

PH1201 : Basic Electricity & Magnetism

Lecture Notes

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Foreword



These are notes of lectures I have been giving at the Department of Physical Sciences (DPS), IISER-Kolkata for the first year undergraduate course on Electricity and Magnetism (Physics 2, PH1201) over the years. I am very thankful to Mr. Ankan Basak for kindly preparing the first chapter of these lectures notes on mathematical requisites. Any errors, typographical or otherwise, can be laid at my door. Please do email me with corrections.

Shankar



The intention of the lectures is to provide a gradual conceptual understanding of the subject. Even as attention has given towards covering the phenomenology of these topics, an effort has also been made to provide adequately the mathematical underpinnings of these subjects. Given their venerable status, almost all the topics covered within these areas are well-known and covered in detail in several places (some of whom are listed below).

Griffiths



A word on prerequisites. A good coverage of school level calculus, as well as the basics of vectors and their algebraic manipulations, is enough for the reader to make progress through these notes. Unless mentioned specifically, no claims of originality are being made in either the content or presentation of material covered in these lectures. Indeed, they were prepared from a thorough consultation of a selection of excellent textbooks already available on the topics I covered, including:

- R. Shankar, *Fundamentals of Physics II: Electromagnetism, Optics and Quantum Mechanics*, Yale Univ. Press, 2020

- David J. Griffiths, *Introduction to Electrodynamics*, 4th Edition, Cambridge Univ. Press, 2017

Morin

- Edward M. Purcell and David J. Morin, *Electricity and Magnetism*, 3rd Edition, Cambridge Univ. Press, 2013

My lectures have relied heavily on the first of these three texts for content as well as flow; I have also taken several pictures from some of these texts. Indeed, I am deeply thankful to these authors

for their excellent texts. I am content with having conveyed some of the material covered within them to the best of my abilities.

Finally, I invite you to begin your journey into these lectures with a quote from a favourite fictional character of mine: to ∞ and beyond!



To ∞ and Beyond! (Source: The internet.)

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Chapter 1

Basic Mathematical Tools

1.1 Volume Integrals

We consider the real 3-dimensional space \mathbb{R}^3 . A volume element in the said space can be expressed non-uniquely in terms of any orthogonal (and preferably *normalized*) basis. (A normalized basis is one all the basis vectors of which have unit norm.) The 3 most commonly used coordinate systems, for this matter, are:

- i. The Cartesian coordinate system – Basis: $\{\hat{x}, \hat{y}, \hat{z}\}$.
- ii. The Cylindrical polar coordinate system – Basis: $\{\hat{\rho}, \hat{\phi}, \hat{z}\}$.
- iii. The Spherical polar coordinate system – Basis: $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$.

The Cartesian coordinate system is the standard coordinate system, where each of the basis vectors is a unit vector lying parallel to the respectively designated axes X, Y and Z.



Descartes

The cylindrical polar coordinate system can be visualized as follows:

The coordinate system is constructed as follows:

- The point P is represented by: $P \equiv (\rho, \phi, z)$.
- $\rho \in \mathbb{R}^+ \cup \{0\}$, $\phi \in [0, 2\pi]$, $z \in \mathbb{R}$.

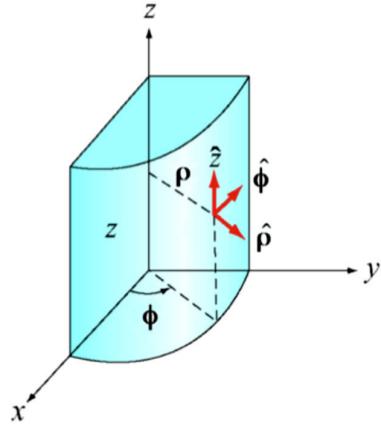


Figure 1.1: A Point in Space Represented By Cylindrical Polar Coordinates

- ρ denotes the length of the *projection* of the position vector of the point P on the XY-plane.
- ϕ denotes the angle the projection of the position vector on the XY-plane makes with respect to the X-axis.
- z denotes the projection of the position vector on the Z-axis, like in case of the Cartesian coordinate system.
- $\hat{\rho}$ points in the direction in which the projection of P on the XY-plane lies with respect to the origin O .
- $\hat{\phi}$ points in the *anti-clockwise* direction, such that it is parallel to the XY-plane and is normal to $\hat{\rho}$.
- \hat{z} is the usual unit vector as in the Cartesian coordinate system.

The spherical polar coordinate system can be visualized as follows:

The coordinate system is constructed as follows:

- The point P is represented by: $P \equiv (r, \theta, \phi)$.
- $r \in \mathbb{R}^+ \cup \{0\}$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$.
- r denotes the distance of the point P from the origin O .
- θ denotes the angle the position vector makes with respect to the Z-axis.

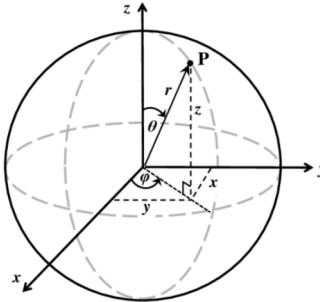


Figure 1.2: A Point in Space Represented By Spherical Polar Coordinates

- ϕ denotes the angle the *projection* of the position vector on the XY-plane makes with respect to the X-axis.
- \hat{r} points in the direction in which P lies with respect to O, the origin.
- $\hat{\phi}$ points in the *anti-clockwise* direction, such that it is parallel to the XY-plane and is normal to the projection of \hat{r} on the XY-plane.
- $\hat{\theta}$ is the unit vector mutually orthogonal to both of \hat{r} and $\hat{\phi}$, and pointing *downward*.

Now, we can check that the following differential relations hold:

- $dx = dr \sin(\theta)\cos(\phi) = d\rho \cos(\phi)$.
- $dy = dr \sin(\theta)\sin(\phi) = d\rho \sin(\phi)$.
- $dz = dr \cos(\theta)$.
- Differential change in position: $d\vec{r} = dr \hat{r} + r d\theta \hat{\theta} + r \sin(\theta) d\phi \hat{\phi} = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{z}$.

We also have the following mathematical definitions of the unit vectors, in terms of the unit vectors of the Cartesian coordinate system, as follows:

- $\hat{\rho} := \cos(\phi) \hat{x} + \sin(\phi) \hat{y}$.
- $\hat{\phi} := -\sin(\phi) \hat{x} + \cos(\phi) \hat{y}$.
- $\hat{r} := \sin(\theta)\cos(\phi) \hat{x} + \sin(\theta)\sin(\phi) \hat{y} + \cos(\theta) \hat{z}$.

- $\hat{\theta} := \cos(\theta)\cos(\phi)\hat{x} + \cos(\theta)\sin(\phi)\hat{y} - \sin(\theta)\hat{z}$.

It's an easy exercise to verify that the above 4 definitions satisfy all of the requirements of the geometrical constructions of the cylindrical and spherical polar coordinate systems, and also that each of the 4 vectors have unit norm, and the bases $\{\hat{r}, \hat{\theta}, \hat{z}\}$ and $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$ are orthogonal. (Hint: Use the results of the inner product relations between \hat{x} , \hat{y} and \hat{z} .) We can also verify the following:

-

$$\hat{r} \times \hat{\theta} = \hat{z}, \quad \hat{\theta} \times \hat{z} = \hat{r}, \quad \hat{z} \times \hat{r} = \hat{\theta}.$$

-

$$\hat{r} \times \hat{\phi} = \hat{\theta}, \quad \hat{\theta} \times \hat{\phi} = \hat{r}, \quad \hat{\phi} \times \hat{r} = \hat{\theta}.$$

Now that we have the necessary coordinate systems constructed, we define the differential volume element in \mathbb{R}^3 , relevant for this course, as follows:

$dV \equiv$ The volume of a 3-dimensional object bounded by 3 length-like differential elements, each orthogonal to each other.

Hence, we get:

$$dV = \begin{cases} dx dy dz & (\text{in Cartesian coordinate system}), \\ d\rho (\rho d\phi) dz = \rho d\rho d\phi dz & (\text{in Cylindrical polar coordinate system}), \\ dr (r d\theta) (r \sin(\theta) d\phi) = r^2 \sin(\theta) dr d\theta d\phi & (\text{in Spherical polar coordinate system}). \end{cases}$$

When any quantity f is integrated over a system with a volume bounded by a given set of values for each of the variables defining the coordinate system being used, we can express the

integral as follows:

$$\int_V f \, dV = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_1(x, y, z) \, dx \, dy \, dz, \quad (\text{in CartesianCoords.}) \quad (1.1)$$

$$= \int_{z_1}^{z_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f_2(\rho, \phi, z) \rho \, d\rho \, d\phi \, dz, \quad (\text{in Cylindrical PolarCoords.}) \quad (1.2)$$

$$= \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f_3(r, \theta, \phi) r^2 \sin(\theta) \, dr \, d\theta \, d\phi. \quad (\text{in Spherical PolarCoords.}) \quad (1.3)$$

Quick points to note here:

- The ordering of these integrals is irrelevant.
- Notice how the function being integrated is labelled differently for each coordinate system. That is not to state that the quantity whose volume integral over the given volume is being evaluated is different for different coordinate systems, but quite simply, that the functional form of the quantity, expressed as a function of variables, is different, in general, in each of the 3 cases.
- The boundaries of the system for each coordinate system should be computed, while switching from one coordinate system to another, in such a way that the total volume of the system is preserved.

(Exercise: Compute the volume of a cone by using what you've learnt above.)

1.2 The $\vec{\nabla}$ Operator and Its Operational Usage

We begin our discussion with the Cartesian coordinate system, and establish the intuitive usefulness of the Nabla ($\vec{\nabla}$) operator, and then aim to obtain the relevant expressions in terms of any general coordinate system with an orthonormal basis.

In the Cartesian coordinate system, the $\vec{\nabla}$ operator is defined as:

$$\vec{\nabla} := \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}.$$

This operator can act on both, scalar-valued functions and vector-valued functions. One such

operation is termed as the *gradient of f*, where: f is a scalar-valued function. We say:

$$\text{grad}(f) := \vec{\nabla}f = \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z}.$$

In order to get an intuitive understanding of the significance of the gradient operation, we check the following result (treating $\vec{\nabla}f$ as a vector object):

$$(\vec{\nabla}f) \cdot d\vec{r} = (\hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z}) \cdot (\hat{x} dx + \hat{y} dy + \hat{z} dz) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz =: df, \text{ where :}$$

$$df \equiv \text{A differential change in the function } f.$$

Thus, the gradient of a scalar valued function of 3-dimensional position can be interpreted as the vector-valued object whose component along any given differential position vector, multiplied by the norm of the differential position vector, gives the differential change in the function in that direction.

Next, we consider the *divergence* operation on a vector-valued function \vec{v} . The *divergence of \vec{v}* is defined as follows:

$$\text{div}(\vec{v}) := \vec{\nabla} \cdot \vec{v} = (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}) \cdot (\hat{x} v_x + \hat{y} v_y + \hat{z} v_z) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

The intuitive interpretation of this operation, in terms of *sources* and *sinks* of vector fields, will be provided later.

Next, we have the *curl* operation on, again, a vector-valued function \vec{v} . The *curl of \vec{v}* is defined as follows:

$$\text{curl}(\vec{v}) := \vec{\nabla} \times \vec{v} = (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}) \times (\hat{x} v_x + \hat{y} v_y + \hat{z} v_z) = \hat{x} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{y} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{z} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right).$$

The intuitive interpretation of this operation, in terms of *flows* of *vector fields*, will, again, be provided later.

1.3 Integrals of Fields

For our discussion, a *field* is defined as a function (be it scalar-valued or vector-valued) of spatial position (and time, in the most general case). We have, for the 3-dimensional space \mathbb{R}^3 , the following 3 types of integrals:

i) Line Integrals

They are, further, of the following 3 broad kinds:

a.

$$\int_{\vec{a}}^{\vec{b}} \phi \, d\vec{r}, \phi \text{ being a scalar-valued function.}$$

b.

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot d\vec{r}, \vec{v} \text{ being a vector-valued function.}$$

c.

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \times d\vec{r}, \vec{v} \text{ being a vector-valued function.}$$

A significant property of line integrals:

$$\int_{\vec{a}}^{\vec{b}} = \int_{\vec{a}}^{\vec{c}} + \int_{\vec{c}}^{\vec{b}} \quad \forall \vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3.$$

For integrals over a closed 1-dimensional path in \mathbb{R}^3 , we use the symbol:

$$\oint_L \cdot$$

ii) Surface Integrals

We define a differential area vector as follows:

$$d\vec{A} := dA \hat{n}, \text{ where : } \hat{n} \text{ denotes the unit vector normal to the differential area element } dA.$$

The differential surface element dA is not as straightforward to compute as is the case for the differential volume element dV , because in this case, the function defining the surface influences the expression of the differential surface element at any point on the surface. On the other hand, the expression for the volume element is universal for the space \mathbb{R}^3 .

Now, we have the following 3 kinds of surface integrals (over a surface, say, S):

a.

$$\int_S \phi d\vec{S}, \phi \text{ being a scalar-valued function.}$$

b.

$$\int_S \vec{v} \cdot d\vec{S}, \vec{v} \text{ being a vector-valued function.}$$

c.

$$\int_S \vec{v} \times d\vec{S}, \vec{v} \text{ being a vector-valued function.}$$

We have a simple method to determine the unit normal vector to a surface at any point on the surface. Let us consider a surface defined as follows:

$$f(x, y, z) = c, \text{ where } c \in \mathbb{R} \text{ is a constant.}$$

Now, we know that: *for any direction of $d\vec{r}$, $df = \vec{\nabla}f \cdot d\vec{r}$* . Thus, if $d\vec{r}$ is along a tangential direction to the surface at any point on the surface, then at that point, and for that $d\vec{r}$, we get:

$df = 0 = \vec{\nabla}f \cdot d\vec{r}$, because: *The value of the function is a constant on the surface defined by the function.*

As a result, we can conclude, in all generality, that $\vec{\nabla}f$ is always normal to the surface defined by: $f(x, y, z) = c$. Using this result, we compute the unit normal vector at any point on the surface as follows:

$$\hat{n} := \pm \frac{\vec{\nabla}f}{\|\vec{\nabla}f\|}.$$

The \pm symbol has been included in the definition because for any surface in \mathbb{R}^3 , the unit normal vector can point in either of the 2 directions, *going into* the surface or *coming out of* the surface, the definitions of which depends on the convention used, for a specific problem. For example, for most problems involving a closed surface, the unit normal vector, at any point on the surface, points in such a way that it appears to come out of the enclosed volume through the surface, thereby necessitating the \hat{n} to be computed accordingly, with the suitable sign. A surface integral involving a closed surface is denoted by the integration symbol:

$$\oint_S.$$

iii) Volume Integrals

We have been introduced to a volume element, already, and as expected (hint: dV is scalar-valued, unlike $d\vec{r}$ and $d\vec{S}$), we have the following 2 kinds of volume integrals (over a volume,

say, V):

a.

$$\int_V \phi \, dV, \phi \text{ being a scalar-valued function.}$$

b.

$$\int_V \vec{v} \, dV, \vec{v} \text{ being a vector-valued function.}$$

1.4 Some Important Theorems

We, now, look at a few theorems involving the operations of the $\vec{\nabla}$ operator, as follows:

i) The Gradient Theorem:

For any reasonably smooth scalar-valued function ϕ ,

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} \phi \cdot d\vec{r} = \phi(\vec{b}) - \phi(\vec{a}) \quad \forall \vec{a}, \vec{b} \in \mathbb{R}^3.$$

The proof of this is really simple; it follows directly from the result we had obtained previously, involving the gradient operation.

ii) Gauss' Theorem (also called The Divergence Theorem):

For any reasonably smooth vector-valued function \vec{v} ,



Gauss

$$\oint_S \vec{v} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{v} \, dV \text{ for any closed surface } S \text{ with volume } V.$$

iii) **Stokes' Theorem:**



For any reasonably smooth vector-valued function \vec{v} ,

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint_L \vec{v} \cdot d\vec{r} \text{ for any } 2\text{-dimensional surface } S \text{ bounded by a closed path } L.$$

Stokes

1.5 Further Discussions on the $\vec{\nabla}$ Operator

We return to the topic of operations of the $\vec{\nabla}$ operator, in order to lay down the framework for finding an expression for each of the operations in terms of any general coordinate system with an orthonormal basis.

We consider a general coordinate system, defined by the variables (a, b, c) and the orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, such that the following is the expression for the differential change in position:

$$d\vec{r} = \hat{e}_1 h_1 da + \hat{e}_2 h_2 db + \hat{e}_3 h_3 dc.$$

Then, the steps to obtain the desired general expressions are:

- Obtain the expressions of the basis vectors \hat{e}_i ($\forall i \in \{1, 2, 3\}$) in terms of the basis $\{\hat{x}, \hat{y}, \hat{z}\}$ of the Cartesian coordinate system. (We have done this, previously, for the cylindrical and the spherical polar coordinate systems.)

- For a scalar valued function f , obtain the expressions of $\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial c}$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$. For a vector-valued function \vec{v} , obtain the expressions of $\frac{\partial v_1}{\partial a}, \frac{\partial v_1}{\partial b}, \dots, \frac{\partial v_1}{\partial c}, \dots, \frac{\partial v_3}{\partial a}, \dots, \frac{\partial v_3}{\partial c}$ in terms of $\frac{\partial v_x}{\partial x}, \frac{\partial v_x}{\partial y}, \dots, \frac{\partial v_y}{\partial x}, \dots, \frac{\partial v_z}{\partial z}$.

(N.B.: By the term *function*, here, we mean the quantity being considered. The notation for a quantity has been kept the same for both, the Cartesian coordinate system variables and those of the general coordinate system, in order to avoid notational jargon.)

- From the expressions obtained in the above-described fashion, obtain the *reverse* expressions, i.e. $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ in terms of $\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial c}$, etc.

- Plug these expressions in those of the previously defined (in terms of the Cartesian coordinate system) expressions of the operations of the $\vec{\nabla}$ operator.

Using the algorithm laid down above, it is a *not-so-easy* exercise to check that the following are the desired expressions:

i.

$$\vec{\nabla}\phi = \hat{e}_1 \frac{1}{h_1} \frac{\partial\phi}{\partial a} + \hat{e}_2 \frac{1}{h_2} \frac{\partial\phi}{\partial b} + \hat{e}_3 \frac{1}{h_3} \frac{\partial\phi}{\partial c}.$$

ii.

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial a} (h_2 h_3 v_1) + \frac{\partial}{\partial b} (h_3 h_1 v_2) + \frac{\partial}{\partial c} (h_1 h_2 v_3) \right).$$

iii.

$$\vec{\nabla} \times \vec{v} = \frac{1}{h_1 h_2 h_3} \left(\hat{e}_1 h_1 \left(\frac{\partial}{\partial b} (h_3 v_3) - \frac{\partial}{\partial c} (h_2 v_2) \right) + \hat{e}_2 h_2 \left(\frac{\partial}{\partial c} (h_1 v_1) - \frac{\partial}{\partial a} (h_3 v_3) \right) + \hat{e}_3 h_3 \left(\frac{\partial}{\partial a} (h_2 v_2) - \frac{\partial}{\partial b} (h_1 v_1) \right) \right).$$

1.5.1 The Laplacian operator ∇^2



Lastly, to finish off our discussion on the utilities of the $\vec{\nabla}$ operator, we consider a new operation, defined as follows:

Laplace

$$\nabla^2 f := \vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \text{ for any smooth scalar-valued function } f.$$

The operator ∇^2 is termed as the *Laplacian* (or the *Laplace operator*). Using names of the previously defined operations, we can also call $\nabla^2 f$ as the *divergence of the gradient of f*. Now, as you might have observed, ∇^2 is a *scalar-valued* operator. Thus, we can (and it is a well-defined operation)

operate this operator on a vector-valued function \vec{v} , as well. The formal definition goes as follows:

$$\nabla^2 \vec{v} := \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{v}).$$

Using the expressions of the 3 originally defined operations of the $\vec{\nabla}$ operator in terms of a general coordinate system, we get the expression of the Laplacian acting on a scalar valued function ϕ as follows:

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial a} \left(\frac{h_2 h_3}{h_1} \left(\frac{\partial \phi}{\partial a} \right) \right) + \frac{\partial}{\partial b} \left(\frac{h_3 h_1}{h_2} \left(\frac{\partial \phi}{\partial b} \right) \right) + \frac{\partial}{\partial c} \left(\frac{h_1 h_2}{h_3} \left(\frac{\partial \phi}{\partial c} \right) \right) \right).$$

(Exercise: Check this result.)

For the case of the Laplacian acting on a vector-valued function, we obtain, in the case of the Cartesian coordinate system, the following:

$$\nabla^2 \vec{v} = \frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} + \frac{\partial^2 \vec{v}}{\partial z^2} = (\vec{\nabla} \cdot \vec{\nabla}) \vec{v}.$$

Thus, the Laplacian, in this case, can be obtained as the *inner product* of the $\vec{\nabla}$ operator with itself. As a result, we can also express the above in the following fashion:

$$\nabla^2 \vec{v} = (\nabla^2 v_x, \nabla^2 v_y, \nabla^2 v_z), \text{ where : } \vec{v} := (v_x, v_y, v_z).$$

We end our section on the $\vec{\nabla}$ operator with one final (interesting) bit of discussion.

1.5.2 Alternative Definitions

We could also have begun with the following definitions of the 3 operations of the $\vec{\nabla}$ operator, and traced our way back to the original definitions, since these definitions are *equivalent* to each other, something which is not at all trivial to show.

i.

$$\vec{\nabla} \phi := \lim_{V \rightarrow 0} \left(\frac{1}{V} \oint_S \phi d\vec{S} \right), \text{ where : } V \text{ and } S \text{ have their usual notational meanings.}$$

ii.

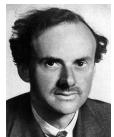
$$\vec{\nabla} \cdot \vec{v} := \lim_{V \rightarrow 0} \left(\frac{1}{V} \oint_S \vec{v} \cdot d\vec{S} \right).$$

iii.

$$(\vec{\nabla} \times \vec{v}) \cdot \hat{n} := \lim_{S \rightarrow 0} \left(\frac{1}{S} \oint_L \vec{v} \cdot d\vec{r} \right), \text{ where: } \hat{n}, L, \text{ and } \vec{r} \text{ have their usual notational meanings.}$$

Observe that the last 2 definitions can lead to the statements of the Divergence theorem and Stokes' theorem, respectively, through analytic manipulations. The first definition, in a similar fashion, leads to the following result:

$$\int_V \vec{\nabla} \phi \, dV = \oint_S \phi \, d\vec{S}.$$



1.6 The Dirac Delta (“Function”) Distribution

With a grain of salt, we define the *scalar-valued Dirac delta function* as follows:

Dirac

$$\delta(x) := \begin{cases} \infty & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

such that it satisfies the following condition:

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1.$$

(N.B.: Math enthusiasts must exercise caution in calling the above a function, because a real-valued function, by definition, can only give values belonging in \mathbb{R} as output. The ensuing discussion, as

a result, will avoid being rigorous as much as possible.)

The point where the *spike* in the value of the Delta function occurs is known as its *origin*. If instead of at $x = 0$, the spike occurs at any arbitrary $x = x_0 \in \mathbb{R}$, then x_0 is termed as the origin of the Delta function, and we have the following *generalized* notational definition:

$$\delta(x - x_0) := \begin{cases} \infty & \text{if } x = x_0, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, we have:

$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1.$$

The following are the properties of the generalized Dirac delta function:

i.

$$\int_{-\infty}^{\infty} f(x) \delta(x - x_0) dx = f(x_0), \text{ for any reasonably well-behaved function } f.$$

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \delta(x - x_0) dx &= \lim_{\epsilon \rightarrow 0} \left(\int_{x_0 - \frac{1}{\epsilon}}^{x_0 + \frac{1}{\epsilon}} f(x) \delta(x - x_0) dx \right) \quad (\epsilon > 0.) \\ &= f(x_0) \lim_{\epsilon \rightarrow 0} \left(\int_{x_0 - \frac{1}{\epsilon}}^{x_0 + \frac{1}{\epsilon}} \delta(x - x_0) dx \right) \\ &= f(x_0) \int_{-\infty}^{\infty} \delta(x - x_0) dx \\ &= f(x_0). \end{aligned}$$

□

ii.

$$\delta(\alpha(x - x_0)) = \frac{1}{|\alpha|} \delta(x - x_0) \quad \forall \alpha \in \mathbb{R}.$$

Proof. We consider a reasonably well-behaved function f .

$$\begin{aligned} \forall \alpha \in \mathbb{R}, \int_{-\infty}^{\infty} f(x) \delta(\alpha(x-x_0)) dx &= \int_{-\infty}^{\infty} f\left(\frac{u}{\alpha} + x_0\right) \delta(u) \frac{du}{|\alpha|} \quad (\text{where: } u := \alpha(x-x_0).) \\ &= \frac{1}{|\alpha|} f(x_0) \\ &= \frac{1}{|\alpha|} \int_{-\infty}^{\infty} f(x) \delta(x-x_0) dx. \end{aligned}$$

Comparing both sides above, we get our desired result. \square

(Exercise: Check that δ is an even function.)

iii.

$$\int_{-\infty}^{\infty} \delta(z-x) \delta(x-y) dx = \delta(y-z) \text{ for any well-behaved variables } y, z.$$

(Proof not included.)

- iv. For any well-behaved function f , we have the following property of the first derivative (defined in the usual way as in the case of any real-valued function) of the δ -function:

$$\int_{-\infty}^{\infty} f(x) \delta'(x-x_0) dx = -f'(x_0).$$

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \delta'(x-x_0) dx &= (f(x) \delta(x-x_0)) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \delta(x-x_0) dx \\ &= 0 - f'(x_0) \\ &= -f'(x_0). \end{aligned}$$

\square

We, furthermore, have the following definition of the *composition* of δ and f (a *continuously differentiable* function with $f' \neq 0$ everywhere):

$$\forall x \in \mathbb{R}, \quad \delta(f(x)) := \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \text{ where: } x_i \text{'s are all of the different simple roots of } f \text{ in } \mathbb{R}.$$

(Exercise: Write down the expression for $\delta(x^2 - a^2)$, $a \in \mathbb{R}$.)

1.6.1 δ -function of Position Vectors

For the final topic of discussion on the basic mathematical tools required for this course, we shall discuss the δ -function as a scalar-valued function of vectors in \mathbb{R}^3 . The various aspects of this discussion will be vastly similar to those of the above one. We begin with the simple (and logically familiar) definition:

$$\delta^3(\vec{r} - \vec{r}_0) := \begin{cases} \infty & \text{if } \vec{r} = \vec{r}_0, \\ 0 & \text{otherwise,} \end{cases}$$

such that:

$$\int_{\mathbb{R}^3} \delta^3(\vec{r} - \vec{r}_0) dV = 1.$$

Like in the previous case, we get the following algebraic properties:

i.

$$\int_{\mathbb{R}^3} f(\vec{r}) \delta^3(\vec{r} - \vec{r}_0) dV = f(\vec{r}_0), \text{ for any reasonably well-behaved function } f.$$

(f can be either scalar-valued or vector-valued.)

ii.

$$\delta^3(\alpha(\vec{r} - \vec{r}_0)) = \frac{1}{|\alpha|^3} \delta^3(\vec{r} - \vec{r}_0) \quad \forall \alpha \in \mathbb{R}.$$

Interestingly, we get the following result (in terms of Cartesian coordinate system):

$$\delta^3(\vec{r} - \vec{r}_0) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0).$$

Proof. Checking that the condition for the definition of $\delta^3(\vec{r} - \vec{r}_0)$ holds:

$$\begin{aligned} \int_{\mathbb{R}^3} \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) dV &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) dx dy dz \\ &= (\int_{\mathbb{R}} \delta(x - x_0) dx)(\int_{\mathbb{R}} \delta(y - y_0) dy)(\int_{\mathbb{R}} \delta(z - z_0) dz) \\ &= 1. \end{aligned}$$

□

(Exercise: Verify that the algebraic properties (i. and ii.) mentioned above hold for this expression of $\delta^3(\vec{r} - \vec{r}_0)$.)

For any general coordinate system, defined by the variables (a, b, c) and the orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, we have the following expression:

$$\delta^3(\vec{r} - \vec{r}_0) = \frac{1}{h_1 h_2 h_3} \delta(a - a_0) \delta(b - b_0) \delta(c - c_0).$$

Proof. Similar to that for the case of Cartesian coordinate system. □

Chapter 2

Electrostatics: Coulomb's law

We will learn the basics of the theory of electricity and magnetism in this course. While you may have been exposed to these ideas before (and know how to solve a few interesting and even difficult problems!), I will assume that we are starting from the very beginning. We will also need to learn some mathematics as we go along in order to strengthen the understanding of the associated physics concepts. For those of you who feel daunted by the maths, I request your patience and effort. I assure you that, given your persistence, everything will fall into place. So let's get started by reviewing the basics.

2.1 Forces



In your course on mechanics last semester (PH1101), you learnt that the net unbalanced force F on a body of mass m is related to its acceleration as follows:

Newton

$$\vec{F} = m \vec{a}. \quad (2.1)$$

This relation is useful for

- (i) defining the notion of mass, i.e., the proportionality constant between a known force and the acceleration it produces. Of course, this needs your being able to measure forces!

One way to proceed is to first borrow a standard 1kg that has been defined by the Bureau of Standards (in whom we trust!), and then use the Force law given above to measure all other masses. How would we do this? For instance, you could connect the 1kg mass to a spring (attached to a wall, say), pull on the mass so that the spring is extended and measure it's extension x (use a ruler!) and it's acceleration a_{1kg} (use a ruler and a clock!). Now, unhook the 1kg mass

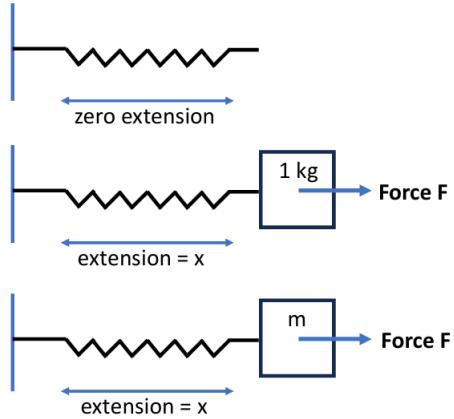


Figure 2.1: Top panel: a spring attached to the wall, but unextended. Middle panel: A mass of 1 kg is attached to the spring, and a force F is applied to the mass such that the spring is extended by a length x . Bottom panel: An unknown mass m is attached to the spring, and a force applied to the mass such that the spring is extended by a length x . By Hooke's law, this force must be the same (F) as that applied the 1 kg mass.



and attach the unknown mass m to the spring instead. Pull on this unknown mass so that you extend the spring by the same extension x and measures its acceleration a_m as well. Now, given that the extension x is the same, we know (thanks to Hooke's Law: $F = -Kx$, where K is the spring constant) that the force exerted by both masses must be the same. Thus,

$$F = 1\text{ kg} \times a_{1\text{ kg}} = m \times a_m. \quad (2.2)$$

Since you've measured both accelerations, you can now determine the mass m from the above (in units of kgs).

(ii) The relation is also useful for determining the unknown force acting on a body of known mass by measuring it's acceleration, and

(iii) determining the acceleration of a body if it's mass is known, as is the force acting on it.

Can we use the notion of a force to determine it's trajectory? Yes, but for this you will need another independent relation that has the force F in it (i.e., different from the $F = ma$ given above). Examples include:

(i) Hooke's law : $\vec{F} = -Kx\hat{i}$, where K is the spring constant, and the $-$ sign tells you that the force is restoring in nature, i.e., it opposes the stretching of the spring, and

(ii) Newton's law of gravitation : $F = -\frac{GMm}{r^2}$, where M and m are the masses of the two bodies feeling the gravitational force, r is the distance between them (i.e., their relative displacement with

Newton



regards to some chosen origin of coordinates) and G is Newton's gravitational constant.

Note that the law of gravitation is unlike Hooke's law in the sense that it needs no contact between the two bodies. The gravitational force is thus "non-local" in nature, and an example of "action at a distance". [Newton was clearly possessed of a formidable personality, as you could have been burned at the stake in those times for such claims of witchcraft!] Also, note that the gravitational force has no "range", i.e., it exists between two masses that are infinitely far away (even if it is infinitely weak!). Consider a force $F \propto F_0 e^{-r/a}$, where F_0 is a constant (with dimensions of force) and the constant a clearly sets a lengthscale that we can think of as the range of the force (i.e., beyond which the force is negligibly small!). Clearly, there is no such constant signifying a range in Newton's law of gravitation. In an earlier course, you would have learnt how Newton was able to use this law to understand the trajectories of planets in our solar system (Kepler's laws). We also use a convention in noting that the gravitational is also purely attractive in nature: the negative sign in F arises from the fact that M, m and r are all positive (as is the proportionality constant G), the force is directed along $-\hat{r}$. We will very soon encounter forces that can be attractive as well as repulsive. As a final point, note that if we take the (say, smaller) body m very close to the (say, bigger) body M (i.e., the surface of the Earth)

$$\vec{F} = -\frac{GMm\hat{r}}{r^2} \Big|_{r \rightarrow R_E} \implies -\frac{GM\hat{r}}{R_E^2} = g, \quad (2.3)$$

i.e., the magnitude acceleration due to gravity very near to, and directed towards, the surface of the Earth is $g = GM/R_E^2$, where R_E is the radius of the Earth.

2.2 The Electric Force

Discovering new forces is an experimental task. If we believe that $F = ma$ holds in all circumstances, then a measurement of a that cannot determine all the forces contributing to F suggests that you've got to search for something new. Here's one possible experiment: comb your hair (on a dry day!), and hold the comb to a small piece of paper. You see that the paper sticks to the comb, resisting gravity's pull on it. Indeed, the "mysterious force" attracting the paper to the comb is resisting (or acting against) the gravitational pull of the entire planet Earth on the paper! So, something new is clearly working on the paper. But what?

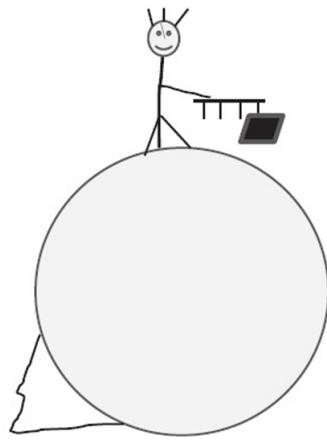


Figure 2.2: The comb is pulling up the piece of paper electrostatically, while the entire Earth is pulling it towards itself via gravitation! Guess who wins? Credit: Shankar, Vol.2

Let's perform a few more experiments.

1. Rub a metal rod on fur (being nice to your friend's dog/ cat will help). Now touch this rod to an isolated metallic sphere. The two will be observed to repel one another. (Make sure that you are wearing silk gloves while doing this!) There is nothing very mysterious about this: the rubbing caused electric charge to jump onto the rod, rendering it negatively charged. Touching the sphere caused some of the charge to migrate onto the sphere, and the rod and sphere thus repelled one another.
2. Slowly move the charged rod near an isolated metallic sphere. Before they touch, they will attract one another. The same happens if you replace the metal sphere by a piece of paper. The paper is a dielectric, and it becomes polarised when the charged rod is brought nearby: there is a slight displacement of the electrons and the positively charged nuclei, such that the positive charges are placed near to the negatively charged rod. Hence, the paper is attracted to the rod.

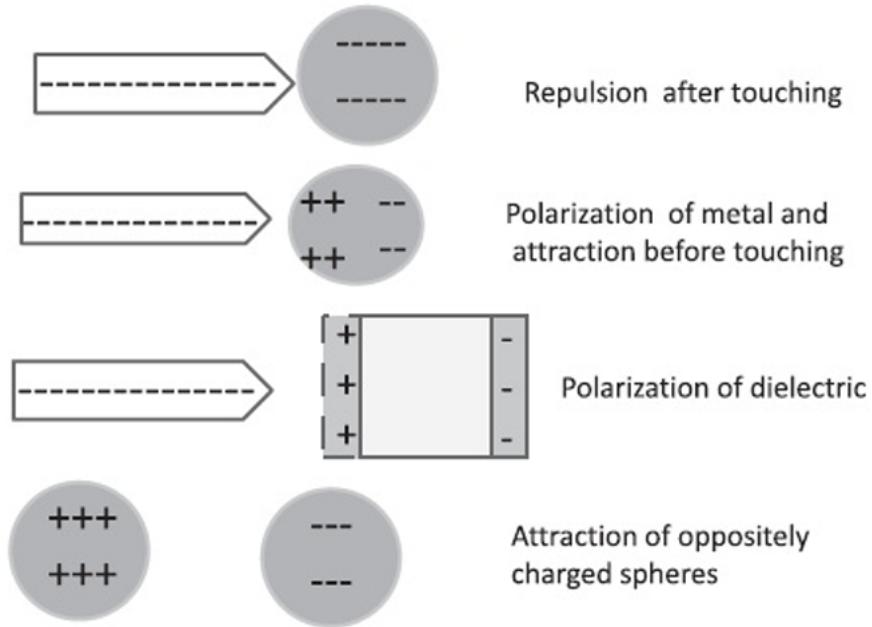


Figure 2.3: Top panel: A negatively charged rod imparts some negative charge to the sphere upon contact and they repel each other. Second panel: The negatively charged rod attracts a neutral sphere by polarizing it without touching it. Third panel: A charged rod polarizes a dielectric. The light region in the middle is the overlap of rectangles with positive (dotted boundary) and negative charges (solid boundary). The light region is neutral and the edges carry the uncanceled charges. Bottom panel: The charged spheres attract because they have been charged oppositely.
Credit: Shankar, Vol.2

3. Repeat experiments 1 and 2 by rubbing the metallic rod on polyester cloth instead of fur. We will observe essentially the same results but with the following difference: rubbing the rod on the polyester causes electric charge to migrate from the rod onto the cloth. This leaves the rod positively charged. The rest of the story as far as experiments 1 and 2 are concerned will be identical, but with signs reversed.
4. Take two rods, rub one against fur and the other against polyester cloth. Bring them close together and watch them attract one another!
5. Connect the two rods from experiment 4 to one another with a metallic wire and note that they stop attracting one another. This is because, as long as the two rods were equally but oppositely charged, the conducting wire transfers excess charge from the negatively charged rod to the positively charged rod until both are rendered charge neutral.

2.3 Coulomb's Law

This law is thanks to Charles Augustine de Coulomb (1736-1806). This law gives us a way by which to quantify charges as well as the forces between static charges



$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}^2} \hat{r}_{12} = -\vec{F}_{21}, \quad (2.4)$$

Coulomb where $r_{12} = |\vec{r}_2 - \vec{r}_1|$ is the spatial distance between the points characterised by the displacements \vec{r}_2 and \vec{r}_1 , $\hat{r}_{12} = \vec{r}_2 - \vec{r}_1 / |\vec{r}_2 - \vec{r}_1|$ is the unit vector associated with \vec{r}_{12} , and the proportionality constant in Coulomb's law is $1/4\pi\epsilon_0 = 9 \times 10^9 \text{ Nm}^2/\text{C}^2$ where C stands for Coulombs (the unit of charge!).

Immediate takeaways : 1. If q_1 and q_2 have the same sign (i.e., both + or both -), they repel one

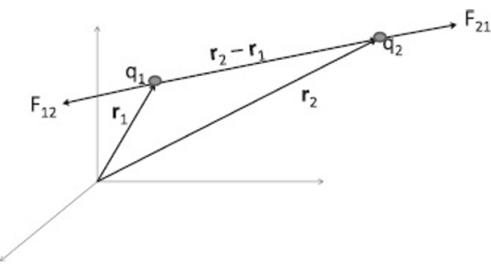


Figure 2.4: The forces between two charges q_1 and q_2 located at \vec{r}_1 and \vec{r}_2 . The force \vec{F}_{12} acts on q_1 due to q_2 and is equal and opposite to \vec{F}_{21} , which is defined similarly. Credit: Shankar, Vol.2

another. On the other hand, if q_1 and q_2 have opposite signs, they attract one another.

2. The force formula also helps us define a charge of 1 Coulomb (1C): if two charges of 1C each are held apart by a distance of 1m, the repulsive force between them is $9 \times 10^9 \text{ N}$ (an enormous force! How many human adults of 100kg each would weigh that much? $10^7!$). This tells us that we won't usually encounter 1C of charge; nature would usually look to neutralise such large charges. Indeed, the charge of the electron is $-e = -1.6 \times 10^{-19} \text{ C}$ and the charge of the proton is $e = 1.6 \times 10^{-19} \text{ C}$; all atoms contain only a few of these charges outside and inside the nucleus respectively.

2.4 A few important points on the electric charge

1. The total charge in any given circumstance is conserved, i.e., it does not change in time, provided you keep track of all signs. This is equally true of chemical and nuclear reactions (taking place inside particle accelerators), as it is of electrical charge in classical electrodynamics. Importantly,

electrical charge is conserved locally, i.e., it cannot disappear and reappear here and there at will. Instead, it can move around, and its motion can be tracked continuously. Mathematically, this is encoded within the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = \sigma , \quad (2.5)$$

where $\rho(x, y, z, t)$ is the local charge density (a scalar quantity) and $\vec{j}(x, y, z, t)$ is the local current density (a vector) which tracks the macroscopic motion of the charge density ($\vec{j} = \rho \vec{v}$, where \vec{v} is the local velocity vector of the current density). Further, $\partial/\partial t$ refers to a partial derivative with respect to time t (keeping x, y , and z fixed), $\vec{\nabla} \cdot \vec{j}$ refers to the divergence of the current density, i.e., the net flux of charge carried by the current density (i.e., the difference between influx and outflux), and σ accounts for the net charge in the system under consideration per unit time per unit volume due to external influences (i.e., sources and sinks). Thus, $\sigma > 0$ refers to a net flux of charge due to sources (influx of charges into the system under study) and $\sigma < 0$ to a net flux of charge due to sinks (outflux of charges from the system).

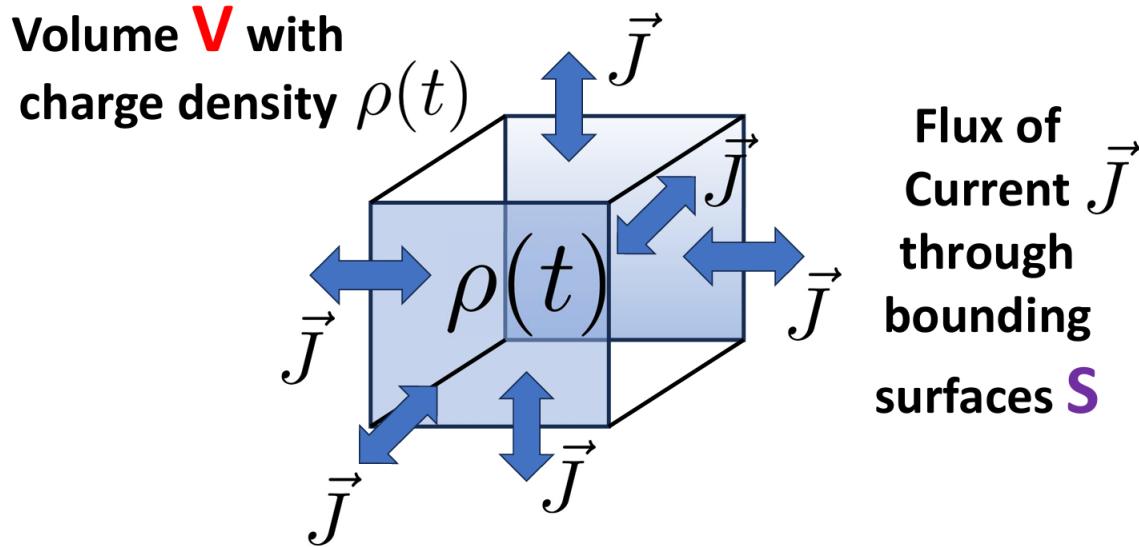


Figure 2.5: The equation of continuity relates the rate of change of charge density $\rho(t)$ within a volume V to the divergence of the current flux \vec{j} (i.e., difference between influx and outflux) through the surfaces S that bound V , and including the presence of sources and sinks.

For the case of charges that are locally conserved (i.e., no sources or sinks, and no intrinsic mechanism by which to create/ destroy charge at will), $\sigma = 0$ and the equation of continuity

simplifies to

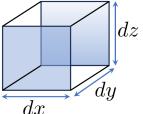
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 . \quad (2.6)$$

What the above tells us is this: in any closed region of space, if you (i) count all charges inside the volume and keep track of how their number changes with time, and (ii) keep track of all charge leaving or entering the volume through its enclosing surfaces (the “flux”), you will find that the two must be equal and opposite to one another in the absence of sources and sinks. In the presence of sources and sinks, the two counts are known upto a value dictated by the external sources.

We can write the total charge within the system of volume V as

$$Q(t) = e\rho(t)V \quad \text{if } \rho \text{ is independent of } x, y, z \quad (2.7)$$

$$= e \int dx \int dy \int dz \rho(x, y, z, t) \quad \text{otherwise ,} \quad (2.8)$$



where I have made the electron charge e explicit, such that the ρ in eq.(2.8) refers to the number density (and $\int_V dV \rho$ to the total number of charge carrying particles inside the volume V). The total change in charge Q is then given by

$$\frac{dQ(t)}{dt} = \frac{d}{dt} \left(e \int dx \int dy \int dz \rho(x, y, z, t) \right) . \quad (2.9)$$

The net change in charge moving across the surfaces S bounding the volume V is the flux

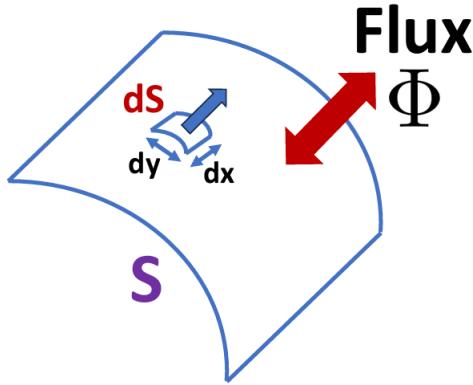


Figure 2.6: The current flux Φ is obtained by integrating the differential flux $\vec{j} \cdot \vec{dS}$ over all small patches of the area $d\vec{S}$ within the surface S .

$$\Phi = \iint_S \vec{j} \cdot \vec{dS} \quad (2.10)$$

where $d\vec{S}$ is the unit area vector whose direction is defined as normal to the patch of area $|dS|$. Finally, the total change in charge within volume V due to external sources and sinks is Σ . Taking

all these three contributions together, we receive the integral version of the equation of continuity

$$\frac{dQ}{dt} + \iint_S \vec{j} \cdot d\vec{S} = \Sigma . \quad (2.11)$$

To see where the differential form of the equation of continuity given above in eq.(2.5) arises from, imagine taking the limit of shrinking all three dimensions of the volume V to be very small, i.e., a differential volume element dV of size $O(\epsilon^3)$ where ϵ is the extent of the volume in each of x, y and z . Then, we can consider a “local” charge density $\rho(x, y, z, t)$ and current density $\vec{j}(x, y, z, t)$ such that

$$\begin{aligned} \frac{dQ}{dt} &= \int_V dV \frac{\partial \rho}{\partial t} , \quad \Sigma = \int_V dV \sigma , \\ \Phi &= \iint_S \vec{j} \cdot d\vec{S} = \int_V dV (\vec{\nabla} \cdot \vec{j}) . \end{aligned}$$

Note that, in the second line above, we’ve used a special relation between an integral over a volume V of the divergence of the current vector $\vec{\nabla} \cdot \vec{j}$ as being equal to an integral of the current \vec{j} passing through the bounding surface S . This is called Gauss’s theorem, attributed to Carl Friedrich Gauss, and we will learn more about this up ahead.

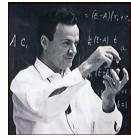


Then, we can rewrite the integral form of the continuity equation as follows:

$$\begin{aligned} \text{Gauss} \quad & \frac{dQ}{dt} + \iint_S \vec{j} \cdot d\vec{S} = \Sigma , \\ & \int_V dV \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} \right] = \int_V dV \sigma , \\ & \Rightarrow \int_V dV \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} - \sigma \right] = 0 , \\ & \Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} - \sigma = 0 , \\ & \text{or, } \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = \sigma . \end{aligned} \quad (2.12)$$

It is worth noting that the approach we have taken above treats the electric current as a “fluid”. Such an approach is called “hydrodynamic” in nature, and relies on densities and currents for the description of the statics and dynamics of the fluid. Indeed, all circuit theory and modern day machinery has emerged from such a hydrodynamic treatment.

2. A second important feature of electrical charge is that it takes values in the form of integer multiples of a discrete value $e = \pm 1.6 \times 10^{-19} \text{ C}$. This value comes from our understanding of the quantum mechanical nature of the universe, and a deeper explanation of its origin lies well beyond the scope of these lectures. Suffice it to say for now that e is what is responsible for all interactions



between light and matter at the quantum level; this is called “quantum electrodynamics” (QED) and won Richard Feynman, Julian Schwinger and Sin-Itiro Tomonaga the Nobel prize in 1964. Perhaps you may wonder on why the value of e is as small as it is: the rough answer is that we’ve learnt from QED and beyond that the value of e depends on the energyscale at which you probe nature with regards to this question. In our present day cold universe, the temperature of the cosmic microwave background radiation (which makes up most of so-called empty space) is 2.726K (i.e., very near absolute zero!) and e is as we find it. If we could go far back in time towards the big bang and a much hotter universe, we would find the value of e to be much larger. This energyscale dependent measure of e goes by the name of the “renormalisation of the electric charge”, and I hope you will learn more about this someday.

Feynman



Schwinger



Tomonaga

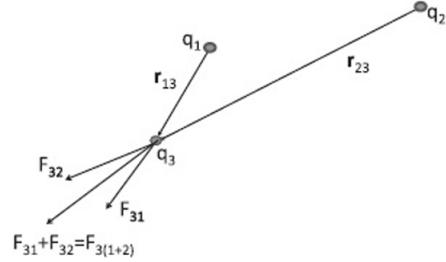


Figure 2.7: The electric force on charge q_3 due to charges q_1 and q_2 is the sum of the forces each would have exerted on q_3 in the absence of the other. This is the superposition principle. Credit: Shankar, Vol.2

An empirically established fact of classical electrodynamics says that the combined or effective Coulomb force that acts on a charge q_1 due to a set of other ($N - 1$) charge q_2, q_3, \dots, q_N is given by a vector sum of all pairwise “two-body” interactions between q_1 and each of q_2, \dots, q_N

$$\vec{F}_{1,eff} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \dots + \vec{F}_{1N}. \quad (2.13)$$

This is called the “superposition principle”. It should be noted that this is not a logical necessity or even an outcome of Coulomb’s law. Indeed, if we include the effects of relativistic quantum mechanics, this would not even be true. However, it seems to work at the level of classical electromagnetic theory just fine.

2.6 How would you verify Coulomb's law?

Here are a series of "thought" experiments that we could consider conducting in order to verify Coulomb's law: $\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}^2} \hat{r}_{12} = -\vec{F}_{21}$.

Experiment 1

Suppose we want to measure the charge carried by a charged sphere, how should we go about it? Here is one way. Take two identical charged spheres each with an unknown charge q and place them at some known separation (say 1m). Then measure the force exerted by the repulsion (by, say, connecting the spheres to calibrated springs and using Hooke's law). Then, use Coulomb's law to evaluate q^2 and thence q .

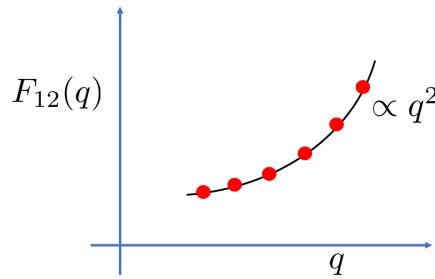


Figure 2.8: A schematic representation for how an experimental verification of F_{12} versus q should look for the case $q_1 = q_2 \equiv q$.

Experiment 2

Verify that $F_{12} \propto q_1$ and $F_{12} \propto q_2$.

Fix, say, q_1 and vary q_2 and track the changing force F_{12} . Then do the opposite.

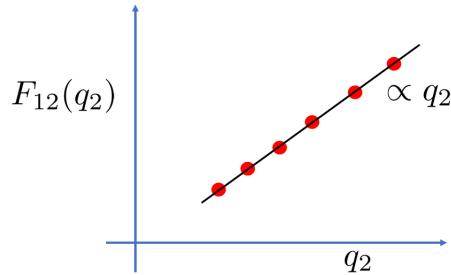


Figure 2.9: A schematic representation for how an experimental verification of F_{12} versus q_2 should look for the case $q_1 \neq q_2$ and the charge q_1 is a known fixed quantity.

Experiment 3

Verify that $F_{12} \propto r_{12}^{-2}$.

For this, we don't even need to know the values of q_1 and q_2 ! Measure F_{12} for a given value of r_{12} . Now repeat the measurement for $r_{12} \rightarrow 2r_{12}$, $r_{12} \rightarrow 3r_{12}$, $r_{12} \rightarrow 4r_{12}$ and so on. Repeat the experiment till you get atleast 7-10 readings for r_{12} and corresponding values of F_{12} . Now, plot the measured values of F_{12} versus r_{12} . Your plot should hopefully verify the relation $F_{12} \propto r_{12}^{-2}$ (which can be confirmed from a best fit and looking at the goodness of fit number!).

Finally, see how close you can get to the known value of $4\pi\epsilon_0$.

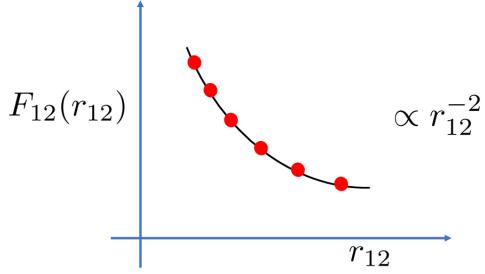


Figure 2.10: A schematic representation for how an experimental verification of F_{12} versus r_{12} should look for the case $q_1 \neq q_2$ and both charges q_1 and q_2 are known fixed quantities.

2.7 Ratio of gravitational and electric forces

For two particles of mass m_1 and m_2 , charges q_1 and q_2 kept at a distance r apart, the ratio of the gravitational (F_G) and electric (F_e) forces is given by

$$\frac{F_G}{F_e} = \frac{Gm_1m_2/r^2}{q_1q_2/4\pi\epsilon_0 r^2} = \frac{Gm_1m_24\pi\epsilon_0}{q_1q_2} . \quad (2.14)$$

For two electrons with mass $m = 9 \times 10^{-31}\text{kg}$, $e = 1.6 \times 10^{-19}\text{C}$, $G = 6.7 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$, we obtain

$$\frac{F_G}{F_e} = \frac{6.7 \times 10^{-11} \times (9 \times 10^{-31})^2}{(1.6 \times 10^{-19})^2 \times 9 \times 10^9} \approx 2.3 \times 10^{-43} ! \quad (2.15)$$

For a proton and an electron, $F_G/F_e \sim 10^{-40}$ and for two protons, $F_G/F_e \sim 10^{-36}$. Thus, we clearly establish that gravity is much weaker than the electric force. Remarkably, given the fact that the electric force can be either repulsive or attractive (as electric charges come in two varieties) whereas gravity is always attractive, nature appears to nullify electrical forces by choosing charge neutral objects. But masses cannot be "hidden" in this way, and a weak gravitational force thus shapes the universe at the biggest scales by acting on large (i.e., very massive) bodies in a big way. Consider the clumping of galaxies, the formation of stars and galaxies etc. However,



electromagnetic forces play a very important role in shaping various materials, molecules and even life: recall that van der Waals forces and Hydrogen bonding are essentially electromagnetic in nature! Indeed, electromagnetic forces dominate in the range of lengthscales lying between 10^{-15}m (Femtometer) and 10^7m . Gravity dominates in the scales above 10^7m while nuclear (weak and strong) forces dominate in the scales below 10^{-15}m . This neat separation of lengthscales for the various fundamental forces demarcates roughly (in increasing scale) the fields of high energy physics, condensed matter physics and gravitation & cosmology.

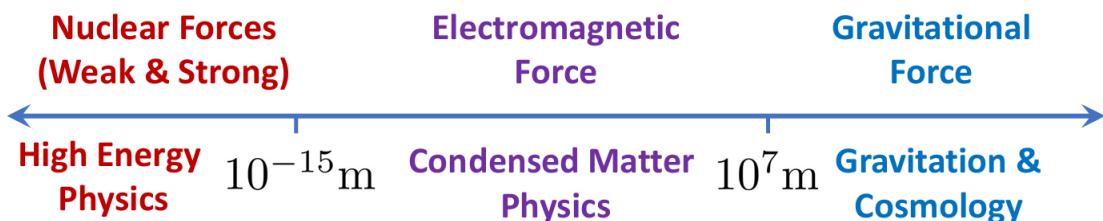


Figure 2.11: The range of lengthscales at which various fundamental forces of nature dominate.

2.8 Coulomb's law for a continuous charge density

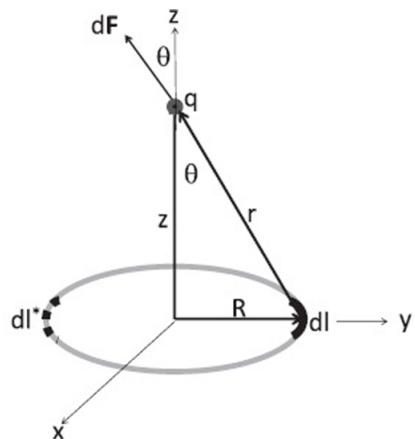


Figure 2.12: The electric force due to a loop in the xy -plane, on a charge located on the z -axis. The highlighted segment of length dl has charge λdl and exerts a force $d\vec{F}$. We keep only the vertical part along the z -axis since the diametrically opposite segment dl^* (shown by a dotted curve) will cancel the horizontal part. Credit: Shankar, Vol.2

For continuous charge aggregates that are described by densities, Coulomb's law will obtain

the net force by replacing discrete sum ($\sum_i F_i$) by integrals over continuous variables ($\int dr F(r)$). Let us consider the following example. A circular wire of radius R with line charge density $\lambda \text{ Cm}^{-1}$ is lying on the XY plane with its centre as the origin. Find the force exerted on a charge q located at a height z on the z -axis. Noting that a line segment of length dl has a charge λdl , the Coulomb force on q due to such a line segment lying on the y -axis is

$$d\vec{F}_{dl} = \frac{q\lambda dl}{4\pi\epsilon_0(R^2+z^2)}(-\sin\theta\hat{j}+\cos\theta\hat{k}) . \quad (2.16)$$

Now, for every such line segment dl (no matter where it is placed on the circle!), there exists a segment dl' lying diametrically opposite such that the horizontal components of the Coulomb force (i.e., in the XY plane, and $\propto \sin\theta = R/(R^2+z^2)^{1/2}$) will cancel one another.

However, the vertical components (i.e., along \hat{k} , $dF_{dl}^z \propto \cos\theta = z/(R^2+z^2)^{1/2}$) will add up

$$\begin{aligned} F^z &= \int dF_{dl}^z = \int_0^{2\pi R} dl \frac{q\lambda}{4\pi\epsilon_0(R^2+z^2)} \frac{z}{(R^2+z^2)^{1/2}} , \\ &= \int_0^{2\pi R} dl \frac{q\lambda z}{4\pi\epsilon_0(R^2+z^2)^{3/2}} = \frac{q\lambda 2\pi R z}{4\pi\epsilon_0(R^2+z^2)^{3/2}} . \end{aligned} \quad (2.17)$$

Here are a few sanity checks. First, for $z = 0$ (i.e., the charge q located at the centre of the loop), the force F^z should vanish as all the electrical forces either precisely cancel one another or vanish identically. Indeed, this is easily seen to be the case by setting $z = 0$ in the result above. Second, for $z \gg R$, the loop should look like a point charge when viewed from the location of q

$$F^z = \frac{q\lambda 2\pi R z}{4\pi\epsilon_0(R^2+z^2)^{3/2}}|_{z \gg R} = \frac{q\lambda 2\pi R z}{4\pi\epsilon_0 z^3} = \frac{(2\pi R \lambda)q}{4\pi\epsilon_0 z^2} , \quad (2.18)$$

i.e., the entire loop acts like a single charge of value $2\pi R \lambda$ when q is placed very far away!

Chapter 3

Electrostatics: the Electric field I

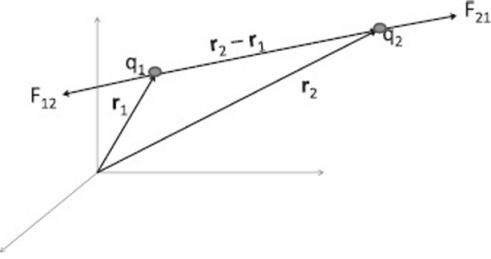


Figure 3.1: The forces between two charges q_1 and q_2 located at \vec{r}_1 and \vec{r}_2 . The force \vec{F}_{12} acts on q_1 due to q_2 and is equal and opposite to \vec{F}_{21} , which is defined similarly. Credit: Shankar, Vol.2

We will now introduce the idea of the electric field. For this, we rewrite Coulomb's law as



$$\vec{F}_{12} = \frac{q_1 q_2 \hat{r}_{12}}{4\pi\epsilon_0 r_{12}^2} = q_2 \frac{q_1 \hat{r}_{12}}{4\pi\epsilon_0 r_{12}^2} = q_2 \vec{E}_{12}(\vec{r}_1), \quad (3.1)$$

where $\vec{E}_{12}(\vec{r}_1)$ is the electric field at \vec{r}_1 due to the charge q_2 at position \vec{r}_2 . Since $\vec{F}_{12} = -\vec{F}_{21}$, we have

Coulomb

$$\begin{aligned} \vec{F}_{21} = -\vec{F}_{12} &= -q_1 \frac{q_2 \hat{r}_{12}}{4\pi\epsilon_0 r_{12}^2}, \\ &= q_1 \frac{q_2 \hat{r}_{21}}{4\pi\epsilon_0 r_{12}^2} \quad (\text{as } \hat{r}_{21} = -\hat{r}_{12}), \\ &= q_1 \vec{E}_{21}(\vec{r}_2), \end{aligned} \quad (3.2)$$

where $\vec{E}_{21}(\vec{r}_2)$ is the electric field at \vec{r}_2 due to the charge q_1 located at \vec{r}_1 .

The upshot is that we can choose to compute the electric field due to any one of the two charges by measuring the force experienced by the other due to it. Notice

(a) while an electric field can be generated by only one charge q (placed at, say, the origin of

coordinates)

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \quad (3.3)$$

at position \vec{r} and $\hat{r} = \vec{r}/|\vec{r}|$ is the unit vector defined in the radial direction from the origin to point \vec{r} , one needs a second charge to experience a force \vec{F} exerted due to this electric field.

(b) The electric field $\vec{E}(\vec{r})$ is non-zero everywhere, and therefore a charge placed anywhere will experience the Coulomb force \vec{F} .

(c) As $\vec{E} \propto \vec{F}$, for the net field at a given point due to the fields emanating from a collection of charges placed at various positions, we can use the superposition principle to compute the vector sum.

(d) Measuring an electric field is simple. Put a test charge q (of known value) at \vec{r} , and use the relation $\vec{F}(\vec{r}) = \vec{E}(\vec{r})q$. For $q = 1C$, we say that the field is the force on a unit charge.

Let's practice this in an example. Given the distribution of charges of q_1 at $(0, y)$ (i.e., on the y-axis), q_2 at $(0, 0)$ (the origin) and q_3 at $(x, 0)$ (i.e., on the x-axis) as shown below in Fig.3.2 (for the special case of $x = a = y$), what is the electric field at $\vec{r} = (x, y)$?

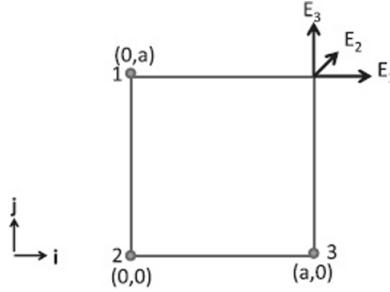


Figure 3.2: The electric fields \vec{E}_1 , \vec{E}_2 and \vec{E}_3 at (a, a) in terms of the unit vectors \hat{i} and \hat{j} , due to three equal charges q located at $(0, a)$, $(0, 0)$ and $(a, 0)$. The total field at (a, a) is the vector sum of the three pieces. A more general case is first treated in the text. Credit: Shankar, Vol.2

Let us compute the electric fields at $\vec{r} = (x, y)$ due to the three charges q_1 , q_2 and q_3 in turn. Thus, the electric field at $\vec{r} = (x, y)$ due to charge q_1 is given by

$$\vec{E}_1 = \hat{i} \frac{q_1}{4\pi\epsilon_0 ((x-0)^2 + (y-a)^2)^{1/2}} = \hat{i} \frac{q_1}{4\pi\epsilon_0 x^2} . \quad (3.4)$$

Similarly, the electric field at $\vec{r} = (x, y)$ due to charge q_3 is given by

$$\vec{E}_3 = \hat{j} \frac{q_3}{4\pi\epsilon_0 ((x-a)^2 + (y-0)^2)^{1/2}} = \hat{j} \frac{q_3}{4\pi\epsilon_0 y^2} . \quad (3.5)$$

Further, the electric field at $\vec{r} = (x, y)$ due to charge q_2 is given by

$$\vec{E}_2 = \frac{q_2}{4\pi\epsilon_0 [(x-0)^2 + (y-0)^2]^{1/2}} (\cos\theta\hat{i} + \sin\theta\hat{j}), \quad (3.6)$$

$$= \frac{q_2}{4\pi\epsilon_0 [x^2 + y^2]} \left(\frac{x}{\sqrt{x^2 + y^2}} \hat{i} + \frac{y}{\sqrt{x^2 + y^2}} \hat{j} \right), \quad (3.7)$$

$$= \frac{q_2}{4\pi\epsilon_0 [x^2 + y^2]^{3/2}} (x\hat{i} + y\hat{j}), \quad (3.8)$$

where the angle θ is defined as that between \vec{E}_2 and \hat{i} , such that

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}. \quad (3.9)$$

Finally, using the superposition principle, we obtain

$$\begin{aligned} \vec{E}(x, y) &= \vec{E}_1 + \vec{E}_2 + \vec{E}_3, \\ &= \frac{1}{4\pi\epsilon_0} \left[\left(\frac{q_1}{x^2} + \frac{q_2 x}{(x^2 + y^2)^{3/2}} \right) \hat{i} + \left(\frac{q_3}{y^2} + \frac{q_2 y}{(x^2 + y^2)^{3/2}} \right) \hat{j} \right]. \end{aligned} \quad (3.10)$$

For the special case of $x = a = y$, $q_1 = q_2 = q_3 \equiv q$ (see Fig.3.2), we obtain

$$\begin{aligned} \vec{E}_1 &= \frac{q}{4\pi\epsilon_0 a^2} \hat{i}, \quad \vec{E}_3 = \frac{q}{4\pi\epsilon_0 a^2} \hat{j}, \quad \vec{E}_2 = \frac{q(\hat{i} + \hat{j})}{4\pi\epsilon_0 2 \sqrt{2} a^2}, \\ \vec{E}(a, a) &= \frac{q}{4\pi\epsilon_0 a^2} \left[1 + \frac{1}{2\sqrt{2}} \right] (\hat{i} + \hat{j}). \end{aligned} \quad (3.11)$$

Further, for the case when $x \gg y$, i.e., the effect of charges q_1 and q_2 are expected to become negligible in comparison to that of q_3 (as it is the closest to the point (x, y)). We find that this is indeed the case as we find

$$\vec{E}(x, y)|_{x \gg y} \rightarrow \frac{q_3}{4\pi\epsilon_0 y^2}. \quad (3.12)$$

Similarly, for the case of $y \gg x$, we find that the effect of charges q_2 and q_3 are expected to become negligible in comparison to that of q_1 (as it is the closest to the point (x, y)):

$$\vec{E}(x, y)|_{y \gg x} \rightarrow \frac{q_1}{4\pi\epsilon_0 x^2}. \quad (3.13)$$

3.1 Visualising the Electric field

Remember that $\vec{E}(\vec{r})$ is a vectorial quantity, and hence any pictorial representation should communicate both numerical values as well as directions. First, it is easy to see that since the field due to a charge q placed at the origin,

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}, \quad (3.14)$$

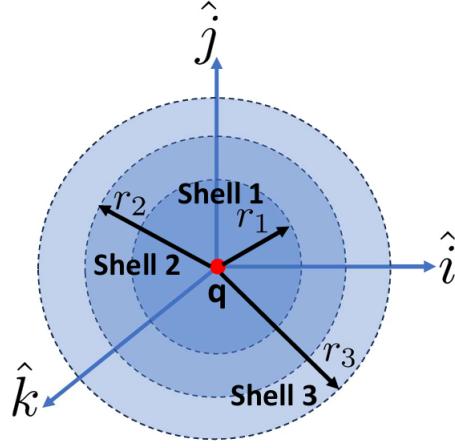


Figure 3.3: The electric fields \vec{E} for a single charge q placed at the origin of coordinates is fixed at various spherical surface contours visualised in terms of the corresponding radius. Three such shells, corresponding to radii r_1 , r_2 and r_3 are shown in the figure.

such that $\vec{E}(\vec{r})$ depends on only the radial coordinate and the radial unit vector (and neither of the two angular coordinates θ and ϕ nor their unit vectors). Hence, the electric field $\vec{E}(\vec{r})$ possesses spherical symmetry, radiates outwards and must be isotropic (i.e., uniformly distributed with respect to θ and ϕ). Further, the strength of $\vec{E}(\vec{r})$ is fixed on a spherical shell (defined with a fixed radius $r = |\vec{r}|$, $0 \leq \theta \leq 180^\circ$ and $0 \leq \phi \leq 360^\circ$).

Further, the strength will fall off as r^{-2} as we go outwards from a given shell to another. Given that $E \propto r^{-2}$,

$$\left| \frac{E(r_2)}{E(r_1)} \right| = \left(\frac{r_1}{r_2} \right)^2 \ll 1 \text{ for } r_2 < r_1. \quad (3.15)$$

As mentioned earlier, given the vectorial nature of the electric field, we need to indicate its direction; for the electric field due to a point charge q , $\vec{E} \propto \hat{r}$. Hence, for a 2D projection of the arrangement of 3D spherical shells discussed above, the arrows indicating the field lines will point outwards from the positive charge $q(> 0)$ (placed at, say, the origin). The magnitude of the field is indicated by the fact that the field lines have a higher density for smaller shell radius and become increasingly dilute as the shell radius is increased. Indeed, it is easy to see that the number of field lines remains fixed across shells (i.e., the flux is conserved): as the shell surface area $\propto R^2$, the field line density $\propto R^{-2}$. Thus, the fact that $\vec{E}(\vec{r}) \propto r^{-2}$ and we live in three spatial dimensions leads to the conservation of flux of electric field lines!

In this way, we may visualise the \vec{E} as being strong in those regions of space where the field lines

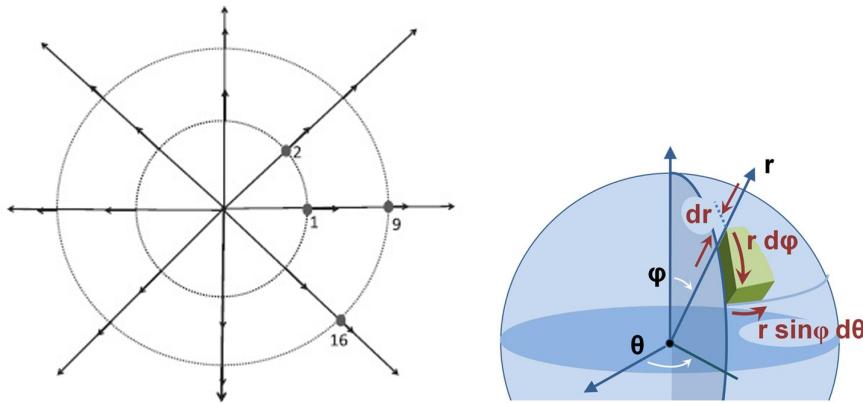


Figure 3.4: (left) The electric field lines due to a charge placed at the origin. The actual charge and lines live in three dimensions, and the figure shows what happens in a representative plane, which are assumed to have (say) 8 lines. Credit: Shankar, Vol.2

(right) The volume element in spherical polar coordinates.

are dense, and weak where they are dilute. Note: the \vec{E} is continuous, i.e., present everywhere in space and not only along the field lines. The field lines are only useful for visualisation. Also, the field line directions will be reversed for a negative charge $q < 0$, i.e., radially moving inwards towards the location of the charge (at, say, the origin). Since $\vec{F}(\vec{r}) = q\vec{E}(\vec{r})$, the field lines also indicate the force that will be felt by a test positive charge placed at \vec{r} .

In Fig.3.5, we see the \vec{E} field “map” for a “dipole” (whose cross section is shown in 2D). The flux line density (and hence the field) is higher near the charges and weaker farther away. The lines indicate the repulsion (attraction) felt by a positive test charge placed near the $+q$ ($-q$ respectively). Further, note that for any surface drawn that encloses both charges of the dipole, every outgoing field line will be matched precisely by an incoming counterpart. This means that the net flux of field lines inside such a surface is precisely zero. This is also precisely equal to the net charge within the surface ($q - q = 0!$), and is an outcome of something known as Gauss’ theorem (which we will explore very soon).

In Fig.3.6, we consider the case of a system of two positive charges placed at a given separation apart from one another. In this case, the net flux of lines enclosed by a surface is a positive number (as all lines are outgoing!), and is again the sum of all lines emanating from both charges. This is easily understood: far away from the locations of these two charges, the field lines must effectively behave as those belonging to a net point charge of value $q_1 + q_2$ placed roughly midway between

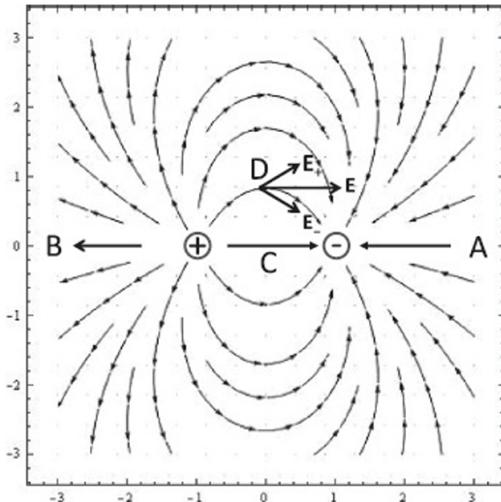


Figure 3.5: The electric field due to a dipole. The two vectors shown at point D are the contributions of \vec{E}_+ and \vec{E}_- to \vec{E} from the two charges. Their vector sum will be horizontal. Credit: Shankar, Vol.2

the two.

Note also that if we would have had $+q_1$ and $-q_2$ (with $q_2 \neq q_1$), there will be a net outward (inward) number of lines within any enclosing surface depending on whether $q_2 < q_1$ (or $q_2 > q_1$ respectively). Again, this is also easily understood: far away from the locations of these two charges, the field lines must effectively behave as those belonging to a net point charge of value $q_1 - q_2$ placed roughly midway between the two.

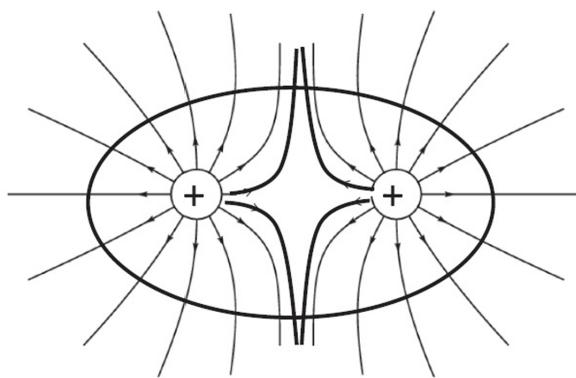
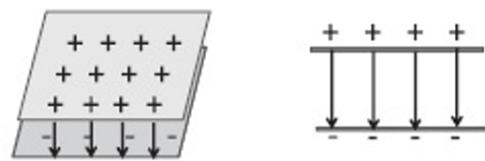


Figure 3.6: The electric field due to two positive charges. Far from both, it looks like the field of a point charge of double the strength. The number of lines crossing a closed surface enclosing both charges is the sum of the lines emanating from both. Credit: Shankar, Vol.2

What about the \vec{E} field arising from a continuous charge distribution? Consider the case of the \vec{E} field for a parallel plate capacitor. The field lines start from the positive charges and end at the negative ones. The pictures are drawn to capture the field lines for a part of the capacitor that is far from the edges (where they lines may bulge out a bit!), i.e., we are assuming the plates to be very large in extent and neglecting all edge effects. What is remarkable about this picture is that the field lines appear to exist only between the plates! For just one conducting plate, the lines would have either emanated from or recede into from both above as well as below. Superposing the two plates, they appear to help one another in the region in between (as shown), but negate exactly (or cancel) one another's fields in all other regions. This suggests that the field due to an infinite plate is independent of the distance from the plate, i.e., a constant, and in both x and y coordinates! We will verify this remarkable result at a later point.

Two views of the parallel plate capacitor



Charged particle in capacitor field

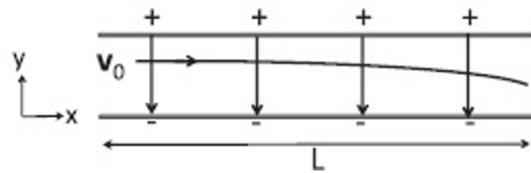


Figure 3.7: The top half shows two views of a parallel plate capacitor and the field inside it. It is uniform except near the edges, where it bulges out (not shown). The bottom shows the trajectory of a positively charged particle shot into it from the left. Credit: Shankar, Vol.2

3.2 Field of a dipole

Consider the \vec{E} field at a point (x, y) due to a dipole placed on the x -axis with $-q$ at $x = -a$ and q at $x = a$. Once we have obtained an answer, we can obtain the full field in 3D by rotating the figure around the x -axis (i.e., by exploiting the cylindrical symmetry of the problem in which the x -axis corresponds to the axis of the cylinder).

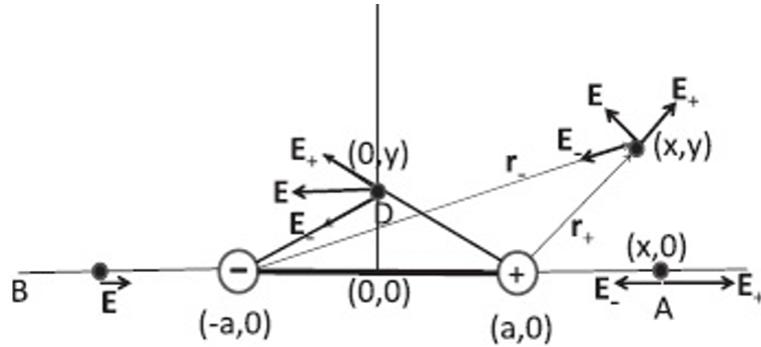


Figure 3.8: Dipole field : \vec{E}_\pm are due to $\pm q$ located at $(\pm a, 0)$. Credit: Shankar, Vol.2

Now, recall that

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}. \quad (3.16)$$

Then, the fields at (x, y) due to the charges q and $-q$ at $(a, 0)$ and $(-a, 0)$ respectively are given by

$$\begin{aligned} \vec{E}_+ &= \frac{q}{4\pi\epsilon_0} \frac{\vec{r}_+}{r_+^3} = \frac{q}{4\pi\epsilon_0} \frac{(x-a)\hat{i} + y\hat{j}}{((x-a)^2 + y^2)^{3/2}}, \\ \vec{E}_- &= -\frac{q}{4\pi\epsilon_0} \frac{\vec{r}_-}{r_-^3} = -\frac{q}{4\pi\epsilon_0} \frac{(x+a)\hat{i} + y\hat{j}}{((x+a)^2 + y^2)^{3/2}}, \end{aligned} \quad (3.17)$$

giving by superposition

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{q}{4\pi\epsilon_0} \left[\frac{(x-a)\hat{i} + y\hat{j}}{((x-a)^2 + y^2)^{3/2}} - \frac{(x+a)\hat{i} + y\hat{j}}{((x+a)^2 + y^2)^{3/2}} \right]. \quad (3.18)$$

We can now learn some important lessons from this general result. First, for a point on the x -axis (i.e., with $y = 0$)

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left[\frac{(x-a)\hat{i}}{|x-a|^3} - \frac{(x+a)\hat{i}}{|x+a|^3} \right], \quad (3.19)$$

such that for a point A with $x > a$ and $x > 0$, we obtain

$$\begin{aligned} \vec{E} &= \frac{q}{4\pi\epsilon_0} \left[\frac{(x-a)}{(x-a)^3} - \frac{(x+a)}{(x+a)^3} \right] \hat{i}, \\ &= \frac{qi}{4\pi\epsilon_0} \left[\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2} \right], \\ &= \frac{qi}{4\pi\epsilon_0} \frac{4ax}{(x^2 - a^2)^2} = \frac{\vec{p}}{4\pi\epsilon_0} \frac{2x}{(x^2 - a^2)^2}, \end{aligned} \quad (3.20)$$

where $\vec{p} = 2aq\hat{i}$ is called the Dipole Moment. For $x \gg a$

$$E(x \gg a) \approx \frac{\vec{p}}{2\pi\epsilon_0 x^3} = \frac{\vec{p}}{2\pi\epsilon_0 r^3}, \quad (3.21)$$

as $x \equiv r$ on the x -axis.

For $x < -a$, we can check that eq.(3.18) is invariant under the relation $x \rightarrow -x$, leading to

$$E(x < -a) = -\frac{\vec{p}}{4\pi\epsilon_0} \frac{2x}{(x^2 - a^2)^2}. \quad (3.22)$$

For a point D on the y -axis, we obtain

$$\begin{aligned} \vec{E} &= \frac{q}{4\pi\epsilon_0} \left[\frac{-a\hat{i} + y\hat{j}}{(y^2 + a^2)^{3/2}} - \frac{a\hat{i} + y\hat{j}}{(y^2 + a^2)^{3/2}} \right], \\ &= -\frac{q}{4\pi\epsilon_0} \frac{2a\hat{i}}{(y^2 + a^2)^{3/2}} = -\frac{\vec{p}}{4\pi\epsilon_0(y^2 + a^2)^{3/2}}, \end{aligned} \quad (3.23)$$

such that for $y \gg a$, we obtain

$$\vec{E}(y \gg a) \simeq -\frac{\vec{p}}{4\pi\epsilon_0|y|^3}, \quad (3.24)$$

indicating that the dipole acts as an effective negatively charged point-like object. Further, if we take the limit of $a \rightarrow 0$, $\vec{E} \rightarrow 0$, i.e., the dipole becomes charge neutral and ceases to exist. We note that the fact that $\vec{E}(x \rightarrow \infty)$ and $\vec{E}(y \rightarrow \infty)$ are not the same arises from the cylindrical symmetry of the dipole (rather than the spherical symmetry of the point charge). The dipole has a preferred axis, lowering the spherical symmetry down to rotational and translational symmetries about the axis of a cylinder.

3.2.1 Far field of the dipole

To obtain the far field contribution of the \vec{E} of the dipole, we must keep the leading order terms in a , i.e., seek the $O(a)$ contribution to the \vec{E} field. Thus,

$$\begin{aligned} \vec{E}_+ &= \frac{q}{4\pi\epsilon_0} \frac{(x-a)\hat{i} + y\hat{j}}{((x-a)^2 + y^2)^{3/2}}, \\ &= \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - a\hat{i}}{((x-a)^2 + y^2)^{3/2}} \text{ where } \vec{r} = x\hat{i} + y\hat{j}, \\ &\simeq \frac{q}{4\pi\epsilon_0} \left[\frac{\vec{r}}{(x^2 + y^2 - 2ax)^{3/2}} - \frac{a\hat{i}}{(x^2 + y^2)^{3/2}} \right] \text{ to } O(a) \text{ in the denominator,} \\ &\simeq \frac{q}{4\pi\epsilon_0} \left[\frac{\vec{r}}{(r^2 - 2ax)^{3/2}} - \frac{a\hat{i}}{r^3} \right]. \end{aligned} \quad (3.25)$$

Similarly, for \vec{E}_- , we set $q \rightarrow -q$ and $a \rightarrow -a$ to obtain

$$\vec{E}_- \simeq -\frac{q}{4\pi\epsilon_0} \left[\frac{\vec{r}}{(r^2 + 2ax)^{3/2}} + \frac{a\hat{i}}{r^3} \right]. \quad (3.26)$$

Putting \vec{E}_+ and \vec{E}_- together, we obtain

$$\begin{aligned}
\vec{E} &\simeq \frac{q}{4\pi\epsilon_0} \left[-\frac{2a\hat{i}}{r^3} + \left(\frac{1}{(r^2 - 2ax)^{3/2}} - \frac{1}{(r^2 + 2ax)^{3/2}} \right) \vec{r} \right], \\
&\simeq \frac{q}{4\pi\epsilon_0} \left[-\frac{2a\hat{i}}{r^3} + \frac{\vec{r}}{r^3} \left\{ \left(1 - \frac{2ax}{r^2} \right)^{-3/2} - \left(1 + \frac{2ax}{r^2} \right)^{-3/2} \right\} \right], \\
&\simeq \frac{q}{4\pi\epsilon_0 r^3} \left[-2a\hat{i} + \vec{r} \left(1 + \frac{3}{2} \frac{2ax}{r^2} \right) - \left(1 - \frac{3}{2} \frac{2ax}{r^2} \right) \right], \quad (\text{Taylor expansion to leading order}) \\
&\simeq \frac{q}{4\pi\epsilon_0 r^3} \left[-2a\hat{i} + 2\vec{r} \frac{3}{2} \frac{2ax}{r^2} \right], \\
&\simeq \frac{q}{4\pi\epsilon_0 r^3} \left[-2a\hat{i} + \frac{3\vec{r} \times 2ax}{r^2} \right], \\
\Rightarrow \vec{E} &\simeq \frac{1}{4\pi\epsilon_0 r^3} \left[-\vec{p} + 3\vec{r} \left(\frac{\vec{p} \cdot \vec{r}}{r^2} \right) \right], \tag{3.27}
\end{aligned}$$

where we have used the fact that $\vec{p} = 2aq\hat{i}$ and $\vec{p} \cdot \vec{r} = (2aq\hat{i}) \cdot (x\hat{i} + y\hat{j}) = 2aqx$. In this way, we can see that \vec{E} has two pieces. The first of these is directly proportional to \vec{p}/r^3 , and corresponds to the field due to the replacement of the dipole by an effective point charge. The second term is proportional to $\vec{r}(\vec{p} \cdot \vec{r}/r^2)$, and is sensitive to the axial symmetry of the dipole.

Exercise: Continue the Taylor expansion of the far field equation for the electric field due to the dipole to the next to leading order to see what it leads to. Can you anticipate what you should get prior to computing the expression? The answer is an example of a so-called “multipole expansion”, where only those terms appear that are consistent with the axial/cylindrical symmetry of the dipole system of charges.

3.3 Response to a field

Let's study a couple of interesting examples of how charges respond to the presence of an electric field.

3.3.1 Charge in a uniform electric field

Consider a positive charge $q > 0$ that is thrown horizontally with initial position $(r_0, 0)$ and initial velocity $\vec{v}(t=0) = v_0\hat{i}$ into a region with a uniform electric field $\vec{E} = -E_0\hat{j}$ arising from a parallel plate capacitor configuration (with positive charges on the upper plate and negative charges on the lower). Now, noting that $\vec{F} = q\vec{E} = m\vec{a}$, we obtain

$$\vec{a} = -\frac{qE_0}{m}\hat{j}. \tag{3.28}$$

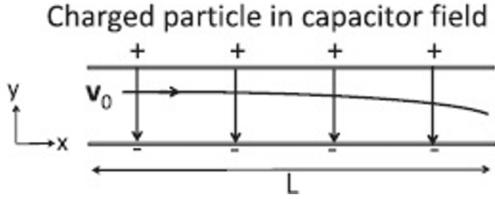


Figure 3.9: The trajectory of a positively charged particle shot into the uniform electric field between a parallel plate capacitor from the left. Credit: Shankar, Vol.2

This then obtains

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t = v_0 \hat{i} - \frac{qE_0 t}{m} \hat{j}, \quad (3.29)$$

$$\vec{r}(t) = r_0 \hat{i} + v_0 t \hat{i} - \frac{qE_0 t^2}{2m} \hat{j}. \quad (3.30)$$

To exit the capacitor (with plates of spatial extension L), the x displacement must be L . Further, since the velocity along \hat{i} is fixed at v_0 , the time of exit is $t^* = L/v_0$. Then, since the particle started from $(r_0, 0)$, the y coordinate after time t^* is

$$y_{exit} = -\frac{eE_0(t^*)^2}{2m} = -\frac{qE_0}{2m} \left(\frac{L}{v_0}\right)^2. \quad (3.31)$$

Such curvilinear trajectories (including those generated in the presence of magnetic fields) are commonly used in Cathode Ray tubes and televisions from the 20th century.

3.3.2 Dipole in a uniform electric field

Consider a dipole with moment $\vec{p} = 2aq$, i.e., comprised of charges $+q$ and $-q$ separated by a distance $2a$. This dipole is placed in a region of space with a uniform electric field $\vec{E} = E_0 \hat{i}$ such that the axis of the dipole makes an angle θ with \vec{E} . Now, given that the forces on the two charges are equal and opposite ($q\vec{E}$ and $-q\vec{E}$ respectively), the dipole does not feel a net force. However, these two forces will act in concert to generate a net torque on the dipole, leading to its rotation. The torque τ is given by

$$\tau = pE_0 \sin \theta = 2qaE_0 \sin \theta, \quad (3.32)$$

where $E_0 \sin \theta$ is the component of \vec{E} acting perpendicular to the dipole's axis. Note also that torque is a vectorial quantity

$$\vec{\tau} = \vec{p} \times \vec{E}, \quad (3.33)$$

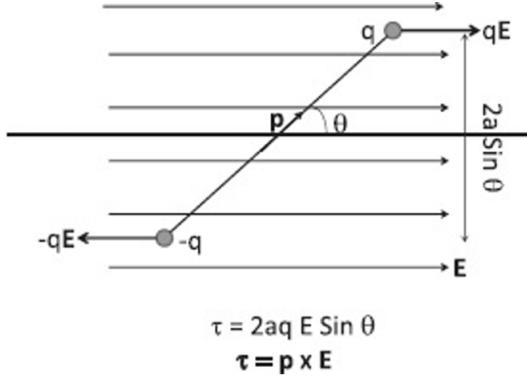


Figure 3.10: The forces and torque τ on a dipole \vec{p} of length a due to a uniform horizontal field \vec{E} . The torque, computed with respect to the negative charge, has a magnitude $\tau = 2aqE \sin \theta$ and tends to align it with the applied field. The vector $\tau = \vec{p} \times \vec{E}$ vanishes only when \vec{p} and \vec{E} are either parallel or anti-parallel. Credit: Shankar, Vol.2

and using the right hand rule, its direction for the present case will be given as the unit vector into the plane of the paper ($-\hat{k}$). Indeed, in this problem $\vec{p} = 2aq(\cos \theta \hat{i} + \sin \theta \hat{j})$ and $\vec{E} = E_0 \hat{i}$, such that

$$\begin{aligned}
\vec{\tau} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2aq \cos \theta & 2aq \sin \theta & 0 \\ E_0 & 0 & 0 \end{vmatrix}, \\
&= \hat{i}(0 - 0) + \hat{j}(0 - 0) + \hat{k}(0 - 2aq \sin \theta E_0), \\
\Rightarrow \vec{\tau} &= -2aqE_0 \sin \theta \hat{k}.
\end{aligned} \tag{3.34}$$

If the two charges are connected by a rigid rod, it will act as an “electric compass needle”, i.e., it will swing about and finally point along the local \vec{E} field. The torque $\vec{\tau}$ is clearly maximised when the dipole is placed perpendicular to \vec{E} , and minimised when the dipole is either aligned or anti-aligned with \vec{E} . The true minimum energy (or equilibrium) configuration of the dipole is the aligned one, while the anti-aligned one corresponds to an unstable equilibrium. To understand this, recalled that for any conservative force $F(x)$, the associated potential energy function $U(x)$ is given by

$$F(x) = -\frac{dU(x)}{dx}, \tag{3.35}$$

$$\Rightarrow U(x_1) - U(x_2) = \int_{x_1}^{x_2} dx F(x). \tag{3.36}$$

Given that torque is analogous to force as angle is analogous to displacement, we can write

$$\begin{aligned}
U(\theta_1) - U(\theta_2) &= \int_{\theta_1}^{\theta_2} d\theta (-pE_0 \sin \theta), \\
&= -pE_0 \int_{\theta_1}^{\theta_2} d\theta \sin \theta, \\
&= -pE_0 (-\cos \theta)|_{\theta_1}^{\theta_2}, \\
&= pE_0 (\cos \theta_2 - \cos \theta_1), \\
\Rightarrow U(\theta) &= -pE_0 \cos \theta = -\vec{p} \cdot \vec{E},
\end{aligned} \tag{3.37}$$

i.e., the component of the dipole moment vector \vec{p} along the electric field vector \vec{E} , and where we have dropped an overall constant of integration. This relation tells us that the potential function $U(\theta)$ is maximised at $\theta = \pm\pi$ (unstable equilibria) and minimised at $\theta = 0$ (stable equilibrium).

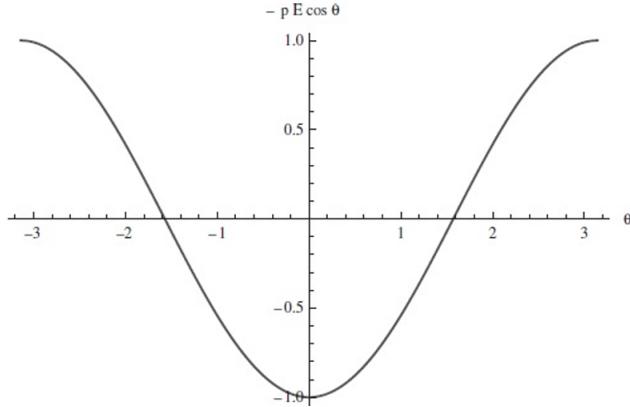


Figure 3.11: The potential energy of a dipole, $U = -pE \cos \theta = -\vec{p} \cdot \vec{E}$, as a function of the angle θ that the dipole moment \vec{p} makes with the electric field vector \vec{E} . Credit: Shankar, Vol.2

Finally, if we expand $\sin \theta$ about $\theta = 0$ in our expression for the torque, we obtain the torque relation as

$$\tau = -pE_0 \sin \theta \simeq -pE_0 \theta. \tag{3.38}$$

The negative sign indicates that the torque acts as a restoring force, and will lead to oscillations of the dipole with a small angle θ about the equilibrium value of $\theta = 0$. The effective “spring constant” of this pendulum is $\kappa = pE_0$, and angular frequency $\omega = \sqrt{\kappa/I} = \sqrt{pE_0/I}$ where I is the moment of inertia of the rigid rod connecting the two charges of the dipole.

Chapter 4

Electrostatics: the Electric field II & Gauss Law



We will now move gradually towards an exposition of Gauss' Law for electrostatics. But first, in order to be able to appreciate the immense simplification that can be obtained from the application of this law, let us compute the electric field of a couple of interesting problems.

Gauss

4.1 Field due to an infinite line charge

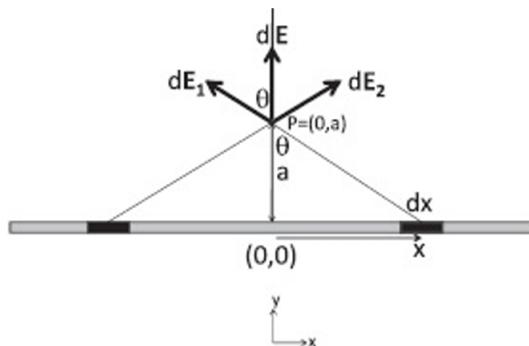


Figure 4.1: The field due to an infinite line charge with linear charge density λ is found by adding the contributions from tiny segments of width dx treated as point charges. The figure shows that the fields dE_1 and dE_2 due to the segments at x and $-x$ have the same y -components and opposite x components. Credit: Shankar, Vol.2

Consider computing the \vec{E} field in the XY plane due to a infinitely long line charge characterised by a uniform linear charge density λ and lying on, say, the x -axis. A careful look at the problem

will tell you that the field at all points at a fixed distance from an infinitely long line must be the same, i.e., the field must be independent of the x coordinate. However, it will depend on the distance from the line, i.e., the y coordinate. This is simply a consequence of the axial symmetry of this cylindrical arrangement of the charge. Indeed, extending this to 3D will tell us that the field must point along the radial unit vector (\hat{r}) in general, whereas for now, we will consider only that which lies along \hat{j} . Clearly, by symmetry, the components of the field in the \hat{i} and \hat{k} directions must vanish: any field generated due to a line segment of charge λdx lying at x will always be cancelled exactly by that from a partner segment (defined with respect to an inversion operation about the origin, i.e., a line segment of charge $\lambda dx'$ lying at $-x$). However, all field lines along the \hat{j} direction in the XY plane (and along \hat{r} in 3D) will add up.

Thus, the differential electric field at a point on the y -axis ($y = a$) will be

$$\begin{aligned}\vec{E}(a) &= \hat{j} \int_{-\infty}^{\infty} dx \frac{\lambda \cos \theta}{4\pi\epsilon_0(x^2 + a^2)} , \text{ where } \cos \theta = \frac{a}{(x^2 + a^2)^{1/2}} , \\ &= \hat{j} \int_{-\infty}^{\infty} dx \frac{\lambda a}{4\pi\epsilon_0(x^2 + a^2)^{3/2}} , \\ &= 2\hat{j} \int_0^{\infty} dx \frac{\lambda a}{4\pi\epsilon_0(x^2 + a^2)^{3/2}} ,\end{aligned}\tag{4.1}$$

as the integrand is an even function in x . Now, substituting $w = x/a$, $dx = adw$, we get the field at the point $(0, a)$ is

$$\begin{aligned}\vec{E}(a) &= \frac{2\lambda a \hat{j}}{4\pi\epsilon_0} \int_0^{\infty} \frac{adw}{a^3(1+w^2)^{3/2}} , \\ &= \frac{\lambda \hat{j}}{2\pi\epsilon_0 a} \times I ,\end{aligned}\tag{4.2}$$

where $I = \int_0^{\infty} \frac{dw}{(1+w^2)^{3/2}}$ is a number (to be evaluated shortly). Note that this result says that $\vec{E}(a) \propto \lambda \hat{j}/a$ (and not $1/a^2$ as could have been expected!). This is because on dimensional grounds, $[\lambda] = C/L$ already has one dimension of L^{-1} in it. In this way, we see that \vec{E} does indeed have the correct dimensions of L^{-2} . For a finite length of wire (L), the answer will also depend on L through the ratio L/a . For $a \gg L$, $E(a) \simeq \lambda L/a^2 = Q/a^2$ where $Q = \lambda L$ is the total charge on the entire finite length of wire acting as a point charge!

To evaluate the integral I , substitute $w = \tan \theta$ and $dw = \sec^2 \theta d\theta$, such that

$$\begin{aligned}I &= \int_0^{\pi/2} d\theta \frac{\sec^2 \theta}{(1+\tan^2 \theta)^{3/2}} , \\ &= \int_0^{\pi/2} d\theta \frac{\sec^2 \theta}{\sec^3 \theta} = \int_0^{\pi/2} d\theta \cos \theta = \sin \theta|_0^{\pi/2} = 1 ,\end{aligned}\tag{4.3}$$

where we have used the trigonometric relation $\sec^2 \theta - \tan^2 \theta = 1$. Finally, we have the electric field in the XY plane as

$$\vec{E}(a) = \frac{\lambda \hat{j}}{2\pi\epsilon_0 a} . \quad (4.4)$$

In 3D, the field will radiate outwards (i.e., point along the radial unit vector \hat{r})

$$\vec{E}(r) = \frac{\lambda \hat{r}}{2\pi\epsilon_0 r} = \frac{\lambda \vec{r}}{2\pi\epsilon_0 r^2} . \quad (4.5)$$

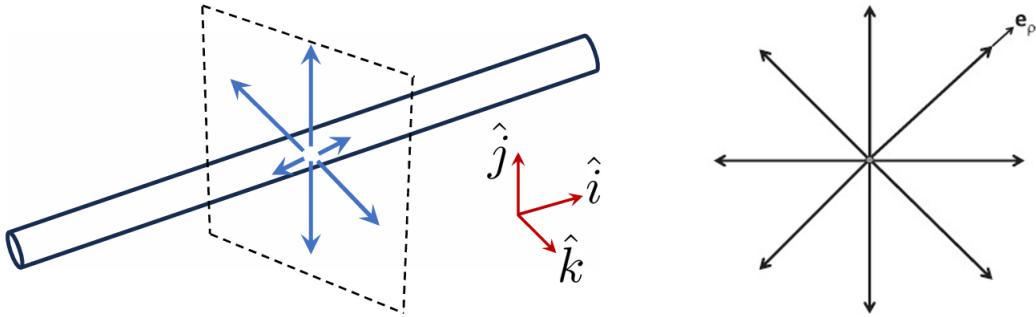


Figure 4.2: Left: the field due to an infinite line charge with linear charge density λ in three dimensions, and right: seen end-on, with the wire perpendicular to the page. The wire and the field distribution it produces are invariant under a rotation of the wire about its axis. Credit: Shankar, Vol.2

4.2 Field due to an infinite planar charge

Consider an infinite plane (coinciding with, say, the XY plane) of uniform areal charge density σ . As above, we can use symmetry arguments to note that the \vec{E} at a point P (at a point distance a above the XY plane) due to a collection of charges on the plane can only lie along a direction perpendicular to the XY plane and must vanish on all planes parallel to it:

$$\vec{E}(\vec{r}) = E(|z|)\hat{k} \text{ for } z > 0 , \quad (4.6)$$

$$= -E(|z|)\hat{k} \text{ for } z < 0 . \quad (4.7)$$

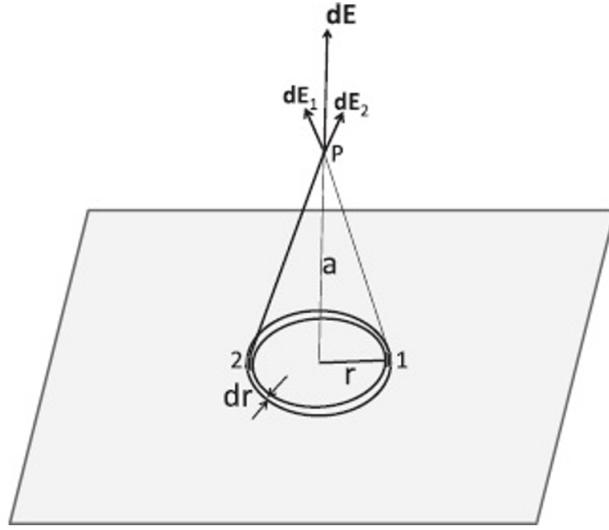


Figure 4.3: The field due to an infinite plane with charge density σ . Shown is one contributing annulus of radius r and thickness dr . It produces a field dE perpendicular to the plane, as shown by the long dark arrow. There is no parallel part due to cancellations between parts of the annulus that are diametrically opposite. Shown are two such contributions, dE_1 and dE_2 , due to the two darkened parts of the annulus. The sum of such vectors due to all parts of the annulus is dE . The integral of dE over all such annuli will give the final electric field E due to the entire plane. Credit: Shankar, Vol.2

Then, the field at point P due to the charge lying on an circular shell-like elemental area $2\pi r dr \times \sigma$ is given by

$$\begin{aligned} d\vec{E}_\perp(a) &= \frac{2\pi r dr \sigma \cos \theta}{4\pi\epsilon_0(r^2+a^2)} \hat{k} \quad \text{where } \cos \theta = \frac{a}{(r^2+a^2)^{1/2}}, \\ \Rightarrow \vec{E}_\perp(a) &= \frac{\sigma a \hat{k}}{2\epsilon_0} \int_0^\infty dr \frac{r}{(r^2+a^2)^{3/2}}. \end{aligned} \quad (4.8)$$

Setting $w = r/a$, $dr = adw$, we obtain

$$\vec{E}_\perp(a) = \frac{\sigma a \hat{k}}{2\epsilon_0} \int_0^\infty dw \frac{a^2 w}{a^3 (1+w^2)^{3/2}} = \frac{\sigma \hat{k}}{2\epsilon_0} \int_0^\infty dw \frac{w}{(1+w^2)^{3/2}}. \quad (4.9)$$

Let $z = w^2$, $dz = 2wdw$ such that

$$\begin{aligned} I &= \int_0^\infty dw \frac{w}{(1+w^2)^{3/2}} = \frac{1}{2} \int_0^\infty \frac{dz}{(1+z)^{3/2}}, \\ &= - \int_0^\infty dz \frac{d}{dz} \left[\frac{1}{(1+z)^{1/2}} \right] = - \frac{1}{(1+z)^{1/2}} \Big|_0^\infty = 0 - (-1) = 1, \end{aligned} \quad (4.10)$$

leading to

$$\vec{E}_\perp(a) = \frac{\sigma \hat{k}}{2\epsilon_0}. \quad (4.11)$$

Note that this result is independent of the lengthscale a ! This is because the dimensions of σ are $[\sigma] = CL^{-2}$. So, $\vec{E}_\perp(a)$ does have the right dimensions after all. Also, note that for $\hat{k} \rightarrow -\hat{k}$, $\vec{E}_\perp(a) \rightarrow -\vec{E}_\perp(-a)$. For a finite sized sheet, we can (and will) obtain corrections to $\vec{E}_\perp(a)$ of $\mathcal{O}(L^2/a^2)$, such that for $a \gg L$, $E_\perp \propto \sigma L^2/a^2 \propto Q/a^2$ (where $Q = \sigma \times L^2$ is the total charge on the finite sized sheet).

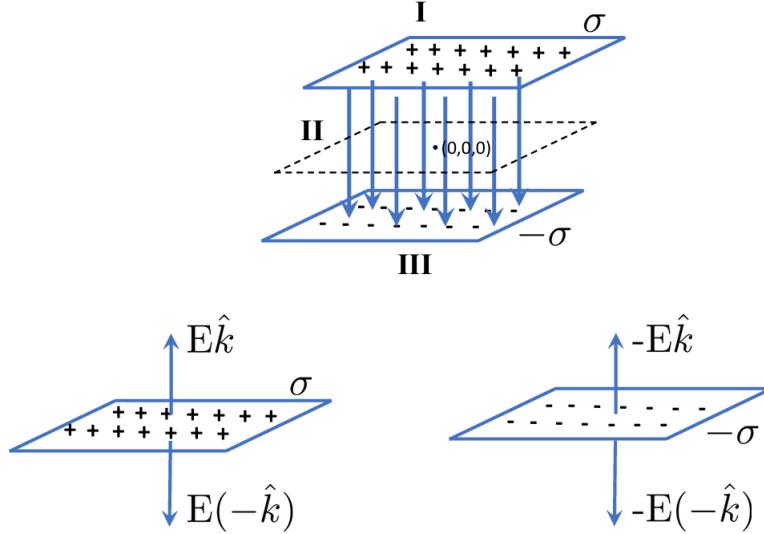


Figure 4.4: The electric field due to an infinite parallel plate capacitor.

We can now easily see how the field of the parallel plate capacitor (with identical but opposite signs for the σ on the two plates) comes about. Given that $\vec{E}_\perp(z > 0) = \frac{\sigma\hat{k}}{2\epsilon_0}$ for the positively charged plate and $\vec{E}_\perp(z > 0) = -\frac{\sigma\hat{k}}{2\epsilon_0}$ for the negatively charged plate, we can see that in the region I (i.e., above the positively charged plate),

$$\vec{E}_{Tot}^I = \frac{\sigma\hat{k}}{2\epsilon_0} - \frac{\sigma\hat{k}}{2\epsilon_0} = 0. \quad (4.12)$$

The maths follows through in precisely the same way for region III (i.e., below the negatively charged plate),

$$\vec{E}_{Tot}^{III} = \frac{\sigma\hat{k}}{2\epsilon_0} - \frac{\sigma\hat{k}}{2\epsilon_0} = 0. \quad (4.13)$$

However, in region II (i.e., in between the two plates), the total \vec{E} will add up from both contributions

$$\vec{E}_{Tot}^{II} = -\frac{\sigma\hat{k}}{2\epsilon_0} - \frac{\sigma\hat{k}}{2\epsilon_0} = -\frac{\sigma\hat{k}}{\epsilon_0}, \quad (4.14)$$

i.e., \vec{E}_{Tot}^{II} points downwards (along $-\hat{k}$, from the positively charged plate to the negatively charged plate) and with a constant magnitude equal to σ/ϵ_0 .

4.3 Areas, Area Vectors and Fluxes

In order to understand Gauss' Law, we will need to understand first the concepts of an area vector and a flux associated with the area on any given surface.

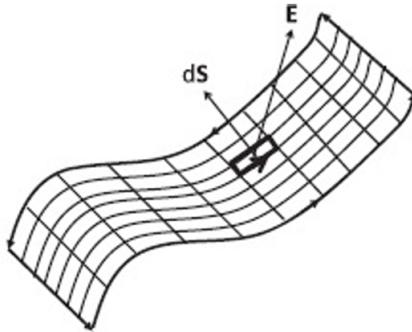


Figure 4.5: A generic surface in three dimensions obtained by gluing together tiny areas or plaquettes. The arrows that used to run around the interior plaquettes have been canceled by the neighbouring plaquettes with counter-propagating arrows. What remains are arrows around the perimeter, which define the boundary of the sum. Also shown for later use is one highlighted interior area dS and the electric field vector \vec{E} at that point. The orientation of this area is indicated by the arrow on one edge. Credit: Shankar, Vol.2

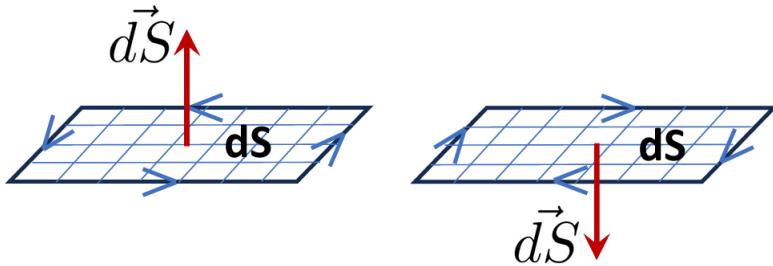


Figure 4.6: The right-hand corkscrew rule for defining \vec{dS} .

For a given small patch of area dS on some given surface S , we note that we can always define a direction normal (or perpendicular) to dS and which can be denoted by the unit vector $d\hat{S}$. Then, we can define the area vector \vec{dS} for the area dS as $\vec{dS} = d\hat{S} dS$ (i.e., with a magnitude $dS = |\vec{dS}|$ and a direction $d\hat{S}$). To choose the orientation of the area vector, we must adopt a convention known as the right hand corkscrew rule (also often referred to as the right hand rule): $d\hat{S}$ points along the thumb of the right hand if we curl the rest of the fingers of that hand into a counter-clockwise loop. It is important to note that only planar areas can be represented as area vectors. Importantly,

all infinitesimal areas can be treated as planar. Thus, any undulating surface can always be treated locally as planar and having a well-defined area vector; the area vector will however vary spatially along with the undulations of the surface.

Very generally, a parallelogram with two adjacent vectors \vec{P} and \vec{Q} (and with an angle θ between them) lead to an area vector

$$\vec{S} = \vec{P} \times \vec{Q} = |PQ \sin \theta| \hat{S}, \quad (4.15)$$

i.e., with magnitude $|PQ \sin \theta|$ and a direction normal (\hat{S}) to the plane containing \vec{P} and \vec{Q} and following the right hand rule. Combining small area patches to create non-planar surfaces can be done by gluing them together, cancelling thereby all edges with opposing arrows on them: the sum of all areas (with the remaining uncancelled edges) corresponds to the non-planar surface.

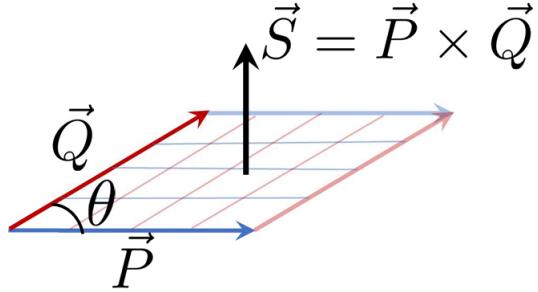


Figure 4.7: Defining the area vector \vec{S} as the vector product of two co-planar vectors \vec{P} and \vec{Q} .

The concept of a “flux” can now also be obtained. Consider a velocity vector \vec{v} denoting the motion of a fictitious fluid that is moving through a tube of varying cross-sectional area vector \vec{A} . Now, the flux Φ corresponds to the volume flow of the fluid per second past the area vector \vec{A} . For the velocity \vec{v} , the flux will be given by

$$\Phi = \vec{A} \cdot \vec{v} = Av \cos \theta. \quad (4.16)$$

Clearly, if the directions of \vec{A} and \vec{v} coincide (such that $\theta = 0$), the flux Φ moving past \vec{A} in one second is maximised. Similarly, it is minimised if \vec{A} is perpendicular to \vec{v} (i.e., $\theta = \pi/2$, $\cos \pi/2 = 0$).

For a general $0 \leq \theta \leq \pi/2$, the flux $\Phi = \vec{A} \cdot \vec{v} = Av \cos \theta$ varies in the range $0 \leq \Phi \leq Av$.

In what follows, we will replace \vec{v} with the electric field \vec{E} and consider the flux of electric field lines, i.e., the number of \vec{E} field lines that move past a given area vector \vec{A} in unit time.

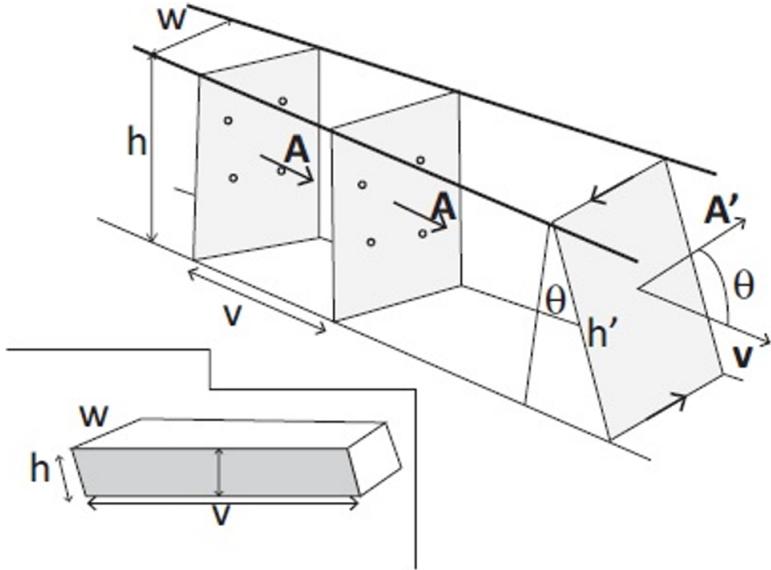


Figure 4.8: A tube of cross-sectional area $A = wh$, carrying a fluid with a velocity v parallel to A . To monitor the flux Φ (volume flow per second) past the area A shown at the left, we sprinkle some beads into the fluid at $t = 0$. One second later, the beads end up at the middle area. The volume between these two fronts is the flow per second, $\Phi = Av = \vec{A} \cdot \vec{v}$. The right most area A' is bigger than A by a factor $1/\cos\theta$ but intercepts the same amount of flux Φ or flow per second. As shown in the text, $\Phi' = \vec{A}' \cdot \vec{v} = A'v \cos\theta = \vec{A} \cdot \vec{v} = \Phi$. The inset shows the volume contained between two tilted areas A' at times $t = 0$ and $t = 1$, separated by $v \cdot 1$ meters. Credit: Shankar, Vol.2

4.4 Towards Gauss' Law

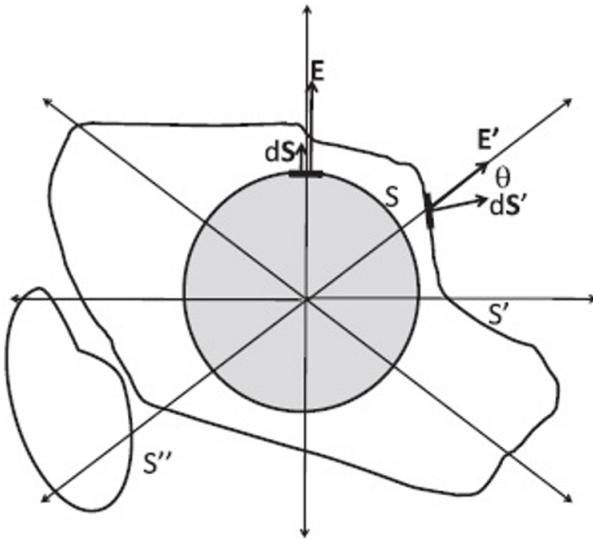


Figure 4.9: The figure shows the two-dimensional cross section of field lines emanating from a charge q . See detailed discussion in text. Credit: Shankar, Vol.2

For a charge q as shown in the Fig.4.9, we can see that

[i.] the number of electric field lines passing through the surface S (centred around q) is independent of its radius and equal to kq (where k is some number), i.e., the number of field lines emanating from q ,

[ii.] the same kq is the number of field lines passing through any surface S' enclosing q . On the other hand, for a surface S'' not enclosing q , the net number of field lines through S'' is zero,

[iii.] For a collection of charges $q_i, (i = 1, \dots, N)$ enclosed by surfaces S and S' (but not S''), the number of field lines is equal to $k \sum_{i=1}^N q_i = kq_{enc}$, where $q_{enc} = \sum_{i=1}^N q_i$.

Now, the number of field lines crossing unit area of a sphere S with radius r is the flux

$$\Phi = \frac{\text{field lines exiting sphere}}{\text{area of sphere}} = \frac{kq}{4\pi r^2} = k\epsilon_0 \frac{q}{4\pi\epsilon_0 r^2} = k\epsilon_0 E(r) . \quad (4.17)$$

Similarly, the number of lines crossing a small surface area patch dS is

$$\begin{aligned} \Phi_{dS} &= k\epsilon_0 E(r)dS , \\ &= k\epsilon_0 E(r)\hat{r} \cdot \hat{dS} , \\ &= k\epsilon_0 \vec{E}(r) \cdot \vec{dS} . \end{aligned} \quad (4.18)$$

Thus, the total electric flux through the entire spherical surface S is

$$\Phi = k\epsilon_0 \oint_S \vec{E} \cdot d\vec{S}, \quad (4.19)$$

where \oint_S refers to the surface integral over the entire surface S . But as the total number of flux lines crossing S has already been shown to be equal to the total number of field lines emanating from the charge enclosed (q_{enc}) by that surface, we can write

$$\begin{aligned} \Phi &= k\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = kq_{enc} = k \sum_{i=1}^N q_i, \\ &\Rightarrow \boxed{\oint_S \vec{E} \cdot d\vec{S} = \frac{q_{enc}}{\epsilon_0}}, \end{aligned} \quad (4.20)$$



Gauss

where \vec{E} is the total electric field at $d\vec{S}$ due to a superposition of all \vec{E} fields due to q_i , $i = 1, \dots, N$. The equation (4.20) is known as **Gauss' Law of electrostatics**. For a surface such as S'' (i.e., which encloses no charge), the net flux of the electric field lines must vanish, i.e., the total number of lines entering S'' (counted as a negative number) must be equal to the total number of lines exiting S'' (counted as a positive number). Indeed, this argument appears to be topological in nature, as no local geometric deformations of the closed surface can affect the flux $\Phi = \oint_S \vec{E} \cdot d\vec{S}$! For an open surface, the flux integral \oint_S will have to be replaced by that over the open surface \int_S .

Finally, for a continuous charge density defined by $\rho(\vec{r})$, the total charge enclosed within an infinitesimal cubic volume $d^3r = dx dy dz$ around the point with displacement vector \vec{r} is given by $\rho(\vec{r})d^3r$ and Gauss' Law becomes

$$\oint_{S=\partial V} \vec{E}(\vec{r}) \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_V d^3r \rho(\vec{r}) \equiv \frac{Q_{enc}}{\epsilon_0}, \quad (4.21)$$

where $S = \partial V$ reflects the closed surface S bounding the volume V .

4.5 Interlude: Gauss' Theorem or the Divergence Theorem

We have just seen that the flux Φ of the electric field is given by

$$\Phi = \oint_S \vec{E} \cdot d\vec{S} = \int_V dV \frac{\rho}{\epsilon_0} = \frac{Q_{enc}}{\epsilon_0}. \quad (4.22)$$

The integral version of Gauss' theorem (or the Divergence theorem) states that

$$\oint_S \vec{E} \cdot d\vec{S} = \int_V dV \vec{\nabla} \cdot \vec{E} = \int_V dV (\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}) \cdot \vec{E}, \quad (4.23)$$

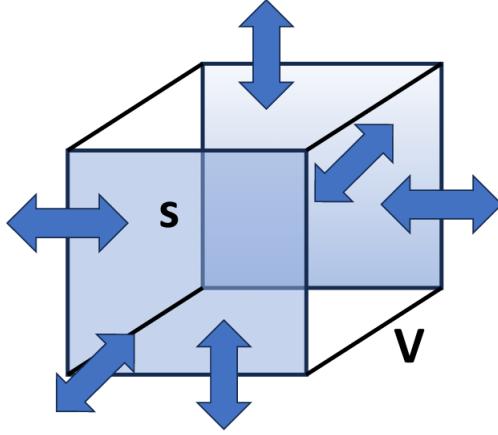


Figure 4.10: A typical volume V and its bounding surface $S \equiv \delta V$ that are used in understanding Gauss' theorem.

where the $\vec{\nabla} \cdot$ is called the “Divergence” of a vectorial quantity (here \vec{E}) from every infinitesimal volume element dV that comprises the volume V . Thus, the divergence refers to the “spreading out” of the \vec{E} vector field lines from all points within the volume dV for the case of a positive sign of $\vec{\nabla} \cdot \vec{E}$; for a negative sign of $\vec{\nabla} \cdot \vec{E}$, we have instead a convergence of the \vec{E} vector field lines into all points within the volume dV . Thus, in differential form, Gauss' law for electrostatics becomes

$$\oint_S \vec{E} \cdot d\vec{S} = \int_V dV \vec{\nabla} \cdot \vec{E} = \int_V dV \frac{\rho}{\epsilon_0}, \quad (4.24)$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}}. \quad (4.25)$$

Indeed, in all generality, Gauss' theorem (in integral form) for any vector field \vec{v} states that

$$\int_V dV \vec{\nabla} \cdot \vec{v} = \oint_{S=\partial V} \vec{v} \cdot d\vec{S}. \quad (4.26)$$

Let us put this to the test. Consider a vector $\vec{v} = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$ and a unit cube volume in the completely positive volume domain in 3D and with the origin as one of its vertices. Then the flux Φ of the vector field \vec{v} can be obtained by first computing

$$\begin{aligned} \vec{\nabla} \cdot \vec{v} &= (\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}) \cdot (y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}), \\ &= \frac{\partial}{\partial x}(y^2)\hat{i} \cdot \hat{i} + \frac{\partial}{\partial y}(2xy + z^2)\hat{j} \cdot \hat{j} + \frac{\partial}{\partial z}(2yz)\hat{k} \cdot \hat{k}, \\ &= 2x + 2y = 2(x + y), \end{aligned} \quad (4.27)$$

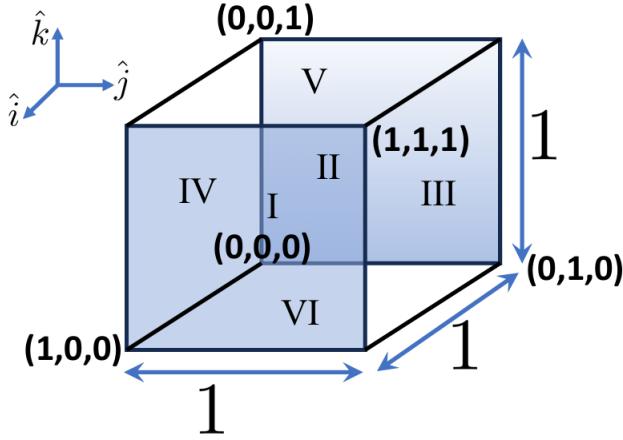


Figure 4.11: A unit cube.

where we have used the fact that many terms disappear simply due to the fact that $\hat{i} \cdot \hat{j} = 0 = \hat{j} \cdot \hat{k} = 0 = \hat{k} \cdot \hat{i}$. Thus, the flux Φ is given by

$$\Phi = \int_V dV 2(x+y) = 2 \int_0^1 dx \int_0^1 dy \int_0^1 dz (x+y), \quad (4.28)$$

$$\Rightarrow 2 \int_0^1 dx (x+y) = 2 \left[\frac{x^2}{2} + xy \right]_0^1 = 2 \left(\frac{1}{2} + y \right) = 1 + 2y, \quad (4.29)$$

$$\Rightarrow \int_0^1 dy (1+2y) = \left[y + \frac{2y^2}{2} \right]_0^1 = (1 + \frac{2}{2}) = 2, \quad (4.30)$$

$$\Rightarrow \int_0^1 dz 2 = [2z]_0^1 = 2, \quad (4.31)$$

$$\Rightarrow \Phi = \int_V dV \vec{\nabla} \cdot \vec{v} = 2. \quad (4.32)$$

Now, to compute the various surface integrals in turn:

$$\text{Face } I : \int_I \vec{v} \cdot d\vec{S} = \int_0^1 dy \int_0^1 dz \vec{v} \cdot \hat{i}, \quad (4.33)$$

$$= \int_0^1 dy \int_0^1 dz y^2 \hat{i} \cdot \hat{i}, \quad (4.34)$$

$$= \left[\frac{y^3}{3} \right]_0^1 [z]_0^1, \quad (4.35)$$

$$= \frac{1}{3} \times 1 = \frac{1}{3}. \quad (4.36)$$

$$\text{Face } II : \int_{II} \vec{v} \cdot d\vec{S} = \int_0^1 dy \int_0^1 dz \vec{v} \cdot -\hat{i}, \quad (4.37)$$

$$= - \int_0^1 dy \int_0^1 dz y^2 \hat{i} \cdot \hat{i}, \quad (4.38)$$

$$= - \left[\frac{y^3}{3} \right]_0^1 [z]_0^1, \quad (4.39)$$

$$= -\frac{1}{3} \times 1 = -\frac{1}{3}. \quad (4.40)$$

$$\text{Face } III : \int_{III} \vec{v} \cdot d\vec{S} = \int_0^1 dx \int_0^1 dz \vec{v} \cdot \hat{j}, \quad (4.41)$$

$$= \int_0^1 dx \int_0^1 dz (2xy + z^2)|_{y=1} \hat{j} \cdot \hat{j}, \quad (4.42)$$

$$= \int_0^1 dx \int_0^1 dz (2x + z^2), \quad (4.43)$$

$$= \int_0^1 dx \left[2xz + \frac{z^3}{3} \right]_{z=0}^1 = \int_0^1 dx \left[2x + \frac{1}{3} \right], \quad (4.44)$$

$$= \left[2 \frac{x^2}{2} + \frac{x}{3} \right]_{x=0}^1, \quad (4.45)$$

$$= 1 + \frac{1}{3} = \frac{4}{3}. \quad (4.46)$$

$$\text{Face } IV : \int_{IV} \vec{v} \cdot d\vec{S} = \int_0^1 dx \int_0^1 dz \vec{v} \cdot -\hat{j}, \quad (4.47)$$

$$= \int_0^1 dx \int_0^1 dz (2xy + z^2)|_{y=0} \hat{j} \cdot -\hat{j}, \quad (4.48)$$

$$= - \int_0^1 dx \int_0^1 dz z^2, \quad (4.49)$$

$$= - \int_0^1 dx \left[\frac{z^3}{3} \right]_{z=0}^1 = - \int_0^1 dx \frac{1}{3}, \quad (4.50)$$

$$= -\frac{x}{3}|_{x=0}^1 = -\frac{1}{3}. \quad (4.51)$$

$$\text{Face } V : \int_V \vec{v} \cdot d\vec{S} = \int_0^1 dx \int_0^1 dy \vec{v} \cdot \hat{k}, \quad (4.52)$$

$$= \int_0^1 dx \int_0^1 dy 2yz|_{z=1} \hat{k} \cdot \hat{k}, \quad (4.53)$$

$$= \int_0^1 dx \int_0^1 dy 2y = \int_0^1 dx \frac{2y^2}{2}|_{y=0}^1, \quad (4.54)$$

$$= \int_0^1 dx 1 = 1. \quad (4.55)$$

$$\text{Face } VI : \int_{VI} \vec{v} \cdot d\vec{S} = \int_0^1 dx \int_0^1 dy \vec{v} \cdot -\hat{k}, \quad (4.56)$$

$$= - \int_0^1 dx \int_0^1 dy 2yz|_{z=0} \hat{k} \cdot \hat{k}, \quad (4.57)$$

$$= - \int_0^1 dx \int_0^1 dy 0 = 0. \quad (4.58)$$

Adding the contributions from all six faces, we obtain the total flux as

$$\Phi = \oint \vec{v} \cdot d\vec{S} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2 = \int_V dV \nabla \cdot \vec{v}. \quad (4.59)$$

Thus, we have shown that \vec{v} does indeed satisfy Gauss' divergence theorem.

4.6 Applications of Gauss' Law

4.6.1 Uniform charged sphere with total charge Q

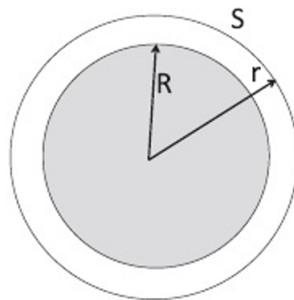


Figure 4.12: Gaussian surface S for computing the electric field outside the charged sphere. Credit: Shankar, Vol.2

Consider a uniformly charged sphere of radius R and with total charge Q . Clearly, the rotational symmetry in 3D for this case means that $\vec{E}(\vec{r}) = E(r)\hat{r}$ will be independent of the polar (θ , with respect to the z -axis) and azimuthal (ϕ , with respect to a chosen line on the XY plane) angles. First, let us compute the \vec{E} field outside the sphere by selecting a spherical Gaussian surface at distance r from

the origin (i.e., center of the sphere)

$$\begin{aligned}
\oint_S d\vec{S} \cdot \vec{E} &= \oint_S dS \hat{r} \cdot \hat{r} E(r), \\
&= E(r) \oint_S dS = E(r) \times 4\pi r^2, \\
&= \frac{Q}{\epsilon_0} \text{ by Gauss' law,} \\
\Rightarrow E(r) &= \frac{Q}{4\pi\epsilon_0 r^2}, \\
\vec{E}(r) &= \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}, \quad r > R.
\end{aligned} \tag{4.60}$$

This indicates that for any $r > R$, the sphere behaves effectively as a point charge Q placed at the origin (i.e., center of the sphere). The relative ease of this calculation was entirely due to the spherical symmetry of the problem, i.e., the spherical symmetry of the charged sphere, which helped us select a spherical Gaussian surface.

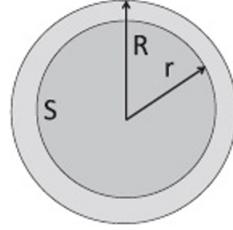


Figure 4.13: Gaussian surface S for computing the electric field inside the charged sphere. Credit: Shankar, Vol.2

Now, for the field inside the sphere. Assuming that the charge density is uniformly spread throughout the sphere, the charge enclosed within the Gaussian surface of radius "r", q_{enc} , is given by the fact that the volume charge density ρ is a constant

$$\begin{aligned}
\rho = \frac{Q}{\frac{4}{3}\pi R^3} &= \frac{q_{enc}}{\frac{4}{3}\pi r^3}, \\
\Rightarrow q_{enc} &= \frac{r^3}{R^3} Q.
\end{aligned} \tag{4.61}$$

Now, from Gauss' law, again taking a spherical Gaussian surface of radius r

$$\begin{aligned}
E(r)4\pi r^2 &= \frac{q_{enc}}{\epsilon_0} = \frac{Qr^3}{\epsilon_0 R^3}, \\
\Rightarrow E(r) &= \frac{Qr}{4\pi\epsilon_0 R^3},
\end{aligned} \tag{4.62}$$

$$\Rightarrow \vec{E}(r) = \frac{Qr}{4\pi\epsilon_0 R^3} \hat{r} = \frac{Q}{4\pi\epsilon_0 R^3} \vec{r}, \quad r < R. \tag{4.63}$$

This shows that \vec{E} varies linearly with r inside the sphere.

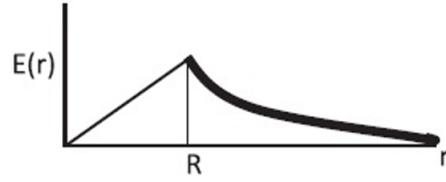


Figure 4.14: The electric field inside as well as outside the charged sphere. Credit: Shankar, Vol.2

A sanity check is that the fields inside as well as outside the sphere should match up at the surface of the sphere:

$$E(r)_{r \rightarrow R^-} = \frac{QR}{4\pi\epsilon_0 R^3} = \frac{Q}{4\pi\epsilon_0 R^2} = E(r)_{r \rightarrow R^+}. \quad (4.64)$$

This result has an interesting corollary in gravity (with the cautionary note that gravity is always attractive). Given that the gravitational force $\propto r^{-2}$, and we live in 3D, there is an analogous Gauss' law for gravity as well. Now, we can see that the gravitational force for a test mass m within the Earth would also vary linearly with the radial displacement

$$F = -\frac{GM_E mr}{R_E^3} = -\kappa r, \quad (4.65)$$

where M_E and R_E are the mass and radius of the Earth respectively, and κ is the effective spring constant for this restoring gravitational force.

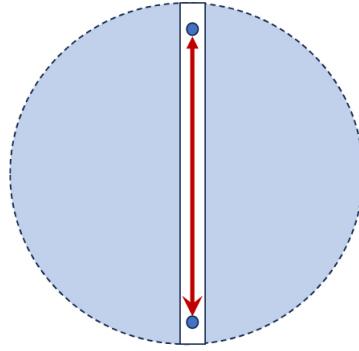


Figure 4.15: “Oscillations” of a small mass in a tunnel dug diametrically across the Earth using gravity.

But how would the “small oscillations” for such a gravitational spring show up? For this, you’ll have to drill a very narrow hole all the way through the Earth from one point on the surface to the diametrically opposite point (and ensure that the mass lost while doing so doesn’t affect the

answer for the field!). Then, drop an object into the hole, and it should oscillate all the way between the two end points of the tunnel with a frequency given by

$$\omega = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{GM_E m}{m R_E^3}} = \sqrt{\frac{GM_E}{R_E^3}}. \quad (4.66)$$

If you put in the numbers for $G = 6.7 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$, $M_E = 5.97 \times 10^{24} \text{ kg}$ and $R_E = 6.37 \times 10^6 \text{ m}$, the effective half-time period corresponding to this oscillation (i.e., the time taken between the two end points) is $\pi/\omega = \pi \sqrt{R_E^3/GM_E} \approx 2525 \text{ s} \approx 42.1 \text{ hours}$. Hmm... do you know of a faster courier service to the other side of the Earth from where you are?

4.6.2 Electric field outside & inside a shell

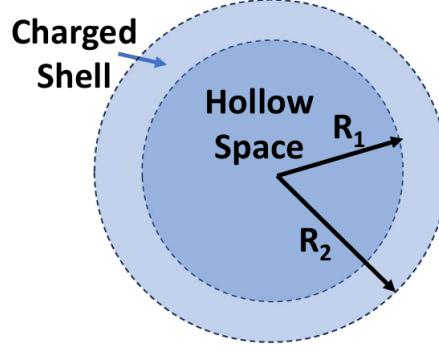


Figure 4.16: The charged spherical shell of inner and outer radii R_1 and R_2 respectively.

Consider a uniformly charged but hollow shell of inner and outer radii R_1 and R_2 ($R_1 < R_2$) respectively and uniform charge density ρ . The total charge on the shell is $Q = 4\pi\rho(R_2^3 - R_1^3)/3$. For the field at a distance r from the center of the shell such that $r > R_2$ (i.e., outside the shell), Gauss' law applied to a spherical Gauss surface (as above) will simply give us

$$\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \quad \text{for } r > R_2. \quad (4.67)$$

Similarly, as there is no charge enclosed within any spherical Gaussian surface completely within the hollow region of the shell (i.e., for $r < R_1$), we have

$$E(r < R_1) = 0 \quad \text{for } r < R_1. \quad (4.68)$$

Now, for $R_1 \leq r \leq R_2$, $q_{enc} = 4\pi\rho(r^3 - R_1^3)/3 = Q(r^3 - R_1^3)/(R_2^3 - R_1^3)$. Then, a spherical Gaussian surface

will now give

$$\vec{E}(r) = E(r)\hat{r} = \frac{Q(r^3 - R_1^3)\hat{r}}{4\pi\epsilon_0 r^2(R_2^3 - R_1^3)} = \frac{Qr}{4\pi\epsilon_0 R_2^3} \frac{(1 - R_1^3/r^3)}{(1 - R_1^3/R_2^3)} \hat{r} \quad \text{for } R_1 < r < R_2. \quad (4.69)$$

This relation shows that there are geometrical corrections to the linear in r behaviour for the \vec{E} within a uniformly charged sphere for the case of a shell; these corrections come in the shape of the function $\frac{(1 - R_1^3/r^3)}{(1 - R_1^3/R_2^3)}$. Note that we have even obtained the correction factor exactly, and not order by order in powers of r through a Taylor expansion. Clearly, using Gauss' law for symmetrical Gaussian surfaces is a really powerful technique!

4.6.3 Field of a uniformly charged infinitely long wire

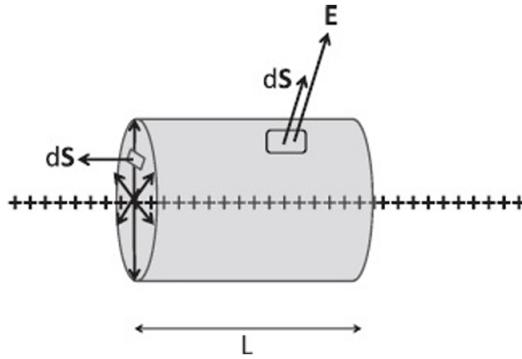


Figure 4.17: By symmetry, the field due to an infinite wire is radial and of constant magnitude at a fixed distance ρ from the wire. The Gaussian surface is a coaxial cylinder of radius ρ and has an arbitrary length L . The charge enclosed is simply λL . The two flat faces make no contribution to the flux since \vec{E} and $d\vec{S}$ are perpendicular. The curved face, on which the flux density is constant, makes a contribution $E(\rho) \cdot 2\pi\rho L$. Credit: Shankar, Vol.2

For the case of an infinitely long wire with uniform line charge density λ , consider a Gaussian surface to be a coaxial cylinder of radius r and length L , such that the charge enclosed is λL . Then, by Gauss' law

$$\begin{aligned} \oint_S \vec{E} \cdot d\vec{S} &= \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}, \\ \Rightarrow \int_0^L dl \int_0^{2\pi} d\phi r \hat{r} \cdot \hat{r} E(r) &= 2\pi r L E(r) = \frac{\lambda L}{\epsilon_0}, \\ \Rightarrow \vec{E}(r) &= \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}. \end{aligned} \quad (4.70)$$

Which is precisely what we obtained earlier.

4.6.4 Field of a uniformly charged infinite plane

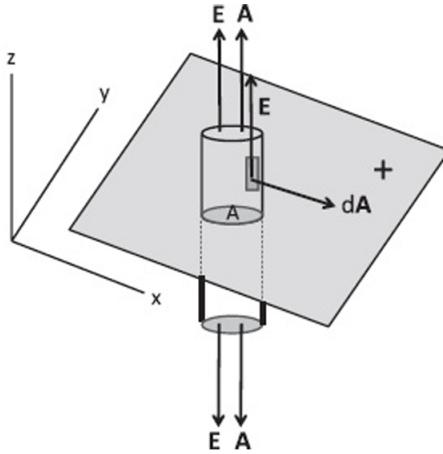


Figure 4.18: Shown is an infinite plane with charge density σ . Symmetry tells us the field is everywhere normal to the plane and constant in magnitude as we move parallel to the plane. The Gaussian surface is a cylinder of area A and height $2z$, symmetrically located with respect to the plane. The charge enclosed is σA . As for the flux, or surface integral of \vec{E} , the curved side makes no contribution because the area vector and field are perpendicular, while the two flat faces make equal contributions of $E(|z|)A$ each, where $E(|z|)$ is the constant value of the field strength at a distance $|z|$ from the charged sheet. Credit: Shankar, Vol.2

As we have argued earlier, the expected electric field \vec{E} for a uniformly charged infinite plane (aligned with the XY plane) and constant areal charge density σ is

$$\vec{E}(z) = E(|z|)\hat{k}, \quad z > 0 \quad (4.71)$$

$$= -E(|z|)\hat{k}, \quad z < 0. \quad (4.72)$$

To find $E(|z|)$, we need to consider a Gaussian surface on whose various parts the \vec{E} is either constant or perpendicular to the corresponding area vector. Such a surface is clearly a cylinder whose axis is aligned along \hat{k} . In that case, the field lines of \vec{E} must exist only through the flat surfaces of the cylinder (of cross section, say, A) as those area vectors are along \hat{k} and $-\hat{k}$ (for the upper and lower flat faces respectively). On the other hand, the area vector $d\vec{A}'$ for the curved surface is

perpendicular to that of \vec{E} . Therefore, from Gauss' law, we obtain

$$\begin{aligned}\oint_S \vec{E} \cdot d\vec{S} &= \hat{k}E(|z|) \cdot \hat{k}A + (-\hat{k}E(|z|) \cdot -\hat{k}A), \\ &= 2AE(|z|) = \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}, \\ \Rightarrow E(|z|) &= \frac{\sigma}{2\epsilon_0},\end{aligned}\tag{4.73}$$

which is the result we had obtained earlier as well.

A few closing remarks on all these cases. The secret to the simplicity with which we obtained the results in all four cases hinged critically on the existing symmetries of the configuration of charges. In a charge configuration with no absolutely no symmetries, obtaining the field everywhere in space may not be quite as easy; perhaps the only statement we can make easily is that if we consider a spherical Gaussian surface that encloses all the charges (whose total is Q), then we can always obtain the electric field on the surface as if Q charge were placed at the center of the sphere.

4.6.5 Field due to a dipole

Gauss' law doesn't yield an explicit form for \vec{E} to this problem as easily as what we have seen above. First, note that this problem does not have spherical symmetry; instead, it has axial (or cylindrical symmetry). Further, the finite extent of the dipole means that \vec{E} can depend on r (the radial coordinate) and z (the axial coordinate), but it cannot depend on ϕ (the azimuthal coordinate): $\vec{E} = \vec{E}(r, z)$. Second, if we consider a Gaussian surface that encloses both the positive and negative charges of the dipole, $Q_{enc} = 0$. Hence, all we can say from Gauss' law involving a (highly symmetrical!) spherical Gaussian surface yields

$$\oint \vec{E} \cdot d\vec{S} = \int d\phi \int d\theta r^2 \sin \theta \vec{E} \cdot \hat{r} = \frac{Q_{enc}}{\epsilon_0} = 0.\tag{4.74}$$

All we can learn from this is that the quantity $\vec{E} \cdot \hat{r}$ cannot be everywhere positive, everywhere negative or everywhere null (as Gaussian surfaces very near either charge will prove!). It doesn't yield an explicit form of \vec{E} . If we consider a system of two positive charges q (or two negative charges $-q$) placed apart by a distance a , the same Gaussian surface will now enclose a charge $Q_{enc} = 2q$ (or $-2q$) in the above expression. This yields the fact that the quantity $\vec{E} \cdot \hat{r}$ can now be everywhere positive (or negative); we will still not find an explicit form of \vec{E} from here.

We will now use Gauss' law to reach some important conclusions about the nature of conductors.

4.7 Some conclusions on Conductors

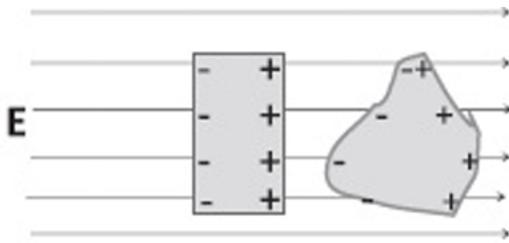


Figure 4.19: Two conductors placed in an external field \vec{E} , which gets screened inside by polarization. In the rectangular slab, the internal field σ/ϵ_0 due to the charges on the two faces neutralizes \vec{E} . Credit: Shankar, Vol.2

I. The electric field inside a conductor is zero.

Given that an electrical field would have caused charges to move around inside a conductor, we see that by definition, the \vec{E} field inside a conductor must vanish for the case of a static situation (i.e., where the charges are stationary). On the other hand, if we place a conductor in an external field \vec{E}_{ext} , there will initially be a field inside the conductor causing the delocalised electrons of the conductor to move in the direction opposite to \vec{E}_{ext} . This charge imbalance will build up until the internally generated field, \vec{E}_{int} (due to the negative and positive charge layers built up at the two extreme ends of the conductor), cancels the external field \vec{E}_{ext} . For an infinitely big slab (i.e., surface area is also infinitely big), we know that $\vec{E}_{int} \propto -\frac{\sigma}{\epsilon_0} \hat{i} \equiv \vec{E}_{ext}$. Even though the finite slab case is complicated by edge effects, we will still find that the net field inside the conductor will vanish. Also, this situation will be achieved irrespective of the shape of the conductor. This is called the “perfect screening” of the \vec{E} field inside a conductor.

II. Net charge on a conductor resides on its surface.

Imagine we place some positive charges on a neutral (and finite sized) conductor; it is easy to see that they will push one another to the surface so as to minimise Coulomb’s law. We can verify this using Gauss’ law. Consider any arbitrary Gaussian surface S inside the conductor. As we already know that there can be no net \vec{E} field inside the conductor, the surface integral of any such \vec{E} over the Gaussian surface will also be zero and hence the enclosed charge will vanish. This means that the additional charges must arrange themselves on the surface so as to lead to a vanishing charge everywhere *inside* the conductor. This means that conductors screen external \vec{E} fields as well as those \vec{E} fields that can arise from additional charges added to them. Again, this happens

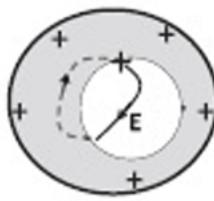


Figure 4.20: A conductor with a hole in it and some positive charges deposited on it. By Gauss's law, these must be on the outer surface and the charge on the inner surface has to be zero. If two cancelling charges reside on the inner surface, they would produce a field \vec{E} , which can do work on a test charge moving from the + to the -. The test charge can then be brought back to the + for free inside the conductor along the dotted line. The cycle violates energy conservation. Credit: Shankar, Vol.2

irrespective of the shape of the conductor.

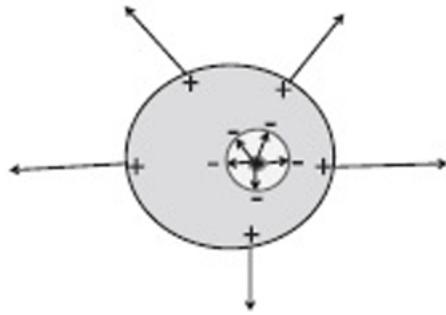


Figure 4.21: A charge q placed inside the hole. The lines it emits terminate on the inner wall (on the negative charges from the conductor that piled up there) and the lines are re-emitted by the positive charges that are on the outer wall. Credit: Shankar, Vol.2

III. Conductor with an internal hole.

Part a: Empty hole Since the screening of the field lines inside the conductor is perfect, any Gaussian surface enclosing the hole must have zero charge. Certainly, we are given that there is no charge inside the hole either. Thus, all charges must again lie on the outer surface of the conductor, and any excess charges inside must vanish by neutralisation.

Part b: Charge inside the hole Can an observer external to the conductor know about the charge q placed inside the inner cavity? Since no fields can enter the conductor, how can Gauss' law yield q ? The way it happens is by the charge q within the inner hole polarising the conductor so that an

effective $-q$ charge sites on the inner surface of the cavity and a concomitant q charge sits on the outer surface. Now, it is simple to argue for the expected results using various kinds of Gaussian surfaces.

Exercise: What would happen if you put more charge inside the hole than the number of free electrons inside the conductor surrounding it? Clearly, the electric field of the charge in the cavity cannot be screened by an equal and opposite charge on the inner surface of the conductor. In such a case, the electric field due to the excess charge will enter the conductor, leading to a breakdown in the perfect screening of electric fields in the interior of the conductor. How much charge would you need to put inside the cavity of a conductor made of 1kg of copper so as to cause the breakdown in perfect screening? What kind of capacitor could store that amount of charge?

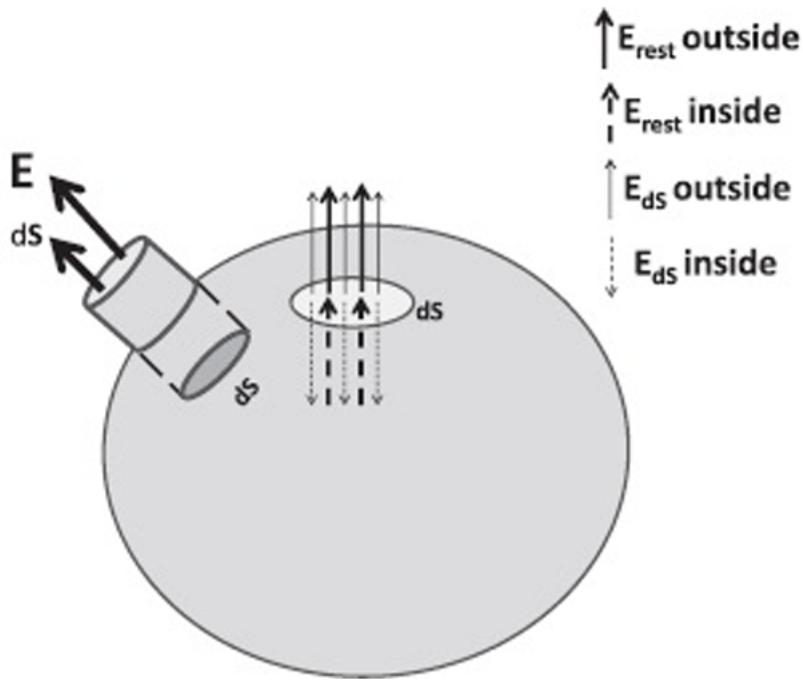


Figure 4.22: The field at the surface of a charged conductor is calculated using a Gaussian cylinder half inside and half outside with its axis normal to the surface. There is non-zero flux only on the flat face outside. There is no field inside and no flux on the curved side outside, which runs parallel to the field. Also shown are the field at a small area dS due to the charges on it (thin arrows, solid outside and dotted inside) and the charges on the rest of the surface (thick arrows, solid outside and dotted inside). The two contributions exactly cancel inside and double up outside. The charge density σ and field \vec{E}_\perp can vary from point to point. Credit: Shankar, Vol.2

IV. Field on surface of conductor.

It is clear thus far that no field can enter the surface for the case of a static charge placed on its surface. Further, given that the charge is static, all components of the \vec{E} locally parallel to the surface must vanish: only the \vec{E} field locally normal to the surface can be non-zero.

Consider a uniform areal charge density σ . For an infinitesimal patch area dS on the arbitrary surface S of the conductor, the total local charge enclosed by a Gaussian surface is clearly σdS . Then, by Gauss' law, the local normal \vec{E}_\perp field must be

$$\vec{E}_\perp \cdot d\vec{S} = \frac{\sigma dS}{\epsilon_0}, \quad (4.75)$$

$$\Rightarrow \vec{E}_\perp = \frac{\sigma}{\epsilon_0} \hat{r}. \quad (4.76)$$

One way to understand this is as follows. We can divide the contributions to the flux from the entire surface S through the tiny areal patch dS into two parts: the first being the E_\perp^{dS} given above from dS itself, and another from the \vec{E}_{rest} from the rest of the surface *but* with a tiny hole in place of the patch. Now, recall that

$$\vec{E}_\perp^{dS} = \frac{\sigma}{2\epsilon_0} \hat{r} \text{ for outward normal ,} \quad (4.77)$$

$$= -\frac{\sigma}{2\epsilon_0} \hat{r} \text{ for inward normal ,} \quad (4.78)$$

if we consider regions within dS whose dimensions are much smaller than the linear dimensions of dS . The discontinuity in \vec{E}_\perp^{dS} arises due to the σ , and is familiar from the physics of the charged infinite plane. Now, for \vec{E}_{rest} . Given that the charges in the rest of S do not reside in the hole (corresponding to dS), they cannot cause any discontinuity in \vec{E}_{rest} : $\vec{E}_{rest} = (\sigma/2\epsilon_0)\hat{r}$ (i.e., along the outward normal). This immediately gives us

$$\begin{aligned} \vec{E} &= \vec{E}_\perp^{dS} + \vec{E}_{rest} , \\ &= \frac{\sigma \hat{r}}{2\epsilon_0} (1+1) = \frac{\sigma}{\epsilon_0} \hat{r} \text{ for outward normal ,} \end{aligned} \quad (4.79)$$

$$= \frac{\sigma \hat{r}}{2\epsilon_0} (1-1) = 0 \text{ for inward normal .} \quad (4.80)$$

Thus, the two fields \vec{E}_\perp^{dS} and \vec{E}_{rest} reinforce one another along the outward normal \hat{r} , but cancel one another precisely along the inward normal $-\hat{r}$.

4.8 Interlude: Curl, Stokes' theorem and some vector identities

The **Curl operator** $\vec{\nabla} \times$ measures the “twist” (equally good words to build your intuition here would be “circulation” or “vorticity” or “swirl”) of a given vector \vec{v} . Thus, an integral $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S}$

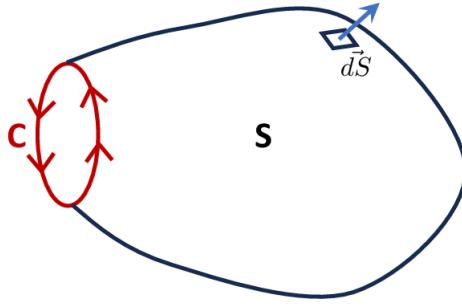


Figure 4.23: A typical surface S and its boundary curve $C \equiv \partial S$ that are used in understanding Stokes' theorem.

represents the flux of the curl throughout the surface S , i.e., the total amount of twist or swirl of \vec{v} in passing through S .

Stokes' theorem states

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint_{C=\partial S} \vec{v} \cdot d\vec{l}, \quad (4.81)$$

Stokes

where $C = \partial S$ represents the boundary curve of the surface S and $\oint_{C=\partial S} \vec{v} \cdot d\vec{l}$ represents the “circulation” or “vorticity” of \vec{v} in going around the curve C with the right hand rule signifying the direction of the area vector $d\vec{S}$ such that $d\vec{l}$ circulates in a anticlockwise direction around the curve C .

Notice that Stokes' theorem indicates

- (i) the value of the integral $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S}$ depends only on the boundary curve C and *not* on any particular surface S it bounds. There are an infinite number of open surfaces (related by local geometric deformations) all of whom possess the same boundary curve C . Stokes' theorem guarantees that $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S}$ for all these surfaces and is given by the same $\oint_{C=\partial S} \vec{v} \cdot d\vec{l}$; this is a topological argument, and
- (ii) for a closed surface integral $\oint (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = 0$, as any boundary line we choose for such a closed surface can always be shrunk down to a point (and hence has zero contribution). You can easily picture this with any loop you can draw on the surface of a sphere.

Let us now test Stokes' theorem. Consider the vector $\vec{v} = (2xy + 3y^2)\hat{j} + 4yz^2\hat{k}$ for a unit square surface on the YZ plane with the origin as the southwest corner. Clearly, $d\vec{S} = dydz\hat{i}$ (right hand rule gives a positive sign with anticlockwise rotation of the boundary curve for the square). Then,



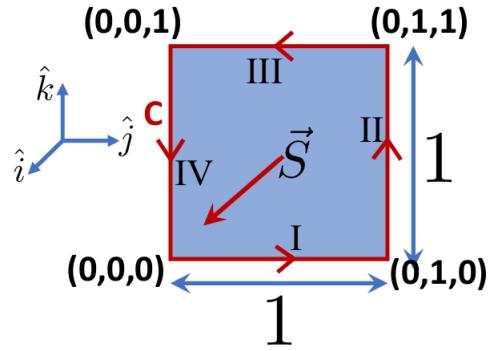


Figure 4.24: A unit square on the YZ plane with the origin at the southwest corner.

the curl of \vec{v} is given by

$$\begin{aligned}
 \vec{\nabla} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (2xy + 3y^2) & 4yz^2 \end{vmatrix}, \\
 &= \hat{i}\left(\frac{\partial}{\partial y}(4yz^2) - \frac{\partial}{\partial z}(2xy + 3y^2)\right) + \hat{j}\left(\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial x}(4yz^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(2xy + 3y^2) - \frac{\partial}{\partial y}(0)\right), \\
 &= (4z^2 - 2x)\hat{i} + (2z)\hat{k}.
 \end{aligned} \tag{4.82}$$

$$\therefore \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \int_0^1 dy \int_0^1 dz \hat{i} \cdot [(4z^2)\hat{i} + (2y)\hat{k}]_{x=0}, \tag{4.83}$$

$$\begin{aligned}
 &= \int_0^1 dy \int_0^1 dz (\hat{i} \cdot \hat{i}) 4z^2, \\
 &= \int_0^1 dy \frac{4z^3}{4} \Big|_{z=0}^1 = \int_0^1 dy \frac{4}{3} = \frac{4y}{3} \Big|_{y=0}^1 = \frac{4}{3}.
 \end{aligned} \tag{4.84}$$

To complete our check on Stokes' theorem, we must compute the line integral of $\vec{v} \cdot d\vec{l}$ as well. This can be done over the four segments of the boundary of the unit square (bearing in mind that the right hand rule gives a positive sign with anticlockwise rotation of the boundary curve for the square):

$$\begin{aligned}
 \textbf{Segment I. } x = 0 = z, \quad \vec{v} \cdot d\vec{l} &= (2xz + 3y^2)|_{x=0=z} \hat{j} \cdot dy \hat{j} = 3y^2 dy, \\
 \Rightarrow \int_I \vec{v} \cdot d\vec{l} &= \int_0^1 dy 3y^2 = \frac{3y^3}{3} \Big|_{y=0}^1 = 1.
 \end{aligned} \tag{4.85}$$

$$\begin{aligned}
 \textbf{Segment II. } x = 0, y = 1, \quad \vec{v} \cdot d\vec{l} &= (4yz^2)|_{x=0,y=1} \hat{k} \cdot dz \hat{k} = 4z^2 dz, \\
 \Rightarrow \int_{II} \vec{v} \cdot d\vec{l} &= \int_0^1 dz 4z^2 = \frac{4z^3}{3} \Big|_{z=0}^1 = \frac{4}{3}.
 \end{aligned} \tag{4.86}$$

Segment III. $x=0, z=1$, $\vec{v} \cdot d\vec{l} = (2xz + 3y^2)|_{x=0=z} \hat{j} \cdot dy(-\hat{j}) = -3y^2 dy$,

$$\Rightarrow \int_{III} \vec{v} \cdot d\vec{l} = \int_0^1 dy - 3y^2 = -\frac{3y^3}{3}|_{y=0}^1 = -1. \quad (4.87)$$

Segment IV. $x=0=y$, $\vec{v} \cdot d\vec{l} = (4yz^2)|_{x=0=y} \hat{k} \cdot dz(-\hat{k}) = 4z^2 dz = 0$,

$$\Rightarrow \int_{IV} \vec{v} \cdot d\vec{l} = - \int_0^1 dz 0 = 0. \quad (4.88)$$

Adding the contributions from segments I through to IV, we obtain

$$\oint_C \vec{v} \cdot d\vec{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3} = \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S}. \quad (4.89)$$

In this way, we have seen an explicit computation that verifies Stokes' theorem.

4.8.1 Two important vector identities

Identity 1. For any vector field $\vec{A} = \vec{A}(x, y, z)$, we have

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0. \quad (4.90)$$

In other words, the “Divergence of the Curl of \vec{A} vanishes”. Writing this out explicitly, we obtain

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A}{\partial y} - \frac{\partial A}{\partial z}\right) + \hat{j}\left(\frac{\partial A}{\partial z} - \frac{\partial A}{\partial x}\right) + \hat{k}\left(\frac{\partial A}{\partial x} - \frac{\partial A}{\partial y}\right), \quad (4.91)$$

$$\begin{aligned} \Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= (\hat{i} \cdot \hat{i}) \frac{\partial}{\partial x} \left(\frac{\partial A}{\partial y} - \frac{\partial A}{\partial z} \right) + (\hat{j} \cdot \hat{j}) \frac{\partial}{\partial y} \left(\frac{\partial A}{\partial z} - \frac{\partial A}{\partial x} \right) + (\hat{k} \cdot \hat{k}) \frac{\partial}{\partial z} \left(\frac{\partial A}{\partial x} - \frac{\partial A}{\partial y} \right), \\ &= \frac{\partial^2 A}{\partial x \partial y} - \frac{\partial^2 A}{\partial x \partial z} + \frac{\partial^2 A}{\partial y \partial z} - \frac{\partial^2 A}{\partial y \partial x} + \frac{\partial^2 A}{\partial z \partial x} - \frac{\partial^2 A}{\partial z \partial y}, \\ &= 0, \end{aligned} \quad (4.92)$$

as all terms cancel in pairs (as the order in which the partial derivatives are taken is interchangeable). Thus, any vector $\vec{v} = \vec{\nabla} \times \vec{A}$ (i.e, can be written as the curl of another vector) is known as “divergence free”. We shall see later that this is the case for a magnetic field \vec{B} , as \vec{B} can be written as the curl of something we call the vector potential (\vec{A}): $\vec{B} = \vec{\nabla} \times \vec{A}$, such that $\vec{\nabla} \cdot \vec{B} = 0$.

Identity 2. For any vector field $\vec{A} = \vec{\nabla} S$ (i.e, can be written as the gradient of a scalar function $S(x, y, z)$), we have

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\vec{\nabla} S) = 0. \quad (4.94)$$

Let us see this explicitly.

$$\begin{aligned}
\vec{\nabla} \times \vec{\nabla} S &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} & \frac{\partial S}{\partial z} \end{vmatrix}, \\
&= \hat{i}\left(\frac{\partial^2 S}{\partial y \partial z} - \frac{\partial^2 S}{\partial z \partial y}\right) + \hat{j}\left(\frac{\partial^2 S}{\partial z \partial x} - \frac{\partial^2 S}{\partial x \partial z}\right) + \hat{k}\left(\frac{\partial^2 S}{\partial x \partial y} - \frac{\partial^2 S}{\partial y \partial x}\right), \\
&= (0)\hat{i} + (0)\hat{j} + (0)\hat{k} = 0.
\end{aligned} \tag{4.95}$$

Any such vector \vec{v} , i.e., for which $\vec{\nabla} \times \vec{v} = 0$, is known as “irrotational”. This can be observed to be the case for the electric field $\vec{E} = -\vec{\nabla}V$ where V is the scalar potential

$$\vec{\nabla} \times \vec{E} = -(\vec{\nabla} \times \vec{\nabla} V) = 0. \tag{4.96}$$

Let us verify this for the case of the point charge: $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}$ and $d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$ in spherical polar coordinates. Then,

$$\vec{E} \cdot d\vec{l} = \frac{q}{4\pi\epsilon_0 r^2} dr, \tag{4.97}$$

$$\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b dr \frac{q}{4\pi\epsilon_0 r^2} = -\frac{q}{4\pi\epsilon_0 r} \Big|_{r_a}^{r_b} = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r_a} - \frac{1}{r_b} \right], \tag{4.98}$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{l} = 0 \text{ as } r_a = r_b. \tag{4.99}$$

$$\Rightarrow \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = \oint_C \vec{E} \cdot d\vec{l} = 0, \tag{4.100}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = 0. \tag{4.101}$$

The take home message from these two vector identities is that the Electric field \vec{E} is irrotational ($\vec{\nabla} \times \vec{E} = 0$), while the Magnetic field \vec{B} is divergence free ($\vec{\nabla} \cdot \vec{B} = 0$). Thus, while \vec{E} can be written as the gradient of a scalar potential (V): $\vec{E} = -\vec{\nabla}V$, \vec{B} can be written as the curl of a vector potential (\vec{A}): $\vec{B} = \vec{\nabla} \times \vec{A}$.

Chapter 5

Electrostatics: the Coulomb Potential

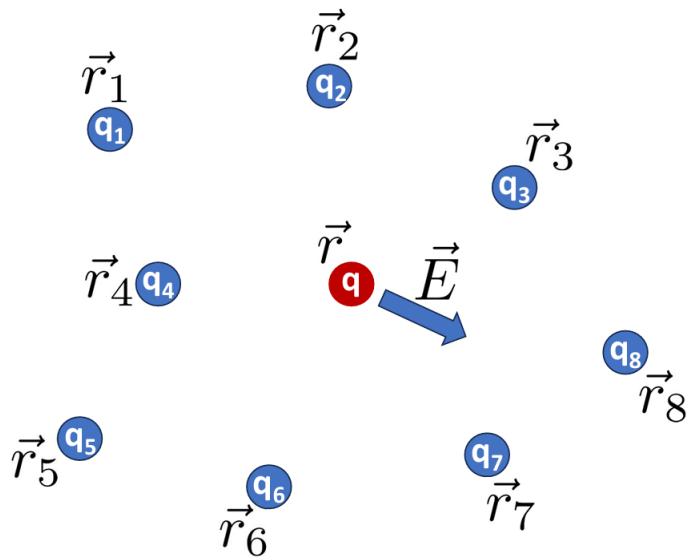


Figure 5.1: The response of a charge q at position \vec{r} due to the electric field \vec{E} produced by charges $q_1 \dots$ held static at positions $\vec{r}_1 \dots$.

The subject of electrodynamics can be considered as being comprised as solving two questions: for a given configuration of charges, find the electric field \vec{E} produced at the location of one of the charges (say q) due to all the others, and second, compute the response of the charge to the force $\vec{F} = q\vec{E}$ applied by the electric field \vec{E} . While this is easy to state, it is a very hard problem to solve in all generality. This is not hard to fathom: each of the charges is producing the field experienced by the others, as well as responding to the field created by them. In keeping with the demands of causality set by special relativity, the fields produced by the charges will depend

of the past positions of all of them. The simplification we were making till now is that we were enforcing stationarity for all the charges; with their positions fixed, all fields are easily computed from Coulomb's law. We now move forward by relaxing this constraint just a little: we keep the positions of all but one charges fixed so as to produce a field \vec{E} given by Coulomb's law, and the one charge q will be permitted to respond to this field. As this charge moves, the electric force it exerts on the other charges will vary; however, we will ignore this "backreaction" as the other charges are held fixed in their place. We can then proceed to track the dynamics of this lone charge q (and mass m) as follows:

$$m \frac{d^2\vec{r}}{dt^2} = m \frac{d\vec{v}}{dt} = q \vec{E}(\vec{r}(t)) , \quad (5.1)$$

$$\implies \vec{v}(dt) = \vec{v}(t=0) + \frac{q \vec{E}(\vec{r}(t=0))}{m} dt , \quad (5.2)$$

$$\text{and } \vec{r}(dt) = \vec{r}(t=0) + \vec{v}(t=0) dt , \quad (5.3)$$

where we have used the velocity ($\vec{v}(t=0)$) and position ($\vec{r}(t=0)$) at time $t=0$ as the two initial conditions needed to solve the second order differential equation at hand.

5.1 Conservative Forces and their related Potentials

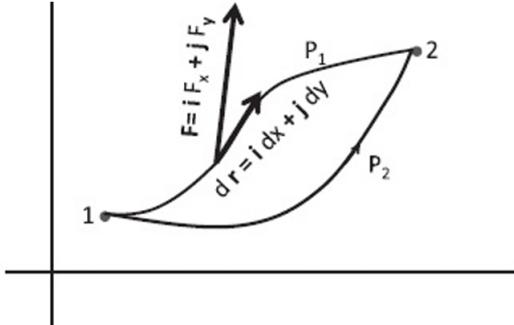


Figure 5.2: Two paths P_1 and P_2 connecting the same end points 1 and 2. The line integral, which is the sum over $\vec{F} \cdot d\vec{r}$, will generally depend on the path. Credit: Shankar, Vol.2

A conservative force is one which can be related to the gradient of a potential function

$$\vec{F} = -\vec{\nabla}U , \quad (5.4)$$

$$\Rightarrow - \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = U(\vec{r}_2) - U(\vec{r}_1) . \quad (5.5)$$

The important point to note is that the value of the line integral $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$ is independent of the path taken in going between the points \vec{r}_1 and \vec{r}_2 . Instead, it is dependent on only the difference of the values of the function U at these two points. Thus,

$$\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z , \quad (5.6)$$

$$= -(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}) \equiv -\vec{\nabla}U(x, y, z) . \quad (5.7)$$

Now, $\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$, such that

$$\vec{F} \cdot d\vec{r} = -\vec{\nabla}U \cdot d\vec{r} , \quad (5.8)$$

$$= -(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz) = -dU \text{ by defintion} , \quad (5.9)$$

$$\Rightarrow - \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} dU = U(\vec{r}_2) - U(\vec{r}_1) . \quad (5.10)$$

The left hand side of the last equation corresponds to the work that must be done *against* the force \vec{F} (hence the $-$ sign) to move the system from point \vec{r}_1 to point \vec{r}_2 . The right hand side, on the other hand, corresponds to the gain in potential energy in doing so.

Thus, for a conservative force, it becomes clear that no work is done (and no potential energy gained) in going around in a closed circuit (i.e., \vec{r}_1 back to \vec{r}_1)

$$\oint \vec{F} \cdot d\vec{r} = 0 . \quad (5.11)$$

Proof: Consider the case of two paths 1 and 2 that exist between the points \vec{r}_1 and \vec{r}_2 . Then, we must have

$$\int_{\text{Path 1, } \vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\text{Path 2, } \vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} , \text{ due to path independence} \quad (5.12)$$

$$\Rightarrow \int_{\text{Path 1, } \vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} - \int_{\text{Path 2, } \vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = 0 ,$$

$$\Rightarrow \int_{\text{Path 1, } \vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} + \int_{\text{Path 2, } \vec{r}_2}^{\vec{r}_1} \vec{F} \cdot d\vec{r} = 0 ,$$

$$\Rightarrow \oint \vec{F} \cdot d\vec{r} = 0 . \quad (5.13)$$

Note the power of this statement: the contour C can be distorted in an infinite number of ways while keeping it a closed loop that begins and ends at the same point. These local deformations of the loop contour C cannot change the result $\oint \vec{F} \cdot d\vec{r} = 0$; this is again a topological argument, and needs only for the loop to be closed.

From Stokes' theorem, for such a conservative force,

$$\oint_{C=\partial S} \vec{F} \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0, \quad (5.14)$$

$$\Rightarrow \vec{\nabla} \times \vec{F} = 0, \quad (5.15)$$

i.e., the force \vec{F} is irrotational.

This, then, offers a test of a conservative force. Consider a $U(x, y, z)$ and \vec{F} in three spatial dimensions. Then, the test of a conservative force is that the force $\vec{F} = -\vec{\nabla}U$ must be irrotational, i.e.,

$$\vec{\nabla} \times \vec{F} = -\vec{\nabla} \times \vec{\nabla}U = 0, \quad (5.16)$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial U}{\partial x} & -\frac{\partial U}{\partial y} & -\frac{\partial U}{\partial z} \end{vmatrix} = 0, \quad (5.17)$$

$$\Rightarrow \hat{i} \text{ component : } \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = -\frac{\partial^2 U}{\partial y \partial z} + \frac{\partial^2 U}{\partial z \partial y} = 0, \quad (5.18)$$

$$\hat{j} \text{ component : } \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = -\frac{\partial^2 U}{\partial z \partial x} + \frac{\partial^2 U}{\partial x \partial z} = 0, \quad (5.19)$$

$$\hat{k} \text{ component : } \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = -\frac{\partial^2 U}{\partial y \partial x} + \frac{\partial^2 U}{\partial x \partial y} = 0. \quad (5.20)$$



If we are given a potential U in 3D, then these are three (the so-called Cauchy-Riemann) conditions that we need to check. Strictly speaking, any continuous and differentiable function $U(x, y, z)$ will have to satisfy these Cauchy-Riemann conditions (as the order of partial derivatives is interchangeable for such functions). Let us consider the following 2D potential as a simple case: $U = 3xy^2$.

Then, we have

$$\begin{aligned} \vec{F} &= -\vec{\nabla}U = -(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})U, \\ &= -(\hat{i}3y^2 + \hat{j}6xy), \end{aligned} \quad (5.21)$$

$$\begin{aligned} \Rightarrow \vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y^2 & -6xy & 0 \end{vmatrix}, \\ &= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(-6y+6y) = 0. \end{aligned} \quad (5.22)$$

Riemann

Thus, we can conclude that the given potential function $U(x, y) = 3xy^2$ does indeed give rise to a conservative force.

5.2 The Electric Field and the Coulomb Potential

Let us check: is the electric field $\vec{E} = -\vec{\nabla}V$, i.e., the gradient of a scalar potential V ?

We now that

$$\vec{E}(r) = \frac{q}{4\pi\epsilon_0 r^2} \hat{r} = \frac{q}{4\pi\epsilon_0 r^3} \vec{r}. \quad (5.23)$$

Now, can the potential function $V(r) = \frac{q}{4\pi\epsilon_0 r}$ (where $r = (x^2 + y^2 + z^2)^{1/2}$) work? Let us put it to the test:

$$-\vec{\nabla}V(r) = -\hat{i}\frac{\partial V(r)}{\partial x} - \hat{j}\frac{\partial V(r)}{\partial y} - \hat{k}\frac{\partial V(r)}{\partial z}, \quad (5.24)$$

$$= -\frac{q}{4\pi\epsilon_0} \left[\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \right], \quad (5.25)$$

$$\text{Now, } \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left[(x^2 + y^2 + z^2)^{-1/2}\right], \quad (5.26)$$

$$= -\frac{1}{2} \frac{2x}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \quad (5.27)$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \quad (5.28)$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}, \quad (5.29)$$

$$\Rightarrow -\vec{\nabla}V = -\frac{q}{4\pi\epsilon_0} \left[\frac{-\hat{i}x - \hat{j}y - \hat{k}z}{(x^2 + y^2 + z^2)^{3/2}} \right] = \frac{q\vec{r}}{4\pi\epsilon_0 r^3} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \equiv \vec{E}(r). \quad (5.30)$$

Note: We can always add a constant to the function $V(r)$ which does not affect the form of $\vec{E} = -\vec{\nabla}V$. This freedom allows us to choose where to set the absolute zero for $V(r)$. By convention, we choose $V(r \rightarrow \infty) = 0$, i.e., work will have to be done to move a test charge from $r \rightarrow \infty$ to a finite r under the influence of the electric field generated by the gradient of the potential $V(r)$, $\vec{E} = -\vec{\nabla}V(r)$.

We now obtain the line integral

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \cdot d\vec{r}, \quad d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}, \quad (5.31)$$

$$= \int_{\vec{r}_1}^{\vec{r}_2} \frac{q}{4\pi\epsilon_0 r^2} dr (\hat{r} \cdot \hat{r}),$$

$$= \frac{q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{r_1}^{r_2},$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_1} - \frac{1}{r_2} \right), \quad (5.32)$$

$$= V(|\vec{r}_1|) - V(|\vec{r}_2|), \quad (5.33)$$

where we have used the most general form of the vector $d\vec{r}$ in three dimensions in spherical polar coordinates (r, θ, ϕ) : $d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$, and \hat{r} and $\hat{\theta}$ are the unit vectors along the radial

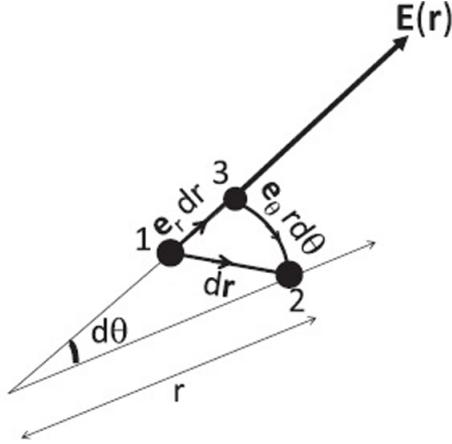


Figure 5.3: The work done by the electric field \vec{E} when the particle moves by a tiny amount \vec{dr} is either given by $\vec{E} \cdot \vec{dr}$ or as a sum of the work done on a radial segment $\hat{e}_r dr$ and an angular segment $\hat{e}_\theta r d\theta$ that connect the same end points. The angular part does not contribute to the work done as \hat{e}_r is perpendicular to \hat{e}_θ . Credit: Shankar, Vol.2

and angular direction respectively. In fact, the quantity $Q \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$ is the work done (an energy cost!) to move a test charge Q against the electric field \vec{E} from \vec{r}_1 to \vec{r}_2 (the left hand side of the above relation, upto the factor of Q), and corresponds to the potential energy gained $V(|\vec{r}_1|) - V(|\vec{r}_2|)$.

The potential at \vec{r} due to a set of discrete charges q_i at points \vec{r}_i ($i = 1, \dots, N$) is given by



$$V(|\vec{r}|) = \sum_{i=1}^N \frac{q_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|}, \quad (5.34)$$

and the potential energy for a test charge \tilde{q} in the potential V (which generates the field \vec{E}) is

Coulomb

$$U(r) = \tilde{q} V(r). \quad (5.35)$$

Both $V(r)$ and $U(r)$ are scalar quantities.

For continuous charge distributions ($\lambda(r)$ in 1D, $\sigma(r)$ in 2D and $\rho(r)$ in 3D), we can write the potential for a lone charge q at \vec{r} as

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_V dV' \frac{\rho(r')}{r'} \text{ in 3D}, \quad (5.36)$$

$$= \frac{1}{4\pi\epsilon_0} \int_V dV' \frac{\sigma(r')}{r'} \text{ in 2D}, \quad (5.37)$$

$$= \frac{1}{4\pi\epsilon_0} \int_V dV' \frac{\lambda(r')}{r'} \text{ in 1D}. \quad (5.38)$$

$$(5.39)$$

5.3 Path independence & Equipotential contours/surfaces

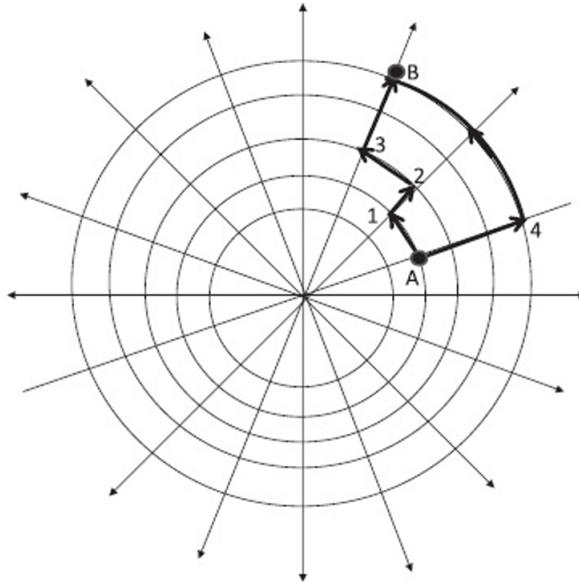


Figure 5.4: The work done in going from A to B by the field of a point charge is path-independent. One path goes radially out from A to 4 and then in the angular direction at fixed radius r to B. The angular part $4 \rightarrow B$ does not contribute since \vec{E} and $d\vec{r}$ are orthogonal. In the other path $A \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow B$, the angular parts $A \rightarrow 1$ and $2 \rightarrow 3$ likewise do not contribute, while the radial parts $1 \rightarrow 2$ and $3 \rightarrow B$ together contribute the same as the radial part $A \rightarrow 4$ in the other path. Credit: Shankar, Vol.2

Here are a couple of points to keep in mind about the line integral of \vec{E} :

1. All paths between two points A and B will be equivalent in terms of the line integral

$$\int_A^B \vec{E} \cdot d\vec{r} = V_B - V_A , \quad (5.40)$$

as only the radial contributions add up (as \vec{E} and \hat{r} are aligned), while the angular parts of the path do not contribute (as \vec{E} is orthogonal to $\hat{\theta}$ and $\hat{\phi}$).

2. The circular contours (in 2D)/ spherical surfaces (in 3D) around a point charge represent the contours/surfaces of constant potential, hence the name “Equipotential” surfaces.

5.4 Superposition Principle extends to Potentials

We have seen earlier that the total force on a test charge Q due to a set of other charges q_i ($i = 1, \dots, N$) is given by the vector sum of all the Coulomb forces

$$\vec{F}_Q = \vec{F}_{q_1} + \vec{F}_{q_2} + \dots + \vec{F}_{q_N} = \sum_{i=1}^N \vec{F}_{q_i}. \quad (5.41)$$

Dividing throughout this expression by Q , we obtain the electric field experienced by the test charge Q

$$\vec{E}_Q = \vec{E}_{q_1} + \vec{E}_{q_2} + \dots + \vec{E}_{q_N} = \sum_{i=1}^N \vec{E}_{q_i}. \quad (5.42)$$

Given that $\vec{E} = -\vec{\nabla}V$, upon integrating from a common reference point (say $r \rightarrow \infty$) to $|\vec{r}|$, we obtain the superposition principle for the Coulomb potential

$$V_Q(r) = V_{q_1}(r) + V_{q_2}(r) + \dots + V_{q_N}(r) = \sum_{i=1}^N V_{q_i}(r). \quad (5.43)$$

5.5 Examples of the Coulomb potential

Let us now compute a couple of examples of the Coulomb potential.

5.5.1 Inside & Outside a spherical shell of radius R with uniform surface charge σ

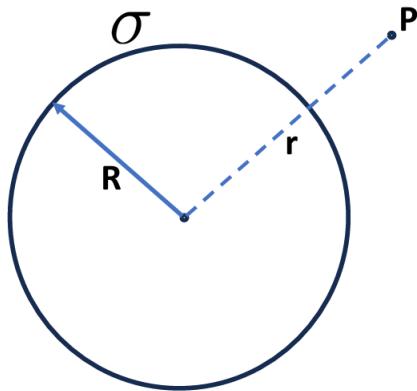


Figure 5.5: The spherical shell of radius R with uniform surface charge σ .

We set the reference point $V(r \rightarrow \infty) = 0$. We saw earlier that

$$\vec{E}(|\vec{r}| > |\vec{R}|) = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}, \text{ where } q = 4\pi R^2 \sigma, \quad (5.44)$$

$$\therefore V(r > R) = - \int_{\infty}^r dr' \frac{q}{4\pi\epsilon_0 r'^2}, \quad (5.45)$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r'} \Big|_{\infty}^r = \frac{q}{4\pi\epsilon_0 r}, \quad (5.46)$$

obtains as the potential outside the shell ($r > R$). Indeed, the potential falls off with the expected r^{-1} form outside the shell, indicating that for all points outside the shell, an effective charge $q = 4\pi R^2 \sigma$ is present at the center of the shell. For the potential inside the shell ($r < R$), recall that the field inside the shell vanishes. Hence,

$$V(r < R) = - \frac{q}{4\pi\epsilon_0} \int_{\infty}^R \frac{dr'}{r'^2} - \frac{q}{4\pi\epsilon_0} \int_R^r dr' \times 0, \quad (5.47)$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r'} \Big|_{\infty}^R + 0 = \frac{q}{4\pi\epsilon_0 R}, \quad (5.48)$$

i.e., the potential is a constant $q/(4\pi\epsilon_0 R)$ inside the shell.

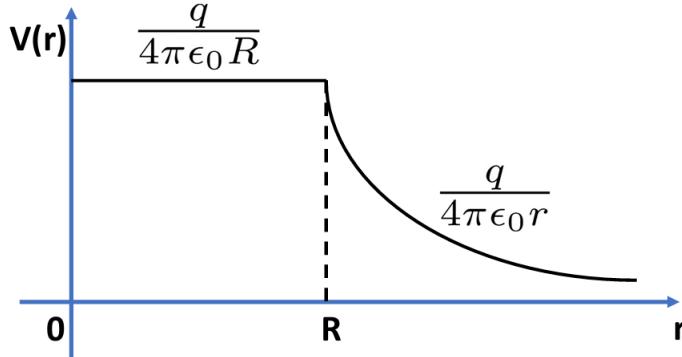


Figure 5.6: The electrical potential for a spherical shell of radius R with uniform surface charge σ .

5.5.2 Potential and Field for a dipole

Using the superposition principle for the potential of a dipole of charges q and $-q$ placed at $(a, 0)$ and $(-a, 0)$ respectively, we obtain at a point $\vec{r} = (x, y)$

$$V(r) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_+} - \frac{1}{r_-} \right), \quad (5.49)$$

$$= \frac{q}{4\pi\epsilon_0} \frac{(r_- - r_+)}{r_+ r_-}, \quad (5.50)$$

where $r_+ = ((x-a)^2 + y^2)^{1/2}$, $r_- = ((x+a)^2 + y^2)^{1/2}$.

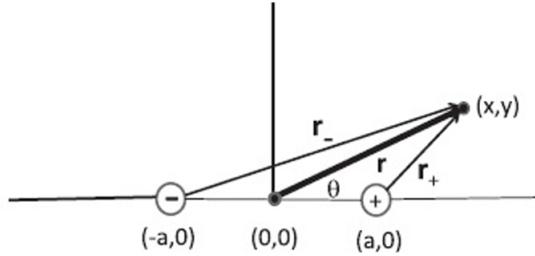


Figure 5.7: The potential at the point (x, y) is simply the sum of the two scalar contributions from charges $+q$ placed at $(\pm a, 0)$. Credit: Shankar, Vol.2

Clearly, when $a = 0$, $r_+ = r_-$ and hence $V(r) = 0$. This is the “no dipole” case, as the two charges are placed on top of one another! We want to study the case when $r \gg a$. Thus, to obtain the leading expression in $O(a)$, we can expand the square roots in $r_- - r_+$ to $O(a)$ in the numerator, and drop all terms of $O(a)$ and above from the product r_+r_- in the denominator

$$r_- - r_+ = \sqrt{(x+a)^2 + y^2} - \sqrt{(x-a)^2 + y^2}, \quad (5.51)$$

$$\approx \sqrt{x^2 + 2ax + y^2} - \sqrt{x^2 - 2ax + y^2} \text{ dropping terms of } O(a^2), \quad (5.52)$$

$$\approx \sqrt{r^2 + 2ax} - \sqrt{r^2 - 2ax}, \text{ as } r^2 = x^2 + y^2, \quad (5.53)$$

$$\approx r\left(1 + \frac{2ax}{r^2}\right)^{1/2} - r\left(1 - \frac{2ax}{r^2}\right)^{1/2}, \quad (5.54)$$

$$\approx r\left(1 + \frac{2ax}{2r^2} + \dots\right) - r\left(1 - \frac{2ax}{2r^2} + \dots\right), \text{ expanding } (1+x)^{1/2} \approx 1 + x/2 + \dots \text{ for } x \ll 1 \quad (5.55)$$

$$\approx \frac{4axr}{2r^2} \approx \frac{2ax}{r}. \quad (5.56)$$

Then, setting $r_+r_- \approx r^2$ in the denominator of $V(r)$ (which is justified in the limit of $r \gg a$), we obtain

$$V(r) = \frac{q}{4\pi\epsilon_0} \frac{r_- - r_+}{r_+r_-} \Big|_{r \gg a} \approx \frac{2axq}{4\pi\epsilon_0 r^3} \quad (5.57)$$

$$\approx \frac{px}{4\pi\epsilon_0 r^3} = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3} \quad (5.58)$$

as $\vec{r} = x\hat{i} + y\hat{j}$ and the dipole moment is $\vec{p} = 2aq\hat{i} = p\hat{i}$. Alternatively, we may write

$$V(r) \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3} = \frac{pr \cos \theta}{4\pi\epsilon_0 r^3} = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}, \quad (5.59)$$

where θ is the angle between the vectors \vec{p} and \vec{r} .

Now, to compute the electric field $\vec{E} = -\vec{\nabla}V(x, y) = E_x \hat{i} + E_y \hat{j}$

$$E_x = -\frac{\partial V}{\partial x} = -\frac{p}{4\pi\epsilon_0} \frac{\partial}{\partial x} \left[\frac{x}{(x^2+y^2)^{3/2}} \right], \quad (5.60)$$

$$= -\frac{p}{4\pi\epsilon_0} \left[\frac{1}{(x^2+y^2)^{3/2}} - \frac{3x}{2} \frac{2x}{(x^2+y^2)^{5/2}} \right], \quad (5.61)$$

$$= \frac{p}{4\pi\epsilon_0 r^3} \left(\frac{3x^2}{r^2} - 1 \right) = \frac{p}{4\pi\epsilon_0 r^3} (3\cos^2\theta - 1). \quad (5.62)$$

Similarly, we obtain

$$E_y = -\frac{\partial V}{\partial y} = -\frac{p}{4\pi\epsilon_0} \frac{\partial}{\partial y} \left[\frac{x}{(x^2+y^2)^{3/2}} \right], \quad (5.63)$$

$$= -\frac{p}{4\pi\epsilon_0} \frac{3x}{2} \frac{2y}{(x^2+y^2)^{5/2}}, \quad (5.64)$$

$$= \frac{p}{4\pi\epsilon_0 r^3} \frac{3x}{r} \frac{y}{r} = \frac{p}{4\pi\epsilon_0 r^3} (3\sin\theta\cos\theta). \quad (5.65)$$

Now, using $\vec{p} = p\hat{i}$ and $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$, we find

$$\vec{E} = E_x \hat{i} + E_y \hat{j} = \frac{p}{4\pi\epsilon_0 r^3} (3\cos^2\theta\hat{i} - \hat{i} + 3\sin\theta\cos\theta\hat{j}), \quad (5.66)$$

$$= \frac{1}{4\pi\epsilon_0 r^3} [3p\cos\theta(\cos\theta\hat{i} + \sin\theta\hat{j}) - p\hat{i}], \quad (5.67)$$

$$= \frac{1}{4\pi\epsilon_0 r^3} [(3\vec{p} \cdot \hat{r})\hat{r} - \vec{p}]. \quad (5.68)$$

This was, if you care to recall, precisely the expression we obtained earlier.

5.6 Visualising the Equipotential Surfaces

The expression $\vec{E} = -\vec{\nabla}V$ means that the \vec{E} field must always lie along directions that are locally normal (perpendicular) to the equipotential surfaces. Thus,

$$dV = -\vec{E} \cdot d\vec{r} = -|\vec{E}| |\vec{dr}| \cos\theta, \quad (5.69)$$

where θ is the angle between \vec{E} and $d\vec{r}$. This tells us that the \vec{E} field points in the direction of the greatest change (here drop) in the potential V : $\theta = 0$. On the other hand, for $\theta = \pi/2$, $\cos\theta = 0$ and there is no change in V , i.e., $dV = 0$, defining thereby the notion of an “equipotential” surface.

5.7 Poisson's & Laplace's Equations

We have already seen that

$$(i) \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad (ii) \vec{\nabla} \times \vec{E} = 0, \quad (iii) \vec{E} = -\vec{\nabla}V. \quad (5.70)$$

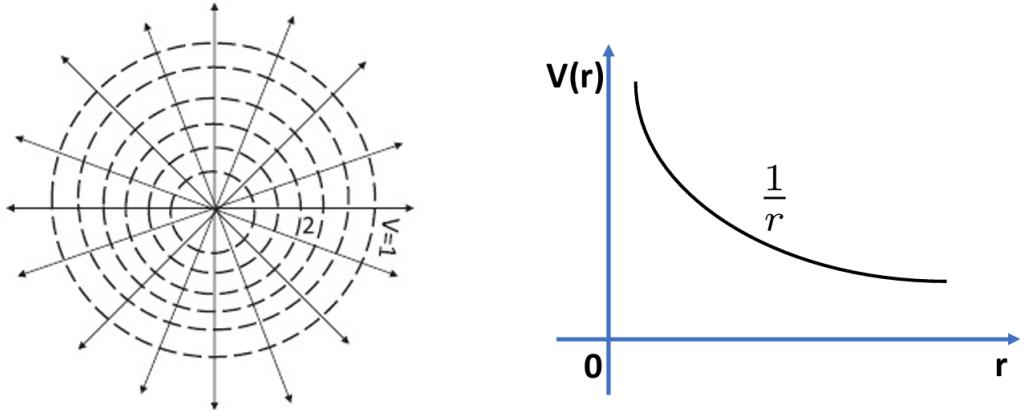


Figure 5.8: Left panel: A two-dimensional cross section of the equipotentials (dotted lines) due to the uniform electric field (solid arrows) of a point charge. Note that the field is always perpendicular to the equipotential surfaces corresponding to concentric spheres. Credit: Shankar, Vol.2.

Right panel: Climbing Mount Coulomb!

Using relations (ii) and (iii), we obtain the fact that $\vec{\nabla} \times (-\vec{\nabla}V) = 0$. However, this is simply a restatement of the fact that $\vec{E} = -\vec{\nabla}V$ is irrotational and corresponds to a conservative field. We have already studied this just above. Now, using relations (i) and (iii), we obtain



Poisson

$$\vec{\nabla} \cdot (-\vec{\nabla}V) = -\vec{\nabla} \cdot \vec{\nabla}V = -\nabla^2V = \frac{\rho}{\epsilon_0}, \quad (5.71)$$

$$\Rightarrow \text{Poisson's equation : } \nabla^2V = -\frac{\rho}{\epsilon_0}, \quad (5.72)$$

$$\text{Laplace's equation : } \nabla^2V = 0 \text{ in regions with no charge .} \quad (5.73)$$



Let's see what $\vec{\nabla} \cdot \vec{\nabla}V$ looks like

$$\vec{\nabla} \cdot \vec{\nabla}V = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \cdot (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})V(x, y, z), \quad (5.74)$$

$$= (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})V(x, y, z) \equiv \nabla^2V, \quad (5.75)$$

$$\Rightarrow \text{Laplacian Operator : } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (5.76)$$

5.8 Boundary Conditions

In order to gauge the importance of boundary conditions when obtaining unique solutions to either of Poisson's or Laplace's equations, let us revisit a problem we solved earlier. Recall that when we solved for the \vec{E} field for the case of a plane with a uniform surface density σ , we found

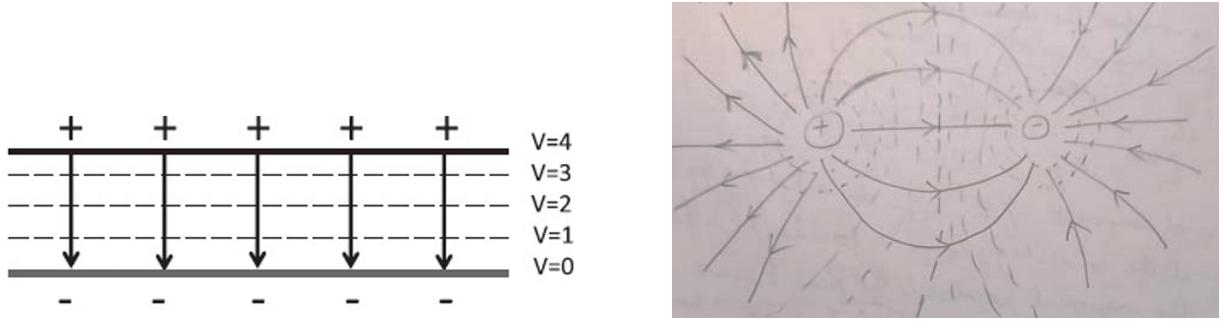


Figure 5.9: A two-dimensional cross section of the equipotentials (dotted lines) due to the uniform electric field (solid arrows) between (left panel) parallel plates carrying uniform surface charges that are equal and opposite and (right panel) a dipole. Note that the field is always perpendicular to the equipotential surfaces, corresponding to planes that are equiplanar with the two parallel plates in the left panel; for the dipole, the direction of the field lines keeps changing. Credit: Shankar, Vol.2

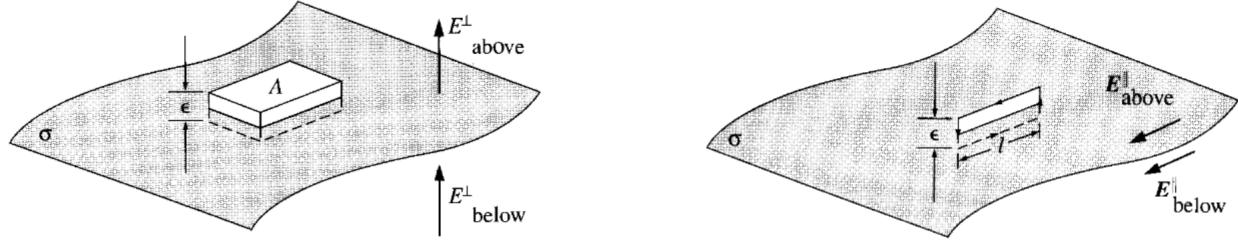


Figure 5.10: Using a Gaussian pillbox to visualise how \vec{E} changes (left panel) across and (right panel) parallel to a boundary. Credit: Griffiths, Fourth Edition.

that

$$\vec{E}_\perp^{\text{above}} = \frac{\sigma \hat{k}}{2\epsilon_0} , \quad \vec{E}_\perp^{\text{below}} = -\frac{\sigma \hat{k}}{2\epsilon_0} , \quad (5.77)$$

$$\vec{E}_\perp^{\text{above}} - \vec{E}_\perp^{\text{below}} = \frac{\sigma \hat{k}}{\epsilon_0} \left(\frac{1}{2} - \left(-\frac{1}{2} \right) \right) = \frac{\sigma \hat{k}}{\epsilon_0} , \quad (5.78)$$

i.e., the normal component of \vec{E} , \vec{E}_\perp , is discontinuous by the amount σ/ϵ_0 at the conducting plane (a boundary in space dividing the upper half from the lower half). On the other hand, for the component parallel to the plane \vec{E}_\parallel , we can argue that since $\oint_C \vec{E} \cdot d\vec{l} = 0$ for the contour C being a thin rectangle cutting through the conducting plane, and the sides of length ϵ don't contribute (as

they are perpendicular to \vec{E}_{\parallel}), we have

$$\begin{aligned} (\vec{E}_{\parallel}^{above} l - \vec{E}_{\parallel}^{below} l) &= 0, \\ \vec{E}_{\parallel}^{above} &= \vec{E}_{\parallel}^{below}. \end{aligned} \quad (5.79)$$

These two relations are boundary conditions on \vec{E}_{\perp} and \vec{E}_{\parallel} respectively, and can be combined into one relation as follows

$$\vec{E}^{above} - \vec{E}^{below} = \frac{\sigma \hat{n}}{\epsilon_0}, \quad (5.80)$$

where \hat{n} is a unit vector perpendicular to the conducting plane's surface. Further, the potential is given by

$$\begin{aligned} V_{above} - V_{below} &= - \int_a^b \vec{E} \cdot d\vec{l}, \\ &\rightarrow 0, \quad \text{as } a \rightarrow b. \end{aligned} \quad (5.81)$$

$$\Rightarrow V_{above} = V_{below}, \quad (5.82)$$

i.e., V is continuous across the conducting plane even if $\vec{E} = -\vec{\nabla}V$ is not! This is a general truth: the potential function V is very generally continuous, but its spatial derivative (i.e., the electric field \vec{E}) need not be.

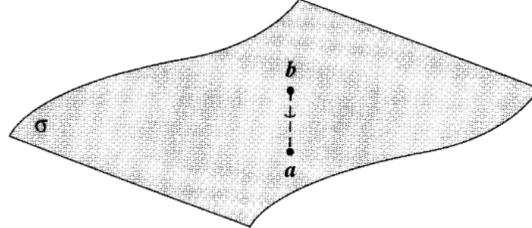


Figure 5.11: Matching the potentials across the boundary. Credit: Griffiths, Fourth Edition.

5.9 Work Done & Coulomb Potential

The work done W to move a test charge Q from point a to point b against an \vec{E} field is given by

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q \int_a^b \vec{E} \cdot d\vec{l}, \quad (5.83)$$

$$= -Q \int_a^b (-\vec{\nabla}V) \cdot d\vec{l}, \quad (5.84)$$

$$= Q \int_a^b d\vec{l} \cdot \vec{\nabla}V, \quad (5.85)$$

$$\Rightarrow W = Q[V(b) - V(a)]. \quad (5.86)$$

Setting the reference point for the potential at ∞ , i.e., $V(r \rightarrow \infty) = 0$, we can see that the work done to bring a test charge Q to a point r from ∞ is given by

$$W = QV(r) . \quad (5.87)$$

5.10 A closer look at Laplace's Equation

The main objective of electrostatics is to be able to compute the electric field $\vec{E}(r)$ of a given stationary charge distribution $\rho(\vec{r})$

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \int_V dV' \frac{\rho(\vec{r}')}{r'^2} \hat{r}' . \quad (5.88)$$

Unless $\rho(\vec{r})$ is simple in its form, the integral to obtain $\vec{E}(r)$ is a difficult one. Celebrated simple examples involve exploiting symmetries and using Gauss' law, but usually the best strategy is to compute the potential

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V dV' \frac{\rho(\vec{r}')}{r'} . \quad (5.89)$$

This is, in itself, still a very hard problem for an arbitrary $\rho(\vec{r})$. For instance, in conductors, charges move around freely and $\rho(\vec{r})$ itself may be difficult to determine, often only the total charge is well determined. Indeed, it is often better to recast the problem in its differential form

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0} , \quad (\text{Poisson's Equation}) \quad (5.90)$$

which, together with appropriate boundary conditions, yields a solution equivalent to the integral form of the potential $V(\vec{r})$. For the special case of $\rho(\vec{r}) = 0$, we have Laplace's equation

$$\nabla^2 V(\vec{r}) = 0 , \quad (5.91)$$

$$\text{or, } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x, y, z) = 0 . \quad (5.92)$$

Let us now try to obtain some general conclusions on this equation.

1D. Here, Laplace's equation becomes an ordinary differential equation (ODE)

$$\frac{d^2 V(x)}{dx^2} = 0 \implies V(x) = ax + b . \quad (5.93)$$

Given that Laplace's equation is second order in x , it is obvious that we would obtain a solution involving two parameters whose values can only be obtained through the information of two

boundary conditions, e.g., $V(x = 1) = 3$ and $V(x = 6) = 0$, such that

$$3 = a + b \quad \& \quad 0 = 6a + b \implies a = -\frac{b}{6}, \quad (5.94)$$

$$3 = -\frac{b}{6} + b = \frac{5b}{6} \implies b = \frac{18}{5}, a = -\frac{3}{5}, \quad (5.95)$$

$$\implies V(x) = -\frac{3x}{5} + \frac{18}{5}. \quad (5.96)$$

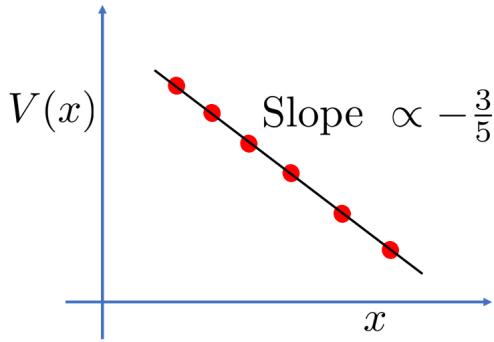


Figure 5.12: The potential $V(x) = -\frac{3x}{5} + \frac{18}{5}$.

Consequences: (a) Writing Laplace's equation in discrete form, we get

$$0 = \frac{d^2V}{dx^2} = \lim_{\epsilon \rightarrow 0} \left[\frac{(V(x+\epsilon) - V(x))}{\epsilon} - \frac{(V(x) - V(x-\epsilon))}{\epsilon} \right] / \epsilon, \quad (5.97)$$

$$= \lim_{\epsilon \rightarrow 0} \frac{V(x+\epsilon) + V(x-\epsilon) - 2V(x)}{\epsilon^2}, \quad (5.98)$$

$$\implies 0 = V(x+\epsilon) + V(x-\epsilon) - 2V(x), \quad (5.99)$$

$$\implies V(x) = \frac{V(x+\epsilon) + V(x-\epsilon)}{2}, \quad (5.100)$$

i.e., Laplace's equation enforces local averaging of the potential.

(b) Laplace's equation leads to solutions that have no local extrema: all extrema occur at the boundaries. This is consistent with the first conclusion, as a local extremum would always be either greater or lesser than the local average.

2D. We now have a partial differential equation (PDE)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \quad (5.101)$$

The solutions to these are known as "harmonic functions" in mathematics. To visualise this 2D problem, try to picture a drumhead stretched out taut over a given surface.

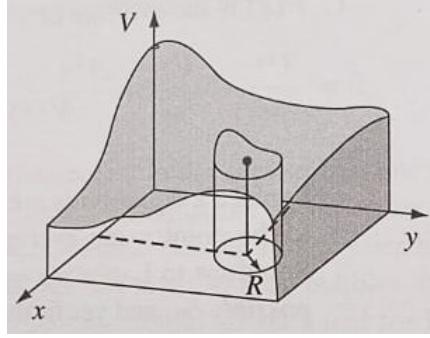


Figure 5.13: The potential $V(x)$ as a drumhead stretched out taut over a given surface. Credit: Griffiths, Fourth Edition.

Here are some noteworthy consequences of the harmonic functions:

- (a) The value of the potential $V(x, y)$ is the average of V at points lying on a circle of radius R around the point (x, y)

$$V(x, y) = \frac{1}{2\pi R} \oint_{Circle} dl V . \quad (5.102)$$

(b) As in 1D, Laplace's equation leads to solutions that have no local extrema: all extrema occur at the boundaries. Consistent with the boundary conditions imposed, Laplace's equation picks out the most featureless solution possible, i.e., one with no local hills or valleys, and only the smoothest surface achievable. In terms of stability, this suggests that test charges must move towards the boundaries and cannot sit still anywhere. This then brings us to Earnshaw's theorem:

"It is impossible to construct an electric field that will hold a charged particle in stable equilibrium in empty space."

Corollary 1: Unstable equilibria are allowed. But they are, as stated, unstable.

Corollary 2: It is possible to hold stably a charged particle with a non-zero charge distribution, i.e., a positive charge at the center of a solid sphere of uniform negative charge.

Corollary 3: It is possible to trap and hold stably a charge particle by electric fields that vary in time.

3D. In 3D, we can only offer the following conclusions

- (a) the value of the potential V at \vec{r} is given by taking the average over a surface of a sphere of radius R surrounding the point \vec{r}

$$V(\vec{r}) = \frac{1}{4\pi R^2} \iint_{Surface} dS V . \quad (5.103)$$

- (b) The potential V can again have no local extrema; the extremal values of V must exist at the boundaries.

Consider the potential at the origin due to a charge placed at $(0,0,z)$. For this, we consider a

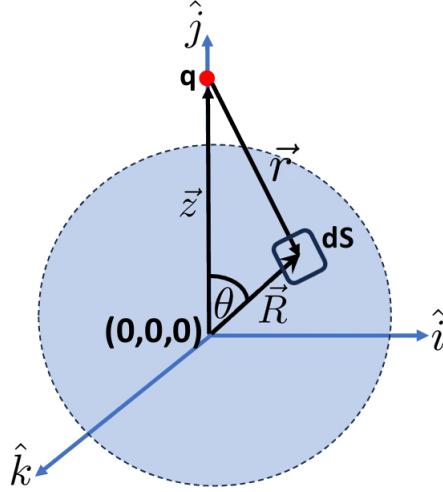


Figure 5.14: The potential $V(x)$ at the origin for a charge placed at $(0,0,z)$.

sphere of radius R centered around the origin. Now, the value of the potential at a point on the surface at a distance r from the charge is

$$V(r) = \frac{q}{4\pi\epsilon_0 r} , \quad (5.104)$$

where the displacement \vec{r} can be written as

$$\vec{r} = \vec{R} - \vec{z} , \quad (5.105)$$

$$r = |\vec{r}| = |\vec{R} - \vec{z}| , \quad (5.106)$$

$$r^2 = |\vec{r}|^2 = (\vec{R} - \vec{z}) \cdot (\vec{R} - \vec{z}) , \quad (5.107)$$

$$\implies r^2 = R^2 + z^2 - 2\vec{R} \cdot \vec{z} = R^2 + z^2 - 2Rz \cos\theta , \quad (5.108)$$

where θ is the angle between the vectors \vec{R} and \vec{z} . Then, the average potential on the surface of the

sphere of radius R is given by

$$V_{ave} = \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int \int dS \frac{1}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}} , \quad (dS = R\sin\theta d\theta \times R d\phi)$$

$$= \frac{qR^2}{16\pi^2\epsilon_0 R^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{\sin\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}} , \quad (5.109)$$

$$= \frac{2\pi qR^2}{16\pi^2\epsilon_0 R^2} \int_0^\pi d\theta \frac{\sin\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}} , \quad (5.110)$$

$$\frac{d}{d\theta}((R^2 + z^2 - 2Rz\cos\theta)^{1/2}) = \frac{Rz\sin\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}} , \quad (5.111)$$

$$\Rightarrow V_{ave} = \frac{2\pi q}{16\pi^2\epsilon_0 Rz} \int_0^\pi d\theta \frac{d}{d\theta}((R^2 + z^2 - 2Rz\cos\theta)^{1/2}) , \quad (5.112)$$

$$= \frac{q}{8\pi\epsilon_0 Rz} (R^2 + z^2 - 2Rz\cos\theta)^{1/2} \Big|_0^\pi , \quad (5.113)$$

$$= \frac{q}{8\pi\epsilon_0 Rz} [(R+z)^{1/2} - (z-R)^{1/2}] , \quad (5.114)$$

$$= \frac{q}{8\pi\epsilon_0 Rz} [(R+z) - (z-R)] , \quad (5.115)$$

$$= \frac{q}{8\pi\epsilon_0 Rz} \times 2R = \frac{q}{4\pi\epsilon_0 z} , \quad (5.116)$$

i.e., the average potential on the surface of a sphere of radius R due to the charge q at a distance z on the z -axis from the center of the sphere is precisely that due to q precisely at the center of the sphere!

There is another elegant way to show this result. Note that the average potential over the surface of a sphere of radius r can be written as

$$V_{ave} = \frac{1}{4\pi} \int d\Omega \hat{r} \cdot \vec{V} , \quad (5.117)$$

$$\Rightarrow \frac{dV_{ave}}{dr} = \frac{1}{4\pi} \int d\vec{\Omega} \cdot \vec{\nabla} V \quad (d\vec{\Omega} = d\Omega \hat{r}) , \quad (5.118)$$

$$= \frac{1}{4\pi r^2} \int d\vec{S} \cdot \vec{\nabla} V \quad (d\vec{S} = r^2 d\Omega \hat{r}) , \quad (5.119)$$

$$= \frac{1}{4\pi r^2} \int dV (\vec{\nabla} \cdot \vec{\nabla} V) \quad (\text{by divergence theorem}) , \quad (5.120)$$

$$= \frac{1}{4\pi r^2} \int dV \vec{\nabla}^2 V , \quad (5.121)$$

$$= 0 \quad (\text{as } \vec{\nabla}^2 V = 0) , \quad (5.122)$$

where $d\Omega$ is the differential element of the solid angle subtended at the origin by the area patch dS on the surface of the sphere. Thus, we have established that V_{ave} does not change with r , i.e., this result does not change with the radius of the sphere r . Now, for a sphere of infinitesimal radius $r \rightarrow 0+$, it is clear that $V_{ave} \equiv V_{center}$, i.e., the value of the potential at the center of the sphere (and

here, the origin of coordinates). Thus, we can see that the value of the average of the potential V_{ave} is fixed everywhere to be that of the potential at the center of the sphere. This was what we set out to establish.

You should have anticipated the final form of the answer obtained above from the simplicity of the problem at hand. Importantly, we have demonstrated that this answer is also the expected result from the solution to the Laplace equation indicated above in eq.(5.103). Further, this extends, via the superposition principle, to any collection of charges outside the sphere: their average potential over the sphere is equal to the net potential they produce at the center of the sphere.

5.11 Boundary Conditions and the Uniqueness Theorems

What are the appropriate boundary conditions that should be applied to obtain unique and physically relevant solutions to Laplace's equation? In 1D problems, choosing the boundary conditions suitably is easy, but the situation is not so obvious in 2D and 3D. The proof that a proposed set of boundary conditions will suffice is usually presented in the form of a "uniqueness theorem". Here, we will present two of them. Note that such theorems only guarantee the existence of unique solutions, not the ease in finding them. Nevertheless, such existence proofs offer the hope that such solutions can also be found in a given context (with enough effort!).

5.11.1 First Uniqueness Theorem

Theorem: The solution to Laplace's equation in some volume \mathcal{V} is uniquely determined if the potential V is specified on the surface S bounding \mathcal{V} .

Proof: Supposing there are two solutions V_1 and V_2 to Laplace's equation, both of which assume the specified value on the bounding surface S , we want to show that they must be equal.

Consider $V_3 = V_1 - V_2$. Since

$$\nabla^2 V_1 = 0 = \nabla^2 V_2, \quad (5.123)$$

$$\implies \nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0, \quad (5.124)$$

such that $V_1 = V_2$ on S , and $V_3 = 0$ on S . Now, given that Laplace's equation can have no local extrema and all extrema must lie on the boundary, both the maxima and minima of V_3 are zero. Thus, $V_3 = 0$ throughout the volume \mathcal{V} : $V_1 = V_2$ everywhere in \mathcal{V} .

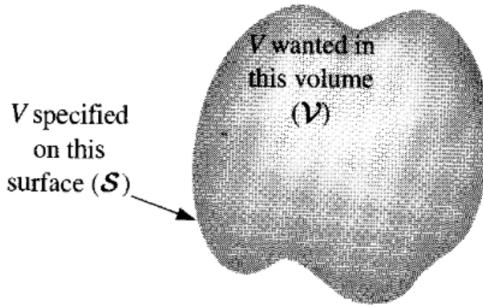


Figure 5.15: The potential $V(x)$ that we seek within the volume \mathcal{V} given that on the surface S bounding \mathcal{V} . If there are holes inside \mathcal{V} , we must specify the potential $V(x)$ on those inner surfaces as well. Credit: Griffiths, Fourth Edition.

Now, to extend this to Poisson's equation

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} = \nabla^2 V_2, \quad (5.125)$$

$$\Rightarrow \nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0. \quad (5.126)$$

Once again, $V_3 = 0$ on surface S (as $V_1 = V_2$ on S) and since all extrema of solutions to Laplace's equation must lie on S , $V_3 = 0$ everywhere in volume \mathcal{V} , such that $V_1 = V_2$ everywhere in \mathcal{V} .

Corollary for Poisson's equation: The potential $V(r)$ in a volume \mathcal{V} is uniquely determined if (a) the charge density $\rho(r)$ throughout \mathcal{V} and (b) the value of the potential V on all boundaries S of \mathcal{V} are specified.

5.11.2 Conductors and the Second Uniqueness Theorem

In electrostatic problems involving conductors, we know we know the charges on conducting surfaces rather than the electrostatic potential V . Can we now determine the electric field uniquely? Or can the charges rearrange themselves on their respective conductors in different ways so as to generate different electric fields? In order to answer these questions, another uniqueness theorem comes in handy. (Note that if batteries are involved, we do know about the potential maintained on the conductor connected to the battery; the same is true if the conductor is connected to the Earth/ is grounded such that $V = 0$.)

Theorem: In a volume \mathcal{V} surrounded by conductors and containing a specified charge density ρ existing between the conductors, The electric field \vec{E} is uniquely determined if the total charge on each conductor (Q_i) is given. (Note that the entire region as a whole may be either bounded by

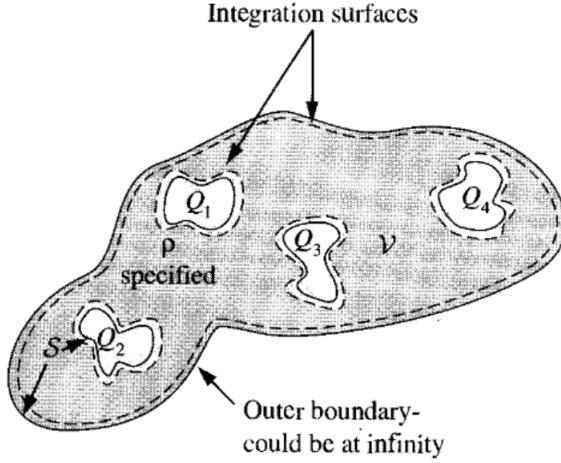


Figure 5.16: In a volume \mathcal{V} surrounded by conductors and containing a specified charge density ρ existing between the conductors, The electric field \vec{E} is uniquely determined if the total charge on each conductor (Q_i) is given. (Note that the entire region as a whole may be either bounded by another conductor or even unbounded. Credit: Griffiths, Fourth Edition.

another conductor or even unbounded.

[Proof:] Suppose there are two fields, \vec{E}_1 and \vec{E}_2 , satisfying the conditions of the problem. In the space between the conductors, any Gaussian surface will lead to Gauss' law in differential form:

$$\vec{\nabla} \cdot \vec{E}_1 = \frac{\rho}{\epsilon_0} = \vec{\nabla} \cdot \vec{E}_2 . \quad (5.127)$$

Further, for Gaussian surfaces enclosing each of the i conductors (with charge Q_i respectively on them), \vec{E}_1 and \vec{E}_2 must satisfy

$$\oint_{S_i} \vec{E}_1 \cdot d\vec{S} = \frac{Q_i}{\epsilon_0} = \oint_{S_i} \vec{E}_2 \cdot d\vec{S} , \quad (5.128)$$

where S_i corresponds to the i th Gaussian surface.

Likewise, for the outer boundary (whether at a finite distance and bounded by an enclosing conductor, or placed at spatial infinity)

$$\oint_{S_{outer}} \vec{E}_1 \cdot d\vec{S} = \frac{Q_{Total}}{\epsilon_0} = \oint_{S_{outer}} \vec{E}_2 \cdot d\vec{S} , \quad (5.129)$$

where $Q_{Total} = \sum_i Q_i + \rho \mathcal{V}'$ (where \mathcal{V}' corresponds to the volume within \mathcal{V} that excludes the total volume of the conductors). Now, we define the electric field $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$, such that in the region between the conductors

$$\vec{\nabla} \cdot \vec{E}_3 = \vec{\nabla} \cdot \vec{E}_1 - \vec{\nabla} \cdot \vec{E}_2 = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0 , \quad (5.130)$$

and over each Gaussian surface S_i (which now includes the outer boundary)

$$\oint_{S_i} \vec{E}_3 \cdot d\vec{S} = \oint_{S_i} \vec{E}_1 \cdot d\vec{S} - \oint_{S_i} \vec{E}_2 \cdot d\vec{S} = \frac{Q_i}{\epsilon_0} - \frac{Q_i}{\epsilon_0} = 0. \quad (5.131)$$

Note that since we already know that no electric field can exist within any of the conductors, this means that $\vec{E}_3 = 0$ within each conductor as well as on their surfaces.

We can also argue along similar lines for the case of the outermost surface S_{outer}

$$\oint_{S_{outer}} \vec{E}_3 \cdot d\vec{S} = \oint_{S_{outer}} \vec{E}_1 \cdot d\vec{S} - \oint_{S_{outer}} \vec{E}_2 \cdot d\vec{S} = \frac{Q_{Total}}{\epsilon_0} - \frac{Q_{Total}}{\epsilon_0} = 0. \quad (5.132)$$

However, things are a little more complicated now. Recall that a vanishing flux, $\oint_{S_{outer}} \vec{E}_3 \cdot d\vec{S} = 0$, shows that the electric field \vec{E}_3 cannot be either everywhere positive or everywhere negative within the volume bounded by S_{outer} . The electric field \vec{E}_3 can be everywhere null, but this is only the simplest possibility and does not rule out the fact that there could be spatial variations in \vec{E}_3 within the volume ' \mathcal{V} ' (i.e., the volume corresponding to the charge density ρ lying between the conductors) that could still lead to the condition $\oint_{S_{outer}} \vec{E}_3 \cdot d\vec{S} = 0$. To see that the electric field \vec{E}_3 is indeed everywhere null, we proceed as follows.

Even if we don't know how the charge Q_i is distributed over the i th conductor, we do know that each conductor is an equipotential, i.e., the potential V_3 is a constant (which doesn't have to be zero!) over each conducting surface. We can then proceed to show that the electric field is the same everywhere. To see this, we use $\vec{E}_3 = -\vec{\nabla}V_3$ and $\vec{\nabla} \cdot \vec{E}_3 = 0$ (as established above!), such that

$$\vec{\nabla} \cdot (V_3 \vec{E}_3) = V_3 (\vec{\nabla} \cdot \vec{E}_3) + \vec{E}_3 \cdot (\vec{\nabla} V_3), \quad (5.133)$$

$$= -\vec{E}_3 \cdot \vec{E}_3 = -(E_3)^2, \quad (5.134)$$

$$\Rightarrow \int_{\mathcal{V}} dV \vec{\nabla} \cdot (V_3 \vec{E}_3) = \oint_S (V_3 \vec{E}_3) \cdot d\vec{S} = - \int_{\mathcal{V}} dV (E_3)^2. \quad (5.135)$$

Since V_3 is a constant for any conducting surface

$$\oint_{S_{outer}} (V_3 \vec{E}_3) \cdot d\vec{S} = V_3 \oint_{S_{outer}} \vec{E}_3 \cdot d\vec{S} = 0 = - \int_{\mathcal{V}} dV (E_3)^2, \quad (5.136)$$

$$\Rightarrow \vec{E}_3 = 0 \text{ everywhere in } \mathcal{V}, \quad (5.137)$$

$$\Rightarrow \vec{E}_1 = \vec{E}_2 \text{ everywhere in } \mathcal{V}. \quad (5.138)$$

5.12 Capacitors

Capacitors are a means by which to store electrical potential energy that can be used later or at will.

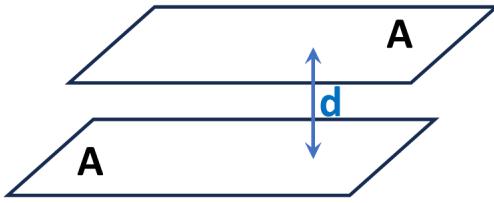


Figure 5.17: The geometry of a parallel plate capacitor in which the plates are of area A and are separated by a distance d .

Consider first the parallel plate variety, i.e., two plates of area A each placed at a uniform distance d apart. Each plate is a conductor with no \vec{E} field in its interior, and hence at a fixed potential. When they are kept neutral initially, they will both be at the same potential (say zero). Now, we transfer some negative charge from the upper plate to the lower plate such that the lower plate has $-Q$ charge on it while the upper plate has $+Q$ charge on it. Then, ignoring finite-size effects related to the existence of boundaries (here, the edges of the plates), the \vec{E} field due to this configuration of oppositely charged plates is

$$\vec{E} = E(-\hat{k}) = \frac{\sigma}{\epsilon_0}(-\hat{k}), \quad (5.139)$$

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0} \equiv -\frac{dV(z)}{dz}, \quad (5.140)$$

$$\implies V_{lower} - V_{upper} = \Delta V = \int_0^d dz E = \frac{\sigma d}{\epsilon_0} = \frac{Qd}{\epsilon_0 A}. \quad (5.141)$$

If we define the “Capacitance” as

$$C = \frac{Q}{\Delta V} \implies C = \frac{\epsilon_0 A}{d} \quad (5.142)$$

for the parallel plate capacitor. When we introduce a dielectric insulating material (with permittivity ϵ_{Medium}) within the two plates, we need to replace ϵ_0 by ϵ_{Medium} , and $C = \epsilon_{Medium}A/d$.

As another example, consider two concentric spheres of radii R_1 and R_2 with $R_1 < R_2$. Now, in the region between the two spheres, the inner sphere acts as an effective point charge located at the origin (this is easily seen by applying Gauss’ law!) and the potential difference between the inner and outer spheres due to a charge Q transferred from outer to inner sphere is given by

$$\Delta V = V_{inner} - V_{outer} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2} \right], \quad (5.143)$$

$$= \frac{Q(R_2 - R_1)}{4\pi\epsilon_0 R_1 R_2} \equiv \frac{Q}{C}, \quad (5.144)$$

$$\implies C = \frac{4\pi\epsilon_0 R_1 R_2}{R_2 - R_1}. \quad (5.145)$$

Here is a sanity check. If $R_2 - R_1 \ll R_1 \& R_2$, i.e., there exists a narrow region between the two spheres, the spheres should appear infinitely large and planar such that the capacitance should reduce effectively to that of the parallel plate capacitor. To see this, set $R = (R_1 + R_2)/2$ and $d = R_2 - R_1$, such that

$$C = \frac{4\pi\epsilon_0 R_1 R_2}{R_2 - R_1}, \quad (5.146)$$

$$= \frac{4\pi\epsilon_0 (R - d/2)(R + d/2)}{d}, \quad (5.147)$$

$$= \frac{4\pi\epsilon_0 (R^2 - d^2/4)}{d}, \quad (5.148)$$

$$\approx \frac{4\pi\epsilon_0 R^2}{d} \equiv \frac{\epsilon_0 A}{d}, \quad (5.149)$$



where the effective area of the equivalent parallel plate capacitor is given by $A = 4\pi R^2$.

The unit of Capacitance is the Farad (F) \equiv Coulomb/Volt, and named after Michael Faraday. 1F is actually a big capacitance, and mF and μF are typical values of capacitors appearing in circuits.

Faraday Finally, to obtain the energy stored in a capacitor, recall that the work (dW) done against transferring charge dQ against a potential difference ΔV is

$$dW = \Delta V dQ = \frac{Q}{C} dQ, \quad (5.150)$$

$$\implies \text{Work } W = \int_0^{\tilde{Q}} dQ \frac{Q}{C}, \quad (5.151)$$

$$= \frac{\tilde{Q}^2}{2C}, \quad (5.152)$$

$$= \frac{C \Delta V^2}{2} \quad (\text{as } \tilde{Q} = C \Delta V). \quad (5.153)$$

The work W corresponds to the energy stored in the capacitor.