Lecture: 27.03.2025

$T : V \longrightarrow W$ linear map.

We have seen that as we change basis the representative matrix changes.

So we ask that whether there any relation betn. the two representative matrix?

So suppose $\mathcal{B}_1, \mathcal{B}_2$ two bases of V

$\mathcal{B}_1', \mathcal{B}_2'$ two bases of W.

Qn: How $[T]_{\mathcal{B}_1'}^{\mathcal{B}_1}$ is related to the matrix $[T]_{\mathcal{B}_2'}^{\mathcal{B}_2}$?

Before answering this question, looking at the formula

$$[Tv]_{\mathcal{B}'} = [T]_{\mathcal{B}'}^{\mathcal{B}} [v]_{\mathcal{B}}$$

it is eminent that we need to figure out how the co-ordinates of vector changes w.r. to different basis of the vector space, that is, the question is how

$[v]_{\mathcal{B}_1}$ and $[v]_{\mathcal{B}_2}$ are related?

For example: Consider $\mathbb{R}^2$, $\mathcal{E} = \{e_1, e_2\}$ std. basis.

$(2, 3) \in \mathbb{R}^2$, $\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{E}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

take $\mathcal{E}' = \{e_2, e_1\}$ – another ordered basis

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 3e_2 + 2e_1$$

Thus $\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{E}'} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Also if $\mathcal{B} = \{(1, 0), (1, 1)\}$ be another ordered basis of $\mathbb{R}^2$

then $\begin{pmatrix} 2 \\ 3 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

So we ask how $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ are related?

In order to answer the question above we discuss when two vector spaces are same!

Defⁿ: Two vector spaces V and W over $\mathbb{F}$ ($\mathbb{R}$) are "same" — ISOMORPHIC if $\exists$ a map (function)

$T : V \longrightarrow W$ such that

i) T is linear

ii) T is bijection

Such T is called an isomorphism or an invertible map.

↳ you may ask why invertible?

For set bijection has a inverse which is a bijection, that is,

for $f : A \longrightarrow B$ — bijection, then $\exists$ a bijection

$g : B \longrightarrow A$ such that

$$g \circ f = Id_A \quad \& \quad f \circ g = I_B.$$

So here for T bijection we will have a function $S : W \longrightarrow V$ such that $S \circ T = Id_V$ and $T \circ S = Id_W$.

Claim: linearity of T implies S is linear.

(obviously uses T – a bijection).

RTP: S linear, i.e,

$$S(\alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 S w_1 + \alpha_2 S w_2.$$

for $w_1, w_2 \in W$ and

$\alpha_1, \alpha_2 \in \mathbb{F}$ $(\mathbb{R})$.

Note $T(S(\alpha_1 w_1 + \alpha_2 w_2))$

$$= T \circ S(\alpha_1 w_1 + \alpha_2 w_2)$$

$$= Id_W(\alpha_1 w_1 + \alpha_2 w_2)$$

$$= \alpha_1 w_1 + \alpha_2 w_2 \quad \text{——} \quad (1)$$

$T(\alpha_1 S w_1 + \alpha_2 S w_2)$

$$= \alpha_1 T S w_1 + \alpha_2 T S w_2$$

$$= \alpha_1 Id_W(w_1) + \alpha_2 Id_W(w_2)$$

$$= \alpha_1 w_1 + \alpha_2 w_2 \quad \text{——} \quad (2).$$

Comparing (1) & (2) we have

$$T(S(\alpha_1 w_1 + \alpha_2 w_2)) = T(\alpha_1 S w_1 + \alpha_2 S w_2)$$

Since T is 1-1, we have

$$S(\alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 S w_1 + \alpha_2 S w_2.$$

The defn above helps us to say that

$$\mathbb{R}^4 \cong M_{2\times 2}(\mathbb{R}) \cong \mathcal{P}_3(\mathbb{R}).$$

$\cong$: isomorphic notn.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a + bt + ct^2 + dt^3$$

check that it is an isomorphism.

$$T : \mathbb{R}^4 \longrightarrow M_{2\times 2}(\mathbb{R}).$$

$$T : \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \longmapsto \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

an isomorphism (check).

In general we have the following theorem:

Thm: If V is an n-dimn vector space over $\mathbb{F}$ $(\mathbb{R})$, then V is isomorphic to $\mathbb{F}^n$ $(\mathbb{R}^n)$.

Pf: Let us restrict us to $n=2$ & $\mathbb{F} = \mathbb{R}$

the proof of the general case is similar.

V — 2-dimn vector space over $\mathbb{R}$.

We shall show V & $\mathbb{R}^2$ are isomorphic.

Let $\mathcal{B} = \{v_1, v_2\}$ be an $\underset{\text{ordered}}{\vee}$ basis of $\mathbb{R}^2$.

Define $T : V \longrightarrow \mathbb{R}^2$

$$T(v) = [v]_{\mathcal{B}} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

where $v = \alpha_1 v_1 + \alpha_2 v_2$.

i) **T is linear**

Take $w \in V$, $\exists\ \beta_1, \beta_2 \in \mathbb{R}$

s.t $w = \beta_1 v_1 + \beta_2 v_2$.

$$\therefore\ [w]_{\mathcal{B}} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$T(v) + T(w) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \end{pmatrix}$$

$$v + w = (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2$$

$$\therefore\ [v + w]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \end{pmatrix}.$$

$$\Rightarrow\ T(v + w) = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \end{pmatrix}$$

So $T(v + w) = Tv + Tw$.

Also $cv = c\alpha_1 v_1 + c\alpha_2 v_2$.

$$\Rightarrow\ [cv]_{\mathcal{B}} = \begin{pmatrix} c\alpha_1 \\ c\alpha_2 \end{pmatrix} = c\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = c[v]_{\mathcal{B}}$$

$$\Rightarrow\ T(cv) = cTv.$$

2. a) **T is 1 - 1**

$$Tv = Tw \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\Rightarrow\ \alpha_1 = \beta_1\ \&\ \alpha_2 = \beta_2$$

So $v = \alpha_1 v_1 + \alpha_2 v_2 = \beta_1 v_1 + \beta_2 v_2 = w$.

Thus $Tv = Tw \Rightarrow v = w$

Hence, T is 1-1.

2. b) **T is onto**.

Take $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2$. define

$$x := \alpha v_1 + \beta v_2 .$$

Then, $Tx = [x]_\beta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

Thus, every vector in $\mathbb{R}^2$ has a pre-image, hence T is onto.

Exc: complete the proof for n-dim^n case.