# Lecture 18/03/25

- Row-reduced echelon form
  - it is an $\underbrace{\text{Echelon matrix}}$
    - zero rows appears last.
    - if $k_i$ - be the column where the first non-zero entry appears in $i^{th}$ row, $1 \leq i \leq r$, ($r$ - non zero rows) then $k_1 < k_2 \ldots < k_r$.

PIVOT

  - Each PIVOT is 1
  - Every other entry in the column containing the PIVOT is zero.

- Every matrix is row equivalent to a unique row-reduced echelon form.

Note if $A$ - $m \times n$ - matrix
& $r$ - no. of non-zero rows in its row reduced echelon form
then $r \leq m$, $r \leq n$.

**Thm:** An $n \times n$ matrix $A$ is row equivalent to Identity
$\Longleftrightarrow$ $Ax = 0$ has only trivial solution ($x = 0$).

Pf: $(\Rightarrow)$ A is row equivalent to I

$\Rightarrow$ $Ax = 0$ & $Ix = 0$ has same solution.

$\Rightarrow$ $Ax = 0$ has only trivial solution.

$(\Leftarrow)$ Suppose $Ax = 0$ has only trivial solution.

We know if R is the row reduced echelon form of the matrix A and $r$ be the no. of non zero rows in R,

then $r \leq n$.

Now if $r < n$, then $\dim N(R) = n - r$

$\Rightarrow$ $Rx = 0$ has non-zero solution

$\Rightarrow$ $Ax = 0$ has non-zero solution $\times$

So $r \not< n \Rightarrow r \geq n \Rightarrow r = n$.

$k_1, k_2 \ldots k_n$ - column where 1 appears

$1 \leq k_1 < k_2 \ldots < k_n \leq n$.

Note $k_1 \geq 1 \Rightarrow k_2 > k_1 \geq 1 \Rightarrow k_2 \geq 2$.

I.H: $k_i \geq i$

$\Rightarrow k_{i+1} > k_i \geq i \Rightarrow k_{i+1} \geq i+1$

So inductively $k_i \geq i \ \forall i$

Claim: In this case $k_i = i$

if not $k_i > i \Rightarrow k_i \geq i+1$.

Now $k_{i+1} - k_i \geq 1$, $k_{i+2} - k_{i+1} \geq 1$.

$\Rightarrow \quad k_{i+2} - k_i \geq 2.$

|||ly $\quad k_n - k_i \geq n - i$

$\Rightarrow k_n \geq k_i + n - i \geq i+1 + n - i = n+1$

$\times\!\!\times$.

So

$$k_i = i \quad \forall\, i = 1, 2 \ldots n.$$

$\Rightarrow R = I.$

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**Thm:** $A$ - $n \times n$ - matrix. TFAE:

1) $A$ is invertible
2) $A$ is row equivalent to Identity
3) $A$ is product of elementary matrices.

**Pf:** 1) $\Rightarrow$ 2)

$A$ invertible $\Rightarrow \exists\, B$ s.t $AB = BA = I$.

Consider $\underline{Ax = 0}$

$$BAx = 0 \Rightarrow x = 0$$

$\Rightarrow Ax = 0$ has only trivial sol$^n$.

$\Rightarrow A$ is row equivalent to $I$. (by prev. thm).

2) $\Rightarrow$ 3)

$E_1 E_2 \ldots E_k A = I$ $\quad$ elementary row op.

$\Rightarrow A = E_k^{-1} \ldots E_1^{-1}$

3) $\Rightarrow$ 1) $\quad A = F_1 \cdots F_k$.

$\Rightarrow$ A is invertible. $\blacksquare$

Remark: This is the technique we used in Gauss-Jordan method for finding inverse as $A^{-1} = E_1 \cdots E_k I$.

Thm: For an $n \times n$ matrix $A$

A is invertible $\Leftrightarrow$ $Ax = 0$ has only trivial solution.

Pf: ($\Rightarrow$) – If B is the inverse of A, then $Ax = 0 \Rightarrow B(Ax) = 0 \Rightarrow x = 0$.

($\Leftarrow$) $Ax = 0$ has trivial solⁿ.

$\Rightarrow$ A is row equivalent to $I$

$\Rightarrow$ A is invertible. $\blacksquare$

Defⁿ:

A matrix A is said to be right invertible if $\exists$ a matrix B (with appropriate order) such that $AB = I$.

On the other hand, A is said to be left invertible if $\exists$ C s.t $CA = I$.

Thm: An $n\times n$ matrix which is either left invertible or right invertible, is invertible.

Pf: Suppose $A$ is left invertible.

(we first take this one).

So $\exists\ C$ s.t $CA = I$.

$\Rightarrow$ $Ax = 0$ has ONLY trivial sol$^n$.

$\Rightarrow$ $A$ is invertible.

Now if $A$ is right invertible

$\Rightarrow$ $AB = I$.

$\Rightarrow$ $B$ is left invertible

$\Rightarrow$ $B$ is invertible.

$\Rightarrow$ $B = A^{-1}$.

We proved earlier if further $BC = I$ $\Rightarrow$ $A = C$.

We will now make some remarks about $m\times n$ matrix in general. (instead of square matrix).

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{right invertible}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

now $\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{}\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{nonzero soln.}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Note: $\left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 0 \end{pmatrix} \neq I_3 \right)$.

$\to$ not left invertible.

as $A$ is left invertible

$\Rightarrow$ $Ax = 0$ has trivial solution only.

(note the earlier thms, we have written for square matrices only).

We introduce some more terminology:

$R(A) :=$ row space of $A$.

$= \{$ all possible linear combinations of rows of $A \}$.

row rank of $A := \dim R(A)$

column rank of $A := \dim C(A)$.

We shall show that for any matrix $A$,

row rank of $A$ = column rank of $A$.