**Problem Set - 1**

MA 1201 Spring Sem, 2025

1. Verify whether the following pairs $(A, B)$ of sets are having same cradinality. If yes, establish an explicit bijection. If not, prove.
   (a) $A := \mathbb{N}$; $B := \{n \in \mathbb{N} : n \text{ is a power of } 2\}$.
   (b) *$A := \mathbb{N}$; $B := \mathbb{Z}$.
   (c) $A := \mathbb{Z}$; $B := \mathbb{N}$.
   (d) $A := \{1, 2\}$; $B := \{x \in \mathbb{R} : x^2 + bx + c = 0\}$, where $b, c \in \mathbb{R}$ are given and $b^2 - 4ac = 0$.
   (e) $A := \{1, 2\}$; $B := \{x \in \mathbb{R} : x^2 + bx + c = 0\}$, where $b, c \in \mathbb{R}$ are given and $b^2 - 4ac > 0$.
   (f) $A := \{1, 2\}$; $B := \{x \in \mathbb{R} : x^2 + bx + c = 0\}$, where $b, c \in \mathbb{R}$ are given and $b^2 - 4ac < 0$.
   (g) $A = \{1, 2, 3, 4\}$; $B := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m^2 + n^2 = 169\}$.
   (h) $A := (0, \infty)$; $B := (-\infty, 0)$.
   (i) *$A := (0, \infty)$; $B := (1, \infty)$.
   (j) $A := (1, \infty)$; $B := (-\infty, -3)$.
   (k) $A := (0, 1)$; $B := (1, \infty)$.
   (l) *$A := (0, 1)$; $B := (a, b)$, where $a < b$, $a, b \in \mathbb{R}$.
   (m) $A := (0, 1)$; $B := (0, \infty)$.
   (n) *$A := (0, 1)$; $B := \mathbb{R}$.
   (o) *$A := (0, 1)$; $B := [0, 1]$.
2. Let $X$ be a set and $A, B \subseteq X$. Let $A \sim B$ if and only if $A$ and $B$ have same cardinality. Show that $\sim$ is an equivalence relation on $\mathcal{P}(X)$.
3. *If $A$ is a finite set and $B \subseteq A$, then show that $B$ is finite and $|B| \leq |A|$.
4. *If $A$ is a finite set and $B$ is a proper subset of $A$, then show that $|B| < |A|$.
5. If $A$ is a finite set and $a \notin A$, then prove $|A \cup \{a\}| = |A| + 1$.
6. *If $A, B$ are finite sets, then prove that $A \cup B$ is a finite set and
$$|A \cup B| = |A| + |B| - |A \cap B|.$$
7. If $A, B$ are finite sets, then prove that $A \times B$ is finite and
$$|A \times B| = |A||B|.$$
8. Let $X$ be a finite set and $f : X \to X$ be a map. Show that the following are equivalent:
   (a) $f$ is a bijection.

(b) $f$ is $1-1$.

(c) $f$ is onto.

9. Let $A$ and $B$ be finite sets and $f : A \to B$ be a map. Prove the following:

   (a) If $f$ is $1-1$, then $|A| \leq |B|$.

   (b) If $f$ is onto, then $|A| \geq |B|$.

   (c) If $f : A \to B$ and $g : B \to A$ are $1-1$, then $|A| = |B|$, and $f$ and $g$ are bijections.

10. Show that every infinite set contains a countable subset.

11. *Prove that any subset of a countable set is atmost countable.

12. *Prove that finite union of countable set is countable.

13. (a) Let $A$ be an infinite set and $B \subseteq A$ a finite set. Show that $A \setminus B$ is infinite.

    (b) Let $A$ be uncountable and $B \subseteq A$ a countable set. Show that $A \setminus B$ is uncountable.

14. *Show that for any infinite set $A$ and a countable set $B$, the sets $A$ and $A \cup B$ are of same cardinality.

15. *For a nonempty subset $A$, prove that the following are equivalent:

    (a) $A$ is atmost countable.

    (b) There exists a $1-1$ map of $A$ to $\mathbb{N}$.

    (c) There exists an onto map of $\mathbb{N}$ to $A$.

16. *Suppose that $A \subseteq B$ then prove that

    (a) $B$ is finite $\implies$ $A$ is finite.

    (b) $A$ is infinite $\implies$ $B$ is infinite.

    (c) $B$ is countable $\implies$ $A$ is atmost countable.

    (d) $A$ is uncountable $\implies$ $B$ is uncountable.

17. *Suppose $f : A \to B$ is injective then prove that

    (a) $B$ is finite $\implies$ $A$ is finite.

    (b) $A$ is infinite $\implies$ $B$ is infinite.

    (c) $B$ is countable $\implies$ $A$ is atmost countable.

    (d) $A$ is uncountable $\implies$ $B$ is uncountable.

18. Suppose $f : A \to B$ is surjective then prove that

    (a) $A$ is finite $\implies$ $B$ is finite.

    (b) $B$ is infinite $\implies$ $A$ is infinite.

    (c) $A$ is countable $\implies$ $B$ is atmost countable.

    (d) $B$ is uncountable $\implies$ $A$ is uncountable.