Solution Set - 2

MA 1201 Spring Sem, 2025

1. For any two finite sets A and B, the cardinality of A^B is $|A|^{|B|}$.

Solution: We have $A^B := \{f | f : B \to A\}$. Suppose $|A| = m$ and $|B| = n$. From the definition of function, for each element $x \in B$, possible number of value of $f(x)$ is m. Therefore, by the multiplication principle (MP) in combinatorics, the total number of function from $B \to A$ is, $\underbrace{m \cdot m \cdot \cdots \cdot m}_{n-\text{copy}} = m^n = |A|^{|B|}$.

Alternative solution

Since B is finite, then there exist a bijection $g : I_n \to B$ for some n. Therefore we have, $|A^B| = |A^{I_n}|$ (prove that $f \mapsto f \circ g$ gives a bijection from A^B to A^{I_n}) . Now, we have $A^{I_n} := \{f | f : I_n \to A\}$. Then A^{I_n} can be thought of as,

$$A^{I_n} = \underbrace{A \times A \times \cdots \times A}_{n-\text{ copy}}$$

Now, we prove that $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|$ by mathematical induction. Now, we know that, for finite sets A_1, A_2, $|A_1 \times A_2| = |A_1||A_2|$, so the statement is true for $n = 2$. Suppose the statement is true for $n = k$, let $A = A_1 \times A_2 \times \cdots \times A_k$, then by induction hypothesis, we have

$$|A \times A_{k+1}| = |A||A_{k+1}| = (|A_1| \cdot |A_2| \cdot \cdots \cdot |A_k|) \cdot |A_{k+1}|$$

So, the statements is true for $k + 1$, Hence by mathematical induction, statement is true fo all n, therefore,

$$|A^B| = |A^{I_n}| = \underbrace{|A| \cdot |A| \cdot \cdots \cdot |A|}_{n-\text{ copy}} = |A|^n = |A|^{|B|}$$

2. Show that countable union of finite set is countable.

Solution: Let $\mathcal{C} := \{A_k\}_{k\in\mathbb{N}}$ be a countable collection of finite sets.
To show: $A := \bigcup_{k\in\mathbb{N}} A_k$ is atmost countable.
Case-I: Only finitely many sets in $\mathcal{C}$ are nonempty.
Let $B_1, B_2, \ldots, B_N$ be the non-empty sets from $\mathcal{C}$. Then, $A := \bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k=1}^{N} B_k$, being finite union of finite sets is finite. Hence, A is atmost countable.
Case-II: Infinitely many sets in $\mathcal{C}$ are nonempty.
Let $\{B_k\}_{k\in\mathbb{N}}$ be the collection of nonempty subsets of $\mathcal{C}$. Define:

$$\begin{aligned} C_1 &:= B_1 \\ C_2 &:= B_2 \setminus B_1 \\ C_3 &:= B_3 \setminus (B_1 \cup B_2) \\ &\vdots \end{aligned}$$

Then $\tilde{\mathcal{C}} := \{C_k\}_{k\in\mathbb{N}}$ is a collection of finite sets, which are pairwise disjoint and also

$$A = \bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k\in\mathbb{N}} B_k = \bigcup_{k\in\mathbb{N}} C_k$$

- If $C_k \neq \phi$ for only finite number of k, then again A, being finite union of finite sets, is finite, and hence atmost countable.

- Assume $C_k \neq \phi$ for inifinitely many k. Let $D_1, D_2, \ldots$ be nonempty subsets from collection $\tilde{\mathcal{C}}$. Then D_k's are pairwise disjoint and

$$A = \bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k\in\mathbb{N}} B_k = \bigcup_{k\in\mathbb{N}} C_k = \bigcup_{k\in\mathbb{N}} D_k.$$

  Since D_k is nonempty finite set, so $\exists\, n_k \in \mathbb{N}$ and a bijective map $f_k : I_{n_k} \to D_k$. Using these maps f_k, we define

$$\begin{aligned} &\tilde{f}_1 : \{1, 2, \ldots, n_1\} \to D_1 : \tilde{f}_1(n) = f_1(n), \\ &\tilde{f}_k : \{(n_1 + \cdots + n_{k-1}) + 1, \ldots, (n_1 + \cdots + n_{k-1}) + n_k\} \to D_k \text{ as} \\ &\qquad\qquad \tilde{f}_k\left((n_1 + \cdots + n_{k-1}) + n\right) = f_k(n), \text{ for } k \geq 2. \end{aligned}$$

  Finally, we define the function $f : \mathbb{N} \to \bigcup_{k\in\mathbb{N}} D_k$ as

$$f(n) = \begin{cases} \tilde{f}_1(n), & \text{if } 1 \leq n \leq n_1, \\ \tilde{f}_2(n - n_1), & \text{if } n_1 < n \leq n_1 + n_2, \\ \tilde{f}_k\left(n - (n_1 + \cdots + n_k)\right), & \text{if } n_1 + \cdots + n_k < n \leq n_1 + \cdots + n_{k+1}, \text{ for } k \geq 2. \end{cases}$$

Check f is bijective.
Hence $A = \bigcup_{k\in\mathbb{N}} D_k$ is countable.

3. Show that $\mathbb{Q}_+$ is countable.

Solution: It has been proved in class that $\mathbb{N}\times\mathbb{N}$ is countable. Thus, there exists a bijective map $f : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$.
Define a map $g : \mathbb{N}\times\mathbb{N} \to \mathbb{Q}_+$ as $g(p,q) = \frac{p}{q}$. Note that g is onto.
Now, consider the composition $h := gof : \mathbb{N} \to \mathbb{Q}_+$. Then the map h, being composition of two onto maps, is onto.
By problem (15) of Problem Sheet-1, the set $\mathbb{Q}_+$ is atmost countable.
Also, the set $\mathbb{Q}_+$ is infinite as the inclusion map

$$\mathbb{N} \hookrightarrow \{1\}\times\mathbb{N} \hookrightarrow \mathbb{Q}_+.$$

is injective. Hence $\mathbb{Q}_+$ is countable.

4. Show that if A is countable, then $A^k = A\times\cdots A\,(k \text{ times }),\ k\in\mathbb{N}$ is countable. As a result $\mathbb{Q}^k$ is countable for any $k\in\mathbb{N}$.

Solution: We shall prove this statement using the principle of mathematical induction. Consider the statement, $P(k)$, for $k\in\mathbb{N}$ by,

$$P(k) = \text{For a countable set } A, A^k \text{ is countable}$$

Let $A = \{a_1, a_2, \cdots, a_n, \cdots\}$ be a given countable set.
Base Case $(k = 1)$:
For $k = 1$, $A^1 = A$ is countable by the assumption.
Inductive Step:
Assume that the statement holds for $k = n$, i.e., $A^n = \underbrace{A\times A\times\cdots A}_{n-\text{times}}$ is countable.
We now prove that $A^{n+1} = \underbrace{A\times A\times\cdots A}_{n+1-\text{times}}$ is also countable. Observe that,

$$\underbrace{A\times A\times\cdots A}_{n+1-\text{times}} = \bigcup_{p=1}^{\infty}\{a_p\}\times\underbrace{A\times A\times\cdots A}_{n-\text{times}}$$

where $\{a_p\} \times \underbrace{A \times A \times \cdots A}_{n-\text{times}}$ is the set of all elements in A^{n+1} whose first coordinate is $\{a_p\}$.

Let, $B_p = \{a_p\} \times A^n$. For each $p \in \mathbb{N}$, define the function $f_p : B_p \to \underbrace{A \times A \cdots A}_{n-\text{times}}$ by, $f_p(a_p, a_1, a_2, \cdots, a_n) = (a_1, a_2, \cdots, a_n)$.

(a) f_p **is one-one**: Let,

$$f_p(a_p, a_1, a_2, \cdots, a_n) = f_p(a_p, b_1, b_2, \cdots, b_n)$$

implies that, $(a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n)$ and hence $a_i = b_i$ for $i \in \{1, \cdots, n\}$ which shows that $(a_p, a_1, a_2, \cdots, a_n) = (a_p, b_1, b_2, \cdots, b_n)$. This conclude that f_p is one-one.

(b) f_p **is onto**: Let, $y = (a_1, a_2, \cdots, a_n) \in A^n$, consider the element $x = (a_p, a_1, \cdots, a_n) \in B_p$, then $f_p(x) = y$, thus every element of A^n has a pre-image in B_p shows that f_p is onto.

This shows that f_p is bijective, A^n is countable by the induction hypothesis, combining both the fact, it follows that B_p is countable for each $p \in \mathbb{N}$.
Now, $A^{n+1} = \bigcup\limits_{p=1}^{\infty} B_p$ is a countable union of countable sets, which is countable by the theorem proved in class.

5. *Prove that the set $\mathcal{P}$ of all polynomials

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

with integral coefficients, that is, where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$, is countable.

Solution: Let $\mathcal{P}_k \subset \mathcal{P}$ be the set of all $k-$degree polynomial, i.e., the coefficients of x^n is zero for $n > k$ and non zero for x^k. Note that $f : \mathcal{P}_n \to \mathbb{Z}^k \times \{\mathbb{Z} \backslash \{0\}\}$ defined as

$$f(a_o + a_1 x + \cdots a_k x^k) = (a_0, a_1, \cdots, a_k).$$

is a bijection. We also have $\mathcal{P} = \bigcup_{k \in \mathbb{N}} \mathcal{P}_k$. We see that $\mathcal{P}$ is countable union of countable set, hence is countable.

6. *A real number r is called an *algebraic number* if r is a solution to a polynomial equation

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

with integral coefficients. For example, integers, $\sqrt{2}, \sqrt[3]{4}$ are algebraic numbers. Prove that the set of algebraic numbers is countable.

Solution:

Note that a polynomial of degree n has at most n real roots. From the solution of Q.5, it follows that the set of all polynomials with integral coefficients is countable. Thus the set of all algebraic numbers, that is, the set of all roots of these polynomials is a countable union of finite sets(the real roots of each polynomial) and hence is countable.

7. *Let $\mathcal{A} = \{A_i : i \in I\}$, for some index set I, be a set of pairwise disjoint intervals in $\mathbb{R}$. Show that $\mathcal{A}$ is atmost countable. You can use $\mathbb{Q}$ is 'dense' in $\mathbb{R}$.

Solution: Let $A_i \in \mathcal{A}$. Since $A_i \subseteq \mathbb{R}$ and A_i is an interval, then there exist $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ such that $(a_i, b_i) \subseteq A_i$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists at least one rational number $q_i \in A_i$.

Now, since the intervals A_i are pairwise disjoint, then for the interval $A_j (i \neq j)$, we can find another rational number $q_j \in A_j$. If not, then $q_i \in A_i \cap A_j$ for some $i \neq j$. This would imply that A_i and A_j are not disjoint, contradicting the assumption.

Now define $f : \mathcal{A} \to \mathbb{Q}$ by $f(A_i) = q_i$. This mapping is injective because each interval corresponds to a unique rational number. Since $\mathbb{Q}$ is countable and any subset of a countable set is atmost countable, the image of f is atmost countable. Therefore, the injective mapping f implies that $\mathcal{A}$ is atmost countable.

8. Prove or disprove: If $\mathcal{B} = \{B_i : i \in I\}$, for some index set I, is a set of pairwise disjoint circles in $\mathbb{R}^2$, then $\mathcal{B}$ is countable.

Solution: The above statement is false. Consider $\mathcal{B} = \{B_i : B_i$ is a circle in $\mathbb{R}^2$ centered at origin and with radius $i, i \in \mathbb{R}\}$. Then $\mathcal{B}$ is a set of pairwise disjoint circles in $\mathbb{R}^2$. We define $f : \mathcal{B} \to \mathbb{R}$ given by $f(B_i) = i$. Now, f is bijective because given any $i \in \mathbb{R}$, there is a unique circle B centered at origin and with radius i, that is $f(B) = i$. So $\mathcal{B}$ is uncountable.

9. Prove that the set of all circles in the plane $\mathbb{R}^2$ having rational radii and centers with rational coordinates is countable.

Solution: Let $\mathcal{B}$ be the set of circles in $\mathbb{R}^2$ with rational radii and centers with rational coordinates. We define a map $f : \mathcal{B} \to \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^+$ by $f(C) = (x, y, r)$, where $(x, y) \in \mathbb{Q} \times \mathbb{Q}$ is the center of $C \in \mathcal{B}$ and $r \in \mathbb{Q}^+$ is the radius of $C \in \mathcal{B}$. Note that for every $(x, y, r) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^+$, there is a unique $C \in \mathcal{C}$ such that the center of C is (x, y) and the radius of C is r, which implies that f is a bijection. So $\mathcal{B}$ and $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^+$ have the same cardinality. Now, $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^+$ is countable as it is a finite product of countable sets. Hence, so is $\mathcal{B}$.

10. Let $\mathcal{C} = \{C_i : i \in I\}$, for some index set I, be a set of pairwise disjoint discs (sets of the form $\{(x, y) : (x - a)^2 + (y - b)^2 < r^2\}$) in $\mathbb{R}^2$. Show that $\mathcal{C}$ is atmost countable.

Solution: Let $C_i \in \mathcal{C}$. Since C_i is a disc in $\mathbb{R}^2$. The set of points in $\mathbb{R}^2$ with rational coordinates, $\mathbb{Q}^2 = \{(p, q) : p, q \in \mathbb{Q}\}$, is dense in $\mathbb{R}^2$. Therefore, there exists at least one point $(p_i, q_i) \in \mathbb{Q}^2$ such that $(p_i, q_i) \in C_i$.

Since the discs C_i are pairwise disjoint, the rational point (p_i, q_i) must be unique to C_i. Suppose, for contradiction, that $(p_i, q_i) \in C_i \cap C_j$ for some $i \neq j$. This would imply that C_i and C_j overlap at (p_i, q_i), contradicting the assumption that the discs are pairwise disjoint. Thus, each C_i contains a unique rational point (p_i, q_i).

Define a mapping $f : \mathcal{C} \to \mathbb{Q}^2$ by $f(C_i) = (p_i, q_i)$. This mapping is injective because each disc C_i is associated with a unique rational point (p_i, q_i). Since $\mathbb{Q}^2$ is countable and any subset of a countable set is atmost countable, the image of f is atmost countable. Therefore, the injective mapping f implies that $\mathcal{C}$ is atmost countable.

11. A real number is called *transcendental* if it is not algebraic. For example, π, e are transcendental numbers. Prove that the set of transcendental numbers is uncountable.

Solution:

Recall(from class notes) that $\mathbb{R}$ is uncountable.

Let A, T denote the set of algebraic and transcendental numbers respectively. Thus, $\mathbb{R} = A \cup T$. If possible let T is atmost countable. Then that will imply that $\mathbb{R}$, being a union of a countable and an atmost countable set, is also countable, which is a contradiction. Therefore, T is uncountable.

12. *Show that the plane $\mathbb{R}^2$ is not a union of countable number of lines.

Solution: We will prove it by contradiction. Assume $\mathbb{R}^2 = \bigcup_1^\infty L_j$ where $L_j = \{a_j + tb_j \mid t \in \mathbb{R}\},\ a_j, b_j \in \mathbb{R}^2$. Now consider the family of vertical line segments $V_x = \{(x,t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$.

Now this set $\{V_x \mid x \in \mathbb{R}\}$ is uncountable as there exists a bijection with $\mathbb{R}$. So that there exists $x_0 \in \mathbb{R}$ such that $V_{x_0} \neq L_j$ for any $j \in \mathbb{N}$. We will now use the property that two different straight lines intersect in atmost one point. This allow us to define a map from $\mathbb{N}$ to $\mathbb{R}$ by,

$$\Phi(j) = \begin{cases} s & \text{if } (x_0, s) \in L_j \cap V_{x_0}. \\ 0 & \text{if } L_j \cap V_{x_0} = \phi. \end{cases}$$

This map is well-defined. Now if this map is onto then $\mathbb{R}$ is atmost countable [As we know A is atmost countable iff there exists an onto map $f : \mathbb{N} \to A$] . But $\mathbb{R}$ is uncountable which contradicts Φ is onto. So there exists a $s_0 \in \mathbb{R}$, which has no preimage in $\mathbb{N}$. So $(x_0, s_0) \notin L_j$ for any j. This gives $(x_0, s_0) \notin \bigcup_1^\infty L_j$ which is a contradiction to $\mathbb{R}^2 = \bigcup_1^\infty L_j$.

13. *Show that no power set can be countable, that is, a power set is either finite or uncountable.

Solution: Let B be a power set of some set A i.e $B = P(A)$. Now if A is finite with cardinality n, then $\mid P(A) \mid = 2^n$.

If A is infinite, then there exists a $1-1$ map f from $\mathbb{N}$ to A. Now $f(\mathbb{N})$ is countable.

We first show that power set of countable set is uncountable.Assume X is a countable set. So that there exists a bijection $g : \mathbb{N} \to X$ which induces a map $\tilde{g} : P(\mathbb{N}) \to P(X)$ given by $\tilde{g}(K) = g(K)$ where $K \in P(\mathbb{N})$. This is well defined. Now for $\tilde{g}(K_1) = \tilde{g}(K_2)$ gives $g(K_1) = g(K_2)$. As g is a bijection $g^-1(g(K)) = K$ for any $K \in P(\mathbb{N})$ which gives $g^-1(g(K_1)) = g^-1(g(K_2)) \Rightarrow K_1 = K_2$.So $\tilde{g}$ is a bijection. As we know that $P(\mathbb{N})$ is uncountable which gives $P(X)$ uncountable. So power set of any countable set is uncountable.

So that $P(f(\mathbb{N}))$ is uncountable.
Again we have $P(f(\mathbb{N})) \subseteq P(A)$. Since $P(f(\mathbb{N}))$ uncountable gives $P(A)$ uncountable.