Problem Set - 10

MA 1201 Spring Sem, 2025

1. *Classify each of the following differential equations as linear or nonlinear, and specify the order.
   (i) $y'' + (\cos x)y = 0$
   (ii) $y'' + x \sin y = 0$
   (iii) $y' = \sqrt{1+y}$
   (iv) $y'' + (y')^2 + y = x$
   (v) $y'' + xy' = \sin y$
   (vi) $\left(x\sqrt{1+x^2}\, y'\right)' = e^x y$

2. Find the differential equation of each of the following families of plane curves. Here $a, b, c \in \mathbb{R}$ denote arbitrary constants:
   (a) $xy^2 - 1 = cy$
   (b) $y = ax + b + c$
   (c) *Circles touching the x-axis with centres on the y-axis.
   (d) $y = a \sin x + b \cos x + b$

3. Verify that the given function on the left is a implicit solution to the corresponding differential equation on the right.
   (i) $x^3 + y^3 = 3cxy$ $\quad x(2y^3 - x^3)y' = y(y^3 - 2x^3)$
   (ii) *$y = ce^{-x} + x^2 - 2x + 4$ $\quad y' + y = x^2 + 2$
   (iii) $y = cx - c^2$ $\quad (y')^2 - xy' + y = 0$

4. Find implicit solutions the following equations by separating variables:
   (a) $\dfrac{dy}{dx} = y^2 - 2y + 2$
   (b) $x\sqrt{1-y^2} + \sqrt{1-x^2}yy' = 0$
   (c) $(x^2 - 1)(y^2 - 1) + xyy' = 0$
   (d) $(y - x\dfrac{dy}{dx}) = a(y^2 + \dfrac{dy}{dx})$

5. Solve the Initial value problem (IVP) $(1 - x^2)\dfrac{dy}{dx} = 2y$ with $y(2) = 1$ implicitly.

6. *Verify that $y = \dfrac{1}{x+c}$ is the implicit/general solution of $y' = -y^2$. Find particular solutions such that:
   (i) $y(0) = 5$
   (ii) $y(2) = -\dfrac{1}{5}$

   In both cases, find the largest interval I on which y is defined.

7. Solve the IVP - $y\frac{dy}{dx} = e^x$, with $y(0) = 1$. Find the largest interval of validity of the solution.

8. (a) If $\frac{dy}{dx} = f(ax + by + c)$, then show that the substitution $ax + by + c = v$ will change it to a separable equation in x and v.

   (b) Using the above, solve the following:

   (i) $\frac{dy}{dx} = \sin(x + y)$

   (ii) $(x - y)^2 \frac{dy}{dx} = a^2$

9. Find out the implicit/general solution of the following homogeneous ODEs:

   (a) $2xy\frac{dy}{dx} = (x^2 - y^2)$

   (b) $(y^4 - 2x^3y) + (x^4 - 2xy^3)y' = 0$

   (c) $3x^2y + (x^3 + y^3)y' = 0$

10. (a) If $\frac{dy}{dx} = f(\frac{ax + by + c}{Ax + By + C})$, and $aB - bA \neq 0$, then show that the substitution $x = h + X, y = k + Y$ will change the differential equation to

$$\frac{dY}{dX} = F(\frac{aX + bY}{AX + BY}),$$

where (h, k) is the intersection point of two lines $ax + by + c = 0$ and $Ax + By + C = 0$ (why there is any?). Further substitution $Y = VX$ will make it to a separable equation in X and V.

   (b) Using the above, solve the following:

   (i) $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$

   (ii) $\frac{dy}{dx} = \frac{2x + 9y - 20}{6x + 2y - 10}$

11. (a) If $\frac{dy}{dx} = f(\frac{ax + by + c}{Ax + By + C})$, and $aB - bA = 0, a \neq 0, A \neq 0$, then show that the substitution

$$v = x + \frac{b}{a}y = x + \frac{B}{A}y$$

will make it to a separable equation in x and v.

   (b) Using the above, solve the following:

   (i) $\frac{dy}{dx} = \frac{3x - 4y - 2}{6x - 8y - 5}$

   (ii) $\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$ with $y(\frac{2}{3}) = \frac{1}{3}$.

12. *Show that the set of solutions of the homogeneous linear equation $y' + P(x)y = 0$ on an interval $I = [a, b]$ form a vector subspace W of the real vector space of continuous functions on I. What is the dimension of W?

13. Solve the linear first-order IVP:

$$y' + y\tan x = \sin(2x), \quad y(0) = 1$$

14. *Let φ_i be a solution of $y' + ay = b_i(x)$ for $i = 1, 2$. Show that $\varphi_1 + \varphi_2$ satisfies $y' + ay = b_1(x) + b_2(x)$. Solve:

$$y' + y = x + 1$$
$$y' + y = \cos(2x)$$

Hence solve: $y' + y = 1 + \dfrac{x}{2} - \cos^2 x$

15. Solve the following linear equations:

    (a) $\dfrac{dy}{dx} + 2xy = 4x$

    (b) $\dfrac{dy}{dx} - y\tan x = \cos x$

    (c) $*x\cos x\dfrac{dy}{dx} + y(x\sin x + \cos x) = 1$

    (d) $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1}y = \dfrac{1}{(x^2+1)^3}$

16. Reduce the following ODEs of Bernoulli'is form to linear equations and solve:

    (a) $xy - \dfrac{dy}{dx} = y^3 e^{-x^2}$

    (b) $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$

    (c) $*y(2xy + e^x) - e^x\dfrac{dy}{dx} = 0$

17. Using appropriate substitution, reduce the following differential equations to linear form and solve:

    (i) $*y^2y' + \dfrac{y^3}{x} = x^{-2}\sin x$

    (ii) $*y'\sin y + x\cos y = x$

    (iii) $y' = y(xy^3 - 1)$

    (iv) $\sec^2 y\dfrac{dy}{dx} + 2x\tan y = x^2$