Solution Set - 1

MA 1201 Spring Sem, 2025

1. Verify whether the following pairs (A, B) of sets are having same cradinality. If yes, establish an explicit bijection. If not, prove.

   (a) $A := \mathbb{N}$; $B := \{n \in \mathbb{N} : n \text{ is a power of } 2\}$.

   (b) *$A := \mathbb{N}$; $B := \mathbb{Z}$.

   (c) $A := \mathbb{Z}$; $B := \mathbb{N}$.

   (d) $A := \{1, 2\}$; $B := \{x \in \mathbb{R} : x^2+bx+c = 0\}$, where $b, c \in \mathbb{R}$ are given and $b^2-4ac = 0$.

   (e) $A := \{1, 2\}$; $B := \{x \in \mathbb{R} : x^2+bx+c = 0\}$, where $b, c \in \mathbb{R}$ are given and $b^2-4ac > 0$.

   (f) $A := \{1, 2\}$; $B := \{x \in \mathbb{R} : x^2+bx+c = 0\}$, where $b, c \in \mathbb{R}$ are given and $b^2-4ac < 0$.

   (g) $A = \{1, 2, 3, 4\}$; $B := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m^2 + n^2 = 169\}$.

   (h) $A := (0, \infty)$; $B := (-\infty, 0)$.

   (i) *$A := (0, \infty)$; $B := (1, \infty)$.

   (j) $A := (1, \infty)$; $B := (-\infty, -3)$.

   (k) $A := (0, 1)$; $B := (1, \infty)$.

   (l) *$A := (0, 1)$; $B := (a, b)$, where $a < b$, $a, b \in \mathbb{R}$.

   (m) $A := (0, 1)$; $B := (0, \infty)$.

   (n) *$A := (0, 1)$; $B := \mathbb{R}$.

   (o) *$A := (0, 1)$; $B := [0, 1]$.

Solution:

(a) Yes. Choose the bijection function $f : A \to B$ as $f(n) = 2^n$.

(b) Yes. Choose the bijection function $f : \mathbb{N} \to \mathbb{Z}$ as $f(n) = \begin{cases} k, & \text{if } n = 2k, \\ 1-k, & \text{if } n = 2k-1. \end{cases}$

Injectivity : To show injectivity, we prove that if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

Case 1: Both n_1 and n_2 are even. If $n_1 = 2k_1$ and $n_2 = 2k_2$, then:

$$f(n_1) = k_1, \quad f(n_2) = k_2.$$

If $f(n_1) = f(n_2)$, then $k_1 = k_2$, so $n_1 = 2k_1 = 2k_2 = n_2$.

Case 2: Both n_1 and n_2 are odd. If $n_1 = 2k_1 - 1$ and $n_2 = 2k_2 - 1$, then:

$$f(n_1) = 1 - k_1, \quad f(n_2) = 1 - k_2.$$

If $f(n_1) = f(n_2)$, then $1-k_1 = 1-k_2$, so $k_1 = k_2$, and $n_1 = 2k_1-1 = 2k_2-1 = n_2$.

Case 3: One of n_1 or n_2 is even, and the other is odd. Suppose $n_1 = 2k_1$ (even) and $n_2 = 2k_2 - 1$ (odd). Then:

$$f(n_1) = k_1, \quad f(n_2) = 1 - k_2.$$

If $f(n_1) = f(n_2)$, then $k_1 = 1 - k_2$, which implies $k_1 + k_2 = 1$. However, this contradicts n_1 and n_2 being distinct natural numbers, as even and odd integers cannot overlap.

Thus, $f(n_1) = f(n_2)$ implies $n_1 = n_2$, and f is injective.

Surjectivity : To show surjectivity, we prove that for every $z \in \mathbb{Z}$, there exists an $n \in \mathbb{N}$ such that $f(n) = z$.

Case 1: $z > 0$. Let $z = k$, where $k \in \mathbb{N}$. Choose $n = 2k$ (even). Then:

$$f(n) = f(2k) = k = z.$$

Case 2: $z \leq 0$. Let $z = 1 - k$, where $k \in \mathbb{N}$. Choose $n = 2k - 1$ (odd). Then:

$$f(n) = f(2k-1) = 1 - k = z.$$

Thus, for every $z \in \mathbb{Z}$, there exists an $n \in \mathbb{N}$ such that $f(n) = z$, and f is surjective.

(c) Yes. Choose the bijection function $f : \mathbb{Z} \to \mathbb{N}$ as $f(z) = \begin{cases} 2z, & \text{if } z > 0, \\ 1 - 2z, & \text{if } z \leq 0. \end{cases}$

(d) No. Since $|A| = 2$ and $|B| = 1$.

(e) Yes. Choose the bijection function $f : A \to B$ as $f(n) = \begin{cases} \frac{-b+\sqrt{b^2-4ac}}{2}, & \text{if } n = 1, \\ \frac{-b-\sqrt{b^2-4ac}}{2}, & \text{if } n = 2. \end{cases}$.

(f) No. Since $|A| = 2$ and $|B| = 0$.

(g) No. Since $|A| = 4$ and $|B| = 8$ as
$B = \{(12, 5), (5, 12), (5, -12), (-12, 5), (12, -5), (-5, 12), (-5, -12), (-12, -5)\}$.

(h) Yes. Choose the bijection function $f : (0, \infty) \to (-\infty, 0)$ as $f(x) = -x$.

(i) Yes. Choose the bijection function $f : (0, \infty) \to (1, \infty)$ as $f(x) = x + 1$.

(j) Yes. Choose the bijection function $f : (1, \infty) \to (-\infty, -3)$ as $f(x) = -x - 2$.

(k) Yes. Choose the bijection function $f : (0, 1) \to (1, \infty)$ as $f(x) = \frac{1}{x}$.

Injectivity: Assume $f(x_1) = f(x_2)$. Then $\frac{1}{x_1} = \frac{1}{x_2}$, so $x_1 = x_2$. Thus, f is injective.

Surjectivity: For any $y \in (1, \infty)$, let $x = \frac{1}{y}$. Since $y > 1$, we have $0 < x < 1$, and $f(x) = \frac{1}{x} = y$. Thus, f is surjective.

(l) Yes. Choose the bijection function $f : (0, 1) \to (a, b)$ as $f(x) = a + (b - a)\, x$.

(m) Yes. Choose the bijection function $f : (0, 1) \to (0, \infty)$ as $f(x) = \frac{1}{x} - 1$.

(n) Yes. Choose the bijection function $f : (0, 1) \to \mathbb{R}$ as $f(x) = \ln\left(\frac{x}{1-x}\right)$.

Injectivity: Assume $f(x_1) = f(x_2)$. Then $\ln\left(\frac{x_1}{1-x_1}\right) = \ln\left(\frac{x_2}{1-x_2}\right)$, so taking exponential, $\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$. Simplifying gives $x_1 = x_2$. Thus, f is injective.

Surjectivity: For any $y \in \mathbb{R}$, let $x = \frac{e^y}{1+e^y}$. Since $e^y > 0$, $0 < x < 1$, and $f(x) = \ln\left(\frac{x}{1-x}\right) = y$. Thus, f is surjective.

(o) Yes. Choose the bijection function $f : (0, 1) \to [0, 1]$ as

$$f(x) = \begin{cases} 0, & \text{if } x = \frac{1}{2}, \\ 1, & \text{if } x = \frac{1}{3}, \\ \frac{1}{n-2}, & \text{if } x \in \left\{\frac{1}{n} : n \in \mathbb{N}, n \geq 4\right\}, \\ x, & \text{if } x \in (0, 1) \setminus \left\{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\right\}. \end{cases}$$

Injectivity:

- If $f(x_1) = f(x_2)$, then:
  - $f(x_1) = f(x_2) = 0 \implies x_1 = x_2 = \frac{1}{2}$,

- $f(x_1) = f(x_2) = 1 \implies x_1 = x_2 = \frac{1}{3}$,
- For $x_1 = \frac{1}{n_1}$ and $x_2 = \frac{1}{n_2}$ (with $n_1, n_2 \geq 4$), $f(x_1) = f(x_2) \implies n_1 = n_2$, so $x_1 = x_2$,
- For $x_1, x_2 \in (0,1) \setminus \left\{\frac{1}{n} : n \geq 2\right\}$, $f(x_1) = f(x_2) \implies x_1 = x_2$.

Thus, f is injective.

Surjectivity:

- For $y = 0$, $f\left(\frac{1}{2}\right) = 0$,
- For $y = 1$, $f\left(\frac{1}{3}\right) = 1$,
- For $y = \frac{1}{n}, n \geq 2$, $f\left(\frac{1}{n+2}\right) = \frac{1}{n}$,
- For $y \in (0,1) \setminus \left\{\frac{1}{n}, n \geq 2\right\}$, $f(y) = y$.

Thus, f is surjective.

2. Let X be a set and $A, B \subseteq X$. Let $A \sim B$ if and only if A and B have same cardinality. Show that $\sim$ is an equivalence relation on $\mathcal{P}(X)$.

Solution: Since $A \sim A$, it follows that $\sim$ is reflexive. If $A \sim B$, then there exists a bijective map $f : A \to B$. By the properties of bijections, the inverse map $g = f^{-1}$ is also bijective, with $g : B \to A$. This implies $B \sim A$. Therefore, $\sim$ is symmetric.

Next, suppose $A \sim B$ and $B \sim C$. Then there exist bijective maps $f_1 : A \to B$ and $f_2 : B \to C$. Define $g = f_2 \circ f_1$, which is a composition of bijections and hence itself bijective, with $g : A \to C$. This implies $A \sim C$. Therefore, $\sim$ is transitive.

Since $\sim$ satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation on $\mathcal{P}(X)$.

3. *If A is a finite set and $B \subseteq A$, then show that B is finite and $|B| \leq |A|$.

Solution: If $A = \phi$, then $B \subseteq A$ gives $B = \phi$. Hence B is finite and $|B| = 0 = |A|$.

Suppose $A \neq \phi$. Then there exists $n \in \mathbb{N}$ and a bijective function $f : A \to I_n$. Consider the inclusion map $i : B \to A$ defined by $i(b) = b$ for all $b \in B$. Then i is injective with range of $i = B \subseteq A$.

Thus the composition map $g = f \circ i : B \to I_n$ is $1 - 1$. This implies that B is finite and $|B| \leq n$ (by the proposition in page 13 in week 01 notes).

4. *If A is a finite set and B is a proper subset of A, then show that $|B| < |A|$.

Solution:

Let $|A| = n$. Since B is a proper subset of A therefore there exists $a \in A$ such that $a \notin B$. Thus $B \subseteq A \setminus \{a\}$. From the solution of the previous question it follows that $|B| \leq |A \setminus \{a\}| = n - 1 < |A|$.

5. If A is a finite set and $a \notin A$, then prove $|A \cup \{a\}| = |A| + 1$.

Solution: We prove that if A is a finite set and $a \notin A$ then $|A \cup \{a\}| = |A| + 1$. Let $B = A \cup \{a\}$. Since $a \notin A$, so $A = B \setminus \{a\}$. From the lemma (in Page 9 of week 1 notes) it follows that $|A| = |B| - 1$. This implies $|A \cup \{a\}| = |A| + 1$.

6. *If A, B are finite sets, then prove that $A \cup B$ is a finite set and

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Solution: Suppose, A contains m elements and $A = \{x_1, x_2, \ldots, x_m\}$. B contains n elements and $B = \{y_1, y_2, \ldots, y_n\}$.
Case-1: If $A \cap B = \emptyset$, then

$$A \cup B = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$$

Therefore, $|A \cup B| = m + n - 0 = |A| + |B| - |A \cap B|$.

Remark: If for $i = 1, 2, \ldots, n$, A_i are finite sets and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Therefore by Mathematical induction we have, $|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|$.
Case-2: If $A \cap B \neq \emptyset$. Suppose $A \cap B$ contains k elements, Therefore, $|A - (A \cap B)| = m - k$. Now, we can write,

$$A \cup B = (A - (A \cap B)) \cup B$$

Since $(A - (A \cap B)) \cap B = \emptyset$, then by **case-1:**

$$|A \cup B| = |(A - (A \cap B))| + |B| = m - k + n = |A| + |B| - |A \cap B|$$

7. If A, B are finite sets, then prove that $A \times B$ is finite and

$$|A \times B| = |A||B|.$$

Solution: Suppose, A contains m elements and $A = \{x_1, x_2, \ldots, x_m\}$. Now we can write,

$$A \times B = \bigcup_{i=1}^{m} A_i, \text{ where } A_i = \{x_i\} \times B$$

Since $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $|A_i| = |B|$ $(\forall i)$, therefore by **Remark** in solution of problem (6), we have

$$|A \times B| = \sum_{i=1}^{m} |A_i| = m|B| = |A||B|$$

8. Let X be a finite set and $f : X \to X$ be a map. Show that the following are equivalent:

(a) f is a bijection.

(b) f is $1 - 1$.

(c) f is onto.

Solution: (a)$\Rightarrow$(b): Obvious, as bijective $\Leftrightarrow$ injective + onto.
(b)$\Rightarrow$(c): To prove onto, we prove that $\mathbf{Im}(f) = f(X) = X$. Suppose not, that is $f(X) \subset X$. Then by Problem 6, $|f(X)| < |X|$. On the other hand, ince $f : X \to X$ is one one, then $f : X \to f(X)$ is a bijection showing $|f(X)| = |X|$, a contradiction. Hence $f(X) = X$, that is, f is onto.
(c)$\Rightarrow$(a): Here, To prove bijective, we need to prove f is one one. Since f is onto, then for each $y \in X = \mathbf{Codomain}(f)$, there exist an element $x \in X = \mathbf{Dom}(f)$ such that $f(x) = y$. Since X is finite, there exists a bijection from I_n to X and we can write $X = \{y_1, \ldots y_n\}$. Since f is onto, $|f^{-1}\{y_i\}| \geq 1$ for all $i, 1 \leq i \leq n$. If f is not $1-1$, then there exist a i such that $|f^{-1}\{y_i\}| > 1$. Then $X = \cup_{i=1}^{n} f^{-1}\{y_i\}$ being the disjoint union, we have $|X| = \sum_{i=1}^{n} |f^{-1}\{y_i\}| > n$ - a contradiction. So, f is one one and hence, f is bijective.

9. Let A and B be finite sets and $f : A \to B$ be a map. Prove the following:
   (a) If f is $1-1$, then $|A| \leq |B|$.
   (b) If f is onto, then $|A| \geq |B|$.
   (c) If $f : A \to B$ and $g : B \to A$ are $1-1$, then $|A| = |B|$, and f and g are bijections.

Solution:

(a) If B is empty, then A has to empty else the function f cannot be defined. Thus, A is finite and $|A| = 0 = |B|$.
Suppose B is non-empty. Then there exists some $n \in \mathbb{N}$ and a bijective map $f_B : B \to I_n$. As $f(A) \subset B$, consider the composition map

$$h_A := f_B o f : A \to I_n.$$

Note that the map h_A being composition fo two $1-1$ map is $1-1$, and hence the set A is finite and $|A| \leq n = |B|$. This proves the result.

(b) If A is empty, then B has to empty and hence $|B| = 0 = |A|$.
Suppose A is non-empty. Then there exits $m \in \mathbb{N}$ and a bijective map $f_A : I_m \to A$. Now, consider the composition map

$$h_B := f o f_A : I_m \to B.$$

Note that the map h_B being composition of two onto function is onto and so the set B is finite and $|B| \leq m = |A|$. This proves the result.

(c) If $f : A \to B$ is $1-1$, then by (a)

$$|A| \leq |B|. \tag{1}$$

Similarly, if $g : B \to A$ is $1-1$, then by (a)

$$|B| \leq |A|. \tag{2}$$

Due to (1) and (2), we have

$$|A| = |B|. \tag{3}$$

For bijectiveness of f and g:
f is given to be $1-1$, so we need to show that f is onto. If possible, suppose f is not onto, i.e, $f(A) \subsetneq B$. By (3), $|f(A)| < |B|$. Now since f is one-one and $|f(A)| = |A|$. This shows $|A| < |B|$, which contradicts (3). Thus, f is onto.

One can similarly argue to show that g is bijective.

10. Show that every infinite set contains a countable subset.

Solution: *Theorem stated in class:* A set A is infinite iff $\exists$ a $1-1$ map $f : \mathbb{N} \to A$. Consider the map $g : \mathbb{N} \to f(\mathbb{N})$ given by $g(n) = f(n)$. Clearly, the map g is bijective and hence the subset $f(\mathbb{N}) \subseteq A$ is countable.

11. *Prove that any subset of a countable set is atmost countable.

Solution: Suppose $A \subseteq B$, where B is countable.

If B is finite, then A is finite.

If B is infinite, $\exists$ a bijection $f : B \to \mathbb{N}$. Note that the map $g : A \to f(A)$, defined by $g(a) = f(a)\ \forall a \in A$, is a bijection. So A and $f(A)$ have the same cardinality. Since

$f(A) \subset \mathbb{N}$, by Theorem stated in class, $f(A)$ is almost countable. Hence, A is almost countable.

12. *Prove that finite union of countable set is countable.

Solution: It is enough to prove that union of two countable sets is countable. Let $A = \{a_1, a_2, \ldots, a_n, \ldots\}$,($a_i$'s are all distinct) $B = \{b_1, b_2, \ldots, b_n, \ldots\}$ (b_i's are all distinct) be two countable sets. We will consider two cases.

Case I : $A \cap B = \phi$

We define a map $g : \mathbb{N} \to A \cup B$ given by

$$g(n) = \begin{cases} a_k & \text{if } n = 2k \\ b_k & \text{if } n = 2k - 1 \end{cases}$$

Let $n \neq m$. If n, m are both even, then $g(n) = a_{\frac{n}{2}} \neq a_{\frac{m}{2}} = g(m)$. If n is even and m is odd, then $g(n) \in A$ and $g(m) \in B$. But $A \cap B = \phi$. So $g(n) \neq g(m)$. Similarly one can show that $g(n) \neq g(m)$, when n, m are both odd. So g is injective. Now let $x \in A \cup B$. Then $x \in A$ or $x \in B$, (not in both as A, B are disjoint). If $x \in A$, then $x = a_k$ for some $k \in \mathbb{N}$. Then $g(2k) = a_k = x$. So x has a preimage under g. Similarly, one can show that if $x \in B$, then x has a preimage under g. So g is surjective. Thus $A \cup B$ is countable.

Case II : $A \cap B \neq \phi$

Write $A \cup B = A \cup (B \setminus A)$. Note that $A \cap (B \setminus A) = \phi$ and $(B \setminus A) \subset B$ is atmost countable by problem no. 13. So by Case I, $A \cup B = A \cup (B \setminus A)$ is countable.

13. Let A be an infinite set and $B \subseteq A$ a finite set. Show that $A \setminus B$ is infinite.

Solution: We know that $A = B \cup \{A \setminus B\}$. We also know that union of two finite set is finite. If $A \setminus B$ is finite then $A = B \cup \{A \setminus B\}$ is also finite contradicting the fact that A is infinite.

14. Let A be uncountable and $B \subseteq A$ a countable set. Show that $A \setminus B$ is uncountable.

Solution: Similar as above.

15. *Show that for any infinite set A and a countable set B, the sets A and $A \cup B$ are of same cardinality.

Solution: Since A is infinite there exists a countable set $A' \subseteq A$. We are given B is countable. We now use the fact that union of two countable set is countable to produce a bijection $f : A' \to A' \cup B$. Now this allows us to define $g : A \to A \cup B$ as

$$g(a) = \begin{cases} f(a) & \text{if } a \in A' \\ a & \text{if } a \in A \setminus A' \end{cases}$$

Hence A and $A \cup B$ have same cardinality.

16. For a nonempty subset A, prove that the following are equivalent:

(a) A is atmost countable.

(b) There exists a $1-1$ map of A to $\mathbb{N}$.

(c) There exists an onto map of $\mathbb{N}$ to A.

Solution: (a)$\Rightarrow$(c) If A is finite, then there exists a bijection f from I_m to A. Now construct $g : \mathbb{N} \to I_m$ by

$$g(j) = \begin{cases} j \text{ if j} \leq \text{m}, \\ m \text{ if j} > \text{m} \end{cases}$$

Here g is onto gives $f \circ g : \mathbb{N} \to A$ is onto.

If A is infinite then there exists a bijection from $\mathbb{N}$ to A. So we are done.
(c)$\Rightarrow$(b) we define $h : A \to \mathbb{N}$, $h(a) = \min\limits_{x \in f^{-1}\{a\}} x$, then h is well defined as each $f^{-1}\{a\}$ has a least element from well ordering principle and injective as $f^{-1}\{a\} \cap f^{-1}\{b\} = \phi$ for $a \neq b$.
(b)$\Rightarrow$(a) If A is finite then we are done.
Since, there exists an $1-1$ map from A into $\mathbb{N}$ so that f is a bijection from A onto $f(A) \subseteq \mathbb{N}$. But we have any subset of $\mathbb{N}$ is countable. So A countable.

17. *Suppose that $A \subseteq B$ then prove that

  (a) B is finite $\implies$ A is finite.
  (b) A is infinite $\implies$ B is infinite.
  (c) B is countable $\implies$ A is countable.
  (d) A is uncountable $\implies$ B is uncountable.

Solution:

(a) As B is finite, we have a bijection f from B to I_m. Define $g : A \to B$ by $g(a) = a$. So, $f \circ g$ is $1-1$ from A into I_m, which gives A finite.

(b) Suppose B is not infinite, that is, finite, then from part (a) it follows that A is finite contradicting the hypothesis that A is infinite. Hence B is finite.

(c) If A is finite then it is atmost countable.Now, B is countable gives a bijection p from B to $\mathbb{N}$.So that $f = p \circ g$ is $1-1$ map from A into $\mathbb{N}$ and a bijection from A onto $f(A) \subseteq \mathbb{N}$. Again we have $f(A)$ countable [As any infinite subset of $\mathbb{N}$ is countable] gives A countable.

(d) Given A is uncountable. We will show this by contradiction. Assume B is finite or countable. Then by part (a) and (c), A has to be atmost countable. So it contradicts the fact that A is uncountable. So B has to be uncountable.

18. *Suppose $f : A \to B$ is injective then prove that

  (a) B is finite $\implies$ A is finite.
  (b) A is infinite $\implies$ B is infinite.
  (c) B is countable $\implies$ A is atmost countable.
  (d) A is uncountable $\implies$ B is uncountable.

Solution:

(a) Since the set B is finite, there exists a bijection, $g : B \to I_m$ for some $m \in \mathbb{N}$, where $I_m = \{1, 2, \cdots, m\}$.
Consider the composition of the map,

$$h := g \circ f : A \to I_m.$$

Both the maps f and g are one-one, implies that the composition map, $h := g \circ f$, is also one-one (composition of two one-one is one-one). We conclude that A is finite, using the proposition discussed in the class, which states that if $f : A \to I_m$ is $1-1$, then A is finite and $|A| \leq m$ (don't confuse this 'f' in the proposition by the 'f' given in the problem).

(b) We will prove this statment by contradiction. Let, if possible, B is not infinite, i.e, B is finite. Recall that a set is either finite or infinite.
From part (a), we know that if B is finite, then A must be finite. However, this contradict the given assumption that A is infinite.
Thus, our assumption that B is finite must be false. Therefore, B must be infinte.

(c) By the definition, the set B being countable means that there exists a bijective map $g : B \to \mathbb{N}$. Now, consider the composition map $g \circ f : A \to \mathbb{N}$, where $f : A \to B$ is the one-one map given in the problem. Since f is one-one and g is bijective, their composition $g \circ f$ is an injective map from A to $\mathbb{N}$.
By Problem 18(b), a set that admits an injective map into $\mathbb{N}$ is atmost countable. Therefore, A is atmost countable.

(d) We will prove this statement by contradiction. Let, if possible, B is not uncountable, i.e, B is either finite or countable. We will give the contradiction separably for both cases,

i. B **is finite**: From part (a), we know that if B is finite, then A must be finite. However, this contradict the given assumption that A is uncountable.

ii. B **is countable**: From part (c), we know that if B is countable, then A is atmost countable. However, this again contradict the given assumption that A is uncountable.

Thus, our assumption that B is atmost countable must be false. Therefore, B must be uncountable.

19. *Suppose $f : A \to B$ is surjective then prove that

(a) A is finite $\implies$ B is finite.

(b) B is infinite $\implies$ A is infinite.

(c) A is countable $\implies$ B is atmost countable.

(d) B is uncountable $\implies$ A is uncountable.

Solution:

(a) Since the set A is finite, there exists a bijection, $g : I_m \to A$ for some $m \in \mathbb{N}$, where $I_m = \{1, 2, \cdots, m\}$.
Consider the composition of the map,

$$h := f \circ g : I_m \to B.$$

Both the maps f and g are surjective, implies that the composition map, $h := f \circ g$, is also surjective (composition of two surjective map is surjective). We conclude that B is finite, using the proposition discussed in the class, which states that if $f : I_m \to A$ is surjective, then A is finite and $|A| \leq m$ (do not confuse this 'f' in the proposition by the 'f' given in the problem).

(b) We will prove this statment by contradiction. Let, if possible, A is not infinite, i.e, A is finite. Recall that a set is either finite or infinite.
From part (a), we know that if A is finite, then B must be finite. However, this contradict the given assumption that B is infinite.
Thus, our assumption that A is finite must be false. Therefore, A must be infinte.

(c) By the definition, the set A being countable means that there exists a bijective map $g : \mathbb{N} \to A$. Now, consider the composition map $f \circ g : \mathbb{N} \to B$, where $f : A \to B$ is the surjective map given in the problem. Since f is surjective and g is bijective, their composition $f \circ g$ is a surjective map from $\mathbb{N}$ to B.
By Problem 18(c), if there exists surjective map from a set $\mathbb{N}$ into B, then the set B is atmost countable. Therefore, B is atmost countable as $f \circ g : \mathbb{N} \to B$ is the required surjective map.

(d) We will prove this statement by contradiction. Let, if possible, A is not uncountable, i.e, A is either finite or countable. We will give the contradiction separably for both cases,

i. A **is finite**: From part (a), we know that if A is finite, then B must be finite. However, this contradict the given assumption that B is uncountable.

ii. A **is countable**: From part (c), we know that if A is countable, then B is atmost countable. However, this again contradict the given assumption that B is uncountable.

Thus, our assumption that A is atmost countable must be false. Therefore, A must be uncountable.