# Cardinality

**MA 1201**

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# References

# Books

- *A Foundation Course in Mathematics* by Ajit Kumar, S. Kumaresan and B. K. Sharma
- *Analysis - I* by T. Tao
- *Schaum's Outline Of Set Theory and Related Topics* by S. Lipschutz
- *Introduction to real analysis* by S. K. Mapa
- *Proof from the book* by Martin Aigner , Gunter M. Ziegler, et al.

# Notations

# Usual notations

$\mathbb{N}$: Natural numbers, $\{1, 2, 3, \ldots, \}$

$\mathbb{Z}$: Integers

$\mathbb{Q}$: Rational numbers

$\mathbb{R}$: Real numbers

$\mathbb{C}$: Complex numbers

Given a set $A$, let $\mathcal{P}(A)$ denotes its power set.

When $A$ contains $n$ many elements, then $\mathcal{P}(A)$ contains $2^n$ many elements.

# Finite set

# Natural intuitions

Finite set: One can count and finish counting!

We say the set contains those many elements!

A natural question: When two sets have same number of elements?

For finite set, the answer could be found by simply counting the number of elements. All the sets below

$$\{a, b, c, d\},\ \{2, 3, 5, 7\},\ \{x, y, z, t\}$$

have 4 elements.

## Alternate way

However, it is not always necessary to know the number of elements in two sets to know that they have same number of elements!

Suppose a number of people board a bus.

When we will say that the number of people is the same as the number of seats in the bus?

Let all the people sit down.

If everyone finds a seat and no seat remains empty, then and only then two sets have same number of elements - basically $1-1$, onto correspondence from people to seat.

# Nothing new

In fact, we learnt to count object by forming an $1-1$ correspondence between objects and fingers or the lines on the finger.

If one asks the question:

How many days are there until Saturday?

The response is often to actually pair the remaining days with one's finger.

Mathematically, we make a $1-1$, onto correspondence with the set of objects to

$$I_n := \{1, 2, \ldots, n\}.$$

This lead us to the following definition.

# Same cardinality

**Definition.** Two sets $A$ and $B$ are said to have *same cardinality* (that is, same no of elements or same size) if $\exists$ a function $f : A \to B$ which is bijective, that is, both $1 - 1$ and onto.

**Exc.** Let $X$ be a set and $A, B \subseteq X$. Let $A \sim B$ if and only if $A$ and $B$ have same cardinality. Show that $\sim$ is an equivalence relation on $\mathcal{P}(X)$.

Why suddenly we put the sets inside a set $X$ - to avoid a logical fallacy.

**Russell's paradox**

A barber in a city shaves only those who do not shave themselves. The questions is whether the barber shaves himself or not!

# Finite set and cardinal number

**Definition.** A set $A$ is said to be *finite* if $A$ is either $\varnothing$ or $\exists\, n \in \mathbb{N}$ and a bijection $f : I_n \to A$ (or equivalently, $\exists$ a bijection $g : A \to I_n$).

In this case, we say $A$ has *cardinality* $n$.

We say cardinality is 0 if $A$ is an empty set.

Though the earlier exercise allows us to define cardinality, there might be one problem with this definition: a set might have two different cardinalities!!! (well-definedness)

BUT, this is not possible!

# A Lemma

**Lemma**

*Suppose $n \in \mathbb{N}$ and $A$ has cardinality $n$. Then $A$ is non-empty, and if $a$ is an element of $A$, then the set $A \setminus \{a\}$ has cardinality $n-1$.*

**Proof.** $A$ is non-empty as there is no bijection (function) from a non-empty set to an empty set.

Let $g : A \to I_n$ be a bijection. As $g(a) \in \mathbb{N}$ and $g$ is $1-1$, the either $g(x) < g(a)$ or $g(x) > g(a)$. Define $h : A \setminus \{a\} \to I_{n-1}$ by

$$h(x) = \begin{cases} g(x), & \text{if } g(x) < g(a), \\ g(x) - 1, & \text{if } g(x) > g(a). \end{cases}$$

# Proof ctd.

Note $h$ is $1-1$ as if $h(y)=h(z)=i$, then for $i<g(a)$, $g(y)=g(z)$ gives $y=z$ and if $i\geq g(a)$ (this case may not appear as well), $g(y)-1=g(z)-1$ gives $y=z$.

$h$ is onto since for $i<g(a)$, $h^{-1}(i)=x$ if $g(x)=i$ and for $i\geq g(a)$, $h^{-1}(i)=x$ if $g(x)=i+1$.

## Uniqueness of cardinality

**Proposition**

*Let A be a set with cardinality n. Then A can not have any other cardinality, that is, if there is another bijection between $I_m$ and A, then $m = n$.*

**Proof.** Induction on $n$. Suppose $n = 1$. If there exist a bijection from $I_m$ for some $m$, then there is a bijection $f$ from $I_m$ to $I_n$. So if $m > 1$, $f(1) = f(m) = 1$ contradicts that $f$ is $1 - 1$. Hence $m = 1$.

Induction hypothesis: the statement is true for $n = k$.

Let $n = k + 1$. Suppose it has cardinality $m$ as well. Then $A$ is non-empty and for $a \in A$, the set $A \setminus \{a\}$ has cardinality $k$ and $m - 1$. By induction hyothesis, $k = m - 1$ and hence, we are done.

# Well Ordering Principle in $\mathbb{N}$

**Theorem**

*Every non-empty subset of $\mathbb{N}$ has a least element.*

**Proof.** Let $A \subseteq \mathbb{N}$ be a non-empty subset.

Proof by contradiction. Suppose $A$ doesn't have a least element.

Consider $S = \mathbb{N} \setminus A$. We shall show $S = \mathbb{N}$.

We will use induction.

$1 \in S$, otherwise $A$ has a least element.

Induction hypothesis: $\{1, \ldots, k\} \subseteq S$

$k + 1 \in S$.

By induction, $S = \mathbb{N}$ contradiction to the fact that $A$ is non-empty.

# Interesting Proposition

Let us denote cardinality of a set $A$ by $|A|$.

**Proposition**

*If $f : A \to I_m$ is $1-1$, then $A$ is finite and $|A| \leq m$.*

**Proof.** By well ordering principle, let

$$m_1 := \min_{a \in A} f(a).$$

Clearly, $m_1 \geq 1$. Then consider

$$m_2 := \min f(A) \setminus \{m_1\} = \min\{y : y \in f(A) \text{ such that } y \neq m_1\}.$$

Note that, we have $m_2 \geq 2$.

# Proof continued

So at $k$-th step, we have $m_k \geq k$ such that

$$m_k := \min f(A) \setminus \{m_1, \ldots, m_{k-1}\}.$$

We claim that $\exists\, \ell$ such that $f(A) = \{m_1, \ldots, m_\ell\}$ with $\ell \leq m$. Otherwise, if $\ell > m$, then $m_\ell \geq \ell > m$ contradicts $m_\ell \in I_m$.

Define $g : A \to I_\ell$ by $g(a) = i$ if $f(a) = m_i$.

$g$ is onto by construction.

Since $f$ is $1-1$, the map $g$ is $1-1$ and hence is a bijection.

Therefore, $A$ is a finite set with $|A| \leq m$.

$$\begin{tikzcd} A \arrow[r, "f", "\text{bij}"'] \arrow[dr, "g"'] & f(A) \arrow[d] & \ni m_\ell \arrow[d, mapsto] \\ & I_\ell & \ell \end{tikzcd}$$

**Corollary**

*If there exists a $1-1$ map $f : I_n \to I_m$, then $n \leq m$, that is, if $n > m$, then there is no $1-1$ map from $I_n$ into $I_m$*

## The proposition on onto

**Proposition**

*If $f : I_n \to A$ is onto, then $A$ is finite and $|A| \leq n$.*

**Proof.** Define $g : A \to I_n$ by setting

$$g(a) = \min f^{-1}(a) = \min_{x \in I_n : f(x) = a} x.$$

Then $g$ is $1-1$ map as $g(a_1) = g(a_2)$ implies $a_1 = f(x) = a_2,$ and we are done by the previous proposition.

Note that in the proof above it is not neccessary to choose the minimum, you can choose exactly one from the pre-image set $f^{-1}(a)$ for each $a \in A$.

**Corollary**

*If there exists a onto map $f : I_n \to I_m$, then $m \leq n$, that is, if $m > n$, then there is no onto map from $I_n$ to $I_m$.*

# Cardinal arithmetic

**Exc.** If $A$ is a finite set and $B \subseteq A$, then $B$ is finite and $|B| \leq |A|$.

*Hint.* Consider the inclusion map of $B$ tp $A$ and compose it with the bijection of $A$ to $I_n$.

**Exc.** If $A$ is a finite set and $B$ is a proper subset of $A$, then $|B| < |A|$.

*Hint.* $\exists\, a \in A$ such that $B \subseteq A \setminus \{a\}$.

**Exc.** If $A$ is a finite set, then $|A \cup \{a\}| = |A| + 1$.

**Exc.** If $A, B$ are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

# Infinite set

# Infinite set

**Definition.** A set is *infinite* if it is not finite.

Now - an example of an infinite set.

**Theorem**

*$\mathbb{N}$ is infinite.*

**Proof.** Proof by contradiction. Suppose $\exists$ a bijection from $f : I_n \to \mathbb{N}$ for some $n \in \mathbb{N}$.

Let $M := \max\{f(1), \ldots, f(n)\}$.

But then the natural number $M + 1 \notin f(I_n)$ contradicting that $f$ is a bijection.

**Theorem**

*The set of prime numbers is infinite.*

# The first type of infinity

**Theorem**

*A set $A$ is an infinite set if and only if $\exists$ a $1-1$ map $f:\mathbb{N}\to A$.*

**Proof.** (Only if) If $A$ is an infinite set, then there exist a $1-1$ map $f:\mathbb{N}\to A$.

Choose an element from $A$, name it $a_1$ and define $f(1)=a_1$.

Choose $a_k \in A\setminus\{a_1,\ldots,a_{k-1}\}$ (we can do so as $A$ is infinite) and define $f(k)=a_k$.

(If) Suppose $A$ is not infinite, then $f(\mathbb{N})\subseteq A$ is finite.

Since $f:\mathbb{N}\to f(\mathbb{N})$ is a bijection, we have $\mathbb{N}$ finite - a contradiction.

# Countable set

**Definition.**

- A set $A$ is said to be *countably infinite* or just *countable* if it has same cardinality with $\mathbb{N}$, that is, $\exists$ a bijection between $\mathbb{N}$ and $A$.
- A set is called *atmost countable* if it is finite or countable.
- An infinite set is called *uncountable* if it is not countable.

The previous Theorem says that every infinite set contains a countable subset.

# Examples of countable sets

- Obviously, $\mathbb{N}$ is countable.
- $\mathbb{N} \setminus \{1\}$ is countable as $f : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ given by $f(n) = n + 1$ is a bijection - example of first departure from finite set - a proper subset having the same cardinality with the set itself.
- $2\mathbb{N}$ - the set of even numbers - countable: $f : \mathbb{N} \to 2\mathbb{N}$ given by $f(n) = 2n$ is a bijection.

# First observation

**Theorem**

*A set is infinite if and only if it has a proper subset with same cardinality to itself.*

**Proof.** Note if a set is finite then it can not have same cardinality with any of its proper subset.

So contrapositively, we have the if part.

(Only if) Let $A$ be an infinite set. We have already seen that $\exists$ a $1-1$ map from $f: \mathbb{N} \to A$.

Let $f(n) = a_n$.

Define $g : A \to A \setminus \{a_1\}$ by

$$g(x) = \begin{cases} x, & \text{if } x \in A \setminus \{a_1, a_2, \ldots, a_n \ldots\}, \\ a_{n+1}, & \text{if } x = a_n \text{ for some } n \in \mathbb{N}. \end{cases}$$

$g$ is a bijection.

## 2nd observation

**Theorem**

*Any infinite subset of $\mathbb{N}$ is countable.*

**Proof.** Let $A$ be an infinite subset of $\mathbb{N}$. Thus, we can construct $m_k \geq k$ such that

$$m_k := \min A \setminus \{m_1, \ldots, m_{k-1}\}.$$

Define $f : \mathbb{N} \to A$ by $f(k) = m_k$. Clearly, $f$ is $1 - 1$.

To prove $f$ onto, we need to show that for each $a \in A$ $\exists n$ such that $a = m_n$.

Note $A \cap I_k \subseteq \{m_1, \ldots, m_k\}$. Since $a \in \mathbb{N}$, there exist a natural number $N$ such that $a \leq N$ (?).

So $a \in A \cap I_N$ and hence we are done.

**Exc.** Any subset of a countable set is atmost countable.

# Adjoining an element

Take $x$ a new element, $\mathbb{N} \cup \{x\}$ is countable: $f : \mathbb{N} \to \mathbb{N} \cup \{x\}$ given by $f(1) = x$ and $f(n) = n - 1$, $n \geq 2$, is a bijection.

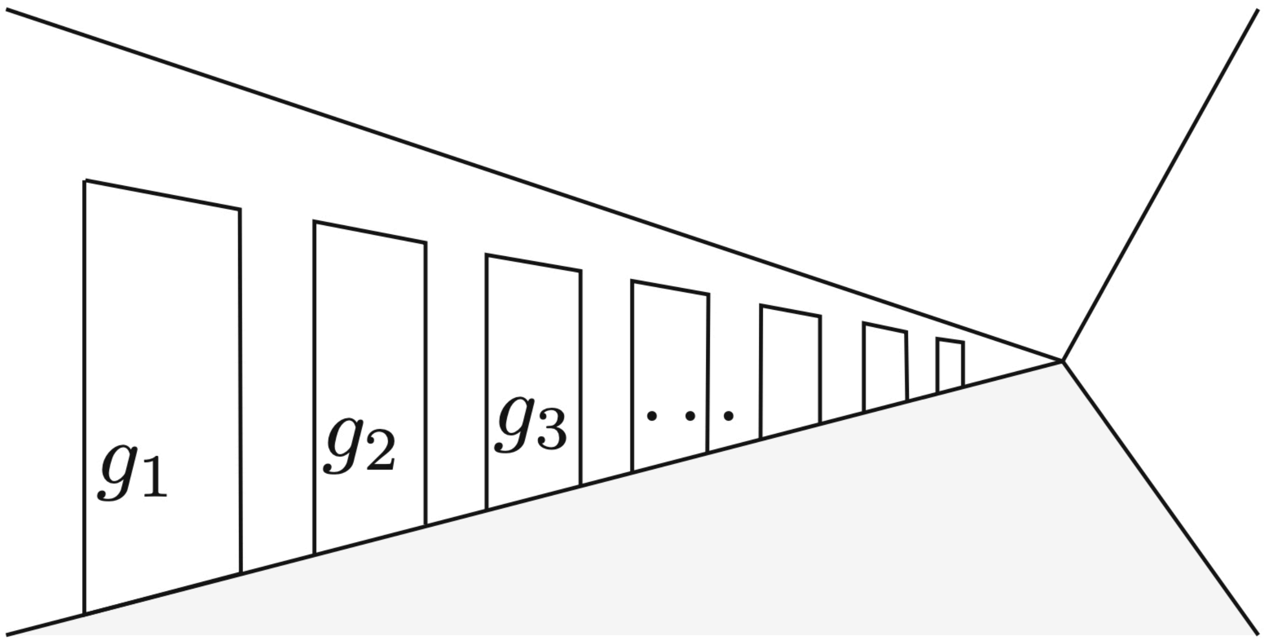
Similarly if $A$ is a countable set, then $A \cup \{x\}$ is countable.

If $f : \mathbb{N} \to A$ is a bijection and $f(n) = a_n$, then $A = \{a_1, \ldots, a_n, \ldots\}$. The map $g : \mathbb{N} \to A \cup \{x\}$ defined by

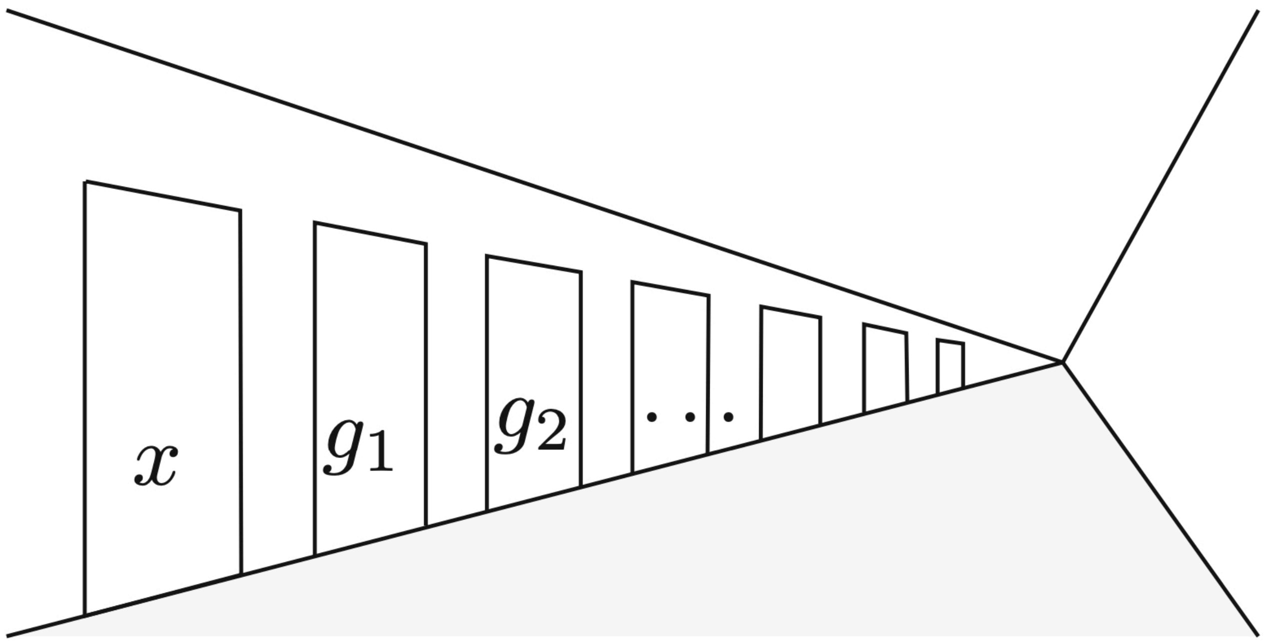
$$g(1) = x \text{ and } g(n) = a_{n-1} \text{ for } n \geq 2,$$

is a bijection.

# Hilbert's hotel



# Hilbert's hotel



# Union of countable sets

Can we accommodate any finite no. of people?

Clearly, for a countable set $A$, the set $A \cup \{x_1, x_2, \ldots, x_k\}$ is countable,since we have the bijection $g(i) = x_i$ for $1 \leq i \leq k$ and $g(n) = a_{n-k}$ for $n \geq k+1$.

What if a countable no. of people come?

Evidently, this question boils down to whether union of two countable set is countable!

## Finite union of countable sets

**Theorem**

*If $A$ and $B$ are countable sets, then $A \cup B$ is a countable set.*

**Proof.** Without loss of generality (WLOG) one can assume $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ and $B = \{b_1, b_2, \ldots, b_n, \ldots\}$. The map $g : \mathbb{N} \to A \cup B$ given by

$$g(n) = \begin{cases} a_k, & \text{if } n = 2k \text{ for some } k \in \mathbb{N}, \\ b_k, & \text{if } n = 2k - 1 \text{ for some } k \in \mathbb{N}. \end{cases}$$

is a bijection. Are we missing something?

- $\mathbb{N} \cup \{0\}, \mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$ are countable.

# Some more observations

**Corollary**

*Finite union of countable sets is countable.*

**Exc.** For any infinite set $A$ and a countable set $B$, the sets $A$ and $A \cup B$ are of same cardinality.

*Hint.* Let $\{a_1, a_2, \ldots, a_n \ldots\} \subseteq A$. Consider the map $g : A \to A \cup B$ given by

$$g(x) = \begin{cases} x, & \text{if } x \in A \setminus \{a_1, a_2, \ldots, a_n \ldots\}, \\ a_k, & \text{if } x = a_{2k} \text{ for some } k \in \mathbb{N}, \\ b_k, & \text{if } x = a_{2k-1} \text{ for some } k \in \mathbb{N}. \end{cases}$$

# Countable union

**Theorem**

*Countable union of countable sets is countable.*

**Proof.** WLOG, let $A_1, A_2, \ldots, A_n, \ldots$ be a countable family of countable sets. Also let WLOG

$$\begin{array}{rcl} A_1 & = & \{a_{11}, a_{12}, \ldots, a_{1n,\ldots}\} \\ A_2 & = & \{a_{21}, a_{12}, \ldots, a_{2n,\ldots}\} \\ \vdots & & \vdots \quad \ldots \quad \vdots \quad \ldots \\ A_n & = & \{a_{n1}, a_{n2}, \ldots, a_{nn,\ldots}\} \\ \vdots & & \vdots \quad \ldots \quad \vdots \quad \ldots \end{array}$$

Let $A = \bigcup_{i=1}^{\infty} A_i$.

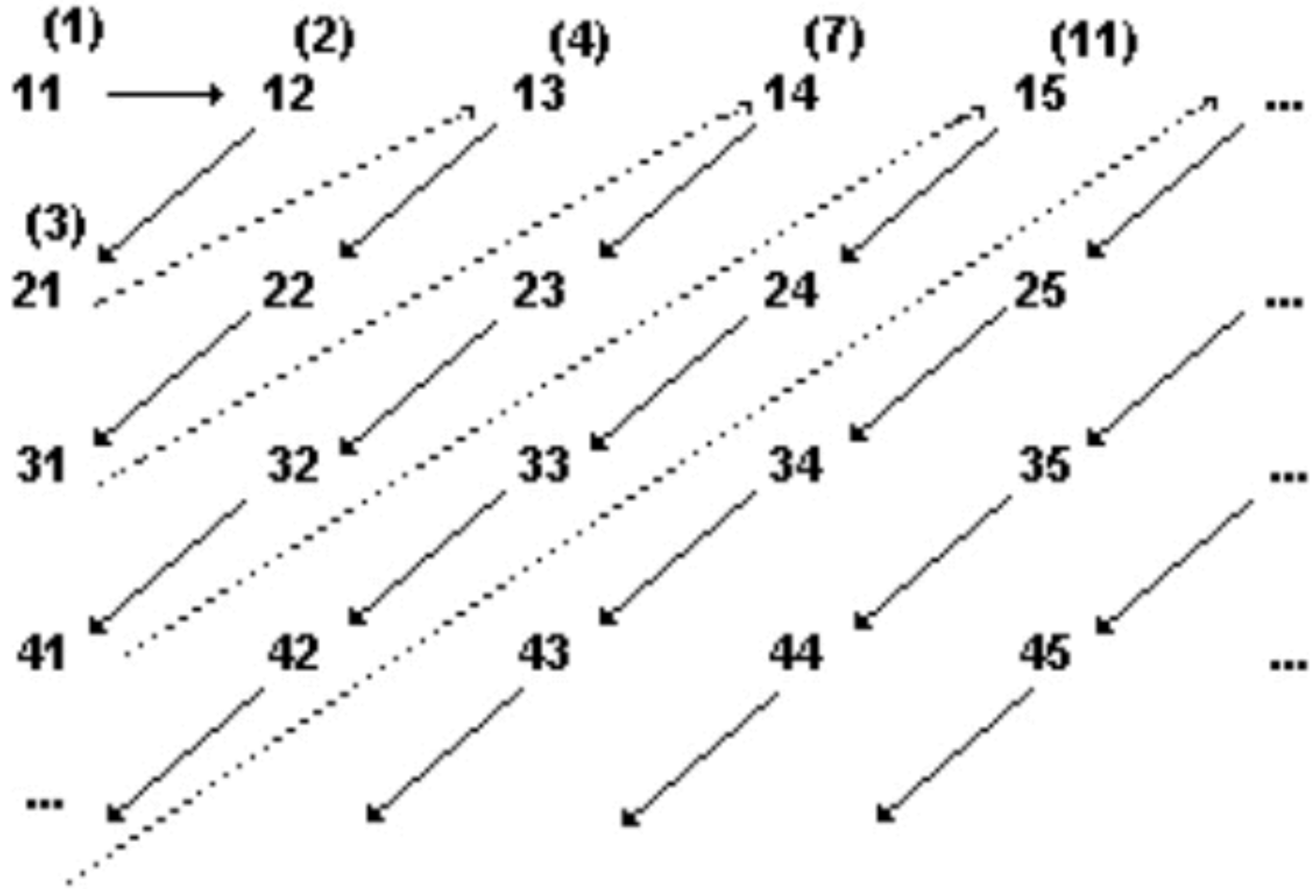
**Case I.** $A_i \cap A_j = \varnothing$ for all $i \neq j$.

$$
\begin{array}{ccccccccc}
a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
a_{21} & & a_{22} & & a_{23} & & a_{24} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
a_{31} & & a_{32} & & a_{33} & & a_{34} & & \cdots \\
 & \swarrow & & \nearrow & & & & & \\
a_{41} & & a_{42} & & a_{43} & & a_{44} & & \cdots \\
\downarrow & \nearrow & & & & & & & \\
a_{51} & & a_{52} & & a_{53} & & a_{54} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

# Idea of bijection

$$
\begin{tikzcd}
1 \arrow[r] & 2 \arrow[dl] & 6 \arrow[r] & 7 \arrow[dl] & 15 \arrow[r] & \cdots \\
3 \arrow[d] & 5 \arrow[ur] & 8 \arrow[dl] & 14 \arrow[ur] & \cdots & \\
4 \arrow[ur] & 9 \arrow[dl] & 13 \arrow[ur] & \cdots & & \\
10 \arrow[d] & 12 \arrow[ur] & \cdots & & & \\
11 \arrow[ur] & \cdots & & & &
\end{tikzcd}
$$

# Tweak a little for explicit bijection



## The bijection

Note that

$$\mathbb{N} = \bigcup_{m\in\mathbb{N}} (\frac{m(m-1)}{2}, \frac{m(m+1)}{2}]$$

and that this union is disjoint.

Also note that the arrows are like slanting lines joining $(1, m)$ to $(m, 1)$ with any point $(r, s)$ of the $m$ points lying on it satisfies $r + s = m + 1$.

So given $n \in \mathbb{N}$, $\exists\ m$ such that

$$\frac{m(m-1)}{2} < n \leq \frac{m(m+1)}{2}$$

Define $f : \mathbb{N} \to A$ by $f(n) = a_{rs}$ with $r + s = m + 1$ and $r = n - \frac{m(m-1)}{2}$, $r \geq 1$.

This proof is not yet complete, however, we observed that this might lead to countability of $\mathbb{Q}$ and $\mathbb{N} \times \mathbb{N}$!