Lecture Feb 20, 2025

Solve

$$x_1 + 2x_2 + 3x_3 + 5x_4 = b_1$$
$$2x_1 + 4x_2 + 8x_3 + 12x_4 = b_2$$
$$3x_1 + 6x_2 + 7x_3 + 13x_4 = b_3$$

not all zero

for some appropriate $b_1, b_2, b_3 \in \mathbb{R}$, such that the system is solvable (non-singular).

$$Ax = b$$

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}.$$

Idea:

STEP I: Solve $Ax = 0$.

Describe $N(A)$. $\dim N(A)$?

STEP II: Find $b_1, b_2, b_3 \in \mathbb{R}$ such that system is solvable.

STEP III: Choose one particular such b and get a particular solution x_0

STEP IV: $\{x : Ax = b\}$

$$= \{x_0 + x : x \in N(A)\}.$$

STEP I :

$A \longrightarrow$ Echelon $\longrightarrow$ row-reduced echelon

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$$

$$\Big\downarrow \; R_2 - 2R_1 \;/\; R_3 - 3R_1$$

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

PIVOT
1st row

$$\Big\downarrow \; R_3 + R_2$$

$k_1 = 1$

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{— Echelon matrix.}$$

Pivot

PIVOT.
$k_2 = 3$

$$\Big\downarrow \; \tfrac{1}{2} R_2$$

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \; R_1 - 3R_2$$

PIVOT VARIBLES

Free variables

PIVOTS

$$R = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Row reduced echelon matrix.

$$N(A) = \{x : Ax = 0\} = \{x : Rx = 0\},$$

$$= \left\{ x : \begin{array}{l} x_1 + 2x_2 + 2x_4 = 0 \\ x_3 + x_4 = 0 \end{array} \right\}.$$

Let $x_2 = \alpha$

$x_4 = \beta$

$x_1 = -2\alpha - 2\beta$

$x_3 = -\beta.$

So $N(A)$

$$= \{ (-2\alpha - 2\beta, \alpha, -\beta, \beta) : \alpha, \beta \in \mathbb{R} \}.$$

You can write as columns as we identified them.

$$\begin{pmatrix} -2\alpha - 2\beta \\ \alpha \\ -\beta \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

sol^n when $\alpha = 1$, $\beta = 0$

$x_1 = -2$, $x_3 = 0$.

sol^n when $\alpha = 0$, $\beta = 1$

$x_1 = -2$, $x_3 = -1$.

1 sitting at different position makes them linearly independent.

— We get here 2 LI - vectors corresponding to 2 free variables

So $\dim N(A)$ = no. of free variables.

$\leq n$

(as dim^n subspace $\leq \dim \mathbb{R}^n = n$

as $\mathbb{R}^n$ – can't have a subset containing more than n elts.).

Here $\dim N(A) = 2$.

This is what we said in yesterday's class:

if the number of pivot is r, then the number of free variables is $n-r$

& $\dim N(A) = n - r$.

<u>Coming back to the soln :</u>

We have completed <u>STEP I</u>.

<u>STEP II</u>

Now to find a particular b :

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array}\right)$$

$$\xrightarrow{R_3 + R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right).$$

So this equation have solution only when $b_3 + b_2 - 5b_1 = 0$.

let us take $b_1 = 0$, $b_2 = 2$, $b_3 = -2$.

so that not all b_1, b_2, b_3 are 0.

STEP III : Find solution of $Ax = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$.

$$\xrightarrow{\frac{1}{2} R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2}(b_2 - 2b_1) = 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Pivot variables.

free variables

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array}$$

$$\xrightarrow{R_1 - 3R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + 2x_2 + 2x_4 = -3$$

$$x_3 + x_4 = 1$$

Take $x_2 = 0 = x_4$ (free variables).

$x_1 = -3 \quad x_3 = 1$.

So a particular solⁿ is $(-3, 0, 1, 0)$, of the eqⁿ.

$$Ax = b \quad \text{where} \quad b = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

STEP IV: All the solutions of the equation $Ax = b$ is $\{ x_0 + y : x_0$ is a particular soln. $Ax = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ and $y \in N(A) \}$.

All ..s are

$$\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Remark:

1. Note if b changes the particular soln changes, and adding it to the null space of A gives all the solution of $Ax = b$.

2. If A is row equivalent to R, by defn. we have that $\exists$ finitely many row operations $E_1, E_2 \ldots E_k$ with $E = E_k \ldots E_1$ such that $EA = R$.

So $Ax = b \iff EAx = Eb$

$\iff Rx = Eb$

Thus $\{x : Ax = b\} = \{x : Rx = Eb\}$.

In the example above, we changed $Ax = b$ to $Rx = Eb$ & then solve

So finding a particular soln of $Ax = b$ is same a finding a particular solution of $Rx = Eb$.