MA 1201 - Mathematics II

ODE - Week 02

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Existence and Uniqueness

The IVP's that we have considered usually have unique solutions. However, this is not always the case.

Example

Example: Consider the IVP

$$\frac{dy}{dx} = y^{\frac{1}{3}};\ y(0) = 0.$$

$$y = \phi(x) = \left[\frac{2x}{3}\right]^{\frac{3}{2}};\ x \geq 0$$

is a solution.

$$y = -\phi(x) = -\left[\frac{2x}{3}\right]^{\frac{3}{2}};\ x \geq 0$$

is also a solution of the IVP.

$$y = \psi(x) \equiv 0$$

is also a solution of the IVP.
For any $a > 0$,

$$y = \phi_a(x) = \begin{cases} 0 & \text{if } x \in [0, a) \\ \pm\left[\frac{2}{3}(x-a)\right]^{\frac{3}{2}} & \text{if } x \geq a \end{cases}$$

is continuous, differentiable, and gives a solution of the given IVP.

Existence and Uniqueness

That is, we get infinitely many solutions of the given IVP.

No solution of IVP

It may happen there exists no differentiable function satisfying the ODE and initial value!

Ex. Solve: $y(t)y'(t) = \frac{1}{2}, y(0) = 0$.

Ans. no solution, because if there exists any solution ϕ, then $\phi(t)\phi'(t) = \frac{1}{2}$ and putting $t = 0$, $\phi(0) = 0$ contradicts the equation.

How to determine if an IVP has a solution? In case a solution exists, when it has to be unique? Existence and uniqueness theorem.

Existence - Uniqueness Theorem

Let R be a rectangle containing (x_0, y_0): $R : |x - x_0| < a, \ |y - y_0| < b$.

- $f(x, y)$ be continuous at all points $(x, y) \in R$ in and
- bounded in R, that is, $|f(x, y)| \leq K, \ \forall (x, y) \in R$.

Then, the IVP $y' = f(x, y), \ y(x_0) = y_0$ has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| < \alpha$, where

$$\alpha = \min\left\{a, \frac{b}{K}\right\}.$$

If, in addition to the above conditions, f also satisfies the Lipschitz condition with respect to y in R, that is,

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2| \ \ \forall (x, y_1), (x, y_2) \text{ in } R,$$

then,the IVP admits a unique solution on the interval $(x_0 - \alpha, x_0 + \alpha)$. [1].

[1]Existence - Peano, Existence & uniqueness -Picard

Summary of Week 01 - First Order Equations

- Definition of ODE - Explicit & Implicit
- Definition of solution of ODE - Explicit & Implicit
- Initial Value Problem
- 1st order ODE
- Separable ODE: $\boxed{M(x) + N(y)y' = 0}$
  - Homogeneous ODE - reducible to separable ODE: $\boxed{M(x,y) + N(x,y)y' = 0 \text{ - } M(x,y), N(x,y) \text{ - homogeneous of degree } d}$
- 1st order Linear Equations: $\boxed{y' + P(x)y = Q(x)}$ - Solution
  - Reducible to linear - Bernoulli: $\boxed{y' + P(x)y = Q(x)y^n}$
- Existence & Uniqueness results for IVP : $\boxed{y' = f(x,y),\ y(x_0) = y_0}$ - statement

Second order linear ODEs

Second order non-homogeneous ODEs

- Method of Undetermined Coefficients
- Method of Variation of Parameters

Second Order Linear ODE's

Recall that a general second order linear ODE is of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

where $a_2(x) \neq 0$ for all x.

Definition

An ODE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called a second order linear ODE in standard form.

Assume $p(\cdot), q(\cdot), r(\cdot)$ are continuous functions on an interval I of $\mathbb{R}$.

Though there is no formula to find all the solutions of such an ODE, we study the existence and number of linearly independent solutions of such ODE's.

Initial Value Problem

An initial value problem of a second order linear ODE is of the form:

$$y'' + p(x)y' + q(x)y = r(x);\ y(x_0) = a, y'(x_0) = b,$$

where $p(\cdot)$, $q(\cdot)$ and $r(\cdot)$ are assumed to be continuous on an interval I with $x_0 \in I$.

Existence-Uniqueness Theorem for IVP

Theorem

Consider the IVP

$$y'' + p(x)y' + q(x)y = r(x), y(x_0) = a, y'(x_0) = b,$$

where p, q and r are continuous on an interval I, x_0 is any point in I, and a, b are real numbers. Then there is a unique solution to the IVP on I.

Example: The IVP

$$y''(x) - 2y'(x) + y(x) = \sin x, \quad y(0) = 1, \quad y'(0) = 0,$$

has $y(x) = \frac{1}{2}(e^x - xe^x + \cos x)$ as the unique solution.

Second Order Linear ODE's

If $r(x) \equiv 0$ in the equation above, i.e.,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

then the ODE is said to be homogeneous. Otherwise it is called nonhomogeneous.

Solving IVP's

Let

$$\begin{aligned} C(I) &= \{f : I \to \mathbb{R} \mid f \text{ is continuous}\} \\ C^1(I) &= \{f : I \to \mathbb{R} \mid f, f' \text{ are continuous}\} \\ C^2(I) &= \{f : I \to \mathbb{R} \mid f, f', f'' \text{ are continuous}\}. \end{aligned}$$

Check: $C(I), C^1(I), C^2(I)$ are vector spaces with addition and scalar multiplication defined as:

$$(f+g)(x) = f(x) + g(x),\ x \in I,$$

$$(k \cdot f)(x) = kf(x),\ k \in \mathbb{R}, x \in I.$$

Solving IVP's

Define

$$L : C^2(I) \to C(I)$$

by

$$L(f) = f'' + p(x)f' + q(x)f.$$

Then L is a linear transformation, i.e.,

$$L(cf + dg) = cL(f) + dL(g),$$

for all $c, d \in \mathbb{R}$ and for all $f, g \in C^2(I)$.
The null space of L, denoted by $N(L)$ is

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

Thus, $N(L)$ consists of solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Qn. How to characterize the space $N(L)$? What is the dimension of the space?

Linearly independent & dependent functions

Definition

The functions f and g are said to be linearly independent on an interval I if

$$c_1 f(x) + c_2 g(x) = 0 \;\; \forall x \in I \implies c_1 = c_2 = 0.$$

The functions are said to be linearly dependent on an interval I if they are not linearly independent on I.

Examples : 1. The functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent on $(-\infty, \infty)$.

2. The functions $f(x) = x$ and $g(x) = |x|$ are linearly dependent on $(0, \infty)$ but are linearly independent on $(-\infty, \infty)$.

Wronskian and Linear Independence

Definition

The Wronskian of any two differentiable functions f and g is defined by

$$W(f, g; x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

Wronskian and Linear Independence

Proposition

Suppose f and g are linearly dependent and differentiable on an interval I. Then, $W(f, g; x) = 0$ on I.
In other words, if two differentiable functions f and g have $W(f, g; x_0) \neq 0$, for some $x_0 \in I$, then the functions f and g are linearly independent on I.

Proof. As f and g are linearly dependent, there exist $c, d \in \mathbb{R}$, not both 0, such that

$$cf(x) + dg(x) = 0.$$

Thus,

$$cf'(x) + dg'(x) = 0.$$

Hence,

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $W(f, g; x) = f(x)g'(x) - f'(x)g(x) = 0$ for all x (Why?) since $\begin{bmatrix} c \\ d \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Wronskian and Linear Independence

Example.

1. Let $m_1 \neq m_2$, two real numbers. Are the functions $f(x) = e^{m_1 x}$ and $g(x) = e^{m_2 x}$ linearly independent on $\mathbb{R}$?
2. Let m be a real number. Are the functions $f(x) = e^{mx}$ and $g(x) = xe^{mx}$ linearly independent on $\mathbb{R}$?
3. Let m be a real number. Are the functions $f(x) = \sin(mx)$ and $g(x) = \cos(mx)$ linearly independent on $\mathbb{R}$?

Note: The converse of the Proposition is not true. For instance, if $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

If $x \geq 0$, $W(x^2, x^2; x) = 0$. If $x < 0$, $W(x^2, -x^2; x) = 0$. Hence,

$$W(f, g; x) = 0 \quad \text{forall} \quad x \in \mathbb{R}$$

but f and g are linearly independent on $\mathbb{R}$. (why? Check using definition)

Wronskian and Linear Independence

Theorem (Abel's Formula)

Let p, q be continuous on an interval I and let f, g be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on I. Let a be any point of I. Then

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt}, x \in I$$

Proof. Set $W(f, g; x) = W(x)$. Then,

$$W(x) = (fg' - f'g)(x)$$
$$W'(x) = (fg'' - f''g)(x).$$

Wronskian and Linear Independence

Now,

$$\begin{aligned} f'' &= -p(x)f' - q(x)f \\ g'' &= -p(x)g' - q(x)g. \end{aligned}$$

Thus,

$$\begin{aligned} W'(x) &= (fg'' - f''g)(x) \\ &= (-fpg' - fqg + gpf' + gqf)(x) \\ &= -p(x)(fg' - f'g)(x) \\ &= -p(x)W(x), \end{aligned}$$

i.e., W is the solution of the IVP

$$y' + p(x)y = 0, \; y(a) = W(a).$$

Hence,

$$W(x) = W(a)e^{-\int_a^x p(t)dt},$$

i.e.,

$$W(f, g; x) = W(f, g; a)e^{-\int_a^x p(t)dt}.$$

Wronskian and Linear Independence

Theorem

Let p, q be continuous on an interval I and let f, g be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on I. Then,

1. *If $W(f, g; a) = 0$ for some $a \in I$, then $W \equiv 0$ on I.*
2. *f and g are linearly dependent on I if and only if $W(f, g; a) = 0$ for some $a \in I$.*

Thus, f and g are linearly independent on I iff $W(f, g; x) \neq 0$ for all $x \in I$.

Wronskian and Linear Independence

Proof of (1): Suppose $W(a) = 0$ for some $a \in I$. Then, for any $x \in I$,

$$W(x) = W(a)e^{-\int_a^x p(t)dt} = 0.$$

Hence, $W \equiv 0$ on I.

Wronskian and Linear Independence

Proof of (2): $\Rightarrow$ Done earlier. Need to do $\Leftarrow$.
Suppose that $W(f, g; a) = 0$ for some $a \in I$. Then the following linear system

$$\begin{aligned} C_1 f(a) + C_2 g(a) &= 0, \\ C_1 f'(a) + C_2 g'(a) &= 0. \end{aligned}$$

has non-zero solution $(C_1, C_2) \neq (0, 0)$.
Set $\psi(x) = C_1 f(x) + C_2 g(x)$ for all $x \in I$, for the C_1, C_2 obtained solving the above linear system. Then note that ψ satisfies the ODE with the initial value:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad y(a) = 0, \quad y'(a) = 0.$$

Also, note that 0 is a solution of the above ODE-IVP. Using the Uniqueness of solution of the second order linear IVP, we get $\psi(x) = 0$ for all $x \in I$. It implies that there exist non-zero C_1 and C_2 such that

$$C_1 f(x) + C_2 g(x) = 0, \quad \forall\, x \in I.$$

Hence f and g are linearly dependent on I.

Existence of set of fundamental solutions

Theorem

There exist two linearly independent solutions of the second order ODE

$$L(y) = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

where $p(\cdot), q(\cdot)$ are continuous on an interval I.

Proof. Let $x_0 \in I$. Recall that for any initial conditions, the 2-nd order linear IVP admits a unique solution.

Let $\phi_1(\cdot)$ be the solution of the ODE-IVP:

$$L(y)(x) = 0, \quad \forall\, x \in I, \quad y(x_0) = 1, \quad y'(x_0) = 0.$$

Let $\phi_2(\cdot)$ be the solution of the ODE-IVP:

$$L(y)(x) = 0, \quad \forall\, x \in I, \quad y(x_0) = 0, \quad y'(x_0) = 1.$$

Claim. The two functions ϕ_1 and ϕ_2 are linearly independent on I.
Check that

$$W(\phi_1, \phi_2; x_0) = \det \begin{pmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{pmatrix} = 1.$$

Hence ϕ_1 and ϕ_2 are linearly independent on I.

Proposition

Let f and g be any two solutions of the second order ODE on I with the property that $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ are two linearly independent vectors in $\mathbb{R}^2$, for some $x_0 \in I$. The the two solutions f and g are linearly independent on I.

Linear combination

Theorem

If y_1 and y_2 are solutions of homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

on an interval I, then any linear combination

$$y = c_1 y_1 + c_2 y_2$$

of y_1 and y_2 is also a solution of (3) on I.

(Why?).

Basis of Solutions

Theorem

Let f, g be two solutions of the homogeneous second order linear ODE

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous on an interval I in $\mathbb{R}$. Let $(f(x_0), f'(x_0))$ and $(g(x_0), g'(x_0))$ be linearly independent vectors in $\mathbb{R}^2$, for some $x_0 \in I$. Then the solution space is the linear span of f and g.

Proof. Let h be a solution of the given ODE. We want to find c and d such that

$$h(x) = cf(x) + dg(x) \text{ for all } \quad x \in I.$$

Basis of Solutions

To do it, first solve the linear system for c, d

$$\begin{aligned} cf(x_0) + dg(x_0) &= h(x_0) \\ cf'(x_0) + dg'(x_0) &= h'(x_0). \end{aligned}$$

Thus,

$$\begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h(x_0) \\ h'(x_0) \end{bmatrix}.$$

As the column vectors

$$\begin{bmatrix} f(x_0) \\ f'(x_0) \end{bmatrix} \;\&\; \begin{bmatrix} g(x_0) \\ g'(x_0) \end{bmatrix}$$

are linearly independent, the matrix

$$W(x_0) = \begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix}$$

is invertible.

Basis of Solutions

Therefore, there exists a unique solution (c, d) to the linear system given by

$$c = \frac{\begin{vmatrix} h(x_0) & g(x_0) \\ h'(x_0) & g'(x_0) \end{vmatrix}}{\det W(x_0)},$$

and

$$d = \frac{\begin{vmatrix} f(x_0) & h(x_0) \\ f'(x_0) & h'(x_0) \end{vmatrix}}{\det W(x_0)}.$$

(What's this method called?) Let

$$u(x) = h(x) - cf(x) - dg(x) \quad \text{for all} \quad x \in I.$$

Check that u is a solution of the IVP

$$y'' + p(x)y' + q(x)y = 0, y(x_0) = 0, y'(x_0) = 0,$$

which implies that $u \equiv 0$ by the uniqueness theorem, i.e.,

$$h(x) = cf(x) + dg(x) \quad \text{for all} \quad x \in I.$$

Note that in the above span of h by f, g, the constants c, d are uniquely determined.

Fundamental Theorem

Given a vector space, what's the most important thing about it? Dimension. The above results give:

Theorem (Dimension Theorem)

Let I be an interval in $\mathbb{R}$, p, q be continuous on I and let

$$L : C^2(I) \to C(I)$$

be defined by

$$L(f) = f'' + p(x)f' + q(x)f$$

and the null space of L

$$N(L) = \{f \in C^2(I) \mid L(f) = f'' + p(x)f' + q(x)f = 0\}.$$

The dimension of $N(L) = 2 =$ order of the ODE.

Second Order Linear Homogeneous ODE's: Summary

Theorem

Suppose p and q are continuous on an interval I and let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

on I. Then the following statements are equivalent:

(i) $\{y_1, y_2\}$ *is linearly independent on I.*

(ii) *Every solution of (4) on I can be written as a linear combination of y_1 and y_2.*

(iii) *The Wronskian of $\{y_1, y_2\}$ is non-zero at some point in I.*

(iv) *The Wronskian of $\{y_1, y_2\}$ is non-zero at all points in I.*

Second Order Linear ODE's

We've been looking at the second order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0.$$

As we remarked earlier, there is no general method to find a basis of solutions. However, if we know one non-zero solution f then we have a method to find another solution g such that f and g are linearly independent.
To find such a g, set

$$g(x) = v(x)f(x).$$

We'll choose v such that $\{f, g\}$ are linearly independent. Can v be a constant? No. Now for g to be a solution of the given ODE

$$g'' + p(x)g' + q(x)g = 0.$$

i.e.,

$$(vf)'' + p(x)(vf)' + q(x)(vf) = 0.$$

Second Order Linear ODE's

$$(vf)'' + p(x)(vf)' + q(x)(vf) = 0.$$

Thus,

$$\begin{aligned} 0 &= (v'f + vf')' + p(v'f + vf') + qvf \\ &= v''f + 2v'f' + vf'' + p(v'f + vf') + qvf \\ &= v(f'' + pf' + qf) + v'(2f' + pf) + v''f. \end{aligned}$$

Thus, get $v'' + \left[\frac{2f'+pf}{f}\right] v' = 0$, and set $u(x) = v'(x)$, get a linear first order ODE

$$u' + \left[\frac{2f' + pf}{f}\right] u = 0$$

Therefore, using integrating factor and solving, get

$$u(x) = \frac{e^{-\int p(x)dx}}{f^2(x)} \quad \forall \quad x \in I,$$

i.e., $v'(x) = \frac{e^{-\int p(x)dx}}{f^2(x)}$ on I, and thus

$$v(x) = \int \frac{e^{-\int p(x)dx}}{f^2(x)} dx, \quad \text{on} \quad I.$$

Second Order Linear ODE's

Claim: f and vf are linearly independent.
Proof. Enough to check Wronskian!

$$\begin{aligned} W(f, vf) &= f(v'f + f'v) - f'vf \\ &= f^2 v' \\ &= f^2 \frac{e^{-\int p dx}}{f^2} \\ &= e^{-\int p dx} \\ &\neq 0. \end{aligned}$$

Second Order Linear ODE's

Example. Find all solutions of

$$y'' - 2y' + y = 0.$$

Note $f(x) = e^x$ is a solution. How do you find another linearly independent solution on $\mathbb{R}$? Let $g(x) = v(x)f(x) = v(x)e^x$ be another solution. Then as shown in the method above $v(x)$ can be obtained as

$$v(x) = \int \frac{e^{-\int p dx}}{f^2} dx,$$

where $p(x) = -2$ and $f(x) = e^x$. Thus, $v(x) = x$.

Two linearly independent solution of the second order ODE: $f(x) = e^x$ and $g(x) = xe^x$ on $\mathbb{R}$.
General solution: $y(x) = c_1 e^x + c_2 x e^x$, for any real numbers c_1 and c_2.

Second Order Linear Homogeneous DE's with constant coefficients

We have developed enough theory to now find all solutions of

$$y'' + py' + qy = 0,$$

where p and q are in $\mathbb{R}$, i.e., a second order homogeneous linear ODE with constant coefficients. Suppose e^{mx} is a solution of this equation. Then,

$$m^2 e^{mx} + pme^{mx} + qe^{mx} = 0,$$

and this implies

$$m^2 + pm + q = 0.$$

This is called the characteristic equation of the linear homogeneous ODE with constant coefficients. The roots of this equation are

$$m_1, m_2 = -\frac{p \pm \sqrt{p^2 - 4q}}{2}.$$

Second Order Linear ODE's

Case I: $m_1, m_2 \in \mathbb{R}, m_1 \neq m_2$.
When $p^2 - 4q > 0$, m_1 and m_2 are distinct real numbers. Moreover,

$$W(e^{m_1 x}, e^{m_2 x}; x) \neq 0, \quad x \in \mathbb{R}.$$

Hence, $e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent on $\mathbb{R}$. So the general solution of

$$y'' + py' - qy = 0$$

is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x},$$

where $c_1, c_2 \in \mathbb{R}$.

Example

Consider $y'' - 3y' + 2y = 0$.

The characteristic equation (CE) is: $m^2 - 3m + 2 = 0$, whose roots are $m_1 = 1$, $m_2 = 2$.

The roots are real and distinct.

The corresponding solutions are e^x and e^{2x}.

They are linearly independent because:

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x \cdot 2e^{2x} - e^{2x} \cdot e^x = e^{3x} \neq 0$$

Thus, the solutions are:

$$y(x) := \alpha_1 e^x + \alpha_2 e^{2x}$$

Example

Consider $y^{(3)} - 4y'' + y' + 6y = 0$.

The CE is $m^3 - 4m^2 + m + 6 = 0$ whose roots are $m_1 = -1, m_2 = 2$ and $m_3 = 3$.

The roots are real and distinct.

The corresponding solutions are e^{-x}, e^{2x} and e^{3x}.

They are linearly independent (exercise!).

Thus, the general solution is $y(x) := \alpha_1 e^{-x} + \alpha_2 e^{2x} + \alpha_3 e^{3x}$.

Second Order Linear ODE's

Case II: $m_1 = m_2 \in \mathbb{R}$.

$$m_1 = m_2 \iff p^2 - 4q = 0,$$

and in this case $m_1 = m_2 = -\frac{p}{2}$. Hence $e^{-\frac{px}{2}}$ is one solution. To find the other solution, let

$$g(x) = v(x)e^{-\frac{px}{2}}.$$

Then,

$$\begin{aligned} v(x) &= \int \frac{e^{-\int p dx}}{e^{-px}} dx \\ &= ax + b, \end{aligned}$$

(note: $p \in \mathbb{R}$) for some $a, b \in \mathbb{R}$. Choose $v(x) = x$. Then, $g(x) = xe^{-\frac{px}{2}}$. Hence the general solution is

$$ae^{-\frac{px}{2}} + bxe^{-\frac{px}{2}},$$

with $a, b \in \mathbb{R}$.

Example

Consider the ODE $y'' - 2y' + y = 0$.

The CE is $m^2 - 2m + 1 = 0$ with two repeated roots $m_1 = m_2 = 1$.

The corresponding solution is e^x.

A linear independent solution is xe^x and the general solution is

$$y(x) := (\alpha_1 + \alpha_2 x)e^x.$$

Example

Consider $y^{(3)} - 4y'' - 3y' + 18y = 0$.

The CE is $m^3 - 4m^2 - 3m + 18 = 0$ whose roots are $m_1 = m_2 = 3$ and $m_3 = -2$.

The two of the roots are repeated.

The corresponding solutions are e^{3x} and e^{-2x}.

Thus, the general solution is $y(x) := (\alpha_1 + \alpha_2 x)e^{3x} + \alpha_3 e^{-2x}$.

Example

Consider $y^{(4)} - 5y^{(3)} + 6y'' + 4y' - 8y = 0$.

The CE is $m^4 - 5m^3 + 6m^2 + 4m - 8 = 0$ whose roots are $m_1 = m_2 = m_3 = 2$ and $m_4 = -1$.

Some roots are repeated!

Thus, the general solution is $y(x) := (\alpha_1 + \alpha_2 x + \alpha_3 x^2)e^{2x} + \alpha_4 e^{-x}$.

Second Order Linear ODE's

Case III: $m_1 \neq m_2 \in \mathbb{C} \backslash \mathbb{R}$.
$m^2 + pm + q = 0$ has distinct complex roots if and only if $p^2 - 4q < 0$.
In this case, let

$$m_1 = a + ib, m_2 = a - ib.$$

(Why not $a_1 + ib_1, a_2 + ib_2$?) Thus,

$$e^{m_1 x} = e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx),$$

and

$$e^{m_2 x} = e^{(a-ib)x} = e^{ax}(\cos bx - i \sin bx).$$

As we are only interested in real valued functions, we take

$$f(x) = \frac{e^{m_1 x} + e^{m_2 x}}{2} = e^{ax} \cos bx,$$

and

$$g(x) = \frac{e^{m_1 x} - e^{m_2}}{2\imath} = e^{ax} \sin bx.$$

Second Order Linear ODE's

Now, check

$$W(f, g; x) \neq 0 \quad x \in \mathbb{R}.$$

Thus the general solution is of the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx),$$

with $c_1, c_2 \in \mathbb{R}$.

Example

Find the general solution of $y'' + 4y' + 13y = 0$.

Characteristic equation is $m^2 + 4m + 13 = (m + 2)^2 + 9 = 0$

Roots of the characteristic equation are $m_1 = -2 + 3i$ and $m_2 = -2 - 3i$.

It is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions. To obtain real valued solutions, consider the sum and difference given by $e^{-2x} \cos 3x$ and $e^{-2x} \sin 3x$ respectively.

Hence the general solution is

$$y(x) = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x),$$

where $c_1, c_2 \in \mathbb{R}$.

Example

Consider $y^{(4)} - 4y^{(3)} + 14y'' - 20y' + 25y = 0$.pause

The CE is $m^4 - 4m^3 + 14m^2 - 20m + 25 = 0$ whose roots are $m_1 = m_2 = 1 + 2i$ and $m_3 = m_4 = 1 - 2i$.pause

The complex pair of roots are repeated.pause

Thus, the general solution is

$$y(x) := e^x \left[(\alpha_1 + \alpha_2 x) \sin 2x + (\alpha_3 + \alpha_4 x) \cos 2x \right].$$

Non-homogeneous Second Order Linear ODE's

Theorem

Let f be any solution of

$$y'' + p(t)y' + q(t)y = r(t), \tag{3}$$

where p, q and r are continuous on an interval I. Let y_1, y_2 be a basis of the solution space of the corresponding homogeneous DE. Then the set of solutions of equation (3) on I is

$$\{c_1 y_1(t) + c_2 y_2(t) + f(t) \mid c_1, c_2 \in \mathbb{R}\}.$$

Proof: Let ϕ be any solution of

$$L(y) = y'' + p(t)y' + q(t)y = r(t)$$

on I. Then,

$$L(\phi(t) - f(t)) = L(\phi(t)) - L(f(t)) = r(t) - r(t) = 0.$$

Non-homogeneous Second Order Linear ODE's

Hence, $\phi(t) - f(t)$ is a solution of the homogeneous DE. Thus,

$$\phi(t) - f(t) = c_1 y_1(t) + c_2 y_2(t),$$

for $c_1, c_2 \in \mathbb{R}$. Hence,

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t) + f(t),$$

for $t \in I$.
Summary: In order to find the general solution of a non-homogeneous DE, we need to

- get one particular solution of the non-homogeneous DE
- get the general solution of the corresponding homogeneous DE.

How to find particular solution

No standard method!

Often useful methods:

- Method of undetermined coefficients,
- Method of variation of parameters.
- Method of particular integral

Method of Undetermined Coefficients

Suppose the non-homogeneous ODE has constant coefficients. In this case, we know how to write down the general solution of the corresponding homogeneous ODE. So we need to find one solution of the non-homogeneous DE. One way to do this is called the method of undetermined coefficients. Thus, we have:

$$y'' + py' + qy = r(t),$$

with $p, q \in \mathbb{R}$, and r is a continuous function on I. The method of undetermined coefficients does not work for any $r(t)$, but only when we know more about $r(t)$. We'll use this method only if $r(t)$ involves $e^{at}, \sin at, \cos at$ or polynomials in t.

Method of Undetermined Coefficients

Example: Find a particular solution of the DE:

$$y'' - 3y' - 4y = 3e^{2t}.$$

We'll search for a solution of the form ae^{2t}, where a is a constant. So put $y = ae^{2t}$. We get:

$$(ae^{2t})'' - 3(ae^{2t})' - 4ae^{2t} = 3e^{2t}.$$

Thus,

$$4ae^{2t} - 6ae^{2t} - 4ae^{2t} = 3e^{2t}.$$

Thus,

$$a = -\frac{1}{2}.$$

Hence $-\frac{1}{2}e^{2t}$ is a particular solution of the DE.
How do you get the general solution? Analyse roots of $m^2 - 3m - 4 = 0$.
So general solution is

$$y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t},$$

where $c_1, c_2 \in \mathbb{R}$.

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t.$$

Make a guess as to functions of which form we'll search for as a solution. $a\sin t$? No. $a\sin t + b\cos t$? Yes. So set

$$y(t) = a\sin t + b\cos t.$$

Thus,

$$y' = a\cos t - b\sin t;\ \ y'' = -a\sin t - b\cos t.$$

Substituting, we get:

$$(-5a + 3b - 2)\sin t + (-3a - 5b)\cos t = 0.$$

Thus,

$$-5a + 3b = 2;\ \ 3a + 5b = 0$$

(Why?). Thus, $a = -\frac{5}{17}$, $b = \frac{3}{17}$, and a particular solution is

$$y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' - 3y' - 4y = 4t^2 - 1.$$

Set

$$y(t) = at^2 + bt + c.$$

Substituting, we get:

$$-4at^2 + (-6a - 4b)t + (2a - 3b - 4c) = 4t^2 - 1.$$

Thus,

$$-4a = 4,\ -6a - 4b = 0,\ 2a - 3b - 4c = -1.$$

Thus,

$$a = -1,\ b = \frac{3}{2},\ c = -\frac{11}{8}.$$

Thus, a particular solution is

$$y(t) = -t^2 + \frac{3}{2}t - \frac{11}{8}.$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

We should search for a solution of the form

$$y(t) = ae^t \cos 2t + be^t \sin 2t.$$

Then,

$$y'(t) = (a + 2b)e^t \cos 2t + (-2a + b)e^t \sin 2t,$$

and

$$y'' = (-3a + 4b)e^t \cos 2t + (-4a - 3b)e^t \sin 2t.$$

Substituting, we get:

$$-10a - 2b = -8, \; 2a - 10b = 0.$$

Thus, a particular solution is

$$y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

Since $r(t) = 3\cos 2t$, you would look for solutions of the form

$$y(t) = a\cos 2t + b\sin 2t.$$

Thus,

$$y'(t) = -2a\sin 2t + 2b\cos 2t,$$

$$y''(t) = -4a\cos 2t - 4b\sin 2t.$$

Substituting in the given DE, we get:

$$(-4a\cos 2t - 4b\sin 2t) + 4(a\cos 2t + b\sin 2t) = 3\cos 2t.$$

But the lhs is 0! So can't solve for a and b.

Method of Undetermined Coefficients

Why this ...? Note that $\sin 2t$ and $\cos 2t$ are also solutions of the associated homogeneous ODE: $y'' + 4y = 0$. So lesson learnt? When we search for solutions of a particular form, we need to make sure that it's not a solution of the associated homogeneous equation.
We now modify the proposed solution as:

$$y(t) = at\cos 2t + bt\sin 2t.$$

Then,

$$y'(t) = (b - 2at)\sin 2t + (a + 2bt)\cos 2t,$$

$$y''(t) = -4at\cos 2t - 4bt\sin 2t - 4a\sin 2t + 4b\cos 2t.$$

Substituting, we get:

$$-4a\sin 2t + 4b\cos 2t = 3\cos 2t.$$

Thus, $a = 0,\ b = \frac{3}{4}$, and a particular solution is $y(t) = \frac{3}{4}t\sin 2t$.

Method of Undetermined Coefficients

If the obvious candidate for a solution, say $y(t) = f(t)$, as well as this one multiplied by t, $y(t) = tf(t)$, turn out to be solutions of the associated homogeneous ODE, then what to do? Modify the proposed solution by multiplying it with t^2; i.e., set

$$y(t) = t^2 f(t).$$

Can this too be a solution of the homogeneous ODE? No, since the solution space is two dimensional.

Method of Undetermined Coefficients

Consider the DE

$$y'' + py' + qy = r(t).$$

where p and q are real numbers. If

$$r(t) = r_1(t) + r_2(t) + \ldots + r_n(t),$$

where $r_i(t)$ are e^{at} or $\sin at$ or $\cos at$ or polynomials in t, consider the n subproblems

$$y'' + py' + qy = r_i(t), 1 \leq i \leq n.$$

If $y_i(t)$ is a particular solution of this problem, then,

$$y(t) = y_1(t) + y_2(t) + \ldots + y_n(t)$$

is a particular solution of

$$y'' + py' + qy = r(t).$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t + 4t^2 - 1 - 8e^t \cos 2t.$$

Here,

$$r(t) = r_1(t) + r_2(t) + r_3(t) + r_4(t).$$

We need to solve

$$y'' - 3y' - 4y = r_i(t),$$

get a particular solution $y_i(t)$, and then

$$y(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t)$$

is a particular solution of the given problem. Thus, a particular solution is

$$y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t - t^2 + \frac{3}{2}t - \frac{11}{8} + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

Method of Undetermined Coefficients

Example: Find a particular solution of

$$y'' + y = x^3 \sin x.$$

What's your candidate? Presence of $\sin x$ indicates both $\sin x$ and $\cos x$ in the answer. Presence of x^3 indicates a generic cubic polynomial. Thus,

$$(a_1x^3 + b_1x^2 + c_1x + d_1)\cos x + (a_2x^3 + b_2x^2 + c_2x + d_2)\sin x?$$

This wouldn't do since $\sin x$ and $\cos x$ are already solutions of the homogeneous part. So work with

$$x(a_1x^3 + b_1x^2 + c_1x + d_1)\cos x + x(a_2x^3 + b_2x^2 + c_2x + d_2)\sin x.$$

Method of Undetermined Coefficients

Consider the DE

$$y'' + py' + qy = r(t).$$

where p and q are real numbers.
Step I: Find the general solution of the homogeneous equation

$$y'' + py' + qy = 0.$$

Step II: If $r(t) = r_1(t) + r_2(t) + \ldots + r_n(t)$, where $r_i(t)$ are e^{at} or $\sin at$ or $\cos at$ or polynomials in t, consider

$$y'' + py' + qy = r_i(t), 1 \leq i \leq n.$$

Step III: Find a particular solution $y_i(t)$ for each of the above n subproblems. Then

$$y(t) = y_1(t) + y_2(t) + \ldots + y_n(t)$$

is a particular solution of

$$y'' + py' + qy = r(t).$$

Step IV: In case, the trial function is same as the solution of the associated homogeneous equation, then we modify that trial function by multiplying it with t or atmost t^2.

Method of Undetermined Coefficients

Table of particular solutions

$r(t)$	Particular solutions $y(t)$	
$P_n(t) = \sum_{i=0}^{n} a_i t^i$	$t^s \left(\sum_{i=0}^{n} A_i t^i\right)$	
$P_n(t)e^{\alpha t}$	$e^{\alpha t} t^s \left(\sum_{i=0}^{n} A_i t^i\right)$	
$P_n(t)e^{\alpha t}\sin\beta t$	$e^{\alpha t} t^s \left(\sum_{i=0}^{n} A_i t^i\right)\cos\beta t$	$+$
	$e^{\alpha t} t^s \left(\sum_{i=0}^{n} B_i t^i\right)\sin\beta t$	
$P_n(t)e^{\alpha t}\cos\beta t$	$e^{\alpha t} t^s \left(\sum_{i=0}^{n} A_i t^i\right)\cos\beta t$	$+$
	$e^{\alpha t} t^s \left(\sum_{i=0}^{n} B_i t^i\right)\sin\beta t$	

Here $s = 0, 1$ or 2 has to be chosen appropriately.

Method of Variation of Parameters

A method to find a particular solution of a non-homogeneous ODE is the method of variation of parameters. Consider the DE

$$y'' + p(x)y' + q(x)y = r(x),$$

where p, q and r are continuous on an interval I. The associated homogeneous DE is

$$y'' + p(x)y' + q(x)y = 0. \tag{4}$$

Suppose that we know the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

of equation (4). In the method of variation of parameters, we vary the constants c_1, c_2 by functions $v_1(x), v_2(x)$, (to be suitably determined) so that

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$y'' + p(x)y' + q(x)y = r(x).$$

Method of Variation of Parameters

Note that

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$y'' + p(x)y' + q(x)y = r(x).$$

Now,

$$y' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2.$$

Let's also assume that

$$v_1' y_1 + v_2' y_2 = 0.$$

Thus,

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

Method of Variation of Parameters

Substituting y, y', y'' in the given non-homogeneous ODE, and rearranging terms, we get:

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = r(x).$$

Thus,

$$v_1'y_1' + v_2'y_2' = r(x).$$

Recall that we also have

$$v_1'y_1 + v_2'y_2 = 0.$$

Thus, we have:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Method of Variation of Parameters

Therefore,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \ v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)}.$$

Thus,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \ v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Hence,

$$\begin{aligned} y &= v_1 y_1 + v_2 y_2 \\ &= y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)} dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)} dx. \end{aligned}$$

Method of Variation of Parameters

Example: Find a particular solution of

$$y'' + y = \csc x.$$

Step I: Find a basis of solutions for the associated homogeneous equation

$$y'' + y = 0.$$

The general solution of this is

$$y(x) = c_1 \sin x + c_2 \cos x.$$

Step II: Calculate the Wronskian $W(y_1, y_2)$:

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Method of Variation of Parameters

Now,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)}\,dx = -\int \frac{\cos x \csc x}{-1}\,dx = \ln|\sin x|,$$

and

$$v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)}\,dx = \int \frac{\sin x \csc x}{-1}\,dx = -x.$$

Hence, a particular solution is given by

$$y(x) = \sin x \ln|\sin x| - x\cos x.$$

Method of Variation of Parameters

Example: Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

Recall that via the method of undetermined coefficients, you had to modify the proposed initial solution by multiplying it by t, and you got the answer as $\frac{3}{4}t\sin 2t$. Now in variation of parameters,

$$y_1 = \cos 2t,\ \ y_2 = \sin 2t,$$

and

$$v_1 = -\int \frac{\sin 2t \cdot 3\cos 2t}{2}\,dt = \frac{3}{16}\cos 4t,$$

$$v_2 = \int \frac{\cos 2t \cdot 3\cos 2t}{2}\,dt = \frac{3}{16}\sin 4t + \frac{3}{4}t.$$

Thus, a particular solution is

$$v_1y_1 + v_2y_2 = \frac{3}{16}\cos 2t + \frac{3}{4}t\sin 2t.$$

Method of Variation of Parameters

Example: Find a particular solution of

$$y'' + 3y' + 2y = \frac{1}{1+e^x}.$$

Step I: Find a basis of solutions for the associated homogeneous equation

$$y'' + 3y' + 2y = 0.$$

The general solution of the homogeneous DE is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}.$$

Step II: We look for a particular solution of the non-homogeneous DE

$$y(x) = v_1 e^{-x} + v_2 e^{-2x}.$$

where

$$v_1' e^{-x} + v_2' e^{-2x} = 0$$

$$-v_1' e^{-x} - 2v_2' e^{-2x} = \frac{1}{1+e^x}.$$

Solve for v_1 and v_2 ....

Method of particular integral

While the method of variation of parameters works in general non-homogeneous linear ODE, method of particular integral, like the method of undetermined coefficients, often found useful in computing 2nd order non-homogeneous linear ODE with constant coefficients.

Let us denote $D := \frac{d}{dx}$.

The differential equation $Dy = f$ has antiderivative of f as solution and defined as $D^{-1}f$ (formal definition will be done later in ODE course), often denoted by $\frac{1}{D}$ as well.

So what is $\frac{1}{D-m}f$?

It is a solution to the 1st order linear differential equation $\frac{dy}{dx} - my = f$ and we know that a solution is

$$\frac{1}{D-m}f = y(x) = e^{mx}\left(\int f(x)e^{-mx}dx\right).$$

Method of particular integral - ctd.

What about $\dfrac{1}{(D-m_1)(D-m_2)}f$?
It is a solution to the 2nd order linear differential equation $(D-m_1)(D-m_2)y = f$ (which is the general form of such type of equation when you allow m_1, m_2 to be complex as well). So we consider two linear equations, At first we solve

$$(D-m_1)u = f,$$

and for which the solutions is

$$u(x) = e^{m_1 x}(\int f(x)e^{-m_1 x}dx.$$

Then we solve for

$$(D-m_2)y = u,$$

and in this case the solution is

$$y = e^{m_2 x}(\int u(x)e^{-m_2 x}dx = e^{m_2 x}(\int e^{(m_1-m_2)x}(\int f(x)e^{-m_1 x}dx)dx$$

The method works even when m_1, m_2 are equal.

Method of particular integral - ctd.

Since $\frac{1}{(D-m_1)(D-m_2)}$ looks so much like an algebraic fraction, it is natural to ask whether one can, as in algebra, resolve it into partial fractions; that is, can we write for example, in the case where m_1, m_2 are distinct, the identity

$$\frac{1}{(D-m_1)(D-m_2)} = \frac{1}{m_1 - m_2}\left(\frac{1}{D-m_1} - \frac{1}{D-m_2}\right)?$$

This turns out to be correct and is easier than the interpretation since it involves only single integrations.

All these calculations involving D can be formalised and proved but are beyond the scope of this course.

Caution: Because of this example, the student must not get the impression that any manipulation of operators like the scalars can be done and will lead to profitable results.

We now give some examples on how to compute!

Example 1

Consider the ODE $(D^2 - 1)y = e^{-x}$.

We may write the equation as:

$$(D-1)(D+1)y = e^{-x} \Rightarrow y = \frac{1}{(D-1)(D+1)} e^{-x}$$

Using partial fractions:

$$\frac{1}{(D-1)(D+1)} = \frac{1}{2}\left(\frac{1}{D-1} - \frac{1}{D+1}\right)$$

Apply this to e^{-x}, we get the particular solution:

$$y = \frac{1}{2}\left[e^{x}\int e^{-x} \cdot e^{-x} dx - e^{-x}\int e^{-x} \cdot e^{x} dx\right] = -\frac{1}{2}xe^{-x} - \frac{1}{4}e^{-x}$$

So the solutions of the ODE are of the form:

$$Ae^{x} + Be^{-x} - \frac{1}{2}xe^{-x} - \frac{1}{4}e^{-x}$$

Example 2

Consider $(D^2 + 4D + 4)y = x^3 e^{-2x}$.

We write $(D+2)^2 y = x^3 e^{-2x}$.

Thus:

$$y = \frac{1}{(D+2)^2} x^3 e^{-2x} = \frac{1}{D+2} \left(\frac{1}{D+2} x^3 e^{-2x} \right)$$

Use:

$$\frac{1}{D+2} f(x) = e^{-2x} \int e^{2x} f(x) dx$$

to have

$$\frac{1}{D+2} x^3 e^{-2x} = e^{-2x} \int e^{2x} x^3 e^{-2x} \, dx = e^{-2x} \int x^3 \, dx = e^{-2x} \left(\frac{x^4}{4} \right)$$

Example 2 ctd.

Now apply $\dfrac{1}{D+2}$ again to get the particular solution:

$$\begin{aligned}\frac{1}{D+2}\left(e^{-2x}\cdot\frac{x^4}{4}\right) &= e^{-2x}\int e^{2x}\cdot\frac{x^4}{4}e^{-2x}\,dx = e^{-2x}\int\frac{x^4}{4}\,dx\\ &= e^{-2x}\cdot\frac{1}{4}\cdot\frac{x^5}{5}\\ &= \frac{1}{20}x^5e^{-2x}\end{aligned}$$

So the solutions of the ODE are of the form:

$$Ae^{-2x}+Bxe^{-2x}+\frac{1}{20}x^5e^{-2x}$$

Example 3

Consider the ODE: $\dfrac{d^2y}{dx^2} + y = \sec x$.

To find the particular integral, we see that

$$\frac{1}{D^2+1} = \frac{1}{2i}\{\frac{1}{D-i} - \frac{1}{D+i}\}$$

Check that

$$\begin{aligned}\frac{1}{D-i}\sec x &= e^{ix}\int e^{-ix}\sec(x)\ dx\\ &= (x\cos x - \sin x\ \ \log\cos x) + i(x\sin x + \cos x\log\cos x).\end{aligned}$$

Example 3 ctd.

Similarly check that

$$\frac{1}{D+i}\sec x = (x\cos x - \sin x\ \log\cos x) + i(x\sin x + \cos x\log\cos x).$$

So the particular integral will be:

$$x\sin x + \cos x\log\cos x$$

and hence all the solutions are of the form:

$$A\sin x + B\cos x + x\sin x + \cos x\log\cos x.$$

Thank you!