## Problem Set - 07

MA 1201 Spring Sem, 2025

Below, "*special solutions*" refer to a set of basis vectors for the null space of the matrix $A$ associated with the system $Ax = b$. Since the number of free variables determines the dimension of the null space, one way to construct these special solutions is by setting one free variable to 1 at a time while assigning 0 to all other free variables.

1. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions.

2. Reduce $A$ and $B$ to echelon form. Which variables are free?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

3. Find the echelon form $\mathcal{E}$, the free variables, and the special solutions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

   Find condition on $b$ so that $Ax = b$ has a solution. Find the complete solution for one such $b$.

4. Carry out the same steps as in the previous problem to find the complete solution of $Mx = b$ :

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

5. Write the complete solutions to these systems:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

6. Describe the set of attainable right-hand sides $b$ (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   by finding the constraints on $b$ that turn the third equation into $0 = 0$ (after elimination). What is a particular solution?

7. Find the value of $c$ that makes it possible to solve $Ax = b$, and solve it:

$$\begin{aligned} u + v + 2w &= 2 \\ 2u + 3v - w &= 5 \\ 3u + 4v + w &= c \end{aligned}$$

8. Under what conditions on $b_1$ and $b_2$ (if any) does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Find two vectors in the nullspace of $A$, and the complete solution to $Ax = b$.

9. (a) Find the special solutions to $Ux = 0$. Reduce $U$ to $R$ and repeat:

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) If the right-hand side is changed from $(0, 0, 0)$ to $(a, b, 0)$, what are all solutions?

10. Find a 2 by 3 system $Ax = b$ whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Find a 3 by 3 system with these solutions exactly when $b_1 + b_2 = b_3$.

11. Write a 2 by 2 system $Ax = b$ with many solutions $Ax = 0$ but no particular solution. (Therefore the system has no solution.) Which $b$ 's allow a particular solution ?

12. Find the row-reduced echelon forms $R$:

(a) The 3 by 4 matrix of all 1 s .

(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.

(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

13. Find $R$ for each of these (block) matrices, and the special solutions:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} A & A \end{bmatrix} \quad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$