Solution Set - 06

MA 1201 Spring Sem, 2025

1. Prove that $\mathbb{Z}_2 = \{0, 1\}$ with addition ('+') defined by

$$0 + 0 = 0, 0 + 1 = 1 + 0 = 1, 1 + 1 = 0$$

and multiplication('.') defined by

$$0.0 = 0, 0.1 = 1.0 = 0, 1.1 = 1$$

is a field.

2. Let $V = \{\theta\}$ and $\theta + \theta = \theta$ and $\alpha.\theta = \theta$ for all $\alpha \in \mathbb{F}$. Prove that V is a vector space over $\mathbb{F}$. (V is called the **zero vector space**)

3. Determine whether the following statements are true or false by giving justifications or counter-examples. Assume usual addition and scalar multiplication unless otherwise stated.

(a) Any non-zero vector space over $\mathbb{F} = \mathbb{R}$ has infinitely many distinct elements.

(b) The set $\mathbb{Q}$ of rational numbers is a vector space over $\mathbb{R}$.

(c) The set $\mathbb{R}_{\geq 0}$ of non-negative real numbers is a vector space over $\mathbb{R}$.

(d) The set $\mathbb{C}$ of complex numbers is a vector space over $\mathbb{R}$.

(e) The set $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - 2x_3 = 0\}$ is a vector space $\mathbb{R}$.

(f) The set $V = \{f : \mathbb{R} \to \mathbb{R} : f(t) = f(-t), \forall t \in \mathbb{R}\}$ (that is, the set of **even functions**) is a vector space $\mathbb{R}$.

(g) The set $V = \{f : \mathbb{R} \to \mathbb{R} : f(t) = -f(-t), \forall t \in \mathbb{R}\}$ (that is, the set of **odd functions**) is a vector space $\mathbb{R}$.

(h) The set $\mathbb{R}^2$ with usual addition and new scalar multiplication defined by

$$\alpha(x_1, x_2) = \begin{cases} (0, 0), & \text{if } \alpha = 0; \\ (\alpha x_1, \frac{x_2}{\alpha}), & \text{if } \alpha \neq 0, \end{cases}$$

is a vector space over $\mathbb{R}$.

(i) The set $\mathbb{R}^3$ with usual addition and new scalar multiplication defined by

$$\alpha(x_1, x_2, x_3) = (\alpha x_1, x_2, x_3)$$

is a vector space over $\mathbb{R}$.

(j) The set $\mathbb{C}$ with usual addition and new scalar multiplication defined by $\alpha.x = \alpha^2 x$ is a vector space over $\mathbb{C}$.

(k) The set $\mathbb{C}$ with usual addition and new scalar multiplication defined by $\alpha.x = (\text{Re } \alpha)x$ is a vector space over $\mathbb{C}$.

(l) The set $\mathbb{F}^2$ with usual addition and new scalar multiplication defined by $\alpha.(\beta,\gamma) = (\alpha\beta, 0)$ is a vector space over $\mathbb{F}$.

(m) Let $V = \mathbb{R}_+ :=$ Set of all positive real numbers, with addition defined by

$$x + y := xy$$

and scalar multiplication defined by

$$\alpha.x := x^{\alpha}$$

is a vector space over $\mathbb{R}$.

(n) Each of the sets $P := \{(x,y) \in \mathbb{R}^2 : y^2 = 4ax, a > 0\}, E := \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a, b > 0\}, H := \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a, b > 0\}$ is a vector space over $\mathbb{R}$.

4. Prove that if $\mathbb{G}$ is a subfield of $\mathbb{F}$ (subset of $\mathbb{F}$ which is a field itself with respect to the addition and multiplication in $\mathbb{F}$), then $\mathbb{F}$ is a vector space over $\mathbb{G}$.

5. Prove that $-(-x) = x$ for every $x \in V$.

6. Prove that if $\alpha \in \mathbb{F}$ and $x \in V$ such that $\alpha x = \theta$, then either $\alpha = 0$ or $x = \theta$. This shows that the singleton set $\{x\}$, for $x \neq \theta$, is lineraly indenpendent.

7. Give examples of nonempty subset S of $\mathbb{R}^2$ such that

   (a) S is closed under addition and under taking additive inverses but not a subspace of $\mathbb{R}^2$.

   (b) S is closed under scalar multiplication but not a subspace of $\mathbb{R}^2$.

   This shows you are required to check closeness under both addition and scalar multiplication to check whether certain subset is a subspace.

8. Determine which of the following subsets S of the vector space V over $F = \mathbb{R}$ are subspaces.

   (a) $S = \{(x_1, x_2, x_3) : x_1 = x_2,\ x_3 = 2x_1\}$, $V = \mathbb{R}^3$.

   (b) $S = \{(x_1, x_2, x_3) : x_1 = 0\}$, $V = \mathbb{R}^3$.

   (c) $S = \{(x_1, x_2, x_3) : x_1 = 1\}$, $V = \mathbb{R}^3$.

   (d) $S = \{(x_1, x_2, x_3) : x_2 x_3 = 0\}$, $V = \mathbb{R}^3$.

   (e) $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 0\}$, $V = \mathbb{R}^3$.

   (f) $S = \{(x_1, x_2, x_3) : 3x_1 - x_2 + x_3 = 0\}$, $V = \mathbb{R}^3$.

   (g) $S = \{(x_1, x_2, x_3) : x_1 + x_2 = 0\}$, $V = \mathbb{C}^3$.

   (h) $S = \{(x_1, x_2, x_3) : x_1 + x_2 \geq 0\}$, $V = \mathbb{C}^3$.

   (i) $S =$ the set of all polynomials whose constant term is zero, $V = \mathcal{P}(\mathbb{R})$.

   (j) $S =$ the set of all polynomials whose degree is equal to 2, $V = \mathcal{P}(\mathbb{R})$.

   (k) $S =$ the set of all polynomials $f(x)$ such that $f'(1) = 0$, $V = \mathcal{P}(\mathbb{R})$.

   (l) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.

   (m) $S = \{A \in M_{n\times n}(\mathbb{R}) : A^T = 2A\}$, $V = M_{n\times n}(\mathbb{R})$.

9. $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a subspace of the vector space $\mathbb{R}$ over $\mathbb{Q}$. (You can construct many such non-trivial subspaces of $\mathbb{R}$ over $\mathbb{Q}$)

10. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

    (For further thinking) What about union of more than two subspaces?

11. Describe the column space and the nullspace of the following matrices:

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

12. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form

   (a) a plane;
   (b) a line;
   (c) a point;
   (d) a subspace;
   (e) the nullspace of A;
   (f) the column space of A.

13. Write an example of a 2 by 2 system $Ax = b$ with many solutions for $Ax = 0$ but no solution $Ax = b$. (Therefore the system has no solution.)

14. The columns of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) the column space of A. Give an example where the column spaces of A and AB are not equal.

15. True or false (with a counterexample if false)?

   (a) The vectors b that are not in the column space $C(A)$ form a subspace.
   (b) If $C(A)$ contains only the zero vector, then A is the zero matrix.
   (c) The column space of $2A$ equals the column space of A.
   (d) The column space of $A - I$ equals the column space of A.

16. Prove that if $a = 0, d = 0$, or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

    If a, d, f are all nonzero, show that the only solution to $Ux = 0$ is $x = 0$. Then U has independent columns.

17. Prove that columns of an upper triangular matrix are linearly independent if and only if all of its diagonal entries (PIVOTS) are non-zero.

18. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1 v_1 + \cdots + c_4 v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

19. (a) Under what conditions on the scalar ξ are the vectors $(\xi, 1, 0), (1, \xi, 1)$ and $(0, 1, \xi)$ in $\mathbb{R}^3/\mathbb{R}$ are linearly dependent?

    (b) What is the answer to (a) for $\mathbb{Q}^3/\mathbb{Q}$ (in place of $\mathbb{R}^3/\mathbb{R}$)?

20. Find all possible values for a for which the vector $(3, 3, a)$ is in the span of the vectors $(1, -1, 1)$ and $(1, 2, -3)$.

21. If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are dependent. Find a combination of the v's that gives zero.

22. If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are independent. [Hint: $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ in terms of the w's. Find and solve equations for the c's.]

23. Find a basis for the column space (in $\mathbb{R}^2$) and nullspace (in $\mathbb{R}^5$) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

24. Find a basis for the subspace $W = \{(x_1, x_2, x_3, x_4) : x_1 - 3x_2 + x_3 = 0\}$ of $\mathbb{R}^4$.

25. Let $V = M_n(\mathbb{R})$, the vector space of all $n \times n$ real matrices and S denote the subset of V of all symmetric matrices, that is, $S = \{A \in M_n(\mathbb{R}) : A^T = A\}$.

    (a) Prove that S is a subspace of V.

    (b) Find a basis for V and S.

    Answer the same set of questions when S denote the subset of all skew symmetric matrices ($A^T = -A$).