5. DIFFERENTIABILITY

> "The calculus is the greatest aid we have to the application of physical truth in the broadest sense of the word."
>
> ~ WILLIAM FOGG OSGOOD

We begin by introducing the notion of derivative of a function.

DEFINITION 5·1 (DERIVATIVE)

Let $-\infty < a < b < \infty$, let $f:(a,b) \to \mathbb{R}$ and let $x_0 \in (a,b)$. We say that f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, we say that

$$\boxed{f'(x_0) \overset{\text{defn}}{:=} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$

is the derivative of f at x_0. Moreover, if f is differentiable at every point of (a,b), we say that f is differentiable in (a,b).

A more common way to think of derivative is to describe it as the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function. A geometric viewpoint is as follows: the derivative is the slope of the tangent line to the graph of the function, if the graph has a tangent.

The following result gives us the relation between differentiability & continuity.

THEOREM 5·2 (DIFFERENTIABILITY IMPLIES CONTINUITY)

Let $-\infty < a < b < \infty$, let $f:(a,b) \to \mathbb{R}$ be differentiable at x_0. Then f is continuous at x_0.

Proof. Define $\varphi : (a,b) \to \mathbb{R}$ as follows:

$$\varphi(x) := \begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x = x_0. \end{cases}$$

Then, $\lim_{x \to x_0} \varphi(x)$ exists as f is differentiable at x_0. Thus, using Theorem 4.1.6 (ii) to

$$\lim_{x\to x_0} \varphi(x) \text{ exists } \& \lim_{x \to x_0} (x-x_0) = 0 \text{ (exists)}$$

we obtain that $\varphi(x)(x-x_0)$ has a limit as x tends to x_0;

$$\lim_{x\to x_0} \varphi(x)(x-x_0) = \left(\lim_{x\to x_0} \varphi(x)\right)\left(\lim_{x\to x_0}(x-x_0)\right) = 0$$

However, as $f(x) - f(x_0) = \varphi(x)(x-x_0)$, this implies that

$$\lim_{x\to x_0} (f(x)-f(x_0)) = 0,$$

whence f is continuous at x_0. ∎

In Bill Thurston's philosophical (& mathematical) paper, "On Proof and Progress in Mathematics", he points out mathematicians often think of a single concept in many different ways. In particular, he provides seven different elementary ways of thinking about derivative (of a function). If the variable x is time t, then $f'(t_0)$ is the instantaneous speed of f at time t_0. A microscopic perspective is to treat the derivative as the limit of what one gets by looking at the function under a microscope of higher and higher power.

We now look at a few examples.

EXAMPLE 5.3

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ for any $x \in \mathbb{R}$. We claim that f is differentiable in $\mathbb{R}$ and that f' is the function that maps x to $2x$ for any $x \in \mathbb{R}$.

To see this, let $a \in \mathbb{R}$. Then,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} (x + a) = 2a.$$

Hence, the limit exists and $f'(a) = 2a$. Therefore, $f'(x) = 2x$ for any $x \in \mathbb{R}$.

EXAMPLE 5.4

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that f is not differentiable at 0, as

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin(1/x)}{x} = \lim_{x \to 0} \sin(1/x)$$

does not exist.

EXAMPLE 5.5

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that f is differentiable at 0, as

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0.$$

Note that $f'(0) = 0$, as computed above.

EXAMPLE 5.6

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$ for any $x \in \mathbb{R}$. We claim that f is not differentiable at 0, as

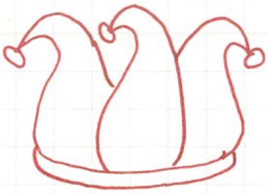
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

are unequal. However, if $a > 0$, then for any $\varepsilon > 0$, choose $\delta = a/2$ and note that if $0 < |x - a| < a/2 = \delta$, then $x \in (a/2, 3a/2)$. We see that

$$\left| \frac{f(x) - f(a)}{x - a} - 1 \right| = \left| \frac{x - a}{x - a} - 1 \right| = |1 - 1| < \varepsilon$$

Thus, f is differentiable at a & $f'(a)=1$. We leave it as an exercise to show that $f'(a)=-1$ if $a<0$. This is also consistent with the viewpoint that the line $y=x$ is the tangent line (with slope 1) to f at any positive real number. Similarly, the line $y=-x$ is the tangent line to f at negative real numbers. At zero, there is no tangent line!



IN JEST

- What did the calculus teacher ask the dazed and confused student?
  "Young man, have you been taking derivatives?"
- I have another calculus joke, but it's a little derivative.

THEOREM 5.7 (PROPERTIES OF DERIVATIVE)

Let $-\infty<a<b<\infty$ and let $f,g:(a,b)\to\mathbb{R}$ be differentiable at $x_0\in(a,b)$. Then,

i) $f+g$ is differentiable at x_0, and
$$(f+g)'(x_0)=f'(x_0)+g'(x_0).$$

ii) for all $\alpha\in\mathbb{R}$, αf is differentiable at x_0, and
$$(\alpha f)'(x_0)=\alpha f'(x_0).$$

iii) fg is differentiable at x_0, and
$$(fg)'(x_0)=f'(x_0)g(x_0)+f(x_0)g'(x_0).$$

iv) if $g(x_0)\neq 0$, f/g is differentiable at x_0, and
$$(f/g)'(x_0)=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. We shall prove (ii) & (iii). The others are left as an exercise. For (ii), with $x\in(a,b)$ and $x\neq x_0$,
$$(f+g)(x)-(f+g)(x_0)=f(x)-f(x_0)+g(x)-g(x_0).$$
Thus,
$$\lim_{x\to x_0}\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0}=\lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}\right)$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0) + g'(x_0).$$

For (iii), with $x \in (a,b)$ and $x \neq x_0$,

$$\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = f(x)\frac{g(x) - g(x_0)}{x - x_0} + g(x_0)\frac{f(x) - f(x_0)}{x - x_0}.$$

As f is differentiable, it is continuous. As g is differentiable,

$$\lim_{x \to x_0} f(x) \frac{g(x) - g(x_0)}{x - x_0} = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}\right)$$

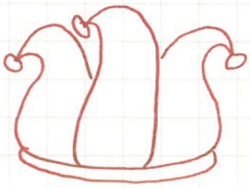
$$= f(x_0)\, g'(x_0).$$

By differentiability of f,

$$\lim_{x \to x_0} g(x_0) \frac{f(x) - f(x_0)}{x - x_0} = g(x_0) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= g(x_0) f'(x_0).$$

Thus, the limit exists and equals $f(x_0)g'(x_0) + f'(x_0)g(x_0)$. ■



IN JEST

$$\frac{d(g(f(x)))}{dx} = \frac{dg}{dx}(f(x))\frac{df(x)}{dx}$$

The fraction gang

vs.

$$(g \circ f)' = g'(f) f'$$

The snobbish elite

THEOREM 5·8 (DIFFERENTIATION & COMPOSITION)

Let $-\infty < a < b < \infty$, $-\infty < c < d < \infty$, let $f: (a,b) \to \mathbb{R}$, $g: (c,d) \to \mathbb{R}$ with $f((a,b)) \subseteq (c,d)$. Let us suppose that f is differentiable at $x_0 \in (a,b)$ and g is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0).$$

The proof is left as an exercise. The following expression

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

may be useful in the proof.

Let $-\infty < a < b < \infty$ and let $f:(a,b) \to \mathbb{R}$ be differentiable. If $f':(a,b) \to \mathbb{R}$, which is a (continuous) function is differentiable at $x_0 \in (a,b)$, then the derivative of f' at x_0 is called the second derivative of f at x_0, and is denoted by $f''(x_0)$. Note that for $f''(x_0)$ to be defined, f' needs to be defined on an open interval around x_0. Typically, one uses $f^{(k)}$ to denote the k^{th} derivative of f rather than using f, f'', f''' etc. Moreover, if the parameter $t \in (a,b)$ is time, then f' is the velocity, f'' is the acceleration and f''' is the "turn."

EXAMPLE 5.9

Consider $f, g: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = |x|, \quad g(x) := x^2 \text{ for any } x \in \mathbb{R}.$$

Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is given by $(g \circ f)(x) = x^2$. This is a differentiable function. However, f is not differentiable at $x = 0$.