Binary repn. of nos. in $(0,1)$

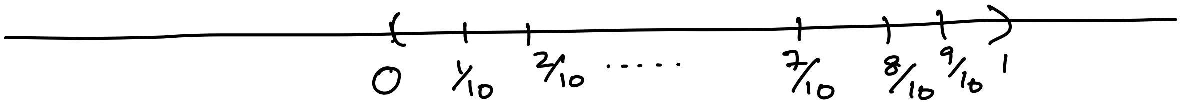
Motivation from Decimal repn:

Suppose $x = 0\cdot 879\ldots\ldots$

question is how do we recover $8, 7, 9\ldots$ from x.

We assume every pt in the a line corresponds to a real number.

Draw $(0,1)$



Divide the interval into 10 parts.

$\left(0, \frac{1}{10}\right), \left[\frac{1}{10}, \frac{2}{10}\right) \; \cdots\cdots \; , \left[\frac{8}{10}, \frac{9}{10}\right), \left[\frac{9}{10}, 1\right)$

$(0,1)$ — disjoint union of these intervals.

Any $x \in (0,1)$ must lie in one of these.

note $x = 0.879\ldots \in \left[\frac{8}{10}, \frac{9}{10}\right)$

So $8 := \max\left\{ k \in \mathbb{N} \cup \{0\} : \frac{k}{10} \leq x \right\}$.

since $x < 1 \Rightarrow \frac{k}{10} < 1 \Rightarrow k = 0, 1, 2 \ldots\ldots 9$.

In general then we define

$$a_1 := \max\left\{ k \in \mathbb{N} \cup \{0\} : \frac{k}{10} \leq x \right\}.$$

We then ask $x = \frac{a_1}{10}$? if yes, then $x = 0.a_1$.

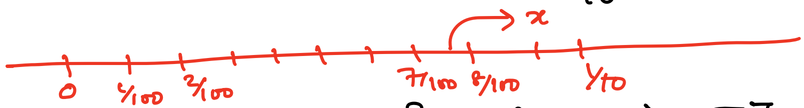
If not, $x \in \left(\frac{a_1}{10}, \frac{a_1 + 1}{10}\right)$.

So $x - \frac{a_1}{10} \in \left(0, \frac{1}{10}\right)$.

now we divide $\left(0, \frac{1}{10}\right)$ into 10 disjoint part:

$$\left(0, \frac{1}{100}\right), \left[\frac{1}{100}, \frac{2}{100}\right), \left[\frac{2}{100}, \frac{3}{100}\right), \cdots \left[\frac{7}{100}, \frac{8}{100}\right), \left[\frac{8}{100}, \frac{9}{100}\right), \left[\frac{9}{100}, \frac{1}{10}\right)$$

we ask in which interval $x - \frac{a_1}{10}$ lies?



in the example $x - \frac{8}{10}$ lies in $\left[\frac{7}{100}, \frac{8}{100}\right)$.

Then $\quad 7 := \max\left\{ k \in \mathbb{N} \cup \{0\} : \frac{k}{100} \le x - \frac{8}{10} \right\}$.

Thus in general,

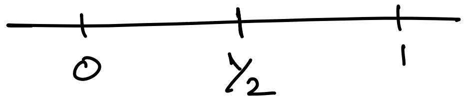
$$a_2 := \max\left\{ k \in \mathbb{N} \cup \{0\} : \frac{k}{100} \le x - \frac{a_1}{10} \right\}$$

$$:= \max\left\{ k \in \mathbb{N} \cup \{0\} : \frac{a_1}{10} + \frac{k}{10^2} \le x \right\}.$$

<u>Inductive step</u>: $\quad a_n := \max\left\{ k \in \mathbb{N} \cup \{0\} : \sum_{i=1}^{n-1} \frac{a_i}{10^i} + \frac{k}{10^n} \le x \right\}$.

Note on each step we are getting closer to x.

Construction of a_i's in binary representation:



divide in two parts -

$$[0, 1/2), \ [\tfrac{1}{2}, 1).$$

$$a_1 := \{ k \in \mathbb{N} \cup \{0\} : \frac{a_1}{2} \leq x \}.$$

since $x < 1$, $a_1 \leq 2x < 2 \Rightarrow a_1 \in \{0, 1\}$.

Inductive step:

$$a_n := \{ k \in \mathbb{N} \cup \{0\} : \sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{k}{2^n} \leq x \}.$$

Claim: $a_n \in \{0, 1\}$.

This follows from induction.

The base case: $a_1 \in \{0, 1\}$ - we have already seen.

Induction hypothesis: $a_n \in \{0,1\}$.

RTP: $a_{n+1} \in \{0,1\}$

if not $a_{n+1} \geq 2$.

Then $$\sum_{i=1}^{n} \frac{a_i}{2^i} + \frac{a_{n+1}}{2^{n+1}} \leq x$$

$$\Rightarrow \sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{a_n}{2^n} + \frac{\overbrace{a_{n+1}-2}^{\geq 0}}{2^{n+1}} + \frac{\cancel{2}}{\cancel{2^{n+1}}_{2^n}} \leq x$$

$$\Rightarrow \sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{a_n+1}{2^n} + \frac{a_{n+1}-2}{2^{n+1}} \leq x, \quad \geq 0$$

$$\Rightarrow \sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{a_n+1}{2^n} \leq x$$

This contradicts that a_n is the maximum natural number K s.t $$\sum_{i=1}^{n-1} \frac{a_i}{2^i} + \frac{K}{2^n} \leq x.$$

This contradiction appears due to the assumption

$$a_{n+1} \geq 2$$

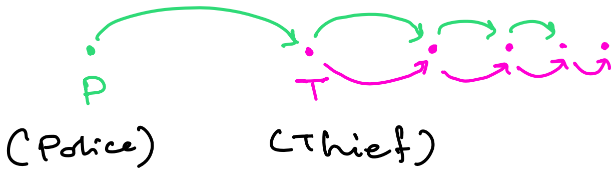
Hence $a_{n+1} \in \{0,1\}$.

Now we come up with a series $\displaystyle\sum_{i=1}^{\infty} \frac{a_i}{2^i}$,

if the construction does not stop.

Will we reach to x by this construction.

→ almost similar question like <u>Zeno's paradox</u>.



If every time police reaches the position of thief, then in that time thief advances to another point and by doing this it seems police will never be able to catch the thief. However, the real computation shows the calculation is basically summing up a series that will give finite time.

We will also reach x; i.e $\sum_{i=1}^{\infty} \frac{a_i}{2^i}$ is convergent (cgt) and cges to x.

Blackbox: Regarding $\sum_{i=1}^{\infty} \frac{a_i}{2^i}$ convergence:

Def^n: A series $\sum_{n=1}^{\infty} c_n$, $c_n \geq 0$, is cgt iff the sequence of of partial sum $s_k = \sum_{n=1}^{k} c_n$ is cgt, and if $\lim_{k \to \infty} s_k = s$, then s is defined as $\sum_{n=1}^{\infty} c_n$.

(A sequence $\{s_n\}$ cges to s iff given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon$ $\forall n \geq N$)

Facts: 1. $\sum_{n=1}^{\infty} r^{n-1}$, $0<r<1$, is cgt and cges to $\frac{1}{1-r}$.

in particular, we use $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1-\frac{1}{2}} = 2$

and $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \cdot \left(\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \right) = \frac{1}{2} \cdot 2 = 1$

2. If $\sum d_n$ is a cgt series and $\sum c_n$ is dominated by $\sum d_n$, that is, if $s_k = \sum_{n=1}^{k} c_n$ & $\tilde{s}_k = \sum_{n=1}^{k} d_n$ and $s_k \le \tilde{s}_k$ $\forall k$

then, $\sum c_n$ is cgt.

$\longrightarrow$ One can use Cauchy seq^n argument to prove the fact above.

A seq^n $\{s_n\}$ is Cauchy iff given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t $|s_{n+p} - s_n| < \varepsilon$ $\forall n \ge N$ & $p \in \mathbb{N}$

Fact. Every Cauchy seq^n is cgt.

We use the above fact to see that

$\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ is cgt as it is dominated by the cgt series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

3. We can say otherway also.

note $s_k = \sum_{n=1}^{k} \frac{a_n}{2^n}$ - the seq^n of partial sum in our case is increasing

and $s_k \leq x$.

We then use the fact that an increasing seq^n which is bounded above is cgt, we have the series $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ is cgt and $\sum_{n=1}^{\infty} \frac{a_n}{2^n} \leq x$. — (1)

Once we note $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ cges,

$$\boxed{\text{We claim: } \quad x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}}$$

If not, from (i) it follows that

$$x - \sum_{n=1}^{\infty} \frac{a_n}{2^n} > 0 .$$

Black box: Archimedean property

given $\varepsilon > 0$, $\exists$ a natural number n such that $\varepsilon > \frac{1}{n}$.

— From this it follows that

given $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ such that $\varepsilon > \frac{1}{2^N}$.

So $\exists$ $N \in \mathbb{N}$ such that $x - \sum_{n=1}^{\infty} \frac{a_n}{2^n} > \frac{1}{2^N}$.

$$\Rightarrow \sum_{n=1}^{N-1} \frac{a_n}{2^n} + \frac{a_N}{2^N} + \sum_{n=N+1}^{\infty} \frac{a_n}{2^n} + \frac{1}{2^N} < x$$

$$\Rightarrow \sum_{n=1}^{N-1} \frac{a_n}{2^n} + \frac{a_N + 1}{2^N} + \sum_{n=N+1}^{\infty} \frac{a_n}{2^n} \overset{\geq 0}{} < x$$

The inequality above contradicts that a_N is maximum among all $k \in \mathbb{N} \cup \{0\}$ such that

$$\sum_{n=1}^{N-1} \frac{a_n}{2^n} + \frac{k}{2^N} \leq x.$$

The contradiction appears due to the assumption

$$x - \sum_{n=1}^{\infty} \frac{a_n}{2^n} > 0$$

Hence $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$.

In notation, x is then written as $0.{}_{2}\, a_1 a_2 a_3 a_4 \ldots\ldots$

denotes the no. w.r.t. which we are considering this representation