1. The columns of A are n vectors from $\mathbb{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbb{R}^m$, what is the rank? If they are a basis for $\mathbb{R}^m$, what then?

2. *Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbb{R}^5$.

   (a) The equation $Ax = 0$ has only the solution $x = 0$ because ______.

   (b) For every $b \in \mathbb{R}^5$, the system $Ax = b$ is solvable because ______ .

   Note: A is invertible. Its rank is 5 .

3. *Suppose **S** is a five-dimensional subspace of $\mathbb{R}^6$. True or false?

   (a) Every basis for **S** can be extended to a basis for $\mathbb{R}^6$ by adding one more vector.

   (b) Every basis for $\mathbb{R}^6$ can be reduced to a basis for **S** by removing one vector.

4. *Prove that if **V** and **W** are three-dimensional subspaces of $\mathbb{R}^5$, then **V** and **W** must have a nonzero vector in common. [Hint: Start with bases for the two subspaces, making six vectors in all.]

5. If A is a 64 by 17 matrix of rank 11, how many independent vectors satisfy $Ax = 0$? How many independent vectors satisfy $A^{\mathrm{T}}y = 0$?

6. Find a basis for each of these subspaces of 3 by 3 matrices:

   (a) All diagonal matrices.

   (b) All symmetric matrices $\left(A^{\mathrm{T}} = A\right)$.

   (c) All skew-symmetric matrices $\left(A^{\mathrm{T}} = -A\right)$.

7. Find the dimension and a basis for the four fundamental subspaces $(C(A), C(A^T), N(A), N(A^T))$ for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

8. Suppose A is an m by n matrix of rank r. Under what conditions on those numbers does

   (a) A have a two-sided inverse: $AA^{-1} = A^{-1}A = I$?

   (b) $Ax = b$ have infinitely many solutions for every b?

9. Why is there no matrix whose row space and nullspace both contain $(1, 1, 1)$?

10. Suppose the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. What is the rank and why? The columns of A are linearly ______.

11. *Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbb{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.

12. If $Ax = 0$ has a nonzero solution, show that $A^{\mathrm{T}}y = b$ fails to be solvable for some right-hand sides b. Construct an example of A and b.

13. *Construct a matrix with the required property, or explain why you can't.

    (a) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

    (b) Column space has basis $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, nullspace has basis $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

    (c) Row space = column space.

14. What 3 by 3 matrices represent the transformations that

    (a) project every vector onto the xy plane?

    (b) reflect every vector through the xy plane?

    (c) rotate the xy plane through $90°$, leaving the z-axis alone?

    (d) rotate the xy plane, then xz plane, then yz plane, through $90°$?

    (e) rotate the xy plane, then xz plane, then yz plane, through $180°$?

15. *If $T : V \to V$ is a linear transformation, then prove that T^2 is also a linear transformation.

16. *The space $M_{2,2}(\mathbb{R})$ of all 2 by 2 matrices has the four basis "vectors"

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

    For the linear transformation T of transposing (that is, $T : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ is defined by $T(P) = P^{\mathrm{T}}$ for every $P \in M_{2,2}(\mathbb{R})$), find its matrix A with respect to the above basis. We know that $T^2(P) = \left(P^{\mathrm{T}}\right)^{\mathrm{T}} = P$, that is, $T^2 = I$. Is $A^2 = I$?

17. With $v = (v_1, v_2) \in \mathbb{R}^2$, suppose $T(v) = v$, except that $T(0, v_2) = (0, 0)$. Show that this transformation satisfies $T(c\,v) = c\,T(v)$ for every $v \in \mathbb{R}^2$ and $c \in \mathbb{R}$, but it need not satisfy $T(v + w) = T(v) + T(w)$ for some $v, w \in \mathbb{R}^2$.

18. Which of these transformations is not linear? The input is $v = (v_1, v_2) \in \mathbb{R}^2$.

    (a) $T(v) = (v_2, v_1)$; (b) $T(v) = (v_1, v_1)$; (c) $T(v) = (0, v_1)$; (d) $T(v) = (0, 1)$.

19. Suppose a linear T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(v)$ when

    (a) $v = (2, 2)$; (b) $v = (3, 1)$; (c) $v = (-1, 1)$; (d) $v = (a, b)$.

20. (a) What matrix transforms $(1, 0)$ and $(0, 1)$ to $(2, 5)$ and $(1, 3)$?

    (b) What matrix transforms $(1, 0)$ and $(0, 1)$ to (r, t) and (s, u)?

    (c) *What matrix transforms $(2, 5)$ and $(1, 3)$ to $(1, 0)$ and $(0, 1)$?

    (d) *Why does no matrix transform $(2, 6)$ and $(1, 3)$ to $(1, 0)$ and $(0, 1)$?