4. Limit & Continuity

> "The only way to discover the limits of the possible is to go beyond them into the impossible."
>
> ~ Arthur C. Clarke

We shall start with the notion of limits, followed by the notion of continuity of a function.

§4.1 Limit of a Function

Let us begin with the definition.

Definition 4.1.1 (Limit of a function)

For $-\infty < a < b < \infty$, let $f : (a,b) \to \mathbb{R}$ be a function. Let $x_0 \in [a,b]$ and $L \in \mathbb{R}$. We say that L is the limit of f at x_0, denoted by

$$L = \lim_{x \to x_0} f(x)$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in (a,b)$

$$\boxed{0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon} \qquad \cdots \quad (4.1)$$

Note that if $\delta > 0$ satisfies (4.1), then any $\delta' < \delta$, $\delta' > 0$ also satisfies (4.1). Therefore, for any given $\varepsilon > 0$, $\delta > 0$ ~~is~~ satisfying (4.1) is not unique.

Examples 4.1.2

Define $f : (-1,1) \to \mathbb{R}$ by $f(x) = x^2$ for any $x \in (-1,1)$.

(i) We claim that

$$\boxed{\lim_{x \to 0} f(x) = 0.}$$

To see this, let $\varepsilon > 0$ be given. Let us choose $\delta := \sqrt{\varepsilon}$. Here $x_0 = 0$ and for any $x \in (-1,1)$,

$$0 < |x - 0| = |x| < \delta = \sqrt{\varepsilon} \ \Rightarrow\ |f(x) - 0| = |x^2| = x^2 < \delta^2 = \varepsilon.$$

Thus, $\lim_{x \to 0} f(x) = 0$.

(ii) We claim that $\boxed{\lim_{x \to 1} f(x) = 1.}$

To see this, let $\varepsilon > 0$ be given. Let us choose $\delta := \varepsilon/2$. Here $x_0 = 1$ and for any $x \in (-1, 1)$, $|x+1| < 2$ as well as

$$0 < |x-1| < \delta = \varepsilon/2 \implies |f(x) - 1| = |x^2 - 1| = |x-1||x+1| < \tfrac{\varepsilon}{2} \cdot 2 = \varepsilon.$$

Thus, $\lim_{x \to 1} f(x) = 1$.

EXAMPLE 4.1.3

Let $a \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(x) := \begin{cases} x \sin(1/x) & \text{if } x \in \mathbb{R} \text{ \& } x \neq 0 \\ a & \text{if } x = 0. \end{cases}$$

We claim that $\boxed{\lim_{x \to 0} g(x) = 0.}$

To see this, let $\varepsilon > 0$ be given. Let us choose $\delta = \varepsilon$. Here $x_0 = 0$ and for any $x \in \mathbb{R}$,

$$0 < |x - 0| = |x| < \delta = \varepsilon \implies |g(x) - 0| = |x \sin(1/x)| \leq |x| < \delta = \varepsilon.$$

Thus, $\lim_{x \to 0} g(x) = 0$. Note that the value of g at 0, which is a, is of no relevance.

EXAMPLE 4.1.4

Let us define $h : \mathbb{R} \to \mathbb{R}$ as follows

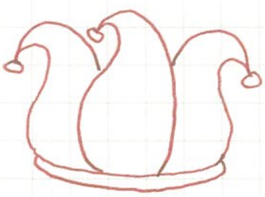
$$h(x) := \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \in \mathbb{R} \text{ \& } x \neq 2 \\ 2024 & \text{if } x = 2 \end{cases}$$

We claim that $\boxed{\lim_{x \to 2} h(x) = 4.}$

To see this, let $\varepsilon > 0$ be given. Let us choose $\delta = \varepsilon$. Here $x_0 = 2$ and for any $x \in \mathbb{R}$,

$$0 < |x - 2| < \delta \implies |h(x) - 4| = \left|\frac{x^2 - 4}{x - 2} - 4\right| = |(x+2) - 4| = |x - 2| < \delta = \varepsilon$$

as $|x - 2| > 0$ forces $x \neq 2$. Thus, $\lim_{x \to 2} h(x) = 4$.



IN JEST

If $f(x) := \begin{cases} \frac{1}{x-8} & \text{if } x \neq 8 \\ 0 & \text{if } x = 8 \end{cases}$, then $\lim_{x \to 8} f(x) = \infty$.

Thus, if $g(x) := \begin{cases} \frac{1}{x-5} & \text{if } x \neq 5 \\ 0 & \text{if } x = 5 \end{cases}$, then $\lim_{x \to 5} g(x) = \omega$.

EXAMPLE 4.1.5

Let us define $\varphi : \mathbb{R} \to \mathbb{R}$ as follows:

$$\varphi(x) := \begin{cases} \sin(1/x) & \text{if } x \in \mathbb{R} \,\&\, x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that $\lim_{x \to 0} \varphi(x)$ does not exist. We shall prove this via contradiction. Let us suppose, to the contrary, that the limit exists; we write

$$L := \lim_{x \to 0} \varphi(x).$$

We have two cases to consider.

CASE 1 : $L \neq 0$

In this case, we have to find $\varepsilon > 0$ for which no δ will work, i.e., find an $\varepsilon > 0$ s.t

$$0 < |x - 0| = |x| < \delta \quad \text{but} \quad |\varphi(x) - L| > \varepsilon.$$

Choose $\varepsilon = \frac{1}{2}|L|$ and $\delta > 0$ be arbitrary. Let us choose $N \in \mathbb{N}$ such that

$$0 < \frac{1}{N\pi} < \delta.$$

If we set $x = 1/N\pi$, then

$$|\varphi(x) - L| = |\sin(N\pi) - L| = |L| > \tfrac{1}{2}|L| = \varepsilon.$$

CASE 2 : $L = 0$

As in the previous case, we set $\varepsilon = \frac{1}{2}$. For any $\delta > 0$, choose $M \in \mathbb{N}$ such that

$$0 < \frac{1}{2M\pi + \pi/2} < \delta.$$

If we set $x = \frac{1}{2M\pi + \pi/2}$, then $0 < |x| < \delta$ but

$$|\varphi(x) - 0| = |\sin(2M\pi + \pi/2)| = 1 > \tfrac{1}{2} = \varepsilon.$$

Therefore, we conclude that $\lim_{x \to 0} \sin(1/x)$ does not exist. It is instructive to draw the graphs of the functions in Example 4.1.3 and Example 4.1.5.

The following result states the elementary properties of the limit of a function.

THEOREM 4.1.6 (PROPERTIES OF LIMIT)

Let $-\infty < a < b < \infty$. Let $x_0 \in [a,b]$ & let $f, g : (a,b) \to \mathbb{R}$ be such that

$$\lim_{x \to x_0} f(x) \text{ exists } \& \lim_{x \to x_0} g(x) \text{ exists.}$$

Then the following hold:

(i) $\lim_{x \to x_0} (f(x) + g(x))$ exist and

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

(ii) $\lim_{x \to x_0} (f(x) g(x))$ exist and

$$\lim_{x \to x_0} (f(x) g(x)) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right).$$

(iii) For any $\alpha \in \mathbb{R}$,

$$\lim_{x \to x_0} (\alpha f(x)) = \alpha \left(\lim_{x \to x_0} f(x)\right).$$

(iv) If $\lim_{x \to x_0} g(x) \neq 0$, then $\lim_{x \to x_0} (f(x)/g(x))$ exist and

$$\lim_{x \to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}.$$

Proof. Let us write

$$L := \lim_{x \to x_0} f(x) \quad \& \quad M := \lim_{x \to x_0} g(x).$$

We shall prove (ii); others are left as exercise. Let $\varepsilon > 0$ be given. By definition of limit, there exists $\delta_1 > 0$ such that

$$0 < |x - x_0| < \delta_1 \Rightarrow |g(x) - M| < 1 \Rightarrow |g(x)| < 1 + |M| \quad \cdots \quad (4.2)$$

With $\varepsilon' = \frac{\varepsilon}{1 + |L| + |M|}$, there exists $\delta_2 > 0$ such that

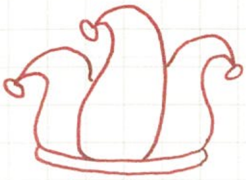
$$0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{1 + |L| + |M|} \quad \cdots \quad (4.3)$$

$$0 < |x - x_0| < \delta_3 \Rightarrow |f(x) - L| < \frac{\varepsilon}{1 + |L| + |M|} \quad \cdots \quad (4.4)$$

Let us set $\delta := \min\{\delta_1, \delta_2, \delta_3\}$. Then for $x \in (a,b)$ with $0 < |x - x_0| < \delta$, we have (using (4.1), (4.2))

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \leq |f(x) - L||g(x)| + |L||g(x) - M| \\ &\leq \frac{\varepsilon}{1 + |L| + |M|} \cdot (1 + |M|) + |L| \frac{\varepsilon}{1 + |L| + |M|} \\ &< \varepsilon. \end{aligned}$$

$\blacksquare$



IN JEST

Two interesting limits:

$$0.9999\ldots = 1$$

$$\lim_{x\to\infty} x^2 = \infty\infty\infty$$

§ 4.2 CONTINUITY

Let us begin with the definition.

DEFINITION 4.2.1 (CONTINUITY AT A POINT)

For $-\infty < a < b < \infty$, let $f : [a,b] \to \mathbb{R}$ be given. We say that f is continuous at $x_0 \in [a,b]$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x \in [a,b]$

$$|x - x_0| < \delta \;\Rightarrow\; |f(x) - f(x_0)| < \varepsilon.$$

If f is continuous at every point, then we say that f is continuous on $[a,b]$.

Although Definitions 4.1.1 & 4.2.1 look similar, note the differences. The following result connects the two notions.

THEOREM 4.2.2 (RELATION BETWEEN LIMIT & CONTINUITY)

Let $-\infty < a < b < \infty$, $f : [a,b] \to \mathbb{R}$ & $x_0 \in [a,b]$. Then, f is continuous at x_0 if and only if

i) $\lim_{x\to x_0} f(x)$ exists, and

ii) $\lim_{x\to x_0} f(x) = f(x_0)$.

Proof. Let f be continuous at x_0. Then for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \;\Rightarrow\; |f(x) - f(x_0)| < \varepsilon.$$

Thus, the above holds even when $0 < |x - x_0| < \delta$ & we may set $L := f(x_0)$ to obtain that the limit exists and

$$\lim_{x\to x_0} f(x) = L \; (= f(x_0)).$$

Conversely, assume (i) & (ii), i.e, $f(x_0)$ is the limit. When $x = x_0$, $f(x) = f(x_0)$, whence $|f(x) - f(x_0)| = 0 < \varepsilon$ in this case. The other cases $(x \neq x_0)$ is taken care of by (i). ▨

Example 4.2.3

Let us define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 \text{ for any } x \in \mathbb{R}.$$

We claim that f is continuous on $\mathbb{R}$, i.e., f is continuous at all points of $\mathbb{R}$.

Case 1: $a = 0$

Given $\varepsilon > 0$, set $\delta := \sqrt{\varepsilon}$. If

$$|x| = |x - 0| < \delta \Rightarrow x^2 = |x|^2 = |x^2 - 0| < \delta^2 = \varepsilon.$$

Thus, f is continuous at 0.

Case 2: $a \neq 0$

Given $\varepsilon > 0$, set $\delta := \min\{\frac{\varepsilon}{3|a|}, |a|\}$. If $|x - a| < \delta$, then

$$|x^2 - a^2| = |x - a| \cdot |x + a| \leq |x - a|(|x - a| + 2|a|) < \delta(\delta + 2|a|)$$
$$\leq \frac{\varepsilon}{3|a|}(|a| + 2|a|) = \varepsilon.$$

Thus, f is continuous at a.

Example 4.2.4

Let us define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

We claim that g is not continuous at 0. If g is continuous at 0, then for $\varepsilon = 1$, there exists $\delta > 0$ such that

$$|x| < \delta \Rightarrow |g(x) - g(0)| = |g(x) - 1| < 1.$$

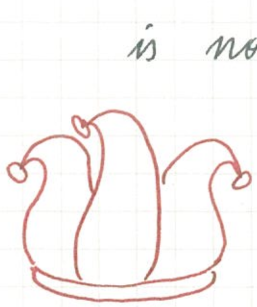
In particular, for $x = -\delta/2$, we obtain

$$2 = |g(-\delta/2) - 1| < 1,$$

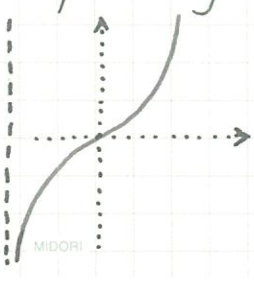
a contradiction. Observe that for $a \in \mathbb{R}$, the function

$$g_a(x) := \begin{cases} 1 & \text{if } x > 0 \\ a & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is not continuous at 0 for any choice of $a \in \mathbb{R}$.



IN JEST



THEOREM 4.2.5

Let $-\infty < a < b < \infty$ and let $f, g : [a,b] \to \mathbb{R}$ be continuous at $x_0 \in [a,b]$. Then,

i) $f+g$ is continuous at x_0.
ii) αf is continuous at x_0 for any $\alpha \in \mathbb{R}$.
iii) fg is continuous at x_0.
iv) if $g(x_0) \neq 0$, then f/g is continuous at x_0.

Proof. We shall prove i) & ii). The others are left as an exercise.
For i), given $\varepsilon > 0$, find $\delta_1 > 0$ & $\delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon/2$$
$$|x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon/2$$

Set $\delta := \min\{\delta_1, \delta_2\}$ & notice that

$$|x - x_0| < \delta \Rightarrow \varepsilon = \varepsilon/2 + \varepsilon/2 > |f(x) - f(x_0)| + |g(x) - g(x_0)|$$
$$\geq |f(x) + g(x) - (f(x_0) + g(x_0))|.$$

Thus, $f+g$ is continuous at x_0.
For ii), given $\varepsilon > 0$ and $\alpha \neq 0$, find $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon/|\alpha|$$

Then, for the same δ,

$$|\alpha f(x) - \alpha f(x_0)| = |\alpha|\,|f(x) - f(x_0)| < |\alpha| \cdot \varepsilon/|\alpha| = \varepsilon.$$

This proves that αf is continuous at x_0. The case $\alpha = 0$ reduces to the zero function, which is continuous. ∎

COROLLARY 4.2.6

Let $-\infty < a < b < \infty$ and let $P : [a,b] \to \mathbb{R}$ be a polynomial, i.e., $P(x) := a_0 + a_1 x + \cdots + a_d x^d$ with $a_d \neq 0$ and $a_i \in \mathbb{R}$. Then, P is continuous.

Proof. We know that $f(x) = x$ and $g(x) = c$ are continuous maps. Using iii) $(k-1)$ times, we conclude that the function $P_k(x) := x^k$ is continuous. Using ii), we conclude that $a_k x^k$ is continuous. Using i) k times (by adding the functions a_0, $a_1 x, \dots, a_d x^d$), we conclude that P_k (the sum) is continuous. ∎