Problem Set - 09

MA 1201 Spring Sem, 2025

Linear Transformation and Matrix Representation

1. In this exercise, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is *not* linear.

   (a) $T(a_1, a_2) = (1, a_2)$

   (b) $T(a_1, a_2) = (a_1, a_2^2)$

   (c) $T(a_1, a_2) = (\sin a_1, 0)$

   (d) $T(a_1, a_2) = (|a_1|, a_2)$

   (e) $T(a_1, a_2) = (a_1 + 1, a_2)$

For Exercises 2 through 6, find the matrix representation of the linear map with respect to standard basis in each case. Note that the matrices E_{ij}- with ij-th entry 1 and all other entries 0, $1 \leq i \leq m, 1 \leq j \leq n$, forms the standard basis of $M_{m\times n}(\mathbb{F})$ and $\{1, x, \ldots, x^k\}$ forms the standard basis of $P_k(\mathbb{R})$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by
$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

3. $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by
$$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2).$$

4. $T : M_{2\times 3}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by
$$T\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\right) = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

5. $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by
$$T(f(x)) = xf(x) + f'(x).$$

6. $T : M_{n\times n}(F) \to F$ defined by $T(A) = \mathrm{tr}(A)$ (called *Trace* of A), where
$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

7. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1, 0) = (1, 4)$, and $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Is T one-to-one?

8. Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. What is $T(8, 11)$?

9. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$

(c) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p'(x)$

(d) $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + 2bx + (c+d)x^2$$

(e) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$$

10. Which of the following pairs of vector spaces are isomorphic? Justify your answers.

(a) $\mathbb{F}^3$ and $P_3(\mathbb{F})$

(b) $\mathbb{F}^4$ and $P_3(\mathbb{F})$

(c) $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$

(d) $V = \{A \in M_{2\times 2}(\mathbb{R}) : \mathrm{tr}(A) = 0\}$ and $\mathbb{R}^4$

11. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \quad \text{and} \quad U(a + bx + cx^2) = (a + b, c, a - b)$$

Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(a) Compute $[U]_\gamma^\beta$, $[T]_\beta^\beta$, and $[UT]_\gamma^\beta$ directly and verify that matrix representation of the composition is matrix multiplication of the individual matrix in the same order.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h]_\beta$ and $[U(h)]_\gamma$. Then verify $[U(h)]_\gamma = [U]_\gamma^\beta [h]_\beta$.

12. Find linear transformations $U, T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $UT = 0$ (the zero transformation) but $TU \neq 0$.

13. Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix},$$

let β be the standard ordered basis for $\mathbb{R}^2$, and let

$$\beta' = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}.$$

Use the change of basis matrix to find $[T]_{\beta'}^{\beta'}$.

14. Let T be the linear operator on $P_1(\mathbb{R})$ defined by $T(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1+x, 1-x\}$. Use the change of basis matrix to find $[T]_{\beta'}^{\beta'}$ and verify by computing independently.

15. If the transformation T is a reflection across the 45° line in the plane, find its matrix with respect to the standard basis $w_1 = (1, 3)$, $w_2 = (2, -1)$, and also with respect to $v_1 = (1, 2)$, $v_2 = (3, -1)$. Show also that those matrices are similar and find the matrix which that gives the similarity.

Determinant

16. Let Q be an orthogonal $n \times n$ matrix, that is, $Q^{\mathrm{T}}Q = I$. Prove that $\det Q$ equals $+1$ or -1.

17. Show that
$$\det \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} = -(a^3 + b^3 + c^3 - 3abc).$$

18. Without expanding the determinant prove that
$$\det \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} = 0.$$

19. Use row operations to verify that the 3 by 3 "Vandermonde determinant" is
$$\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (b-a)(c-a)(c-b).$$

20. Use the problem above to show that if $s_r = a^r + b^r + c^r$, then
$$\det \begin{pmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix} = (b-a)^2(c-a)^2(c-b)^2.$$

21. True or false, with reason if true and counterexample if false:
    (a) If A and B are identical except that $b_{11} = 2a_{11}$, then $\det B = 2 \det A$.
    (b) The determinant is the product of the pivots.
    (c) If A is invertible and B is singular (*not invertible*), then $A + B$ is invertible.
    (d) If A is invertible and B is singular, then AB is singular.
    (e) The determinant of $AB - BA$ is zero.

22. (a) If every row of A adds to zero, prove that $\det A = 0$.
    (b) If every row of A adds to 1, prove that $\det(A - I) = 0$. Show by example that this does not imply $\det A = 1$.

23. Find these 4 by 4 determinants by Gaussian elimination:
$$\det \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix} \quad \text{and} \quad \det \begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{pmatrix}.$$

24. Given $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$, find the determinants of $A - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$, for every $\lambda \in \mathbb{R}$.

    For which values of λ is $A - \lambda I$ a singular matrix (*not invertible*)?

25. Let $A = (a_{ij}) \in M_n(\mathbb{R})$.

    (a) For any $n \geq 3$, if $a_{ij} = i + j$, show that $\det A = 0$.

    (b) For any $n \geq 2$, if $a_{ij} = ij$, show that $\det A = 0$.

26. If $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$, find A^{-1}.

27. Find the matrix A, if $A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ 1 & -2 & 3 \\ 3 & -3 & 4 \end{pmatrix}$.

28. Solve the following system of equations by Cramer's rule:

$$\begin{aligned} x + 2y - 3z &= 1 \\ 2x - y + z &= 4 \\ x + 3y &= 5. \end{aligned}$$

29. Solve by Cramer's rule:

$$\text{(i)} \quad \begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 14 \\ x - y + z &= 2, \end{aligned} \qquad\qquad \text{(ii)} \quad \begin{aligned} x + y + z &= 1 \\ ax + by + cz &= 1 \\ a^2x + b^2y + c^2z &= 1, \quad a \neq b \neq c. \end{aligned}$$

Eigen-values and Eigen-vectors

30. Find the rank and all four eigenvalues for the checker board matrix $C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. Which eigenvectors correspond to nonzero eigenvalues?

31. (a) Construct 2 by 2 matrices A and B such that the eigenvalues of AB are not the products of the eigenvalues of A and B, and the eigenvalues of $A+B$ are not the sums of the individual eigenvalues.

    (b) Construct 3 by 3 matrices A and B with same properties as in part (a).

32. Prove that the real eigenvalues of A equal the eigenvalues of A^{T}. Show by an example that the eigenvectors of A and A^{T} need not be the same.

33. Every permutation matrix leaves $x = (1, 1, \dots, 1)$ unchanged. Thus, $\lambda = 1$ is an eigenvalue for every permutation matrix. Find two more eigenvalues for the following permutations:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

34. Find the 2 by 2 matrix A whose eigenvalues are 1 and 4, with the corresponding eigenvectors being $(3, 1)$ and $(2, 1)$ respectively.

35. Which of these matrices cannot be diagonalized? Recall that diagonalizable means that there exists a basis consisting of eigen vectors.

$$A_1 = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 \\ 2 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}.$$

36. Describe all matrices S that diagonalize the matrix $A = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix}$. Then describe all matrices that diagonalize A^{-1}.

37. If the eigenvalues of a 3 by 3 matrix A are $1, 1, 2$, which of the following are certain to be true? Give a reason if true or a counterexample if false:

    (a) A is invertible.

    (b) A is diagonalizable.

    (c) A is not diagonalizable.

38. Suppose the only eigenvectors of a 3 by 3 matrix A are multiples of $(1, 0, 0)$. True or false:

    (a) A is not invertible.

    (b) A has a repeated eigenvalue.

    (c) A is not diagonalizable.

39. Diagonalize $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, that is, find a matrix S such that $S^{-1}AS = D$ is a diagonal matrix. Use it to prove that for any $k \in \mathbb{N}$, we have that $A^k = \frac{1}{2} \begin{pmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{pmatrix}$.

40. Given $A \in M_n(\mathbb{R})$, explain why A is never similar to $A + I$.

41. Let $A, B \in M_n(\mathbb{R})$. If A or B is invertible then show that BA is similar to AB. Give an example of matrices A and B such BA is not similar to AB.

42. Let $A, B \in M_n(\mathbb{R})$. Prove that AB has the same eigenvalues as BA.