Problem Set - 2

MA 1201 | Spring Sem, 2025

1. For any two finite sets A and B, the cardinality of A^B is $|A|^{|B|}$.
2. Show that countable union of finite set is countable.
3. Show that $\mathbb{Q}_+$ is countable.
4. Show that if A is countable, then $A^k = A \times \cdots A\,(k \text{ times })$, $k \in \mathbb{N}$ is countable. As a result $\mathbb{Q}^k$ is countable for any $k \in \mathbb{N}$.
5. *Prove that the set $\mathcal{P}$ of all polynomials
$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
with integral coefficients, that is, where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$, is countable.
6. *A real number r is called an *algebraic number* if r is a solution to a polynomial equation
$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$
with integral coefficients. For example, integers, $\sqrt{2}$, $\sqrt[3]{4}$ are algebraic numbers. Prove that the set of algebraic numbers is countable.
7. *Let $\mathcal{A} = \{A_i : i \in I\}$, for some index set I, be a set of pairwise disjoint intervals in $\mathbb{R}$. Show that $\mathcal{A}$ is countable. You can use $\mathbb{Q}$ is 'dense' in $\mathbb{R}$.
8. Prove or disprove: If $\mathcal{B} = \{B_i : i \in I\}$, for some index set I, is a set of pairwise disjoint circles in $\mathbb{R}^2$, then $\mathcal{B}$ is countable.
9. Prove that the set of all circles in the plane $\mathbb{R}^2$ having rational radii and centers with rational coordinates is countable.
10. Let $\mathcal{C} = \{C_i : i \in I\}$, for some index set I, be a set of pairwise disjoint discs (sets of the form $\{(x, y) : (x - a)^2 + (y - b)^2 < r^2\}$) in $\mathbb{R}^2$. Show that $\mathcal{C}$ is countable.
11. A real number is called *transcendental* if it is not algebraic. For example, π, e are transcendental numbers. Prove that the set of transcendental numbers is uncountable.
12. *Show that the plane $\mathbb{R}^2$ is not a union of countable number of lines.
13. *Show that no power set can be countable, that is, a power set is either finite or uncountable.

End of problem sheet