Lecture : Mar 26, 2025.

$T : \underset{n\text{-dim}}{V} \longrightarrow \underset{m\text{-dim}^n}{W}$

$\{v_1, v_2 \dots v_n\}$ basis of V

pick any vector $u_1 \dots u_n \in W$
& define
$Tv_i = u_i$

Note u_i could be any thing in W. It may happen $u_i = u_0 \ \forall i$ ↳ fixed vector. ⊛

Then $Tv = \alpha_1 u_1 + \dots + \alpha_n u_n$ — gives the linear map.
where $v = \alpha_1 v_1 + \dots + \alpha_n v_n$.

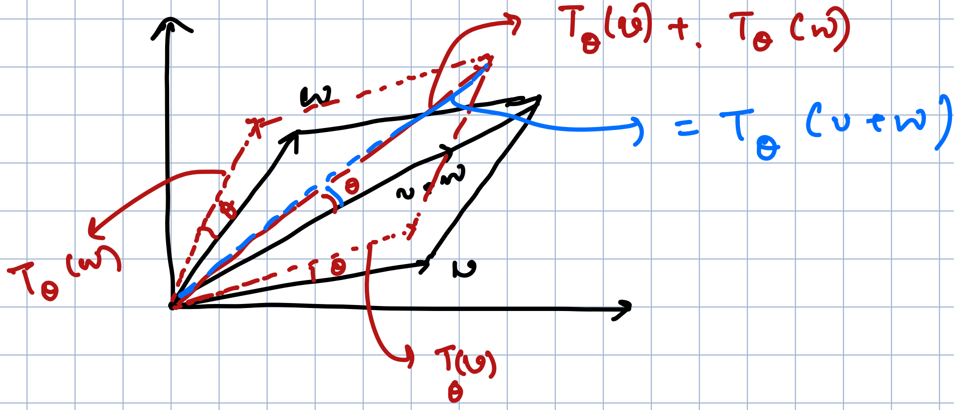
⊛ in that case $Tv = (\alpha_1 + \dots + \alpha_n) u_0$.

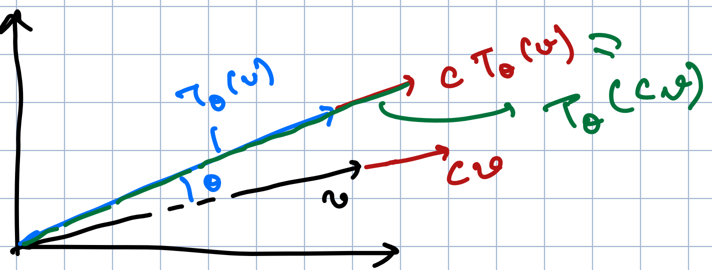
(This is different from the map
$T : V \xrightarrow[\text{lin.}]{} W$ s.t $T(v) = u_0 \ \forall v \in V$.).

We come back to the linear map - rotation by $90°$
$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

s.t $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \doteq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

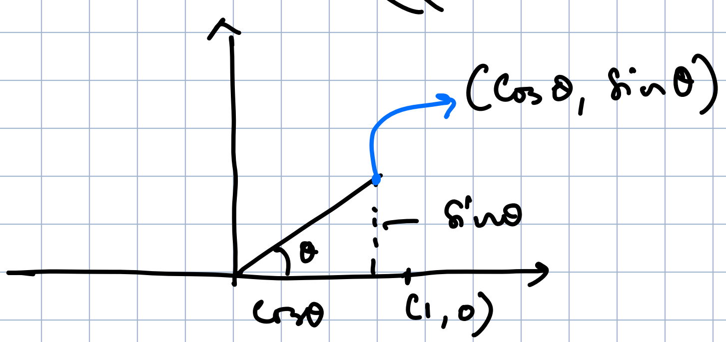
In fact, one can deal with linear map - rotation by angle θ.
$T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



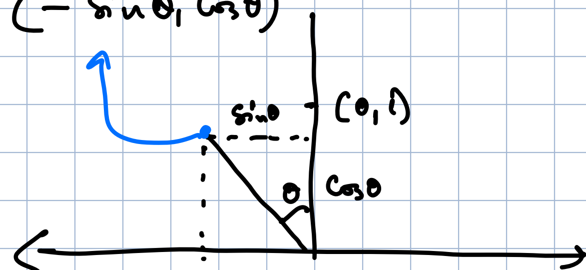


This geometrically gives linearity of the map (function) T_θ.

$T_\theta(1,0)$ $\parallel$



$T_\theta(0,1)$
$\parallel$
$(-\sin\theta, \cos\theta)$



$$
\begin{aligned}
T_\theta(x,y) &= T_\theta\big(x(1,0) + y(0,1)\big) \\
&= x\,T_\theta(1,0) + y\,T_\theta(0,1) \\
&= x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta) \\
&= (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)
\end{aligned}
$$

If we write everything as column vectors, we have

$$T_\theta\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \qquad T_\theta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

$$T_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} | & | \\ T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ | & | \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$T_\theta(e_1) = \underbrace{(\cos\theta)}_{a_{11}} e_1 + \underbrace{(\sin\theta)}_{a_{21}} e_2$$

$$T_\theta(e_2) = \underbrace{(-\sin\theta)}_{a_{12}} e_1 + \underbrace{(\cos\theta)}_{a_{22}} e_2$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$ — The matrix we obtained.

Consider another example:

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

$$= \begin{array}{c c} & \overset{a_{11}}{} \\ \begin{array}{c} \\ a_{21} \\ a_{31} \\ a_{41} \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ a_2 \\ x_3 \end{pmatrix}$$

$\mathcal{B} = \{e_1, e_2, e_3\}$ Std basis of $\mathbb{R}^3$ (ordered)

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\mathcal{B}' = \{f_1, f_2, f_3, f_4\}$ Std basis of $\mathbb{R}^4$ (ordered)

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \cdots \quad f_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \underset{a_{11}}{\underset{\parallel}{1}} \cdot f_1 + \underset{a_{21}}{\underset{\parallel}{0}} \cdot f_2 + \underset{a_{31}}{\underset{\parallel}{0}} \cdot f_3 + \underset{a_{41}}{\underset{\parallel}{1}} \cdot f_4$$

$$T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \underset{a_{12}}{\underset{\parallel}{0}} \cdot f_1 + \underset{a_{22}}{\underset{\parallel}{1}} \cdot f_2 + \underset{a_{32}}{\underset{\parallel}{0}} \cdot f_3 + \underset{a_{42}}{\underset{\parallel}{1}} \cdot f_4$$

$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \underset{a_{13}}{\underset{\parallel}{0}} \cdot f_1 + \underset{a_{23}}{\underset{\parallel}{0}} \cdot f_2 + \underset{a_{33}}{\underset{\parallel}{1}} \cdot f_3 + \underset{a_{43}}{\underset{\parallel}{1}} \cdot f_4$$

For the linear map given by matrix multiplication

that is,

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$T_A(x) = Ax. \qquad A = \begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix}$$

We observe that if

$\{e_1 \dots e_n\}$ std basis (ordered) of $\mathbb{R}^n$.

& $\{f_1 \dots f_m\}$ " of $\mathbb{R}^m$,

Then the first column

$$(T_A e_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \text{ — 1st column of } A$$

can be thought as a co.efficient corr. to rep^n. of $\mathbb{R}^3$.

$$= a_{11} f_1 + \dots + a_{m1} f_m$$

IIIly the jth column

$$\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} = A e_j = T_A(e_j)$$

So $\{T(e_1), \dots T(e_j), \dots T(e_n)\}$ gives the columns of A.

We mimic these examples and show how we can associate a matrix to a linear map T.

Let V be an n-dimensional vector space

& $\mathcal{B} = \{v_1, \dots v_n\}$ – be an ordered basis of V.

'ordered' means if you change the order of the vectors, it will represent a different basis.

For example the std. basis $\mathcal{E} = \{e_1, e_2\}$ of $\mathbb{R}^2$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is different from the basis $\{e_2, e_1\}$.

Given any $v \in V$, $\exists\, \alpha_1, \alpha_2 \dots \alpha_n \in \mathbb{F}$ s.t

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

Not^n: We write $[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$.

Ex: 1) $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ $\quad \mathcal{E} = \{e_1, e_2\}$ std. basis

$\mathcal{B} = \{(1,0), (1,1)\}$ another basis

& $\mathcal{E}' = \{e_2, e_1\}$.

Note $\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{E}} = \begin{pmatrix} x \\ y \end{pmatrix}$.

as $\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x e_1 + y e_2$.

However $\begin{pmatrix} x \\ y \end{pmatrix} = y e_2 + x e_1$

$\Rightarrow \left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{E}'} = \begin{pmatrix} y \\ x \end{pmatrix}$.

So $\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{E}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{E}'} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$\begin{pmatrix} x \\ y \end{pmatrix} = (x-y)(1,0) + y(1,1)$

$\Rightarrow \left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} x-y \\ y \end{pmatrix}$.

Remark: Note that if we take $v \in \mathbb{R}^n$

(A) $\mathcal{E} = \{e_1 \dots e_n\}$ denotes the std. basis with usual order

then any $v \in \mathbb{R}^n$

$$v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 e_1 + \dots + \alpha_n e_n$$

So $[v]_{\mathcal{E}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

Thus for $v \in \mathbb{R}^n$, $v = [v]_{\mathcal{E}}$.

Now we take linear map $T : V \longrightarrow W$

where V — n-dimensional

W — m-dimensional.

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be an ordered basis of V

& $\mathcal{B}' = \{w_1, \dots, w_m\}$ be an ordered basis of W.

Apply T to the first basis vector v_1,

$Tv_1 \in W$ & $\{w_1 \dots w_m\}$ basis of W

$\Rightarrow \exists$ scalars $a_{11}, \dots a_{m1} \in \mathbb{F}$ s.t

$$Tv_1 = a_{11} w_1 + \dots + a_{m1} w_m.$$

Thus $[Tv_1]_{\mathcal{B}'} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$

IIly

$$Tv_j = a_{1j} w_1 + \cdots + a_{mj} w_m \Rightarrow [Tv_j]_{\mathcal{B}} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

↓

applying to j^{th} basis vector

$$Tv_n = a_{1n} w_1 + \cdots + a_{mn} w_m \Rightarrow$$

& IIly $[Tv_n] = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$ for some $a_{ij} \in \mathbb{R} / \mathbb{F}$.

Define

$$A := \begin{pmatrix} | & | & & | \\ [Tv_1]_{\mathcal{B}'} & [Tv_2]_{\mathcal{B}'} & \cdots & [Tv_n]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$ $m \times n$ – matrix.

$$= [T]^{\mathcal{B}}_{\mathcal{B}'}$$ – notation

Some examples again:

(1) $T(x, y) = (-y, x)$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\mathbb{R}^2$ ‖ Std. Basis $\mathcal{E} = \{e_1, e_2\}$

$\mathbb{R}^2 = \mathcal{E} = \{e_1, e_2\}$

↳ mentioning this means with usual order OR the order it is getting stated.

$Te_1 = (0, 1)$

$= e_2$

$= 0 \cdot e_1 + 1 \cdot e_2$

$Te_2 = -e_1 = -1 \cdot e_1 + 0 \cdot e_2$.

So $[T]_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

maintaining the order

Now suppose we change the codomain basis to $\mathcal{E}' = \{e_2, e_1\}$ just changing the order

Then the same T will yield

$Te_1 = e_2 = 1 \cdot e_2 + 0 \cdot e_1$

$Te_2 = -e_1 = 0 \cdot e_2 + (-1) \cdot e_1$

So $[T]_{\mathcal{E}'}^{\mathcal{E}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

what abt $[T]^{\mathcal{E}'}_{\mathcal{E}'}$?

This is now the first vector in the basis

$$Te_2 = -e_1 = 0 \cdot e_2 + (-1)\, e_1$$

$$Te_1 = e_2 = 1 \cdot e_2 + 0 \cdot e_1$$

$$[T]^{\mathcal{E}'}_{\mathcal{E}'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In general, let $T : V \longrightarrow W$ be a linear map

- let $\dim V = n$ & $\mathcal{B} = \{v_1, \dots v_n\}$ be a basis of V.
- let $\dim W = n$ & $\mathcal{B}' = \{w_1, \dots, w_n\}$ be a basis of W.

now take a vector $v \in V$

Since $\mathcal{B}$ is a basis, $\exists\, \alpha_1, \dots \alpha_n \in \mathbb{F}/\mathbb{R}$ such that $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

that is, $[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

So $Tv = \alpha_1 Tv_1 + \cdots + \alpha_n Tv_n$.

Now if $[T]^{\mathcal{B}}_{\mathcal{B}'} = A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

by definition of $[T]_{\mathcal{B}'}^{\mathcal{B}}$

Then $T v_1 = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$

$T v_2 = a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m$

$\vdots$

$T v_n = a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m$

So $$\begin{aligned} T v = {} & \alpha_1 (a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m) \\ & + \alpha_2 (a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m) \\ & \vdots \\ & + \alpha_n (a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m) \end{aligned}$$

$$\begin{aligned} = {} & (\alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_n a_{1n}) w_1 \\ & + (\alpha_1 a_{21} + \alpha_2 a_{22} + \cdots + \alpha_n a_{2n}) w_2 \\ & \vdots \\ & + (\alpha_1 a_{m1} + \alpha_2 a_{m2} + \cdots + \alpha_n a_{mn}) w_m \end{aligned}$$

$$\begin{aligned} = {} & (a_{11} \alpha_1 + a_{12} \alpha_2 + \cdots + a_{1n} \alpha_n) w_1 \\ & + (a_{21} \alpha_1 + a_{22} \alpha_2 + \cdots + a_{2n} \alpha_n) w_2 \\ & \vdots \\ & + (a_{m1} \alpha_1 + a_{m2} \alpha_2 + \cdots + a_{mn} \alpha_n) w_m \end{aligned}$$

So by notn

$$[Tv]_{\mathcal{B}'} = \begin{pmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n \\ \vdots \\ a_{m1}\alpha_1 + a_{m2}\alpha_2 + \cdots + a_{mn}\alpha_n \end{pmatrix}$$

$$= A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = [T]^{\mathcal{B}}_{\mathcal{B}'} [v]_{\mathcal{B}}$$

$$\Rightarrow \boxed{[Tv]_{\mathcal{B}'} = [T]^{\mathcal{B}}_{\mathcal{B}'} [v]_{\mathcal{B}}}$$

The highlights some way justifies the notn that $\mathcal{B}$ - $\mathcal{B}$ gets cancelled & $\mathcal{B}'$ remains on RHS

$\mathcal{E}$ – Std. basis of $\mathbb{R}^2$

$\mathcal{E}'$ – std. basis of $\mathbb{R}^3$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = [T]^{\mathcal{E}}_{\mathcal{E}'}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{E}}$$

② $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\left[T\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{E}'} = T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ x+y \\ x-y \end{pmatrix}$$

Co-ordinates usually → understood w.r.t std basis

So we have $\left[T\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{E}'} = [T]^{\mathcal{E}}_{\mathcal{E}'} \left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{E}}$ for std. bases.

However, we would like to find out $[T]^{\mathcal{B}}_{\mathcal{B}'}$

where

$\mathcal{B} = \{ \underbrace{(1, 1)}_{v_1}, \underbrace{(1, -1)}_{v_2} \}$ basis of $\mathbb{R}^2$

& $\mathcal{B}' = \{ \underbrace{(1, 0, 0)}_{w_1}, \underbrace{(1, 1, 0)}_{w_2}, \underbrace{(1, 1, 1)}_{w_3} \}$ basis of $\mathbb{R}^3$.

$$T\begin{pmatrix}1\\1\end{pmatrix}_{v_1} = \begin{pmatrix}2\\2\\0\end{pmatrix} = \underset{a_{11}}{0}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}_{w_1} + \underset{a_{21}}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}_{w_2} + \underset{a_{31}}{0}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}_{w_3}$$

$$T\begin{pmatrix}1\\-1\end{pmatrix}_{v_2} = \begin{pmatrix}2\\0\\2\end{pmatrix} = \underset{a_{12}}{2}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}_{w_1} + \underset{a_{22}}{(-2)}\begin{pmatrix}1\\1\\0\end{pmatrix}_{w_2} + \underset{a_{32}}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}_{w_3}$$

So $$[T]^{\mathcal{B}}_{\mathcal{B}'} = \begin{pmatrix}0 & 2\\2 & -2\\0 & 2\end{pmatrix}.$$

Verification of the fact $[Tv]_{\mathcal{B}'} = [T]^{\mathcal{B}}_{\mathcal{B}'}[v]$ in this example:

Any $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$ $\quad \begin{pmatrix}x\\y\end{pmatrix} = \frac{x+y}{2}(1,1)$

$v = \alpha_1 v_1 \dots + \alpha_n v_n$, $\mathcal{B} = \{v_1 \dots v_n\}$ basis $\qquad + \frac{x-y}{2}(1,-1)$

$[v]_{\mathcal{B}} = \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$

$$\Rightarrow \quad \left[\begin{pmatrix}x\\y\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}\frac{x+y}{2}\\ \frac{x-y}{2}\end{pmatrix}$$

$$\left[T\begin{pmatrix}x\\y\end{pmatrix}\right]_{\mathcal{B}'} = \left[\begin{pmatrix}2x\\x+y\\x-y\end{pmatrix}\right]_{\mathcal{B}'}$$

Recall

$$\left[\begin{pmatrix}a\\b\\c\end{pmatrix}\right]_{\mathcal{B}'} = \begin{pmatrix}a-b\\b-c\\c\end{pmatrix} \Bigg| \; \underline{(a,b,c)} = (a-b)(1,0,0) + (b-c)(1,1,0)$$

$$+ c\,(1,1,1)$$

$$\left[\begin{pmatrix} 2x \\ x+y \\ x-y \end{pmatrix}\right]_{\beta'} = \begin{pmatrix} 2x - (x+y) \\ x+y-(x-y) \\ x-y \end{pmatrix} = \begin{pmatrix} x-y \\ 2y \\ x-y \end{pmatrix}.$$

$$[T]^{\beta}_{\beta'} \left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\beta} = \begin{pmatrix} 0 & 2 \\ 2 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{pmatrix}$$

$$= \begin{pmatrix} x-y \\ 2y \\ x-y \end{pmatrix} = \left[T\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\beta'}$$

<u>Remark:</u>

$$T : \overset{\mathcal{E}}{\mathbb{R}^n} \longrightarrow \overset{\mathcal{E}'}{\mathbb{R}^m}$$

linear.

Then $T = T_A$ for some $m \times n$ matrix A

$: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$v \longmapsto Av$

i.e $T_A v = Av$.

$$v = [v]_{\mathcal{E}}$$

$$[Tv]_{\mathcal{E}'} = Tv.$$

$$[Tv]_{\mathcal{E}'} = [T]^{\mathcal{E}}_{\mathcal{E}'} [v]_{\mathcal{E}}$$

$$\Rightarrow \quad Tv = \underset{\substack{\parallel \\ A}}{[T]^{\mathcal{E}}_{\mathcal{E}'}} v$$

$$= Av.$$