# Lecture Mar 20, 2025

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{2\times 3}$

Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

$$Av = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ dx + ey + fz \end{pmatrix} \in \mathbb{R}^2.$$

Clearly $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ - is a fn $v \longmapsto Av$.

In general, if $A \in M_{m\times n}(\mathbb{R})$
↪ $m \times n$ matrix

then it can be viewed as a function from
$\underline{\mathbb{R}^n \text{ to } \mathbb{R}^m}$.

<u>Note</u>: The following properties hold true for these fn s :

$$\begin{cases} A(v+w) = Av + Aw \\ A(\alpha v) = \alpha Av \quad \forall \alpha \in \mathbb{R}. \end{cases}$$

↪ that is, A is preserving the vector space structure (<u>the linearity</u>).

**Defⁿ (Linear transformation / linear map).**

Let $V$ and $W$ be vector spaces over $\mathbb{F}$.

A function $T : V \longrightarrow W$ is said to be a linear transformation or a linear map if

i) $T(v_1 + v_2) = Tv_1 + Tv_2 \quad \forall\, v_1, v_2 \in V.$

ii) $T(\alpha v) = \alpha\, Tv\,, \quad \forall\, v \in V,\ \alpha \in \mathbb{F}.$

Examples:

1) $A$ - $m \times n$ - matrices

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t. $v \longmapsto Av$

— linear map.

2) The identity fn $Id : V \longrightarrow V$ defined by $Id(v) = v \quad \forall\, v \in V$

— linear map.

$$Id(v_1 + v_2) = v_1 + v_2 = Id(v_1) + Id(v_2)$$

$$Id(\alpha v) = \alpha v = \alpha\, Id(v).$$

3) The zero map $T : V \longrightarrow W$ defined by

$$Tv = 0 \quad \forall\, v \in V.$$

is a linear map.

$$T(v + w) = 0 = 0 + 0 = Tv + Tw$$

$$T(\alpha v) = 0 = \alpha . 0 = \alpha\, Tv.$$

**Note** any other non-zero const. map is not a linear map!

Let $w_0 \in W$ be a fix nonzero vector &

let $T : V \to W$ defined by

$$T v = w_0 \quad \forall\, v \in V.$$

— NOT A LINEAR MAP.

$T(v_1 + v_2) = w_0$

$Tv_1 + Tv_2 = 2w_0$

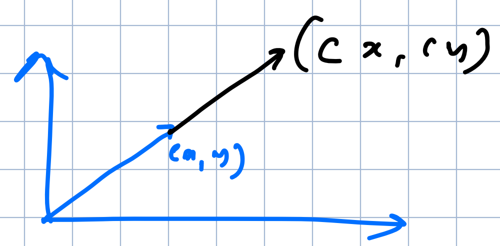
$w_0 \neq 2w_0$ as $w_0 \neq 0$.

4) $T : V \to V$, $Tv = cv$, $c \in \mathbb{F}$.

$$T(v_1 + v_2) = c(v_1 + v_2) = cv_1 + cv_2 = Tv_1 + Tv_2.$$

$$T(\alpha v) = c(\alpha v) = (c\alpha) v = (\alpha c) v = \alpha(cv)$$
$$= \alpha T v.$$

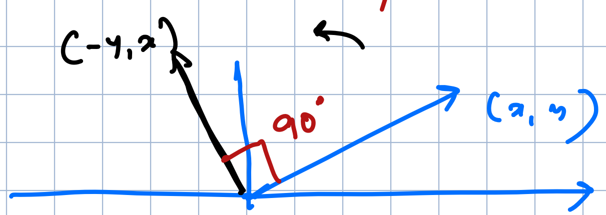
$V = \mathbb{R}^2$ (say) $\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$




This linear map which is constant multiple of identity, — stretches every vector by the same factor $c$.

## 5) Rotation :

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$




Rotating each vector anti clockwise by 90°.

$$(x, y) = (r\cos\theta, r\sin\theta)$$

$$\left( r\cos(\theta + 90^\circ), r\sin(\theta + 90^\circ) \right)$$

$$\|$$

$$(-r\sin\theta, r\cos\theta) = (-y, x).$$

$$T\left\{ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\} = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} -(y_1 + y_2) \\ x_1 + x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -y_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} -y_2 \\ x_2 \end{pmatrix}$$

$$= T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

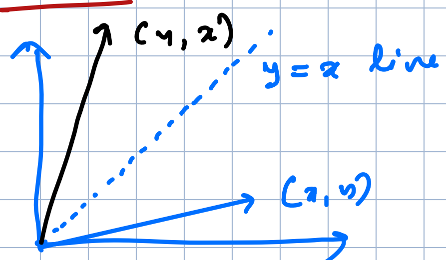
$$T\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} = \begin{pmatrix} -\alpha y \\ \alpha x \end{pmatrix} = \alpha \begin{pmatrix} -y \\ x \end{pmatrix} = \alpha\, T\begin{pmatrix} x \\ y \end{pmatrix}.$$

— T is linear.

## 6) Reflection :

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Check: T is linear.

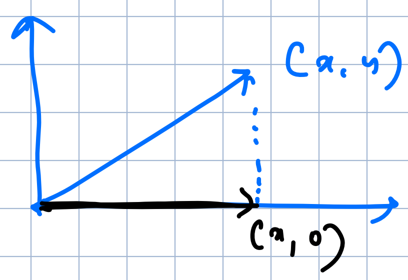



Reflection along the line $y = x$.

7) Projection:

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

Check: T is linear.




Projection onto x-axis.

Note: some of these operators can be defined in much more generalities – you will see this later in linear algebra I.

8) Consider $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = (x_2, x_1, x_1 + x_3).$$

— T is a linear map.

• $T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big)$

$= T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$= (x_2 + y_2, x_1 + y_1, x_1 + y_1 + x_3 + y_3)$

$= (x_2, x_1, x_1 + x_3) + (y_2, y_1, y_1 + y_3)$

$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3).$

• $T(\alpha(x_1, x_2, x_3))$

$= T(\alpha x_1, \alpha x_2, \alpha x_3)$

$= (\alpha x_2, \alpha x_1, \alpha x_1 + \alpha x_3)$

$= \alpha (x_2, x_1, x_1 + x_3)$

$= \alpha T(x_1, x_2, x_3)$.

9) $\underbrace{\mathcal{P}_k(\mathbb{R})}_{\text{vector sp of dim}^{n} (k+1)} :=$ set of all polynomials of degree $\leq k$ with $\underset{\mathbb{R}}{\underline{\text{real}}}$ co-efficients

$= \{a_0 + a_1 x + \cdots + a_k x^k : a_j \in \mathbb{R}, 0 \leq j \leq k\}$.

$D : \mathcal{P}_k(\mathbb{R}) \longrightarrow \mathcal{P}_k(\mathbb{R})$.

$Df = f'$ — Differential operator.

$\longrightarrow$ linear map. (Check!)

10) $I : \mathcal{P}_k(\mathbb{R}) \longrightarrow \mathcal{P}_{k+1}(\mathbb{R})$

$$If(x) = a_0 x + \frac{a_1}{2} x^2 + \cdots + a_k \frac{x^{k+1}}{k+1}$$

<u>Check</u>: $I$ - linear map.

— Integral operator.

# Properties of linear maps:

$T : V \longrightarrow W$ - a linear map.

Then

i) $T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T v_1 + \alpha_2 T v_2$

$\forall\, \alpha_1, \alpha_2 \in \mathbb{F}$,

$v_1, v_2 \in V$.

ii) $\underbrace{T(0)}_{\substack{\text{zero vector} \\ \text{in } V}} = 0$ → zero vector in $W$

Pf: i) $T(\alpha_1 v_1 + \alpha_2 v_2)$

$= T(\alpha_1 v_1) + T(\alpha_2 v_2)$

$= \alpha_1 T v_1 + \alpha_2 T v_2$.

ii) $T(0) = T(\underbrace{0}_{\text{scalar}} . \underbrace{0}_{\substack{\text{zero vector} \\ \text{in } V}})$

$= \underbrace{0}_{\mathbb{F}} . T(\underbrace{0}_{\text{zero vector in } V}) = 0 \in W$.

Alternatively,

$T(0) = T(0+0) = T(0) + T(0)$.

$\Rightarrow T(0) = 0$.

Note $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

So it seems linear maps are again can be viewed as matrix

— we explore this in next class.