Problem Set - 3

MA 1201 Spring Sem, 2025

Recall each set A is assigned a symbol in such a way that two sets A and B are assigned same symbol if and only if there is a bijection between them. This symbol is called the *cardinality* or *cardinal number* of A and is denoted by $|A|$.

α is a cardinal number if there exists a set A such that $|A| = \alpha$.

Notation: $|\mathbb{N}| = \aleph_0$, $|\mathbb{R}| = \aleph_1$.

1. Let α and β be two cardinal numbers, let A and B be two disjoint sets with $|A| = \alpha$ and $|B| = \beta$. Then sum of α and β is denoted and defined by

$$\alpha + \beta := |A \cup B|.$$

   (a) Show that there exists two such disjoint sets. (Hint: A and $A \times \{1\}$ are in bijection)

   (b) Show that the sum is well-defined.

2. The product of two cardinal numbers α and β is denoted and defined by

$$\alpha\beta := |A \times B|$$

   where A and B are two sets with $|A| = \alpha$ and $|B| = \beta$. Show that the product is well defined.

3. *Let α and β be two cardinal numbers and A and B are two sets with $|A| = \alpha$ and $|B| = \beta$. Then α^β is defined by

$$\alpha^\beta := |A^B| = |\{f | f : B \to A\}|.$$

   Show that taking the exponent of a cardinal number is well defined.

4. Prove that $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0\aleph_0 = \aleph_0$.

5. Let α be an infinite cardinal number. Prove that $\aleph_0 + \alpha = \alpha$.

6. *Prove that $\aleph_0\aleph_1 = \aleph_1$.

7. *Show that $\mathbb{R}$ and $\mathbb{R}^2$ have same cardinality, in other words, $\aleph_1\aleph_1 = \aleph_1$. More generally, $|\mathbb{R}^n| = \aleph_1$ for all $n \in \mathbb{N}$.

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8. Prove that $2^{\aleph_0} = \aleph_1$.

9. Suppose α and β are cardinal numbers such that $\alpha \leq \beta$. Show that there exists a set S with a subset A such that $|A| = \alpha$ and $|S| = \beta$.

10. *Let X, Y, X_1 be sets such that $X \supseteq Y \supseteq X_1$ and X and X_1 are in bijection.

    (a) Prove using Schroeder-Bernstein Theorem that there exists a bijection between X and Y.

    (b) Suppose it is known that whenever the hypothesis holds, the conclusion in part (a) is true. Using this prove Schroeder-Bernstein Theorem.

11. Show that for cardinal numbers α, β, γ,

    (a) $\alpha + \beta = \beta + \alpha$

    (b) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

    (c) $\alpha\beta = \beta\alpha$

    (d) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

    (e) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

    (f) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$

    (g) $(\alpha\beta)^\gamma = \alpha^\gamma\beta^\gamma$

    (h) $\alpha^\beta\alpha^\gamma = \alpha^{\beta+\gamma}$

    (i) *If $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$

    (j) If $\alpha \leq \beta$, then $\alpha^\gamma \leq \beta^\gamma$ and $\gamma^\alpha \leq \gamma^\beta$

12. Prove that $\aleph_0^{\aleph_0} = \aleph_1$ and $*\aleph_1^{\aleph_0} = \aleph_1$.

13. Show that $|\{f : \mathbb{R} \to \mathbb{R} | f \text{ is continuous}\}| = \aleph_1$.

    (Assume that if f and g are such continuous functions and $f(q) = g(q)$ for all rational numbers $q \in \mathbb{Q} \subset \mathbb{R}$, then $f = g$, that is, $f(x) = g(x)$ for all $x \in \mathbb{R}$)

14. *Let $\mathcal{C}$ be the collection of all circles in the plane $\mathbb{R}^2$. Show that $|\mathcal{C}| = \aleph_1$.