Lecture 04: February 04, 25.

Recall the example

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 5 & 8 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} U \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

This means inter change of row 2 and row 3

This lead us to another kind of elementary matrix known as 'PERMUTATION MATRIX'

Permutation matrix :

An $n \times n$ matrix is said to be a permutation matrix if it has exactly one 1 and all other entries 0 in every row and every column.

It is easy to note that if $P = (a_{ij})_{1 \le i, j \le n}$

and if $a_{pq} = 1$ for some p, q such that $1 \le p, q \le n$

then

$$\begin{array}{cc} & \\ p\text{th row} & \begin{pmatrix} & & | & & \\ & & | & & \\ - & - & a_{pq} & - & - \\ & & | & & \\ & & | & & \end{pmatrix} \\ & q\text{th column} \end{array} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

implies $x_q = y_p$.

$\{I, P_{12}, P_{23}, P_{31}, P_{123}, P_{132}\}$

$\leftarrow$ 3x3 permutation matrix

Example:

$$\underset{R_1 \leftrightarrow R_2}{P_{12}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\rightarrow$ basically interchanging row 1 (R_1) and row 2 (R_2) of identity matrix.

All 3x3 permutations.

$$\underset{R_2 \leftrightarrow R_3}{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \underset{R_3 \leftrightarrow R_1}{P_{31}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\underset{R_1 \to R_2 \to R_3 \to R_1}{P_{123}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \underset{R_1 \to R_3 \to R_2 \to R_1}{P_{132}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Basically, a permutation matrix is nothing but reordering of rows of the identity matrix.

Coming back to the example

$$A \xrightarrow[\text{pre-multiplying by}]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 5 & 8 \end{pmatrix} \xrightarrow[\text{pre-multiplying by}]{R_3 \to R_3 - 2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow[\substack{\text{pre-multiplying} \\ \text{permutation} \\ \text{matrix obtained} \\ \text{by interchange} \\ \text{of row 2 \& row 3}}]{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix} = U.$$

So if $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\underline{PFEA = U.}$$

So when we perform "Gauss elimination" together with "permutation matrix" - we get an upper triangular matrix U.

- that is, basically pre multiplying by elementary matrices - of two types:

- Elementary matrix $E_{ij}(r)$
  - obtained by replacing i^{th} row by i^{th} row $+ \; r \times j^{th}$ row.

- Permutation matrix P_{ij}
  - obtained by interchanging row i and row j.

So we have

$$\cdots P_3 P_2 E_3 P_1 E_2 E_1 A = U$$

However, we can do the permutation, that is, interchanging the rows altogether at first, and then do the Gaussian Elimination.

— why and how? - This requires a proof.
- we will see this later if time permits.

So we obtain an LU-decomposition of the permuted matrix of A, that is

$$\boxed{PA = LU}$$

product of permutation matrices

known as permuted LU-decomposition.

We have mentioned that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ doesn't have LU-decomposition

but $P_{12} A = I$ — $I . I$ - LU-decomposition

But note A - may not have LU-decomposition but to solve equation $Ax = b$, it is enough to have permuted LU - decomposition.

Remark: Is permuted LU-decomposition is always possible? Ans: NO (In general, $PAQ = LU$).
permutation matrix

YES, when you have A is invertible (we discuss later)

Altogether, only one of the two possibilities are possible:

(i) <u>Non-singular case</u>:

There is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. In this case $PA = LU$ is possible and $Ax = b$ has a unique solution.

(ii) <u>Singular case</u>:

No such P produces a full set of pivots

$\Rightarrow$ elimination fails.

$$\left.\begin{aligned} u + v + w &= 1 \\ 2u + 2v + 2w &= 2 \\ 3u + 3v + 4w &= 4 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} u + v + w &= 1 \\ 0 &= 0 \\ w &= 1 \end{aligned}\right\}$$

without permutation

OR (<u>with permutation</u>)

$$\left.\begin{aligned} u + v + w &= 1 \\ 3u + 3v + 4w &= 4 \\ 2u + 2v + 2w &= 2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} u + v + w &= 1 \\ w &= 1 \\ 0 &= 0 \end{aligned}\right\}$$

— But it does not produce full set of PIVOTS.

Now we formally introduce A^{-1} (After addition & multiplication — natural next operation)

- definition
- how to calculate — Gauss-Jordan method

motivated by the Gauss elimination method.

- Given an $n\times n$ matrix, if there exists an $n\times n$ matrix B such that $AB = BA = I$, then we say that A is invertible.

Qn: Is such a B unique?

Claim: If $\exists$ B and C such that $AB = BA = I$ and $AC = CA = I$, then $B = C$.

Proof: $B = B.I = B(AC) = (BA)C = I \cdot C = C$.

We define such a B as "the inverse" of A and is denoted by A^{-1}.

Exc. verify G^{-1}, F^{-1}, E^{-1} are actually inverses of G, F, E in this sense

Remark: Note in the proof, we have showed that if $BA = I = AC$, then $B = C$.

In other words, one can't have two candidates either side of A producing I.

Question: Can we have two candidates on same side of A producing I?

(In other words, can we have B, C – $n \times n$ matrices with $B \neq C$, such that $AB = I$ and $AC = I$?)

Ans: NO – One can check with determinants or rank-nullity theorem both of which we learn as we progress.