Lecture 10: Feb 18, 2025.

Basis: A set $\mathcal{B} = \{x_1, x_2, \dots x_k\}$ is a basis of $V/\mathbb{F}$ if

i) $\text{span}\, \mathcal{B} = V$

ii) $\mathcal{B}$ is linearly independent.

If such a set exists for a $V/\mathbb{F}$, then we call V is finite dimensional over $\mathbb{F}$.

If no such $\mathcal{B}$ exists, then V is called infinite dimensional.

A bit of digression:

Note

- Superset of a linearly dependent set is linearly dependent.
  Thus any set containing 0 is linearly dependent.

- subset of a linearly independent set is linearly independent.

Def^n: $S \subseteq V/\mathbb{F}$.

Span $S := \{ \alpha_1 x_1 + \dots \alpha_k x_k \mid x_1, x_2 \dots x_k \in S$
$\& \ \alpha_1 \dots \alpha_k \in \mathbb{F}$
$\& \ k \in \mathbb{N} \}.$

$= \{$ all linear combinations (necc. finite) of vectors of $S \}$.

S is said to be linearly independent if every finite subset of S is linearly independent.

— Note here S is arbitrary – not necc. finite!

General def^n of basis (Hamel) of $V/\mathbb{F}$:

A subset S of $V/\mathbb{F}$ is said to be a basis of $V/\mathbb{F}$ if.

i) span $S = V$

ii) S is linearly independent.

Example: $\mathbb{R}[x]$ — real polynomials of arbitrary degree.

$$S = \{1, x, x^2 \dots \} = \{ x^n : n \in \mathbb{N} \cup \{0\} \}$$

is a basis of $\mathbb{R}[x] / \mathbb{R}$.

ZORN'S LEMMA $\Rightarrow$ Every vector space has a basis.

(Hamel).

<u>Fact:</u> If $\mathcal{B}$ & $\mathcal{B}'$ are two bases of a $V/\mathbb{F}$, then $|\mathcal{B}| = |\mathcal{B}'|$.

— We will not prove it here in this generality.

(in Linear Alg course).

— We shall prove this when $V/\mathbb{F}$ is finite dimensional.

Before proving this, let us see why basis is important to us:

<u>Lemma:</u> $\mathcal{B} = \{x_1, \dots x_k\}$ is a basis of $V/\mathbb{F}$ if and only if for each $v \in V$, $\exists$ unique $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{F}$ such that

$$v = \alpha_1 x_1 + \dots + \alpha_k x_k.$$

<u>Pf:</u> ($\Rightarrow$) $v \in V$ & span $\mathcal{B} = V$

$\Rightarrow v \in \text{span}\,\mathcal{B}$

$\Rightarrow \exists\ \alpha_1, \alpha_2, \dots, \alpha_k$ s.t $v = \alpha_1 x_1 + \dots + \alpha_k x_k$

Suppose $\exists\ \beta_1, \beta_2, \dots, \beta_k$ s.t $v = \beta_1 x_1 \dots + \beta_k x_k$

$\Rightarrow (\alpha_1 - \beta_1) x_1 + \dots + (\alpha_k - \beta_k) x_k = 0$

Since $\{x_1 \dots x_k\}$ is LI

This implies that $\alpha_1 = \beta_1, \ldots, \alpha_k = \beta_k$

Thus every vector is unique represented.

($\Leftarrow$) Clearly span $\mathcal{B} = V$ by hypothesis

Now consider a linear combination of the vectors in $\mathcal{B}$, i.e

$$\alpha_1 x_1 + \alpha_2 x_2 \ldots + \alpha_k x_k = \theta \cdot (\text{Say}).$$

$$= 0 \cdot x_1 + \ldots + 0 \cdot x_k$$

unique repⁿ $\Rightarrow \alpha_1 = 0 = \alpha_2 \ldots = \alpha_k$.

Hence $\mathcal{B}$ is linearly independent.

* We have also seen that there can be plenty of basis

$\{(1,0), (1,r)\}$, $r \neq 0$, forms a basis of $\mathbb{R}^2/\mathbb{R}$.

<u>Theorem</u>: Let $\mathcal{B} = \{x_1, x_2, \ldots x_k\}$ be a basis of $V/\mathbb{F}$. Let $S \subseteq V/\mathbb{F}$ be a linearly independent subset of $V/\mathbb{F}$.

Then $|S| \leq K = |\mathcal{B}|$.

<u>Corollary</u>: If $\mathcal{B} = \{x_1, x_2, \ldots x_m\}$ & $\mathcal{B}' = \{y_1, \ldots y_n\}$ are two bases of $V/\mathbb{F}$, then $m = n$.

An approach to the proof of the theorem:

Let $|S| > k$.

Since S is linearly independent, $\exists\, y_1, y_2, \ldots y_l$, $l > k$ such that $\{y_1, \ldots, y_l\}$ is linearly independent. (by defn).

Since span $\{x_1, x_2 \ldots x_k\} = V$

$$\Rightarrow \quad y_j = a_{1j} x_1 + \cdots + a_{kj} x_k.$$

for some $a_{1j}, \ldots a_{kj} \in \mathbb{F}$.

$$\begin{aligned}
y_1 &= a_{11} x_1 + a_{21} x_2 + \cdots + a_{k1} x_k \\
y_2 &= a_{12} x_1 + a_{22} x_2 \cdots + a_{k2} x_2 \qquad (A) \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
y_l &= a_{1l} x_1 + a_{2l} x_2 + \cdots + a_{kl} x_k.
\end{aligned}$$

Consider the matrix

$$A = \left(\left(a_{ij} \right)\right)_{\substack{1 \le i \le k \\ 1 \le j \le l}}.$$

— $k \times l$ matrix $k < l$.

i.e $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\ell} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{k\ell} \end{pmatrix}$.

RTP

<u>Claim:</u> $\exists\ (\alpha_1 \ldots \alpha_\ell) \neq (0, \ldots 0)$ s.t

$$A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_\ell \end{pmatrix} = 0.$$

Suppose such $\alpha_1, \ldots \alpha_\ell$ exists.

$$A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_\ell \end{pmatrix} = 0 \Rightarrow \begin{array}{l} \alpha_1 a_{11} + \alpha_2 a_{12} \cdots + \alpha_\ell a_{1\ell} = 0 \\ \alpha_1 a_{21} + \alpha_2 a_{22} \cdots + \alpha_\ell a_{2\ell} = 0 \\ \vdots \\ \alpha_1 a_{k1} + \alpha_2 a_{k2} \cdots + \alpha_\ell a_{k\ell} = 0 \end{array} \quad \text{—} (*)$$

So $\alpha_1 y_1 + \cdots + \alpha_\ell y_\ell$

$$= \alpha_1 (a_{11} x_1 + \cdots + a_{k1} x_k)$$
$$+ \alpha_2 (a_{12} x_1 + \cdots + a_{k2} x_k)$$
$$\cdots + \alpha_\ell (a_{1\ell} x_1 + \cdots + a_{k\ell} x_k)$$

$$= (\alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_\ell a_{1\ell}) x_1$$
$$+ \cdots + (\alpha_1 a_{k1} + \cdots + \alpha_\ell a_{k\ell}) x_k$$

(*)

$$\Longrightarrow \quad \alpha_1 y_1 + \cdots + \alpha_\ell y_\ell = 0$$

$\cancel{\times}$ $\{y_1 \ldots y_\ell\}$ linearly independent.

Thus $|S| \leq k = |\mathcal{B}|$. $\blacksquare$

Idea of the proof:

Symbolically one can write the equations (A) as follows:

$$(y_1 \ y_2 \ \cdots \ y_\ell)_{1\times \ell} = (x_1, x_2 \ldots x_k)_{1\times k} \, A_{k\times \ell}.$$

— thinking of usual matrix multiplication

So by claim if $\exists$ a nonzero $(\alpha_1 \ldots \alpha_\ell)$ (that is, $(\alpha_1, \alpha_2, \ldots \alpha_\ell) \neq (0, 0, \ldots 0)$)

such that $A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

So

$$\alpha_1 y_1 + \cdots + \alpha_\ell y_\ell = (y_1 \ldots y_\ell) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_\ell \end{pmatrix}$$

$$= (x_1 \ldots x_k) \, A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_\ell \end{pmatrix}$$

$$= 0.$$