Lecture 02.04.2025

Let $V/_{\mathbb{F}}$ - finite dimensional vector space.

$$T : V \xrightarrow[\text{linear}]{} V$$

Def^n:

A scalar $\lambda \in \mathbb{F}$ is said to be an eigen-value if $\exists$ a non-zero vector $v \in V$ such that

$$Tv = \lambda v.$$

Such a non-zero vector v is called an eigen-vector of T.

Trivial examples:

'0' map has every nonzero vector as eigen-vector with 0 as eigen-value.

'Id'-map has every non-zero vector as eigen vector with 1 as eigen value.

Remark: Knowing eigen values helps us working with simpler matrices.

Suppose there exists a basis $\mathcal{B}$ of V such that all the vectors in $\mathcal{B}$ are eigenvectors of T, that is if $\mathcal{B} = \{v_1, \dots v_n\}$

then $\exists\ \lambda_i$ s.t $Tv_i = \lambda_i v_i$,

Then $[T]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$

& Conversely if $[T]^{\mathcal{B}}_{\mathcal{B}}$ is diagonal matrix, then every vector in $\mathcal{B}$ is an eigen vector. (CHECK!!)

(follows from the observations below)

<u>Some observations</u>:

$T : V \to V$ linear, λ e.v & v - corr. e.vector

$$(T - \lambda I)v = 0$$

$$\Leftrightarrow [(T - \lambda I)v]_{\mathcal{B}} = 0 \quad / \text{ where } \mathcal{B} \text{ - basis of } V$$

$$\Leftrightarrow [T - \lambda I]^{\mathcal{B}}_{\mathcal{B}} [v]_{\mathcal{B}} = 0. \quad — (*)_1$$

<u>Check</u>: $[T - \lambda I]^{\mathcal{B}}_{\mathcal{B}} = [T]^{B}_{B} - \lambda [Id]^{B}_{B}$

$= A - \lambda I_n$ where $[T]^{\mathcal{B}}_{B} = A$ ↳ $n \times n$ <u>square matrix</u> <u>where $\dim V = n$.</u>

Follows from the fact that If $T, S : V \to W$ linear maps then $[T + S]^{\mathcal{B}}_{\mathcal{B}'} = [T]^{\mathcal{B}}_{\mathcal{B}'} + [S]^{\mathcal{B}}_{\mathcal{B}'}$

$$\Leftrightarrow (A - \lambda I_n)[v]_{\mathcal{B}} = 0. \quad — (*)_2$$

<u>Def^n</u>: $\text{Ker } T = \{v \in V : Tv = 0\}$.

<u>Fact</u>: T is 1-1 $\Leftrightarrow \text{Ker } T = \{0\}$.

Pf: ($\Rightarrow$) Suppose $x \in \ker T \Rightarrow Tx = 0 = T0$

$\Rightarrow x = 0$ as T is 1–1

$\Rightarrow \ker T \subseteq \{0\}$

& $\{0\} \subseteq \ker T \Rightarrow \ker T = \{0\}$

($\Leftarrow$) If $\ker T = \{0\}$, consider x_1, x_2

s.t $Tx_1 = Tx_2 \Rightarrow T(x_1 - x_2) = 0$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

& hence T is 1–1.

Now similar to our observations earlier we have

$v \in \ker T \Leftrightarrow Tv = 0 \Leftrightarrow [Tv]_{\mathcal{B}} = 0$

$\Leftrightarrow [T]_{\mathcal{B}}^{\mathcal{B}} [v]_{\mathcal{B}} = 0$

$\Leftrightarrow A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ where $[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in N(A)$

$\Leftrightarrow [v]_{\mathcal{B}} \in N(A)$

So we have

$v \in \ker T \Leftrightarrow [v]_{\mathcal{B}} \in N(A)$ where $[T]_{\mathcal{B}}^{\mathcal{B}} = A$.

Consider the isomorphism

$$\left.\begin{array}{rcl} V & \longrightarrow & \mathbb{F}^n \\ v & \longmapsto & [v]_{\mathcal{B}} \end{array}\right\} \text{ Say } S.$$

Since $\ker T$ is a subspace of V

Then S restricted to $\ker T$, that is

$$S\big|_{\ker T} : \ker T \longrightarrow N(A) \qquad \text{(Check!)}$$

is an isomorphism

i.e $\quad \ker T \cong N(A)$

Exc: If V, W f.d spaces such that V & W are isomorphic, then $\dim V = \dim W$.

Hint: $\{v_1 \dots v_n\}$ basis of V & $T: V \xrightarrow[\text{iso}]{} W$

(hence 1-1)

then $\{Tv_1, \dots, Tv_n\}$ is LI in W.

$$\text{Image}(T) = \text{Range}(T)$$
$$= \text{span}\{Tv_1 \dots Tv_n\}$$

Since T is onto, $\text{Range}(T) = W$

$$\Rightarrow W = \text{span}\{Tv_1, \dots Tv_n\}$$

Hence $\dim W = n = \dim V$.

So T is 1-1 $\iff \ker T = \{0\}$

$\iff N(A) = \{0\}$

$\iff Ax = 0$ has only zero solⁿ.

$\iff A$ is invertible (as A is a square matrix)

$\iff T$ invertible.

So an linear map from a space to itself is 1-1 iff T is invertible.

(So onto-ness is automatic if T is 1-1).

Remark: The same thing happens with range T

it is a subspace

& $\operatorname{ran}(T) \cong C(A)$

& $\dim \operatorname{ran}(T) = \dim C(A) = \operatorname{rank} A$.

& hence the rank-nullity for T will give

$\dim \ker T + \dim \operatorname{ran} T = \dim V$.

Note this gives the onto $\iff$ T invertible.

However, we would not elaborate on the proof as you will do it in Lin. Alg course next semester.

So continuing (infact repeating) with $\circledast_1$ & $\circledast_2$ we have

$$Tv = \lambda v \iff (T - \lambda I) v = 0$$

$$\iff [T - \lambda I]_{\mathcal{B}}^{\mathcal{B}} [v]_{\mathcal{B}} = 0$$

Any one of this also referred as defn. of eigenvalue of the matrix A

$$\iff (A - \lambda I_n) [v]_{\mathcal{B}} = 0$$

$$\iff [v]_{\mathcal{B}} \in N(A - \lambda I_n)$$

$$\iff (A - \lambda I_n) \text{ is not invertible}$$

$$\Updownarrow$$

here DETERMINANT makes entry.

$$\det(A - \lambda I_n) = 0.$$

we will see ↳ this will be an n-degree polynomial eqn. in λ.

This gives a way to calculate the eigen values of T,

otherwise, impossibly difficult to try out all $\lambda \in \mathbb{F}$ in such that

$$(A - \lambda I_n) x = 0$$

will have only 0 solution!

<u>Remark</u>: One might ask A is the matrix w.r.t a particular basis what happens w.r.t other basis?

note if $[T]_{\mathcal{B}'}^{\mathcal{B}'} = B$ for some other basis $\mathcal{B}'$ of V

then $B = PAP^{-1}$ for some invertible matrix

So $B - \lambda I_n = P(A - \lambda I_n)P^{-1}$

Thus $B - \lambda I_n$ invertible

$\iff A - \lambda I_n$ invertible.

So any matrix representative would do the job.

Also note

$$\det(B - \lambda I_n) = \det P(A - \lambda I_n)P^{-1}$$
$$= \cancel{\det P} \det(A - \lambda I_n) \cancel{\det P^{-1}}$$
$$= \det(A - \lambda I_n)$$

follows from properties of determinant that $\det(AB) = \det(A)\det(B)$ & $\det(I) = 1$.

So the polynomial $\det(A - \lambda I_n)$ is independent of the representative matrix.

The polynomial $\det(xI_n - A)$ is known as characteristic polynomial of T as well as A denoted by $\mathrm{char}(T)$ (or $\mathrm{char}\,A$ respectively), and the polynomial eq^n $\det(xI_n - A) = 0$ know as characteristic eq^n of T (or A).

So now we will learn what is DETERMINANT and then come back to eigen-values/vectors computations.

DETERMINANT of 2x2 matrices

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det A := ad - bc.$

determinant of A, denoted by $\det A$; defined by

it is a fn from $\underbrace{M_2(\mathbb{F})}_{\text{set of } 2\times 2 \text{ matrices with entries in } \mathbb{F}.} \longrightarrow \mathbb{F}$

Goal: To define DETERMINANT for matrices in $M_n(\mathbb{F})$ for any n.

So we would like to study the properties of the determinant fn. $M_2(\mathbb{F}) \longrightarrow \mathbb{F}$, look for some characterization which we would like to lift to $M_n(\mathbb{F})$.

Since we were studying linear maps, we ask whether determinant is a linear map from $M_n(\mathbb{F}) \longrightarrow \mathbb{F}$?

In particular

$$\det(A+B) = \det(A) + \det(B) \; ?$$

Take $A = I_2$, $B = -I_2$

$A + B = 0$

$\det(A+B) = 0$

On the other hand

$\det(A) = 1 = \det(B)$.

$\Rightarrow \det(A) + \det(B) = 2$

$\therefore \det(A+B) \neq \det(A) + \det(B)$ — in general.

(there are plenty of such examples).

Also $\det(\lambda A) = \det\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$

$= \lambda^2 (ad - bc) = \lambda^2 \det(A)$.

— scalar mult. is also not "respected"

Properties of Determinant (2x2):

- $\det(I_2) = 1$ ~ "normalization"

- $\det(A) = \det(A^T)$

  $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $\det A^T = ad - bc$.

- $\det(AB) = \det(A)\det(B)$ (check!)

- A – 2 × 2 matrix having same rows

  $\det\begin{pmatrix} a & b \\ a & b \end{pmatrix} = ab - ba = 0$.

  $\therefore \det A = 0$.

we shall show these two properties are same in mathematical word "are equivalent"

If we interchange the two rows of A the determinant changes sign.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = bc - ad = -\det(A).$$

If we write A as two rows

where R_1 – first row

R_2 – 2nd row

then the blue highlighted text just above

says $\det \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = -\det \begin{pmatrix} R_2 \\ R_1 \end{pmatrix}$.

using $\det(A) = \det(A^T)$

& writing $A = (C_1 \; C_2)$ C_1, C_2 - column

we have $\det(C_2 \; C_1) = -\det(C_1 \; C_2)$.

- $A = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ Let R be any other row, i.e, a 1×2 - matrix.

  & α be a scalar i.e, $\alpha \in \mathbb{F} \, (\mathbb{R})$.

Consider $\det \begin{pmatrix} R_1 + \alpha R \\ R_2 \end{pmatrix}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, Then $R_1 = (a, b)$
$R_2 = (c, d)$.

Let $R = (a', b')$.

Then $\begin{pmatrix} R_1 + \alpha R \\ R_2 \end{pmatrix} = \begin{pmatrix} a + \alpha a' & b + \alpha b' \\ c & d \end{pmatrix}$.

$$\det \begin{pmatrix} R_1 + \alpha R \\ R_2 \end{pmatrix}$$

$$= \det \begin{pmatrix} a + \alpha a' & b + \alpha b' \\ c & d \end{pmatrix}$$

$$= (a + \alpha a') d - (b + \alpha b') c$$

$$= (ad - bc) + \alpha (a'd - b'c)$$

$$= \det \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \alpha \det \begin{pmatrix} R \\ R_2 \end{pmatrix}.$$

So $\det \begin{pmatrix} R_1 + \alpha R \\ R_2 \end{pmatrix} = \det \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \alpha \det \begin{pmatrix} R \\ R_2 \end{pmatrix}$. — (A')

if we further break the above we have

$$\det \begin{pmatrix} R_1 + R \\ R_2 \end{pmatrix} = \det \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \det \begin{pmatrix} R \\ R_2 \end{pmatrix}$$

Choose $\alpha = 1$.

$$\det \begin{pmatrix} \alpha R_1' \\ R_2' \end{pmatrix} = \alpha \det \begin{pmatrix} R_1' \\ R_2' \end{pmatrix}$$

choose $R_1 = 0$, $R = R_1'$, $R_2 = R_2'$.

So if you fix R_2, then determinant is a linear fn on first row

Note

$$\det\begin{pmatrix} R_1 + \alpha R_2 \\ R_2 \end{pmatrix} = \det\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \alpha \underset{\overset{\shortparallel}{0}}{\det\begin{pmatrix} R_2 \\ R_2 \end{pmatrix}}$$

$$= \det\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

This type of elementary row op. will keep determinant unchanged.

follows from (A).

Also

$$\det\begin{pmatrix} R_1 + R_2 \\ R_1 + R_2 \end{pmatrix}$$

$$= \det\begin{pmatrix} R_1 \\ R_1 + R_2 \end{pmatrix} + \det\begin{pmatrix} R_2 \\ R_1 + R_2 \end{pmatrix}$$

using linearity of first row

$$= \det\begin{pmatrix} R_1 \\ R_1 \end{pmatrix} + \det\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \det\begin{pmatrix} R_2 \\ R_1 \end{pmatrix} + \det\begin{pmatrix} R_2 \\ R_2 \end{pmatrix}.$$

The properties

So if two rows of a matrix are same implies the det is 0. $\longrightarrow$ (C)

implies $\det\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = -\det\begin{pmatrix} R_2 \\ R_1 \end{pmatrix}$

i.e interchange of two rows results in a sign change in determinant. — (D')

On the otherhand,

(E)
$$\det\begin{pmatrix} R_1 \\ R_1+R_2 \end{pmatrix} = \det\begin{pmatrix} R_1 \\ R_1 \end{pmatrix} + \det\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

$$\|$$

$$-\det\begin{pmatrix} R_1+R_2 \\ R_1 \end{pmatrix} = -\det\begin{pmatrix} R_1 \\ R_1 \end{pmatrix} - \det\begin{pmatrix} R_2 \\ R_1 \end{pmatrix}$$

$$\Rightarrow \quad \det\begin{pmatrix} R_1 \\ R_1 \end{pmatrix} = 0.$$

— gives that statement (D') $\Rightarrow$ Statement (C)

We also not here that

$$\det\begin{pmatrix} 0 \\ R \end{pmatrix} = 0$$

as $$0 = \det\begin{pmatrix} R \\ R \end{pmatrix} = \det\begin{pmatrix} 0 \\ R \end{pmatrix} + \underbrace{\det\begin{pmatrix} R \\ R \end{pmatrix}}_{0}$$

$$\Rightarrow \quad \det\begin{pmatrix} 0 \\ R \end{pmatrix} = 0$$

(Also one can think that keeping 2nd row fix, det is a lin. fn. on first row gives $\det\begin{pmatrix} 0 \\ R \end{pmatrix} = 0.$).

<u>Remark:</u> Note that if δ is a fn from $M_2(\mathbb{F}) \longrightarrow \mathbb{F}$ satisfying

$$\delta\begin{pmatrix} R_1 + \alpha R \\ R_2 \end{pmatrix} = \delta\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \alpha\,\delta\begin{pmatrix} R \\ R_2 \end{pmatrix}$$

gives that

$$\delta\begin{pmatrix} R_1 + \alpha R_2 \\ R_2 \end{pmatrix} = \delta\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

if we assume

$$\delta\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = -\delta\begin{pmatrix} R_2 \\ R_1 \end{pmatrix}.$$

(as by (E) above it will $\Rightarrow \delta\begin{pmatrix} R_2 \\ R_2 \end{pmatrix} = 0$).

Now we claim that the properties highlighted in blue is enough to characterize determinant $\longrightarrow$ as we can see in the theorem below:

<u>THM:</u> Let $\delta : M_2(\mathbb{R}) \longrightarrow \mathbb{R}$ be a fn such that

1) $\delta(I_2) = 1$
2) $\delta(B) = -\delta(A)$, if B is obtained by interchanging two rows of A.
3) δ is linear on each row, keeping the other row(s) fixed.

Then $\delta(A) = \det(A)$.

(So this says any fn satisfying these three properties must be the determinant fn and hence satisfies all other properties of determinant as well).

Pf: RTP

$\hookrightarrow \delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Case 1 $\quad a = 0 \quad \det A = 0.d - bc = -bc$

$$\delta\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -bc$$

$$\delta\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = b\,\delta\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$$ follows from prop ③

$$= -b\,\delta\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$$ follows from ②

R_2

$$= -b\,\delta\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$$ $R_1 \leftrightarrow R_1 - dR_2$ follows from ③ as suggested in the remark.

$$= -bc\,\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ — from ③

I_2.

follows from ①. $\delta(I_2)=1$

$$= -bc.$$

Case II $a \neq 0$

$$\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,\delta\begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix}.$$ follows from (3')

$$= a\,\delta\begin{pmatrix} 1 & b/a \\ 0 & d - \frac{bc}{a} \end{pmatrix}.$$ $R_2 \leftrightarrow R_2 - cR_1$ follows from (3'')

$$= a\left(d - \frac{bc}{a}\right)\delta\begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$$ follows from (3)

$$= (ad - bc)\,\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ $R_1 \leftrightarrow R_1 - \frac{b}{a} R_2$ follows from (3'').

follows from (1)

$$= ad - bc.$$ $\blacksquare$

We finish this note by another observation:

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

we can define "adjoint" (Known as classical adjoint).

$$\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Note

$$\operatorname{adj}(A)\cdot(A)\;A = A\cdot\operatorname{adj}(A) = \det(A)\, I_2.$$

(Check!!) — ☆

One can see that if A is invertible,

then $\quad AB = I_2$

$$\Rightarrow \det(A)\det(B) = 1$$

$$\Rightarrow \det(A) \neq 0 .$$

On the other hand if $\det(A) \neq 0$

from $\star$ it follows A is invertible

and $\dfrac{1}{\det(A)}\operatorname{adj}(A)$ is the inverse.

In other words;

$$A \text{ is not invertible} \iff \det A = 0$$

Recall this we used CRUCIALLY to define char poly / eq^{n}. for finding eigen values.

So we would like to generalize this kind of formulation as well to higher order matrices.