1. The columns of A are n vectors from $\mathbb{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbb{R}^m$, what is the rank? If they are a basis for $\mathbb{R}^m$, what then?

Solution: Recall that if A is an $m \times n$ matrix then

$$rank(A) = \text{row rank of } A = \text{column rank of } A = r,$$

where $r \leq m$ and $r \leq n$.

Let $C_1, C_2, \ldots, C_n$ be the columns of the $m \times n$ matrix A.

(a) If $C_1, C_2, \ldots, C_n$ are linearly independent then the column rank of $A = n$.

Hence, $rank(A) = n$, and $n \leq m$.

(b) If $C_1, C_2, \ldots, C_n$ span $\mathbb{R}^m$, that is, the column space of A is equal to $\mathbb{R}^m$, then the column rank of $A = m$.

Hence, $rank(A) = m$, and $m \leq n$.

(c) Assume that $\{C_1, C_2, \ldots, C_n\}$ is a basis of $\mathbb{R}^m$. Then, the column rank of $A = n$.

We also know that the dimension of $\mathbb{R}^m$ is m, which is the same as saying that every basis of $\mathbb{R}^m$ must have exactly m number of vectors. So, in the present case we must have that $m = n$. Hence, we get that $rank(A) = m = n$.

2. Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbb{R}^5$.

   (a) The equation $Ax = 0$ has only the solution $x = 0$ because ______.

   (b) For every $b \in \mathbb{R}^5$, the system $Ax = b$ is solvable because ______ .

   Conclusion: A is invertible. Its rank is 5 .

Solution:

- **Part (a):** Suppose there exists another solution $u \neq 0$ such that $Au = 0$, then $N(A) \neq \{0\}$.

  Then, Nullity of $A \geq 1$.

  By Rank-Nullity theorem, we have

  $$5 = \text{Rank of } A + \text{Nullity of } A \geq \text{Rank of } A + 1 \implies \text{Rank of } A \leq 4.$$

  We also have that Column space of A is $\mathbb{R}^5$ as columns of A are linearly independent and this implies that Rank of $A = 5 > 4$, which is a contradiction.

  Hence $Ax = 0$ has only the solution $x = 0$.

- **Part (b):** Let $b \in \mathbb{R}^5$. Since, the columns of A spans $\mathbb{R}^5$, we have that $b \in C(A)$, where $C(A)$ is the column space of A, i.e. there exists $u_1, u_2, u_3, u_4, u_5 \in \mathbb{R}$ such that

  $$b = \sum_{i=1}^{5} u_i C_i,$$

  where C_1, C_2, C_3, C_4, C_5 are the columns of A. The above identity implies that

  $$A\boldsymbol{u} = b, \text{ where } \boldsymbol{u} = (u_1, u_2, u_3, u_4, u_5).$$

  Thus $Ax = b$ is solvable for every $b \in \mathbb{R}^5$.

- **Rank of A is 5:** Since Row rank of A = Column rank of $A = 5$, we have Rank of $A = 5$.

- **A is invertible:** We know that for a $m \times n$ matrix, right inverse exists if and only if Rank of the matrix $= m \leq n$. Hence, the right inverse of A exists since Rank of $A = m = n = 5$. By a similar reasoning A is also left invertible. Hence A in invertible.

3. Suppose $\mathbf{S}$ is a five-dimensional subspace of $\mathbb{R}^6$. True or false?

   (a) Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbb{R}^6$ by adding one more vector.

   (b) Every basis for $\mathbb{R}^6$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

Solution:

(a) **TRUE**.

Note: We need to find one vector which when included in the basis of $\mathbf{S}$ forms a basis of $\mathbb{R}^6$.

Let $\{u_1 u_2, u_3, u_4, u_5\}$ be a basis for $\mathbf{S}$. We know that dimension of $\mathbb{R}^6 = 6$. Therefore, u_1, u_2, u_3, u_4, u_5 cannot span $\mathbb{R}^6$. Hence, there exists a vector $v \in \mathbb{R}^6$ which cannot be written as a linear combination of u_1, u_2, u_3, u_4, u_5.

Claim: $\{u_1, u_2, u_3, u_4, u_5, v\}$ are linearly independent.

Suppose not, that is, $\exists\, c_i \in \mathbb{R}$, $i = 1, 2, 3, 4, 5, 6$ (at least one c_i is non zero) such that

$$\sum_{i=1}^{5} c_i u_i + c_6 v = 0.$$

Note that c_6 can not be zero. Because if $c_6 = 0$ then we would have

$$\sum_{i=1}^{5} c_i u_i = 0,$$

which would force $c_1 = c_2 = c_3 = c_4 = c_5 = 0$, thanks to $\{u_1, u_2, u_3, u_4, u_5\}$ being linearly independent. But, it will contradict the fact that at least one of the six c_i's are non-zero.

Now, with $c_6 \neq 0$, we get

$$v = \sum_{i=1}^{5} \left(-c_6^{-1} c_i\right) u_i.$$

But, we had taken v from outside of the span of vectors u_1, u_2, u_3, u_4, u_5, so we get a contradiction.

Thus, we get that vectors $u_1, u_2, u_3, u_4, u_5, v$ are linearly independent, proving our claim.

Finally, $\{u_1, u_2, u_3, u_4, u_5, v\}$ being linearly independent, must form a basis of the six dimensional space $\mathbb{R}^6$.

(b) **FALSE**.

Let $e_1 = (1,0,0,0,0,0)$, $e_2 = (0,1,0,0,0,0)$, $e_3 = (0,0,1,0,0,0)$, $e_4 = (0,0,0,1,0,0)$, $e_5 = (0,0,0,0,1,0)$, $e_6 = (0,0,0,0,0,1)$ be the standard basis of $\mathbb{R}^6$.

Consider the subspace $\mathbf{S} = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 :\ x_1 = x_2\}$.

It is not difficult to see that $\{(1,1,0,0,0,0), e_3, e_4, e_5, e_6\}$ forms a basis of $\mathbf{S}$. Thus, $\mathbf{S}$ is a five-dimensional subspace of $\mathbb{R}^6$.

Note that $u = (1,1,1,1,1,1) \in \mathbf{S}$. We now show that u does not belong to the span of any of the five vectors from $e_1, e_2, e_3, e_4, e_5, e_6$.

Fix an $i_0 \in \{1,2,3,4,5,6\}$. Let $\mathbf{S}_{i_0}$ be the subspace spanned by $\{e_1, e_2, e_3, e_4, e_5, e_6\} \setminus \{e_{i_0}\}$, then for any $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbf{S}_{i_0}$, $x_{i_0} = 0$ and thus, $(1,1,1,1,1,1) \notin \mathbf{S}_{i_0}$.

4. Prove that if **V** and **W** are three-dimensional subspaces of $\mathbb{R}^5$, then **V** and **W** must have a nonzero vector in common. [Hint: Start with bases for the two subspaces, making six vectors in all.]

Solution: Let $\{u_1, u_2, u_3\}$ be a basis of **V** and $\{u_4, u_5, u_6\}$ be a basis of **W**.

Since dimension of $\mathbb{R}^5$ is 5, the vectors $u_1, u_2, u_3, u_4, u_5, u_6$ are linearly dependent. This means there exists $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$, at least one of c_i is non-zero, such that

$$c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + c_5u_5 + c_6u_6 = 0.$$

Without loss of generality, we assume $c_1 \neq 0$. Then $c_1u_1 + c_2u_2 + c_3u_3 \neq 0$ as u_1, u_2, u_3 are linearly independent.

Let us write $v = c_1u_1 + c_2u_2 + c_3u_3$ and $w = c_4u_4 + c_5u_5 + c_6u_6$. Then, we have that $v \in V$, $w \in W$ and $v \neq 0$.

Moreover, $v + w = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + c_5u_5 + c_6u_6 = 0$, implying that $v = -w \in W$.

Thus, the non-zero vector v belongs to both **V** and **W** and we are done.

5. If A is a 64 by 17 matrix of rank 11, how many maximum independent vectors satisfy $Ax = 0$? How many maximum independent vectors satisfy $A^{\mathrm{T}}y = 0$?

Solution: We will find the maximum number of linearly independent vectors x which satisfy $Ax = 0$ and also the maximum number of linearly independent vectors y satisfying $A^{\mathrm{T}}y = 0$.

Let M be an m by n matrix with rank r. Recall that

$$\begin{aligned}
&\text{the maximum number of linearly independent vectors } u \text{ satisfying } Mu = 0\\
=\;&\text{the maximum number of linearly independent vectors in the nullspace of } M\\
=\;&\text{the dimension of the nullspace, that is, the nullity of } M\\
=\;&\text{the number of columns of } M - \text{the rank of } M \qquad \text{(by the rank-nullity theorem)}\\
=\;&n - r,
\end{aligned}$$

and by the similar arguments, we also have

$$\begin{aligned}
&\text{the maximum number of linearly independent vectors } v \text{ satisfying } M^{\mathrm{T}}v = 0\\
=\;&\text{the number of columns of } M^{\mathrm{T}} - \text{the rank of } M^{\mathrm{T}}.
\end{aligned}$$

But since the columns of M^{T} are exactly the same as the rows of M, note that

$$\begin{aligned}
&\text{the rank of } M^{\mathrm{T}}\\
=\;&\text{the dimension of the column space of } M^{\mathrm{T}}\\
=\;&\text{the dimension of the row space, that is, the row rank of } M\\
=\;&\text{the (column) rank of } M\\
=\;&r,
\end{aligned}$$

and this guarantees that

$$\begin{aligned}
&\text{the maximum number of linearly independent vectors } v \text{ satisfying } M^{\mathrm{T}}v = 0\\
=\;&\text{the number of columns of } M^{\mathrm{T}} - \text{the rank of } M\\
=\;&\text{the number of rows of } M - \text{the rank of } M\\
=\;&m - r.
\end{aligned}$$

Hence, in our case, we obtain that

$$\begin{aligned}
&\text{the maximum number of linearly independent vectors } x \text{ satisfying } Ax = 0\\
=\;&\text{the number of columns of } A - \text{the rank of } A\\
=\;&17 - 11\\
=\;&6,
\end{aligned}$$

and that

$$\begin{aligned}
&\text{the maximum number of linearly independent vectors } y \text{ satisfying } A^{\mathrm{T}}y = 0 \\
= \;& \text{the number of rows of } A - \text{the rank of } A \\
= \;& 64 - 11 \\
= \;& 53.
\end{aligned}$$

6. Find a basis for each of these subspaces of 3 by 3 matrices:

   (a) All diagonal matrices.

   (b) All symmetric matrices $\left(A^{\mathrm{T}} = A\right)$.

   (c) All skew-symmetric matrices $\left(A^{\mathrm{T}} = -A\right)$.

Solution:

Note: Observe that each set mentioned in the question is indeed a subspace of the vector space $M_{3\times 3}(\mathbb{R})$.

For $i, j \in \{1, 2, 3\}$, let Q_{ij} denote the 3 by 3 matrix whose $(i, j)^{\text{th}}$ entry is one and the rest of the entries are zero.

For example, $Q_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $Q_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Let $\mathbf{0}$ denote the 3 by 3 zero matrix.

(a) Let W_1 denote the subspace of all 3 by 3 diagonal matrices.

Note that $W_1 = \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} : a_{11}, a_{22}, a_{33} \in \mathbb{R} \right\}$.

Now, given any diagonal matrix $D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \in W_1$, we have

$$D = a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = a_{11}Q_{11} + a_{22}Q_{22} + a_{33}Q_{33}.$$

In other words, the set $\mathcal{B}_1 = \{Q_{11}, Q_{22}, Q_{33}\} \subseteq W_1$ spans the space W_1.

As the three diagonal entries (are necessary and sufficient to) "determine" any matrix in W_1, we could guess that any basis of W_1 should have 3 elements. The above-mentioned set $\mathcal{B}_1$ is a spanning set of W_1 and it has size 3, so we expect that to be linearly independent. Let us prove that this is indeed the case.

Let $a, b, c \in \mathbb{R}$ be arbitrary such that $aQ_{11} + bQ_{22} + cQ_{33} = \mathbf{0}$, that is,

$$a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which clearly implies that $a = b = c = 0$. Hence, the set $\mathcal{B}_1$ is linearly independent. Combining this with the fact that it also spans W_1, we conclude that $\mathcal{B}_1$ is a basis of W_1.

(b) Let W_2 denote the subspace of all 3 by 3 symmetric matrices.

Suppose $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3 by 3 symmetric matrix. Then, by the definition of a symmetric matrix, we have $M^{\mathrm{T}} = M$, that is,

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

and thus, $a_{12} = a_{21}$, $a_{13} = a_{31}$ and $a_{23} = a_{32}$.

Also, note that for any $a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33} \in \mathbb{R}$, the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ lies in W_2.

Hence, we have the explicit description of W_2 as follows.

$$W_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} : a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33} \in \mathbb{R} \right\}$$

Note that the matrices in W_2 are determined by the entries at the $(1,1)^{\text{th}}$, $(1,2)^{\text{th}}$, $(1,3)^{\text{th}}$, $(2,2)^{\text{th}}$, $(2,3)^{\text{th}}$ and $(3,3)^{\text{th}}$ positions, and these positions are "free" to take any real number as their value. This allows us to guess that any basis of W_2 consists of precisely 6 symmetric matrices.

Further, given any matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \in W_2$, note that

$$\begin{aligned} A = {} & a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & + a_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + a_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = {} & a_{11}Q_{11} + a_{12}Q_{12} + a_{13}Q_{13} + a_{12}Q_{21} + a_{22}Q_{22} + a_{23}Q_{23} + a_{13}Q_{31} + a_{23}Q_{32} + a_{33}Q_{33} \\ = {} & a_{11}Q_{11} + a_{12}(Q_{12} + Q_{21}) + a_{13}(Q_{13} + Q_{31}) + a_{22}Q_{22} + a_{23}(Q_{23} + Q_{32}) + a_{33}Q_{33}, \end{aligned}$$

where all of the matrices Q_{11}, $Q_{12} + Q_{21}$, $Q_{13} + Q_{31}$, Q_{22}, $Q_{23} + Q_{32}$ and Q_{33} are symmetric. This shows that the subset $\mathcal{B}_2 = \{Q_{11}, Q_{12}+Q_{21}, Q_{13}+Q_{31}, Q_{22}, Q_{23}+Q_{32}, Q_{33}\}$ of W_2 (having 6 elements) spans W_2.

We claim that this subset $\mathcal{B}_2$ of W_2 is a basis of W_2. We will be done once we show that $\mathcal{B}_2$ is linearly independent, as we have already shown that it spans W_2.

Let $a, b, c, d, e, f \in \mathbb{R}$ be arbitrary such that

$$aQ_{11} + b(Q_{12} + Q_{21}) + c(Q_{13} + Q_{31})dQ_{22} + e(Q_{23} + Q_{32}) + fQ_{33} = \mathbf{0},$$

that is,

$$a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+b\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}+c\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}+d\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+e\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}+f\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}a&b&c\\b&d&e\\c&e&f\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

which gives us that $a=b=c=d=e=f=0$, and hence, $\mathcal{B}_2$ is a linearly independent subset of W_2.

(c) Let W_3 denote the subspace of all 3 by 3 skew-symmetric matrices.

If the matrix $M=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ lies in W_3, by the definition of a skew-symmetric matrix, we have $M^{\mathrm{T}}=-M$, that is,

$$\begin{bmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{bmatrix}=\begin{bmatrix}-a_{11}&-a_{12}&-a_{13}\\-a_{21}&-a_{22}&-a_{23}\\-a_{31}&-a_{32}&-a_{33}\end{bmatrix},$$

and this forces us to have $a_{11}=a_{22}=a_{33}=0$, $a_{21}=-a_{12}$, $a_{31}=-a_{13}$ and $a_{32}=-a_{23}$.

Also, for any $a_{12},a_{13},a_{23}\in\mathbb{R}$, the matrix $\begin{bmatrix}0&a_{12}&a_{13}\\-a_{12}&0&a_{23}\\-a_{13}&-a_{23}&0\end{bmatrix}$ is clearly skew-symmetric.

Therefore, we obtain that

$$W_3=\left\{\begin{bmatrix}0&a_{12}&a_{13}\\-a_{12}&0&a_{23}\\-a_{13}&-a_{23}&0\end{bmatrix}:a_{12},a_{13},a_{23}\in\mathbb{R}\right\}.$$

Note that only three of the entries, namely $(1,2)^{\text{th}}$, $(1,3)^{\text{th}}$ and $(2,3)^{\text{th}}$, are "free", and they completely determine the matrices in W_3. So, we conjecture that any basis of W_3 is of size 3.

Further, given any matrix $C=\begin{bmatrix}0&a_{12}&a_{13}\\-a_{12}&0&a_{23}\\-a_{13}&-a_{23}&0\end{bmatrix}\in W_3$, we can write

$$\begin{aligned}C&=a_{12}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}+a_{13}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}+(-a_{12})\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\\&\quad+a_{23}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}+(-a_{13})\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}+(-a_{23})\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\\&=a_{12}Q_{12}+a_{13}Q_{13}+a_{12}(-Q_{21})+a_{23}Q_{23}+a_{13}(-Q_{31})+a_{23}(-Q_{32})\\&=a_{12}(Q_{12}-Q_{21})+a_{13}(Q_{13}-Q_{31})+a_{23}(Q_{23}-Q_{32}),\end{aligned}$$

where the matrices $Q_{12} - Q_{21}$, $Q_{13} - Q_{31}$ and $Q_{23} - Q_{32}$ are skew-symmetric. Hence, the subset $\mathcal{B}_3 = \{Q_{12} - Q_{21}, Q_{13} - Q_{31}, Q_{23} - Q_{32}\}$ of W_3 spans W_3. Since it has 3 elements, we guess that the set $\mathcal{B}_3$ is a basis of W_3. To prove this guess, we just need to show that it is linearly independent. Let us show that.

Let $a, b, c \in \mathbb{R}$ be arbitrary such that $a(Q_{12} - Q_{21}) + b(Q_{13} - Q_{31}) + c(Q_{23} - Q_{32}) = \mathbf{0}$, that is,

$$a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, we get that $a = b = c = 0$, and it follows that the subset $\mathcal{B}_3$ of W_3 is linearly independent.

7. Find the dimension and a basis for the four fundamental subspaces $(C(A), C(A^T), N(A), N(A^T))$ for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: The four fundamental subspaces associated to a matrix A are $C(A), C(A^{\mathrm{T}}), N(A)$ and $N(A^{\mathrm{T}})$. We know that for an m by n matrix A of rank r,

$$\dim(C(A)) = \dim(C(A^{\mathrm{T}})) = r, \qquad \dim(N(A)) = n - r \qquad \text{and} \qquad \dim(N(A^{\mathrm{T}})) = m - r.$$

So, to find the dimension of these subspaces we need to calculate the rank of the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the matrix U is an echelon form of A, which is obtained from A by replacing the third row of matrix A with the difference between the third and the first row of A and we have $U = BA$, where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Since B is invertible, the rank of matrix A is the same as the rank of U.

Also, $C(A^{\mathrm{T}}) = C(U^{\mathrm{T}})$ and $N(A) = N(U)$.

Now, U is in echelon form, its rank is equal to the number of pivots. Hence, we have $\operatorname{rank}(A) = \operatorname{rank}(U) = 2$. Therefore, we have

$$\dim(C(A)) = 2, \qquad \dim(N(A)) = 2, \qquad \dim(C(A^{\mathrm{T}})) = 2 \quad \text{and} \quad \dim(N(A^{\mathrm{T}})) = 1,$$

and

$$\dim(C(U)) = 2, \qquad \dim(N(U)) = 2, \qquad \dim(C(U^{\mathrm{T}})) = 2 \quad \text{and} \quad \dim(N(U^{\mathrm{T}})) = 1.$$

Next, we work towards finding bases for these subspaces.

Basis of C(A) and C(U): Let us first find basis for $C(A)$. Since, we know $\dim(C(A)) = 2$, it is enough to find two linearly independent vectors in the column space of A. As columns of a matrix are always in the column space, hence we can just pick any two linearly independent column vectors. **Note that two vectors are linearly independent if and only if they are not multiple of each other.** As the first and third columns of matrix A are not multiple of each other, they are linearly independent. Hence,

$$\text{Basis of C(A)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Similarly, we can see that the first and third columns of matrix U are linearly independent, and also $\dim(C(U)) = 2$, we get that

$$\text{Basis of C(U)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Basis of N(A) and N(U): We know that $N(A) = N(U)$. We will find a basis of the null space of U. Since, $\dim(N(U)) = 2$, we need to find two linearly independent vectors in the null space of U.

Repeating the idea of part (b) of problem 6, one can show that

$$\text{Basis of } N(A) = \text{Basis of } N(U) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Note that

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We leave it as an exercise to find bases of $C(A^{\mathrm{T}})$, $C(U^{\mathrm{T}})$, $N(A^{\mathrm{T}})$ and $C(A^{\mathrm{T}})$.

(Remember their dimensions and the fact that $C(A^{\mathrm{T}}) = C(U^{\mathrm{T}})$.)

8. Suppose A is an m by n matrix of rank r. Under what conditions on those numbers does

   (a) A have a two-sided inverse: $AA^{-1} = I_m$ and $A^{-1}A = I_n$?

   (b) $Ax = b$ have infinitely many solutions for every b?

Solution: Consider an m by n matrix A of rank r.

(a) We have to give conditions on m, n and r such that A has two sided inverse.

We know that A has a right inverse if and only if

$$r = m \leq n.$$

Also, A has a left inverse if and only if

$$r = n \leq m.$$

Combining the above two conditions, we get that A has a two-sided inverse if and only if

$$r = m = n.$$

(b) We have to give conditions on m, n and r such that $Ax = b$ has infinitely many solutions for every b.

First of all, given b, to have a solution for $Ax = b$, we must have that b is in the column space of A. Thus, to have solution for every b, we must have that every b is in the column space of A, which implies that the dimension of the column space is m. Recall that the dimension of column space is equal to the rank of the matrix. Thus, we get $r = m$.

Now, the rank of a matrix is less than or equal to n. So, we have $m = r \leq n$.

If $r = n$, then A becomes a non-singular square matrix and we will get a unique solution for every b, which is given by $x = A^{-1}b$. Thus, to get infinitely many solutions for every b, we must have $r < n$.

Indeed, if $r < n$, then by the rank-nullity theorem, we get that the nullity of A is positive (equal to $n - r$), and we know that to get infinitely many solutions we should have a positive dimension of null space (nullity).

Altogether, the required condition is $r = m < n$.

9. Why is there no matrix whose row space and nullspace both contain $(1, 1, 1)$?

Solution: Suppose there exists an $m \times 3$ matrix A whose row space and null space both contain $(1, 1, 1)$. Let

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix},$$

where each row $R_j \in \mathbb{R}^3$.

Now, by $(1, 1, 1) \in N(A)$, we mean

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

which is same as saying that

$$R_j \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \tag{1}$$

for every $1 \leq j \leq m$.

On the other hand, if $(1, 1, 1) \in R(A)$, then one would have real numbers $c_1, c_2, \ldots, c_m$ such that

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = c_1 R_1 + c_2 R_2 + \cdots + c_m R_m.$$

But then

$$\begin{aligned} 3 &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= (c_1 R_1 + c_2 R_2 + \cdots + c_m R_m) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \sum_{j=1}^{m} c_j R_j \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= 0, \end{aligned}$$

which is not possible.

Hence we can not have $(1, 1, 1)$ in the row space and nullspace of a single matrix.

10. Suppose the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. What is the rank and why? The columns of A are linearly ______.

Solution: Let us assume that for an $m \times n$ matrix A, we have that the only solution to $Ax = 0$ is $x = 0$.

Thus, $N(A)$ does not contain any non-zero vector, that is, $N(A) = \{0\}$, and $\text{nullity}(A) = 0$.

Recall the Rank-Nullity Theorem, which states that

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns of } A.$$

Since $\text{nullity}(A) = 0$, we get

$$\text{rank}(A) = \text{column rank of } A = n.$$

Since the number of columns in A is n, and we already know that the column rank of A equals to the maximum number of linearly independent columns in A, we conclude that the columns of A are **linearly independent**.

11. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbb{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.

Solution:

- Let $S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 + 4x_3 = 0\right\}$.

  We need to find a 1×3 matrix A such that $N(A) = S$.

  Consider

  $$A = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}.$$

  Now,

  $$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in N(A) &\Leftrightarrow \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \\ &\Leftrightarrow x_1 + 2x_2 + 4x_3 = 0 \\ &\Leftrightarrow (x_1, x_2, x_3) \in S \end{aligned}$$

  Therefore, $N(A) = S$.

- Now, observe from the above example that if we take a 1×3 matrix as above then we have that equation $x_1 + 2x_2 + 4x_3 = 0$ is always satisfied. With this we can take the 3×3 matrix B who every row is equal to $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$.

  For the 3×3 matrix $B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$, note that

  $$\begin{aligned} B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ x_1 + 2x_2 + 4x_3 \\ x_1 + 2x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &\Leftrightarrow x_1 + 2x_2 + 4x_3 = 0 \\ &\Leftrightarrow (x_1, x_2, x_3) \in S. \end{aligned}$$

  That is, $N(B) = S = N(A)$.

12. If $Ax = 0$ has a nonzero solution, show that $A^{\mathrm{T}}y = b$ fails to be solvable for some right-hand sides b. Construct an example of A and b.

Solution: We shall make use of the following fact: For any $k \times l$ matrix B, the system $Bx = c$ has a solution $x \in \mathbb{R}^l$ if and only if $c \in \mathbb{R}^k$ belongs to $C(B)$.

Let A be an $m \times n$ matrix. Now, having a nonzero solution of the system $Ax = 0$ is equivalent to the fact that $N(A)$ is non-trivial, that is, $N(A) \neq \{0\}$. Equivalently, $\text{nullity}(A) > 0$.

By the rank-nullity theorem:

$$n = \text{rank}(A) + \text{nullity}(A),$$

which implies that

$$\text{rank}(A) = n - \text{nullity}(A),$$

and thus,

$$\text{rank}(A) < n.$$

Recall that the set of columns of A^{T} are nothing but the the set of rows of A. More precisely, j^{th} column of A^{T} is exactly the j^{th} row of A. Thus, A^{T} is an $n \times m$ matrix and $\text{rank}(A^{\mathrm{T}}) = \text{rank}(A) < n$. Consequently, $C(A^{\mathrm{T}})$ is a proper subspace of $\mathbb{R}^n$.

Thus, the set $\mathbb{R}^n \setminus C(A^{\mathrm{T}})$ is non-empty, and for any $b \in \mathbb{R}^n \setminus C(A^{\mathrm{T}})$, $A^{\mathrm{T}}y = b$ fails to be solvable.

Example:

Consider the 2×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix A has a nontrivial null space. In fact,

$$A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $Ax = 0$ has a nonzero solution $x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Now, the transpose of A is $A^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$.

For $A^{\mathrm{T}}y = b$ to be solvable, b must lie in the span of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

By direct checking, one can verify that if we take $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then there is no solution to $A^{\mathrm{T}}y = b$.

13. Construct a matrix with the required property, or explain why you can't.

(a) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

(b) Column space has basis $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, nullspace has basis $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

(c) Row space = column space.

Solution:

(a) Let A be an $m \times n$ matrix. Recall that $C(A)$ is a subspace of $\mathbb{R}^m$ and $R(A)$ is a subspace of $\mathbb{R}^n$. So, in this problem we have to analyse the possibilities within the space of 3×2 matrices.

Claim: Such 3×2 matrices exist whose column space contain $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and whose row space contain $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

- Observe that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent vectors (since none of these is a multiple of the other), so they form a basis of $C(A)$ for such a 3×2 matrix A. In particular, the rank of A is 2. Now, the row space is a subspace of $\mathbb{R}^2$ and row rank is 2, therefore the row space for such a matrix A will automatically be full $\mathbb{R}^2$. Consequently, it will contain vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.
- For example, we can simply consider the matrix with columns exactly the given two vectors, namely $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, and all the requirements are fulfilled.

(b) Let A be an $m \times n$ matrix. Recall that $C(A)$ is a subspace of $\mathbb{R}^m$ and $N(A)$ is a subspace of $\mathbb{R}^n$. So, in this problem we have to analyse the possibilities within the space of 3×3 matrices.

Let A be a 3×3 matrix such that $C(A)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $N(A)$ has basis $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

We thus have that

$$\operatorname{rank}(A) = \operatorname{nullity}(A) = 1.$$

But, by the rank-nullity theorem we should have

$$3 = \text{number of columns of } A = \operatorname{rank}(A) + \operatorname{nullity}(A) = 2,$$

which is not possible.

Hence, no such matrix A exists.

(c) Let A be an $m \times n$ matrix. Recall that $C(A)$ is a subspace of $\mathbb{R}^m$ and $R(A)$ is a subspace of $\mathbb{R}^n$. So, in this problem we have to analyse the possibilities within the space of square $n \times n$ matrices.

We also know that

$$\text{row space of } A = \text{column space of } A^{\mathrm{T}}.$$

If we take a symmetric matrix A, that is, $A^{\mathrm{T}} = A$, then

$$\text{row space of } A = \text{column space of } A^{\mathrm{T}} = \text{column space of } A.$$

Thus, for any symmetric matrix A, the row space of A equals the column space of A.

For example, consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, A is symmetric, and therefore the row space of A is equal to the column space of A.

Remark: We also have examples of non-symmetric matrices whose row and column spaces coincide. In fact, for any invertible $n \times n$ matrix, the row space and the column space are $\mathbb{R}^n$.

14. What 3 by 3 matrices represent the transformations that

(a) project every vector onto the xy plane?

(b) reflect every vector through the xy plane?

(c) rotate the xy plane through 90°, leaving the z-axis alone?

(d) rotate the xy plane, then xz plane, then yz plane, through 90°?

(e) rotate the xy plane, then xz plane, then yz plane, through 180°?

Solution:

- Throughout the solution we consider the set $\{e_1, e_2, e_3\}$ as the usual basis of $\mathbb{R}^3$, where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$.
- By convention, we take rotations to be anti-clockwise unless otherwise stated.

(a) Let, T be the transformation that projects every vector onto the xy plane, that is,

$$T(x,y,z) = (x,y,0)$$

for all $(x,y,z) \in \mathbb{R}^3$.

Now observe that $T(e_1) = e_1$, $T(e_2) = e_2$ and $T(e_3) = (0,0,0)$, that is,

$$\begin{aligned} T(e_1) &= e_1 = 1\,e_1 + 0\,e_2 + 0\,e_3, \\ T(e_2) &= e_2 = 0\,e_1 + 1\,e_2 + 0\,e_3, \\ \text{and} \quad T(e_3) &= (0,0,0) = 0\,e_1 + 0\,e_2 + 0\,e_3. \end{aligned}$$

Therefore, the associated matrix to this linear transformation T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) Let, T be the transformation that reflects every vector of $\mathbb{R}^3$ through the xy plane, that is,

$$T(x,y,z) = (x,y,-z)$$

for all $(x,y,z) \in \mathbb{R}^3$.

Now observe that $T(e_1) = e_1$, $T(e_2) = e_2$ and $T(e_3) = -e_3$, that is,

$$\begin{aligned} T(e_1) &= e_1 = 1\,e_1 + 0\,e_2 + 0\,e_3, \\ T(e_2) &= e_2 = 0\,e_1 + 1\,e_2 + 0\,e_3, \\ \text{and} \quad T(e_3) &= (0,0,0) = 0\,e_1 + 0\,e_2 + (-1)\,e_3. \end{aligned}$$

Therefore the associated matrix to this linear transformation T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(c) Consider T_θ as the linear transformation that rotates the xy plane through an angle θ, leaving the z-axis alone. Here $0 \le \theta < 360$.

Observe that any unit vector in $\mathbb{R}^3$ on the plane xy can be written as $(\cos\phi, \sin\phi, 0)$, for some $0 \le \phi < 360$.

Notice that $e_1 = (1,0,0) = (\cos 0, \sin 0, 0)$, $e_2 = (1,0,0) = (\cos 90, \sin 90, 0)$.

Now, if we rotate e_1 by θ degree in the anticlockwise direction (in xy plane), we get the vector $(\cos(0+\theta), \sin(0+\theta), 0)) = (\cos\theta, \sin\theta, 0)$, that is, $T_\theta(e_1) = (\cos\theta, \sin\theta, 0)$.

Similarly, $T_\theta(e_2) = (\cos(\theta+90), \sin(\theta+90), 0) = (-\sin\theta, \cos\theta, 0)$.

Since T_θ does not affect the z-axis, $T_\theta(e_3) = e_3$.

Thus,

$$\begin{aligned} T_\theta(e_1) &= (\cos\theta, \sin\theta, 0) = \cos\theta\, e_1 + \sin\theta\, e_2 + 0\, e_3, \\ T_\theta(e_2) &= (-\sin\theta, \cos\theta, 0) = (-\sin\theta)\, e_1 + \cos\theta\, e_2 + 0\, e_3, \\ \text{and} \quad T_\theta(e_3) &= (0,0,1) = 0\, e_1 + 0\, e_2 + 1\, e_3. \end{aligned}$$

Therefore, the associated matrix to the transformation T_θ is $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

In particular, the matrix associated with the transformation T_{90} is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) Let us denote the matrices associated to the linear transformation that rotates the xy plane (respectively yz plane and zx plane) through an angle $\theta°$, leaving the z-axis (respectively x-axis and y-axis) alone, by $T_{\theta,xy}$ (respectively $T_{\theta,yz}$ and $T_{\theta,zx}$).

Thus, we have from part (c) (and following similar ideas) that

$$T_{\theta,xy} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_{\theta,yz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad T_{\theta,zx} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Therefore the matrix, which represents the transformation that rotates the xy plane, then xz plane, then yz plane, through an angle θ degree is $T_\theta = T_{\theta,yz}\, T_{\theta,zx}\, T_{\theta,xy}$.

In particular,

$$T_{90} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(e) Put $\theta = 180$ in the matrices $T_{\theta,yz}, T_{\theta,zx}$, and $T_{\theta,xy}$, and then the required matrix is $T_{180} = T_{180,yz}\, T_{180,zx}\, T_{180,xy}$

15. If $T : V \to V$ is a linear transformation, then prove that T^2 is also a linear transformation.

Solution: We have that $T : V \to V$ is a linear transformation.

Therefore, $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$, and $T(c\,v) = c\,T(v)$ for all $c \in \mathbb{R}$ and $v \in V$.

Now, for given $v_1, v_2 \in V$, let us write $v_1' = T(v_1)$ and $v_2' = T(v_2)$. Then,

$$\begin{aligned} T^2(v_1 + v_2) &= T(T(v_1 + v_2)) \\ &= T(T(v_1) + T(v_2)) \\ &= T(v_1' + v_2') \\ &= T(v_1') + T(v_2') \\ &= T(T(v_1)) + T(T(v_2)) \\ &= T^2(v_1) + T^2(v_2). \end{aligned}$$

Similarly, for a given $v \in V$ and $c \in \mathbb{R}$, let us write $w = T(v)$. Then,

$$\begin{aligned} T^2(c\,v) &= T(T(c\,v)) \\ &= T(c\,T(v)) \\ &= T(c\,w) \\ &= c\,T(w) \\ &= c\,T(T(v)) \\ &= c\,T^2(v). \end{aligned}$$

Thus, the map T^2 satisfies $T^2(v_1 + v_2) = T^2(v_1) + T^2(v_2)$ for all $v_1, v_2 \in V$ and $T^2(c\,v) = c\,T^2(v)$ for all $v \in V$ and $c \in \mathbb{R}$. Hence, T^2 is also a linear transformation.

16. The space $M_{2,2}(\mathbb{R})$ of all 2 by 2 matrices has the four basis "vectors"

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the linear transformation T of transposing (that is, $T : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ is defined by $T(P) = P^{\mathrm{T}}$ for every $P \in M_{2,2}(\mathbb{R})$), find its matrix A with respect to the above basis. We know that $T^2(P) = \left(P^{\mathrm{T}}\right)^{\mathrm{T}} = P$, that is, $T^2 = I$. Is $A^2 = I$?

Solution: The given basis of $M_{2,2}(\mathbb{R})$ is

$$\mathcal{B} = \left\{ v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

- Note that

$$\begin{aligned} T(v_1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = v_1, \\ T(v_2) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = v_3, \\ T(v_3) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = v_2, \\ T(v_4) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = v_4. \end{aligned}$$

  So, we can write the following:

$$\begin{aligned} T(v_1) &= 1.v_1 + 0.v_2 + 0.v_3 + 0.v_4, \\ T(v_2) &= 0.v_1 + 0.v_2 + 1.v_3 + 0.v_4, \\ T(v_3) &= 0.v_1 + 1.v_2 + 0.v_3 + 0.v_4, \\ T(v_4) &= 0.v_1 + 0.v_2 + 1.v_3 + 1.v_4. \end{aligned}$$

  Therefore, the associated matrix is $A = [T]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

- Note that $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix}$ for any column vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. That is, the position of x_2 and x_3 is interchanged under the action of A, whereas x_1 and x_4 remain unchanged.

Therefore, under the action of A^2, the position of these two entries (x_2 and x_3) are interchanged twice and we get back the original column vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

That is,

$$A^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Therefore, $A^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, that is, $A^2 = I$.

17. With $v = (v_1, v_2) \in \mathbb{R}^2$, suppose $T(v) = v$, except that $T((0, v_2)) = (0, 0)$. Show that this transformation satisfies $T(c\,v) = c\,T(v)$ for every $v \in \mathbb{R}^2$ and $c \in \mathbb{R}$, but it need not satisfy $T(v + w) = T(v) + T(w)$ for some $v, w \in \mathbb{R}^2$.

Solution: For any $v = (v_1, v_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$, we have

$$T(cv) = T((cv_1, cv_2)) = \begin{cases} (cv_1, cv_2) & \text{if } v_1 \neq 0 \\ (0, 0) & \text{if } v_1 = 0, \end{cases}$$

and

$$T(v) = T((v_1, v_2)) = \begin{cases} (v_1, v_2) & \text{if } v_1 \neq 0 \\ (0, 0) & \text{if } v_1 = 0 \end{cases}$$

Hence, $T(cv) = cT(v)$ for every $v \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

It need not satisfy $T(v + w) = T(v) + T(w)$ for some $v, w \in \mathbb{R}^2$.

For example, take $v = (1, 0)$ and $w = (0, 1)$ so that $v + w = (1, 1)$.

Now, $T(v) = T((1, 0)) = (1, 0)$, $\;T(w) = T((0, 1)) = (0, 0)$ and $T(v + w) = T((1, 1)) = (1, 1)$.

Clearly, $T(v) + T(w) = (1, 0) + (0, 0) = (1, 0) \neq T(v + w)$.

18. Which of these transformations is not linear? The input is $v = (v_1, v_2) \in \mathbb{R}^2$.

(a) $T(v) = (v_2, v_1)$; (b) $T(v) = (v_1, v_1)$; (c) $T(v) = (0, v_1)$; (d) $T(v) = (0, 1)$.

Solution: A transformation $T : V \to W$ is linear if it satisfies the following two properties:

- $T(v + w) = T(v) + T(w)$ for all $v, w \in V$.
- $T(c\,v) = c\,T(v)$ for all $c \in \mathbb{R}$ and $v \in V$.

(a) $T(v) = (v_2, v_1)$.

Let $v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2$ and consider

$$\begin{aligned} T(v+w) &= T((v_1, v_2) + (w_1, w_2)) \\ &= T((v_1 + w_1, v_2 + w_2)) \\ &= (v_2 + w_2, v_1 + w_1) \\ &= (v_2, v_1) + (w_2, w_1) \\ &= T((v_1, v_2)) + T((w_1, w_2)) \\ &= T(v) + T(w). \end{aligned}$$

Now, let $c \in \mathbb{R}$ and $v = (v_1, v_2) \in \mathbb{R}^2$ and consider

$$\begin{aligned} T(cv) &= T(c(v_1, v_2)) \\ &= T((cv_1, cv_2)) \\ &= (cv_2, cv_1) \\ &= c(v_2, v_1) \\ &= cT((v_1, v_2)) \\ &= cT(v). \end{aligned}$$

Thus, T satisfies the above two properties and hence it is a linear transformation.

(b) $T(v) = (v_1, v_1)$.

Let $v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2$ and consider

$$\begin{aligned} T(v+w) &= T((v_1, v_2) + (w_1, w_2)) \\ &= T((v_1 + w_1, v_2 + w_2)) \\ &= (v_1 + w_1, v_1 + w_1) \\ &= (v_1, v_1) + (w_1, w_1) \\ &= T((v_1, v_2)) + T((w_1, w_2)) \\ &= T(v) + T(w). \end{aligned}$$

Now, let $c \in \mathbb{R}$ and $v = (v_1, v_2) \in \mathbb{R}^2$ and consider

$$\begin{aligned} T(cv) &= T(c(v_1, v_2)) \\ &= T((cv_1, cv_2)) \\ &= (cv_1, cv_1) \\ &= c(v_1, v_1) \\ &= cT(v_1, v_2) \\ &= cT(v). \end{aligned}$$

Thus, T satisfies the required two properties and hence it is a linear transformation.

(c) $T(v) = (0, v_1)$.

Let $v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2$ and consider

$$\begin{aligned} T(v + w) &= T((v_1, v_2) + (w_1, w_2)) \\ &= T((v_1 + w_1, v_2 + w_2)) \\ &= (0, v_1 + w_1) \\ &= (0, v_1) + (0, w_1) \\ &= T((v_1, v_2)) + T((w_1, w_2)) \\ &= T(v) + T(w). \end{aligned}$$

Now, let $c \in \mathbb{R}$ and $v = (v_1, v_2) \in \mathbb{R}^2$ and consider

$$\begin{aligned} T(cv) &= T(c(v_1, v_2)) \\ &= T(cv_1, cv_2) \\ &= (0, cv_1) \\ &= c(0, v_1) \\ &= cT(v_1, v_2) \\ &= cT(v). \end{aligned}$$

Thus, T satisfies the required two properties and hence it is a linear transformation.

(d) $T(v) = (0, 1)$.

Consider $v = (1, 0)$ and $w = (0, 1) \in \mathbb{R}^2$. Then, $v + w = (1, 1)$ and we have

$$T(v + w) = T(1, 1) = (0, 1),$$

and

$$T(v) + T(w) = (0, 1) + (0, 1) = (0, 2).$$

Thus, there exists $v, w \in \mathbb{R}^2$ such that $T(v + w) \neq T(v) + T(w)$.

Hence, T is not a linear transformation.

19. Suppose a linear T transforms $(1,1)$ to $(2,2)$ and $(2,0)$ to $(0,0)$. Find $T(v)$ when

(a) $v=(2,2)$; (b) $v=(3,1)$; (c) $v=(-1,1)$; (d) $v=(a,b)$.

Solution: Let us first find the linear transformation T which transforms $(1,1)$ to $(2,2)$ and $(2,0)$ to $(0,0)$.

Note that $\{(1,1),(2,0)\}$ is a basis of $\mathbb{R}^2$. Let us take $v=(v_1,v_2)\in\mathbb{R}^2$, then

$$(v_1,v_2)=\alpha_1(1,1)+\alpha_2(2,0),$$

for some $\alpha_1,\alpha_2\in\mathbb{R}$, which implies $v_1=\alpha_1+2\alpha_2$ and $v_2=\alpha_1$.

Solving for α_1 and α_2, we get $\alpha_1=v_2$ and $\alpha_2=(v_1-v_2)/2$. Thus, we have

$$(v_1,v_2)=v_2(1,1)+\frac{(v_1-v_2)}{2}(2,0).$$

Therefore,

$$\begin{aligned}T(v_1,v_2)&=T\left(v_2(1,1)+\frac{(v_1-v_2)}{2}(2,0)\right)\\&=v_2T(1,1)+\frac{(v_1-v_2)}{2}T(2,0)\qquad\text{(Using linearity of T)}\\&=v_2(2,2)+\frac{(v_1-v_2)}{2}(0,0)=(2v_2,2v_2).\end{aligned}$$

Thus, we have $T(v_1,v_2)=(2v_2,2v_2)$.

Therefore,

(a) $T(2,2)=(4,4)$.

(b) $T(3,1)=(2,2)$.

(c) $T(-1,1)=(2,2)$.

(d) $T(a,b)=(2b,2b)$.

20. (a) What matrix transforms $(1,0)$ and $(0,1)$ to $(2,5)$ and $(1,3)$?

(b) What matrix transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u)?

(c) What matrix transforms $(2,5)$ and $(1,3)$ to $(1,0)$ and $(0,1)$?

(d) Why does no matrix transform $(2,6)$ and $(1,3)$ to $(1,0)$ and $(0,1)$?

Solution: Consider the usual basis $\mathcal{B} = \{(1,0),(0,1)\}$ of $\mathbb{R}^2$.

In order to obtain the required matrices in parts (a), (b) and (c), we shall write the image vectors $T(1,0)$ and $T(0,1)$ as a linear combination of $(1,0)$ and $(0,1)$.

(a) Let T be the linear transformation that transforms $(1,0)$ to $(2,5)$ and $(0,1)$ to $(1,3)$.

Now,

$$T(1,0) = (2,5) = 2(1,0) + 5(0,1),$$
$$T(0,1) = (1,3) = 1(1,0) + 3(0,1).$$

Therefore, the associated matrix is,

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix},$$

which is the required matrix.

(b) We shall find the matrix that transforms $(1,0)$ to (r,t) and $(0,1)$ to (s,u).

Now,

$$T(1,0) = (r,t) = r(1,0) + t(0,1),$$
$$T(0,1) = (s,u) = s(1,0) + u(0,1).$$

Therefore, the associated matrix is,

$$A = \begin{bmatrix} r & s \\ t & u \end{bmatrix},$$

which is the required matrix.

(c) Note that the vectors $(2,5)$ and $(1,3)$ form a basis $S = \{(2,5),(1,3)\}$ of $\mathbb{R}^2$.

Let us take $v = (v_1, v_2) \in \mathbb{R}^2$, then

$$(v_1, v_2) = \alpha_1 (2,5) + \alpha_2 (1,3),$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$, which implies $v_1 = 2\alpha_1 + \alpha_2$ and $v_2 = 5\alpha_1 + 3\alpha_2$.

Solving for α_1 and α_2, we get $\alpha_1 = 3v_1 - v_2$ and $\alpha_2 = -5v_1 + 2v_2$. Thus, we have

$$(v_1, v_2) = (3v_1 - v_2)(2,5) + (-5v_1 + 2v_2)(1,3).$$

In particular,

$$(1,0) = 3\,(2,5) - 5\,(1,3) \quad \text{and} \quad (0,1) = -(2,5) + 2\,(1,3).$$

Now,

$$\begin{aligned} T(1,0) &= T(3\,(2,5) - 5\,(1,3)) = 3\,T(2,5) - 5\,T(1,3) = 3\,(1,0) - 5\,(0,1), \\ T(0,1) &= T(-(2,5) + 2\,(1,3)) = -T(2,5) + 2\,T(1,3) = -(1,0) + 2\,(0,1). \end{aligned}$$

Therefore, the associated matrix is,

$$A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix},$$

which is the required matrix.

(d) We know that every 2×2 matrix A can be seen as a linear transformation from $A : \mathbb{R}^2 \to \mathbb{R}^2$. In particular, we have $A(c\,v) = c\,A(v)$ for all $v \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

So, if there is a matrix A which transforms $(1,3)$ to $(0,1)$, then it will transform $(2,6)$ to $(0,2)$ because $(2,6) = 2(1,3)$, that is,

$$A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad A \begin{bmatrix} 2 \\ 6 \end{bmatrix} = A\left(2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 2\left(A \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Thus, there is no matrix that transforms $(2,6)$ and $(1,3)$ to $(1,0)$ and $(0,1)$.