Recall each set A is assigned a symbol in such a way that two sets A and B are assigned same symbol if and only if there is a bijection between them. This symbol is called the *cardinality* or *cardinal number* of A and is denoted by $|A|$.

α is a cardinal number if there exists a set A such that $|A| = \alpha$.

Notation: $|\mathbb{N}| = \aleph_0$, $|\mathbb{R}| = \aleph_1$.

1. Let α and β be two cardinal numbers, let A and B be two disjoint sets with $|A| = \alpha$ and $|B| = \beta$. Then sum of α and β is denoted and defined by

$$\alpha + \beta := |A \cup B|.$$

   (a) Show that there exists two such disjoint sets. (Hint: A and $A \times \{1\}$ are in bijection)

   (b) *Show that the sum is well-defined.

Solution:

- Let A and B be two sets with cardinalities α and β. Consider the sets,

$$A^{'} := A \times \{1\} \text{ and } B^{'} := B \times \{2\}$$

where $A \times \{1\} = \{(a, 1)|\ a \in A\}$ and $B \times \{2\} = \{(b, 2)|\ b \in B\}$.
These two sets $A^{'}$ and $B^{'}$ has same cardinality as of the sets A and B, using the bijection f and g, defines as,

$$f : A^{'} \to A, \ \ f((a, 1)) = a$$

and

$$g :: B^{'} \to B, \ \ g((b, 2)) = b$$

Thus by the definition of cardinality, $|A^{'}| = |A| = \alpha$ and $|B^{'}| = |B| = \beta$. Moreover, the sets $A^{'} := A \times \{1\}$ and $B^{'} := B \times \{2\}$ is disjoint (the second component is different).

- Note:- *The definition of sum $\alpha + \beta$ depends on the choice of the two disjoint set A and B having the cardinality α and β, but there can be many such sets having the same cardinality. In this problem what we will show that the sum is actually independent of the choice of the disjoint set A and B*

  Let, A and $A^{'}$ be two sets with same cardinality α, B and $B^{'}$ be a two sets of cardinality β such that A, B and $A^{'}, B^{'}$ are disjoint. (*to show,* $|A \cup B| = |A^{'} \cup B^{'}|$, *we will construct a bijective function between the set,* $A \cup B$ *and* $A^{'} \cup B^{'}$)

  Now, define the map $h : A \cup B \to A^{'} \cup B^{'}$ by

  $$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$$

  where $f : A \to A^{'}$ and $g : B \to B^{'}$ is the bijection, which exists because of the assumption $|A| = |A^{'}|$ and $|B| = |B^{'}|$

  Now we will prove h is bijective.

  (a) **one-one**: Let $x, y \in A \cup B$

    i. **Case 1**: $x, y \in A$ and $h(x) = h(y) \implies f(x) = f(y)$, using the fact that f is one-one implies that $x = y$.

    ii. **Case 2**: $x, y \in B$ and $h(x) = h(y) \implies g(x) = g(y)$, using the fact that g is one-one implies that $x = y$.

    iii. **Case 3**: Let $x \in A, y \in B$ such that, $h(x) = h(y)$. This shows that, $f(x) = g(y)$ but $f(x) \in A^{'}$, $g(y) \in B^{'}$. Thus, $f(x) = g(y) \in A^{'} \cap B^{'}$, which is a contradiction to the assumption that $A^{'} \cap B^{'} = \phi$ and hence this case is not possible.

  (b) **onto**: Let $y \in A^{'} \cup B^{'}$, i.e., $y \in A^{'}$ or $y \in B^{'}$

    i. **Case 1**: If $y \in A^{'}$, since f is bijective and hence onto, there exists $x \in A$ such that $f(x) = y$. Now by the definition of h, we have $h(x) = f(x) = y$.

    ii. **Case 2**: If $y \in B^{'}$, since g is bijective and hence onto, there exists $x \in B$ such that $g(x) = y$. Now by the definition of h, we have $h(x) = g(x) = y$.

Thus h is both one-one and onto and hence bijective. Now, by the definition of cardinality we have, $|A \cup B| = |A^{'} \cup B^{'}|$. Thus,

$$\alpha + \beta = |A \cup B| = |A^{'} \cup B^{'}|$$

Which shows that the sum of two cardinal number is well-defined and we are done.

2. The product of two cardinal numbers α and β is denoted and defined by

$$\alpha\beta := |A \times B|$$

where A and B are two sets with $|A| = \alpha$ and $|B| = \beta$. Show that the product is well defined.

Solution: Let $A, A^{'}$ and $B, B^{'}$ be a pair of sets with cardinality α and β. Define the map,

$$h : A \times B \to A^{'} \times B^{'}, \quad h(a, b) = (f(a), g(b))$$

where $f : A \to A^{'}$ and $g : B \to B^{'}$ is a bijection, which exists because of the assumption $\alpha = |A| = |A^{'}|$ and $\beta = |B| = |B|^{'}$. Now we will show h is bijective,

(a) **one-one**: Let $(a, b), (c, d) \in A \times B$ and $h((a, b)) = h((c, d))$; $(f(a), g(b)) = (f(c), g(d))$ which gives $f(a) = f(c)$ and $g(b) = g(d)$. Using the fact that f, g is one-one, it follows that $a = c, b = d$. Thus, $(a, b) = (c, d)$.

(b) **Onto**: Let, $(c, d) \in A^{'} \times B^{'}$; $c \in A^{'}$ and $d \in B^{'}$. Since f and g is onto there exists, $a \in A$ and $b \in B$, such that $f(a) = c, g(b) = d$. Then $h(a, b) = (f(a), g(b)) = (c, d)$. This shows that h is onto.

Thus, $|A \times B| = |A^{'} \times B^{'}|$ and the product is well defined.

3. Let α and β be two cardinal numbers and A and B are two sets with $|A| = \alpha$ and $|B| = \beta$. Then α^{β} is defined by

$$\alpha^{\beta} := |A^B| = |\{f | f : B \to A\}|.$$

Show that taking the exponent of a cardinal number is well defined.

Solution: Let $A, A^{'}$ and $B, B^{'}$ be two pairs of sets with cardinality α and β respectively. Then there exist bijections $h : A \to A^{'}$ and $g : B \to B^{'}$. We want to verify $|A^B| = |A^{'B^{'}}|$ equality holds. Now define

$$\begin{aligned}
h_* :& A^B \to A^{'B}, \quad h_*(f) := h \circ f, \text{ for } f : B \to A \\
\bar{h}_* :& A^{B^{'}} \to A^{'B^{'}}. \quad \bar{h}_*(f) := h \circ f, \text{ for } f : B^{'} \to A \\
g^* :& A^B \to A^{B^{'}}, \quad g^*(f) := f \circ g^{-1}, \text{ for } f : B \to A \\
\bar{g}^* :& A^{'B} \to A^{'B^{'}}, \quad \bar{g}^*(f) := f \circ g^{-1}, \text{ for } f : B \to A^{'}
\end{aligned}$$

which fits into the commutative diagram

$$\begin{tikzcd}
A^B \arrow[r, "h_*"] \arrow[d, "g^*"] & A^{'B} \arrow[d, "\bar{g}^*"] \\
A^{B^{'}} \arrow[r, "\bar{h}_*"] & A^{'B^{'}}
\end{tikzcd}$$

Note that each of the above maps has an inverse, for example define ${h_*}^{-1} : A^{'B} \to A^B$ defined as ${h_*}^{-1}(f) := h^{-1} \circ f$, similarly we can define the inverse of other maps. Hence $\bar{g}^* \circ h_* = \bar{h}_* \circ g^* : A^B \to A^{'B^{'}}$ is a bijection. **Work out the details**.

4. Prove that $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 \aleph_0 = \aleph_0$.

Solution:

- Let $A = \{2n : n \in \mathbb{N}\}$ and $B = \{2n - 1 : n \in \mathbb{N}\}$. Then $|A| = |B| = \aleph_0$.
  Also, $A \cup B = \mathbb{N}$, so $|A \cup B| = \aleph_0$.
  Using the definition of the sum of cardinals, we have
  $$\aleph_0 + \aleph_0 = |A \cup B| = \aleph_0.$$
- Let $A = B = \mathbb{N}$. Then $|A| = |B| = \aleph_0$.
  Also, $A \times B = \mathbb{N} \times \mathbb{N}$ is countable, and so $|A \times B| = \aleph_0$. Using the definition of the product of cardinals, we have
  $$\aleph_0 \aleph_0 = |A \times B| = \aleph_0.$$

5. Let α be an infinite cardinal number. Prove that $\aleph_0 + \alpha = \alpha$.

Solution: Let A and B be two disjoint sets such that $|A| = \aleph_0$ and $|B| = \alpha$. Then

$$\aleph_0 + \alpha = |A \cup B| \tag{1}$$

Since, α is an infinite cardinal number, so the set B is infinite.
Note that the set A is countable and B is infinite, so the sets B and $A \cup B$ have the same cardinality. (Q-14 from P.S.1), i.e,

$$|A \cup B| = |B| = \alpha \tag{2}$$

Thus, using (1) and (2), we get

$$\aleph_0 + \alpha = \alpha.$$

6. *Prove that $\aleph_0 \aleph_1 = \aleph_1$.

Solution: We know that there exist a bijection from $\mathbb{Z} \to \mathbb{N}$ and a bijection from $[0,1) \to \mathbb{R}$. Let us consider the map $f : \mathbb{Z} \times [0,1) \to \mathbb{R}$ which is defined by $f(n,x) = n+x$. We show that f is bijective. Since, every real number x, we have $x = \lfloor x \rfloor + \{x\}$, where $\{x\}$ is the fractional part of x. Therefore, $f(\lfloor x \rfloor, \{x\}) = x$, so f is surjective. Suppose $(x,n) \neq (y,m)$,then either $(x \neq y, m = n)$ or $(x = y, m \neq n)$ or $(x \neq y, m \neq n)$.
Case-1: If $(x \neq y, m = n)$, then $n + x \neq m + y$, so f is one-one.
Case-2: If $(x = y, m \neq n)$, then $n + x \neq m + y$, so f is one-one.
Case-3: If $(x \neq y, m \neq n)$, Since $\{n + x : x \in [0,1)\}$ and $\{m + y : y \in [0,1)\}$ are disjoint (?). So $n + x \neq m + y$, so f is one-one. Therefore, f is bijective and $\aleph_0 \aleph_1 = \aleph_1$.

Alternative solution

We have $|\mathbb{R}| = \aleph_1$ and $|\mathbb{N}| = \aleph_0$. Then $|\mathbb{N} \times \mathbb{R}| = \aleph_0\aleph_1$. Consider the map, $i : \mathbb{R} \hookrightarrow \mathbb{N} \times \mathbb{R}$, defined by $i(x) = (1, x)$, clearly which is one one, hence $\aleph_1 \leqslant \aleph_0\aleph_1$. Again, consider the inclusion map $\mathbb{N} \times \mathbb{R} \hookrightarrow \mathbb{R} \times \mathbb{R}$, which is one-one, so $\aleph_0\aleph_1 \leqslant \aleph_1\aleph_1$. We know that $\aleph_1 = 2^{\aleph_0}$. Also by problem (4), we have $\aleph_0 + \aleph_0 = \aleph_0$. Therefore,

$$\aleph_0\aleph_1 \leqslant \aleph_1\aleph_1 = 2^{\aleph_0}2^{\aleph_0} = 2^{\aleph_0+\aleph_0} = 2^{\aleph_0} = \aleph_1$$

The 2nd equality above follows from Problem 11(h). Hence, there exist a injection from $\mathbb{N} \times \mathbb{R} \to \mathbb{R}$, so by *Schroeder-Bernstein Theorem*, there exist a bijection from $\mathbb{N} \times \mathbb{R} \to \mathbb{R}$ and $\aleph_0\aleph_1 = \aleph_1$.

7. *Show that $\mathbb{R}$ and $\mathbb{R}^2$ have same cardinality, in other words, $\aleph_1\aleph_1 = \aleph_1$. More generally, $|\mathbb{R}^n| = \aleph_1$ for all $n \in \mathbb{N}$.

Solution: As done in the last proof,

$$|\mathbb{R} \times \mathbb{R}| = \aleph_1\aleph_1 = 2^{\aleph_0}2^{\aleph_0} = 2^{\aleph_0+\aleph_0} = 2^{\aleph_0} = \aleph_1 = |\mathbb{R}|.$$

We prove by induction that $|\mathbb{R}^n| = |\mathbb{R}|$ for all $n \in \mathbb{N}$:

Base Case ($n = 1$): Trivially, $|\mathbb{R}^1| = |\mathbb{R}|$.

Inductive Step: Assume $|\mathbb{R}^k| = |\mathbb{R}|$ for some $k \geq 1$. Consider:

$$\mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R}.$$

Using problem (2) and $|\mathbb{R}^k| = |\mathbb{R}|$(induction hypothesis), we have:

$$|\mathbb{R}^{k+1}| = |\mathbb{R}^k \times \mathbb{R}| = |\mathbb{R}^k||\mathbb{R}| = |\mathbb{R}||\mathbb{R}| = |\mathbb{R}|.$$

Hence, by induction, $|\mathbb{R}^n| = |\mathbb{R}|$ for all $n \in \mathbb{N}$.

Alternative solution

First part. Since $|(0,1)| = |\mathbb{R}|$, as $f:(0,1) \to \mathbb{R}$ defined by $f(x) = \ln\left(\frac{x}{1-x}\right)$ is a bijection, and we know that if $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$, it suffices to prove $|(0,1)\times(0,1)| = |(0,1)|$. To do so, we apply the Schröder–Bernstein theorem by constructing injections in both directions:

Injection $f : (0,1) \to (0,1)\times(0,1)$: Define $f(x) = (x, \frac{1}{2})$. This is injective because $f(x_1) = f(x_2) \implies x_1 = x_2$. Thus, $|(0,1)| \leq |(0,1) \times (0,1)|$.

Injection $g : (0,1) \times (0,1) \to (0,1)$: Represent $x, y \in (0,1)$ with unique decimal expansions that avoid an infinite string of 9's:

$$x = 0.a_1a_2a_3\ldots, \quad y = 0.b_1b_2b_3\ldots,$$

where $a_i, b_i \in \{0,1,2,\ldots,9\}$. Define:

$$g(x,y) = 0.a_1b_1a_2b_2a_3b_3\ldots.$$

This mapping interleaves the digits of x and y, creating a unique $g(x,y)$. Hence, g is injective, and $|(0,1) \times (0,1)| \leq |(0,1)|$.

By the Schröder–Bernstein theorem, $|(0,1) \times (0,1)| = |(0,1)|$.

8. Prove that $2^{\aleph_0} = \aleph_1$.

Solution:

Note that, from the solution of Q.1(i) of problem set 1, we have that $|\mathbb{R}| = |(0,1)|$. Since $(0,1) \subseteq [0,1) \subseteq \mathbb{R}$, therefore $|(0,1)| \leq |[0,1)| \leq |\mathbb{R}|$. Thus $|\mathbb{R}| = |[0,1)| = \aleph_1$.

Recall that $2^{\aleph_0} = |2^{\mathbb{N}}| = |\{0,1\}^{\mathbb{N}}|$ (as discussed in the class). We then proved in the class via binary representation that $|\{0,1\}^{\mathbb{N}}| = |[0,1)|$. Here we give an alternative proof.

Now we shall produce an injective map from $\{0,1\}^{\mathbb{N}}$ to $[0,1)$ and an injective map from $[0,1)$ to $\{0,1\}^{\mathbb{N}}$ in order to apply the Schr$\ddot{o}$eder-Bernstein theorem. Consider the map $f : \{0,1\}^{\mathbb{N}} \to [0,1)$ given by $f((a_1,a_2,\cdots)) = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$. We now show that f is injective.

Let $f((a_1,a_2,\cdots)) = f((b_1,b_2,\cdots))$ for $(a_1,a_2,\cdots),(b_1,b_2,\cdots) \in \{0,1\}^{\mathbb{N}}$. Then we have $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \sum_{n=1}^{\infty} \frac{b_n}{10^n}$, which implies that $a_n = b_n$ for all $n \in \mathbb{N}$ and hence $(a_1,a_2,\cdots) = (b_1,b_2,\cdots)$. Thus f is injective.

For the other way, for a real number x in $[0,1)$, let $0.b_1b_2\cdots$ be the unique binary expansion of x (as discussed in the class). Therefore the map $x \mapsto (b_1, b_2, \cdots)$ from $[0,1)$ to $\{0,1\}^{\mathbb{N}}$ is injective. Thus by Schröeder-Bernstein theorem we have that $|\{0,1\}^{\mathbb{N}}| = |[0,1)|$. Hence $2^{\aleph_0} = \aleph_1$.

9. Suppose α and β are cardinal numbers such that $\alpha \leq \beta$. Show that there exists a set S with a subset A such that $|A| = \alpha$ and $|S| = \beta$.

Solution: Let A_0 and B_0 be two sets such that

$$|A_0| = \alpha, \quad |B_0| = \beta.$$

Since $\alpha \leq \beta$, so there exists an injective map $f : A_0 \to B_0$. Define a map g as

$$g : A_0 \to f(A_0) \subset B_0 : g(a) = f(a).$$

Then, g is a bijective map, and so $|f(A_0)| = |A_0| = \alpha$.
Define $S := B_0$ and $A = f(A_0) \subset S$. Then, $|A| = \alpha$ and $|S| = \beta$.

10. *Let X, Y, X_1 be sets such that $X \supseteq Y \supseteq X_1$ and X and X_1 are in bijection.

(a) Prove using Schroeder-Bernstein Theorem that there exists a bijection between X and Y.

(b) Suppose it is known that whenever the hypothesis holds, the conclusion in part (a) is true. Using this prove Schroeder-Bernstein Theorem.

Solution:

- Let $h : X \to X_1$ be the bijection given in the assumption and consider the inclusion map,

$$k : X_1 \to Y,\ k(x) = x$$

  Now, define the inclusion of Y in X, i.e., $f : Y \to X$ by $f(y) = y$ (as $Y \subseteq Y$) and $g : X \to Y$ as $g(x) = (k \circ h)(x)$ which is injective as both h and k are injective.

Hence we have two injective map $f : Y \to X$ and $g : X \to Y$, using the *Schroeder-Bernstein Theorem* we conclude the result.

- Let $f : X \to Y$ and $g : Y \to X$ be two injective maps.
  Define the composition,

$$h := g \circ f : X \to X, h(x) := g(f(x))$$

both f and g are one-one and hence h is one-one (composition of one-one map is one-one). If we restrict the co-domain of h to its range $h(X)$, i.e.,
$h := g \circ f : X \to h(X)$. On its range, $h(X)$, then this map is also onto and hence $h : X \to h(X)$ is bijective.
We have the containment of the sets: $h(X) = g(f(X)) \subseteq g(Y) \subseteq X$.
Since X and $h(X)$ are in bijection by the map h. Then, by the part (a), there exists a bijection k from X to $g(Y)$. As, $g : Y \to g(Y)$ is a bijection, the composition map,

$$g^{-1} \circ k : X \to Y$$

where g^{-1} is the inverse of the map g restricting the co-domain to $g(Y)$, i.e., $g : Y \to g(Y)$. Now, both k and g^{-1} is a bijection, shows that $g^{-1} \circ k$ is bijection from X to Y and we are done. .

11. Show that for cardinal numbers α, β, γ,

(a) $\alpha + \beta = \beta + \alpha$

Solution: Let A and B be two disjoint sets with $|A| = \alpha$ and $|B| = \beta$. Then sum of α and β is defined by

$$\alpha + \beta := |A \cup B|.$$

Since $A \cup B = B \cup A$, therefore $\alpha + \beta = |A \cup B| = |B \cup A| = \beta + \alpha$.

(b) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

Solution: Let A, B and C be three pairwise disjoint sets with $|A| = \alpha$, $|B| = \beta$ and $|C| = \gamma$. Since $A \cup (B \cup C) = (A \cup B) \cup C$, therefore it follows that

$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(c) $\alpha\beta = \beta\alpha$

Solution: Let A, B be two sets with $|A| = \alpha, |B| = \beta$. Now, $|A \times B| = \alpha\beta$ and $|B \times A| = \beta\alpha$. We define $f : A \times B \to B \times A$ by $f(a, b) = (b, a)$. Then f is a bijection. So $\alpha\beta = \beta\alpha$.

(d) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

Solution: Let A, B, C be three sets with $|A| = \alpha, |B| = \beta, |C| = \gamma$. We define a function $f : A \times (B \times C) \to (A \times B) \times C$ by $f(a, (b, c)) = ((a, b), c)$. Then f is a bijection. Hence, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

(e) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Solution: Let A, B, C be three sets with $|A| = \alpha, |B| = \beta, |C| = \gamma$ with disjoint B and C. Now prove that $A\times(B\cup C) = (A\times B)\cup(A\times C)$ and $(A\times B)\cap(A\times C) = \varnothing$. Complete the proof.

(f) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$

Solution: We have $\alpha \leq \beta$. So there exists a $1 - 1$ map $f : A \to B$. For any cardinal no. γ there exists a set C such that A, B, C pairwise disjoint and $|C \cup B| = \beta + \gamma$ and $|C \cup A| = \alpha + \gamma$. [From question 1($i$)]

Now, construct $h : A \cup C \to B \cup C$ by $h(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in C \end{cases}$

Here h is an one-one map. So, $|A \cup C| \leq |B \cup C|$ gives $\alpha + \gamma \leq \beta + \gamma$.

(g) $(\alpha\beta)^\gamma = \alpha^\gamma\beta^\gamma$

Solution: Let A, B, C be sets such that $|A| = \alpha$, $|B| = \beta$ and $|C| = \gamma$. Define $f : (A \times B)^C \to A^C \times B^C$ as $f(g) = (p_1 \circ g, p_2 \circ g)$ where $p_1(a,b) = a$ and $p_2(a,b) = b$. We also have $h(g_1, g_2)(x) := (g_1(x), g_2(x))$, note that $h \circ f(g)(x) = f(p_1(g), p_2(g))(x) = g(x)$, i.e., $h \circ f = Id_{(A\times B)^C}$. Similarly we have $f \circ H = Id_{A^C \times B^C}$. Hence f is a bijection. Complete the details.

(h) $\alpha^\beta \alpha^\gamma = \alpha^{\beta+\gamma}$

Solution: Let A, B, C be sets such that $|A| = \alpha$, $|B| = \beta$ and $|C| = \gamma$ with B and C being disjoint. Define $f : A^B \times A^C \to A^{B\cup C}$ such that given $x \in B \cup C$ we have

$$f(g_1, g_2)(x) = \begin{cases} g_1(x) & \text{if } x \in B \\ g_2(x) & \text{if } x \in C \end{cases}$$

The map above is well defined as B and C are disjoint. We also have $h(g) := (g_{|_B}, g_{|_C})$, note that $h \circ f(g_1, g_2) = h(g_1 \bigsqcup g_2) = (g_1, g_2)$, where

$$(g_1 \bigsqcup g_2)(x) = \begin{cases} g_1(x) & \text{if } x \in B \\ g_2(x) & \text{if } x \in C \end{cases}$$

Hence we have $h \circ f = Id_{A^B \times A^C}$, similarly see that $f \circ h = Id_{A^{B\cup C}}$. Hence f is a bijection.

(i) *If $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$

Solution: We have $\alpha \leq \beta$. So for two sets A and B if $|A| = \alpha$ and $|B| = \beta$ then there exists an one-one map $h : A \to B$. Let C be a set with $|C| = \gamma$, then we have $|A \times C| = \alpha\gamma$, $|B \times C| = \beta\gamma$.
Construct $\tilde{h} : A \times C \to B \times C$ by $\tilde{h}(a, c) = (h(a), c)$. If $\tilde{h}(a_1, c_1) = \tilde{h}(a_2, c_2)$ implies $(h(a_1), c_1) = (h(a_2), c_2) \Rightarrow c_1 = c_2, a_1 = a_2$ [As h is one-one]. So $\tilde{h}$ is one-one. So, $|A \times C| \leq |B \times C|$ gives $\alpha\gamma \leq \beta\gamma$.

(j) If $\alpha \leq \beta$, then $\alpha^\gamma \leq \beta^\gamma$ and $\gamma^\alpha \leq \gamma^\beta$

Solution: We have $\alpha \leq \beta$. So for two sets A and B if $|A| = \alpha$ and $|B| = \beta$ then there exists an one-one map $h : A \to B$.
For any set C with $\gamma = |C|$ we have $\alpha^\gamma = |A^C|$ and $\beta^\gamma = |B^C|$.
Now construct $\Phi : A^C \to B^C$ by $\Phi(f) = h \circ f$, this is an one-one map as $\Phi(f_1) = \Phi(f_2)$ gives $h \circ f_1 = h \circ f_2$, so $h \circ f_1(c) = h \circ f_2(c)$ for all $c \in C$.
Since h one-one $h(f_1(c)) = h(f_2(c))$ implies $f_1(c) = f_2(c)$. So, $f_1 = f_2$.
This gives $|A^C| \leq |B^C| \Rightarrow \alpha^\gamma \leq \beta^\gamma$.

12. Prove that $\aleph_0^{\aleph_0} = \aleph_1$ and $^*\aleph_1^{\aleph_0} = \aleph_1$.

Solution: 1st Part: We have, from problem (8), $2^{\aleph_0} = \aleph_1$ and from problem (4), $\aleph_0\aleph_0 = \aleph_0$. Note that, $2 \leqslant \aleph_0$, therefore by problem 11(j), $\aleph_1 = 2^{\aleph_0} \leqslant \aleph_0^{\aleph_0}$. Again, we have $\aleph_0 \leqslant \aleph_1 = 2^{\aleph_0}$, therefore by problem 11(j),

$$\aleph_0^{\aleph_0} \leqslant (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0} = \aleph_1$$

Hence, by *Schroeder-Bernstein Theorem*, $\aleph_0^{\aleph_0} = \aleph_1$.
2nd Part: $\aleph_1^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0} = \aleph_1$, by problem (8), (4).

Main result used. For any three cardinalities α, β and γ,

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}.$$

Let A, B, C be sets such that $|A| = \alpha$, $|B| = \beta$, and $|C| = \gamma$. We consider the two quantities

$$(\alpha^\beta)^\gamma = |\{f \mid f : C \to A^B\}|$$

and

$$\alpha^{\beta\gamma} = |\{g \mid g : B \times C \to A\}|.$$

Define the sets

$$X = \{g \mid g : B \times C \to A\} \quad \text{and} \quad Y = \{f \mid f : C \to A^B\}.$$

Our goal is to establish that $|X| = |Y|$ by constructing a bijection between them.

Defining $G : X \to Y$

For $g \in X$, define

$$g_c(b) = g(b, c) \quad \text{for } b \in B, c \in C.$$

Clearly, for each $c \in C$, the function g_c belongs to A^B. Now define $\tilde{g} : C \to A^B$ by

$$\tilde{g}(c) = g_c.$$

Thus, we set $G(g) = \tilde{g}$, which maps g to the corresponding function in Y.

Defining $F : Y \to X$

For $f \in Y$, note that $f(c) \in A^B$ for each $c \in C$, meaning $f(c)$ is itself a function from B to A. Define

$$\hat{f}(b, c) = f(c)(b) \quad \text{for } b \in B, c \in C.$$

Then, $\hat{f}$ is a function in X, so we set $F(f) = \hat{f}$.

Verifying Bijection

To show that G and F are inverses, we first verify $G \circ F = I_Y$. For $f \in Y$,

$$G \circ F(f) = G(F(f)) = G(\hat{f}).$$

For each $c \in C$, we have

$$\tilde{\hat{f}}(c) = \hat{f}_c.$$

For $b \in B$,

$$\hat{f}_c(b) = \hat{f}(b, c) = f(c)(b),$$

which shows that $\hat{f}_c = f(c)$. Thus,

$$\tilde{\hat{f}}(c) = f(c) \quad \text{for all } c \in C.$$

Hence, $G \circ F(f) = f$, proving that $G \circ F = I_Y$.

Similarly, as shown in office hours, $F \circ G = I_X$, completing the proof that G and F are bijections. Thus, $|X| = |Y|$, as required. **Work out the details that are left.**

13. Show that $|\{f : \mathbb{R} \to \mathbb{R} | f \text{ is continuous}\}| = \aleph_1$.

(Assume that if f and g are such continuous functions and $f(q) = g(q)$ for all rational numbers $q \in \mathbb{Q} \subset \mathbb{R}$, then $f = g$, that is, $f(x) = g(x)$ for all $x \in \mathbb{R}$)

Solution: Let $C(\mathbb{R},\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$. We want to show that $|C(\mathbb{R},\mathbb{R})| = \aleph_1$.

Each continuous function $f \in C(\mathbb{R},\mathbb{R})$ is uniquely determined by its values on the rationals $\mathbb{Q} \subset \mathbb{R}$, because $\mathbb{Q}$ is dense in $\mathbb{R}$ and continuity ensures that knowing $f(q)$ for all $q \in \mathbb{Q}$ completely determines $f(x)$ for all $x \in \mathbb{R}$.

Thus, the set of continuous functions $C(\mathbb{R},\mathbb{R})$ has the same cardinality as the set of functions $g : \mathbb{Q} \to \mathbb{R}$. Since $|\mathbb{Q}| = \aleph_0$ and $|\mathbb{R}| = \aleph_1$, the cardinality of all such functions is:

$$|\{g \mid g : \mathbb{Q} \to \mathbb{R}\}| = |\mathbb{R}^{\mathbb{Q}}| = \aleph_1^{\aleph_0}.$$

Using the fact that $\aleph_1^{\aleph_0} = \aleph_1$, we conclude:

$$|C(\mathbb{R},\mathbb{R})| = \aleph_1.$$

14. *Let $\mathcal{C}$ be the collection of all circles in the plane $\mathbb{R}^2$. Show that $|\mathcal{C}| = \aleph_1$.

Solution: We define a map $\phi : \mathcal{C} \to \mathbb{R}^2 \times \mathbb{R}^+$ by $\phi(C) = (x, y, r)$, where $(x, y) \in \mathbb{R}^2$ is the center of C and $r \in \mathbb{R}^+$ is the radius of C. Now, every $(x, y, r) \in \mathbb{R}^3$ determines a unique circle $C \in \mathcal{C}$ with center (x, y) and radius r. So f is a bijection. Hence, $|\mathcal{C}| = |\mathbb{R}^2 \times \mathbb{R}^+| = \aleph_1$, (by problem no. 7).