Lecture 09: Feb 13, 2025

Given finitely many vectors $x_1, x_2, \ldots x_k \in V/\mathbb{F}$, consider the vector

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k,$$

known as linear combination of the vectors.

Let S be the collection of all linear combinations of the vectors, that is,

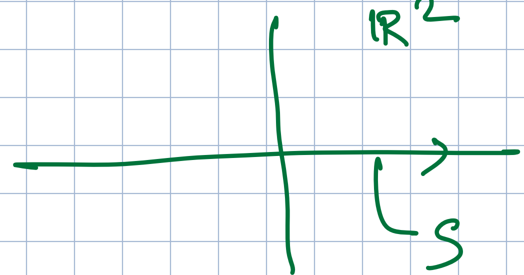
$$S := \{\alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \mid \alpha_1, \alpha_2, \ldots \alpha_k \in \mathbb{R}\}$$

Then S is a subspace of $V/\mathbb{F}$ - called subspace generated by $x_1, x_2, \ldots x_k$

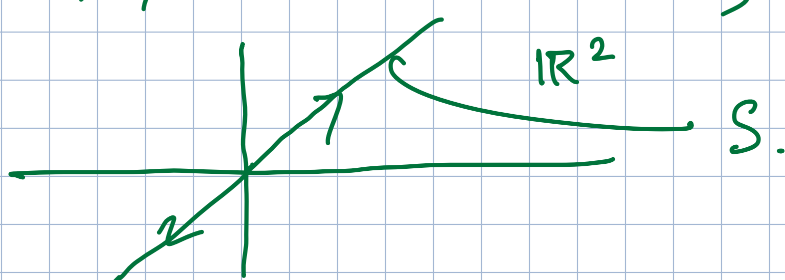
Examples:

$x_1 = (1,0) \in \mathbb{R}^2/\mathbb{R}$

$$S = \{\alpha x_1 : \alpha \in \mathbb{R}\}$$
$$= \{\alpha(1,0) : \alpha \in \mathbb{R}\}$$
$$= \{(\alpha, 0) : \alpha \in \mathbb{R}\} - x\text{-axis}.$$



or $x_1 = (1,1)$, $S = \{(\alpha, \alpha) : \alpha \in \mathbb{R}\}$



$x_1 = (0,1)$, $S = \{(0,\alpha) : \alpha \in \mathbb{R}\}$ — y axis.

See these are lines passing through origin.

— this will always be the case

as $0 \in S$
↳ zero vector

Suppose $x_1 = (1,0)$, $x_2 = (1,1)$

Then $S = \{\alpha(1,0) + \beta(1,1) : \alpha, \beta \in \mathbb{R}\}$

$= \{(\alpha+\beta, \beta) : \alpha, \beta \in \mathbb{R}\}$

Any $(x,y) \in \mathbb{R}^2$, consider $\beta = y$

& $\alpha = x - y$.

$\Rightarrow (x,y) \in S$

So $\mathbb{R}^2 \subseteq S \subseteq \mathbb{R}^2 \Rightarrow S = \mathbb{R}^2$.

Now we come back to the system of equations of the form

$$Ax = b$$

where $A \longrightarrow m \times n$ matrix

$x \longrightarrow n \times 1$ matrix

$b \longrightarrow m \times 1$ matrix

Let us consider the example

where $A = \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix}$ 3×2

and x – is the 2×1 vector $\begin{pmatrix} u \\ v \end{pmatrix}$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$Ax=b$

here $\rightarrow$ becomes $\begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\Rightarrow \quad \begin{aligned} u &= b_1 \\ 5u + 4v &= b_2 \\ 2u + 4v &= b_3 \end{aligned}$$

Note

$$u \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + v \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

So if $\exists$ a solution, then $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is in the span of the vectors $\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$

This happens in general.

Let $A = \left(\begin{array}{cccc} c_1 & c_2 & & c_n \\ \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{pmatrix} & \cdots & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} \right)$

$\therefore$ can be thought as n column vectors C_i - each of which is a $m \times 1$ matrix

where $$C_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \quad 1 \le i \le n.$$

So if $Ax = b$ has a solution

$$x = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

Then $b = u_1 C_1 + u_2 C_2 + \dots u_n C_n.$ — (*)

Define the column space $C(A)$ of A as the subspace generated by the column vectors of A,

i.e $C(A) := \{ \alpha_1 C_1 + \dots + \alpha_n C_n : \alpha_1, \alpha_2 .. \alpha_n \in \mathbb{R} \}.$

(*) Says that if $Ax = b$ admits a solution then $b \in C(A)$.

On the other hand if $b \in C(A)$ then $\exists\ \alpha_1 \dots \alpha_n \in \mathbb{R}$ s.t

$$b = \alpha_1 C_1 + \alpha_2 C_2 \dots + \alpha_n C_n.$$

$$\Rightarrow \quad A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = b.$$

Thus $Ax = b$ admits a solution $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$.

We summarise this as follows:

Lemma: $Ax = b$ solvable if and only if $b \in C(A)$.

Remark:

An alternate way to define $C(A)$ is

$$C(A) = \{ b : \exists\, x \in \mathbb{R}^n \text{ satisfying } Ax = b \}$$

i.e $Ax = b$ solvable.

However, in earlier def^n. it is a subspace by construction, but if you define this way, you have to prove $C(A)$ is a subspace. However, it is not to difficult to see!

Let $b_1, b_2 \in C(A)$, then $\exists\, x_1, x_2 \in \mathbb{R}^n$ such that

$$A x_1 = b_1$$
$$A x_2 = b_2$$

Then $A(x_1 + x_2) = b_1 + b_2 \Rightarrow b_1 + b_2 \in C(A)$

& $A(\alpha x_1) = \alpha A x_1 = \alpha b_1 \Rightarrow \alpha b_1 \in C(A)$.

Example:

Consider $A = \begin{pmatrix} 1 & 0 & 2 \\ 7 & 0 & 5 \\ 9 & 0 & 11 \end{pmatrix}$

What is the column space of A?

$$C(A) = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 7 \\ 9 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$$= \left\{ \alpha_1 \begin{pmatrix} 1 \\ 7 \\ 9 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix} : \alpha_1, \alpha_3 \in \mathbb{R} \right\}$$

Thus $C(A)$ is the plane in $\mathbb{R}^3$ passing through $(1,7,9)$ and $(2,5,11)$

[Note that $(1,7,9)$ & $(2,5,11)$ are not on the same line passing through origin].

Qn: In an $n \times n$ matrix A, when is $C(A) = \mathbb{R}^n$?

(when A is invertible — we will see later)

For now, take A - invertible,

Then $Ax = b \Rightarrow x = A^{-1} b$

So we get $Ax = b$ solvable for every $b \in \mathbb{R}^n$.

Coming back to the system of equations,

$$Ax = b,$$

we assume $b \in C(A)$,

then as mentioned $Ax = b$ has a solution

Suppose x_1, x_2 are two solⁿ. of

$$Ax = b,$$

that is, $Ax_1 = b$ & $Ax_2 = b$.

$$\Rightarrow A(x_1 - x_2) = 0.$$

So any two solⁿ $Ax = b$ are related by the fact that $x_1 - x_2 \in N(A)$.

So any solⁿ of $Ax = b$

can be written as $x_1 + y$

where $y \in N(A)$ and

x_1 is a particular solⁿ. of $Ax = b$.

So $\{x : Ax = b\}$

$$= \{x_1 + y : x_1 \text{ is a particular sol}^n \text{ of } Ax = b \text{ \& } y \in N(A)\},$$

that is,

If we have any one solution x_1 of $Ax = b$, then all other solⁿ. are precisely of the form $x_1 + y$ where $y \in N(A)$.

$0 \in N(A)$ and if $\exists$ a nonzero vector $x_0 \in N(A)$, then $\alpha \cdot x_0 \in N(A)$

Since $\mathbb{R}$ is infinite, in this case $N(A)$ has infinitely many vectors, that is, $Ax = 0$ has infinitely many solutions.

So $Ax = b$ has unique solution or infinitely many solⁿ according $Ax = 0$ has only zero solution or $Ax = 0$ has a nonzero solution.

<u>Qn:</u> When $Ax = 0$ has non-zero solution?

So suppose $\exists$ $\alpha_1\ \alpha_2 \dots \alpha_n \in \mathbb{R}$

NOT ALL ZERO

such that

$$A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = 0 \quad \text{---} \quad (**)$$

Now $A = (c_1\ c_2 \dots c_i \dots c_n)$

$\hookrightarrow$ i^{th} - column.

$(**) \Rightarrow \quad \alpha_1 c_1 + \alpha_2 c_2 \cdots + \alpha_n c_n = 0.$

Since not all α_i's are zero,
without loss of generality, we assume
$\alpha_1 \neq 0$

$$\Rightarrow c_1 = -\frac{1}{\alpha_1}(\alpha_2 c_2 + \cdots \alpha_n c_n)$$

$$= \left(-\frac{\alpha_2}{\alpha_1}\right) c_2 + \cdots + \left(\frac{-\alpha_n}{\alpha_1}\right) c_n$$

Clearly c_1 is dependent on $c_2 \ldots c_n$
infact for any i s.t $\alpha_i \neq 0$,
c_i is dependent on other c_j's $j \neq i$

$\downarrow$

how? - we write in linear combination of others

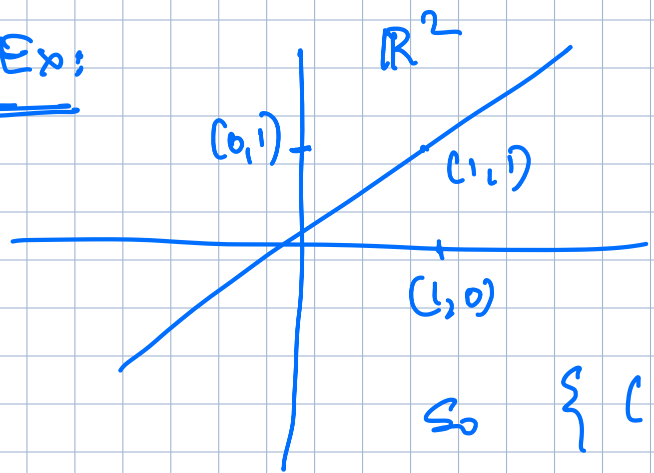
So we say that the set $\{c_1, \ldots c_n\}$ is a linearly dependent set.

Defn: A finite set of vectors $\{x_1, x_2, \ldots x_k\}$ in a vector space $V/\mathbb{F}$ is said to be **LINEARLY DEPENDENT** if $\exists$ scalars $\alpha_1\ \alpha_2 \cdots \alpha_k \in \mathbb{F}$ with at least one of the scalars being non-zero, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0.$$

Thus, for a matrix A, a nonzero sol^n for $Ax = 0$ exists if and only if the columns of A are linearly dependent.

Ex:



$(1,1) = (1,0) + (0,1)$

$\Rightarrow 1.(1,0) + 1.(0,1) - 1(1,1) = 0.$

So $\{(1,0), (0,1), (1,1)\}$ are linearly dependent.

Now we ask whether the set $\{(1,0), (0,1)\}$ is linearly dependent?

that is, does $\exists\ \alpha, \beta$ – not all zero such that $\alpha(1,0) + \beta(0,1) = 0$?

But this $\Rightarrow (\alpha, \beta) = (0,0)$

$\Rightarrow \alpha = 0 = \beta$

So we see that $\nexists$ such α, β !

i.e $\{(1,0), (0,1)\}$ are not linearly dependent

We call such a set linearly independent.

Def^n: A finite set of vectors $\{x_1, x_2, \dots x_k\}$ in $V/_{\mathbb{F}}$ is said to be LINEARLY INDEPENDENT if the linear combination

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_k x_k = 0$$

implies $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$.

Example:

(1) Let $A = \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix}$

$$u_1 \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} u_1 &= 0 \\ 5u_1 + 4u_2 &= 0 \\ 2u_1 + 4u_2 &= 0 \end{aligned}\right\} \Rightarrow u_1 = 0 = u_2.$$

Thus the columns of A are linearly independent, equivalently, $Ax = 0$ has only 0 solution.

(2) Add the columns of matrix A and create a 3rd column.

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{pmatrix}$$

then $B \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Longleftrightarrow \quad \begin{aligned} u_1 + u_3 &= 0 \\ 5u_1 + 4u_2 + 9u_3 &= 0 \\ 2u_1 + 4u_2 + 6u_3 &= 0. \end{aligned}$$

$$\Longleftrightarrow \quad \begin{aligned} u_1 &= -u_3 \\ u_2 &= -u_3 \end{aligned}$$

$$\Longleftrightarrow \quad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} c \\ c \\ -c \end{pmatrix} \quad \text{for some } c \in \mathbb{R}$$

So $N(B) = \left\{ \begin{pmatrix} c \\ c \\ -c \end{pmatrix} : c \in \mathbb{R} \right\}.$

We see that columns of B are linearly dependent by construction and null space of B has nonzero vector.

(3) Consider $\mathcal{P}_2 := \{ p \mid p \text{ is a polynomial with } \deg p \leq 2 \}$.

$\{1, x, x^2\}$ is a linearly independent set in $\mathcal{P}_2$.

Consider $\alpha + \beta x + \gamma x^2 = 0$ for some $\alpha, \beta, \gamma \in \mathbb{R}$.

0-polynomial.

if at least one of α, β, γ is nonzero, then this eq^{n} is either absurd or has one or two solutions for x & in that case apart from those solutions, when we evaluate LHS at other x, the LHS is not zero but RHS zero leading to a contradiction!

Thus $\alpha = \beta = \gamma = 0$

$\Rightarrow \{1, x, x^2\}$ is linearly independent.

(4) In a triangular matrix, all columns are linearly independent if and only if every diagonal entry (PIVOT) is non-zero.

WLOG, let us take A – an $n \times n$ matrix upper triangular.

Say, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots\ 0 & a_{nn} \end{pmatrix}$.

($\Leftarrow$) Suppose every diagonal entry are non-zero, that is $a_{ii} \neq 0 \ \forall i, 1 \leq i \leq n$.

Let $x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ be such that $Ax = 0$.

Then $\alpha_1 \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Last eq$^{\underline{n}}$ is $\alpha_n a_{nn} = 0$

$\Rightarrow \alpha_n = 0$ as $a_{nn} \neq 0$

Back substitution to $(n-1)^{th}$ eq$^{\underline{n}}$ we have

$\alpha_{n-1} a_{n-1\, n-1} = 0 \Rightarrow \alpha_{n-1} = 0$

as $a_{n-1, n-1} \neq 0$.

& then continue back substitution to see that $\alpha_1 = \cdots = \alpha_n = 0$

This shows all columns of A are linearly independent.

One can also give the argument that as $a_{ii} \neq 0 \Rightarrow A$ is invertible (done earlier)

So $Ax = 0 \Rightarrow A^{-1}(Ax) = 0$

$$\Rightarrow x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0.$$

($\Rightarrow$) We shall show that if $\exists$ at least one diagonal entry is 0, then all columns are linearly dependent by showing there exists a non-zero solution.

Contrapositive statement would give us then the desired result.

We have seen in previous lecture that if $a_{jj} = 0$ for the least j, then $\exists$ $u_1, u_2 \ldots u_{j-1}$ s.t

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{j-1} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ is a solution - in fact a non-zero one.

Basis of a vector space:

Def^n: A finite set of vectors $\mathcal{B} = \{v_1 \dots v_k\}$ in $V/\mathbb{F}$ is said to be a basis V if

i) $\mathcal{B}$ is linearly independent

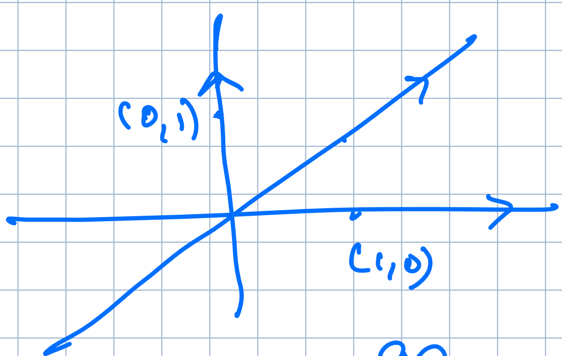
ii) $\mathcal{B}$ spans V, that is, span $\mathcal{B} = V$.

Example:

By previous calculations

i) $\{(1,0), (0,1)\}$ is a basis of $\mathbb{R}^2/\mathbb{R}$.

ii) $\{1, x, x^2\}$ is a basis of $\mathcal{P}_2/\mathbb{R}$.



In fact $\{(1,0), (1,r)\}$ $\forall\, r \neq 0$ is also a basis of $\mathbb{R}^2$

as. $\alpha(1,0) + \beta(1,r) = 0$

$\Rightarrow \alpha + \beta = 0$

& $\beta r = 0 \Rightarrow \beta = 0$

$\Rightarrow \alpha = 0$

$\Rightarrow \{(1,0), (1,r)\}$ is linearly independent.

— properties (5)

$(x,y) \in \mathbb{R}^2$, $(\alpha+\beta, \beta r) = (x,y)$

$$\Rightarrow \quad \beta = \frac{y}{r} \qquad \text{as } r \neq 0$$

$$\alpha = x - \frac{y}{r}.$$

$$\Rightarrow \quad (x,y) \in \text{span}\{(1,0),(1,r)\}$$

$$\Rightarrow \quad \mathbb{R}^2 = \text{span}\{(1,0),(1,r)\} \text{ :- proofs (ii)}$$

Thus for each fix $r \neq 0$ in $\mathbb{R}$

$\{(1,0),(1,r)\}$ is a basis of $\mathbb{R}^2$.

Infact for any $r \neq 0$, $\{r\}$ is a basis of $\mathbb{R}/\mathbb{R}$.

So we see there can exists plenty of bases.
However, we shall show that number of elements (cardinality) of two bases are always same.