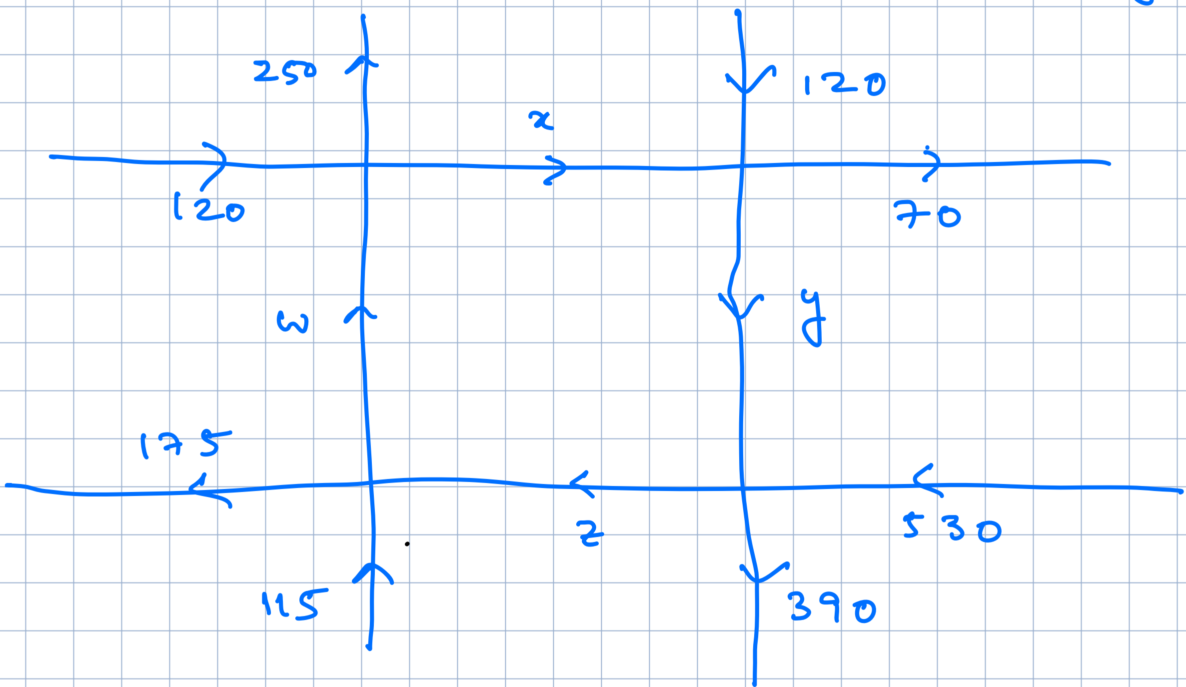
Lecture 01 :

System of linear equations :

It is a natural question to solve a linear equation with certain number of unknowns.

Examples:

1. Traffic flow – all roads are one way.



$$w + 120 = x + 250$$

$$x + 120 = y + 70$$

$$y + 530 = z + 390$$

$$z + 175 = w + 115$$

⟶ Solving this linear equation we get to know traffic flow.

- <u>Chemical equations</u>:

$$x\, C_2H_6 + y\, O_2 \longrightarrow z\, CO_2 + w\, H_2O$$

no. of carbon, hydrogen, oxygen atoms in the LHS should be same as RHS.

So

$$2x = z$$
$$6x = 2w$$
$$2y = 2z + w$$

The most common situation which we deal first is that of a system of n-linear equations with n-unknowns

For example with $n=2$,

$$x + y = 2 \quad \text{------} \quad (i)$$
$$3x + 2y = 5 \quad \text{------} \quad (ii)$$

An usual trick is to eliminate 'x' from eq^n (ii) using eq^n (i) ...

So we do $(ii) - 3(i)$ to obtain.

$$3x + 2y - 3(x+y) = 5 - 3.2$$
$$\Rightarrow 3x + 2y - 3x - 3y = 5 - 6$$
$$\Rightarrow -y = -1$$
$$\Rightarrow y = 1$$

and then substituting $y=1$ back into eq^n (i) to get $\underline{x = 1}$.

This is the most common method to solve solve such a system - known as

GAUSSIAN ELIMINATION

- Goal is to eliminate 1 variable at a time from each of the equation one after other to reach the step of back substitution.

Let us consider another system of eqn:

$n = 3$ — for this example

$$u + v - w = 5 \quad \text{—— (i)}$$

$$-u + v - w = 7 \quad \text{—— (ii)}$$

$$3u + 2v + 2w = 21 \quad \text{—— (iii)}$$

We first fix u, v, w - as our first, 2nd, 3rd variable/unknown here.

By interchanging equations (if necessary) keep the first variable ('u' in this case) in equation i).

Replace (ii) by a linear combination of (i) & (ii) in such a way that the term having "u" is cancelled.

Continue this process downwards.

STEP I

(i) $\longrightarrow$ $u + v - w = 5$ .... (a) remains as it is

(ii) + (i) $\xrightarrow[\text{eqn get replaced}]{\text{2nd}}$ $2v - 2w = 12$ .... (b)

(iii) $\longrightarrow$ $3u + 2v + 2w = 21$ .... (c) - remains as it is

STEP - II

(a) $\longrightarrow$ $u + v - w = 5$ — remains as it is .... (I)

(b) $\longrightarrow$ $2v - 2w = 12$ — remains as it is in the 2nd step. —(II)

(c) $\xrightarrow[\text{changed by}]{\text{gets}}$ $-v + 5w = 6$ —(III)

(c) - 3 × (a)

STEP - III

(I) — $1.u + v - w = 5$

(II) — $2v - 2w = 12$

2(III) + (II) — $8w = 24$

Clearly $w = 3$, substituting back in the 2nd eqn, we get $v = 9$.

Then substituting back w, v in the first eqn we get $u = -1$.

Leading coefficients (NON-ZERO) of u, v, w (1, 2, 8 in previous example) are called $\boxed{\text{PIVOTS}}$

Remark:

- multiplication by scalar doesn't change the solution:

Let (u, v, w) satisfies the equation

$$u + v - w = 5 \text{ if and only if}$$

it satisfies the equation

$$\alpha u + \alpha v - \alpha w = 5\alpha$$

for any nonzero $\alpha \in \mathbb{R}$.

- summing up two equ. preserve the solution as well:

If (u, v, w) satisfies the equations

$$u + v - w = 5 \quad \& \quad -u + v - w = 7$$

then it satisfies the equation

$$(1-1)u + (1+1)v + (-1-1)w = 5+7$$

that is, $2v - 2w = 12$.

Thus, the Gaussian elimination method transforms a given system of equations into a new system of equation, but both of these systems has exactly same solutions.

The breakdown of elimination :

Example (1)

$$\begin{aligned} u + v + w &= 1 \\ 2u + 2v + 5w &= 2 \\ 4u + 6v + 8w &= 3 \end{aligned} \implies \begin{aligned} u + v + w &= 1 \\ 3w &= 0 \\ 2v + 4w &= -1 \end{aligned}$$

no non-zero coeff of v in 2nd eqn.

BUT — we can interchange 2nd & 3rd eqn. to tackle the problem

$$\implies \begin{aligned} u + v + w &= 1 \\ 2v + 4w &= -1 \\ 3w &= 0 \end{aligned}$$

— CURABLE. (unique soln. case)

Example (2):

$$\begin{aligned} u + v + w &= 1 \\ 2u + 2v + 5w &= 2 \\ 4u + 4v + 8w &= 3 \end{aligned} \implies \begin{aligned} u + v + w &= 1 \\ 3w &= 0 \\ 4w &= -1 \end{aligned}$$

— no solution can exist in this case.

On the other hand, if $4u + 4v + 8w = 4$, we would have $w = 0$ & $u + v = 1$, showing the system of equations have plenty of solutions.

— <u>SINGULAR</u> (INCURABLE)

— either inconsistent, that is, no solutions or plenty of solutions.

The process of elimination becomes more & more difficult when one deals with more number of equations.

⟶ This brings the concept of matrices.

Look at the system

$$u + v - w = 5$$
$$-u + v - w = 7$$
$$3u + 2v + 2w = 21$$

and represent it as

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 21 \end{pmatrix}$$

that is, in $\boxed{Ax = b}$ form

where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 3 & 2 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \;\&\; b = \begin{pmatrix} 5 \\ 7 \\ 21 \end{pmatrix}$$

and ⊛ $A.x$ is defined as

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u + v - w \\ -u + v - w \\ 3u + 2v + 2w \end{pmatrix}$$

(**)

(*) $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Longleftrightarrow a_1 = b_1, \; a_2 = b_2 \;\&\; a_3 = b_3$

In general, we write an $m \times n$ matrix A as

$$A = \begin{array}{r} \\ \\ \\ i\text{th row} \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

(i,j)th entry. or simply ijth entry.

jth column

with each entry $a_{ij} \in \mathbb{R}$, $1 \le i \le m$, $1 \le j \le n$.

In short, we write $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$.

Sometimes $[a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

& sometimes $((a_{ij}))$.

(∗∗) suggest to define the action of A on $n \times 1$ matrix $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ by .

$$A \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} u_1 + a_{12} u_2 + \cdots + a_{1n} u_n \\ a_{21} u_1 + a_{22} u_2 + \cdots + a_{2n} u_n \\ \vdots \\ a_{m1} u_1 + a_{m2} u_2 + \cdots + a_{mn} u_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j} u_j \\ \sum_{j=1}^{n} a_{2j} u_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} u_j \end{pmatrix}$$

Clearly, this is a $m \times 1$ matrix

So $(m \times n)$ matrix $\times$ $(n \times 1)$ matrix gives $(m \times 1)$ matrix.

The advantage of doing this can be in seen in the next result.

Proposition: Every system of linear equations of the form $Ax = b$ has either, no solution, one solution or infinitely many solution.

Proof. Suppose there exists two solutions x & y that is, $Ax = b$ & $Ay = b$, & $x \neq y$.

$$\Rightarrow \quad A(x-y) = 0$$

$$\Rightarrow \quad A\{\alpha(x-y)\} = 0 \quad \forall \alpha \in \mathbb{R}$$

$$\Rightarrow \quad x + \alpha(x-y) \text{ is a solution for } \forall \alpha \in \mathbb{R}$$

Corollary: For, $b = 0$,

that is, consider $Ax = 0$.

— known as homogeneous system of equation.

If a homogeneous system of equations has a non-zero solution, then it has infinitely many solutions.

From these results, it is clear that we have two types of system of equations.

- either of the type of example 1 CURABLE
- or of the type of example 2 SINGULAR (INCURABLE)