We want to find $p(x), q(x)$ such that

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \quad y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{3}$$

Solving we get:

$$p(x) = -\left(y_1 y_2'' - y_2 y_1''\right) / W\left(y_1, y_2\right) = -\frac{d}{dx}\left(W\left(y_1, y_2\right)\right) / W\left(y_1, y_2\right)$$

and $q(x) = \left(y_1' y_2'' - y_2' y_1''\right) / W\left(y_1, y_2\right)$. They are continuous on I since $W\left(y_1, y_2\right)$ never zero on I.
[Note that $q(x)$ can also be written as $q(x) = -\frac{1}{y_1}\left(y_1'' + p(x) y_1'\right)$.]
Converse follows from the fact Wronskian is never zero for independent solutions of (**).
(ii) Not true. Consider $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ on $I = [-1, 1.]$ Then they are independent on I, but they cannot be solutions of any ($**$) on I as y_2 is not even differentiable at 0.
(b) Write $y = a y_1(x) + b y_2(x)$ and eliminate a and b. $y = e^{-x}(a + bx) \Longrightarrow e^x y = a + bx$. Differentiating w.r.t. x twice we find

$$e^x\left(y' + y\right) = b \Longrightarrow e^x\left(y'' + 2y' + y\right) = 0 \Longrightarrow y'' + 2y' + y = 0$$

18. Find a particular solution of each of the following equations by method of undetermined coefficients and hence find its general solution:

   (a) $y'' - 3y' + 2y = e^x$
   (b) $y'' + 4y = \cos(2x)$
   (c) $y'' + y = x^2$
   (d) $y'' - 4y' + 4y = xe^{2x}$

Solution: (a) The associated homogeneous equation

$$y'' - 3y' + 2y = 0$$

Characteristic equation: $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

The solution to homogeneous part:

$$y_h = C_1 e^x + C_2 e^{2x}$$

Since 1 is already a root of characteristic equation, for particular solution we try: $y_p = Axe^x$

$$y_p' = Ae^x + Axe^x, \quad y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$$

Substitute into the equation:

$$(2Ae^x + Axe^x) - 3(Ae^x + Axe^x) + 2(Axe^x) = e^x$$

$$-Ae^x = e^x \Rightarrow A = -1$$

$$y_p = -xe^x$$

Hence the general solution is

$$y = C_1 e^x + C_2 e^{2x} - xe^x$$

(b) The associated homogeneous equation:

$$y'' + 4y = 0$$

Characteristic equation: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$ So the solution for the homogeneous part:

$$y_h = C_1 \cos(2x) + C_2 \sin(2x)$$

Since $2i$ is a root, for particular solution, we try: $y_p = x(A\cos(2x) + B\sin(2x))$

After differentiating and substituting, we get:

$$A = 0, \quad B = \frac{1}{4}$$

$$y_p = \frac{x}{4}\sin(2x)$$

Thus the general solution is

$$y = C_1 \cos(2x) + C_2 \sin(2x) + \frac{x}{4}\sin(2x)$$

(c) The associated homogeneous equation:

$$y'' + y = 0$$

Characteristic equation: $m^2 + 1 = 0 \Rightarrow m = \pm i$ The solution to the homogeneous part is

$$y_h = C_1 \cos x + C_2 \sin x$$

For particular solution we try: $y_p = Ax^2 + Bx + C$

$$y_p'' + y_p = 2A + Ax^2 + Bx + C = x^2 \Rightarrow A = 1, \quad B = 0, \quad C = -2$$

$$y_p = x^2 - 2$$

Therefore the general solution is:

$$y = C_1 \cos x + C_2 \sin x + x^2 - 2$$

(d) The associated homogeneous equation:

$$y'' - 4y' + 4y = 0$$

Characteristic equation: $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2$ (repeated root)

The general solution to the homogeneous part is:

$$y_h = (C_1 + C_2 x)e^{2x}$$

Since 2 is a repeated root, for particular solution we try: $y_p = (Ax^3 + Bx^2)e^{2x}$

After computing derivatives and substituting:

$$A = \frac{1}{6}, \quad B = -\frac{1}{3}$$

$$y_p = \left(\frac{1}{6}x^3 - \frac{1}{3}x^2\right) e^{2x}$$

Thus the general solution is:

$$y = (C_1 + C_2 x)e^{2x} + \left(\frac{1}{6}x^3 - \frac{1}{3}x^2\right) e^{2x}$$

19. By using the method of variation of parameters, find the general solution of:
(i) $y'' + 4y = 2\cos^2 x + 10e^x$
(ii) $y'' + y = x \sin x$
(iii) $y'' + y = \cot^2 x$
(iv) $x^2 y'' - x(x+2)y' + (x+2)y = x^3, \quad x > 0.$
[Hint. $y = x$ is a solution of the homogeneous part]

Solution: If y_1, y_2 are independent solutions of the homogeneous part of the ODE

$$y'' + p(x)y' + q(x)y = r(x),$$

then the general solution is $y = Ay_1 + By_2 + uy_1 + vy_2$, where A, B are arbitrary constants and

$$u = -\int \frac{ry_2}{W} dx, \quad v = \int \frac{ry_1}{W} dx, \quad [W(y_1, y_2) \text{ is the Wronskian }]$$

(i) $y_1 = \cos 2x, y_2 = \sin 2x, W(y_1, y_2) = 2, r(x) = 2\cos^2 x + 10e^x = \cos 2x + 1 + 10e^x$.

Now

$$u = -\int y_2 r/W \, dx = \frac{\cos 4x}{16} + \frac{\cos 2x}{4} - e^x(\sin 2x - 2\cos 2x)$$

$$v = \int y_1 r/W \, dx = \frac{\sin 4x}{16} + \frac{x}{4} + \frac{\sin 2x}{4} + e^x(2\sin 2x + \cos 2x)$$

Thus

$$y_p = \frac{\cos 2x}{16} + \frac{x \sin 2x}{4} + \frac{1}{4} + 2e^x$$

General solution: (absorbing first term of y_p in the homogeneous solution)

$$y = A\cos 2x + B\sin 2x + \frac{x\sin 2x}{4} + \frac{1}{4} + 2e^x$$

(ii) $y_1 = \cos x, y_2 = \sin x, W(y_1, y_2) = 1, r(x) = x\sin x$. Now

$$u = -\int y_2 r/W\,dx = -\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}$$
$$v = \int y_1 r/W\,dx = -\frac{x\cos 2x}{4} + \frac{\sin 2x}{8}$$

Thus

$$y_p = \frac{\cos x}{8} + \frac{x\sin x}{4} - \frac{x^2\cos x}{4}$$

General solution: (absorbing first term of y_p in the homogeneous solution)

$$y = A\cos x + B\sin x + \frac{x\sin x}{4} - \frac{x^2\cos x}{4}$$

(iii) $y_1 = \cos x, y_2 = \sin x, W(y_1, y_2) = 1, r(x) = \cot^2 x$. Now

$$u = -\int y_2 r/W\,dx = -\ln(\operatorname{cosec} x - \cot x) - \cos x$$
$$v = \int y_1 r/W\,dx = -\operatorname{cosec} x - \sin x$$

Thus

$$y_p = -2 - \cos x\ln(\operatorname{cosec} x - \cot x)$$

General solution:

$$y = A\cos x + B\sin x - 2 - \cos x\ln(\operatorname{cosec} x - \cot x)$$

(iv) $y_1 = x$ is a solution of the homogeneous part. To find another linearly independent solution we assume $y = xu$. This gives

$$u'' - u' = 0 \Longrightarrow u' - u = 1 \Longrightarrow u = e^x - 1 \Longrightarrow y = xe^x - x$$

Since $y_1 = x$, we take $y_2 = xe^x$. The nonhomogeneous part is written as

$$y'' - \frac{x+2}{x}y' + \frac{(x+2)}{x^2}y = x$$

Thus $r(x) = x$ and $W(y_1, y_2) = x^2 e^x$. Now

$$u = -\int y_2 r / W dx = -x$$

and

$$v = \int y_1 r / W dx = -e^{-x}$$

Thus $y_p = -x - x^2$.
General solution: (absorbing first term of y_p in the homogeneous solution)

$$y = x\left(A + Be^x\right) - x^2$$

20. Find the general solution of a 7th-order homogeneous linear differential equation with constant coefficients whose characteristic polynomial is $p(m) = m\left(m^2 - 3\right)^2\left(m^2 + m + 2\right)$.

Solution: $m = 0, \pm\sqrt{3}, \pm\sqrt{3}, -1/2 \pm i\sqrt{7}/2$. So general solution:
$y = c_1 + c_2 e^{\sqrt{3}x} + c_3 x e^{\sqrt{3}x} + c_4 e^{-\sqrt{3}x} + c_5 x e^{-\sqrt{3}x} + c_6 e^{-x/2}\cos(\sqrt{7}x/2) + c_7 e^{-x/2}\sin(\sqrt{7}x/2)$.

21. Solve: (i) $x^2 y'' + 2xy' - 12y = 0$ (ii)(T) $x^2 y'' + 5xy' + 13y = 0$ (iii) $x^2 y'' - xy' + y = 0$
[Note: The ODE of the form $x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$, where a, b are constants, is called the Cauchy-Euler equation. Under the transformation $x = e^t$ (when $x > 0$) for the independent variable, the above reduces to $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0$, which is an equation with constant coefficients.]

Solution: (i) Using the substitution $x = e^t$, the given equation reduces to

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 12u = 0 \Longrightarrow m^2 + m - 12 = 0 \Longrightarrow m = -4, 3 \Longrightarrow u(t) = Ae^{-4t} + Be^{3t} = y\left(e^t\right).$$

The general solution is thus

$$y(x) = \frac{A}{x^4} + Bx^3.$$

(ii) Using the substitution $x = e^t$, the given equation reduces to,

$$\frac{d^2u}{dt^2} + 4\frac{du}{dt} + 13u = 0 \Longrightarrow m^2 + 4m + 13 = 0 \Longrightarrow m = -2 \pm 3i.$$

Thus

$$u(t) = e^{-2t}(A\cos 3t + B\sin 3t) = y\left(e^t\right).$$

The general solution is

$$y(x) = \frac{1}{x^2}(A\cos(3\ln x) + B\sin(3\ln x)).$$

(iii) Using the substitution $x = e^t$, the given equation reduces to

$$\frac{d^2u}{dt^2} - 2\frac{du}{dt} + u = 0 \Longrightarrow m^2 - 2m + 1 = 0 \Longrightarrow m = 1, 1 \Longrightarrow u(t) = e^t(A + Bt) = y\left(e^t\right)$$

The general solution is thus

$$y(x) = e^x(A + B\ln x).$$