Linear Transformation and Matrix Representation

1. In this exercise, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is *not* linear.

   (a) $T(a_1, a_2) = (1, a_2)$

   (b) $T(a_1, a_2) = (a_1, a_2^2)$

   (c) $T(a_1, a_2) = (\sin a_1, 0)$

   (d) $T(a_1, a_2) = (|a_1|, a_2)$

   (e) $T(a_1, a_2) = (a_1 + 1, a_2)$

Solution:

(a) A linear transformation must map the zero vector to the zero vector. But here:

$$T(0, 0) = (1, 0) \neq (0, 0)$$

Since $T(\mathbf{0}) \neq \mathbf{0}$, the function violates a fundamental property of linearity. Therefore, T is not linear.

(b) T is not linear as $T(1, 1) + T(1, 1) = (2, 2) \neq (2, 4) = T(2, 2)$.

(c) We have $T(\pi/2, 0) = (1, 0)$, $T(\pi, 0) = (0, 0)$, $T(3\pi/2, 0) = (-1, 0)$, Therefore, $T(3\pi/2, 0) = T((\pi/2, 0) + (\pi, 0)) \neq T(\pi/2, 0) + T(\pi, 0)$. Hence, T is not a linear map.

(d)

(e) T is not linear in the first component. For instance,

$$T(1, 0) + T(-1, 0) \neq T(0, 0),$$

as $T(1, 0) + T(-1, 0) = (2, 0)$ but $T(0, 0) = 0$.

For Exercises 2 through 6, find the matrix representation of the linear map with respect to standard basis in each case. Note that the matrices E_{ij}- with ij-th entry 1 and all other entries 0, $1 \leq i \leq m, 1 \leq j \leq n$, forms the standard basis of $M_{m\times n}(\mathbb{F})$ and $\{1, x, \ldots, x^k\}$ forms the standard basis of $P_k(\mathbb{R})$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

Solution: Let $\mathcal{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $\mathcal{A} = ((1,0),(0,1))$ be the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Applying T on $\mathcal{B}$ gets us

$$\begin{aligned} T((1,0,0)) &= 1(1,0) + 0(0,1) \\ T((0,1,0)) &= -1(1,0) + 0(0,1) \\ T((0,0,1)) &= 0(1,0) + 2(0,1) \end{aligned}$$

Hence the linear transformation T written in terms of the standard basis is as follow

$$[T]_{\mathcal{A}}^{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3. $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2).$$

Solution: $T(1,0) = (1,0,2) = e_1 + 2e_3$ $T(0,1) = (1,0,-1) = e_1 - e_3$
The matrix representation for T with respect to standard ordered bases is

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -1 \end{pmatrix}$$

4. $T : M_{2\times 3}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\right) = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

Solution: Fix the standard ordered basis $\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ for $M_{2\times 3}(\mathbb{R})$ and $\mathcal{B}' = \{E_{11}, E_{12}, E_{21}, E_{24}\}$.
Apply T to each basis vector in $\mathcal{B}$:

$$T(E_{11}) = T\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{12}) = T\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = -1E_{11} + 2E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{13}) = T\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0E_{11} + 1E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{21}) = T\left(\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{22}) = T\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{23}) = T\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$$

Therefore the matrix representation of the linear map with respect to the standard is:

$$[T]_{\mathcal{B}}^{\mathcal{B}'} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5. $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by

$$T(f(x)) = xf(x) + f'(x).$$

Solution: We find the matrix representation of T with respect to the standard bases:

$$\mathcal{B} = \{1, x, x^2\} \text{ for } P_2(\mathbb{R}), \quad \mathcal{B}' = \{1, x, x^2, x^3\} \text{ for } P_3(\mathbb{R}).$$

Apply T to each basis vector in $\mathcal{B}$:

$$\begin{aligned} T(1) &= x \cdot 1 + 0 = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3, \\ T(x) &= x \cdot x + 1 = x^2 + 1 = 1 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3, \\ T(x^2) &= x \cdot x^2 + 2x = x^3 + 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 1 \cdot x^3. \end{aligned}$$

Therefore, the matrix representation of T with respect to $\mathcal{B}$ and $\mathcal{B}'$ is:

$$[T]_{\mathcal{B}'}^{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. $T : M_{n\times n}(F) \to F$ defined by $T(A) = \operatorname{tr}(A)$ (called *Trace* of A), where

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Solution: Note that the matrix of T is of order $1 \times n^2$, that is, it has 1 row and n^2 columns. Sinnce $T(E_{ij}) = tr(E_{ij}).1$, therefore the entry in the $((i-1)n+j)$-th column is $tr(E_{ij})$. Also $tr(E_{ij}) = 1$ if $i = j$ and is 0 otherwise. Thus the entry in $((i-1)n+i)$-th column is 1 for $1 \leq i \leq n$ and the entries in other columns are all 0s.

7. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1,0) = (1,4)$, and $T(1,1) = (2,5)$. What is $T(2,3)$? Is T one-to-one?

Solution: Clearly, $\{(1,0),(1,1)\}$ and $\{(1,4),(2,5)\}$ are linearly independent subset of $\mathbb{R}^2$. Now we have,

$$(2,3) = -1 \cdot (1,0) + 3(1,1)$$

Therefore,

$$T(2,3) = -1 \cdot T(1,0) + 3T(1,1) = -1 \cdot (1,4) + 3(2,5) = (5,11)$$

Again,

$$\ker(T) = \{(a,b) : T(a,b) = 0\}$$

Since $\{(1,0),(1,1)\}$ is a basis of $\mathbb{R}^2$, so for $(a,b) \in \ker(T)$, there exist unique real numbers c_1, c_2 such that,

$$(a,b) = c_1(1,0) + c_2(1,1)$$

Therefore,

$$T(a,b) = 0 \Rightarrow c_1(1,4) + c_2(2,5) = 0$$

Since, $\{(1,4),(2,5)\}$ is linearly independent set, so $c_1 = c_2 = 0$. Hence $(a,b) = 0$. Therefore $\ker(T) = \{0\}$ and T is one one.

8. Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

Solution: Note that the vectors $\{(1,1),(2,3)\}$ spans $\mathbb{R}^2$. In fact, any $(x,y) \in \mathbb{R}^2$ can be written as

$$(x,y) = (3x-2y)\,(1,1) + (y-x)\,(2,3).$$

For linear transform T to exist, we must have

$$\begin{aligned} T(x,y) &= (3x-2y)\,T(1,1) + (y-x)\,T(2,3) \\ &= (3x-2y)\,(1,0,2) + (y-x)\,(1,-1,4) \\ &= (2x-y, x-y, 2x). \end{aligned}$$

Thus, $T(8,11) = (5,-3,16)$.

9. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$

(c) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p'(x)$

(d) $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + 2bx + (c + d)x^2$$

(e) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$$

Solution:

(a) The matrix representation of the linear map T has size 3×2, which cannot be invertible. Hence, T is not invertible.

(b) We show that T is an onto map. Note that

$$\begin{aligned} T((0, 0, -\frac{1}{2})) &= (1, 0, 0) \\ T(-(\frac{4}{3}), 1, -2) &= (0, 1, 0) \\ T(\frac{1}{3}, 0, \frac{1}{2}) &= (0, 0, 1) \end{aligned}$$

See that T maps to a basis, hence T is onto. Using the rank nullity theorem we get

$$\begin{aligned} &\mathrm{Rank}(T) + \dim(N(T)) = \dim(\mathbb{R}^3) \\ &3 + \dim(N(T)) = 3 \\ &\implies \dim(N(T)) = 0 \end{aligned}$$

Hence T is an invertible linear transformation.

(c) A linear transformation T is invertible if and only if matrix representation $[T]_{\mathcal{B}}^{\mathcal{B}'}$ is invertible. Here $\mathcal{B} = \{1, x, x^2\}$ for $P_2(\mathbb{R})$, $\quad \mathcal{B}' = \{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$. So, $dim(P_2(\mathbb{R})) = 3$, $dim(P_3(\mathbb{R})) = 4$. So the matrix representation of T with respect to any basis is a 3×4 matrix which cannot be invertible.so T is not invertible.

(d) A linear transformation T is invertible if and only if matrix representation $[T]_{\mathcal{B}}^{\mathcal{B}'}$ is invertible. Howevever, we know that $dim(M_{2\times 2}(\mathbb{R})) = 4$ and $dim(P_2(\mathbb{R})) = 3$. The order of the matrix $[T]_{\mathcal{B}}^{\mathcal{B}'}$ with respect to the standard basis is 3×4; which cannot be invertible. (For a matrix A to be invertible, it should be a square matrix)

(e) Since $M_{2\times 2}(\mathbb{R})$ is a 4-dimensional vector space, we can examine whether the matrix representation of T with respect to the standard basis is invertible.

Let us consider the standard basis:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Applying T to each basis element, we get:

$$T(E_1) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = E_1 + E_2, \quad T(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_1,$$

$$T(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = E_3 + E_4, \quad T(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_4.$$

Thus the matrix representation of T with respect to the basis $\{E_1, E_2, E_3, E_4\}$ is:

$$[T] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

This matrix has full rank (4), and since it is a square matrix, it is invertible. Therefore, the linear transformation T is invertible.

10. Which of the following pairs of vector spaces are isomorphic? Justify your answers.

(a) $\mathbb{F}^3$ and $P_3(\mathbb{F})$

(b) $\mathbb{F}^4$ and $P_3(\mathbb{F})$

(c) $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$

(d) $V = \{A \in M_{2\times 2}(\mathbb{R}) : \mathrm{tr}(A) = 0\}$ and $\mathbb{R}^4$

Solution:

(a) Note that the dimensions of $\mathbb{F}^3$ and $P_3(\mathbb{F})$ are 3 and 4 respectively. Hence they are not isomorphic.

(b) We define a map $\phi : \mathbb{F}^4 \to P_3(\mathbb{F})$ by

$$\phi(a_0, a_1, a_2, a_3) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Clearly, this map is injective and surjective. Also, the above map is a linear map as,

$$\begin{aligned}\phi(ca_0 + b_0, \ldots, ca_3 + b_3) &= (ca_0 + b_0) + \cdots + (ca_3 + b_3)x^3 \\ &= c(a_0 + \cdots + a_3 x^3) + (b_0 + \cdots + b_3 x^3) \\ &= c\phi(a_0 \ldots, a_3) + \phi(b_0, \ldots, b_3)\end{aligned}$$

for all $c \in \mathbb{F}$ and $(a_0, \ldots, a_3) \in \mathbb{F}^4$. Hence, ϕ is a linear isomorphism and $\mathbb{F}^4$ is isomorphic to $P_3(\mathbb{F})$.

(c)

(d) Note that any matrix $A \in V$ can be rewritten as

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Also, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent.

Thus, the set $\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ forms a basis for V, and thus $dim(V) = 3$.

However, $dim(\mathbb{R}^4) = 4$, and hence V is not isomorphic to $\mathbb{R}^4$.

11. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \quad \text{and} \quad U(a + bx + cx^2) = (a + b, c, a - b)$$

Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(a) Compute $[U]_\gamma^\beta$, $[T]_\beta^\beta$, and $[UT]_\gamma^\beta$ directly and verify that matrix representation of the composition is matrix multiplication of the individual matrix in the same order.

Solution: Let $\gamma = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $\beta = \{1, x, x^2\}$ be the standard basis of $P_2(\mathbb{R})$, R^3 respectively.

i. Computation of $[U]_\gamma^\beta$:

$$\begin{aligned} U(1) &= 1(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1) \\ U(x) &= 1(1, 0, 0) + 0(0, 1, 0) + (-1)(0, 0, 1) \\ U(x^2) &= 0(1, 0, 0) + 1(0, 1, 0) + 0(0, 0, 1) \end{aligned}$$

Hence we get

$$[U]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

ii. Computation of $[T]_\beta^\beta$:

$$\begin{aligned} T(1) &= 2(1) + 0(x) + 0(x^2) \\ T(x) &= 3(1) + 3(x) + 0(x^2) \\ T(x^2) &= 0(1) + 6(x) + 4(x^2) \end{aligned}$$

Hence we get

$$[T]_\beta^\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$$

iii. Computation of $[UT]_\gamma^\beta$:

$$UT(a + bx + cx^2) = U(2a + 3b + 3(b + 2c)x + 4cx^2) = (2a + 6b + 6c, 4c, 2a - 6c)$$

Evaluating UT on γ gives us

$$\begin{aligned} UT(1) &= 2(1,0,0) + 0(0,1,0) + 2(0,0,1) \\ UT(x) &= 6(1,0,0) + 0(0,1,0) + 0(0,0,1) \\ UT(x^2) &= 6(1,0,0) + 4(0,1,0) + (-6)(0,0,1) \end{aligned}$$

Hence we get

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$$

An immediate check gives us $[UT]_\gamma^\beta = [U]_\gamma^\beta [T]_\beta^\beta$

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h]_\beta$ and $[U(h)]_\gamma$. Then verify $[U(h)]_\gamma = [U]_\gamma^\beta [h]_\beta$.

Solution: Given $h(x) = 3 - 2x + x^2$, we get $U(h) = (1, 1, 5)$. Immediately we get

$$[h]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$[U(h)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

Then we get

$$[U]_\gamma^\beta [h]_\beta = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = [U(h)]_\gamma$$

12. Find linear transformations $U, T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $UT = 0$ (the zero transformation) but $TU \neq 0$.

Solution: Define linear transformations $U, T : \mathbb{R}^2 \to \mathbb{R}^2$ by $U(x_1, x_2) = (x_2, 0)$
$T(x_1, x_2) = (x_1, 0)$,
$T(U(x_1, x_2)) = (x_2, 0)$,$U(T(x_1, x_2)) = (0, 0)$.

13. Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix},$$

let β be the standard ordered basis for $\mathbb{R}^2$, and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Use the change of basis matrix to find $[T]_{\beta'}^{\beta'}$.

Solution: We know from the result proved in class:
Let V, W be a vector space, and $T : V \to W$ be a linear transformation, if $\mathcal{B}, \mathcal{B}_1$ and $\mathcal{B}', \mathcal{B}_1'$ be a bases for V and W, then

$$[T]_{\mathcal{B}'}^{\mathcal{B}} = [Id]_{\mathcal{B}'}^{\mathcal{B}_1'} \; [T]_{\mathcal{B}_1'}^{\mathcal{B}_1} \; [Id]_{\mathcal{B}_1}^{\mathcal{B}}$$

In our problem, choose $\beta = \mathcal{B}_1 = \mathcal{B}_1'$ and $\beta' = \mathcal{B} = \mathcal{B}'$.

(a) **Computation of** $[Id]_{\beta}^{\beta'}$:

$$Id\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Id\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, $[Id]_{\beta}^{\beta'} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

(b) **Computation of** $[T]_{\beta}^{\beta}$:

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-3)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, $[T]_{\beta}^{\beta} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$.

(c) **Computation of** $[Id]_{\beta'}^{\beta}$:

$$Id\begin{pmatrix}1\\0\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\2\end{pmatrix}$$

$$Id\begin{pmatrix}0\\1\end{pmatrix} = (-1)\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\end{pmatrix}$$

Thus, $[Id]_{\beta'}^{\beta} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}$. Note that, $[Id]_{\beta'}^{\beta} = \left([Id]_{\beta}^{\beta'}\right)^{-1}$

By using the above stated theorem we have;

$$\begin{aligned}[T]_{\beta'}^{\beta'} &= \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\1 & -3\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\\ &= \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}3 & 4\\-2 & -5\end{pmatrix}\\ &= \begin{pmatrix}8 & 13\\-5 & -9\end{pmatrix}\end{aligned}$$

14. Let T be the linear operator on $P_1(\mathbb{R})$ defined by $T(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1+x, 1-x\}$. Use the change of basis matrix to find $[T]_{\beta'}^{\beta'}$ and verify by computing independently.

Solution: We know from the result proved in class:
Let V, W be a vector space, and $T : V \to W$ be a linear transformation, if $\mathcal{B}, \mathcal{B}_1$ and $\mathcal{B}', \mathcal{B}_1'$ be a bases for V and W, then

$$[T]_{\mathcal{B}'}^{\mathcal{B}} = [Id]_{\mathcal{B}'}^{\mathcal{B}_1'}\ [T]_{\mathcal{B}_1'}^{\mathcal{B}_1}\ [Id]_{\mathcal{B}_1}^{\mathcal{B}}$$

In our problem, choose $\beta = \mathcal{B}_1 = \mathcal{B}_1'$ and $\beta' = \mathcal{B} = \mathcal{B}'$.

$$Id(1+x) = 1 + x$$

$$Id(1-x) = 1 - x$$

$$[Id]_{\beta}^{\beta'} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$Id(1) = \frac{1}{2}(1+x) + \frac{1}{2}(1-x)$ $Id(x) = \frac{1}{2}(1+x) - \frac{1}{2}(1-x)$

$$[Id]_{\beta'}^{\beta} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}$$

Here we have $T(1) = 0, T(x) = 1$

$$[T]_{\beta}^{\beta} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

So,

$$[T]_{\beta'}^{\beta'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

15. If the transformation T is a reflection across the $45°$ line in the plane, find its matrix with respect to the standard basis $w_1 = (1,3)$, $w_2 = (2,-1)$, and also with respect to $v_1 = (1,2)$, $v_2 = (3,-1)$. Show also that those matrices are similar and find the matrix which that gives the similarity.

Solution: The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(x,y) = (y,x)$ (Done in class). Let $\beta = \{w_1, w_2\}, \beta' = \{v_1, v_2\}$

$T(w_1) = x_1 w_1 + y_1 w_2 \implies T(1,3) = x_1(1,3) + y_1(2,-1) \implies (3,1) = (x_1 + 2y_1, 3x_1 - y_1)$. So we have two equations $x_1 + 2y_1 = 3, 3x_1 - y_1 = 1$. Solving these two equations, we have $x_1 = \frac{5}{7}, y_1 = \frac{16}{7}$. Now, $T(w_2) = x_2 w_1 + y_2 w_2$. By same calculations as before, we have $x_2 = \frac{3}{7}, y_2 = -\frac{2}{7}$. So,

$$[T]_{\beta}^{\beta} = \begin{pmatrix} \frac{5}{7} & \frac{3}{7} \\ \frac{16}{7} & -\frac{2}{7} \end{pmatrix}$$

Similarly as above, we have

$$[T]_{\beta'}^{\beta'} = \begin{pmatrix} \frac{5}{7} & \frac{8}{7} \\ \frac{3}{7} & -\frac{5}{7} \end{pmatrix}$$

These two matrices are similar and the similarity matrix is given by $[Id]_{\beta'}^{\beta} = \begin{pmatrix} \frac{10}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{5}{7} \end{pmatrix}$.

Determinant

16. Let Q be an orthogonal $n \times n$ matrix, that is, $Q^{\mathrm{T}}Q = I$. Prove that $\det Q$ equals $+1$ or -1.

Solution: 1=det(I)=det(Q^TQ) = det(Q^T)det(Q) = (det(Q))^2, this implies that det(Q) equals $+1$ or -1.

17. Show that

$$\det \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} = -(a^3 + b^3 + c^3 - 3abc).$$

Solution: Denote the given matrix by A. The minors along the first row are,

$$M_{11} = \begin{pmatrix} c & a \\ a & b \end{pmatrix}, M_{12} = \begin{pmatrix} b & a \\ c & b \end{pmatrix}, M_{13} = \begin{pmatrix} b & c \\ c & a \end{pmatrix}.$$

The corresponding cofactors are,
$A_{11} = (-1)^{(1+1)}\det(M_{11}) = cb - a^2$,

$A_{12} = (-1)^{(1+2)}\det(M_{12}) = -(b^2 - ac)$,

$A_{13} = (-1)^{(1+3)}\det(M_{13}) = ba - c^2$.
Expanding along the first row we have,

$$\begin{aligned} \det(A) &= aA_{11} + bA_{12} + cA_{13} \\ &= a(cb - a^2) - b(b^2 - ac) + c(ba - c^2) \\ &= acb - a^3 - b^3 + bac + cba - c^3 \\ &= -(a^3 + b^3 + c^3 - 3abc) \end{aligned}$$

18. Without expanding the determinant prove that

$$\det \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} = 0.$$

Solution: Let A be the given matrix. If $a = 0$ then $\det(A) = 0$ as the first and second column of A are linearly independent. Let us assume $a \neq 0$ and apply row operation.

$$\begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - \frac{b}{a}R_2} \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ 0 & -c & \frac{-cb}{a} \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{c}{a}R_1} \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

Hence A has a row of zero after doing row operations, hence $\det(A) = 0$.

19. Use row operations to verify that the 3 by 3 "Vandermonde determinant" is

$$\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (b-a)(c-a)(c-b).$$

Solution: Using the operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have

$$\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{pmatrix} = (b-a)(c-a) \det \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & (b+a) \\ 0 & 1 & (c+a) \end{pmatrix}.$$

Now, expanding along the column C_1, we get

$$\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (b-a)(c-a)(c-b).$$

20. Use the problem above to show that if $s_r = a^r + b^r + c^r$, then

$$\det \begin{pmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix} = (b-a)^2(c-a)^2(c-b)^2.$$

Solution: Let us denote the matrix in the previous problem as

$$V = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}.$$

Then note that

$$V^tV = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = \begin{pmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{pmatrix}$$
$$= \begin{pmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix}.$$

Thus, we have

$$\det \begin{pmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix} = \det(V^tV) = (\det V)^2 = (b-a)^2(c-a)^2(c-b)^2.$$

21. True or false, with reason if true and counterexample if false:

(a) If A and B are identical except that $b_{11} = 2a_{11}$, then $\det B = 2 \det A$.

Solution: The given statement is false. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

Note that $A = B$ except at $2a_{11} = b_{11}$, but $\det(A) = -1$ and $\det(B) = 0$.

(b) The determinant is the product of the pivots.

Solution: The statement is false. Consider the following example

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The row echelon for of A is the $I_{2\times 2}$ whose product of pivot is 1, but we know the determinant of A is -1. We can still conclude that $\det(A) = \pm(\text{product of pivots})$, as permuting the matrix at best changes sign of determinant.

(c) If A is invertible and B is singular (*not invertible*), then $A + B$ is invertible.

Solution: The given statement is false. Consider the following matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \implies A + B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

See that A is invertible and B is singular, but $A + B$ is also singular.

(d) If A is invertible and B is singular, then AB is singular.

Solution: We know that $\det(AB) = \det(A)\det(B) = 0$ as $\det(B) = 0$, hence AB is singular.

(e) The determinant of $AB - BA$ is zero.

Solution: The following statement is false. Consider the following two matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then $AB - BA =$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then we see that $\det(AB - BA) = -1 \neq 0$.

22. (a) If every row of A adds to zero, prove that $\det A = 0$.

(b) If every row of A adds to 1, prove that $\det(A - I) = 0$. Show by example that this does not imply $\det A = 1$.

Solution: (a). Let $v = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ be the n-dimensional vector all of whose entries are 1. Then we compute Av. Note that Av is an n-dimensional vector, and the i-th entry of Av is the sum of the i-th row of the matrix A for $i = 1, \dots, n$. It follows from the assumption that the sum of elements in each row of A is 0 that we have $Av = 0$. As v is a nonzero vector, this equality implies that A is a singular matrix.

(b) If every row of a A adds to 1, then every row of $(A-I)$ adds to zero. Then by (a), $\det(A-I) = 0$. Consider the following martrix,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $\det(A - I) = 0$, but $\det(A) \neq 1$.

23. Find these 4 by 4 determinants by Gaussian elimination:

$$\det \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix} \quad \text{and} \quad \det \begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{pmatrix}.$$

Solution: To find the determinant of a 4×4 matrix using *Gaussian elimination*, we row-reduce the matrix to upper triangular form, keeping track of any row swaps or scaling that would affect the determinant. The determinant of an upper triangular matrix is the product of its diagonal elements.

Let

$$A = \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{pmatrix}$$

(a)

$$A = \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix} \xrightarrow[R_3:R_3-R_1]{R_2:R_2-R_1, R_4:R_4-R_1} \begin{pmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 20 & 20 & 20 & 20 \\ 30 & 30 & 30 & 30 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 20 & 20 & 20 & 20 \\ 30 & 30 & 30 & 30 \end{pmatrix} \xrightarrow[R_3:R_3-2R_2]{R_4:R_4-3R_2} \begin{pmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$$

Since, R has one zero row, then $\det R = 0$, Hence $\det A = 0$.

(b) Assume $t \neq +1, -1$, **Step 1: Use Row 1 to eliminate entries below the first pivot.**

$$R_2 \leftarrow R_2 - t \cdot R_1 = (0,\ 1-t^2,\ t-t^3,\ t^2-t^4)$$
$$R_3 \leftarrow R_3 - t^2 \cdot R_1 = (0,\ t-t^3,\ 1-t^4,\ t-t^5)$$
$$R_4 \leftarrow R_4 - t^3 \cdot R_1 = (0,\ t^2-t^4,\ t-t^5,\ 1-t^6)$$

New matrix:

$$\begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1-t^2 & t-t^3 & t^2-t^4 \\ 0 & t-t^3 & 1-t^4 & t-t^5 \\ 0 & t^2-t^4 & t-t^5 & 1-t^6 \end{pmatrix}$$

Step 2: Use Row 2 to eliminate below pivot in column 2.

Then:

$$R_3 \leftarrow R_3 - t \cdot R_2$$
$$R_4 \leftarrow R_4 - t^2 \cdot R_2$$

Continue this process — each step introduces another factor of $(1-t^2)$. Eventually, the matrix becomes upper triangular with diagonal entries:

$$1,\ (1-t^2),\ (1-t^2),\ (1-t^2)$$

Hence,

$$\boxed{\det B = 1 \cdot (1-t^2)^3 = (1-t^2)^3}$$

24. Given $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$, find the determinants of $A - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$, for every $\lambda \in \mathbb{R}$.

For which values of λ is $A - \lambda I$ a singular matrix (*not invertible*)?

Solution:

Given
$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix},$$
we compute the determinant of
$$A - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$$
for all $\lambda \in \mathbb{R}$.

Using the determinant formula for a 2×2 matrix:
$$\det(A - \lambda I) = (4-\lambda)(3-\lambda) - (1)(2) = \lambda^2 - 7\lambda + 10.$$

To find when $A - \lambda I$ is singular (not invertible), we solve:
$$\lambda^2 - 7\lambda + 10 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda - 5) = 0.$$

Final Answer:

- $\det(A - \lambda I) = \lambda^2 - 7\lambda + 10$.
- $A - \lambda I$ is singular when $\lambda = 2$ or $\lambda = 5$.

25. Let $A = (a_{ij}) \in M_n(\mathbb{R})$.

(a) For any $n \geq 3$, if $a_{ij} = i + j$, show that $\det A = 0$.

Solution: Replacing R_1 by $R_1 - R_2$ and R_2 by $R_2 - R_3$, we note that the each entry of the first two rows of the matrix becomes -1. So the first two rows are linearly dependent, and hence the determinant is 0.

(b) For any $n \geq 2$, if $a_{ij} = ij$, show that $\det A = 0$.

Solution: We note that in the given matrix, $R_2 = 2.R_1$. So R_1, R_2 are linearly dependent. Hence, the determinant of the matrix is 0.

26. If $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$, find A^{-1}.

27. Find the matrix A, if $A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ 1 & -2 & 3 \\ 3 & -3 & 4 \end{pmatrix}$.

Solution:

Let $B = A^{-1}$, therefore $A = B^{-1}$.

Compute Determinant $D = \det(B)$:

$$D = \begin{vmatrix} 3 & -1 & 1 \\ 1 & -2 & 3 \\ 3 & -3 & 4 \end{vmatrix} = 3\begin{vmatrix} -2 & 3 \\ -3 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 3 & -3 \end{vmatrix}$$

$$= 3(1) + 1(-5) + 1(3) = 1.$$

Compute Adjoint Adj(B):

$$\mathrm{Adj}(B) = \begin{pmatrix} \begin{vmatrix} -2 & 3 \\ -3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ -3 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 3 & -3 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 \\ 5 & 9 & -8 \\ 3 & 6 & -5 \end{pmatrix}$$

Now, $A = B^{-1} = \frac{\mathrm{Adj}(B)}{D} = \begin{pmatrix} 1 & 1 & -1 \\ 5 & 9 & -8 \\ 3 & 6 & -5 \end{pmatrix}$.

28. Solve the following system of equations by Cramer's rule:

$$\begin{aligned} x + 2y - 3z &= 1 \\ 2x - y + z &= 4 \\ x + 3y &= 5. \end{aligned}$$

Solution:

We write the system in matrix form $AX = B$, where

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 1 & 3 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

Compute Determinant $D = \det(A)$:

$$D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-3) - 2(-1) - 3(6+1) = -3 + 2 - 21 = -22.$$

Compute D_x, D_y, D_z:

For D_x:

$$D_x = \begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ 5 & 3 & 0 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} - 3\begin{vmatrix} 4 & -1 \\ 5 & 3 \end{vmatrix}$$

$$= -3 + 10 - 3(12+5) = -3 + 10 - 51 = -44.$$

For D_y:

$$D_y = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 4 & 1 \\ 1 & 5 & 0 \end{vmatrix} = 1\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix}$$

$$= -5 + 1 - 3(10-4) = -5 + 1 - 18 = -22.$$

For D_z:

$$D_z = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 1 & 3 & 5 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ 3 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}$$

$$= (-5-12) - 2(10-4) + (6+1) = -17 - 12 + 7 = -22.$$

Apply Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-44}{-22} = 2, \quad y = \frac{D_y}{D} = \frac{-22}{-22} = 1, \quad z = \frac{D_z}{D} = \frac{-22}{-22} = 1.$$

Final Answer:

$$\boxed{x = 2, \quad y = 1, \quad z = 1.}$$

29. Solve by Cramer's rule:

(i)
$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 14 \\ x - y + z &= 2, \end{aligned}$$

(ii)
$$\begin{aligned} x + y + z &= 1 \\ ax + by + cz &= 1 \\ a^2x + b^2y + c^2z &= 1, \quad a \neq b \neq c. \end{aligned}$$

Solution: (i)

We write the system in matrix form $AX = B$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 6 \\ 14 \\ 2 \end{pmatrix}.$$

Compute Determinant $D = \det(A)$:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 1(2+3) - 1(1-3) + 1(-1-2) = 5 + 2 - 3 = 4.$$

Compute D_x, D_y, D_z:

For D_x:

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 6\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 14 & 3 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 14 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= 6(2+3) - 1(14-6) + 1(-14-4) = 30 - 8 - 18 = 4.$$

For D_y:

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} 14 & 3 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 6 & 1 \\ 14 & 3 \end{vmatrix}$$

$$= 1(14-6) - 1(6-2) + 1(18-14) = 8 - 4 + 4 = 8.$$

For D_z:

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 14 \\ 1 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & 14 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 6 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 6 \\ 2 & 14 \end{vmatrix}$$

$$= 1(4+14) - 1(2+6) + 1(14-12) = 18 - 8 + 2 = 12.$$

Apply Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{4}{4} = 1, \quad y = \frac{D_y}{D} = \frac{8}{4} = 2, \quad z = \frac{D_z}{D} = \frac{12}{4} = 3.$$

Final Answer:

$$\boxed{x = 1, \quad y = 2, \quad z = 3.}$$

(ii)

We write the system in matrix form $AX = B$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Compute Determinant $D = \det(A)$:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 1\begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - 1\begin{vmatrix} a & c \\ a^2 & c^2 \end{vmatrix} + 1\begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$$

$$= 1(bc^2 - cb^2) - 1(ac^2 - ca^2) + 1(ab^2 - ba^2) = bc(c-b) - ac^2 + ab^2 + ca^2 - ba^2$$

$$= bc(c-b) + a(b^2-c^2) + a^2(c-b) = (c-b)(bc-ab-ac+a^2) = (c-b)(a-c)(a-b) = (a-b)(b-c)(c-a).$$

Compute D_x, D_y, D_z:

For D_x:

$$D_x = \begin{vmatrix} 1 & 1 & 1 \\ 1 & b & c \\ 1 & b^2 & c^2 \end{vmatrix} = 1\begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ b^2 & c^2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix}$$

$$= 1(bc^2 - cb^2) - 1(c^2 - b^2) + 1(c-b) = (c-b)(bc - c - b + 1) = (c-b)(c-1)(b-1).$$

For D_y:

$$D_y = \begin{vmatrix} 1 & 1 & 1 \\ a & 1 & c \\ a^2 & 1 & c^2 \end{vmatrix} = 1\begin{vmatrix} 1 & c \\ 1 & c^2 \end{vmatrix} - 1\begin{vmatrix} a & c \\ a^2 & c^2 \end{vmatrix} + 1\begin{vmatrix} a & 1 \\ a^2 & 1 \end{vmatrix}$$

$$= c^2 - c - ac^2 + ca^2 + a - a^2 = (a-c)(1-a)(1-c).$$

For D_z:

$$D_z = \begin{vmatrix} 1 & 1 & 1 \\ a & b & 1 \\ a^2 & b^2 & 1 \end{vmatrix} = 1\begin{vmatrix} b & 1 \\ b^2 & 1 \end{vmatrix} - 1\begin{vmatrix} a & 1 \\ a^2 & 1 \end{vmatrix} + 1\begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$$

$$= b - b^2 - a + a^2 + ab^2 - ba^2 = (b-a)(1-a)(1-b).$$

Apply Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{(c-b)(c-1)(b-1)}{(a-b)(b-c)(c-a)} = \frac{(b-1)(c-1)}{(b-a)(c-a)},$$

$$y = \frac{D_y}{D} = \frac{(a-c)(1-a)(1-c)}{(a-b)(b-c)(c-a)} = \frac{(a-1)(c-1)}{(a-b)(c-b)},$$

$$z = \frac{D_z}{D} = \frac{(b-a)(1-a)(1-b)}{(a-b)(b-c)(c-a)} = \frac{(a-1)(b-1)}{(a-c)(b-c)}.$$

Final Answer:

$$\boxed{x = \frac{(b-1)(c-1)}{(b-a)(c-a)}, \quad y = \frac{(a-1)(c-1)}{(a-b)(c-b)}, \quad z = \frac{(a-1)(b-1)}{(a-c)(b-c)}.}$$

Eigen-values and Eigen-vectors

30. Find the rank and all four eigenvalues for the checker board matrix $C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

Which eigenvectors correspond to nonzero eigenvalues?

Solution: We are given the checker board matrix :

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Our aim is the following :

1. Find the **rank** of C.

2. Compute the **eigenvalues** of C.

3. Determine the **eigenvectors** corresponding to the **nonzero eigenvalues** of C.

1. Rank of C : The rank of the matrix C is the dimension of its column space, which is the maximum number of linearly independent columns in C. Observe that the columns of C are :

$$\mathbf{C}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{C}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

It is easy to see that

$$\mathbf{C}_3 = \mathbf{C}_1 \quad \text{and} \quad \mathbf{C}_4 = \mathbf{C}_2.$$

Thus, the matrix has only two distinct columns $\mathbf{C}_1$ and $\mathbf{C}_2$. Furthermore, the columns $\mathbf{C}_1$ and $\mathbf{C}_2$ are linearly independent because no scalar multiple of $\mathbf{C}_1$ can produce $\mathbf{C}_2$, and vice versa. Since there are two linearly independent columns in C, we conclude that

$$\text{rank}\ (C) = 2.$$

2. Eigenvalues of C : We seek for all such λ's for which $\det(C - \lambda I) = 0$, where I is the 4×4 identity matrix. More precisely, we compute the determinant of the following matrix :

$$C - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{bmatrix}.$$

Expanding this determinant (using cofactor expansion along the first row), we simplify to get

$$\det(C - \lambda I) = \lambda^2(\lambda^2 - 4).$$

Observe that the roots of the polynomial above are $\lambda = 0, 0, 2, -2$. Thus, the eigenvalues of C are $0, 0, 2, -2$.

3 (a) Eigenvectors for $\lambda = 2$: To find the eigenvectors for $\lambda = 2$, we seek for all non-zero vectors $\mathbf{v} \in \mathbb{R}^4$ such that $C\mathbf{v} = 2\mathbf{v}$. More precisely, we solve :

$$(C - 2I)\mathbf{v} = 0,$$

where

$$C - 2I = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}.$$

Consider

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Then $(C - 2I)\mathbf{v} = 0$ yields

$$\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the system of equations :

$$\begin{aligned} -2v_1 + v_2 + v_4 &= 0, \\ v_1 - 2v_2 + v_3 &= 0, \\ v_2 - 2v_3 + v_4 &= 0, \\ v_1 + v_3 - 2v_4 &= 0. \end{aligned}$$

From the first equation :

$$v_2 = 2v_1 - v_4.$$

Substituting into the other equations and simplifying yields :

$$v_1 = v_2 = v_3 = v_4.$$

In conclusion, the set of all eigenvectors corresponding to the eigenvalue $\lambda = 2$ is given by

$$\left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \setminus \{0\} \right\}.$$

3 (b) Eigenvectors for $\lambda = -2$: For $\lambda = -2$, we solve

$$(C + 2I)\mathbf{v} = 0,$$

where

$$C + 2I = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

Set

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Then $(C + 2I)\mathbf{v} = 0$ yields

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the system of equations :

$$2v_1 + v_2 + v_4 = 0,$$
$$v_1 + 2v_2 + v_3 = 0,$$
$$v_2 + 2v_3 + v_4 = 0,$$
$$v_1 + v_3 + 2v_4 = 0.$$

From the first equation :

$$v_2 = -2v_1 - v_4.$$

Substituting into the other equations and simplifying yields :

$$v_1 = -v_2 = v_3 = -v_4.$$

In conclusion, the set of all eigenvectors corresponding to the eigenvalue $\lambda = -2$ is given by

$$\left\{ \beta \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} : \beta \in \mathbb{R} \setminus \{0\} \right\}.$$

This completes the proof.

31. (a) Construct 2 by 2 matrices A and B such that the eigenvalues of AB are not the products of the eigenvalues of A and B, and the eigenvalues of $A+B$ are not the sums of the individual eigenvalues.

(b) Construct 3 by 3 matrices A and B with same properties as in part (a).

Solution:

(a) Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. We now compute eigenvalues of A, B and $A+B$.

(i) **Eigenvalues of** A: Consider the equation $\det(A - \lambda I_2) = 0$. Then, we have

$$\det\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = \lambda^2 = 0.$$

Thus, the eigenvalues of A are 0 and 0.

(ii) **Eigenvalues of** B: To find eigenvalues of B, we need to consider $\det(B - \lambda I_2) = 0$. Then,

$$\det\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 = 0.$$

Hence, the eigenvalues of B are 0 and 0.

(iii) **Eigenvalues of** AB: We have, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Consider the characteristic equation $\det(AB - \lambda I_2) = 0$ of AB. On simplifying, we get

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) = 0.$$

Thus, the eigenvalues of AB are 0 and 1.

(iv) **Eigenvalues of** $A+B$: We have, $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. To determine eigenvalues of $A+B$, consider $\det((A+B) - \lambda I_2) = 0$. Then,

$$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = 0.$$

Therefore, the eigenvalues of $A+B$ are $1, -1$.

Conclusion:

- All the eigenvalues of A and B are 0. However, 1 is an eigenvalue of AB. Hence, eigenvalues of AB are not the product of the individual eigenvalues of A and B.

- The eigenvalues of $A+B$ are $1, -1$. It is clear that sum of the individual eigenvalues of A and B are not equal to the eigenvalues of $A+B$.

(b) Take $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Next, we compute eigenvalues of the matrices A, B and $A+B$.

(i) **Eigenvalues of** A: Consider $\det(A - \lambda I_3) = 0$. This implies that

$$\det\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda^3 = 0.$$

Thus, the eigenvalue of A are 0, 0 and 0.

(ii) **Eigenvalues of** B: We need to solve $\det(B - \lambda I_3) = 0$. We have,

$$\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1) = 0.$$

Thus, the eigenvalues of B are 0, 1 and -1.

(iii) **Eigenvalues of** AB: Note that

$$AB = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We need to find solutions of $\det(AB - \lambda I_3) = 0$. We have,

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = (1-\lambda)\lambda^2 = 0.$$

Thus, the eigenvalues of AB are 1, 0 and 0.

(iv) **Eigenvalues of** $A+B$: First, we compute the matrix $A+B$. We have,

$$A + B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Also,

$$
\begin{aligned}
\det\left(\begin{bmatrix}0 & 1 & 2\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) &= \det\begin{bmatrix}-\lambda & 1 & 2\\ 0 & -\lambda & 1\\ 1 & 0 & -\lambda\end{bmatrix}\\
&= -\lambda^3 + (1+2\lambda) \quad \text{(expanding along first column)}\\
&= -(\lambda+1)(\lambda^2-\lambda-1)\\
&= -(\lambda+1)\left(\lambda - \frac{\sqrt{5}+1}{2}\right)\left(\lambda + \frac{\sqrt{5}-1}{2}\right).
\end{aligned}
$$

Clearly, aforementioned equation has three solutions, namely, -1, $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Conclusion:

- All eigenvalues of A are zero. But, AB has one non-zero eigenvalue namely $\lambda = 1$, hence eigenvalues of AB are not the product of the individual eigenvalues of A and B.

- It is clear that the eigenvalues of $A+B$ are not the sum of the individual eigenvalues of A and B.

32. Prove that the real eigenvalues of A equal the eigenvalues of A^{T}.

Show by an example that the eigenvectors of A and A^{T} need not be the same.

Solution:

Part I: First we recall that for any matrix A, $\det A = \det A^T$. Thus for any $\lambda \in \mathbb{R}$, we have

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I).$$

Now, $\lambda \in \mathbb{R}$ is an eigenvalue of $A \iff \det(A - \lambda I) = 0 \iff \det(A^T - \lambda I) = 0 \iff \lambda \in \mathbb{R}$ is an eigenvalue of A^T.

Hence, the real eigenvalues of A are equal to the eigenvalues of its transpose.

Part II: Let A be the 2×2 matrix given by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

In this case we have $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

The only eigenvalue of A and A^T is $\lambda = 1$. (Note that these are triangular matrices, so the eigenvalues are precisely the diagonal entries.)

We now compute the eigenvector of A and A^T corresponding to the eigenvalue $\lambda = 1$.

$$A \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \iff \begin{bmatrix} x + y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \iff y = 0.$$

Thus the eigenvectors of A corresponding to eigenvalue $\lambda = 1$ are given by $\begin{bmatrix} x \\ 0 \end{bmatrix}$ for any $x \neq 0$.

Similarly, one can verify that the eigenvectors of A^T corresponding to eigenvalue $\lambda = 1$ are given by $\begin{bmatrix} 0 \\ y \end{bmatrix}$ for any $y \neq 0$.

This clearly shows that A and A^T have no eigenvectors in common.

33. Every permutation matrix leaves $x = (1, 1, \ldots, 1)$ unchanged. Thus, $\lambda = 1$ is an eigenvalue for every permutation matrix. Find two more eigenvalues for the following permutations:

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution:

- Let P be an arbitrary n by n permutation matrix with columns $C_1, C_2, \ldots, C_n \in \mathbb{R}^n$.

  By the very definition of a permutation matrix, we know that every row of the matrix P has exactly one entry 1 and all other entries 0. Thus, we have

  $$P \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = P(e_1 + e_2 + \cdots + e_n) = Pe_1 + Pe_2 + \cdots + Pe_n = C_1 + C_2 + \cdots + C_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

  where $e_1, e_2, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.

  This also shows that 1 is an eigenvalue for the permutation matrix P.

- Given that $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, the characteristic polynomial of P_1 is

  $$\chi_{P_1}(\lambda) = \det(P_1 - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} = -(\lambda^3 - 1) = -(\lambda - 1)(\lambda^2 + \lambda + 1).$$

  Note that the equation $\chi_{P_1}(\lambda) = -(\lambda - 1)(\lambda^2 + \lambda + 1)$ has only one solution over the real numbers, namely $\lambda = 1$ (because the equation $\lambda^2 + \lambda + 1 = 0$ has no real solution).

  So, P_1 has **no real eigenvalue** other than 1.

- Given that $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, the characteristic polynomial of P_2 is

  $$\chi_{P_2}(\lambda) = \det(P_2 - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1 - \lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = -(\lambda - 1)^2(\lambda + 1).$$

  Clearly, the three eigenvalues of P_2 are $1, 1, -1$.

34. Find the 2 by 2 matrix A whose eigenvalues are 1 and 4, with the corresponding eigenvectors being $(3,1)$ and $(2,1)$ respectively.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the matrix whose eigenvalues are 1 and 4, with the corresponding eigenvectors being $(3,1)$ and $(2,1)$ respectively.

Then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

From the above system, we get

$$3a + b = 3,$$
$$3c + d = 1,$$

and

$$2a + b = 8,$$
$$2c + d = 4.$$

Solving the above system of equations, we get $a = -5, b = 18, c = -3$ and $d = 10$. Thus, the required matrix is

$$A = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}.$$

35. Which of these matrices cannot be diagonalized?

$$A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}.$$

Solution: Recall that

- $A \in M_n(\mathbb{R}^n)$ is diagonalizable if and only if there exists a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ such that each $v_j \in \mathcal{B}$ is an eigenvector of A.
- A non-zero vector v is an eigen vector corresponding to the eigen value λ of A if and only if v is in the null space of $A - \lambda I$. So, nullity of $A - \lambda I$ is equal to the geometric multiplicity of λ.

(a) Given that $A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$, the characteristic polynomial of A_1 is

$$\chi_{A_1}(\lambda) = \det(A_1 - \lambda I) = \det \begin{bmatrix} 2-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = (2-\lambda)(-2-\lambda) + 4 = \lambda^2.$$

So, 0 is the only eigenvalue of A_1, and the geometric multiplicity of $\lambda = 0$ is the nullity of A_1.

Observe that the two rows of A_1 are dependent which gives that rank of A_1 is 1.

So, by rank-nullity theorem, nullity of A_1 is 1.

Therefore, A_1 has only one independent eigenvector. In fact, the 0-eigenspace is the line $y = x$.

Hence, A_1 is **not diagonalizable**.

(b) Given that $A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$, the characteristic polynomial of A_2 is

$$\chi_{A_2}(\lambda) = \det(A_2 - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 \\ 2 & -2-\lambda \end{bmatrix} = (2-\lambda)(-2-\lambda).$$

So, A_2 has two distinct eigenvalues, namely, 2 and -2.

Furthermore, observe that

$$\begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Clearly, A_2 has two independent eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Hence, A_2 is a **diagonalizable** matrix.

(c) Given that $A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$, the characteristic polynomial of A_3 is

$$\chi_{A_3}(\lambda) = \det(A_3 - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 \\ 2 & 2-\lambda \end{bmatrix} = (2-\lambda)^2.$$

So, 2 is the only eigenvalue of A_3, and

$$A_3 - 2I = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

As in the case of A_1, observe that nullity of $A_3 - 2I$ is 1, that is, A_3 has only one independent eigenvector.

Hence, A_3 is **not diagonalizable**.

36. Describe all matrices S that diagonalize the matrix $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$.

Then describe all matrices that diagonalize A^{-1}.

Solution: We are given that $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$.

Being triangular, we know that the eigenvalues of A are its diagonal entries, that is, $4, 2$.

Since the eigen values of the matrix A are distinct, we get that A is diagonalizable. (The matrix has two independent eigenvectors which form the eigen basis.)

Now, the eigen space of A associated to the eigen value 2 is $N(A-2I)$, which is given by the solutions of

$$\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $N(A-2I) = \left\{ c \begin{bmatrix} 0 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$.

Proceeding similarly, we can show that the eigen space of A associated to the eigen value 4 is $N(A-4I) = \left\{ d \begin{bmatrix} 2 \\ 1 \end{bmatrix} : d \in \mathbb{R} \right\}$.

Hence the set of matrices that diagonalize A is

$$\left\{ \begin{bmatrix} 0 & 2d \\ c & d \end{bmatrix} : c, d \in \mathbb{R} \setminus \{0\} \right\} \cup \left\{ \begin{bmatrix} 2d & 0 \\ d & c \end{bmatrix} : c, d \in \mathbb{R} \setminus \{0\} \right\}.$$

It follows from a general fact that the set of matrices that diagonalize A^{-1} is the same set whose members diagonalize A.

General fact: Let $A \in M_n(\mathbb{R})$ be invertible. Eigen values of A^{-1} are exactly the reciprocal of the eigen values of A. In fact, we have more knowledge, namely, $N(A-\lambda I) = N(A^{-1} - \lambda^{-1} I)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$.

To see this, note that

$$Ax = \lambda x \iff A^{-1}Ax = \lambda A^{-1}x \iff \lambda^{-1}x = A^{-1}x.$$

37. If the eigenvalues of a 3 by 3 matrix A are $1, 1, 2$, which of the following are certain to be true? Give a reason if true or a counterexample if false:

(a) A is invertible.

(b) A is diagonalizable.

(c) A is not diagonalizable.

Solution:

(a) TRUE.

Note that A is non-invertible $\iff$ $\det(A) = 0$ $\iff$ 0 is an eigenvalue of A.

(b) FALSE.

Take, for example, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Since, A is a triangular matrix, we know that its eigenvalues are exactly the diagonal entries, that is, $1, 1, 2$.

We claim that A is not diagonalizable.

Recall that A is diagonalizable $\iff$ it has an eigen-basis. In other words for being diagonalizable, A must have three independent eigenvectors.

But, one can easily check that nullity of $A - I$ is 1 and nullity of $A - 2I$ is 1. That is, we only have at the most two independent eigenvectors of A.

(c) FALSE.

Take, for example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

The above matrix is a diagonal matrix (and hence it is automatically diagonalizable), with eigenvalues $1, 1, 2$.

38. Suppose the only eigenvectors of a 3 by 3 matrix A are multiples of $(1, 0, 0)$. True or false:

(a) A is not invertible.

(b) A has a repeated eigenvalue.

(c) A is not diagonalizable.

Solution:

(a) FALSE.

Counterexample: Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

This matrix has eigenvalue 1 with algebraic multiplicity 3 and it can be easily verified that the only eigenvectors of A are non-zero multiples of $v = (1, 0, 0)$.

Since $\det(A) = 1 \neq 0$, the matrix A is invertible.

(b) TRUE.

Suppose to the contrary that A has at least two distinct eigenvalues, say λ and μ.

Then, there exists eigenvectors v_1 (corresponding to λ-eigenvalue) and v_2 (corresponding to μ-eigenvalue).

But then we must have that v_1, v_2 are linearly independent.

This is contradicting to the given assumption that the only eigenvectors of A are non-zero multiples of a single vector $(1, 0, 0)$.

(c) TRUE.

If A is diagonalizable, then thee exists a basis $\mathcal{B}$ of $\mathcal{R}^3$ with each member of $\mathcal{B}$ being an eigenvector of A (see Lectures-28 & 29).

But, we are given that the only eigenvectors of A are multiples of $(1, 0, 0)$, and hence we conclude that A is not diagonalizable.

39. Diagonalize $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, that is, find a matrix S such that $S^{-1}AS = D$ is a diagonal matrix.

Use it to prove that for any $k \in \mathbb{N}$, we have that $A^k = \frac{1}{2} \begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}$.

Solution: To diagonalize $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, we start by finding its eigenvalues and eigenvectors.

The eigenvalues λ of A satisfy the characteristic polynomial $\det(A - \lambda I) = 0$.

Now,

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

Solving $\det(A - \lambda I) = 0$, we get:

$$\lambda = 3 \quad \text{and} \quad \lambda = 1.$$

Eigenvectors for $\lambda = 3$:

Substituting $\lambda = 3$ into $A - \lambda I$, we get $A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

Note that the both the rows of $A - 3I$ are non-zero, but they multiples of each other, so rank of $A - 3I$ is 1, and therefore the nullity of $A - 3I$ is also 1.

It is easy to check that the vector $v_1 = (1, 1)$ forms a basis of $N(A - 3I)$.

Eigenvectors for $\lambda = 1$:

Substituting $\lambda = 1$ into $A - \lambda I$, we get $A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Note that the both the rows of $A - I$ are non-zero, but they equal to each other, so rank of $A - I$ is 1, and therefore the nullity of $A - I$ is also 1.

It is easy to check that the vector $v_2 = (1, -1)$ forms a basis of $N(A - I)$.

Let S be the matrix with columns v_1 and v_2, that is, $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Then $S^{-1}AS = D$, where D is the diagonal matrix of eigenvalues, $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

Since $A = SDS^{-1}$, we have $A^k = SD^kS^{-1}$.

Now, $D^k = \begin{bmatrix} 3^k & 0 \\ 0 & 1^k \end{bmatrix} = \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix}$, and thus

$$\begin{aligned} A^k &= SD^kS^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 3^k & 1 \\ 3^k & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}. \end{aligned}$$

40. Given $A \in M_n(\mathbb{R})$, explain why A is never similar to $A + I$.

Solution:

- For any $A_1, A_2 \in M_n(\mathbb{R})$, we have
$$\operatorname{trace}(A_1\, A_2) = \operatorname{trace}(A_2\, A_1).$$

- Let $A, B \in M_n(\mathbb{R})$ be similar matrices, then there exists an invertible matrix $C \in M_n(\mathbb{R})$ such that
$$B = C\, A\, C^{-1}.$$
Consequently,
$$\begin{aligned}\operatorname{trace}(B) &= \operatorname{trace}(C\, A\, C^{-1})\\ &= \operatorname{trace}(A\, C^{-1}\, C) \quad \text{(take } A_1 = C \text{ and } A_2 = A\, C^{-1}\text{)}\\ &= \operatorname{trace}(A).\end{aligned}$$

As shown above, similar matrices have equal trace.

Now,
$$\operatorname{trace}(A + I) = n + \operatorname{trace}(A),$$
which proves that A is never similar to $A + I$.

41. Let $A, B \in M_n(\mathbb{R})$. If A or B is invertible then show that BA is similar to AB. Give an example of matrices A and B such BA is not similar to AB.

Solution:

(a) Let $A, B \in M_n(\mathbb{R})$, and assume that A is invertible.

We can write

$$\begin{aligned} BA &= I(BA) \\ &= (A^{-1}A)(BA) \\ &= A^{-1}(AB)A, \end{aligned}$$

which proves that BA is similar to AB.

In a similar manner as above, we can show that if B is invertible then BA is similar to AB.

(b) Consider

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Clearly, both A and B are singular.

Now,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \text{and} \quad BA &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Recall that zero matrix is similar to only itself. This is because, for any invertible matrix C, we have $C\,0\,C^{-1} = 0$).

Since in our example, BA is a non-zero matrix whereas AB is the zero matrix, we get that BA is not similar to AB.

42. Let $A, B \in M_n(\mathbb{R})$. Prove that AB has the same eigenvalues as BA.

Solution: Let $\lambda \in \mathbb{R}$. We shall show that

$$\lambda \text{ is an eigenvalue of } AB \iff \lambda \text{ is an eigenvalue of } BA.$$

We shall prove the claim in two parts, one for $\lambda = 0$ and the other for $\lambda \neq 0$.

(a) **Case 1:** $\lambda = 0$.

Recall that 0 is an eigenvalue of a matrix if and only if the determinant of the matrix is 0.

Also, we know that $\det(AB) = \det(A)\det(B) = \det(BA)$.

Thus, we have

$$\begin{aligned} 0 \text{ is an eigenvalue of } AB &\iff \det(AB) = 0 \\ &\iff \det(BA) = 0 \\ &\iff 0 \text{ is an eigenvalue of } BA. \end{aligned}$$

(b) **Case 2:** $\lambda \neq 0$.

Our claim will be proved if we can show that

$$\lambda I - AB \text{ is invertible } \iff \lambda I - BA \text{ is invertible }.$$

- Let us write $A_1 = \lambda^{-1}A$. Then,

$$\det(\lambda I - AB) = \det\left(\lambda\left(I - A_1 B\right)\right) = \lambda^n \det\left(I - A_1 B\right),$$

and since $\lambda \neq 0$, we get that

$$\lambda I - AB \text{ is invertible } \iff I - A_1 B \text{ is invertible }.$$

- Similarly, we can show that

$$\lambda I - BA \text{ is invertible } \iff I - BA_1 \text{ is invertible }.$$

- Finally, we recall from Problem-07 of Assignment-04 that for any $B_1, B_2 \in M_n(\mathbb{R})$, we have

$$I - B_1 B_2 \text{ is invertible } \iff I - B_2 B_1 \text{ is invertible }.$$

Combining the above facts, we can conclude that

$$\lambda I - AB \text{ is invertible } \iff \lambda I - BA \text{ is invertible }.$$