Solution Set - 5

1. Given an example of two 2 by 2 matrices B and C such that $B \neq C$ but $AB = AC$, where $A = \begin{bmatrix} 1 & 5 \\ 3 & 15 \end{bmatrix}$.

Solution: Consider the following matrices:

$$B = \begin{bmatrix} 5 & 0 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 5 \\ 0 & -1 \end{bmatrix}.$$

Clearly, $B \neq C$, but

$$AB = BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. If the inverse of A^2 is B, show that the inverse of A is AB. (Thus A is invertible whenever A^2 is invertible.)

Solution: We are given that A^2 is invertible and its inverse is B, that is, $A^2B = BA^2 = I$.

Then, it follows that

$$A(AB) = (AA)B = A^2B = I,$$

and

$$\begin{aligned} (AB)A &= IABA \\ &= (BA^2)ABA \\ &= BA^3BA \\ &= BA(A^2B)A \\ &= BAIA \\ &= BA^2 \\ &= I. \end{aligned}$$

Thus, we have seen that $A(AB) = I$ and $(AB)A = I$. Hence, AB is the inverse of A.

3. Find three 2 by 2 matrices, other than I and $-I$, that are their own inverses: $A^2 = I$.

Solution: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & (a+d)b \\ (a+d)c & d^2+bc \end{bmatrix}.$$

In particular, take $d = -a$ so that

$$A^2 = \begin{bmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{bmatrix} = (a^2+bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are many ways to choose $a, b, c \in \mathbb{R}$ such that they satisfy $a^2 + bc = 1$. With any such choice we will have $A^2 = I$.

It is thus easy to see that the square of any of the following matrices is the identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 5 & -1 \end{bmatrix}, \ldots$$

4. Give examples of 2 by 2 matrices matrices A and B such that

(a) $A + B$ is not invertible although A and B are invertible.

(b) $A + B$ is invertible although A and B are not invertible.

(c) All of A, B, and $A + B$ are invertible.

Solution:

(a) Take $A = I$ and $B = -I$. These matrices A and B are invertible. In fact, $A^{-1} = A = I$ and $B^{-1} = B = -I$. But, the matrix $A + B = \mathbf{0}$ is not invertible. Here $\mathbf{0}$ denotes the 2 by 2 zero matrix.

(b) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
Clearly, A and B are not invertible, but $A + B = I$ is invertible.

(c) Take $A = B = I$ so that $A + B = 2I$. Clearly, all of A, B, and $A + B$ are invertible.

5. Let A and B be n by n matrices such that all of A, B, and $A+B$ are invertible. In this case, show that $C = A^{-1} + B^{-1}$ is also invertible, and find a formula for C^{-1}.

Solution:

Recall: The product of finitely many invertible matrices is invertible, with the inverse being the product in the reverse order of the individual inverses. One can see it from a simple check up, for example, for three invertible matrices, say, P, Q and R. We claim that PQR is invertible with its inverse being $R^{-1}Q^{-1}P^{-1}$. In fact,

$$\begin{aligned}(PQR)(R^{-1}Q^{-1}P^{-1}) &= PQ(RR^{-1})Q^{-1}P^{-1} \\ &= PQIQ^{-1}P^{-1} \\ &= P(QQ^{-1})P^{-1} \\ &= PIP^{-1} \\ &= I,\end{aligned}$$

and similarly one can check that $(R^{-1}Q^{-1}P^{-1})(PQR) = I$.
This proves that $R^{-1}Q^{-1}P^{-1}$ is the inverse of PQR.

Coming back to our main question, note that

$$\begin{aligned}A^{-1}(B+A)B^{-1} &= A^{-1}BB^{-1} + A^{-1}AB^{-1} \\ &= A^{-1}I + IB^{-1} \\ &= A^{-1} + B^{-1} = C.\end{aligned}$$

Thus, we see that $C = A^{-1} + B^{-1}$ is the product of three invertible matrices, namely, A^{-1}, $B+A$ and B^{-1}. Therefore, the matrix C is invertible with

$$C^{-1} = B(B+A)^{-1}A.$$

6. Under what conditions on their entries are A and B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}.$$

Solution:

(a) Let P_{13} denote the 3 by 3 permutation matrix which interchanges rows 1 and 3. We know that P_{13} is invertible, which implies that A is invertible if and only if $C = P_{13}A$ is invertible.

Note that

$$C = P_{13}A = \begin{bmatrix} f & 0 & 0 \\ d & e & 0 \\ a & b & c \end{bmatrix}.$$

So, we see that C is a lower triangular matrix, and we know that a triangular matrix (upper or lower) is invertible if and only if every diagonal entry is non-zero.

We conclude that A is invertible if and only if all of c, e and f are non-zero.

(b) We claim that B is invertible if and only if $(ad - bc)e \neq 0$.

(a) **<u>Case 1</u>:** If $b = 0$.

In this case, $B = \begin{bmatrix} a & 0 & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$ is a lower triangular matrix which is invertible if and only if $ade \neq 0$, which in this case is same as $(ad - bc)e = ade \neq 0$.

(b) **<u>Case 2</u>:** If $d = 0$.

Let P_{12} denote the 3 by 3 permutation matrix which interchanges rows 1 and 2. We know that P_{12} is invertible, which implies that B is invertible if and only if $D = P_{12}A$ is invertible. Note that

$$D = P_{12}A = \begin{bmatrix} c & 0 & 0 \\ a & b & 0 \\ 0 & 0 & e \end{bmatrix}.$$

Again, D being a lower triangular matrix, is invertible if and only if $cbe \neq 0$, which in this case is same as $(ad - bc)e = -bce \neq 0$.

(c) **<u>Case 3</u>:** If $bd \neq 0$.

Let E_{12} denote the elimination matrix corresponding to "replacing row 1 by the subtraction of $(-b/d)$ times of row 2 from row 1". That is, $E_{12} = \begin{bmatrix} 1 & -b/d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We know that E_{12} is invertible, which implies that B is invertible if and only if $F = E_{12}B$ is invertible.

Note that

$$F = E_{12}B = \begin{bmatrix} a - \frac{bc}{d} & 0 & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}.$$

As earlier, F being a lower triangular matrix, is invertible if and only if $\left(a - \frac{bc}{d}\right) de \neq 0$, which in this case is same as $(ad - bc)e \neq 0$.

7. **(Remarkable)** Let A and B be n by n matrices. Prove that $I - BA$ is invertible if and only if $I - AB$ is invertible. [Hint: One can make use of the identity $B(I - AB) = (I - BA)B$.]

Solution: We shall prove that if $I - AB$ is invertible, then $I - AB$ is invertible. Since the role of A and B is interchangeable, the other implication is automatic.

So, let us assume that $I - AB$ is invertible. We shall show that $(I - BA)^{-1} = I + B(I - AB)^{-1}A$.

We shall make use of the following interesting identities:

$B(I - AB) = (I - BA)B$ (both sides are equal to $B - BAB$) and $A(I - BA) = (I - AB)A$ (both sides are equal to $A - ABA$).

Note that

$$\begin{aligned}
I &= (I - BA) + BA \\
&= (I - BA) + BIA \\
&= (I - BA) + B\left((I - AB)(I - AB)^{-1}\right)A \\
&= (I - BA) + (B(I - AB))\left((I - AB)^{-1}A\right) \\
&= (I - BA) + ((I - BA)B)\left((I - AB)^{-1}A\right) \qquad \text{(using the identity } B(I - AB) = (I - BA)B\text{)} \\
&= (I - BA) + (I - BA)B(I - AB)^{-1}A \\
&= (I - BA)\left(I + B(I - AB)^{-1}A\right).
\end{aligned}$$

Repeating the similar calculations as above, we also have

$$\begin{aligned}
I &= (I - BA) + BA \\
&= (I - BA) + BIA \\
&= (I - BA) + B\left((I - AB)^{-1}(I - AB)\right)A \\
&= (I - BA) + \left(B(I - AB)^{-1}\right)((I - AB)A) \\
&= (I - BA) + \left(B(I - AB)^{-1}\right)(A(I - BA)) \qquad \text{(using the identity } A(I - BA) = (I - AB)A\text{)} \\
&= (I - BA) + B(I - AB)^{-1}A(I - BA) \\
&= \left(I + B(I - AB)^{-1}A\right)(I - BA).
\end{aligned}$$

We have shown that $(I - BA)\left(I + B(I - AB)^{-1}A\right) = \left(I + B(I - AB)^{-1}A\right)(I - BA) = I$, which proves that $I - BA$ is invertible and $(I - BA)^{-1} = I + B(I - AB)^{-1}A$.

8. Invert these matrices A by the Gauss-Jordan method starting with $[A\ I]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

Solution:

(a) We will find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

To use the Gauss-Jordan method, we will write the matrix A and the identity matrix together as follows:

$$[\,A \mid I\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Our goal is to transform the left side (the matrix A) into the identity matrix while performing the same row operations on the right side (the identity matrix).

Step 1: We shall eliminate the 2 appearing in the $(2,1)^{\text{th}}$ position by performing the operation $R_2 \to R_2 - 2R_1$. This gives us:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Step 2: We shall eliminate the 3 appearing in the $(2,3)^{\text{th}}$ position by performing the operation $R_2 \to R_2 - 3R_3$. This gives us:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus, the inverse of the matrix A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Now, consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

As in part (a), we put together the matrix A and the identity matrix as follows:

$$[\,A \mid I\,] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Our aim is to transform the left side matrix into the identity matrix while performing the same operations on the right side.

Step 1: We first eliminate all the entries of the first column except the $(1,1)^{\text{th}}$ entry by performing the following operations one-by-one:

$$R_2 \to R_2 - \frac{1}{4}R_1, \quad R_3 \to R_3 - \frac{1}{3}R_1 \quad \text{and} \quad R_4 \to R_4 - \frac{1}{2}R_1.$$

This results in:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 & -\frac{1}{3} & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right].$$

Step 2: We now eliminate the entries below the 1 appearing in the $(2,2)^{\text{th}}$ position by exercising the following operations:

$$R_3 \to R_3 - \frac{1}{3}R_2 \quad \text{and} \quad R_4 \to R_4 - \frac{1}{2}R_2.$$

This yields:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & -\frac{3}{8} & -\frac{1}{2} & 0 & 1 \end{array}\right].$$

Step 3: Eliminate the $\frac{1}{2}$ appearing in the $(4,3)^{\text{th}}$ position by performing the operation $R_4 \to R_4 - \frac{1}{2}R_3$. This gives us:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \end{array}\right].$$

Thus, the inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}.$$

9. True or false (with a counterexample if false and a reason if true):

   (a) A 4 by 4 matrix with a row of zeros is not invertible.

   (b) A matrix with 1s down the main diagonal is invertible.

Solution:

(a) The statement is **TRUE**.

Let A be any 4 by 4 matrix and $i \in \{1, 2, 3, 4\}$ be arbitrary. Suppose that the i^{th} row of A has all the entries equal to zero. Let B be any 4 by 4 matrix. After multiplying the matrices A and B, observe that the entry in the $(i, i)^{\text{th}}$ position of the matrix AB is zero. This implies that AB is not the identity matrix. Since B was chosen arbitrarily, we conclude that the matrix A is not invertible.

(b) The statement is **FALSE**.

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Clearly, each entry of A below the main diagonal is 1, but A is not invertible.