1. SET THEORY

> "A set is a Many that allows itself
> to be thought of as a One."
>
> ~ GEORG CANTOR

A set is a collection of objects. These objects are known as elements or members of the set. Elements of a set can be anything: numbers, lines, students, birds, trees and even sets! Informally, we can think of a set as a box with objects inside the box being the elements.

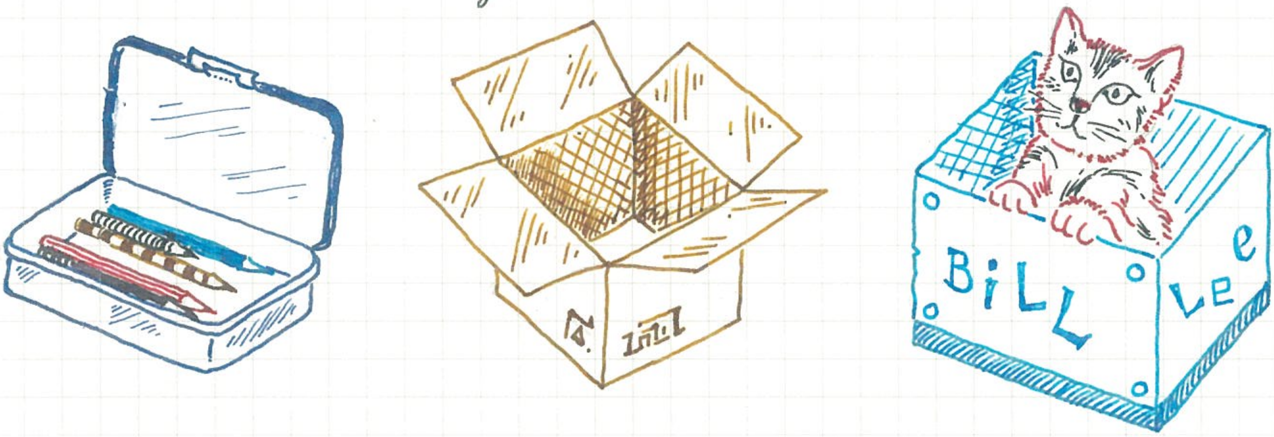


Fig 1.1: Boxes as sets - three instances

We shall denote empty set by the (uppercase) Greek letter ϕ, pronounced as Phi. Such a set is characterized uniquely by its emptiness, i.e., there is only one empty set.

NOTATION: Traditionally, we use uppercase letters to denote sets and lowercase letters to denote elements in a set. So, if A is a set and x is an object, then we write

i) $x \in A$ if x is an element of A

ii) $x \notin A$ if x is not an element of A.

IN JEST

- For us, in MA1101, a set is not a verb or an adjective.
- It is also not to be confused with sets in tennis.
- It is also not the popular card game SET!

As a set is defined (or characterized) by its elements, we write a set by listing its elements. This is traditionally done in one of two ways:

A) ROSTER METHOD: In this method, we write a set by listing all its elements.

i) $A = \{1\}$, $B = \{2\}$, $C = \{\pi\}$; these are called singleton sets. Note that A is different from 1, so is B from 2 and so on.

ii) $D = \{$Mon, Tue, Wed, Thu, Fri, Sat, Sun$\}$; it is a set whose elements are the days of a week.

iii) $E = \{1, 2, \ldots, 100\}$; the three dots signify all positive integers from 3 through 99.

iv) $F = \{$set of all BS-MS 2024 students in IISER-K$\}$; note that F is different from the actual set of all BS-MS 2024 students in IISER-K.

v) $\mathbb{N} = \{1, 2, 3, \ldots\}$ (= set of all positive integers)

vi) $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (= set of all integers)

B) SET-BUILDER METHOD: Not all collections can be listed. This indicates the necessity of the second method in certain cases. In this method, we define the set by describing the properties of its elements only satisfied by them.

i) A = set of all real numbers between 0 & 1
$= \{x \in \mathbb{R} \mid x > 0, x < 1\}$

ii) $\mathbb{Q}$ = set of all rational numbers (fractions)
$= \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(p,q) = 1\}$

iii) Let S be the set of all three letter words in English uppercase alphabet. Thus, S consists of 26^3 elements.

It is quite cumbersome to write S using the roster method. A standard way, even within this method, would be to use the lexicographic ordering. Note that

$$S = \{xyz \mid x, y, z \in \{A, B, C, \ldots, Z\}\}$$

IN JEST

- If we go by words officially allowed in Scrabble, then 'ZZZ' is a valid English word, all whose permutations are equally valid.
- 'EEW' is a valid word and an alternate spelling for 'EW'. All its permutations 'WEE' & 'EWE' are also valid words.

DEFINITION 1.1.1 (NON-EMPTY SETS)

We call a set A non-empty if A has at least one element in it. We denote this as $A \neq \emptyset$.

DEFINITION 1.1.2 (SUBSET)

Given two sets A & B, we say that A is a subset of B, denoted by $A \subseteq B$, if every element of A is also an element of B.

DEFINITION 1.1.3 (EQUALITY)

We say that two sets A & B are equal, denoted by $A = B$, if $A \subseteq B$ and $B \subseteq A$.

DEFINITION 1.1.4 (PROPER SUBSET)

Given two sets A & B, we say that A is a proper subset of B, denoted by $A \subset B$ or $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$, i.e., every element of A is an element of B and there is an element of B that is not in A.

EXAMPLES 1.1.5

i) Consider the following sets:

$$A = \{0, 1\}, \quad B = \{x \in \mathbb{R} \mid x^2 - x = 0\}, \quad C = \{x \in \mathbb{R} \mid x^4 - 3x^3 + 2x^2 = 0\}$$

Verify that $A = B$ & $A \subsetneq C = \{0, 1, 2\}$.

ii) $\mathbb{N} \subsetneq \mathbb{Z}$ & $\mathbb{Z} \subsetneq \mathbb{Q}$.

iii) $\emptyset \subsetneq A$ for any set A.

We shall introduce various operations that allow us to generate new sets from old ones.

DEFINITION 1.2.1

Let A and B be two sets.

UNION: The union of A and B, denoted by $A\cup B$, is defined as
$$A\cup B := \{x \mid x\in A \text{ or } x\in B\}$$

INTERSECTION: The intersection of A & B, denoted by $A\cap B$, is defined as
$$A\cap B := \{x \mid x\in A \text{ and } x\in B\}$$

DIFFERENCE: The difference of B from A, denoted by $A\setminus B$, is defined as
$$A\setminus B := \{x \mid x\in A \text{ and } x\notin B\}$$

SYMMETRIC DIFFERENCE: The symmetric difference of A & B, denoted by $A\Delta B$, is defined as $A\Delta B := \{x \mid x\in A \text{ and } x\notin B\}\cup\{x \mid x\notin A \text{ and } x\in B\}$

Note that if $A\subseteq B$ and $A\subseteq C$, then $A\subseteq B\cap C$. If $A\subseteq C$ & $B\subseteq C$, then $A\cup B\subseteq C$. If $B\subseteq C$, then $A\setminus C\subseteq A\setminus B$. All of these statements are quite evident through Venn diagrams.

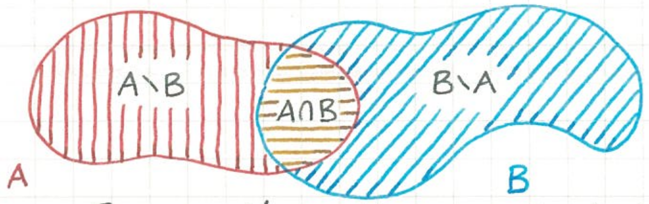


Fig 1.2 : Venn diagram viewpoint

EXAMPLES 1.2.2

i) Let $A=\{-2,-1,0,1,2\}$, $B=\{0,1,2,3\}$. Then
$A\cup B=\{-2,-1,0,1,2,3\}$, $A\cap B=\{0,1,2\}$, $A\setminus B=\{-2,-1\}$, $B\setminus A=\{3\}$.

ii) Let A = all even integers & B = all odd integers. Then
$A\cup B=\mathbb{Z}$, $A\cap B=\emptyset$, $A\setminus B=A$, $B\setminus A=B$, $A\Delta B=\mathbb{Z}$.

THEOREM 1.2.3 Let A, B & C be three sets. The following hold:

i) (ASSOCIATIVITY): $A\cup(B\cup C)=(A\cup B)\cup C$, $A\cap(B\cap C)=(A\cap B)\cap C$

ii) (COMMUTATIVITY): $A\cup B=B\cup A$, $A\cap B=B\cap A$

iii) (DISTRIBUTIVITY): $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

DEFINITION 1·2·4 (DISJOINT SETS)

We say that two sets A and B are disjoint if $A \cap B = \phi$, i.e., A and B have no elements in common.

DEFINITION 1·2·5 (COMPLEMENT)

Let $A \subseteq X$ be two sets. The complement of A (relative to X) is

$$A^c := X \setminus A = \{x \in X \mid x \notin A\}.$$

Often, the set X that contains all elements under study is clear in the context. In such a case we simply say "the absolute complement" or "the complement" of A to mean A^c. This X would be called the "universal set" in that context. Note that

$$(A^c)^c := X \setminus A^c = \{x \in X \mid x \notin A^c\} = A.$$

THEOREM 1·2·6 (DE MORGAN'S LAWS)

Let X be the universal set and $A, B \subseteq X$. Then,

i) $(A \cup B)^c = A^c \cap B^c$

ii) $(A \cap B)^c = A^c \cup B^c$

IN JEST

- The headquarter of the London Mathematical Society is called De Morgan House.
- De Morgan has a lunar crater named after him.

Proof. i) Let $x \in (A \cup B)^c = X \setminus (A \cup B)$. Then $x \in X$ and $x \notin A \cup B$. Thus, $x \notin A$ and $x \notin B$. This implies $x \in X \setminus A$ and $x \in X \setminus B$. Thus, $x \in A^c \cap B^c$ & we conclude that $(A \cup B)^c \subseteq A^c \cap B^c$.

Now let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. This implies that $x \in X$, $x \notin A$ and $x \notin B$. Thus, $x \notin A \cup B$ or equivalently $x \in (A \cup B)^c$. We conclude that $A^c \cap B^c \subseteq (A \cup B)^c$.

The two claims above imply $(A \cup B)^c = A^c \cap B^c$.

ii) Using i) we get

$$(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c = A \cap B$$

$$\Rightarrow \quad A^c \cup B^c = ((A^c \cup B^c)^c)^c = (A \cap B)^c.$$

▨

DEFINITION 1.2.7 (POWER SET)

Let A be a set. The power set of A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A.

The power set of A is also denoted by 2^A. For any A, as $\emptyset \subseteq A$, it follows that $\emptyset \in \mathcal{P}(A)$ and $\mathcal{P}(A)$ is non-empty.

EXAMPLES 1.2.8

i) If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$.

ii) If $A = \{\emptyset\}$, then $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}\} = \{\emptyset, A\}$.

iii) If $A = \{a, b\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, A\}$$

iv) If $A = \{x, y, z\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, A\}.$$

We need to introduce the notion of cartesian product. Towards that, we need the definition of an ordered pair. Informally, if A & B are non-empty sets, then for $a \in A, b \in B$, the ordered pair (a,b) is a symbol, specifying a and b, in that order. This is descriptive and rather unsatisfactory. We would prefer a formal definition that captures the main property of an ordered pair, i.e.,

$$(a,b) = (c,d) \text{ if and only if } a = c \text{ and } b = d.$$

We shall follow the most commonly used definition, due to Kuratowski (1921).

DEFINITION 1.2.9 (ORDERED PAIR)

Let A and B be non-empty sets. If $a \in A$ and $b \in B$, then the ordered pair (a,b) is defined as

$$(a,b) := \{\{a\}, \{a,b\}\}.$$

LEMMA 1.2.10 Let A & B be non-empty sets. Let $a, x \in A$ & $b, y \in B$. Then $(a,b) = (x,y)$ if and only if $a = x$ and $b = y$.

Proof. If $a=x$ and $b=y$, then

$$(a,b)=\{\{a\},\{a,b\}\}=\{\{x\},\{x,y\}\}=(x,y).$$

If $(a,b)=(x,y)$, then we consider two cases.

Case 1: $a=x$

As $\{\{a\},\{a,b\}\}=\{\{x\},\{x,b\}\}=\{\{x\},\{x,y\}\}$, either $\{x,y\}=\{x\}$ or $\{x,b\}$

If $\{x,y\}=\{x\}$, then $y=x$ & $(x,y)=\{\{x\}\}$, forcing $\{a,b\}=\{a\}$. Thus, $b=a=x=y$. If $\{x,y\}=\{x,b\}$, then $b=y$.

Case 2: $a\neq x$

As $\{a\}\in(a,b)=(x,y)=\{\{x\},\{x,y\}\}$, we conclude that $\{a\}=\{x,y\}$. This forces $x=y=a$, a contradiction. Thus, case 2 can't occur.

Thus, in both cases we conclude that the only possibility is $x=a$ and $y=b$. ▨

Definition 1·2·11 (Cartesian Product)

Let A & B be two sets. The cartesian product of A and B, denoted by $A\times B$, is defined as

$$A\times B := \begin{cases} \{(a,b) \mid a\in A,\ b\in B\} & \text{when both } A \text{ \& } B \text{ are non-empty} \\ \emptyset & \text{when } A=\emptyset \text{ or } B=\emptyset. \end{cases}$$

Examples 1·2·12

i) If $A=\{e,f\}$ and $B=\{1,2\}$, then

$$A\times B=\{(e,1),(e,2),(f,1),(f,2)\}.$$

ii) If $A=\{a,b,c\}$, then

$$A\times A=\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}.$$

iii) If $A=\{*\}$, then

$$\mathcal{P}(A)\times\mathcal{P}(A)=\{(\emptyset,\emptyset),(\emptyset,\{*\}),(\{*\},\emptyset),(\{*\},\{*\})\}.$$

§1·3 Relations

We start with the definition of this important concept.

Definition 1·3·1 (Relation)

Let X & Y be non-empty sets. A relation R between X & Y is a subset $R\subseteq X\times Y$. If $X=Y$, then we say R is a relation on X.

Given a relation $R \subseteq X \times Y$, if $(x,y) \in R$ then we say that "x is R-related to y" and write xRy.

EXAMPLES 1·3·2

i) Let $X = \{a,b,c,d\}$, $Y = \{1,2\}$ and

$$R = \{(a,1), (b,2), (c,1), (c,2)\}.$$

Then $aR1$, $bR2$, $cR1$ and $cR2$ while $a \not R 2$, $d \not R 1$, $d \not R 2$ & so on.

ii) Let $X = Y$ be the set of all human beings on Earth. Define $R := \{(x,y) \in X \times X \mid x \text{ is the father of } y\}$. Note that R is a relation. Moreover $x \not R x$ for any $x \in X$. If xRy, then $y \not R x$.

iii) Let $X = Y = \mathbb{Z}$, the set of integers. Define a relation

$$R := \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid m-n \text{ is divisible by } 2\}.$$

In other words, mRn if and only if $m-n$ is even. Note that nRn for any $n \in \mathbb{Z}$. If mRn, then nRm. Also, if mRn & nRp, then mRp.

DEFINITION 1·3·3

Let X be a non-empty set and $R \subseteq X \times X$ be a relation on X.

i) REFLEXIVE: R is said to be reflexive if xRx for all $x \in X$.

ii) SYMMETRIC: R is said to be symmetric if xRy then yRx.

iii) TRANSITIVE: R is said to be transitive if xRy & yRz then xRz.

EXAMPLES 1·3·4

i) The relation in Eg 1·3·2 (ii) is not reflexive, not symmetric and not transitive.

ii) The relation in Eg 1·3·2 (iii) is reflexive, symmetric & transitive.

iii) Let $X = Y = \mathbb{R}$ and let $R := \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid xy = 0\}$. Then R is symmetric but neither reflexive nor transitive.

iv) Let $X = \{a,b\}$ & $R := \{(a,a), (a,b), (b,a), (b,b)\}$. Then R is symmetric & transitive but not reflexive.

v) Let $X = Y = \mathbb{R}$ and define

$$R_1 := \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x \geq y\}$$

$$R_2 := \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x > y\}.$$

Then R_1 is reflexive and transitive but not symmetric. Note that R_2 is transitive but neither reflexive nor symmetric. Moreover $R_2 \subseteq R_1$, i.e., if $x R_2 y$, then $x R_1 y$.

vi) EQUALITY RELATION*: For any non-empty X, define

$$\Delta(X) := \{(x,x) \in X\times X \mid x \in X\}.$$

This relation is reflexive, symmetric and transitive.

vii) CONGRUENCE MODULO n: Let $X = Y = \mathbb{Z}$ and let $n \in \mathbb{N}$. Let us define

$$\equiv_n := \{(m_1, m_2) \in \mathbb{Z}\times\mathbb{Z} \mid n \text{ divides } (m_1 - m_2)\}.$$

Note that for $n=2$, this relation is that described in Eg 1·3·2(iii). Similar to that example, $\equiv_n$ is reflexive, symmetric & transitive.

viii) UNIVERSAL RELATION: For any non-empty X, define $R := X\times X$. It is a relation which is reflexive, symmetric & transitive. Any relation R' on X is a subset of the universal relation.

DEFINITION 1·3·5 (EQUIVALENCE RELATION)

A relation $\sim$ on a non-empty set X is called an equivalence relation (on X) if $\sim$ is reflexive, symmetric & transitive.

For $x \in X$, the equivalence class of x is defined as

$$[x] := \{y \in X \mid y \sim x\}.$$

If $y \in [x]$, then we say that y is a representative of x.

The set of all equivalence classes of X is denoted by $X/\sim$

i.e., $$X/\sim := \{[x] \mid x \in X\}.$$

Note that $[x]$, by definition, is a subset of X. It is non-empty as $x \in [x]$.

* Equality relation is also called the identity relation.

If $y \in [x]$ and $z \in [y]$, then $z \sim y$ and $y \sim x$. By transitivity, $z \sim x$, i.e., $z \in [x]$. This implies $[y] \subseteq [x]$. In a similar manner, we conclude that, using $y \in [x]$ implies $x \in [y]$ via symmetry, $[x] \subseteq [y]$. Therefore $[x] = [y]$ if $y \in [x]$, or equivalently if $[x] \cap [y] \neq \varnothing$.

EXAMPLES 1·3·6

i) For the equality relation $\Delta(X)$ (see Eg 1·3·4 (vi)), for any $x \in X$,
$$[x] = \{y \in X \mid y \,\Delta(X)\, x\} = \{y \in X \mid y = x\} = \{x\}.$$
Therefore,
$$X/\sim = \{\{x\} \mid x \in X\}.$$

ii) For the universal relation (on X), for any $x \in X$
$$[x] = \{y \in X \mid y R x\} = \{y \in X\} = X$$
Therefore, $X/\sim = \{X\}$, is the set containing X as a single element.

iii) For the congruence modulo n relation (see Eg 1·3·4 (vii)), denoted by $\equiv_n$, let us compute $[a]$ for $a \in \mathbb{Z}$. We claim that
$$[a] = \{a + kn \mid k \in \mathbb{Z}\} = \{\dots, a-2n, a-n, a, a+n, a+2n, \dots\}.$$
To prove the claim, let S denote the set $\{a+kn \mid k \in \mathbb{Z}\}$. If $b \in [a]$, then n divides $b-a$. Thus, $b - a = kn$ for some $k \in \mathbb{Z}$. Consequently, $b = a + kn \in S$. Now, if $b' \in S$, then $b' = a + tn$ for some $t \in \mathbb{Z}$. Thus, n divides $b' - a$, implying that $b' \equiv_n a$. Hence, $S \subseteq [a]$. Putting the two conclusions together, we obtain $S = [a]$, proving our claim.

We now find the distinct equivalence classes for $\equiv_n$. We claim that
$$\boxed{\mathbb{Z}/\equiv_n = \{[0], [1], \dots, [n-1]\}.}$$
We shall prove this in two steps.

<u>STEP 1</u> For all $0 \leq r < s \leq n-1$, $[s] \cap [r] = \varnothing$.

<u>STEP 2</u> For any $a \in \mathbb{Z}$, there exists some $r \in \{0, 1, \dots, n-1\}$ such that $[a] = [r]$.

For step 1, we use the observation preceeding Eg 1.3.6. If $[s]\cap[r]\neq\phi$, then $[s]=[r]$, which implies that n divides $s-r$. As $0<s-r\leq n-1$, this is a contradiction. This proves that $[s]\cap[r]=\phi$. For step 2, using the division algorithm (divide a by n), we obtain $q\in\mathbb{Z}$ and $r\in\{0,1,\dots,n-1\}$ such that $a=qn+r$. Thus, n divides $a-r$. This forces $a\in[r]$ and $[a]=[r]$.

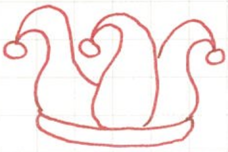
iv) On $\mathbb{R}$ consider the relation

$$\sim \;:=\{(x,y)\in\mathbb{R}\times\mathbb{R} \mid xy>0\}\cup\{(0,0)\}.$$

Verify that $\sim$ is an equivalence relation on $\mathbb{R}$. Note that

$$[x]=\begin{cases}(0,\infty) & \text{if } x>0\\ \{0\} & \text{if } x=0\\ (-\infty,0) & \text{if } x<0\end{cases}$$

Therefore, $\mathbb{R}/\sim \;=\{(-\infty,0),\{0\},(0,\infty)\}$.



IN JEST

- Declare two words in the English dictionary to be related if one rhymes with the other. The relation of rhyming is reflexive, symmetric & almost transitive! (Think of words having more than one pronunciation.)

We record the observation preceeding Eg 1.3.6 formally as well as an effective converse to it.

PROPOSITION 1.3.7

a) Let $\sim$ be an equivalence relation on a non-empty set X. For $x,y\in X$, exactly one of the following holds:
   i) $[x]=[y]$
   ii) $[x]\cap[y]=\phi$.

b) Let $\sim$ be a reflexive relation on a non-empty set X such that for $x,y\in X$ either $[x]=[y]$ or $[x]\cap[y]=\phi$ but not both. Then $\sim$ is an equivalence relation.

Proof. a) It is left as an exercise.

b) If yRx, then $y\in[x]$. As $y\in[y]$, $[x]\cap[y]\neq\phi$ forces $[x]=[y]$. Thus, $x\in[y]$, or equivalently xRy, implying symmetry. If xRy, yRz, then zRy by symmetry. Thus, $z\in[y]$ & $x\in[y]$ This implies $[z]=[y]$ & $[x]=[y]$, whence $[x]=[z]$ or xRz. $\boxtimes$

Note that on $X=\{a,b\}$, the relation $R=\{(a,b),(b,a)\}$ enforces

$$[a]=\{b\} \quad , \quad [b]=\{a\}.$$

Thus, $[a]=[a]$ (& $[a]\cap[a]\neq\emptyset$), $[b]=[b]$ (& $[b]\cap[b]\neq\emptyset$), $[a]\cap[b]=\emptyset$ (& $[a]\neq[b]$). However, R is not reflexive & certainly not an equivalence relation. This illustrates the necessity of reflexivity in the hypothesis of part (b) in Prpn 1·3·7.

DEFINITION 1·3·8 (PARTITION OF A SET)

A partition of a non-empty set X is a collection of pairwise disjoint, non-empty subsets of X whose union is X. Equivalently, a partition of X is a collection $\{A_i \mid i\in I\}$ of subsets of X so that

i) $A_i\neq\emptyset$ for $i\in I$,

ii) $A_i\cap A_j=\emptyset$ if $i,j\in I$ and $i\neq j$, and

iii) $X=\bigcup\limits_{i\in I} A_i$.

EXAMPLES 1·3·9

i) Let $X=\mathbb{R}$ and consider the following partitions.

a) $\{\{x\} \mid x\in\mathbb{R}\}$ (think of the equality relation on $\mathbb{R}$)

b) $\{(-\infty,0),\{0\},(0,\infty)\}$ (think of Eg 1·3·6(iv))

c) $\{\mathbb{Q},\mathbb{R}\setminus\mathbb{Q}\}$

ii) Let $X=\mathbb{Z}$ and consider the following partitions.

a) $\{\mathbb{Z}\}$ (think of the universal relation)

b) $\{\{\ldots,-2,0,2,4,\ldots\},\{\ldots,-3,-1,1,3,\ldots\}\}$ (think of $\equiv_2$ from Eg 1·3·6(iii))

c) $\{\{\ldots,-2,-1\},\{0,1,2,\ldots\}\}$

The significance of equivalence relations lie in the following result.

THEOREM 1·3·10

Let X be a non-empty set.

i) If $\sim$ is an equivalence relation on X, then $X/\sim$ forms a partition of X.

ii) If $\mathcal{J}=\{A_i \mid i\in I\}$ is a partition of X, then there exists an equivalence relation $\sim_{\mathcal{J}}$ on X such that $X/\sim=\mathcal{J}$.

Proof. i) We know that $x \in [x]$ and $X = \bigcup_{x \in X} [x]$. As exactly one of $[x] \cap [y] = \emptyset$ or $[x] = [y]$ hold for $x, y \in X$, choose elements $x_i \in X$ such that $[x_i]$'s represent the distinct equivalence classes. Here $i \in I$ is indexed by any set I which has the same size as the number of distinct equivalence classes. Note that X is a disjoint union of $[x_i]$'s. Thus, $\{[x_i] \mid i \in I\}$ forms a partition of X.

ii) Given $\mathcal{F}$, define $\sim_{\mathcal{F}}$ as follows:

$$x \sim_{\mathcal{F}} y \text{ if and only if for some } i \in I,\ x, y \in A_i.$$

It is clear that $\sim_{\mathcal{F}}$ is reflexive and symmetric. Moreover,

$$[x] = \{y \in X \mid y \sim_{\mathcal{F}} x\} = \{y \in X \mid y \in A_i\} = A_i,$$

where A_i is the unique element of $\mathcal{F}$ that contains x. Applying Proposition 1.3.7(b) to $\sim_{\mathcal{F}}$, we conclude that it is an equivalence relation with A_i's being the equivalence classes. ∎

EXAMPLES 1.3.11

It follows from the theorem that any partition of X gives rise to an equivalence relation. If the partition is efficiently describable, then it is often possible to define the associated equivalence relation in terms of explicit formulae.

i) Let $X = \mathbb{R}$ and for $i \in [0, 1)$, i.e., $0 \le i < 1$, set

$$A_i := \{i + n \mid n \in \mathbb{Z}\}.$$

As any x can be written as $[x] + \{x\}$, where $[x]$ is the greatest integer less than or equal to x & $\{x\}$ is the fractional part of x, we conclude that $x \in A_{\{x\}}$. Moreover A_i's are mutually disjoint. The associated relation $\sim_{\mathcal{F}}$ for $\mathcal{F} = \{A_i\}_{i \in [0,1)}$ is given by $x \sim_{\mathcal{F}} y$ if and only if $\{x\} = \{y\}$.

ii) Let $X = \mathbb{R}\times\mathbb{R}$ and set

$$A_i := \{(x,y)\in\mathbb{R}\times\mathbb{R} \mid x^2+y^2 = i^2\}.$$

for $0\le i<\infty$. Note that A_i's are mutually disjoint and if $(x,y)\in\mathbb{R}\times\mathbb{R}$, then $(x,y)\in A_{x^2+y^2}$. Thus, $\mathcal{F}=\{A_i\}_{i\ge 0}$ forms a partition. It follows that if $(x,y)\in\mathbb{R}\times\mathbb{R}$, then

$$[(x,y)]_{\sim_{\mathcal{F}}} = \{(x',y')\in\mathbb{R}\times\mathbb{R} \mid (x')^2+(y')^2 = x^2+y^2\},$$

i.e., equivalence classes are concentric circles on the plane with the origin as the common centre.

§1.4 FUNCTIONS

Let us start with the definition.

DEFINITION 1.4.1 (FUNCTION)

A relation $R\subseteq X\times Y$ between two non-empty sets X and Y is said to be a function if for every $x\in X$, there exists a unique $y_x\in Y$ such that xRy_x or, equivalently $(x,y_x)\in R$. Moreover, we say X is the domain of R and Y the codomain of R.

EXAMPLES 1.4.2

i) Let $X=\{a,b,c\}$ & $Y=\{1,2\}$. Then

  a) $R_1=\{(a,1),(b,2),(c,1)\}$ is a function;

  b) $R_2=\{(a,2),(b,2),(c,2)\}$ is a function;

  c) $R_3=\{(a,1),(b,1)\}$ is not a function;

  d) $R_4=\{(a,1),(a,2),(b,2),(c,1)\}$ is not a function.

ii) Let $X=\mathbb{R}$ & $Y=\mathbb{R}$. Then

  a) $R_1=\{(x,x)\mid x\in\mathbb{R}\}$ is a function;

  b) $R_2=\{(x,\sin x)\mid x\in\mathbb{R}\}$ is a function;

  c) $R_3=\{(x,y)\mid y^2=x\}$ is not a function.

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- WHAT IS THE MATHEMATICAL FUNCTION CONDEMNED BY CHURCH? It's sin.
- DID YOU KNOW THAT OLD MATH TEACHERS NEVER DIE? They just lose some of their functions.

If $R \subseteq X \times Y$ is a function, then we write $R: X \to Y$ to denote this function. For each $x \in X$, $R(x)$ denotes $y_x \in Y$ which is uniquely defined by $x R y_x$. The functions in example 1.4.2 can be written as follows:

i) $R_1(a) = 1$, $R_1(b) = 2$, $R_1(c) = 1$;

$R_2(a) = 2$, $R_2(b) = 2$, $R_2(c) = 2$.

ii) $R_1(x) = x$ is the identity function;

$R_2(x) = \sin x$ is the (trigonometric) sine function.

DEFINITION 1.4.3

Let $f: X \to Y$ be a function. We say that

a) X is the domain of f;

b) Y is the codomain of f;

c) for each $x \in X$, $f(x)$ is the image of x;

d) $f(X) := \{f(x) \mid x \in X\}$ is the range of f.

EXAMPLES 1.4.4

i) CONSTANT FUNCTION: Let $X \neq \emptyset$ & choose $y_0 \in Y$. Define

$$f_{y_0}: X \to Y, \quad f_{y_0}(x) = y_0 \text{ for any } x \in X.$$

Note that the range $f_{y_0}(X)$ is $\{y_0\}$.

ii) IDENTITY FUNCTION: Let $X \neq \emptyset$ and define

$$Id_X: X \to X, \quad Id_X(x) = x \text{ for any } x \in X.$$

Note that the range $Id_X(X)$ is X.

iii) INCLUSION FUNCTION: Let $X, Y \neq \emptyset$ and $X \subseteq Y$. Define

$$i: X \to Y, \quad i(x) = x \text{ for any } x \in X.$$

This is called the inclusion function & its range is X.

iv) MODULUS FUNCTION: Define $|\cdot|: \mathbb{R} \to \mathbb{R}$ as follows

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

This is called the modulus function & its range is all non-negative real numbers.

DEFINITION 1.4.5 (EQUALITY OF FUNCTIONS)

Let $f: X \to Y$ and $g: A \to B$ be two functions. We say that f and g are equal (and write $f = g$) if $X = A$, $Y = B$ and

$$f(x) = g(x) \text{ for any } x \in X.$$

EXAMPLES 1.4.6

i) Let $f: \mathbb{N} \to \mathbb{Z}$ & $g: \mathbb{Z} \to \mathbb{Z}$ be given by

$$f(n) := n \quad \text{for all } n \in \mathbb{N}$$
$$g(m) := m \quad \text{for all } m \in \mathbb{Z}.$$

Even though $f(n) = g(n) = n$ for $n \in \mathbb{N}$, we have $f \neq g$ as the domains of f & g are not equal.

ii) Let $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to [0, \infty)$ be given by

$$f(x) = g(x) = x^2 \text{ for all } x \in \mathbb{R}.$$

Then $f \neq g$ as codomains of f and g are not equal. Note that $f(\mathbb{R}) = g(\mathbb{R}) = [0, \infty)$.

DEFINITION 1.4.7

Let $f: X \to Y$ be a function.

i) We say f is one-one (or injective) if for $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

ii) We say f is onto (or surjective) if $f(X) = Y$, i.e., for each $y \in Y$ there exists some $x \in X$ such that $f(x) = y$.

iii) We say f is bijective if f is both one-one and onto.

EXAMPLES 1.4.8

i) Let $f: (0,1) \to \mathbb{R}$ be defined as $f(x) = 1/x$. Then f is one-one but $f((0,1)) = (1, \infty)$, implying that f is not onto.

ii) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^3$. This is a bijection.

iii) Let f, g, h be defined as follows:

$$f: \mathbb{R} \to \mathbb{R},\ f(x) = x^2$$
$$g: \mathbb{R} \to [0, \infty),\ g(x) = x^2$$
$$h: [0, \infty) \to [0, \infty),\ h(x) = x^2.$$

Then f is neither one-one nor onto, g is onto but not one-one and h is a bijection.

iv) For $a \neq 0$ & $b \in \mathbb{R}$ let $f: \mathbb{R} \to \mathbb{R}$ be defined as
$$f(x) = ax + b \quad \text{for any } x \in \mathbb{R}.$$
Then f is a bijection.

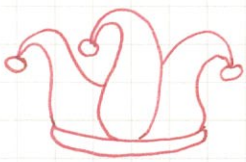
DEFINITION 1.4.9 (COMPOSITION OF FUNCTIONS)

Let $f: X \to Y$ and $g: Y \to Z$ be functions. The composition of f and g is the function $g \circ f: X \to Z$ defined as
$$(g \circ f)(x) := g(f(x)) \quad \text{for any } x \in X.$$

EXAMPLES 1.4.10

i) Let $f, g: \mathbb{R} \to \mathbb{R}$ be defined as
$$f(x) = 2x \;, \; g(x) = x^3 \quad \text{for any } x \in \mathbb{R}.$$
Then
$$(g \circ f)(x) = g(f(x)) = g(2x) = 8x^3$$
$$(f \circ g)(x) = f(g\ ) = f(x^3) = 2x^3.$$

ii) Let $X = \{a, b, c\}$ & $Y = \{1, 2\}$. Define $f: X \to Y$, $g: Y \to X$ as
$$f(a) = 1,\ f(b) = 1,\ f(c) = 2$$
$$g(1) = a \;,\; g(2) = c.$$
Then $f \circ g = Id_Y$ is a bijection but
$$(g \circ f)(a) = a,\ (g \circ f)(b) = a \;,\; (g \circ f)(c) = c.$$
Thus, the range of $g \circ f$ is $\{a, c\}$.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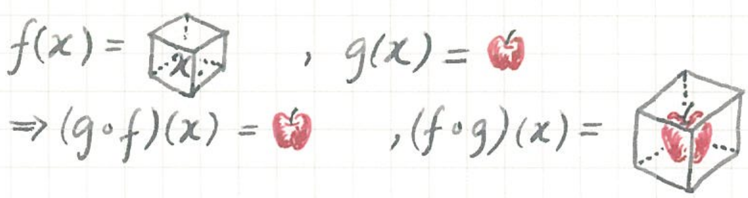

DEFINITION 1.4.11 (INVERSE OF A FUNCTION)

Let $f: X \to Y$ be a bijection. The inverse of f is the function $f^{-1}: Y \to X$ defined as $f^{-1}(y) = x$, where $y = f(x)$, for any $y \in Y$.

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Note that the ontoness of f is needed to define f^{-1} for each $y \in Y$. The property of f being one-one is needed to ensure that y gets mapped to exactly one point in X. Observa that

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) \quad (\text{such that } f(x) = y)$$
$$= y \quad \text{for any } y \in Y.$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \quad \text{for any } x \in X.$$

Thus, $f \circ f^{-1} = Id_Y$ and $f^{-1} \circ f = Id_X$.

EXAMPLES 1.4.12

i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then $f^{-1} : \mathbb{R} \to \mathbb{R}$ is

$$f^{-1}(y) = y^{1/3} \quad \text{for any } y \in \mathbb{R}.$$

ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = ax + b$ for $a \neq 0$. Then $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by

$$f^{-1}(y) = \frac{y - b}{a} \quad \text{for any } y \in \mathbb{R}.$$

iii) Let E be the set of even integers, i.e., $E = \{2m \mid m \in \mathbb{Z}\}$, and let O be the set of odd integers, i.e., $O = \{2m+1 \mid m \in \mathbb{Z}\}$. Define $f : E \to O$ as follows

$$f(2m) = 2m + 1 \quad \text{for any } 2m \in E.$$

Then f is a bijection.

THEOREM 1.4.13 (CHARACTERIZATION OF INVERSE)

Let $f : X \to Y$, $g : Y \to X$, $h : Y \to X$ be functions such that

$$f \circ g = Id_Y, \quad h \circ f = Id_X.$$

Then f is a bijection and $g = h = f^{-1}$.

Proof. We shall prove this in three steps.

STEP 1 f is one-one

Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then, using $h \circ f = Id_X$,

$$x_1 = Id_X(x_1) = h(f(x_1)) = h(f(x_2)) = Id_X(x_2) = x_2.$$

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STEP 2 f is onto

Let $y \in Y$. Using $f \circ g = Id_Y$, we see that

$$f(g(y)) = Id_Y(y) = y.$$

Step 1 and step 2 imply that f is a bijection. Thus, f^{-1} exists and is defined.

STEP 3 $g = h = f^{-1}$

It follows from observations preceeding Eg 1.4.12 and $f \circ g = Id_Y$, $h \circ f = Id_X$ that

$$g = Id_X \circ g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g) = f^{-1} \circ Id_Y = f^{-1}$$

$$h = h \circ Id_Y = h \circ (f \circ f^{-1}) = (h \circ f) \circ f^{-1} = Id_X \circ f^{-1} = f^{-1}.$$

This completes the proof of the theorem. ▨

DEFINITION 1.4.14 (IMAGE & INVERSE IMAGE)

Let $f: X \to Y$ be a function. For $A \subseteq X$, $B \subseteq Y$, we define

i) the image of A to be $f(A) := \{f(x) \mid x \in A\}$

ii) the inverse image of B to be $f^{-1}(B) := \{x \in X \mid f(x) \in B\}$.

Note that in ii) f^{-1} does not stand for the inverse of the function f. In fact, f^{-1} need not exist as f is not assumed to be bijective; $f^{-1}(B)$ is just a notation.

EXAMPLES 1.4.15

i) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. Note that f is neither one-one nor onto.

$$f([-1,1]) = [0,1], \quad f([-1,2]) = [0,4], \quad f([-3,2]) = [0,9]$$

$$f^{-1}([0,1]) = [-1,1], \quad f^{-1}([0,4)) = (-2,2), \quad f^{-1}([-1,4]) = f^{-1}([-2,4]) = [-2,2]$$

ii) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 2024 & \text{if } x = 0 \end{cases}$$

$$f([0,\infty)) = (0,\infty) = f((0,\infty)), \quad f^{-1}(\{2024\}) = \{0, \tfrac{1}{2024}\}.$$

iii) Let $f: \mathbb{R} \to \mathbb{R}$ be the sine function, i.e., $f(x) = \sin x$ for any $x \in \mathbb{R}$.

$f(\mathbb{R}) = [-1,1]$, $f^{-1}([-1,1]) = \mathbb{R}$, $f^{-1}(\{0\}) = \{n\pi \mid n \in \mathbb{Z}\}$.

THEOREM 1.4.16

Let $f: X \to Y$ be a function. If $A, B \subseteq X$ and $C, D \subseteq Y$ then

i) $f(A \cup B) = f(A) \cup f(B)$, $f(A \cap B) \subseteq f(A) \cap f(B)$;

ii) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Proof. i) As $A, B \subseteq A \cup B$, we obtain $f(A), f(B) \subseteq f(A \cup B)$, implying that $f(A) \cup f(B) \subseteq f(A \cup B)$. Conversely, let $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$. If $x \in A$, then $y \in f(A)$ & if $x \in B$, then $y \in f(B)$. Hence, $f(A \cup B) \subseteq f(A) \cup f(B)$ and we have equality: $f(A \cup B) = f(A) \cup f(B)$.

Since $A \cap B \subseteq A$ & $A \cap B \subseteq B$, we obtain

$$f(A \cap B) \subseteq f(A) \quad , \quad f(A \cap B) \subseteq f(B).$$

This implies that $f(A \cap B) \subseteq f(A) \cap f(B)$.

ii) Let $x \in f^{-1}(C \cup D)$, i.e., $f(x) \in C \cup D$. If $f(x) \in C$, then $x \in f^{-1}(C)$. If $f(x) \in D$, then $x \in f^{-1}(D)$, implying that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$. Conversely, if $x \in f^{-1}(C)$, then $f(x) \in C \subseteq C \cup D$, whence $x \in f^{-1}(C \cup D)$. Similarly, $f^{-1}(D) \subseteq f^{-1}(C \cup D)$ & we obtain $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$. This gives us the required equality.

Using arguments as above we conclude that $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$. Now, let $x \in f^{-1}(C) \cap f^{-1}(D)$, i.e., $f(x) \in C$ as well as $f(x) \in D$. This means $f(x) \in C \cap D$ or $x \in f^{-1}(C \cap D)$. Therefore, $f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$ and we have the required equality. ▨

In general, $f(A \cap B) = f(A) \cap f(B)$ need not hold. One can envision a function such that A & B are disjoint subsets but $f(A) \cap f(B) \neq \emptyset$. More specifically, if f is not one-one, then there exists $x_1 \neq x_2$ in X such that $f(x_1) = f(x_2)$. Set $A = \{x_1\}$, $B = \{x_2\}$ & we conclude that

$$\emptyset = f(\emptyset) = f(A \cap B) \quad , \quad f(A) \cap f(B) = \{f(x_1)\}.$$

In fact, if f is one-one, then

$$f(A \cap B) = f(A) \cap f(B)$$

for any $A, B \subseteq X$.

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A FOUNDATION COURSE IN MATHEMATICS

A. Kumar, S. Kumaresan, B.K. Sarma