Constructing a function from $(0,1) \longrightarrow \{0,1\}^{\mathbb{N}}$

By previous construction,

map $x \in (0,1)$ to its binary representation.

say $x = 0._2 a_1 a_2 \ldots\ldots\ldots$

Construct a fn. $f_x : \mathbb{N} \longrightarrow \{0,1\}$ by

$$f(n) = a_n \quad \forall n \in \mathbb{N}.$$

For example,

$x = \frac{3}{4} \in (0,1)$

$a_1 = \max \left\{ k \in \mathbb{N} \cup \{0\} : \frac{k}{2} \leq \frac{3}{4} \right\} = 1.$

$x - \frac{1}{2} = \frac{1}{4} \neq 0$

$a_2 = \max \left\{ k \in \mathbb{N} \cup \{0\} : \frac{k}{4} \leq x - \frac{1}{2} = \frac{1}{4} \right\} = 1.$

$x - \frac{1}{2} - \frac{1}{4} = 0.$

$\therefore \quad x = \frac{1}{2} + \frac{1}{4} = 0._2 11.$

So $a_3 = 0$

$a_4 = 0 \ldots\ldots$

and so on.

So it maps to the fn. $f_{3/4} : \mathbb{N} \longrightarrow \{0,1\}$ defined by

$$f_{3/4}(1) = 1, \quad f_{3/4}(2) = 1$$

$$f_{3/4}(3) = \dots = f_{3/4}(n) = \dots = 0.$$

So we define $(0,1) \xrightarrow{H} \{0,1\}^{\mathbb{N}}$ by

$$H(x) = f_x.$$

H is 1-1

If $H(x) = H(y)$, then $x = y$.

If $x = 0.{}_2 a_1 a_2 \dots$ $\quad H(x) = f_x$, then $f_x(n) = a_n$

$y = 0.{}_2 b_1 b_2 \dots$ $\quad H(y) = f_y$, then $f_y(n) = b_n$.

$$H(x) = H(y) \Rightarrow f_x = f_y$$

$$\Rightarrow f_x(n) = f_y(n) \quad \forall n \in \mathbb{N}$$

$$\Rightarrow a_n = b_n \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow \quad x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = y.$$

H is onto (??)

To show H is onto, we have to prove that if $f \in \{0,1\}^{\mathbb{N}}$, then $\exists\, x \in (0,1)$ such that

$$H(x) = f.$$

Qn is that how we can find such x!

In fact, we observe that given $a_1 a_2 \ldots\ldots$ we have $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$.

So given $f : \mathbb{N} \longrightarrow \{0,1\}$ natural candidate for x is $\sum_{n=1}^{\infty} \frac{f(n)}{2^n}$.

now note if we take $g \in \{0,1\}^{\mathbb{N}}$ such that

$$g(1) = 0, \quad g(2) = g(3) \cdots = g(n) = 1, \ldots$$

the x we associate is $\displaystyle\sum_{n=1}^{\infty} \frac{g(n)}{2^n}$

$$= \sum_{n=2}^{\infty} \frac{1}{2^n} \quad \text{since } g(1) = 0$$

$$= \frac{1}{2^2} \cdot \frac{1}{1-1/2} = \frac{1}{2}.$$

However, $H\left(\frac{1}{2}\right) = 0 \cdot_2 1$

i.e $f_{1/2}(1) = 1$

& $f_{1/2}(2) = \cdots = f_x(n) = \cdots = 0.$

Clearly, $f_{1/2} \neq g$.

We need to figure out what are the elements for which this association fails, i.e

we would like to find out f such that

$$H\left(\sum \frac{f(n)}{2^n}\right) \neq f.$$

So this means we need to find out

$x = \sum \frac{f(n)}{2^n}$ has binary repn.

$0 :_2 a_1 \ldots a_n : \ldots$

but $\exists\, n$ s.t $f(n) \neq a_n$.

So we need to find out that

when for two different $\{a_n\}$ & $\{b_n\}$.

repn. of 0 & 1's (that is, two diff fns $: \mathbb{N} \longrightarrow \{0,1\}$)

such that $\sum_{n=1}^{\infty} \frac{a_n}{2^n} = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$?

$\{a_n\}, \{b_n\}$ are different.

$\Rightarrow \exists n$ s.t $a_i = b_i \quad 1 \le i \le n$

& $a_{n+1} \neq b_{n+1}$ (This is the first time they differ)

WLOG $a_{n+1} > b_{n+1}$.

Qn: when then $\sum_{n=1}^{\infty} \frac{a_n}{2^n} = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$?

The following lemma answers this question.

Lemma: If $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$, $y = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$

and $\exists n \in \mathbb{N}$ s.t $a_i = b_i \quad 1 \le i \le n$ & $a_{n+1} > b_{n+1}$,

then $x \geq y$ and equality holds iff

$a_{n+1} = 1, \; b_{n+1} = 0$ &

$a_k = 0 \;\forall\, k \geq n+2, \; b_k = 1 \;\forall\, k \geq n+2.$

Proof. $x - y = \sum_{k=n+1}^{\infty} \frac{a_k - b_k}{2^k}$

$$= \frac{a_{n+1} - b_{n+1}}{2^{n+1}} + \sum_{k=n+2}^{\infty} \frac{a_k - b_k}{2^k}.$$

$$a_{n+1} > b_{n+1} \Rightarrow a_{n+1} = 1, b_{n+1} = 0.$$

and $a_k - b_k \in \{-1, 0, 1\} \quad \forall k.$

This implies

$$x - y = \frac{1}{2^{n+1}} + \sum_{k=n+2}^{\infty} \frac{a_k - b_k}{2^k}$$

$$\geq \underbrace{\frac{1}{2^{n+1}} + \sum_{k=n+2}^{\infty} \frac{-1}{2^k}}$$

$$= \frac{1}{2^{n+1}} - \frac{1}{2^{n+2}} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k}$$

$$= \frac{1}{2^{n+1}} - \frac{1}{2^{n+2}} \cdot \frac{1}{1-1/2}$$

$$= \frac{1}{2^{n+1}} - \frac{1}{2^{n+1}}$$

$$= 0.$$

Thus $x - y \geq 0$, that is, $x \geq y$.

& $x = y \iff a_{n+1} = 1,\ b_{n+1} = 0$

& $\underbrace{a_k - b_k = -1 \quad \forall\, k \geq n+2}$

$\Updownarrow$

$a_k = 0,\ b_k = -1 \quad \forall\, k \geq n+2.$

Check!

as equality holds when $a_k - b_k$ takes minimum in $\{-1, 0, 1\}$

Define $X, Y \subseteq \{0,1\}^{\mathbb{N}}$ by

$$X := \left\{ f : \mathbb{N} \to \{0,1\} \;\middle|\; \begin{array}{l} f(m) = 1 \text{ for some } m \in \mathbb{N} \\ \& \ f(k) = 0 \quad \forall\, k > m \end{array} \right\}.$$

$$Y := \left\{ f : \mathbb{N} \to \{0,1\} \;\middle|\; \begin{array}{l} f(m) = 0 \text{ for some } m \in \mathbb{N} \\ \& \ f(k) = 1 \quad \forall\, k > m \end{array} \right\}$$

Define a bijection $h : X \to Y$.

$f \in X \implies \exists\, r, a_1, a_2, \dots a_{r-1}$ s.t. $f = f_r^{a_1 \dots a_{r-1}}$

where.
$$f_r^{a_1 \dots a_{r-1}}(n) = a_i \qquad 1 \le i \le r-1$$
$$= 1 \qquad i = r$$
$$= 0 \qquad i > r.$$

map
$$h\left(f_r^{a_1\, a_2 \dots a_{r-1}} \right) = g_r^{a_1\, a_2 \dots a_{r-1}}$$

where $g_r^{a_1 a_2 \dots a_{r-1}}(n) = a_i \quad 1 \le i \le r-1$

$= 0 \quad i = r$

$= 1 \quad i > r.$

Check: h is a bijection. (Exc).

Claim: X is countable.

$$X = \cup X_n$$

where $X_n := \{ f : \mathbb{N} \to \{0,1\} : f(n) = 1$ & $f(k) = 0 \ \forall k \geq n+1 \}$

clearly $|X_n| = 2^{n-1}$.

X, being countable union of finite set, is countable.

Thus Y is countable.

Since the constant fn. $\underline{0}$ and $\underline{1}$ in $\{0,1\}^{\mathbb{N}}$

associates to $\displaystyle\sum_{n=1}^{\infty} \frac{0}{2^n} = 0$

& $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$

We consider the map

$$H : (0,1) \longrightarrow \{0,1\}^{\mathbb{N}} \setminus Y \cup \{\underline{0}\} \cup \{\underline{1}\}$$

We claim this H is a bijection.

$x \in (0,1)$, $f_x \notin Y$.

as any $g \in Y$ $\underbrace{g = g_r^{a_1 \ldots a_{r-1}}}$

the corresponding x has image $\underline{f_x = f_r^{a_1 \ldots a_{r-1}}}$.

$f_x \neq \underline{0}, \underline{1}$

So H is a well defined fn. which is 1-1 as proved previously.

<u>Claim:</u> H is onto.

$f \in \{0,1\}^{\mathbb{N}} \setminus Y \cup \{\underline{0}\} \cup \{\underline{1}\}$.

$x = \sum_{n=1}^{\infty} \frac{f(n)}{2^n}$, let $H(x) \neq f$.

$H(x), f \in \{0,1\}^{\mathbb{N}}$ s.t $\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \sum_{n=1}^{\infty} \frac{H(x)(n)}{2^n}$

$\Leftrightarrow$ One of $H(x)$ or f must lie in Y

$\rightarrow$ which is a contradiction

$\Rightarrow H(x) = f$.

Thus we have $\quad |(0,1)| = |\{0,1\}^{\mathbb{N}} \setminus Y \cup \{\underline{0}\} \cup \{\underline{1}\}|$

Since $Y \cup \{\underline{0}\} \cup \{\underline{1}\}$ is countable, & $\{0,1\}^{\mathbb{N}}$ is uncountable, $\{0,1\}^{\mathbb{N}} \setminus Y \cup \{\underline{0}\} \cup \{\underline{1}\}$ is uncountable.

in fact, $|\{0,1\}^{\mathbb{N}} \setminus Y \cup \{\underline{0}\} \cup \{\underline{1}\}|$

$$= \left| \left(\{0,1\}^{\mathbb{N}} \setminus Y \cup \{\underline{0}\} \cup \{\underline{1}\} \right) \cup Y \cup \{\underline{0}\} \cup \{\underline{1}\} \right|$$

$$= |\{0,1\}^{\mathbb{N}}|.$$

Thus

$$|\mathbb{R}| = |(0,1)| = \{0,1\}^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$$

So we have learnt $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ — countable

$\mathbb{R}$ — uncountable

$\mathbb{R} \setminus \mathbb{Q}$ — irrational nos. uncountable.

Set of $\underbrace{\text{algebraic nos.}}_{}$ countable. (Done in tutorial).

real nos. that are solutions of polynomial equation with integer coefficient.

Hence set of $\underbrace{\text{transcendental number}}_{}$ is uncountable.

real nos. which are not algebraic.

So there are transcendental number in abundance.
However, it is really difficult to prove a number is transcental.

For example, π, e, e^{π}, $\sqrt{2}^{\sqrt{2}}$, $\log 2$, $\sin 1$ — transcendental
— difficult to prove.

It is not yet known whether $\pi + e$, πe are transcendental.
— known that one of them must be.

NOT KNOWN → π^e, π^{π} — transcendental?