Lecture 11 : Feb 19, 2025

Basis — uniqueness of representation of vectors.

Def^n: If $\mathcal{B} = \{x_1 \dots x_k\}$ is a basis for $V/\mathbb{F}$, then the DIMENSION of V over $\mathbb{F}$, denoted by $\dim_{\mathbb{F}} V$, is $|\mathcal{B}|$ = cardinality of a basis.

- $\dim_{\mathbb{R}} \mathbb{R} = 1$, $\{1\}$ basis of $\mathbb{R}/\mathbb{R}$
- $\dim_{\mathbb{R}} \mathbb{R}^n = n$, $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$ a basis of $\mathbb{R}^n/\mathbb{R}$.

Now we solve $Ax = 0$, we already know $0 \in N(A)$.

$\longrightarrow$ We will describe $N(A)$ in terms of dimension and basis.

Rephrasing the claim:

Columns of a $m \times n$ matrix with $m < n$ are linearly dependent.

Now we see whether for $m < n$,

$$N(A) \neq \{0\}, \text{ where } A \text{ is an } m \times n \text{ matrix},$$

enables us to answer the question.

Let us formalise a bit more what we have learnt towards solving a system of equations.

$$\underline{Ax = b.}$$

3 - elementary row operations

1. multiplication of one row of A by a non-zero scalar c

$\longrightarrow$ denoted by. $E_j(c)$.

$$\sim_{j\text{th row}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & c & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

$$D - \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} = \underline{E_1(d_1) \dots E_i(d_i) \dots E_n(d_n)}.$$

2. replacement of ith row by row i plus c times j th row ~ denoted by $E_{ij}(c)$

$$3\times 3 \quad E_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$E_{ij}(c)$, $i > j$ – gives lower triangular matrix

$E_{ij}(c)$, $i < j$ – gives upper triangular matrix.

3. interchange of two rows of A (P_{ij})

3×3 - matrix $P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem:

To each elementary row operation e, there corresponds an elementary row op e_1, of the same type as e, such that

$$e(e_1(A)) = e_1(e(A)) = A \text{ for each } A.$$

- In other words, the inverse operation of an elementary row operation exists and is an elementary row operation of the same type.

1. $E_i\left(\frac{1}{c}\right)E_i(c) = I = E_i(c)\,E_i\left(\frac{1}{c}\right)$.

$c \neq 0$

2. $E_{ij}(-c)\,E_{ij}(c) = I = E_{ij}(c)\,E_{ij}(-c)$.

3. $P_{ij}^{\,2} = I$.

Defn: A & B - $m\times n$ matrices

We say B is row-equivalent to A if B can be obtained from A by a finite sequence of row operations.

Theorem: If A & B are row equivalent, then $Ax=0$ and $Bx=0$ have exactly same solutions.

Pf: B row equivalent to A

$$\Rightarrow B = E_k \cdots E_3 E_2 E_1 A.$$

So $Ax=0 \Rightarrow E_k \cdots E_3 \; E_1 \; A x = 0 \Rightarrow Bx = 0$

on the other hand

$$Bx=0 \Rightarrow E_1^{-1} \cdots E_k^{-1} \, B \, x = 0$$

$$\Rightarrow Ax = 0.$$

Echelon form: (staircase form).

A $m \times n$ matrix $\mathcal{E}$ is called an Echelon matrix if

i) Every row of $\mathcal{E}$ in which all the entries are zero occurs below every row which has non-zero entry.

ii) If rows $1, 2, \ldots r$ are nonzero rows of $\mathcal{E}$ and if the leading entry of row i occurs in column k_i, $1 \le i \le r$, then

$$k_1 < k_2 \cdots < k_r.$$

Tweaking a little from what we have called earlier, we say **PIVOTS** are the first nonzero entries in each row of an Echelon form.

So by definition,

the ik_i th entry, $1 \le i \le r$, are called the **PIVOTS**

Example:

$$E = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

PIVOT

PIVOT

PIVOT

1. $$\begin{bmatrix} 1 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

PIVOT

- here - 2 - nonzero rows.
- zero row at below

$k_1 = 1$, $k_2 = 5$

k_i : column where the first non-zero entry appears in row i

— Echelon matrix.

Echelon matrix.

2.

PIVOTS
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- 3 - non zero rows
- zero row below
- $k_1 = 1$, $k_2 = 2$, $k_3 = 3$.
- Echelon matrix.

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 3 & 4 & 5 & 1 & 1 \\ 4 & 5 & 7 & 2 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{pmatrix}$$

- 4×5 matrix.
- We shall find its Echelon form via finitely many row operations.

- 4 rows - are nonzero to start with.

Let n_i be the column where first non-zero entry appear in row i.

Here $n_1 = 1 = n_2 = n_3 = n_4$. $\min\{n_1, \dots, n_4\} = 1$.

We can choose any row with whose entry at column 1 is non-zero, that is, the first entry is non-zero - this property here is satisfied by all 4 rows.

- We can pick anyone - here, we pick the first one.
- So $(1,1)$ entry is not zero and we make all other entries below it, that is all entries in the first column 0.

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 3 & 4 & 5 & 1 & 1 \\ 4 & 5 & 7 & 2 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{pmatrix} \xrightarrow[\substack{R_3 - 4R_1 \\ R_4 - R_1}]{R_2 - 3R_1} \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Now 1st row - we are done with.

remain 3 non-zero rows,

n_i : column in which the first nonzero entry of row i appears.

$i = 2, 3, 4.$

$n_2 = 2, \; n_3 = 2, \; n_4 = 4$

$\min \{n_2, n_3, n_4\} = 2.$

So we can choose 2nd or 3rd row $\longrightarrow$

We choose 2nd row - and make every thing below the first non zero entry in 2nd row that appears in the 2nd column, ie below 22 th position, 0.

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Since the 3rd row becomes a zero row, we shift it to last row by interchanging with the 4th row

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 4 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \text{Echelon matrix.}$$

PIVOTS

<u>Thm</u>: Every $m \times n$ matrix is row equivalent to an echelon matrix. This proof is algorithmic.

<u>Pf.</u> We shall first identify the existing zero rows and use the permutation matrix to keep it below every non zero row.

Let $1, 2, \ldots r$ be the non zero rows.

Let n_i be the column where first non-zero entry appear for i^{th} row, $1 \leq i \leq r$.

Consider $\min\{n_i : 1 \leq i \leq r\}. = k_1$ (say)

Use permutation matrix to make it first row. and then use $E_{i1}(c)$ $1 < i \leq r$ to make all the entries below the k_1 column zero.

Clearly $k_1 \geq 1$.

Now we repeat the same to the rows $2, \ldots, r$, to arrive at an Echelon matrix after finitely many steps. ∎

For an $n \times n$ matrix, the echelon form is an upper triangular matrix.

We have seen that an upper triangular matrix is invertible iff the PIVOTS appear in the diagonal.

Earlier, when we consider such cases we called those diagonal entries as PIVOTS.

Non-example:

$$\begin{array}{c} \\ k_2 = 1 \\ \\ \end{array} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad k_1 = 2$$

— NOT an Echelon matrix

$k_1 = 2 \qquad k_2 = 1, \ k_3 = 3$

So $k_1 > k_2$

BUT to be echelon form 1st row non-zero entry should appear in the left of the column where the 2nd row non-zero entry appears!

Defn - Row-reduced Echelon matrix:

An $m \times n$ matrix A is called a ROW-REDUCED ECHELON MATRIX if

i) A is an Echelon matrix

ii) In a non-zero row, the PIVOT equals 1 and it is the only non-zero element in its column.

Example:

PIVOTS

$$R = \begin{bmatrix} 1 & 0 & * & 0 & * & * & 0 & * \\ 0 & 1 & * & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

PIVOTS

another example

$$\begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is the next set of operation we did to invertible upper triangular matrix to get identity - in terms of equations, we made the diagonal entries 1 and then use back substitution - GAUSSIAN ELIMINATION

Some non-examples:

$k_1 > k_2, k_3$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

it is not an echelon matrix violating i)

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

zero row above nonzero row

it is not an echelon matrix violating i)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

nonzero entry in the column containing the leading non-zero entry of some row violating 2nd part in ii)

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

leading non-zero entry of first row is not 1, violating the 1st part in ii).

Theorem: Every $m \times n$ matrix A is row-equivalent to a row reduced echelon matrix.

Such a row reduce echelon matrix is called ROW-REDUCED ECHELON FORM OF A.

— We shall see later that such form is unique for a matrix.

Pf: This proof is algorithmic.

We know that A is row equivalent to an echelon matrix. We then multiply each PIVOT by its inverse to make it 1, that is, if ik_i-th entry is denoted by a_{ik_i} then multiply by the matrix by $E_i\left(\frac{1}{a_{ik_i}}\right)$ as PIVOT-entry is non-zero for every non-zero row i, $1 \le i \le r$.

Next we use E_{ji}-type matrices for $1 \le j < i$ so that the column above the ik_i-th entry becomes 0. We do this for each non-zero row to achieve the row-reduced echelon matrix. ∎

Remark: For any $m \times n$ matrix A, we can find a matrix E which is a product of elementary matrix, such that

$$EA = R \text{ (row-reduced echelon matrix)}$$

Coming back to the eqn. $Ax = 0$, we now know that if R is a row-reduced echelon form of A, then $Ax = 0$ and $Rx = 0$ have same solutions.

Now consider $RX = 0$, where R is row reduced echelon matrix with r nonzero rows.

The variables corresponding to the $k_1, \ldots, k_r$-th columns — $x_{k_1}, \ldots x_{k_r}$

— we call them BASIC / PIVOT variables. (r — many)

Rest of the variables — x_j such that $1 \le j \le n$ and $j \ne k_i$ $\forall i$ with $1 \le i \le r$,

— Called FREE variables

So we have $(n-r)$ free variables.

Let us take $\alpha_1, \ldots \alpha_{n-r}$ fixed but arbitrary scalars from the scalar field (which here is $\mathbb{R}$),

and let x_j's $= \alpha_1 \ldots \alpha_{n-r}$.

We can write the equations as follows:

$$x_{k_1} + \sum_{j=1}^{n-r} c_{1j} \alpha_j = 0 \Rightarrow x_{k_1} = -\sum_{j=1}^{n-r} c_{1j} \alpha_j$$

$$\vdots$$

$$x_{k_r} + \sum_{j=1}^{n-r} c_{rj} \alpha_j = 0 \Rightarrow x_{k_r} = -\sum_{j=1}^{n-r} c_{rj} \alpha_j$$

– Note that many c_{ij}'s could be zero.

All the solution of the system of equations $Rx = 0$ are obtained by assigning any values to the free variables x_j's, $1 \le j \le n$, $j \ne k_1, k_2, \ldots, k_r$, and then solving the basic / pivot variables $x_{k_1}, \ldots, x_{k_r}$ in terms of $\alpha_1, \ldots, \alpha_{n-r}$.

Example:

Consider the row reduced echelon matrix

$$R = \begin{pmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$RX = 0 \iff \begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 &= 0 \\ x_4 + 2x_5 &= 0. \end{aligned}$$

x_2, x_4 – basic / pivot variables.

x_1, x_3, x_5 – free variables

Let $x_1 = \alpha_1$, $x_3 = \alpha_2$, $x_5 = \alpha_3$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

$$x_2 - 3\alpha_2 + \frac{1}{2}\alpha_3 = 0$$

$$x_4 + 2\alpha_3 = 0$$

Consider $\cdot\ \alpha_1 = 1,\ \alpha_2 = 0,\ \alpha_3 = 0$

$$x_2 = 0 = x_4$$

$\cdot\ \alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = 0$

$$x_2 = 3,\ x_4 = 0$$

$\cdot\ \alpha_1 = 0 = \alpha_2,\ \alpha_3 = 1$

$$x_2 = -1/2,\ x_4 = -2.$$

We can write the solution

$$\begin{pmatrix} \alpha_1 \\ 3\alpha_2 - \frac{1}{2}\alpha_3 \\ \alpha_2 \\ -2\alpha_3 \\ \alpha_3 \end{pmatrix}$$

So the null space i.e the sol^n. of $RX = 0$ is spanned by 3 - vectors

$$= \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ -1/2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Since these vectors has 1's and 0's at different positions of the same co-ordinates, the vectors are linearly independent. (one can just check directly to understand).

So $\dim N(R) = 3 = \underset{\text{no. of columns}}{5} - \underset{\text{no. of rows}}{2}$

In general $N(R)$?

the pivot variables $x_{k_1}, \ldots x_{k_r}$

let the free variables take the values $\alpha_1 \ldots \alpha_{n-r}$, fixed but arbitrary in scalar field. (here it is $\mathbb{R}$).

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ x_{k_1} \\ \vdots \end{pmatrix} \qquad \vdots$$

$$= \alpha_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ -c_{11} \\ \vdots \\ -c_{21} \end{pmatrix}}_{v_1} + \alpha_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ -c_{21} \\ , \\ \vdots \end{pmatrix}}_{v_2}$$

$$+ \quad \alpha_{n-r} \begin{pmatrix} \cdot \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

$$v_{n-r}.$$

The vectors $v_1 \ldots v_{n-r}$ are linearly independent

as $\sum_{i=1}^{n-r} \alpha_i v_i = 0$,

& they span $N(R)$.

So $\{v_1 \ldots v_{n-r}\}$ forms a basis

of $N(R)$ and $\dim N(R) = n-r$.

Theorem: The row-reduced echelon form of a matrix is unique.

Pf: The proof goes by induction on the number n of columns of the $m \times n$ matrix A.

For $n = 1$, we have only 1 column – the row-reduced echelon matrix is $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ if A is nonzero.

So the result holds for $n=1$.

Assume that $n>1$.

Induction hypothesis: The result holds for $(n-1)$.

Let A be an $m\times n$ matrix and A' be the matrix obtained from A by deleting n^{th} column. Suppose B and C are two distinct row-reduced echelon form for A.

Since any sequence of elementary row operations bringing A to row-reduced echelon form also bring A' to a row-reduced echelon form, by applying Induction Hypothesis B and C can differ only in the n^{th} column.

Let j be such that $b_{jn} \neq c_{jn}$.

Since A is row-equivalent to both B and C $\{x : Bx=0\}$ & $\{x : Cx=0\}$ is same as $\{x : Ax=0\}$.

So for such x, $(B-C)x=0$.

Since B and C differ in the n-th column only, the j-th eq^n of the system $(B-C)x=0$

reads $(b_{jn} - c_{jn})\, x_n = 0$

and hence $x_n = 0$.

So all the solutions of $Ax = 0$ have $x_n = 0$.

We claim that the n^{th} column of B and C must be a column containing pivot.

If not then all the rows below the non-zero rows in A' are zero,

So B & C would look like

$$B = \left(\begin{array}{c|c} \tilde{A} & b \\ \hline 0 & 0 \end{array}\right)$$

$$C = \left(\begin{array}{c|c} \tilde{A} & c \\ \hline 0 & 0 \end{array}\right)$$

which shows that there exists solutions with $x_n = 1$ — contradiction.

Thus B & C must have pivot in the n^{th} column.

Again, since B and C only differ in the last column and since they are in row-reduced echelon form, the row in which the pivot in the last column appears is the same for B and C.

As B & C are row-reduced echelon matrices all the other entries in the last column of B and C are equal to zero, and hence B and C have the same n^{th} column, contradicting the fact that $b_{jn} \neq c_{jn}$.

Thus $B = C$ and hence by induction, the proof is complete. ∎

Theorem: If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

Pf: Let R be a row-reduced echelon matrix which is row equivalent to A.

Then $AX = 0$ & $RX = 0$ has same solutions.

If r is the no. of nonzero rows in R, then certainly $r \leq m$ & as $m < n$, we have $r < n$.

So $\dim N(R) = n - r > 0$

Hence it has non-trivial solutions. ∎

Theorem: If A is an $n \times n$ (square) matrix, then A is row equivalent to identity matrix iff the system of equations $AX = 0$ have trivial solution.

Pf: A is row equivalent to I

$\Rightarrow$ $AX = 0$ & $IX = 0$ have same solution

$\Rightarrow$ $AX = 0$ has trivial solution only.

$(\Leftarrow)$

$AX = 0$ has only trivial solution.

Let R be a row-reduced echelon form of the matrix A.

$RX = 0$ has only trivial solution.

$\Rightarrow$ r can't be less than n.

r → no. of non-zero rows

$\Rightarrow$ $r \geq n$.

On the other hand keeping the number of non-zero rows $r \leq n$

$\Rightarrow r = n$.

So each diagonal entry is either 1 and all others are 0 $\Rightarrow R = I$. ∎

<u>Remark</u>: We have noted that

$$\{x : Ax = 0\} = \{x : Rx = 0\}$$

where R is 'the' row-reduced echelon form of A.

So it is enough to solve $Rx = 0$ to solve the homogeneous system of equations

$$Ax = 0.$$

Now to solve $Ax = b$, we apply the elementary row operations that transforms A to R, that is, $EA = R$ for some E which is product of elementary matrices.

Since E is invertible,

$$\{x : Ax = b\} = \{x : EAx = Eb\}$$

$$= \{x : Rx = Eb\}.$$

So to solve the non-homogeneous system of equations

$$Ax = b, \quad \text{where } A \text{ is } m \times n\text{-matrix}$$

We consider the 'Augmented matrix' of the system $[A \mid b]$ - $m \times (n+1)$ matrix and apply the elementary row operations to get the matrix

$$(R \mid Eb)$$

and solve for $Rx = 0$

& find a particular sol^n. x_0 of $Rx = b$

to get all the solutions of

$$\underline{Ax = b.}$$

that is, since $N(A) = N(R)$

we have

$$\{x : Ax = b\}$$

$$= \{x : Rx = Eb\}$$

$$= \{x : x = x_0 + y \text{ for a fix } x_0 \text{ with } Rx_0 = Eb \text{ \& } y \in N(R)\}.$$