Lecture 05 : Feb 05, 2025.

Computation of A^{-1} : Gauss-Jordan Method

Given a matrix A, by Gauss-Elimination we have elementary matrices $E_1, E_2, E_3 \ldots$ and permutation matrices $P_1, P_2, P_3 \ldots$ such that

$$\cdots P_3 P_2 E_3 P_1 E_2 E_1 A = U \quad - (*)$$

where U is upper triangular.

It is to note that A is invertible if and only if U is so.

This follows from the fact that

Lemma: If A & B are invertible, then so is AB (or BA).

Pf: $AB(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})AB$.

So from $(*)$ it follows that U is invertible and if we rewrite $(*)$ as

$$A = \underbrace{E_1^{-1} E_2^{-1} P_1^{-1} E_3^{-1} P_2^{-1} P_3^{-1}}_{\text{invertible}} \cdots U.$$

then it follows that A is invertible if U is so.

Qn: So we start asking when is an upper triangular matrix is invertible?

– what happens if U has zero(s) in pivot position(s)?

CLAIM: An upper triangular matrix is invertible if and only if all of its diagonal entries are non-zero.

Let a_{ii} – denotes the diagonals of U, $1 \le i \le n$.

Let j be the first index for which $a_{jj} = 0$, that is $a_{11}, a_{22} \cdots a_{j-1,j-1} \neq 0$ and $a_{jj} = 0$ for some j, $1 \le j \le n$.

Consider the equation

$$U = \left(\begin{array}{ccccc|ccc}
a_{11} & & * & & a_{1j} & & & \\
 & a_{22} & & & \vdots & & * & \\
 & & \ddots & & \vdots & & & \\
 & 0 & & a_{j-1,j-1} & a_{j-1,j} & & & \\
 & & & & 0 & & & \\
\hline
 & & & & & a_{j+1,j+1} & & \\
 & & 0 & & & & \ddots & \\
 & & & & & 0 & & a_{nn}
\end{array}\right)
\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{j-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (**)$$

This is same as solving the eq^n

upper triangular system of eq^n

$$\left\{\begin{array}{l}
a_{11}u_1 + a_{12}u_2 + \cdots + a_{1j-1}u_{j-1} = -a_{1j} \\
\qquad\quad a_{22}u_2 + \cdots + a_{2j-1}u_{j-1} = -a_{2j} \\
\qquad\qquad\qquad \vdots \\
\qquad\qquad\qquad a_{j-1,j-1}u_{j-1} = -a_{j-1,j}
\end{array}\right.$$

Since each $a_{11}, a_{22}, \dots a_{j-1,j-1}$ are not zero, by back substitution we find $u_1, u_2, \dots u_{j-1}$ satisfying the eq^n above.

Thus $\exists\ u_1, u_2 \dots u_{j-1}$ s.t

$$\begin{pmatrix} a_{11} & a_{12} & & a_{1,j-1} & a_{1j} \\ & a_{22} & & & \\ & & \ddots & & \\ & & & a_{j-1,j-1} & a_{j-1,j} \\ & & & & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{j-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

& hence $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{j-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ satisfies $(**)$.

Now if U is invertible, then $U^{-1}U = I$.

$\Rightarrow U^{-1}Ux = x \quad \forall x \in \mathbb{R}^n$.

A vector in $\mathbb{R}^n$ is treated as a column vector.

Thus if we take $x = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{j-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then

$$\Rightarrow \quad U^{-1}(0) = x$$

$$\Rightarrow \quad x = 0$$ which is clearly a contradiction.

This contradiction appears as we have assumed U to be invertible.

Hence if there are zeros on the pivot position U is not invertible.

Contrapositively, if U is invertible, then none of the entries in pivot position, that is on diagonal is zero.

Now we assume suppose each of the entries on the diagonal of U is non-zero. We shall show that U is invertible.

<u>Case I:</u>

Suppose U is just diagonal only, say

$$U = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

Then

since $\underline{\underline{a_{ii} \neq 0}}$

$$\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \begin{pmatrix} 1/a_{11} & & 0 \\ & \ddots & \\ 0 & & 1/a_{nn} \end{pmatrix} = I.$$

and diagonal matrix commute.

<u>Case II</u>: for general upper triangular.

$$\begin{pmatrix} 1/a_{11} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1/a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ & 1 & a_{23}/a_{22} \cdots & a_{2n}/a_{22} \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

now $F_k \cdots F_2 F_1 \begin{pmatrix} 1/a_{11} & & \\ & \ddots & \\ & & 1/a_{nn} \end{pmatrix} U = I$.

$$\Rightarrow \quad U = \underbrace{\begin{pmatrix} a_{11} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} F_1^{-1} \cdots F_k^{-1}}_{\text{product of invertible matrices}}.$$

$\Rightarrow$ U is invertible and the inverse is
$F_k F_{k-1} \cdots F_1 \begin{pmatrix} 1/a_{11} & \cdots & 1/a_{nn} \end{pmatrix}$.

Gauss - Jordan Method:

$$\underbrace{\cdots P_3 P_2 E_3 P_1 E_2 E_1}_{E} A = U \text{ — } \boxed{\text{invertible}}$$

$$\underbrace{F_k \cdots F_1 \begin{pmatrix} 1/a_{11} & & \\ & \ddots & \\ & & 1/a_{nn} \end{pmatrix} \cdots P_3 P_2 E_3 P_1 E_2 E_1 A} = I.$$

$$\left(A \mid I \right)$$

— apply first row op.

$$\left(E_1 A \mid E_1 \right)$$

— apply 2nd row op

$$\left(E_2 E_1 A \mid E_2 E_1 \right)$$

$\vdots$ — apply permutation

$$\left(P E_2 E_1 A \mid P_1 E_2 E_1 \right)$$

$\vdots$

$$\left(U \mid \underbrace{\cdots P_3 P_2 E_3 P_1 E_2 E_1}_{E} \right)$$

$\vdots$

$$\left(\underbrace{F_k \cdots F_1 \begin{pmatrix} 1/a_{11} & & \\ & \ddots & \\ & & 1/a_{nn} \end{pmatrix} U}_{I} \;\middle|\; \underbrace{(\quad)\cdots}_{A^{-1}} \right)$$

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix}$$

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 2 & 5 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_2 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 \\ 0 & 3 & 6 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 6 & -2 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 0 \end{array}\right)$$

$$(= U \mid E)$$

$$\xrightarrow[R_3 \to \frac{1}{2} R_3]{R_2 \to \frac{1}{3} R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right)$$

$$R_2 \longrightarrow R_2 - 2R_3 \quad \xrightarrow{\hspace{3em}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3}+1 & -1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right)$$

$$R_1 \longrightarrow R_1 \rightarrow R_3 \quad \xrightarrow{\hspace{3em}} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right)$$

$$R_1 \longrightarrow R_1 - R_2 \quad \xrightarrow{\hspace{3em}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2}-\frac{1}{3} & -\frac{1}{2}+1 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right)$$

$$= \underline{\underline{\left(I \mid A^{-1}\right)}}.$$

Thus $A^{-1} = \begin{pmatrix} \frac{7}{6} & \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

<u>Exc</u>: verify

$A \cdot A^{-1} = I = A^{-1} \cdot A$

Now we come back to uniqueness of LU decomposition

$$LU$$

$$= \underbrace{L \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}}_{L'} \underbrace{\begin{pmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{pmatrix} U}_{U'}.$$

$$= L'U'$$ – So LU decomp. is not unique.

However, a sufficient condn. if we take either of U or L, the diagonals are all 1, then it becomes unique.

Suppose in our case all the diagonal of the upper triangular are 1.

$$LU = L_1 U_1$$

$$\Rightarrow L_1^{-1} L = U_1 U^{-1}.$$

Note if U has all elts in the diagonal 1, then $F_k F_{k-1} \cdots F_1$ inverse of U, where each F_i is upper triangular with the diagonal entries 1.

$$\Rightarrow \begin{bmatrix} \diagdown & 0 \\ & \diagdown \end{bmatrix} = \begin{pmatrix} 1 & a_{11} & & a_{1n} \\ & 1 & & \\ & & \ddots & a_{n-1\,n} \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & b_{n-1\,n} \\ & & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

This has to be zero

$$= I.$$

Hence U^{-1} is upper triangular with diagonal entries 1.

$\Rightarrow$ $\underline{U = U_1}$

& hence $L = L_1$.

Regarding the proof that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ doesn't have LU decomposition, we assume the fact that if A is invertible and $A = LU$, then both L and U are invertible. (determinant /rk multiplis gives a proof).

Note $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \underset{R_1 \leftrightarrow R_2}{P_{12}} A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\text{upper triangular invertible}}{I}$

$\Rightarrow$ A is invertible.

$$A = LU = \begin{pmatrix} l_{11} & & 0 \\ & \ddots & \\ * & & l_{nn} \end{pmatrix} \begin{pmatrix} u_{11} & & * \\ & \ddots & \\ 0 & & u_{nn} \end{pmatrix}$$

Since U is invertible

$$\implies u_{ii} \neq 0 \quad \forall i, \ 1 \leq i \leq n.$$

On the other hand we define transpose of a matrix

$$A^T = (b_{ij}) \quad \text{where } b_{ij} = a_{ji}$$

Exc: 1. $(AB)^T = B^T A^T$

2. A is invertible $\iff$ A^T is invertible

and $(A^T)^{-1} = (A^{-1})^T$.

So L is invertible $\iff$ L^T is invertible

upper triangular

$$\iff l_{ii} \neq 0 \quad \forall i, \ 1 \leq i \leq n.$$

Thus comparing the first entry we have

$$0 = l_{11} u_{11} \neq 0 \quad \text{which is a contradiction}$$

Hence no such L, U exists.