Lecture Mar 19, 2025

So we have seen that if $AB = I_n$ for a $n \times n$ matrix A, then $BA = I_n$ as well.

On the other hand if A is an $m \times n$ matrix & $\exists\, B$ s.t. $AB = I_m$, then it is not true that $BA = I_n$!

(we shall not see the proof here!)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{pmatrix}$$

$AB = I_2$ but $BA \neq I_3$.

In today's class, we first address this question of rectangular matrices (in general).

- along the way, we learn column rank = row rank. & rank-nullity thm.

Let A be a $m \times n$ matrix.

$R(A) := \{$ linear span of the rows of $A \}$

↪ Row space of A.

$R(A) \subseteq \mathbb{R}^n \Rightarrow \dim^{?} R(A) \leq n.$

This is because we know that if $\mathcal{B}$ is a basis of $R(A)$, then $\mathcal{B}$ is LI.
& any LI subset of $\mathbb{R}^n$ has cardinality $\leq n$ & thus $|\mathcal{B}| \leq n$.

$\dim R(A) :=$ row rank of A.

$\dim C(A) :=$ column rank of A.

NOTE:
$R(A)$ can be considered same as $C(A^T)$ &
$C(A)$ can be considered same as $R(A^T)$.

We shall show
row rank of A = column rank of A.

Let R be a row reduced echelon matrix and let r be the no. of non-zero rows of R,
then row rank $R \leq r$.

Now if $R_1, R_2, \ldots R_r$ be the rows & consider the combination

$$\alpha_1 R_1 + \cdots + \alpha_r R_r = 0. \quad -(*)$$

Then the k_1^{th}- component of $\alpha_1 R_1 \ldots + \alpha_r R_r$

is α_1, k_i^{th} - component of $\alpha_1 R_1 \dots + \alpha_r R_r$ is α_i

$(*) \Rightarrow \quad \alpha_1 = \alpha_2 \dots = \alpha_r = 0$

$\Rightarrow$ row rank $R = r$.

What about column rank of R?

$k_1 < k_2 \dots < k_r$

If $k_1 \neq 1$, then any column prior to k_1^{th} - column is a zero column.

For any other column j $\exists$ i s.t $k_i < j < k_{i+1}$
(than the k_i^{th} - columns) or $k_r < j$

Case 1: Suppose $\exists$ i s.t $k_i < j < k_{i+1}$.

then all ℓj-th entries with $\ell > i$ are 0.

Let $a_{\ell j}$ - be the entries $\ell \leq i$.

$$\Rightarrow \quad C_j' = \sum_{\ell=1}^{i} a_{\ell j} C_{k_\ell}$$

So $\{C_{k_i}\}_{i=1}^{r}$ spans $C(R)$

and they have 1's at different position with all other entries are 0

i.e are subset of std basis of $\mathbb{R}^m$;

& hence linearly independent.

$\Rightarrow$ $\dim C(R) = r$

So column rank of $R = r$.

So in a nutshell:

For a RREM $-$ R

column rank of R = row rank of R

$= r$ - the no. of nonzero rows.

Now if R be the row-reduced echelon form of a matrix A.

Then rows of $R \subseteq R(A)$

$\Rightarrow$ row space of $R \subseteq R(A)$.

Also we know $\underbrace{E_1 \cdots E_k}_{\text{finitely many elementary matrices}} A = R$

$\Rightarrow$ $A = \underbrace{E_k^{-1} \cdots E_1^{-1}}_{\text{again finitely many elementary matrices}} R$

Thus rows of $A \subseteq R(R)$.

$\Rightarrow$ $R(A) \subseteq R(R)$.

So $\quad R(A) = R(R)$

row rank of $A = r$ = the no. of non-zero rows of the RREF of A.

What can we say about columns?

<u>Thm</u> (A): Let A be a $m \times n$ matrix and D be an invertible matrix $m \times m$ - matrix.

Let $B = DA$. Then

1) $N(A) = N(B)$

2) Column rk of A = column rank of B.

Note as a result,

$$A = E_k^{-1} \cdots E_1^{-1} R$$

$\Rightarrow$ Column rank of A

= Column rank of R

= row rank of R

= row rank of A.

<u>Corollary</u>: For any $m \times n$ matrix A, column rank of A

= row rank of A.

(This is defined as rank of a matrix A)

Also,

no. of columns of A $= n$

$$= \underbrace{r}_{\text{nonzero rows of } R} + (n-r)$$

$=$ rank of A $+$ $\underbrace{\text{Nullity of A}}_{\dim^{n}. N(A)}$.

— Known as the rank-nullity Thm.

<u>Thm:</u> For any $m \times n$ matrix A,

$$n = \text{rank of } A + \text{nullity of } A.$$

<u>Pf. of the thm (A):</u>

1) $B = DA$

$x \in N(A) \Rightarrow Ax = 0 \Rightarrow DAx = 0 \Rightarrow Bx = 0$

$\Rightarrow x \in N(B)$

$\Rightarrow N(A) \subseteq N(B)$.

Now $A = D^{-1}B \Rightarrow N(B) \subseteq N(A)$

& hence $N(A) = N(B)$.

2) $B = DA$

If c_j's are columns of A &

c_j''s are columns of B,

then $c_j' = D c_j$

Let column rank of $A = k$

and $\{v_1, \dots v_k\}$ be a basis of $C(A)$.

Clearly $v_i = \sum_{j=1}^{n} \alpha_{ij} c_j \quad \forall i$

Define $w_i = \sum_{j=1}^{n} \alpha_{ij} c_j' = D v_i$

Then $\{w_1, \dots w_k\}$ is a basis of $C(B)$.

& hence column rk of $B = k$ (we are done with the proof).

First, we claim : $\{w_1 \dots w_k\}$ are LI.

Suppose $\sum \alpha_i w_i = 0$

$\Rightarrow D (\sum \alpha_i v_i) = 0$

$\Rightarrow \sum \alpha_i v_i = 0$ as D is invertible

$\Rightarrow \alpha_i = 0 \quad \forall i$

So $\{w_1 \dots w_k\}$ is LI.

Next we show $\{w_1 \dots w_k\}$ spans $C(B)$

Let $w \in C(B)$.

Now $\omega = \sum \beta_j c_j'$

$$= \sum \beta_j D c_j'$$

$$= D \underbrace{\left(\sum_{j=1}^{n} \beta_j c_j \right)}_{\in C(A)}.$$

$\sum_{j=1}^{n} \beta_j c_j \in C(A)$

$\Rightarrow \sum_{j=1}^{n} \beta_j c_j = \sum_{j=1}^{k} \gamma_j v_j$

$$= D \left(\sum_{j=1}^{k} \gamma_j v_j \right) = \sum_{j=1}^{k} \gamma_j D v_j$$

$$= \sum_{j=1}^{k} \gamma_j w_j$$

$\Rightarrow \{w_1, \dots w_k\}$ spans $C(B)$.

Hence $\{w_1, \dots w_k\}$ is a basis of $C(B)$.

Therefore,

column rank of $B = k =$ column rank of A. ∎

<u>Remark</u>: The system of eqn. $Ax = b$ is solvable if and only if rank $(A) =$ rank $(A|b)$.

Coming back to the question when an $m \times n$ matrix A - is left invertible or right invertible?

We have seen for square matrix it is same as 'invertibility'.

Here we give condition in terms of rank.

Thm: Let A be an $m \times n$ matrix.

A is right invertible $\Longleftrightarrow$ $\text{rank}(A) = r = m \leq n$.

Pf: ($\Rightarrow$) Suppose $\exists$ a matrix C s.t $AC = I_m$

C — $n \times m$ matrix.

If $C_1, \ldots C_m$ - columns of C, then

we have $\underbrace{A C_j}$ $= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_j$ — j^{th} - std basis of $\mathbb{R}^m$.

j^{th} position

what does this mean?

if $C_j = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ $\quad A C_j$

$= d_1 A_1 + \cdots + d_n A_n$.

where $A_1, A_2 \ldots A_n$ are columns of A.

So it says $e_j \in C(A)$. $\forall j$

$\Rightarrow \mathbb{R}^m \subseteq C(A) \subseteq \mathbb{R}^m$

$\Rightarrow C(A) = \mathbb{R}^m$

$\Rightarrow \dim C(A) = m$

$\Rightarrow \text{rank}(A) = m$

We already know rank(A) is the number of nonzero rows in RREF of A and hence

rank (A) $\leq n$.

So we get rank (A) $= m \leq n$.

($\Leftarrow$) Conversely,

If rank (A) $= m \leq n$ given

then $\mathcal{C}(A) = \mathbb{R}^m$

$\Rightarrow \exists\, C_j$ s.t $AC_j = e_j$ $j=1,\dots m$.

$(n\times 1)$ vectors

$\Rightarrow$ $AC = I_m$.

Thm: A - $m\times n$ matrix

A is left invertible $\Leftrightarrow$ rank(A) $= n \leq m$.

Pf: A left invertible

$\Leftrightarrow A^T$ right invertible

$\hookrightarrow n\times m$

$\Leftrightarrow$ rank $(A^T) = n \leq m$

$\quad\quad\parallel$

$\Leftrightarrow$ rank(A) $= n \leq m$.

So for an $n \times n$ matrix A

A is invertible $\Longleftrightarrow$ A is left invertible

$\Longleftrightarrow$ A is right invertible

$\Longleftrightarrow$ $\operatorname{rank} A = n$.

$\Longleftrightarrow$ $Ax = 0$ has ONLY $x = 0$ solution.