Lecture 25.03.2025.

We have seen that

Rotation by 90° is the operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

can be written as $T(x, y) = (-y, x)$ as well.

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\hookrightarrow$ same as the action of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $\begin{pmatrix} x \\ y \end{pmatrix}$.

Illy when we take projection

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

— the action is given by matrix multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

<u>Question</u>: Whether every linear map can be represented by a matrix — somewhat like the above.

Let $T : V \longrightarrow W$ be linear & $\{v_1, \dots, v_n\}$ be a an ORDERED basis of V.

($\{v_2, v_1, \dots, v_n\}$ is also a basis — but in our consideration it is a different basis.

Take $v \in V$, $\exists\, \alpha_1 \ldots \alpha_n \in \mathbb{F}$ such that

$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n.$$

(For example consider $\mathbb{R}^2$ & let
$\{(1,0), (1,1)\}$ is a basis of $\mathbb{R}^2$
As told earlier $\{(1,1), (1,0)\}$ is a different basis of $\mathbb{R}^2$

$(2, 3) \in \mathbb{R}^2$, then

$$(2,3) = -1.(1,0) + 3.(1,1).$$

So here $\alpha_1 = -1$, $\alpha_2 = 3$.

but if you take $\{(1,1), (1,0)\}$ a basis of $\mathbb{R}^2$

Then $(2,3) = 3.(1,1) + (-1)(1,0)$

so $\alpha_1 = 3$, $\alpha_2 = -1$ here.)

Now $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$.
a vector in V.

T - linear

$$\Rightarrow Tv = T(\alpha_1 v_1 + \ldots + \alpha_n v_n)$$
$$= T(\alpha_1 v_1) + \cdots + T(\alpha_n v_n)$$
$$= \alpha_1 Tv_1 + \cdots + \alpha_n Tv_n.$$

Since we know $\alpha_1 \ldots \alpha_n$ – the co.eff of v.

To know the linear map T that how it

acts on an arbitrary vector v, it is enough to know $Tv_1, \dots, Tv_n$.

So a map on $\mathbb{R}^m$ (– uncountably many vectors) to a linear map it is enough to know the action of the linear on $\underline{\underline{m}}$ many vectors.

$= \dim \mathbb{R}^m$

So from $T : V \longrightarrow W$ define

$$\left.\begin{array}{l} Tv_1 = u_1 \\ \vdots \\ Tv_n = u_n \end{array}\right\}$$

where $u_1, u_2, \dots, u_n \in W$ are any n-vectors in W.

$$Tv = \alpha_1 u_1 + \cdots + \alpha_n u_n$$

where $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.

gives you the linear map.

<u>Example:</u> ① Find the linear map

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

which take
$$(1,0) \longmapsto (2,3,4)$$
$$(0,1) \longmapsto (5,9,2)$$

that is
$$T(1,0) = (2,3,4)$$
$$T(0,1) = (5,9,2).$$

Note any vector in $\mathbb{R}^2$ is (x, y)

$$(x, y) = x(1,0) + y(0,1)$$

$$\Rightarrow \quad T(x,y) = x\,T(1,0) + y\,T(0,1)$$

$$= x(2,3,4) + y(5,9,2)$$

$$= (2x+5y,\ 3x+9y,\ 4x+2y).$$

$\longrightarrow$ linear map.

Example: 2 Find the linear map

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that

$$T(1,0,0) = (4,3)$$

$$T(1,1,0) = (2,-1)$$

$$T(1,1,1) = (1,0).$$

First note $\mathcal{B} = \{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of $\mathbb{R}^3$.

We are required to show

$\mathcal{B}$ - is LI and $\mathcal{B}$ span $\mathbb{R}^3$.

To show $\mathcal{B}$ is LI,

consider the scalar α, β, γ such that

$$(*) \quad \alpha(1,0,0) + \beta(1,1,0) + \gamma(1,1,1) = (0,0,0)$$

R.T.P $\alpha = 0 = \beta = \gamma$.

Now $(*) \Rightarrow (\alpha+\beta+\gamma, \beta+\gamma, \gamma) = (0,0,0)$

$\Rightarrow \gamma = 0$, & $\beta+\gamma = 0 \Rightarrow \beta = 0$

& $\alpha+\beta+\gamma = 0 \Rightarrow \alpha = 0$.

Remark: We can directly say that

$\text{span}\, \mathcal{B} = \mathbb{R}^3$.

This is because if $\{v_1 \dots v_k\}$ LI vectors in V and $\text{span}\{v_1, \dots v_k\} \neq V$, then if we take $v \in V$ s.t $v \notin \text{span}\{v_1 \dots v_k\}$

Then $\{v_1, \dots v_k, v\}$ is LI.

(Pf. Consider the linear combination

$\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha v = 0$.

Now if $\alpha \neq 0$, then $v = -\frac{1}{\alpha}(\alpha_1 v_1 + \dots + \alpha_k v_k)$

$\Rightarrow v \in \text{span}\{v_1, \dots v_k\}$ $\#$.

So $\alpha = 0$ & hence $\alpha_1 v_1 + \dots + \alpha_k v_k = 0$

and $\{v_1 \dots v_k\}$ LI $\Rightarrow \alpha_1 = 0 = \alpha_2 = \dots = \alpha_k$

Hence $\{v_1, v_2, \dots v_k, v\}$ is LI.).

So now if $\text{span}\{\mathcal{B}\} \subsetneq \mathbb{R}^3$

then $\exists\, v \in \mathbb{R}^3$ s.t $v \notin \text{span}\, \mathcal{B}$.

So $\mathcal{B} \cup \{v\}$ is LI & having cardinality 4.

but any LI subset must have cardinality $\leq \dim \mathbb{R}^3 = 3$. $\#$

Hence $\operatorname{Span} \mathcal{B} = \mathbb{R}^3$.

But we want to see how an arbitrary vector can be written in terms of basis vectors explicitly, to know the linear map.

So take $(x, y, z) \in \mathbb{R}^3$

$$(x, y, z) = \alpha (1, 0, 0) + \beta (1, 1, 0) + \gamma (1, 1, 1)$$

$$\Rightarrow (\alpha + \beta + \gamma, \beta + \gamma, \gamma) = (x, y, z)$$

$$\Rightarrow \gamma = z, \quad \beta = y - z, \quad \gamma = x - y.$$

So $T(x, y, z)$

$$= T \{ (x - y)(1, 0, 0) + (y - z)(1, 1, 0) + z (1, 1, 1) \}$$

$$= (x - y) T(1, 0, 0) + (y - z) T(1, 1, 0) + z T(1, 1, 1)$$

$$= (x - y)(4, 3) + (y - z)(2, -1) + z (1, 0)$$

$$= (4x - 4y + 2y - 2z + z, 3x - 3y - y + z)$$

$$= (4x - 2y - z, 3x - 4y + z)$$