Lecture 02 : 29/01/25

Let us return to elimination method.

$$2u + v + w = 5 \quad \text{— (i)}$$

$$4u - 6v = -2 \quad \text{— (ii)}$$

$$-2u + 7v + 2w = 9 \quad \text{— (iii)}$$

Exc: Write it in $Ax = b$ form.

STEP 0 : $\longrightarrow$ u — first variable
v — 2nd variable
w — 3rd variable

In all the equations have u have nonzero co-efficient — So we keep the order of the equation as it is.

STEP 1 :

$$\text{i)} \longrightarrow 2u + v + w = 5 \quad \text{— (I)}$$

$$\text{ii)} - 2\,\text{i)} \longrightarrow -8v - 2w = -12 \quad \text{— (II)}$$

$$\text{iii)} \longrightarrow -2u + 7v + 2w = 9 \quad \text{— (III)}$$

The process above can be demonstrated with the help of the "Elementary matrix"

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To understand better let us consider the following matrix :

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ — Known as identity matrix

Note that

$$Ix = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = x \quad \text{— } (*)$$

So I gives back the same element
'almost like multiplying by 1.

Observe that if we replace 2nd row of I by row 2 − 2× row 1, then we get E.

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b-2a \\ c \end{pmatrix} \quad = I\cdot\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

looking at $(*)$ you can think of E acting on I - making the row actions.

coming back to our equations

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2u+v+w \\ 4u-6v \\ -2u+7v+2w \end{pmatrix} = \begin{pmatrix} 2u+v+w \\ -8v-2w \\ -2u+7v+2w \end{pmatrix}$$

- This is exactly what we did to our system of equations (i), (ii), (iii) to go to (I), (II), (III).

- Also note that

$$E \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ 9 \end{pmatrix}$$

Altogether we see that

$$E\left(Ax\right) = Eb.$$

This suggest – we must know matrix multiplication in more general setup so that these calculations become more formal.

$$E\,(Ax) = \underbrace{(EA)}_{\text{need to know}}\,(x).$$

— Someway also says about associativity of matrix multiplication.

It turns out that this can be achieved by defining product of two matrices $A_{m\times n}$ and $B_{n\times k}$ as follows:

If $b_1, b_2, \dots b_k$ – denotes the columns ($n\times 1$-matrices) of B, then AB is an $m\times k$ matrix whose columns are

$$\underbrace{Ab_1 \quad Ab_2 \quad \cdots\cdots \quad Ab_k}_{k\text{ – columns}}.$$

Each of them are $m\times 1$ matrix

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}_{2\times 3}, \quad B = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & -1 & 2 & -2 \\ 3 & 0 & 1 & 0 \end{pmatrix}_{3\times 4}$$

$AB =$

$$\begin{pmatrix} 1\cdot 0 + 2\cdot 1 + 3\cdot 3 & 1\cdot 1 + 2\cdot(-1) + 3\cdot 0 & 1\cdot 2 + 2\cdot 2 + 3\cdot 1 & 1\cdot 3 + 2\cdot(-2) + 3\cdot 0 \\ 0\cdot 0 + 1\cdot 1 + 2\cdot 3 & 0\cdot 1 + 1\cdot(-1) + 2\cdot 0 & 0\cdot 2 + 1\cdot 2 + 2\cdot 1 & 0\cdot 3 + 1\cdot(-2) + 2\cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -1 & 9 & -1 \\ 7 & -1 & 4 & -2 \end{pmatrix}$$

So if $A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$ — $m\times n$ matrix

& $B = (b_{ij})_{\substack{1\le i\le n \\ 1\le j\le k}}$ — $n\times k$ matrix.

Then $AB = (c_{ij})_{\substack{1\le i\le m \\ 1\le j\le k}}$ — $m\times k$ matrix

$AB =$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & & & \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nk} \end{pmatrix} = \begin{pmatrix} * & \cdots & * & \cdots & * \\ * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & c_{ij} & \cdots & * \\ \vdots & & \vdots & & \\ * & \cdots & * & \cdots & * \end{pmatrix}$$

i^{th} row

j^{th} column (b_j)

i^{th} row

j^{th} colm.

The entry c_{ij} is i^{th} entry of the column Ab_j

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

$$= \sum_{l=1}^{n} a_{il}b_{lj}$$

(you find the ij^{th} entry of product matrix by multiplying i^{th} row of A with j^{th} column of B)

Facebook algorithm

A, B, C, D, E — Five characters

The following table indicates whether one character is friend of any other character!

	A	B	C	D	E
A	0	0	1	1	1
B	0	0	0	1	1
C	1	0	0	1	0
D	1	1	1	0	0
E	1	1	0	0	0

Here if X is friend of Y, then we assume Y is also friend of X - we give the value 1 accordingly, otherwise 0 in relevant place.

- X is not considered friend of X — so we have 0 on the diagonals.

look at F^2, that is, multiply F by F, the table with F^2 would look like.

Let F denote the matrix $\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$

$$F^2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}$$

now if we encode this in table

	A	B	C	D	E
A	3	2	1	1	0
B	2	2	1	0	0
C	1	1	2	1	1
D	1	0	1	3	2
E	0	0	1	2	2

the diagonal entries denotes no. of friends for each character and the off diagonal entries denotes the no. of mutual friends between two characters.

Facebook replicate this computation for very high order matrices

Properties of matrices:

Now that we have defined the product of two matrices A and B of orders $m \times n$ and $n \times K$

— we discuss some properties of matrix multiplication.

But before that we observe one can define addition of two matrices A & B of same order $m \times n$ as well:

If $A = (a_{ij})$ and $B = (b_{ij})$

then $A + B = (c_{ij})$

where $c_{ij} = a_{ij} + b_{ij}$ for every $1 \leq i \leq m$, $1 \leq j \leq n$.

Example:

If $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 7 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 4 \\ 5 & 6 & 7 \end{pmatrix}$

then

$$A + B = \begin{pmatrix} 1+0 & 2+1 & 3+4 \\ -1+5 & 7+6 & 0+7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 7 \\ 4 & 13 & 7 \end{pmatrix}.$$

Easily verifiable facts:

1) **Associativity:**

multiplication - $A(BC) = (AB)C$

— So we just write ABC.

addition $A + (B+C) = (A+B) + C$

— So we just write $A+B+C$.

2) **Distributivity:**

of multiplication over addition

$$A(B+C) = AB + AC$$

$$(B+C)D = BD + CD.$$

3) **commutativity:**

Addition - $A+B = B+A$. - commutative

Multiplication - There exists square $n \times n$ matrices A and B such that

$$AB \neq BA$$

Matrix multiplication is NON-COMMUTATIVE

Example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

Coming back to our equation again:

The aim is to write the equations

$$\left.\begin{array}{rrrrrrr} \cdot\, u & + & \cdot\, v & + & \cdot\, w & = & \_\_ \\ & & \cdot\, v & + & \cdot\, w & = & \_\_ \\ & & & + & \cdot\, w & = & \_\_ \end{array}\right\}$$

– Called a triangular system.

We want these all to be non zero. $\boxed{\text{PIVOT}}$

↳ corresponding matrix have the form

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix}$$

diagonal.

The entries in the diagonal are called PIVOT

– So Try to write the system of eqn's in triangular form – The entries in the diagonal of its associated matrix are called PIVOTS.

Upper triangular matrix:

An $n \times n$ matrix $A = (a_{ij})$ is said to be upper triangular if all the entries below the diagonal are zero, that is

$$a_{ij} = 0 \quad \text{for all} \quad 1 \le j < i \le n.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

diagonal.

So we have seen that when we perform the Gaussian elimination, the final system of equations (in n-equations with n-unknowns) turns out to be in upper triangular form.

Similarly, one defines lower triangular matrix to be the one where all the entries above the diagonal are zero, that is,

$$a_{ij} = 0 \quad \text{for all} \quad 1 \le i < j \le n.$$

Example:

$$\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

upper triangular

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{pmatrix}$$

lower triangular.

Recall the system of linear equations:

$$2u + v + w = 5 \quad \text{— (i)}$$

$$4u - 6v = -2 \quad \text{— (ii)}$$

$$-2u + 7v + 2w = 9. \quad \text{— (iii)}$$

which is written in matrix form as below:

$$Ax = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} = b.$$

Recall the elimination steps:

STEP. I:

Subtract twice equation i) from equation ii) and put it in place of equation (ii).

$$2u + v + w = 5$$

$$-8v - 2w = -12$$

$$-2u + 7v + 2w = 9$$

& the corresponding elimination matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the new system of equation in matrix form:

$$(EA)(x) = Eb.$$