7. Maxima & Minima

> "Nothing takes place in the world whose meaning is not that of some maximum or minimum."
>
> ~ Leonhard Euler

In this chapter, we shall employ techniques of calculus to find points of maximum/minimum of a function. Pierre de Fermat was one of the first mathematicians to propose a general technique for finding maxima and minima of functions. Quite often, we distinguish between global (or absolute) maximum and local maximum.

§7.1. Local Maximum / Minimum

Let us start with a definition.

Definition 7.1.1 (Local maximum/minimum)

Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function. A point $c \in I$ is called a point of local maximum of f if there exists $\delta > 0$ such that

$$\boxed{f(c) \geq f(x) \text{ for all } x \in (c-\delta, c+\delta) \cap I.}$$

A point $c \in I$ is called a point of local minimum of f if there exists $\delta > 0$ such that

$$\boxed{f(c) \leq f(x) \text{ for all } x \in (c-\delta, c+\delta) \cap I.}$$

If $I \subseteq \mathbb{R}$ is an open interval, i.e., $I = (a, b)$ for some $a < b$, then for any $c \in I = (a, b)$, there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq I$. One may choose $\delta = \min\{c-a, b-c\}$, for instance. Thus, if c is a point of locale extremum (maximum or minimum), then we may choose $\delta_0 > 0$ such that

i) $(c-\delta_0, c+\delta_0) \subseteq I$

ii) $f(c) \geq f(x)$ for all $x \in (c-\delta_0, c+\delta_0)$ [for maximum]

($f(c) \leq f(x)$ for all $x \in (c-\delta_0, c+\delta_0)$ [for minimum])

hold.

THEOREM 7.1.2 (NECESSARY CONDITION)

Let $-\infty \leq a < b \leq \infty$, let $f : (a,b) \to \mathbb{R}$ be differentiable at $c \in (a,b)$. If c is a point of local maximum or local minimum, then $f'(c) = 0$.

Note that the derivative vanishing is a necessary condition only if the function is differentiable. For example, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ has a local (in fact, global) minimum at 0 but it is not differentiable at 0.

Proof. Let us suppose that c is a point of local maximum. As (a,b) is open, we may find a $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq (a,b)$ and $f(c) \geq f(x)$ for all $x \in (c-\delta, c+\delta)$. Given $\varepsilon > 0$, there exists $\delta_1 > 0$ (as f is differentiable at c) such that

$$0 < |x-c| < \delta_1 \Rightarrow \left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < \varepsilon.$$

We may choose $\delta_1 < \delta$. In other words,

$$f'(c) - \varepsilon < \frac{f(x)-f(c)}{x-c} < f'(c) + \varepsilon \quad \text{for all } x \in (c-\delta_1, c) \cup (c, c+\delta_1).$$

In particular,

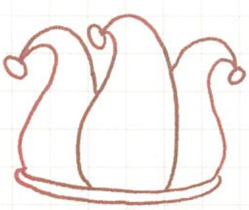
$$f'(c) - \varepsilon < \frac{f(x)-f(c)}{x-c} \leq 0 \quad \text{for all } x \in (c, c+\delta_1)$$

$$0 \leq \frac{f(x)-f(c)}{x-c} < f'(c) + \varepsilon \quad \text{for all } x \in (c-\delta_1, c).$$

This implies that

$$-\varepsilon < f'(c) < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this forces $f'(c) = 0$. ▨



IN JEST

- What do you call the mum with the most kids in the world? Maximum!
- Why is Optimus Prime worthless? The derivative at the maximum is zero!

THEOREM 7.1.3 (SUFFICIENT CONDITION)

Let $-\infty \leq a < b \leq \infty$, let $f:(a,b) \to \mathbb{R}$ be such that f, f', f'' exist and are continuous on (a,b). Let $c \in (a,b)$ be such that $f'(c) = 0$ and $f''(c) \neq 0$. Then,

i) c is a point of strict local maximum if $f''(c) < 0$

ii) c is a point of strict local minimum if $f''(c) > 0$.

Note the following points:

a) c is a strict local extremum means $f(c) > f(x)$ (or $f(c) < f(x)$) for all $x \in (c-\delta, c+\delta) \subseteq (a,b)$ for some appropriate $\delta > 0$.

b) The derivative vanishing and the second derivative not vanishing form a sufficient condition only when f is twice differentiable and f'' is continuous.

c) If $f''(c) = 0$, then the test is inconclusive. Consider, for example, $h_1(x) := x^3$, $h_2(x) = x^4$ and $h_3(x) := -x^4$. Note that $x_0 = 0$ is a point of (global) minimum for h_2, a point of (global/strict) maximum for h_3 and a point of neither maximum or minimum for h_1.

Proof. We shall prove (i). The proof of (ii) is similar and will be left as an exercise. Since f'' is continuous and $f''(c) < 0$, there exists $\delta > 0$ such that

$$[c-\delta, c+\delta] \subseteq (a,b) \quad \& \quad f''(x) \ \ 0 \text{ for all } x \in (c-\delta, c+\delta).$$

For each $x \in (c-\delta, c+\delta)$ & $x \neq c$, apply Theorem 6.2.1 to $f:[x,c]$ (if $x < c$) or $f:[c,x]$ (if $x > c$) to obtain d_x lying between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(d_x)}{2}(x-c)^2$$

$$= f(c) + \tfrac{1}{2} f''(d_x)(x-c)^2 < f(c)$$

as $f''(d_x) < 0$. Thus, c is a point of strict local maximum. ▨

EXAMPLE 7.1.4

Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) := x^5 - 5x^4 + 5x^3 + 10$$

As any polynomial is continuous and derivative of a polynomial is another polynomial, we may apply Theorems 7.1.2 & 7.1.3 to f. Note that

$$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x-1)(x-3),$$
$$f''(x) = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3).$$

Thus, $\{x \in \mathbb{R} \mid f'(x) = 0\} = \{0, 1, 3\}$. Moreover,

$$f''(0) = 0 \quad , \quad f''(1) = -10 < 0 \quad , \quad f''(3) = 90 > 0.$$

Hence, 1 is a point of strict local maximum, 3 is a point of strict local minimum but we may deduce nothing about 0.

§7.2. TAYLOR'S THEOREM

In calculus, Taylor's expansion provides an approximation of a k-times differentiable function around a given point by a polynomial of degree k. There are several versions of Taylor's Theorem, named after Brook Taylor, who stated a version in 1715. Taylor's Theorem is one of main elementary tools in analysis.

THEOREM 7.2.1 (TAYLOR'S THEOREM)

Let $-\infty < a < b < \infty$, let $n \in \mathbb{N}$ and let $f: [a,b] \to \mathbb{R}$ be such that the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous in $[a,b]$ and differentiable in (a,b). Then, for some $c \in (a,b)$,

$$\boxed{f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(c)}{n!}(b-a)^n} \qquad (7.1)$$

$$= f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b-a)^n.$$

Note the following:

i) When $k=0$, $f^{(k)}(a) = f(a)$, $k! = 1$ and $(b-a)^k = 1$.

ii) When $n=1$, (7.1) becomes

$$f(b) = f(a) + f'(c)(b-a).$$

When $n=2$, (7.1) becomes

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2.$$

iii) The assumption that $f^{(n-1)}$ is differentiable in (a,b) implies continuity of $f^{(n-2)}$ if $n \geq 2$.

Proof. We shall prove this using Rolle's Theorem. Let us define an auxilliary function

$$\varphi : [a,b] \to \mathbb{R}$$

$$\varphi(x) := f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(b-x)^k - A(b-x)^n$$

where wes set A such that $\varphi(a) = \varphi(b)$. Note that $\varphi(b) = 0$, whence

$$0 = \varphi(a) = f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k - A(b-a)^n$$

$$\Rightarrow \quad A = \frac{1}{(b-a)^n}\left[f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k\right]. \qquad (7.2)$$

Applying Rolle's Theorem to φ, we obtain $c \in (a,b)$ such that $\varphi'(c) = 0$. But

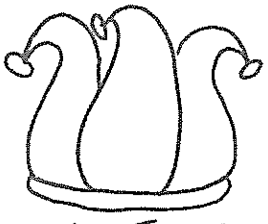
$$\varphi'(x) = -\sum_{k=0}^{n-1}\left[\frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{f^{(k)}(x)}{k!}k(b-x)^{k-1}\right] + An(b-x)^{n-1}$$

$$= -\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1} + An(b-x)^{n-1}$$

As $\varphi'(c) = 0$, we obtain $A = \frac{f^{(n)}(c)}{n!}$. Combining this with (7.2), we obtain

$$\frac{f^{(n)}(c)}{n!}(b-a)^n = f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k$$

which simplifies to the stated expansion. ▨



IN JEST

Why would Taylor never run for US president?

- He doesn't want to be limited to one term!

[Note that US president Z. Taylor died during his first term.]

EXAMPLE 7.2.2

Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function, i.e., $f(x) = e^x$. We assume that $f(0) = 1$, $f(x) > 0$ for any $x \in \mathbb{R}$ and $f'(x) = e^x$. It follows that $f^{(k)}(x) = e^x$ & $f^{(k)}(0) = 1$ for any $k \in \mathbb{N}$. Thus, the Taylor expansion of k^{th} order is of the form

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \frac{e^c}{(k+1)!} x^{k+1}$$

for some $c \in (0, x)$, possibly depending on x.

EXAMPLE 7.2.3

Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree $n \geq 0$, i.e., $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_n \neq 0$. As polynomials are differentiable and the derivative of a polynomial is also a polynomial, $P(x)$ is k-times differentiable for any $k \in \mathbb{N}$. Moreover, $P^{(k)}(x) = 0$ for all $x \in \mathbb{R}$ if $k > n$. The Taylor expansion of k^{th} order (if $k > n$) is of the form

$$\begin{aligned} P(x) &= P(0) + P'(0)x + P''(0)\frac{x^2}{2!} + \cdots + P^{(n)}(0)\frac{x^n}{n!} \\ &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \end{aligned}$$

THEOREM 7.2.4 (MAXIMA & MINIMA)

Let $-\infty < a < b < \infty$, let $n \in \mathbb{N}$ and let $f : (a,b) \to \mathbb{R}$ be n times differentiable and the n^{th} derivative $f^{(n)}$ is continuous in (a,b). Let $c \in (a,b)$ be such that

i) $f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$

ii) $f^{(n)}(c) \neq 0$.

Then

i) if n is odd, then c is not a point of maxima or minima

ii) if n is even, then c is a point of maximum (respectively minimum) if $f^{(n)}(c) < 0$ (respectively $f^{(n)}(c) > 0$).

Proof. Using the continuity of $f^{(n)}$ & $f^{(n)}(c) \neq 0$, obtain $\delta > 0$ such that $f^{(n)}(x) f^{(n)}(c) > 0$ for all $x \in [c-\delta, c+\delta]$ and $[c-\delta, c+\delta] \subseteq (a,b)$. Using Taylor's Theorem (Theorem 7.2.1) obtain $d \in (c,x)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n)}(d)}{n!}(x-c)^n.$$

This implies

$$f(x) - f(c) = \frac{f^{(n)}(d)}{n!}(x-c)^n.$$

Case 1: n is odd

In this case,

$$(x-c)^n < 0 \quad \text{if} \quad c-\delta < x < c$$
$$(x-c)^n > 0 \quad \text{if} \quad c < x < c+\delta.$$

If $f^{(n)}(d) > 0$, then

$$f(x) - f(c) < 0 \quad \text{if} \quad c-\delta < x < c$$
$$f(x) - f(c) > 0 \quad \text{if} \quad c < x < c+\delta.$$

If $f^{(n)}(d) < 0$, then

$$f(x) - f(c) > 0 \quad \text{if} \quad c-\delta < x < c$$
$$f(x) - f(c) < 0 \quad \text{if} \quad c < x < c+\delta.$$

Hence, c is not a point of extremum.

Case 2: n is even

In this case,

$$(x-c)^n > 0 \quad \text{if} \quad x \in (c-\delta, c+\delta), x \neq c.$$

If $f^{(n)}(d) > 0$, then

$$f(x) - f(c) > 0 \quad \text{if} \quad x \in (c-\delta, c+\delta), x \neq c$$

and c is a point of minimum.

If $f^{(n)}(d) < 0$, then

$$f(x) - f(c) < 0 \quad \text{if} \quad x \in (c-\delta, c+\delta), x \neq c$$

and c is a point of maximum. ▨

EXAMPLE 7.2.5

Let $f(x) = x^4$, defined on $\mathbb{R}$. As $f'(0) = f''(0) = f'''(0) = 0$ but $f^{(4)}(0) = 4! = 24 > 0$, by Theorem 7.2.4, 0 is a local minima. The same is true for any polynomial of the form x^{2k}, i.e., $g(x) = x^{2k}$ has a minima (in fact, it is strict & global) at 0. However, functions of the form $h(x) = x^3$ or, more generally, $h(x) = x^{2k+1}$ does not have a maxima or minima at 0.