6. Mean Value Theorems

"The Mean Value Theorem is the midwife of calculus - not very important or glamorous by itself, but often helping to deliver other theorems that are of major significance."

~ Edward Mills Purcell

In this chapter, we shall study a family of theorems, known as mean value theorems. These, informally speaking, state that for a given planar arc between two points, there is at least one point on the arc at which the tangent to the arc is parallel to the secant through its end points. While these theorems, in some form, were known for many years, it was Cauchy who proved the mean value theorem in its modern form.

§6.1. Mean Value Theorems (First-Order)

We shall begin with the following theorem, the proof of which is beyond the scope of this course.

Theorem 6.1.1 (Rolle's Theorem)

Let $-\infty < a < b < \infty$ and let $f : [a,b] \to \mathbb{R}$ be such that

i) f is continuous in $[a,b]$

ii) f is differentiable in (a,b), and

iii) $f(a) = f(b)$.

Then, there exists $c \in (a,b)$ such that $f'(c) = 0$.

Note that differentiability of f in (a,b) is essential. For instance, consider $[a,b] = [-1,1]$, $f(x) = |x|$. It is not differentiable at $x = 0$ & there is no $x \in (-1,1)$ such that $f'(x) = 0$.

THEOREM 6.1.2 (LAGRANGE'S MEAN VALUE THEOREM)

Let $-\infty < a < b < \infty$ and let $f : [a,b] \to \mathbb{R}$ be continuous, and let f be differentiable in (a,b). Then, there exists $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)\,(b-a).$$

Proof. Define $\phi : [a,b] \to \mathbb{R}$ as follows:

$$\phi(x) := x\,(f(b)-f(a)) - f(x)\,(b-a) \quad , x \in [a,b].$$

Note that ϕ is continuous, as it is a sum of two continuous functions ($x(f(b)-f(a))$ and $f(x)(b-a)$). On (a,b), ϕ is differentiable, as it is a sum of the same two differentiable functions. Applying Rolle's Theorem (Thm 6.1.1), we obtain $c \in (a,b)$ such that $\phi'(c) = 0$, i.e.,

$$0 = f(b) - f(a) - f'(c)(b-a).$$

This proves the theorem. ∎

THEOREM 6.1.3 (CAUCHY'S MEAN VALUE THEOREM)

Let $-\infty < a < b < \infty$ and let $f, g : [a,b] \to \mathbb{R}$ be such that

i) f, g are continuous, and

ii) f, g are differentiable in (a,b).

Then, there exists $c \in (a,b)$ such that

$$(g(b)-g(a))\,f'(c) = (f(b)-f(a))\,g'(c).$$

Proof. We shall indicate the ideas in the proof; details will be left as an exercise. It is prudent to consider the function $\phi : [a,b] \to \mathbb{R}$

$$\phi(x) := (g(b)-g(a))\,f(x) - (f(b)-f(a))\,g(x).$$

Check that ϕ is continuous in $[a,b]$, differentiable in (a,b) and satisfies $\phi(a) = \phi(b)$. Applying Thm 6.1.1, then leads us to some $c \in (a,b)$ such that

$$0 = \phi'(c) = (g(b)-g(a))\,f'(c) - (f(b)-f(a))\,g'(c).$$ ∎

Note that Theorem 6.1.3 is a generalization of Theorem 6.1.2; set $g = id$ to obtain Theorem 6.1.2.

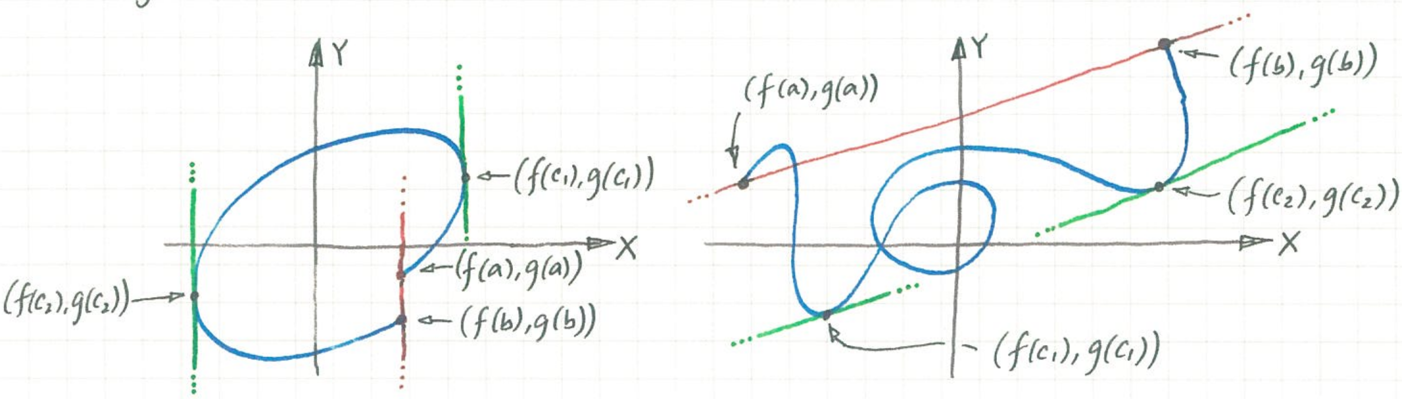


Fig 6.1: Visual interpretation of Cauchy's Mean Value Theorem

If $f(a) \neq f(b)$ and $f'(c) \neq 0$, then the conclusion of Theorem 6.1.3 can be restated as

$$\frac{g'(c)}{f'(c)} = \frac{g(b)-g(a)}{f(b)-f(a)}.$$

Notice that the chord joining the points $(f(a),g(a))$ & $(f(b),g(b))$ has slope $(g(b)-g(a))/(f(b)-f(a))$. In Fig 6.1, the curves are traced out by points of the form $(f(t),g(t))$ for $t \in [a,b]$. The tangent to this curve at $t = c_0$ has slope $g'(c_0)/f'(c_0)$. Thus, Theorem 6.1.3 is asserting (when $f(a) \neq f(b)$ & $f'(c) \neq 0$) that there is a point on this curve where the slope of the tangent line is equal to the slope of the chord joining $(f(a),g(a))$ and $(f(b),g(b))$.

We now consider a few applications of these theorems.

THEOREM 6.1.4

Let $-\infty < a < b < \infty$ and let $f:(a,b) \to \mathbb{R}$ be differentiable. Then, f is a constant if and only if $f'(x) = 0$ for all $x \in (a,b)$.

Proof. If f is a constant function, then f' is identically zero. For the converse, let f be such that $f'(x) = 0$ for all $x \in (a,b)$. Let $t < s$ in (a,b). Now f is continuous in $[t,s]$ and differentiable in (t,s). By Theorem 6.1.2, there exists $r \in (t,s)$ such that

$$f(s) - f(t) = (s-t)f'(r) = 0.$$

This implies $f(s)=f(t)$. As s & t were arbitrary, we conclude that f is a constant. ▨

THEOREM 6·1·5

Let $-\infty<a<b<\infty$ and let $f:(a,b)\to\mathbb{R}$ be differentiable satisfying $f'(x)>0$ for any $x\in(a,b)$. Then, f is strictly increasing, i.e., $f(x)>f(y)$ for all $x,y\in(a,b)$ with $x>y$.

The proof is left as an exercise.

EXAMPLES 6·1·6 (APPLICATIONS TO INEQUALITIES)

i) For all $x\in(0,\pi/2)$,

$$(2/\pi)x<\sin x<x<\tan x.$$

Define the functions $f,g,h:(0,\pi/2)\to\mathbb{R}$ as follows:

$$f(x):=\tan x-x\ ,\quad g(x):=x-\sin x\ ,\quad h(x):=\frac{\sin x}{x}.$$

Now, $f'(x)=\sec^2 x-1$, $g'(x)=1-\cos x$, $h'(x)=\cos x\left(\frac{x-\tan x}{x^2}\right)$. As $f'(x)>0$ and $g'(x)>0$ for all $x\in(0,\pi/2)$, by Theorem 6·1·5, both f & g are strictly increasing, i.e.,

$$f(x)>f(y)\quad\text{and}\quad g(x)>g(y)\ \text{ for all } x,y\in(0,\pi/2)\ \&\ x>y.$$

Thus, $f(x)>\lim\limits_{y\to0^+}f(y)=0$ & $g(x)>\lim\limits_{y\to0^+}g(y)=0$. Hence,

$$\tan x-x>0\quad\&\quad x-\sin x>0\quad\text{for all } x\in(0,\pi/2).$$

As $h'(x)=\cos x\left(-\frac{f(x)}{x^2}\right)<0$ on $(0,\pi/2)$, it follows that h is a strictly decreasing function. Therefore,

$$h(x)>h(y)\quad\text{for all } x,y\in(0,\pi/2)\ \&\ x<y$$

Thus, $h(x)>\lim\limits_{y\to\frac{\pi}{2}^-}h(y)=2/\pi$. Hence,

$$\frac{\sin x}{x}>\frac{2}{\pi}\quad\Rightarrow\quad\sin x>\frac{2}{\pi}x\quad\text{for all } x\in(0,\pi/2).$$

ii) For all $x\in(0,\infty)$,

$$x>\ln(1+x)>\frac{x}{1+x}.$$

Consider the functions $f,g:(0,\infty)\to\mathbb{R}$ as follows:

$$f(x)=x-\ln(1+x)\ ,\quad g(x)=\ln(1+x)-\frac{x}{1+x}.$$

Check that f' & g' are always positive & apply Theorem 6·1·5.

THEOREM 6.1.7 (L'HÔPITAL'S RULE)

Let $a<c<b$ and let $f, g : (a,b) \to \mathbb{R}$ be continuous and let f, g be differentiable in $(a,b) \setminus \{c\}$. Suppose that

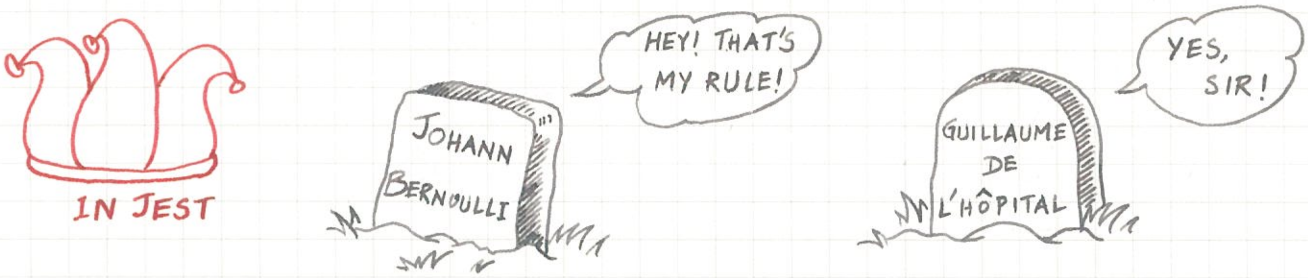
i) $f(c) = g(c) = 0$

ii) $g(x) \neq 0$ for all $x \in (a,b) \setminus \{c\}$

iii) $g'(x) \neq 0$ for all $x \in (a,b) \setminus \{c\}$.

If $\lim\limits_{x \to c} f'(x)/g'(x)$ exists, then $\lim\limits_{x \to c} f(x)/g(x)$ exists and

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$



Proof. Let L be the limit defined by

$$L := \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

We shall prove that $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = L$. Let $\varepsilon > 0$ be given. By definition of L, there exists $\delta > 0$ such that for any $x \in (a,b)$,

$$\text{if } 0 < |x-c| < \delta, \text{ then } \left| f'(x)/g'(x) - L \right| < \varepsilon. \qquad \cdots \quad (6.1)$$

<u>CASE 1</u>: $x \in (c, c+\delta)$

Using Theorem 6.1.3, we find $c_1 \in (c,x) \subseteq (c, c+\delta)$ such that

$$\left| \frac{f(x)}{g(x)} - L \right| \overset{(i)}{=} \left| \frac{f(x)-f(c)}{g(x)-g(c)} - L \right| \overset{6.1.3}{=\!=} \left| \frac{f'(c_1)}{g'(c_1)} - L \right| \overset{(6.1)}{<} \varepsilon.$$

<u>CASE 2</u>: $x \in (c-\delta, c)$

Using Theorem 6.1.3, we find $c_2 \in (x,c) \subseteq (c-\delta, c)$ such that

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{-f(x)}{-g(x)} - L \right| \overset{(i)}{=} \left| \frac{f(c)-f(x)}{g(c)-g(x)} - L \right| \overset{6.1.3}{=\!=} \left| \frac{f'(c_2)}{g'(c_2)} - L \right| \overset{(6.1)}{<} \varepsilon.$$

Thus, for any $x \in (a,b)$

$$\text{if } \; < |x-c| < \delta, \text{ then } \left| f(x)/g(x) - L \right| < \varepsilon.$$

Hence,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L. \qquad \blacksquare$$

Examples 6.1.8

i) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x^2+x} = \lim_{x\to 0} \frac{e^x}{2x+1} = 1$

[Apply Theorem 6.1.7 to $f(x) = e^x - 1$, $g(x) = x^2+x$, $c = 0$, $a = -\frac{1}{2}$, $b = \frac{1}{2}$. You should think why we chose a & b as above.]

ii) $$\lim_{x\to 0} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \qquad [\text{Thm } 6.1.7]$$

$$= \lim_{x\to 0} \frac{-\sin x}{\cos x - x\sin x + \cos x} \qquad [\text{Thm } 6.1.7]$$

$$= 0.$$

[Think about the pair of functions and their common domains for both instances of Theorem 6.1.7.]

§6.2. Mean Value Theorems (Second-Order)

We conclude this chapter with a mean value theorem involving second order derivatives. This theorem will have applications in finding maxima or minima of a function.

Theorem 6.2.1 (Taylor's Theorem of Second Order)

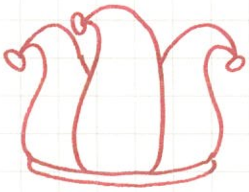
Let $-\infty < a < b < \infty$ and let $f : [\ .b]$ be such that

i) f & f' are continuous in $[a,b]$

ii) f' is differentiable in (a,b).

Then, there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \tfrac{1}{2} f''(c)(b-a)^2.$$



IN JEST

- Q: Did you ever hear that joke about Taylor series?

  A: I don't remember it precisely, but I can tell it pretty close. In fact, it becomes better every time I add something to it.

- Q: Why did Taylor never run for US president?

  A: Because he does not want to be limited to two terms.

Although we shall not discuss it here, there are Taylor theorems of higher order.

Proof. Let us define $\varphi : [a,b] \to \mathbb{R}$ by
$$\varphi(x) := f(b) - f(x) - f'(x)(b-x) - A(b-x)^2 \text{ for any } x \in [a,b],$$
where $A \in \mathbb{R}$ will be chosen later. Note that φ is continuous in $[a,b]$ and differentiable in (a,b). We choose $A \in \mathbb{R}$ such that $\varphi(a) = \varphi(b)$, i.e.,
$$f(b) - f(a) - f'(a)(b-a) - A(b-a)^2 = \varphi(a) = \varphi(b) = 0$$
$$\Rightarrow \quad A = \frac{1}{(b-a)^2}[f(b) - f(a) - f'(a)(b-a)].$$
Using Theorem 6.1.1 to φ, we find $c \in (a,b)$ such that
$$0 = \varphi'(c) = -f'(c) - f''(c)(b-c) + f'(c) + 2A(b-c)$$
$$\Rightarrow \quad A = \tfrac{1}{2} f''(c).$$
Thus, we have (by comparing the two values of A)
$$f(b) - f(a) - f'(a)(b-a) = \tfrac{1}{2} f''(c)(b-a)^2$$
$$\Rightarrow \quad f(b) = f(a) + f'(a)(b-a) + \tfrac{1}{2} f''(c)(b-a)^2.$$
This proves the theorem. ∎

EXAMPLE 6.2.2

Apply Theorem 6.2.1 to the function $f(x) = \sin x$. With $a = 0$ and any $b > 0$, we get c (possibly depending on b) such that
$$f(b) = f(0) + f'(0)b + \tfrac{1}{2} f''(c) b^2$$
$$\Rightarrow \sin b = b - \tfrac{1}{2}(\sin c) b^2, \text{ for some } c \in (0,b).$$
If we set $b \leq \pi/2$, then $c < b \leq \pi/2$ and $\sin c$ lies between 0 & 1. Thus, we obtain $\sin x < x$ for any $x \in (0, \pi/2]$.

EXAMPLE 6.2.3

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Suppose $a \in \mathbb{R}$ such that $f'(a) = 0$ and $f''(a) < 0$. By continuity of f'', there exists $\varepsilon > 0$ such that $f''(b) < 0$ for $b \in (a-\varepsilon, a+\varepsilon)$. Thus, for any such b, by using Theorem 6.2.1, we get $c \in (a,b)$ such that
$$f(b) = f(a) + f'(a)(b-a) + \tfrac{1}{2} f''(c)(b-a)^2$$
$$\Rightarrow f(b) = f(a) + \tfrac{1}{2} f''(c)(b-a)^2 < f(a).$$
Hence, $f(a)$ is a local maxima of f.

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