Lecture : 01/04/2025

We would like to discuss the relation between the co-ordinates $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{B}'}$ w.r.to two different bases $\mathcal{B}$ and $\mathcal{B}'$ of the vector space V.

Consider the identity linear map:

$$Id : \underset{\mathcal{B}}{V} \longrightarrow \underset{\mathcal{B}_1}{V}.$$

$T : \underset{\mathcal{B}}{V} \xrightarrow{\text{lin}} \underset{\mathcal{B}'}{V}$

$[Tv]_{\mathcal{B}'} = [T]^{\mathcal{B}}_{\mathcal{B}'} [v]_{\mathcal{B}}$

We consider the matrix representation of Id w.r.to the bases $\mathcal{B}$ and $\mathcal{B}_1$ where $\mathcal{B}$ is considered to basis of the domain space V & $\mathcal{B}_1$ is the basis of codomain space V.

So $$[Id(v)]_{\mathcal{B}_1} = [Id]^{\mathcal{B}}_{\mathcal{B}_1} [v]_{\mathcal{B}}.$$

$$\Rightarrow \boxed{[v]_{\mathcal{B}_1} = [Id]^{\mathcal{B}}_{\mathcal{B}_1} [v]_{\mathcal{B}}.}$$

Note : how $[Id]^{\mathcal{B}}_{\mathcal{B}_1}$ look like ?

Let $V = \mathbb{R}^2$ $\mathcal{B} = \mathcal{E} = \mathcal{B}_1$ So $\mathcal{B} = \{e_1, e_2\}$

$\hookrightarrow$ std ORDERED basis $\mathcal{B}_1 = \{e_1, e_2\}$

$$Id(e_1) = e_1 = 1.e_1 + 0.e_2$$

$$Id(e_2) = e_2 = 0.e_1 + 1.e_2.$$

So $$[Id]^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

However, if $\mathcal{B} = \mathcal{E} = \{e_1, e_2\}$ — Std ORDERED basis

$\mathcal{B}_1 = \mathcal{E}' = \{e_2, e_1\}$. — ordered basis

Then $Id(e_1) = e_1 = 0 \cdot e_2 + 1 \cdot e_1$

$Id(e_2) = e_2 = 1 \cdot e_2 + 0 \cdot e_1$

So $[Id]_{\mathcal{E}'}^{\mathcal{E}} = \underline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ — not the identity matrix.

Exc: Check if V is a n-dim^n v. space & $\mathcal{B}$ be a basis of V, then

$$[Id]_{\mathcal{B}}^{\mathcal{B}} = I_n \quad \text{— } n \times n \text{ identity matrix.}$$

But as shown in the example above, if you change basis, ie consider $[Id]_{\mathcal{B}_1}^{\mathcal{B}}$, it need not be identity matrix.

One can see, continuing the last example

$$[v]_{\mathcal{E}'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} [v]_{\mathcal{E}}.$$

So if $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow [v]_{\mathcal{E}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$$\Rightarrow [v]_{\mathcal{E}'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

which we also know by computation w.r.t ordered basis as

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Consider another ordered basis $\mathcal{B}_2 = \left\{ \underset{w_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\}$.

$Id(e_1) = e_1 = w_1 = 1 \cdot w_1 + 0 \cdot w_2$

$Id(e_2) = e_2 = w_2 - w_1 = (-1) w_1 + 1 \cdot w_2$

So $[Id]^{\mathcal{E}}_{\mathcal{B}_2} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

$v = [v]_{\mathcal{E}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ (say).

then $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (\alpha_1 - \alpha_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= (\alpha_1 - \alpha_2) w_1 + \alpha_2 w_2$

$\therefore [v]_{\mathcal{B}_2} = \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 \end{pmatrix}$.

Note $[Id]^{\mathcal{E}}_{\mathcal{B}_2} [v]_{\mathcal{E}} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

$= \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 \end{pmatrix}$

So we verified that $[v]_{\mathcal{B}_2} = [Id]^{\mathcal{E}}_{\mathcal{B}_2} [v]_{\mathcal{E}}$.

Also this gives a way to compute co-ordinates w.r.t a basis in the co-domain space.

However, the question is that is there any thing common between $[Id]^{\mathcal{E}}_{\mathcal{E}}$, $[Id]^{\mathcal{E}}_{\mathcal{E}'}$, $[Id]^{\mathcal{E}}_{\mathcal{B}_2}$??

Note Identity map by def^n. is an isomorphism of a space with itself.

– So to know how the matrices behave, we need to know how the matrices of isomorphism behave!

— To know this we need to know the how matrices of composition of linear map behaves?

(as isomorphism satisfies

$$T \circ S = Id_W \quad \& \quad S \circ T = Id_V \text{)}.$$

So let us consider the following linear maps:

$$X \xrightarrow{T} Y \xrightarrow{S} Z \qquad \left(\text{So } S \circ T : X \longrightarrow Z \right).$$

where X – vector space with basis $\mathcal{B}$ – n-dim^n

Y – " " " " $\mathcal{B}'$ – m dim^n

Z – " " " " $\mathcal{B}''$. – l dim^n

We would like to relate

$$[S \circ T]_{\mathcal{B}''}^{\mathcal{B}} \quad \text{with} \quad [T]_{\mathcal{B}'}^{\mathcal{B}}, \ [S]_{\mathcal{B}''}^{\mathcal{B}'}$$

Let $\mathcal{B} = \{x_1, \ldots, x_n\}$

$\mathcal{B}' = \{y_1, \ldots, y_m\}$

$\mathcal{B}'' = \{z_1, \ldots, z_l\}$

$$[S \circ T]_{\mathcal{B}''}^{\mathcal{B}} \qquad [S]_{\mathcal{B}''}^{\mathcal{B}'} \qquad [T]_{\mathcal{B}'}^{\mathcal{B}}$$

$l \times n$ matrix $\qquad$ $l \times m$ matrix $\qquad$ $m \times n$ matrix

ANY GUESS ??

THM: $[S \circ T]_{\mathcal{B}''}^{\mathcal{B}} = [S]_{\mathcal{B}''}^{\mathcal{B}'} \cdot [T]_{\mathcal{B}'}^{\mathcal{B}}$

matrix multiplication

(So one can define matrix multiplication by this way:

Given A - $m \times n$ - matrix, B - $l \times m$ matrix,

let $X \xrightarrow{T} Y \xrightarrow{S} Z$, T, S - linear maps

n-dim, m-dim, l-dim

β, β', $\mathcal{B}''$ - bases.

s.t $[T]_{\mathcal{B}'}^{\mathcal{B}} = A$ & $[S]_{\mathcal{B}''}^{\mathcal{B}'} = B$.

* On is how we do it?

Given $\mathcal{B} = \{x_1, x_2 \dots x_n\}$

$\mathcal{B}' = \{y_1, y_2 \dots y_m\}$

Define T be the linear map given by

$$T : x_j \longmapsto a_{1j} y_1 + a_{2j} y_2 + \dots + a_{mj} y_m$$

where $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ — jth column of A.

Thus $[T]^{\mathcal{B}}_{\mathcal{B}'} = A$.

Illy given $\mathcal{B}'$, $\mathcal{B}''$, we can construct linear map $S : Y \longrightarrow Z$ s.t

$$[S]^{\mathcal{B}'}_{\mathcal{B}''} = B.$$

Define $B \cdot A := [S \circ T]^{\mathcal{B}}_{\mathcal{B}''}$.

$$\parallel$$

$$\left([S]^{\mathcal{B}'}_{\mathcal{B}''} [T]^{\mathcal{B}}_{\mathcal{B}'} \right)$$

Remark: In words we say that matrix of composition of two linear maps is multiplication of matrix of the individual linear maps.

Pf. of the THM:

in this case, general case will follow similarly.

$$\left\{ \begin{array}{ll} X - 2\dim^{n}. & \mathcal{B} = \{x_1, x_2\} \\ Y - 2\dim^{n} & \mathcal{B}' = \{y_1, y_2\} \\ Z - 3\dim^{n} & \mathcal{B}'' = \{z_1, z_2, z_3\} \end{array} \right\} \text{respective bases.}$$

$$[T]_{\mathcal{B}'}^{\mathcal{B}} \text{ — } 2\times 2 \text{ matrix} = A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$[S]_{\mathcal{B}''}^{\mathcal{B}'} \text{ — } 3\times 2 \text{ - matrix} = B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$[S\circ T]_{\mathcal{B}''}^{\mathcal{B}} \text{ — } 3\times 2 \text{ - matrix}.$$

$$= C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}.$$

$$\begin{aligned}
(S\circ T)(x_1) &= S(Tx_1) \\
&= S(a_{11} y_1 + a_{21} y_2) \\
&= a_{11} S y_1 + a_{21} S y_2 \\
&= a_{11}(b_{11} z_1 + b_{21} z_2 + b_{31} z_3) \\
&\quad + a_{21}(b_{12} z_1 + b_{22} z_2 + b_{32} z_3) \\
&= (a_{11} b_{11} + a_{21} b_{12}) z_1 \\
&\quad + (a_{11} b_{21} + a_{21} b_{22}) z_2 \\
&\quad + (a_{11} b_{31} + a_{21} b_{32}) z_3. \\
&= \underbrace{(b_{11} a_{11} + b_{12} a_{21})}_{c_{11}} z_1 \\
&\quad + \underbrace{(b_{21} a_{11} + b_{22} a_{21})}_{c_{21}} z_2 \\
&\quad + \underbrace{(b_{31} a_{11} + b_{32} a_{21})}_{c_{31}} z_3.
\end{aligned}$$

$$[S]^{\mathcal{B}'}_{\mathcal{B}''}\,[T]^{\mathcal{B}}_{\mathcal{B}'} = B \cdot A$$

$$= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3\times 2} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$$

$$= \begin{pmatrix} b_{11} a_{11} + b_{12} a_{2} & b_{11} a_{12} + b_{12} a_{22} \\ b_{21} a_{11} + b_{22} a_{21} & b_{21} a_{12} + b_{22} a_{22} \\ b_{31} a_{11} + b_{32} a_{21} & b_{31} a_{12} + b_{32} a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} = C = [S \circ T]^{\mathcal{B}}_{\mathcal{B}''}$$

as $(S \circ T)(x_2) = S(T x_2)$

$$= S(a_{12} y_1 + a_{22} y_2)$$

$$= a_{12} S y_1 + a_{22} S y_2$$

Compare the colour pallete.

$$= a_{12}(b_{11} z_1 + b_{21} z_2 + b_{31} z_3) + a_{22}(b_{12} z_1 + b_{22} z_2 + b_{32} z_3)$$

$$= \underbrace{(b_{11} a_{12} + b_{12} a_{22})}_{c_{12} =} z_1 + \underbrace{(b_{21} a_{12} + b_{22} a_{22})}_{c_{22} =} z_2 + \underbrace{(b_{31} a_{12} + b_{32} a_{22})}_{c_{32} =} z_3$$

General proof :

$$C = BA \to m \times n$$

C: $l \times n$. B: $l \times m$

$$c_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$$

Note we have seen this :

$c_{11} = b_{11} a_{11} + b_{12} a_{21}$ | $c_{12} = b_{11} a_{12} + b_{12} a_{22}$

$c_{21} = b_{21} a_{11} + b_{22} a_{21}$ | $c_{22} = b_{21} a_{12} + b_{22} a_{22}$

$c_{31} = b_{31} a_{11} + b_{32} a_{21}$ | $c_{32} = b_{31} a_{12} + b_{32} a_{22}$

Exc: Prove the general case.

So we have

$$\boxed{[S \circ T]^{\mathcal{B}}_{\mathcal{B}''} = [S]^{\mathcal{B}'}_{\mathcal{B}''} \cdot [T]^{\mathcal{B}}_{\mathcal{B}'}}$$

So when T is an isomorphism, i.e $\exists$ another S - linear s.t

$$S \circ T = Id_V \quad \& \quad T \circ S = Id_W .$$

Id_V: $id : V \to V$

Id_W: $id : W \to W$.

$$\underset{\mathcal{B}}{V} \xrightarrow{T} \underset{\mathcal{B}'}{W} \xrightarrow{S} \underset{\mathcal{B}}{V}$$

& $\mathcal{B}$ – basis of V – n-dim

$\mathcal{B}'$ – basis of W. – m-dim^n

We have

$$[S]_{\mathcal{B}}^{\mathcal{B}'} [T]_{\mathcal{B}'}^{\mathcal{B}} = [Id]_{\mathcal{B}}^{\mathcal{B}} = \underline{I_n}.$$

$$[T]_{\mathcal{B}'}^{\mathcal{B}} [S]_{\mathcal{B}}^{\mathcal{B}'} = [Id]_{\mathcal{B}'}^{\mathcal{B}'} = \underline{I_m}.$$

So if $[T]_{\mathcal{B}'}^{\mathcal{B}} = A$ & $[S]_{\mathcal{B}}^{\mathcal{B}'} = B$

we have $BA = I_n$ and $AB = I_m$.

$\hookrightarrow m \times n$ $\qquad$ $\hookrightarrow$ right invertible.

left. invertible $\qquad$ $\hookrightarrow$ $\underline{n \geq m}$.

$\to n \leq m$

Recall $BA = I_n$.

A – left invertible.

$$Ax = 0 \implies BAx = 0 \implies x = 0$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$ unique sol^n.

rank $\to r = n \leq m$.

$\implies n \leq m$.

Consider $B^T A^T = I_m$

$\implies \underline{m = n}$.

Remark: There can be isomorphisms between same dimensional spaces only.

So we have $\dim V = n = \dim W$.

& $BA = AB = I_n$

$\Rightarrow A, B$ — invertible matrices.

So if $\underset{\beta}{V} \xrightarrow{Id} \underset{\beta_1}{V} \xrightarrow{Id} \underset{\beta}{V}$

$$\Rightarrow \underbrace{[Id]_{\beta}^{\beta_1}}_{Q} \; \underbrace{[Id]_{\beta_1}^{\beta}}_{P.} = [Id]_{\beta}^{\beta} = I_n.$$

$\Rightarrow \exists\, Q$ s.t $QP = PQ = I_n$.

& $[v]_{\beta_1} = P\,[v]_{\mathcal{B}}$ — This give relation b/t ^{n} co-ordinates.

CALLED CHANGE OF BASIS MATRIX FROM $\mathcal{B}$ TO $\mathcal{B}_1$

an invertible matrix.

In fact, given an invertible matrix P and a basis $\mathcal{B}$ of V,
We can find a basis $\mathcal{B}_1$ of V such that $[v]_{\mathcal{B}_1} = P\,[v]_{\mathcal{B}}$.

Note if $\mathcal{B}_1 = \{w_1, \ldots w_n\}$ $\qquad$ $[v]_{\mathcal{B}_1} = P[v]_{\mathcal{B}}$

then $[w_1]_{\mathcal{B}_1} = P\,[w_1]_{\mathcal{B}}$

$\Rightarrow Q \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = [w_1]_{\mathcal{B}}$. where $Q = P^{-1}$

$\Rightarrow \quad [w_1]_{\mathcal{B}} = \text{first column of } Q$

$\downarrow$ inverse of P.

$|||ly \quad [w_j]_{\mathcal{B}} = j^{th}$ column of Q

Thus if $\mathcal{B} = \{v_1 \dots v_n\}$

then $w_j = q_{1j} v_1 + \dots + q_{nj} v_n$.

So you know $w_1 \dots w_n$.

It is enough to show $w_1 \dots w_n$ are L.I.

Let $\alpha_1 w_1 + \dots + \alpha_n w_n = 0$.

$$\Rightarrow \sum_{j=1}^{n} \alpha_j w_j = 0$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{n} q_{ij} y_i = 0$$

$$\Rightarrow \sum_{i=1}^{m} \left(\sum_{j=1}^{n} q_{ij} \alpha_j \right) y_i = 0 \quad \text{—} \; (*)$$

Since $\{y_i\}$'s are linearly independent

$$(*) \Rightarrow \sum_{j=1}^{n} q_{ij} \alpha_j = 0 \quad \forall j$$

$$\Rightarrow Q\alpha = 0 \quad \text{for } \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$\Rightarrow \alpha = 0$ as Q is invertible.

This shows $\{w_1, \dots w_n\}$ are LI.

Since $\dim V = n$, we have $\{w_1, \dots w_n\}$ is a basis of $\mathcal{B}_1$

and $[v]_{\mathcal{B}_1} = P\,[v]_{\mathcal{B}}$.

Computation in an example $\circledast$:

Let $\mathbb{R}^2$ is given with the ORDERED basis $\mathcal{B} = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\}$

Let $P = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ be an invertible matrix (?)

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \xrightarrow[\frac{1}{2}R_1]{} \left(\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow R_1 - \tfrac{3}{2}R_2$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1/2 & -3/2 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

So

$$Q = P^{-1} = \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix}$$

We would like to have a basis $\mathcal{B}'$ such that for any $v \in \mathbb{R}^2$

$$[v]_{\mathcal{B}'} = P\,[v]_{\mathcal{B}}. \quad \forall v \in \mathbb{R}^2 \quad \text{—}(*)$$

Let $\mathcal{B}' = \{ w_1 , w_2 \}$.

Then $[w_1]_{\mathcal{B}'} = P\,[w_1]_{\mathcal{B}}$

$$\Rightarrow \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P\,[w_1]_{\mathcal{B}}$$

as $w_1 = 1.w_1 + 0.w_2$

$$\Rightarrow \quad Q \begin{pmatrix} 1 \\ 0 \end{pmatrix} = [w_1]_{\mathcal{B}}$$

$$\Rightarrow \quad \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = [w_1]_{\mathcal{B}}.$$

$$\Rightarrow \quad \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = [w_1]_{\mathcal{B}}.$$

So

$$w_1 = \frac{1}{2} v_1 + 0.v_2 = \frac{1}{2} v_1 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

Now putting w_2 in (*) we have

$$[w_2]_{\mathcal{B}'} = P\,[w_2]_{\mathcal{B}}$$

$$\Rightarrow \quad Q \begin{pmatrix} 0 \\ 1 \end{pmatrix} = [w_2]_{\mathcal{B}}$$

$$\Rightarrow \quad \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} = [w_2]_{\mathcal{B}}$$

So $w_2 = -\frac{3}{2} v_1 + 1 \cdot v_2 = \begin{pmatrix} -3/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}.$$

So $\mathcal{B}' = \left\{ \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \right\}$ — $\underline{\text{LI}}$.

& hence a basis.

<u>Claim:</u> $[v]_{\mathcal{B}'} = P\,[v]_{\mathcal{B}}$.

Let $v = \begin{pmatrix} x \\ y \end{pmatrix} = [v]_{\mathcal{E}}$ $\hookrightarrow$ Std. basis.

$$\begin{pmatrix} x \\ y \end{pmatrix} = (x-y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\Rightarrow [v]_{\mathcal{B}} = \left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} x-y \\ y \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = (2x+y)\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}.$$

So $\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{B}'} = \begin{pmatrix} 2x+y \\ y \end{pmatrix}.$

Now $P\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x-y \\ y \end{pmatrix}$

$$= \begin{pmatrix} 2x - 2y + 3y \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y \end{pmatrix}.$$

$$= \left[\begin{pmatrix} x \\ y \end{pmatrix} \right]_{\mathcal{B}'}$$

– So P is the change of basis matrix from $\underbrace{\mathcal{B}}_{\text{given}}$ to $\underbrace{\mathcal{B}'}_{\text{constructed newly to get } P \text{ as change of basis matrix}}$.

Now we come back to the question of how how $[T]^{\beta}_{\beta'}$ & $[T]^{\beta_1}_{\beta_1'}$ is related?

where β, β_1 – bases of V

β', β_1' – bases of W

& $T: \underset{\beta_1}{\underset{\beta}{V}} \xrightarrow{\text{linear}} \underset{\beta_1'}{\underset{\beta'}{W}}$

Can be written like Considering the linear map w.r.t two different set of bases.

$T: (V, \mathcal{B}) \longrightarrow (W, \mathcal{B}')$

& $T: (V, \mathcal{B}_1) \longrightarrow (W, \mathcal{B}_1')$

Let $[Id]^{\mathcal{B}}_{\mathcal{B}_1} = P$ – change of basis (from $\mathcal{B}$ to $\mathcal{B}_1$) matrix on V

$[Id]^{\mathcal{B}'}_{\mathcal{B}_1'} = P'$ – change of basis (from $\mathcal{B}'$ to $\mathcal{B}_1'$) matrix on W.

The matrix repn. of $T: (V, \mathcal{B}) \longrightarrow (W, \mathcal{B}')$ is given by

$$[Tv]_{\mathcal{B}'} = [T]^{\mathcal{B}}_{\mathcal{B}'} [v]_{\mathcal{B}} \quad \text{— } (1).$$

Illy for $T : (V, \mathcal{B}_1) \longrightarrow (W, \mathcal{B}_1')$ we have

$$[Tv]_{\mathcal{B}_1'} = [T]_{\mathcal{B}_1'}^{\mathcal{B}_1} [v]_{\mathcal{B}_1} \quad \text{--- (2)}$$

<u>Now</u>

$$[Tv]_{\mathcal{B}_1'} = [Id]_{\mathcal{B}_1'}^{\mathcal{B}'} [Tv]_{\mathcal{B}'}$$

$$= P' [Tv]_{\mathcal{B}'} \quad \text{--- (3)}$$

On the other hand

$$[v]_{\mathcal{B}_1} = [Id]_{\mathcal{B}_1}^{\mathcal{B}} [v]_{\mathcal{B}} = P[v]_{\mathcal{B}}. \quad \text{--- (4)}$$

(2) & (3)

$$\Rightarrow \quad P'[Tv]_{\mathcal{B}'} = [T]_{\mathcal{B}_1'}^{\mathcal{B}_1} [v]_{\mathcal{B}_1}$$

$$= [T]_{\mathcal{B}_1'}^{\mathcal{B}_1} P[v]_{\mathcal{B}} \quad \text{from (4)}$$

$$\Rightarrow \quad [Tv]_{\mathcal{B}'} = (P')^{-1} [T]_{\mathcal{B}_1'}^{\mathcal{B}_1} P [v]_{\mathcal{B}}$$

Comparing with (1) we have

$$[T]_{\mathcal{B}'}^{\mathcal{B}} = (P')^{-1} [T]_{\mathcal{B}_1'}^{\mathcal{B}_1} P.$$

$$= \left([Id]_{\mathcal{B}_1'}^{\mathcal{B}'}\right)^{-1} [T]_{\mathcal{B}_1'}^{\mathcal{B}_1} [Id]_{\mathcal{B}_1}^{\mathcal{B}}$$

$$= [Id]_{\mathcal{B}'}^{\cancel{\mathcal{B}_1'}} [T]_{\cancel{\mathcal{B}_1'}}^{\cancel{\mathcal{B}_1}} [Id]_{\cancel{\mathcal{B}_1}}^{\mathcal{B}}$$

So $$[T]^{\mathcal{B}}_{\mathcal{B}'} = [Id]^{\mathcal{B}_1'}_{\mathcal{B}'} \, [T]^{\mathcal{B}_1}_{\mathcal{B}_1'} \, [Id]^{\mathcal{B}}_{\mathcal{B}_1}$$

or $$[T]^{\mathcal{B}_1}_{\mathcal{B}_1'} = \left([Id]^{\mathcal{B}_1'}_{\mathcal{B}'}\right)^{-1} [T]^{\mathcal{B}}_{\mathcal{B}'} \left([Id]^{\mathcal{B}}_{\mathcal{B}_1}\right)^{-1}$$

$$= [Id]^{\mathcal{B}'}_{\mathcal{B}_1'} \, [T]^{\mathcal{B}}_{\mathcal{B}'} \, [Id]^{\mathcal{B}_1}_{\mathcal{B}}.$$

Computation in an example:

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by

$$T(x_1, x_2) = (x_2, -5x_1, -7x_1 + 16x_2).$$

Case I: Take $\mathcal{E} = \{ e_1 = (1,0), e_2 = (0,1) \}$ std. basis of $\mathbb{R}^2$

& $\mathcal{E}' = \{ e_1' = (1,0,0), e_2' = (0,1,0), e_3' = (0,0,1) \}$ std basis of $\mathbb{R}^3$.

$$Te_1 = T(1,0) = (0,-5,-7)$$
$$= 0 \cdot e_1' - 5e_2' - 7e_3'$$

$$Te_2 = T(0,1) = (1, 0, 16) = 1 \cdot e_1' + 0 \cdot e_2' + 16 \cdot e_3'$$

So $$[T]^{\mathcal{E}}_{\mathcal{E}'} = \begin{pmatrix} 0 & 1 \\ -5 & 0 \\ -7 & 16 \end{pmatrix}.$$

Case II: Take $\mathcal{B} = \{v_1, v_2\}$ where

$v_1 = (3, 1)$ & $v_2 = (5, 2)$.

& $\mathcal{B}' = \{w_1, w_2, w_3\}$

where $w_1 = (1, 0, 1)$

$w_2 = (-1, 2, 2)$

$w_3 = (0, 1, 2)$.

Exc: Verify that $\mathcal{B}$ is a basis of $\mathbb{R}^2$ and $\mathcal{B}'$ is a basis of $\mathbb{R}^3$.

We would like to compute $[T]_{\mathcal{B}'}^{\mathcal{B}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ ↪ 2×3 matrix

$$Tv_1 = T(3, 1) = (1, -15, -5)$$

$$= a_{11} w_1 + a_{21} w_2 + a_{31} w_3.$$

$$= a_{11}(1, 0, 1) + a_{21}(-1, 2, 2) + a_{31}(0, 1, 2)$$

$$= (a_{11} - a_{21},\ 2a_{21} + a_{31},\ a_{11} + 2a_{21} + 2a_{31})$$

$$\Rightarrow \left.\begin{aligned} a_{11} - a_{21} &= 1 \\ 2a_{21} + a_{31} &= -15 \\ a_{11} + 2a_{21} + 2a_{31} &= -5 \end{aligned}\right\}$$

Solve eq^n to get a_{11}, a_{21}, a_{31} -- (A)

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & -15 \\ 1 & 2 & 2 & -5 \end{array}\right)$$

$$\xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & -15 \\ 0 & 3 & 2 & -6 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2} R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1/2 & -15/2 \\ 0 & 3 & 2 & -6 \end{array}\right)$$

$$\xrightarrow[R_3 - 3R_2]{R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1/2 & -13/2 \\ 0 & 1 & 1/2 & -15/2 \\ 0 & 0 & 1/2 & -6 + \frac{45}{2} \end{array}\right)$$

$$\xrightarrow{2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1/2 & -13/2 \\ 0 & 1 & 1/2 & -15/2 \\ 0 & 0 & 1 & 33 \end{array}\right)$$

$$\xrightarrow[R_2 - \frac{1}{2} R_3]{R_1 - \frac{1}{2} R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & -24 \\ 0 & 0 & 1 & 33 \end{array}\right)$$

So $a_{11} = -23$, $a_{21} = -24$, $a_{31} = 33$.

$$T v_2 = T(5,2) = (2, -25, -3)$$

$$= a_{12}\, w_1 + a_{22}\, w_2 + a_{32}\, w_3 .$$

Since only the b part of the previous system of eqn. (A) changes, we just apply the same steps to the new $b = (2, -25, -3)$

$$\begin{pmatrix} 2 \\ -25 \\ -3 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 2 \\ -25 \\ -5 \end{pmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{pmatrix} 2 \\ -25/2 \\ -5 \end{pmatrix} \xrightarrow[R_3 - 3R_2]{R_1 + R_2} \begin{pmatrix} -21/2 \\ -25/2 \\ \frac{65}{2} \end{pmatrix}$$

$$\begin{pmatrix} -21/2 \\ -25/2 \\ \frac{65}{2} \end{pmatrix} \xrightarrow{2R_3} \begin{pmatrix} -21/2 \\ -25/2 \\ 65 \end{pmatrix} \xrightarrow[R_2 - \frac{1}{2} R_3]{R_1 - \frac{1}{2} R_3} \begin{pmatrix} -43 \\ -45 \\ 65 \end{pmatrix}$$

So $a_{12} = -43$, $a_{22} = -45$, $a_{32} = 65$

So $$[T]^{\mathcal{B}}_{\mathcal{B}'} = \begin{pmatrix} -23 & -43 \\ -24 & -45 \\ 33 & 65 \end{pmatrix} \quad \text{— } (*)_1$$

Now we need to verify

$$[T]^{\mathcal{B}}_{\mathcal{B}'} = [Id]^{\mathcal{E}'}_{\mathcal{B}'} \, [T]^{\mathcal{E}}_{\mathcal{E}'} \, [Id]^{\mathcal{B}}_{\mathcal{E}} \quad \text{— } (B)$$

$$\mathcal{B} = \{ v_1 = (3,1),\ v_2 = (5,2) \}$$

$$Id(v_1) = (3,1) = 3e_1 + 1.e_2$$

$$Id(v_2) = (5,2) = 5e_1 + 2.e_2 .$$

$$\therefore \quad [Id]_{\mathcal{E}}^{\mathcal{B}} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} . = P$$

$$\mathcal{E}' = \{ e_1', e_2', e_3' \}$$

$$\mathcal{B}' = \{ w_1 = (1,0,1),\ w_2 = (-1,2,2),\ w_3 = (0,1,2) \}$$

$$[Id]_{\mathcal{B}'}^{\mathcal{E}'} = \left([Id]_{\mathcal{E}'}^{\mathcal{B}'} \right)^{-1}$$

$$Id(w_1) = w_1 = 1.e_1' + 0.e_2' + 1.e_3'$$

$$Id(w_2) = w_2 = (-1).e_1' + 2.e_2' + 2.e_3'$$

$$Id(w_3) = w_3 = 0.e_1' + 1.e_2' + 2e_3'$$

So $[Id]_{\mathcal{E}'}^{\mathcal{B}'} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} = P'$

So we need to P'^{-1} to verify (B)

$$[T]_{\mathcal{B}'}^{\mathcal{B}} = (P')^{-1} [T]_{\mathcal{E}'}^{\mathcal{E}} P$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right)$$

we repeat the steps as earlier as
This is the same matrix on left side of the divider.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\Big\downarrow \tfrac{1}{2} R_2$$

$$\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ -1 & -3/2 & 1 \end{pmatrix} \xleftarrow[R_3 - 3R_2]{R_1 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\Big\downarrow 2R_3$$

$$\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ -2 & -3 & 2 \end{pmatrix} \xrightarrow[R_2 - \frac{1}{2} R_3]{R_1 - \frac{1}{2} R_3} \underline{\begin{pmatrix} 2 & 2 & -1 \\ 1 & 2 & -1 \\ -2 & -3 & 2 \end{pmatrix}}.$$

So $[Id]^{E'}_{B'} = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 2 & -1 \\ -2 & -3 & 2 \end{pmatrix}$

So

RHS of B

$$= \begin{pmatrix} 2 & 2 & -1 \\ 1 & 2 & -1 \\ -2 & -3 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -5 & 0 \\ -7 & 16 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & -1 \\ 1 & 2 & -1 \\ -2 & -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -15 & -25 \\ -5 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2.1 + 2(-15) + (-1)(-5) & 2.2 + 2(-25) + (-1).(-3) \\ 1.1 + 2(-15) + (-1)(-5) & 1.2 + 2(-25) + (-1).(-3) \\ -2.1 + (-3)(-15) + 2(-5) & -2.2 + (-3)(-25) + 2.(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 2 - 30 + 5 & 4 - 50 + 3 \\ 1 - 30 + 5 & 2 - 50 + 3 \\ -2 + 45 - 10 & -4 + 75 - 6 \end{pmatrix}$$

$$= \begin{pmatrix} -23 & -43 \\ -24 & -45 \\ 33 & 65 \end{pmatrix}$$ — B verified.

(comparing with (*1)).

Now suppose $V = W$.

we take $\beta = \beta'$

$\beta_1 = \beta_1'$.

that is $T : V \longrightarrow V$ with different choice of bases β, β_1,

that is, we consider $T : (V, \beta) \xrightarrow[\text{linear}]{} (V, \beta)$ & $T : (V, \beta_1) \xrightarrow[\text{linear}]{} (V, \beta_1)$.

$$\Rightarrow \quad [T]_{\beta}^{\beta} = \left([Id]_{\beta_1}^{\beta}\right)^{-1} [T]_{\beta_1}^{\beta_1} [Id]_{\beta_1}^{\beta}$$

$$= P^{-1} [T]_{\beta_1}^{\beta_1} P.$$

So if you consider

$T : V \xrightarrow[\text{linear}]{} V$, once β with both domain and co-domain space and once β_1 with both domain and co-domain space, then

$$[T]_{\beta}^{\beta} \quad \& \quad [T]_{\beta_1}^{\beta_1}$$

are related by

$$[T]_{\beta}^{\beta} = \left([Id]_{\beta_1}^{\beta}\right)^{-1} [T]_{\beta_1}^{\beta_1} [Id]_{\beta_1}^{\beta}$$

or $\quad [T]_{\beta_1}^{\beta_1} = P\, [T]_{\beta}^{\beta}\, P^{-1}$, where $P = [Id]_{\beta_1}^{\beta}$

Such matrices are called similar matrices.

Def^n: Two matrices A & B are called SIMILAR MATRICES iff $\exists$ invertible matrix P such that $B = PAP^{-1}$.

Remark: given a basis $\mathcal{B}$ of V and an invertible matrix P, $\exists$ a basis $\mathcal{B}_1$ of V such that

$$[T]_{\mathcal{B}_1}^{\mathcal{B}_1} = P\,[T]_{\mathcal{B}}^{\mathcal{B}}\,P^{-1}.$$

as — $\mathcal{B}_1$ — be the basis whose co-ordinates are column vector of P^{-1}.
(as constructed earlier).

In continuation with example ✱ :

$$\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\& \quad \mathcal{B}' = \left\{ w_1 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},\ w_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \right\}$$

— are two bases of $\mathbb{R}^2$

$$\& \quad [v]_{\mathcal{B}'} = [Id]_{\mathcal{B}'}^{\mathcal{B}}\,[v]_{\mathcal{B}}$$

where $[Id]_{\mathcal{B}'}^{\mathcal{B}} = P = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$

Consider the linear map

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

We would like to verify

$$[T]_{\mathcal{B}}^{\mathcal{B}} = [Id]_{\mathcal{B}}^{\mathcal{B}'} [T]_{\mathcal{B}'}^{\mathcal{B}'} [Id]_{\mathcal{B}'}^{\mathcal{B}}$$

$$= \left([Id]_{\mathcal{B}'}^{\mathcal{B}}\right)^{-1} [T]_{\mathcal{B}'}^{\mathcal{B}'} [Id]_{\mathcal{B}'}^{\mathcal{B}}$$

$$= P^{-1} [T]_{\mathcal{B}'}^{\mathcal{B}'} P.$$

So to calculate $[T]_{\mathcal{B}}^{\mathcal{B}}$,

we note

$$T v_1 = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_2 = 0 . v_1 + 1 . v_2$$

$$T v_2 = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 . v_1 + 0 . v_2$$

$$\Rightarrow \quad [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Now for $[T]_{\mathcal{B}'}^{\mathcal{B}'}$

$$T\omega_1 = T\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

$$T\omega_2 = T\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -3/2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

So $[T]_{\mathcal{B}'}^{\mathcal{B}'} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix}$

Now we compute P^{-1}

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - \frac{3}{2}R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & -3/2 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

So $P^{-1} = \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix}$

Now $P^{-1} [T]_{\mathcal{B}'}^{\mathcal{B}'} P$

$$= \begin{pmatrix} 1/2 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & \frac{9}{2} - \frac{1}{2} \\ 1 & \frac{3}{2} - \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} - \frac{3}{2} & 2 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = [T]_{\mathcal{B}}^{\mathcal{B}} .$$