Cardinality

MA 1201

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Countable sets

For checking countability

Definition.

- A set A is said to be *countably infinite* or just *countable* if it has same cardinality with $\mathbb{N}$, that is, $\exists$ a bijection between $\mathbb{N}$ and A.
- A set is called *atmost countable* if it is finite or countable.
- An infinite set is called *uncountable* if it is not countable.

Exc.

- If $A \subseteq B$ and B is countable, then A is atmost countable.
- If $f : A \to B$ is $1-1$ and B is countable, then $f(A)$ and hence A is atmost countable.
- If $f : A \to B$ is onto and A is countable, then B is atmost countable.

Countable union

Theorem

Countable union of countable sets is countable.

Proof. WLOG, let $A_1, A_2, \ldots, A_n, \ldots$ be a countable family of countable sets. Also let WLOG

$$\begin{array}{lcl} A_1 & = & \{a_{11}, a_{12}, \ldots, a_{1n,\ldots}\} \\ A_2 & = & \{a_{21}, a_{12}, \ldots, a_{2n,\ldots}\} \\ \vdots & & \quad\vdots \qquad \ldots \quad \vdots \;\; \ldots \\ A_n & = & \{a_{n1}, a_{n2}, \ldots, a_{nn,\ldots}\} \\ \vdots & & \quad\vdots \qquad \ldots \quad \vdots \;\; \ldots \end{array}$$

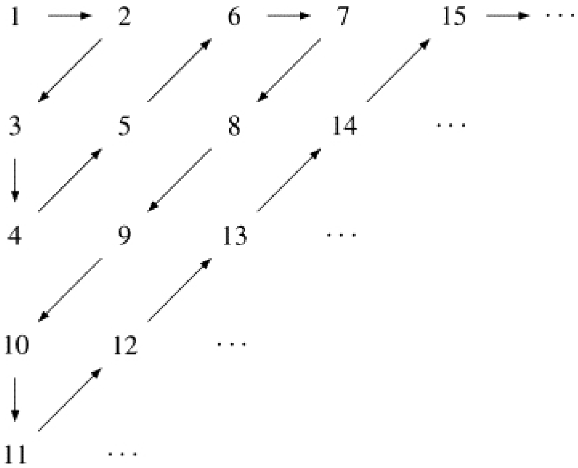
Let $A = \bigcup_{i=1}^{\infty} A_i$.

Case I. $A_i \cap A_j = \varnothing$ for all $i \neq j$.

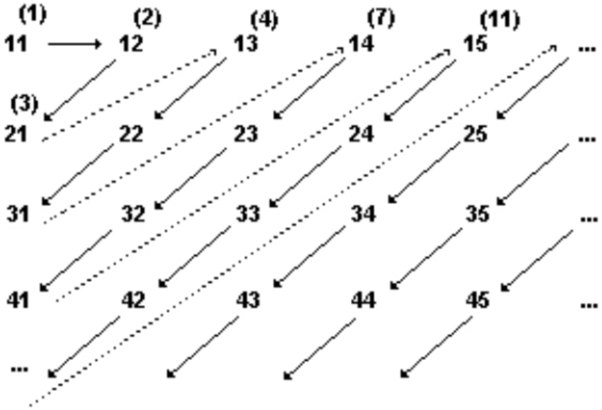
Proof idea

$$
\begin{array}{ccccccccc}
a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
a_{21} & & a_{22} & & a_{23} & & a_{24} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
a_{31} & & a_{32} & & a_{33} & & a_{34} & & \cdots \\
 & \swarrow & & \nearrow & & & & & \\
a_{41} & & a_{42} & & a_{43} & & a_{44} & & \cdots \\
\downarrow & \nearrow & & & & & & & \\
a_{51} & & a_{52} & & a_{53} & & a_{54} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Idea of bijection



Tweak a little for explicit bijection



The bijection

Note that

$$\mathbb{N} = \bigcup_{m\in\mathbb{N}} (\frac{m(m-1)}{2}, \frac{m(m+1)}{2}]$$

and that this union is disjoint.

Also note that the arrows are like slanting lines joining $(1, m)$ to $(m, 1)$ with any point (r, s) of the m points lying on it satisfies $r + s = m + 1$.

So given $n \in \mathbb{N}$, $\exists\, m$ such that

$$\frac{m(m-1)}{2} < n \leq \frac{m(m+1)}{2}$$

Define $f : \mathbb{N} \to A$ by $f(n) = a_{rs}$ with $r + s = m + 1$ and $r = n - \frac{m(m-1)}{2},\ r \geq 1$.

The other way

Now, given rs, we want to associate an integer to it. We first look at which slating line it is lying on that is, what is $r+s$?

Say $r+s=k+1$ for some k. So it is lying on the kth slanting line that has k many elements.

To count upto rs, then we first cross $1+2+\ldots+(k-1)$ many elements, that is, $\frac{k(k-1)}{2}$ elements and then r elements.

So we map

$$rs \mapsto r+\frac{k(k-1)}{2}=r+\frac{(r+s-1)(r+s-2)}{2}.$$

Define $g: A \rightarrow \mathbb{N}$ by $g(a_{rs})=r+\frac{(r+s-1)(r+s-2)}{2}$.

To show bijection between A and $\mathbb{N}$, we shall show this g is a bijection instead of showing f is so.

g is $1-1$

Suppose $g(a_{rs}) = g(a_{ij})$.

$$r + \frac{(r+s-1)(r+s-2)}{2} = i + \frac{(i+j-1)(i+j-2)}{2}.$$

We claim $r+s = i+j$.If not, say $r+s < i+j$, then

$$\begin{aligned} g(a_{ij}) - g(a_{rs}) &= i - r + \sum_{k \geq r+s-1}^{i+j-2} k \\ &\geq i + s - 1 \\ &> 0, \text{ as } i, s \geq 1. \end{aligned}$$

Similarly for $r+s > i+j$.

So $g(a_{rs}) = g(a_{ij})$ implies $r+s = i+j$ and hence from the first equation $r = i$.

Subsequently, $s = j$. Therefore $a_{rs} = a_{ij}$ and g is $1-1$.

g is onto

We look back to f where given n, we associate a_{rs}.

Clearly $g(a_{rs}) = r + \frac{m(m-1)}{2}$ where $r + s = m + 1$.

Now we know $r = n - \frac{m(m-1)}{2}$, and thus $g(a_{rs}) = n$.

So, the proof of **Case I** is complete.

This association, that is, the map g of uniquely encoding two natural numbers into a single natural number is known as Cantor's pairing function (a very slight modification)

Case II

Here A_i need not be pairwise disjoint.

Let us define the sets $B_i,\ i \in \mathbb{N}$ by

$$B_1 = A_1,\, B_2 = A_2 \setminus A_1,\, B_3 = A_3 \setminus (A_1\ \cup A_2), \ldots$$

$$B_k = A_k \setminus \left(A_1 \cup A_2 \ldots \cup A_{k-1}\right)$$

Then $B_i \subseteq A_i$ and hence atmost countable for all i, with B_1 countable, and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i \ (?)$$

and $B_i \cap B_j = \varnothing$ for all $i \neq j$.

This completes the proof. (?)

Confession

I made a mistake - **Snehamayee** corrected me.

Question. Given an $n \in \mathbb{N}$, how to find m such that

$$\frac{m(m-1)}{2} < n \leq \frac{m(m+1)}{2} \ ?$$

Note that $n = r + \frac{m(m-1)}{2}$ with $0 < r \leq m$.

If $\frac{m(m-1)}{2} = \ell$, then $m = \frac{1 \pm \sqrt{1+8\ell}}{2}$ and as m should be positive,

$$m = \frac{1+\sqrt{1+8\ell}}{2}.$$

Now $\ell = \frac{m(m-1)}{2} < n \leq \frac{m(m+1)}{2}$ gives

$$m = \frac{1+\sqrt{1+8\ell}}{2} < \frac{1+\sqrt{1+8n}}{2} \leq m+1.$$

In quest of m - Snehamayee's correction

So we have

$$m = \begin{cases} [\frac{1+\sqrt{1+8n}}{2}], & \text{if } \frac{1+\sqrt{1+8n}}{2} \notin \mathbb{N}, \\ \frac{1+\sqrt{1+8n}}{2} - 1, & \text{if } \frac{1+\sqrt{1+8n}}{2} \in \mathbb{N}. \end{cases}$$

Now I asserted in the class that I prove the function g is $1-1$ instead of proving f is $1-1$ in order to avoid this deduction for m.

However, Snehamayee demonstrated that it is not necessary to determine m to prove that f is $1-1$; the mere existence of such an m suffices. In fact, proving this is significantly easier than showing that g is $1-1$, which I had missed.

Proving f is a bijection

Let $f(n_1) = a_{rs}$, $f(n_2) = a_{ij}$ and $f(n_1) = f(n_2)$. Thus $r = i$ and $s = j$.

Existence of m_1 for n_1 and m_2 for n_2 and the construction of r, s would give $r + s = m_1 + 1$ and $i + j = m_2 + 1$. Therefore, $m_1 = m_2$. Then

$$n_1 = r + \frac{m_1(m_1 - 1)}{2} = i + \frac{m_2(m_2 - 1)}{2} = n_2.$$

On the other hand, f is onto as for any $a_{rs} \in A$, if we take $r + s = m + 1$ and $n = r + \frac{m(m-1)}{2}$,then n satisfies

$$\frac{m(m-1)}{2} < n \leq \frac{m(m+1)}{2},$$

and hence, $f(n) = a_{rs}$.

An alternate proof of Case I

Observe that by what ever we have learnt, it was enough to show g is $1-1$

since A is infinite, $g(A)$ is infinite subset of $\mathbb{N}$and hence countable

and so is A.

Define $g : A \to \mathbb{N}$ by $g(a_{rs}) = 2^r 3^s$.

Exc. Show that g is $1-1$.

We are then done by the observation.

Some more observations

Proposition

$\mathbb{N} \times \mathbb{N}$ *is countable.*

Let $\mathbb{Q}_+$ be the set of all positive rational numbers.

Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}_+$ by $f(p,q) = \frac{p}{q}$. Note f is onto.

Exc. $\mathbb{Q}_+$ is countable.

Proposition

$\mathbb{Q}$ *is countable.*

Proof. $\mathbb{Q} = -\mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_+$.

Cartesian product of countable sets

Exc. Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ be non-empty countable sets, then

$$A_\times \ldots \times A_n$$

is countable.

Question. What about countable product of countable sets? What is countable product?

The finite cartesian product $A_\times \ldots \times A_n$ can be thought of as

$$\{f | f : I_n \to \cup_{i=1}^n A_i \text{ such that } f(i) \in A_i \, \forall i \in I_n\}.$$

Realization of cartesian product of two sets as functions

For two sets A_1 and A_2,

$$A_1 \times A_2 = \{(a_1, a_2) : a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

Let $F := \{f | f : \{1, 2\} \to A_1 \cup A_2 \text{ such that } f(1) \in A_1 \text{ and } f(2) \in A_2\}.$

We claim that there is a natural bijection between $A_1 \times A_2$ and F.

Let $(a_1, a_2) \in A_1 \times A_2$, define a function $f_{a_1 a_2} : \{1, 2\} \to A_1 \cup A_2$ by

$$f_{a_1 a_2}(1) = a_1 \text{ and } f_{a_1 a_2}(2) = a_2.$$

Define $h : A_1 \times A_2 \to F$ by $h(a_1, a_2) = f_{a_1 a_2}$.

h is $1 - 1$ since if $h(a_1, a_2) = h(b_1, b_2)$ for $(a_1, a_2), (b_1, b_2) \in A_1 \times A_2$, we have $f_{a_1 a_2} = f_{b_1 b_2}$ and thus $a_1 = f_{a_1 a_2}(1) = f_{b_1 b_2}(1) = b_1$ and $a_2 = f_{a_1 a_2}(2) = f_{b_1 b_2}(2) = b_2$.

h is onto since if $f \in F$, then $(f(1), f(2)) \in A_1 \times A_2$ and $h(f(1), f(2)) = f_{f(1) f(2)} = f$ (?).

Cartesian product of countably many sets

Let $\{A_\alpha\}$ be a collection of sets indexed by $\alpha \in I$ for some index set I.

$$\prod_{\alpha \in I} A_\alpha := \{f | f : I \to \cup_{\alpha \in I} A_\alpha \text{ such that } f(\alpha) \in A_\alpha \, \forall \alpha \in I\}.$$

Axiom of choice. The cartesian product of non-empty family of sets is non-empty.

Let $\{A_n : n \in \mathbb{N}\}$ be a countable family of sets..

$$\prod_{n \in \mathbb{N}} A_n := \{f | f : \mathbb{N} \to \cup_{n \in \mathbb{N}} A_n \text{ such that } f(n) \in A_n \, \forall n \in \mathbb{N}\}.$$

Question. Suppose each A_i is countable, then is $\prod_{n \in \mathbb{N}} A_n$ countable?

A weaker question

Suppose we take all $A_n = A$, then $\prod A_n$ is denoted by $A^{\mathbb{N}}$. Note that

$$A^{\mathbb{N}} = \{f | f : \mathbb{N} \to A\}.$$

This notation is motivated from the notation $\mathbb{R}^n$ which, by reasons given previously, can be thought of as $\{f | f : \{1, 2, \ldots, n\} \to \mathbb{R}\}$.

Question. Suppose A is countable, then is $A^{\mathbb{N}}$ countable?

If A is a singleton, say $A = \{0\}$. Then $A^{\mathbb{N}} = \{f | f(n) = 0 \forall\, n \in \mathbb{N}\}$ is a singleton.

However, this result is FALSE if $|A| > 1$.

Uncountable sets

Example of uncountable sets

Theorem

$\{0, 1\}^{\mathbb{N}}$ *is uncountable.*

Proof. Note

$$\{0, 1\}^{\mathbb{N}} = \{f | f : \mathbb{N} \to \{0, 1\}\}.$$

The map $n \mapsto \delta_n$ from $\mathbb{N}$ to $\{0, 1\}^{\mathbb{N}}$ is $1 - 1$ (?), where

$$\delta_n(i) = \begin{cases} 0 & \text{if } i \neq n, \\ 1 & \text{if } i = n. \end{cases}$$

Hence, $\{0, 1\}^{\mathbb{N}}$ is infinite.

To prove $\{0, 1\}^{\mathbb{N}}$, we shall show that it is not countable. On the contrary , we assume it to be countable and arrive at a contradiction.

Let $\{0, 1\}^{\mathbb{N}} = \{f_1, f_2, \ldots, f_n \ldots\}$. We shall show that there exists a function $g : \mathbb{N} \to \{0, 1\}$ so that $g \neq f_n$ for any $n \in \mathbb{N}$ - a contradiction.

Idea and proof - Cantor's diagonal argument

Listing of f_n's as sequence when we evaluate them at $1, 2, 3, \dots$:

$$
\begin{array}{ccccccccc}
f_1 & \to & \boldsymbol{f_1(1)} & f_1(2) & f_1(3) & \cdots & & \\
f_2 & \to & f_2(1) & \boldsymbol{f_1(2)} & f_2(3) & \cdots & & \\
f_3 & \to & f_3(1) & f_3(2) & \boldsymbol{f_3(3)} & \cdots & & \\
\vdots & & \vdots & & \vdots & \dots & & \\
f_n & \to & f_n(1) & f_n(2) & f_n(3) & \cdots & \boldsymbol{f_n(n)} & \cdots \\
\vdots & & & \vdots & & \vdots & &
\end{array}
$$

For example, the zero function is given by the sequence

$$0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \; \dots$$

Consider $g : \mathbb{N} \to \{0, 1\}$ given by $g(n) \neq \boldsymbol{f_n(n)}$ (diagonal entry) for all $n \in \mathbb{N}$, that is, $g(n) = 1$ if $f_n(n) = 0$ and $g(n) = 0$ if $f_n(n) = 1$.

Hence $g \neq f_n$ for any $n \in \mathbb{N}$.

Power set of $\mathbb{N}$

Theorem

There exists a bijection between $\mathcal{P}(\mathbb{N})$ and $\{0,1\}^{\mathbb{N}}$.

Proof. Let $A \in \mathcal{P}(\mathbb{N})$, that is, $A \subseteq \mathbb{N}$. Define the characteristics function $\chi_A : \mathbb{N} \to \{0,1\}$ of A by

$$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases}$$

Observe that if $A = \varnothing$ iff χ_A is the constant function 0 and $A = \mathbb{N}$ iff χ_A is the constant function 1.

Define $h : \mathcal{P}(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ by $h(A) = \chi_A$. We claim that h is a bijection.

Let $A, B \subseteq \mathbb{N}$. Then $h(A) = h(B)$ implies $\chi_A = \chi_B$. Now

$$n \in A \iff \chi_A(n) = 1 \iff \chi_B(n) = 1 \iff n \in B$$

The first and last implication follows from definition whereas the middle one follows from the assumption. Thus $A = B$ and hence h is $1-1$.

Uncountability of $\mathcal{P}(\mathbb{N})$

Proof ctd. To prove h is onto, let $f : \mathbb{N} \to \{0, 1\}$. Define

$$A := \{n : f(n) = 1\}.$$

Note that $\chi_A(i) = 1$ if $i \in A$ and by definition, then $f(i) = 1$.

On the other hand, $\chi_A(i) = 0$ if $i \notin A$ by definition, then $f(i) = 0$.

So $\chi_A(i) = f(i)$ for all $i \in \mathbb{N}$. Thus $h(A) = \chi_A = f$ and hence, h is onto.

Corollary

$\mathcal{P}(\mathbb{N})$ *is uncountable.*

Cantor's Theorem

So we just have observed that $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ do not have same cardinality.

What happens in general?

Theorem

For any set A, there does not exists a bijection between A and $\mathcal{P}(A)$.

Proof. We assume on the contrary that there exists a bijection $f : A \to \mathcal{P}(A)$.

Proof of Cantor's Theorem - Russell's paradox

Consdier the set

$$B := \{x \in A : x \notin f(x)\}$$

Since f is onto, $\exists\, b \in A$ such that $f(b) = B$.

Now note either $b \in B$ or $b \notin B$.

If $b \in B$, then by definition, $b \notin f(b) = B$ - contradiction.

If $b \notin B$, then by definition, $b \in f(b) = B$ - contradiction. The proof is complete.

Remark

There does not exists an onto map from A to $\mathcal{P}(A)$.

Rethinking cardinality

Definition. Each set A is assigned with a symbol in such a way that two sets A and B are assigined with the same symbol if and only if there is a bijection between them. This symbol is called *cardinality* or *cardinal number* of A and is denoted by $|A|$.

0 is assigned to $\varnothing$ and n to I_n. Thus

$$|A| = n \text{ if and only if } \exists \text{ a bijection between } A \text{ and } I_n.$$

We just learnt that $\mathcal{P}(\mathbb{N})$ and $\{0, 1\}^{\mathbb{N}}$ has same cardinality, that is $|\mathcal{P}(\mathbb{N})| = |\{0, 1\}^{\mathbb{N}}|$ and hence $|\mathcal{P}(\mathbb{N})| \neq |\mathbb{N}|$.

Similarly Cantor's theorem says that $|A| \neq |\mathcal{P}(A)|$.

Ordering cardinality

Following definition extends the notion of ordering of cardinality of finite sets.

Definition. Let A and B are two sets. We say that cardinality of A is less than or equal to the cardinality of B, denoted by

$$|A| \leq |B|,$$

if A has the same cardinality as of a subset of B, that is, $\exists$ a $1-1$ map from A to B.

Furthermore,

$|A|$ is less than $|B|$, denoted by $|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$,

$|A|$ is greater than or euqal to $|B|$, denoted by $|A| \geq |B|$ if $|B| \leq |A|$,

$|A|$ is greater than $|B|$, denoted by $|A| > |B|$ if $|A| \geq |B|$ but $|A| \neq |B|$.

Rephrasing Cantor's Theorem

For any set, the function $g : A \to \mathcal{P}(A)$ given by $g(a) = \{a\}$ is $1 - 1$.

Thus $|A| \leq |\mathcal{P}(A)|$.

Cantor's Theorem

For any set A, we have $|A| < |\mathcal{P}(A)|$.

Question. If for any two sets A and B with $|A| \leq |B|$ and $|B| \leq |A|$, does it imply

$$|A| = |B|?$$

Answer is YES - this is known as Schroeder - Bernstein Theorem.

We will prove this next week.

A justification for notation

Notation. $\mathcal{P}(A)$ is also denoted by 2^A and $|\mathcal{P}(A)|$ by $2^{|A|}$.

For any finite set A with $|A| = n$, we have $|\mathcal{P}(A)| = 2^n$. A primary motivation for this notation.

Going by previous comments on realizing cartesian products $A^{\mathbb{N}}$ as functions $f : \mathbb{N} \to A$, for any two sets A and B,

let A^B denotes the set of all functions from B to A.

Exc. For any two finite sets A and B, the cardinality of A^B is $|A|^{|B|}$.

We have $|\mathcal{P}(A)| = |\{0, 1\}^A|$ (?). This also somewhat justifies the notation above.