Lecture - 03/04/2025

THM: Let $\delta : M_n(\mathbb{F}) \longrightarrow \mathbb{F}$ be a fn such that

1) $\delta(I_n) = 1$,
2) $\delta(B) = -\delta(A)$, if B is obtained from A by interchanging two rows,
3) δ is linear on each row, while keeping the other rows fixed.

Suppose $\gamma : M_n(\mathbb{F}) \longrightarrow \mathbb{F}$ be another such fn. Then $\gamma = \delta$.

Note linearity on each row means,

Fix rows $R_1, R_2, \dots R_{i-1}, R_{i+1}, \dots R_n$.

define

$$\delta_{R_1, R_2, \dots R_{i-1}, R_{i+1} \dots R_n}(R) = \delta \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_{i-1} \\ R \\ R_{i+1} \\ \vdots \\ R_n \end{pmatrix}$$

Then $\delta_{R_1, R_2, \dots R_{i-1}, R_{i+1}, \dots R_n}(\alpha R + \beta R')$

$$= \alpha\, \delta_{R_1, R_2, \dots R_{i-1}, R_{i+1}, \dots R_n}(R) + \beta\, \delta_{R_1, R_2, \dots R_{i-1}, R_{i+1} \dots R_n}(R')$$

Remark:

The theorem above says that if such a fn. exists, then it must be unique.

Defn: Such a fn. evaluated at each matrix $A \in M_n(\mathbb{R})$ is called DETERMINANT of A.

The question is why such a fn. exists and what is the process to compute it corresponding to a matrix A.

Our approach:

We propose a fn. & show that it satisfies all the three properties.

- We shall show that the proposed fn. (due to LAPLACE) useful in lifting the properties of determinant from 2×2 to arbitrary $n \times n$ - matrix.

- To show that it satisfies the three properties, the idea is to show the proposed fn. is equal another fn. (due to Leibniz). which in turn satisfies three properties.
  — This part we will not give any proof.

Let us examine 2×2-matrix determinant a bit further

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$= a(d) + b(-c)$$

try to figure out how we arrive at it!

expanding along 1st row

11th entry

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

cut the row containing a & column containing a & what remains is d.

12th entry

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

cut the row & column containing b
— what remains is c.

Now we need to take account of the sign!

First we note the process above can be done not just breaking along one row!

Expanding along 2nd row

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

21th - entry

Sign $(-1)^{2+1}$

$$= c(-b) + d(a)$$

$= -$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

sign $(-1)^{2+2}$

$+$

22th entry.

How about columns?

Expanding along 1st column

11th entry

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A) = a(d) + c(-b)$$

sign $(-1)^{1+1}$ $= +$

sign $(-1)^{2+1}$ $= -$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

21 th entry.

Expanding along 2nd column

12th entry

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A) = b(-c) + d(a).$$

Sign $(-1)^{1+2}$ $= -$

sign $(-1)^{2+2}$ $+$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let us look at 3×3 matrix (Expansion along 1st row)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$a_{11} \rightarrow M_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

– 1st term

$$a_{11} \cdot (-1)^{1+1} \det M_{11}.$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$a_{12} \rightarrow M_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

– 2nd term

$$a_{12} \cdot (-1)^{1+2} \det M_{12}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$a_{13} \rightarrow M_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

– 3rd term

$a_{13} \cdot (-1)^{1+3} \det M_{13}$.

So proposed determinant for 3x3 matrix A

$$= a_{11} (-1)^{1+1} \det M_{11} + a_{12} \cdot (-1)^{1+2} \det M_{12} + a_{13} \cdot (-1)^{1+3} \det M_{13}$$

$$= a_{11} \cdot (a_{22} a_{33} - a_{23} a_{32}) - a_{12} (a_{21} a_{33} - a_{23} a_{31}) + a_{13} (a_{21} a_{32} - a_{22} a_{31}) \quad - (*)$$

as mentioned earlier based on this expansion we will define another fn. satisfying conditions 1), 2) & 3) and equal to the proposed fn. above, and as a result the above proposed fn. gives the determinant of A.

↳ a formula for determinant of A.

The formula above gives a recursive way to determine the determinant

– i.e to know determinant of n×n matrix, we need to know determinant of (n-1) × (n-1).

A — $n \times n$ matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ & & & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

$M_{12} \to 12^{th}$ - minor

$$A_{12} = (-1)^{1+2} \det M_{12}$$

12^{th} - cofactor.

Proposed formula for $\det A$

$$\det A = a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n}.$$

— by expanding along first row.

but why first row ??

you can expand along any row —

expansion along i^{th} row

$$\to a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in} = \det(A)$$

$\longrightarrow$ This is true for each i.

Now why about rows?

you can expand along any column as well

$$a_{1j} A_{1j} + a_{2j} A_{2j} + \cdots + a_{nj} A_{nj} = \det(A)$$

for each j

Qn: $\alpha_1 A_{11} + \alpha_2 A_{12} + \cdots + \alpha_n A_{1n} = ?$

$$A_1(\alpha) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \xleftarrow{\text{replaced by}} A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\alpha_1 A_{11} + \alpha_2 A_{12} + \cdots + \alpha_n A_{1n} = \det A_1(\alpha).$$

$\hookrightarrow$ 1st row of A replaced by $(\alpha_1, \alpha_2 \ldots \alpha_n)$.

$$\alpha_1 A_{i1} + \alpha_2 A_{i2} + \cdots + \alpha_n A_{in} = \det A_i(\alpha)$$

ith row cofactors.

$\hookrightarrow$ matrix obtained by replacing ith row of A by $(\alpha_1 \ldots \alpha_n)$.

$$\underbrace{\alpha_1 A_{1j}} + \alpha_2 \underbrace{A_{2j}} + \cdots + \alpha_n \underbrace{A_{nj}}$$

jth column co-factors.

$$= \det \underbrace{A^j(\alpha)}$$

matrix obtained by replacing jth column of A by $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$.

So what is

$$a_{21} A_{11} + a_{22} A_{12} + \cdots + a_{2n} A_{1n}$$

$=$ det of a matrix obtained by replacing the first row of A by $(a_{21}\ a_{22} \cdots a_{2n})$

$$= \det \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = 0.$$

as two rows are identical.

Illy if we take kth row entries as coefficient with ith row cofactors for $i \neq k$

— we get 0, i.e.,

$$a_{k1} A_{i1} + a_{k2} A_{i2} + \cdots + a_{kn} A_{in} = 0 \text{ for } k \neq i$$

$$\& \quad = \det(A) \quad \text{if } k = i \qquad -(*),$$

I^{ly} $a_{1k} A_{1j} + a_{2k} A_{2j} + \dots + a_{nk} A_{nj} = 0$ for $k \neq j$

& $= \det(A)$ for $k = j$ — $(*)_2$

note $(*)_1$ gives

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$$

with first row cofactors as first column.

→ in general i^{th} row cofactors as i^{th} column.

$$= \begin{pmatrix} \underbrace{a_{11} A_{11} + \dots + a_{1n} A_{1n}}_{\det(A)} & \underbrace{a_{11} A_{21} + \dots + a_{1n} A_{2n}}_{= 0} & \dots \\ & & \end{pmatrix}$$

the ij th entry of this matrix would be

$a_{i1} A_{j1} + a_{i2} A_{j2} + \dots + a_{in} A_{jn}$

which is 0 if $i \neq j$ & $= \det(A)$ if $i = j$

So by $(*)$,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & 0 \cdots & 0 \\ & & & \\ 0 & \cdots & \cdots & \det(A) \end{pmatrix}$$

$$= \det(A) I_n .$$

We define the classical adjoint of a matrix A as

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{21} & & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$$

i.e, the ij^{th} entry of $\operatorname{adj}(A)$, i.e $\operatorname{adj}(A)_{ij} = A_{ji}$.

Illy using $(*)_2$ we can show that

$$A \cdot \operatorname{adj}(A) = \det(A)\, I_n.$$

So altogether we have

$$A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A)\, I_n.$$

This shows if $\det(A) \neq 0$, then A is invertible

and $$\boxed{A^{-1} = \frac{1}{\det A} \operatorname{adj} A}$$

formula to compute inverse of an $n \times n$ matrix A.

Conversely, if A is invertible then $\exists$ B s.t

$$AB = I_n \Rightarrow \det(AB) = 1$$

$$\Rightarrow \det(A)\det(B) = 1$$

$$\Rightarrow \det(A) \neq 0.$$

This would follow from properties of determinant.

So we have proved the following theorem:

THM: A is invertible $\Longleftrightarrow$ $\det(A) \neq 0$.

Remark:

In other words,

A is not invertible $\Longleftrightarrow$ $\det A = 0$.

Another impact of this formulae.

Sol^n. to $Ax=b$ where A - $n\times n$ invertible matrix.
computation.

$$Ax = b$$

$$\Rightarrow x = A^{-1} b$$

$$= \frac{1}{\det(A)} \operatorname{adj}(A)\, b.$$

$$= \frac{1}{\det(A)} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \frac{1}{\det(A)} \begin{pmatrix} b_1 A_{11} + b_2 A_{21} + \cdots + b_n A_{n1} \\ b_1 A_{12} + b_2 A_{22} + \cdots + b_n A_{n2} \\ \\ b_1 A_{1n} + b_2 A_{2n} + \cdots + b_n A_{nn} \end{pmatrix}$$

So $$x_j = \frac{\det A^j(b)}{\det(A)}$$

This way of getting the sol^n is known as CRAMER'S RULE

The jth column of A replaced by $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Exc: Check $\det(I_n) = 1$

det upper/lower triangular matrix

= product of diagonal entries.

Now we briefly go over the formula by Leibniz:

— as mentioned earlier the method is to show the Laplace formula is same as Leibniz & for Leibniz, it is relatively easy to verify the conditions 1), 2), 3).

However, we will just give the Leibniz formula & leave the rest of the proofs for interested students to figure out from the determinant chapter in the book

LINEAR ALGEBRA by HOFFMAN - KUNZE.

Motivation behind Leibniz formula:

2x2 case

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$= a_{1\phi(1)}a_{2\phi(2)} - a_{1\varphi(1)}a_{2\varphi(2)}$$

where $\phi : \{1,2\} \longrightarrow \{1,2\}$ given by

$$\phi(1) = 1 \quad \& \quad \phi(2) = 2$$

& $\psi : \{1, 2\} \longrightarrow \{1, 2\}$ given by

$$\psi(1) = 2 \ \& \ \psi(2) = 1.$$

Note ϕ, ψ - are two bijective functions

$$\{1, 2\} \longrightarrow \{1, 2\}$$

— known as permutations of the symbols $\{1, 2\}$

(we know there are only 2 permutations of $\{1, 2\}$).

$\frac{3\times 3}{\text{case}}$ $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ Recall from (*)

$$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32}$$

$$- a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31}$$

$$+ a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}.$$

$$= a_{1\phi_1(1)} a_{2\phi_1(2)} a_{3\phi_1(3)}$$

$$- a_{1\phi_2(1)} a_{2\phi_2(2)} a_{3\phi_2(3)}$$

$$- a_{1\phi_3(1)} a_{2\phi_3(2)} a_{3\phi_3(3)}$$

$$+ a_{1\phi_4(1)} a_{2\phi_4(2)} a_{3\phi_4(3)}$$

$$+ a_{1\phi_5(1)}\, a_{2\phi_5(2)}\, a_{3\phi_5(3)}$$

$$- a_{1\phi_6(1)}\, a_{2\phi_6(2)}\, a_{3\phi_6(3)}.$$

where $\phi_i : \{1,2,3\} \longrightarrow \{1,2,3\}$ bijective fns

s.t

$\phi_1(1) = 1$	$\phi_2(1) = 1$	$\phi_3(1) = 2$
$\phi_1(2) = 2$	$\phi_2(2) = 3$	$\phi_3(2) = 1$
$\phi_1(3) = 3$	$\phi_2(3) = 2$	$\phi_3(3) = 3$
$\phi_4(1) = 2$	$\phi_5(1) = 3$	$\phi_6(1) = 3$
$\phi_4(2) = 3$	$\phi_5(2) = 1$	$\phi_6(2) = 2$
$\phi_4(3) = 1$	$\phi_5(3) = 2$	$\phi_6(3) = 1$

— these are permutation on 3. Symbols.

— 6 many.

<u>Now we explain the signs</u>:

We call permutation on two symbols as <u>transposition</u>. (This keeps the other symbols fixed).

Note ϕ_2, ϕ_3, ϕ_6 are transpositions.

Now let us see

$$\phi_2 \circ \phi_3 (1) = \phi_2 (2) = 3$$

$$\phi_2 \circ \phi_3 (2) = \phi_2 (1) = 1$$

$$\phi_2 \circ \phi_3 (3) = \phi_2 (3) = 2$$

So $\phi_5 = \phi_2 \circ \phi_3$ — product of two transposition.

$$\phi_2 \circ \phi_6 (1) = \phi_2 (3) = 2$$

$$\phi_2 \circ \phi_6 (2) = \phi_2 (2) = 3$$

$$\phi_2 \circ \phi_6 (3) = \phi_2 (1) = 1$$

$\Rightarrow \phi_2 \circ \phi_6 = \phi_4$ — product of two transposition.

Check $\phi_3^2 = \phi_1$ — product of two transposition

In general, any permutation on n-symbols is either product of even number of transposition - called an even permutation or odd number of transposition - called an odd permutation

So if $\sigma \in S_n$, i.e, $\sigma : \{1,2,\dots n\} \xrightarrow[\text{bijection}]{} \{1,2,\dots n\}$

Set of permutation on n-symbols,

then either σ is even permutation,

then $sgn(\sigma) = +$

or σ is an odd permutation

$sgn(\sigma) = -$.

KNOWN as SIGN F^N on S_n

So in the S_3 – permutation on 3 symbols,

for ϕ_1, ϕ_4, ϕ_5 sgn is $+$

$\phi_2, \phi_3\ \phi_6$ sgn is $-$

& now compare with the expansion to see they are the same.

So in general the Leibniz def^n is

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

The properties 1), 2), 3) are now comparatively to easy to check, BUT we will not do it here!

We shall now give a proof of some properties of determinant using the uniqueness of determinant function.

Property 1: $\det(AB) = \det(A)\det(B)$.

Pf: Case I: If $\det B = 0$

$\Rightarrow$ B is not invertible

So $\det(A)\det(B) = 0$

& B is not invertible & hence

AB is not invertible

(as if AB invertible, $(AB)^{-1}A \cdot B = I_n$.

$\Rightarrow$ B - invertible.).

So $\det(AB) = 0$.

$\therefore \det(AB) = 0 = \det(A)\det(B)$.

Case II: If B is invertible, $\det B \neq 0$.

Define $\delta(A) = \dfrac{\det(AB)}{\det(B)}$.

1) $\delta(I_n) = \dfrac{\det(I_n B)}{\det(B)} = \dfrac{\det(B)}{\det(B)} = 1$.

2) $\delta(P_{ij}A) = \dfrac{\det(P_{ij}AB)}{\det(B)} = \dfrac{-\det(AB)}{\det(B)}$

$= -\delta(A)$

3) $A = \begin{pmatrix} R_1 + \lambda R \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$

$\delta\begin{pmatrix} R_1 + \lambda R \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$

$= \dfrac{\det\left(\begin{pmatrix} R_1 + \lambda R \\ R_2 \\ \vdots \\ R_n \end{pmatrix} B\right)}{\det B.}$

Let $b_1\ b_2 \dots b_n$ be n-columns of B.

$\therefore \delta\begin{pmatrix} R_1 + \lambda R \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \dfrac{\det\left(\begin{pmatrix} R_1 + \lambda R \\ R_2 \\ \vdots \\ R_n \end{pmatrix} (b_1\ b_2 \dots b_n)\right)}{\det B}$

$\det\begin{pmatrix} R_1b_1 + \lambda Rb_1 & R_1b_2 + \lambda Rb_2 & \dots & R_1b_n + \lambda Rb_n \\ R_2b_1 & R_2b_2 & \dots & R_2b_n \\ R_nb_1 & R_nb_2 & \dots & R_nb_n \end{pmatrix}$

$$\|$$

$$\det\begin{pmatrix} R_1 b_1 & R_1 b_2 & \cdots & R_1 b_n \\ R_2 b_1 & R_2 b_2 & & R_2 b_n \\ \vdots & \vdots & & \vdots \\ R_n b_1 & R_n b_2 & \cdots & R_n b_n \end{pmatrix}$$

$$+ \det\begin{pmatrix} \lambda R b_1 & \lambda R b_2 & \cdots & \lambda R b_n \\ R_2 b_1 & R_2 b_2 & \cdots & R_2 b_n \\ \vdots & & & \vdots \\ R_n b_1 & R_n b_2 & \cdots & R_n b_n \end{pmatrix}$$

$$= \det\left(\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} B\right) + \lambda \det\left(\begin{pmatrix} R \\ R_2 \\ \vdots \\ R_n \end{pmatrix} B\right).$$

$$\delta\begin{pmatrix} R_1 + \lambda R \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \delta\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} + \lambda\, \delta\begin{pmatrix} R \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

So δ satisfies properties 1), 2), 3)

by uniqueness $\delta(A) = \det(A)$.

$$\therefore \quad \det(A) = \frac{\det(AB)}{\det(B)}.$$

Hence, $\det(AB) = \det(A)\det(B)$.

Exc: Prove similarly $\det(A) = \det(A^T)$.

Now that we know how to compute determinant we come back to computing eigenvalues.

Ex. 1 Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

One can think this as the linear map

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$

Clearly $[T]_{\mathcal{E}}^{\mathcal{E}} = A$ where $\mathcal{E}$ - std. basis (ordered).

or can directly consider

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

To find out its eigenvalues

We consider $\det(A - \lambda I)$.

$$= \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$$

$$= \lambda^2 + 1$$

$\lambda^2 + 1 = 0$ has no root in $\mathbb{R}$.

Hence $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has no eigenvalues in $\mathbb{R}$.

Alternatively one can see it can't have eigenvector.

Suppose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an eigenvector for some λ.

Then $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} x_2 &= \lambda x_1 \\ -x_1 &= \lambda x_2. \end{aligned}$$

So we get $(\lambda^2+1)x_1 = 0$

& $(\lambda^2+1)x_2 = 0$.

Since $\lambda \in \mathbb{R}$, $\lambda^2+1 \neq 0 \Rightarrow x_1 = 0 = x_2$

So $\nexists$ a nonzero vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$.

Remark: However if we consider the matrix over Complex numbers $\mathbb{C}$;

i.e
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix}.$$

$\longrightarrow$ $\det(A - \lambda I) = \lambda^2 + 1 = 0$

$\Rightarrow$ $\lambda = \pm i$ are eigen-values.

In fact, any matrix over complex nos. will have eigen values, as.

Fundamental theorem of algebra

say every complex polynomial has a root in complex numbers.

Ex: 2) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (\lambda - 1)^2 - 4$$

$$= \lambda^2 - 2\lambda + 1 - 4$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

So $3, -1$ are two eigenvalues of A.

To find the eigenvector, we need to find $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that

$$(A - 3I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2x_2 - 2x_1 \\ 2x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 = x_2. = \alpha \text{ (say)}$$

So any eigenvector corr. to 3 is of the form $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}. = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\alpha \in \mathbb{R}$.

So dim of eigenspace corr. to 3

$= \text{nullity}(A - 3I)$

Now the null space

$$N(A - 3I) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Hence dim e.sp corr. to 3 is 1.

Though there are plenty of eigenvectors corr. to 3 as we can choose α at our wish,

but it has only 1 - lin. ind. e.v.

meaning $\underline{\underline{\text{nullity } (A-3I) = 1}}$.

Similarly for $\lambda = -1$

$$(A+I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1+1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2 = \alpha \text{ (say)}$$

$$\therefore N(A+I) = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$\underline{\text{nullity } (A+I) = 1}$.

Note the two eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are LI.

In general, eigen vectors corr. to distinct eigen values are always linearly independent.
— you will see the proof of this FACT next semester.

Hence $\mathcal{B} \doteq \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^2$.

So if we consider $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$Tx = Ax$$

& then $[T]^{\mathcal{B}}_{\mathcal{B}} = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{diagonal matrix}}$

Such T's are called diagonalizable

So for this example, we not only have eigen values/vectors, but have a basis consisting of eigen vectors.

Ex. 3 $\quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$

$$\det(A - \lambda I) = \lambda^2.$$

So it has only 0 as eigenvalue.

To find eigen vector corr. to 0, we look at

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_2 = 0$$

So $N(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$

So $\dim N(A) = 1$.

& hence there is only one LI e.vectors corr. to 0 & since there are no more eigen values,

$\nexists$ a basis consisting of eigen vectors.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $Tx = Ax$ is not diagonalizable in this example.

Ex. 4 Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$

3x3 real matrix

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{pmatrix}$$

expanding along 1st row we get

$$= (1-\lambda)\left[\lambda^2 + 1\right] \ldots$$

$\longrightarrow$ So it has 1 as eigenvalue.

$$(A - I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -x_2 + x_3 = 0 \\ -x_2 + x_3 = 0 \end{array}\right\} \Rightarrow x_2 = 0 = x_3$$

So $N(A - I) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$

$\therefore$ nullity $(A - I) = 1.$

Since there are no more eigen values, the corresponding linear map is not diagonalizable.

Remark: The point to note here is that every $T: \mathbb{R}^3 \to \mathbb{R}^3$ will always have eigen-value in $\mathbb{R}$ as it's characteristic polynomial is of degree 3 and any cubic real eqn. has a root in $\mathbb{R}$.

Ex. 5 Let A be the 3×3 real matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}.$$

$$\det(A - \lambda I)$$

$$= \det \begin{pmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{pmatrix}$$

$$\overset{R_3 - R_2}{=} \det \begin{pmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 0 & \lambda & 1-\lambda \end{pmatrix}$$

expanding along first column

$$= (3-\lambda)\left\{(2-\lambda)(1-\lambda) + \lambda\right\} - 2\left\{(1-\lambda) + \lambda\right\}$$

$$= (3-\lambda)\left\{\lambda^2 - 3\lambda + 2 + \lambda\right\} - 2.$$

$$= (3-\lambda)(\lambda^2 - 2\lambda + 2) - 2.$$

$$= 3\lambda^2 - 6\lambda + 6 - \lambda^3 + 2\lambda^2 - 2\lambda - 2$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

$$= -(\lambda^3 - 5\lambda^2 + 8\lambda - 4)$$

<u>Remark:</u> Usually, when we write (customary to) the polynomial we write the top degree term, known as the leading term, with +ve co.eff., in particular with 1,

This is why we define **characteristic polynomial** to be **$\det(\lambda I - A)$** instead of $\underbrace{\det(A - \lambda I)}$

(as we notice λ^n will have -ve sign if n is odd).

So here $\det(\lambda I - A) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$

$$= \lambda^3 - \lambda^2 - 4\lambda^2 + 4\lambda + 4\lambda - 4$$

$$= \lambda^2(\lambda - 1) - 4\lambda(\lambda - 1) + 4(\lambda - 1)$$

$$= (\lambda - 1)(\lambda^2 - 4\lambda + 4)$$

$$= (\lambda - 1)(\lambda - 2)^2.$$

So the roots of characteristic eqn. are $1, 2, 2$.

i.e, the eigen values are $1, 2$. $\quad$ 2 - double root.

Now we look for the eigen vectors for A considered as a linear map $: \mathbb{R}^3 \to \mathbb{R}^3$

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow (A - I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_3 - R_1]{R_2 \leftrightarrow R_2 - R_1} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\big\downarrow R_2 \leftrightarrow R_3$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow{R_1 \leftrightarrow R_1 - R_2} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\big\downarrow \tfrac{1}{2} R_1$$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the eq^n becomes

$$x_1 - x_3/2 = 0.$$

$$x_2 = 0.$$

$$\Rightarrow x_3 = 2x_1$$

$$N(A - I) = \left\{ \begin{pmatrix} x_1 \\ 0 \\ 2x_1 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

$\dim N(A - I) = 1.$ & $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ is one eigen vector associated with 1.

For the eigenvalue 2,

$$A - 2I = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{pmatrix} \xrightarrow[R_3 \Longleftrightarrow R_3 - 2R_1]{R_2 \Longleftrightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2} R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \Longleftrightarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

So the eq^{n}s become

$$\left.\begin{aligned} x_1 - \frac{x_3}{2} &= 0 \\ x_2 - \frac{x_3}{2} &= 0. \end{aligned}\right)$$

$$N(A - 2I) = \left\{ \begin{pmatrix} x_3/2 \\ x_3/2 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$\dim N(A - 2I) = 1.$

So even though 2 is a double root, the no. of linearly independent eigen vector associated to 2 is 1.

for example $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ one such eigen vector.

So we can not have a basis consisting of eigen vectors as we require 3 linearly independent eigenvectors whereas we have only 2.

— The op. $x \longmapsto Ax$ from $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is <u>not</u> <u>diagonalizable</u>.

Ex. 6 Consider

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

We shall compute $\det(\lambda I - A)$

Let us indicate how one might compute the characteristic polynomial using various row operations

(you can use column operations as well since we have learnt $\det(A) = \det(A^T)$.)

$$\det \begin{pmatrix} \lambda - 5 & 6 & 6 \\ 1 & \lambda - 4 & -2 \\ -3 & 6 & \lambda + 4 \end{pmatrix}$$

$$\overset{R_3 + 3R_2}{=} \det \begin{pmatrix} \lambda - 5 & 6 & 6 \\ 1 & \lambda - 4 & -2 \\ 0 & 3\lambda - 6 & \lambda - 2 \end{pmatrix}$$

$$= \det \begin{pmatrix} \lambda - 5 & 6 & 6 \\ 1 & \lambda - 4 & -2 \\ 0 & 3(\lambda - 2) & \lambda - 2 \end{pmatrix}$$

$$= (\lambda - 2) \det \begin{pmatrix} \lambda - 5 & 6 & 6 \\ 1 & \lambda - 4 & -2 \\ 0 & 3 & 1 \end{pmatrix}$$

expanding along 1st column.

$$= (\lambda - 2) \left\{ (\lambda - 5) \det \begin{pmatrix} \lambda - 4 & -2 \\ 3 & 1 \end{pmatrix} \right.$$

$$- 1 \det \begin{pmatrix} 6 & 6 \\ 3 & 1 \end{pmatrix} \Big] \Big\}$$

$$= (\lambda - 2) \Big\{ (\lambda - 5)(\lambda - 4 + 6) - 1 \cdot (6 - 18) \Big\}$$

$$= (\lambda - 2) \Big\{ (\lambda - 5)(\lambda + 2) + 12 \Big\}$$

$$= (\lambda - 2)(\lambda^2 - 3\lambda - 10 + 12)$$

$$= (\lambda - 2)(\lambda^2 - 3\lambda + 2)$$

$$= (\lambda - 2)(\lambda - 1)(\lambda - 2) = (\lambda - 1)(\lambda - 2)^2.$$

The eigenvalues are $1, 2$.

Now we would like to find eigenvectors associated to $1, 2$ and the dimension of the eigen-space that is, the null spaces

$$N(A - I) \;\&\; N(A - 2I).$$

$$A - I = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}$$

Observe that $\operatorname{rank}(A - I) \geq 2$. (hence nullity $(A - I) \geq 1$ by rank-nullity.)

(as no column is multiple of any other column).

$\longrightarrow$ (||ly row).

$$\begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 3 & 2 \\ 4 & -6 & -6 \\ 3 & -6 & -5 \end{pmatrix}$$

$$\Big\downarrow R_1 \leftrightarrow -R_1$$

$$\begin{pmatrix} 1 & -3 & -2 \\ 0 & 6 & 2 \\ 0 & 3 & 1 \end{pmatrix} \xleftarrow[R_3 \leftrightarrow R_3 - 3R_1]{R_2 \leftrightarrow R_2 - 4R_1} \begin{pmatrix} 1 & -3 & -2 \\ 4 & -6 & -6 \\ 3 & -6 & -5 \end{pmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow \frac{1}{6} R_2$$

$$\begin{pmatrix} 1 & -3 & -2 \\ 0 & 1 & 1/3 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_3 - 3R_2]{R_1 \leftrightarrow R_1 + 3R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$$

So the equ^ns are

$$x_1 - x_3 = 0$$

$$x_2 + \frac{x_3}{3} = 0$$

$$N(A - I) = \left\{ \begin{pmatrix} x_3 \\ -\frac{x_3}{3} \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

So dimⁿ. of eigenspace corr. to 1 is 1

and $v_1 = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$ is an eigen vector corr. to 1.

$$A - 2I = \begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix},$$

Clearly rank $(A - 2I) = 1$ as 1st & 3rd row are multiple of 2nd row.

So $N(A - 2I) = 2$

So we will have 2 L.I. eigenvectors corr. to 2.

Let us find:

$$\begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow -R_2} \begin{pmatrix} 3 & -6 & -6 \\ 1 & -2 & -2 \\ 3 & -6 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow[R_3 \leftrightarrow R_3 - 3R_1]{R_2 \leftrightarrow R_2 - 3R_1} \begin{pmatrix} 1 & -2 & -2 \\ 3 & -6 & -6 \\ 3 & -6 & -6 \end{pmatrix}$$

$R_1 \leftrightarrow R_2$

So the associated eq^{n}s are

$$x_1 - 2x_2 - 2x_3 = 0.$$

$$N(A - 2I) = \left\{ \begin{pmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

Clearly $v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ are two linearly independent eigen vectors corresponding to 2.

Thus as stated earlier that corr. to distinct eigen values associated eigenvectors are linearly independent.

$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent and hence a basis of $\mathbb{R}^3$.

So if $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear map given by $Tx = Ax$, then

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ - diagonal.}$$

diagonalizable.

If $\mathcal{E}$ denotes the standard basis of $\mathbb{R}^3$ then $[T]_{\mathcal{E}}^{\mathcal{E}} = A$.

So $$[T]_{\mathcal{B}}^{\mathcal{B}} = \underbrace{\left([Id]_{\mathcal{E}}^{\mathcal{B}}\right)^{-1}}_{\substack{\parallel \\ Id_{\mathcal{B}}^{\mathcal{E}}.}} [T]_{\mathcal{E}}^{\mathcal{E}} \, [Id]_{\mathcal{E}}^{\mathcal{B}}.$$

Change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$.

$Id(v_1) = 3e_1 - e_2 + 3e_3$

$Id(v_2) = 2e_1 + e_2$

$Id(v_3) = 2e_1 + e_3$.

Thus $[Id]_{\mathcal{E}}^{\mathcal{B}} = \begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = P$ (say)

is the change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$

& $P^{-1} A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.