Lecture 03 : Jan 30.

Triangular factorization :

Recall the system of linear equations -

$$2u + v + w = 5 \quad \text{--- (i)}$$
$$4u - 6v = -2 \quad \text{--- (ii)}$$
$$-2u + 7v + 2w = 9 \quad \text{--- (iii)}$$

Which was written in matrix form as :

$$Ax = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} = b.$$

Recall the three elimination steps.

STEP I : Subtract twice of equation (i) from equⁿ (ii) & put in place of equⁿ (ii)

$$2u + v + w = 5 \quad \text{--- (I)}$$
$$-8v - 2w = -12 \quad \text{--- (II)}$$
$$-2u + 7v + 2w = 9 \quad \text{--- (III)}$$

& the corresponding elimination matrix :

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{\text{row2} \\ \text{replaced by} \\ \text{row 2} - 2 \times \text{row 1.}}]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

STEP II: After step I, add eqn (I) and III and put it in place of eqn (III) –

$$
\begin{aligned}
2u + v + w &= 5 \\
-8v - 2w &= -12 \\
8v + 3w &= 14
\end{aligned}
$$

& the corresponding elimination matrix is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = F$$

This means row 3 replaced by row 3 + row 1

STEP III: After step II, add the 2nd eqn and 3rd eqn in step II and put it in place of 3rd eqn

$$
\begin{aligned}
2u + v + w &= 5 \\
-8v - 2w &= -12 \\
w &= 2
\end{aligned}
$$

& the corresponding elimination matrix is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = G$$

Note that in Gaussian elimination, E acts on Ax, then F acts on EAx, and finally G acts on $FEAx$.

So let us look a the matrices GFE and $GFEA$.

$$FE = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$GFE = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \longrightarrow \text{lower triangular}$$

&

$$GFEA = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow$$ upper triangular

$$= U \text{ (say)}.$$

Also,

$$Ux = GFEAx = GFEb$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -12 \\ 2 \end{pmatrix} = c$$ — Just the name.

- If we now invert/reverse the steps III, II, I, then we would get back to the original system.

What is inverting / reversing Step III ?

We have added 2nd eqⁿ to 3rd eqⁿ of STEP III

So the inverting it would be then subtracting back the 2nd eqⁿ

In terms of elementary matrix it would mean :

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

this we denote by $\underset{\sim}{\vec{G}}$

notation has a meaning — we will explain in next class.

IIIly

$$F^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

now

$$E^{-1} F^{-1} G^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = L \quad \text{(say)}.$$

↳ lower triangular

Reversing the steps would mean

$$\underbrace{E^{-1} F^{-1} G^{-1}}_{L} (\underbrace{G F E A}_{U}) = A$$

$$\Rightarrow \quad A = L U$$

U → upper triangular

L ↳ lower triangular

— known as LU - decomposition.

Note that every square matrix may not have LU decomposition.

Thanks to Arkadip Paul for pointing this out

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ — doesn't have LU - decompos$^{\underline{n}}$.

— we will learn the proof a bit later.

— we will also discuss some cases when this decomposition is unique (!)

whenever it is possible to have the LU-decomposition, that is,

$$A = LU,$$

it is extremely useful to solve $Ax = b$.

In fact, given the eqn. $Ax = b$,

and given that $A = LU$

we have $LUx = b$

$\Leftrightarrow$ $Ux = c$ whenever $Lc = b$.

Solve $Lc = b$ — by forward substitutions to get c.

Solve $Ux = c$ - by backward substitutions to get x.

Remark:

One might think that we obtained LU decomposition while solving the equation, then how is it useful?

— is it a circular argument?

NO — because we have obtained LU-decomp. in one way, but this need not be the only way!

This is why we have written given an LU-decomp. of A it is useful.

So, we have seen a type of elimination in one example — what else can happen?

- Now, while performing elimination, if 0 appears in a pivot, then as we have said earlier, we would interchange rows to bring non-zero number in pivot position, an then continue with elimination as earlier.

Let us see an example:

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 5 & 8 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} U \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

This means inter change of row 2 and row 3

This lead us to another kind of elementary matrix known as '<u>PERMUTATION MATRIX</u>'