Solution Set - 06

MA 1201 Spring Sem, 2025

1. Prove that $\mathbb{Z}_2 = \{0, 1\}$ with addition ('+') defined by

$$0 + 0 = 0, 0 + 1 = 1 + 0 = 1, 1 + 1 = 0$$

and multiplication('.') defined by

$$0.0 = 0, 0.1 = 1.0 = 0, 1.1 = 1$$

is a field.

2. Let $V = \{\theta\}$ and $\theta + \theta = \theta$ and $\alpha.\theta = \theta$ for all $\alpha \in \mathbb{F}$. Prove that V is a vector space over $\mathbb{F}$. (V is called the **zero vector space**)

Solution: We show that $(V, +, .)$ satisfies the defining properties of a vector space.

We have the following:

(a) $\theta + \theta = \theta + \theta$

(b) $\theta + (\theta + \theta) = \theta + \theta = \theta$, also $(\theta + \theta) + \theta = \theta + \theta = \theta$.

(c) $\theta + \theta = \theta + \theta = \theta$, therefore θ is the additive identity for V.

(d) $\theta + \theta = \theta$ implies that θ is its own additive inverse.

(e) $1.\theta = \theta$.

(f) $\alpha.(\beta.\theta) = \alpha.\theta = \theta = (\alpha\beta).\theta$.

(g) $\alpha.(\theta + \theta) = \alpha.\theta = \theta$, also $\alpha.\theta + \alpha.\theta = \theta + \theta = \theta$.

(h) $(\alpha + \beta).\theta = \theta$, also $\alpha.\theta + \beta.\theta = \theta + \theta = \theta$.

3. Determine whether the following statements are true or false by giving justifications or counter-examples. Assume usual addition and scalar multiplication unless otherwise stated.

(a) Any non-zero vector space over $\mathbb{F} = \mathbb{R}$ has infinitely many distinct elements.

Solution: True. As $V \neq \{0\}$ gives there exists $x(\neq 0) \in V$. So $cx \in V, \forall c \in \mathbb{R}$ and if $c_1 \neq c_2$ then $c_1 x \neq c_2 x$ because if $(c_1 - c_2)x = 0$ gives $\frac{1}{(c_1 - c_2)}(c_1 - c_2)x = 0$, So $x = 0$ gives contradiction to $x \neq 0$. As $\mathbb{R}$ uncountable gives $\{cx : c \in \mathbb{R}\}$ uncountable.

(b) The set $\mathbb{Q}$ of rational numbers is a vector space over $\mathbb{R}$.

Solution: False. Although $1 \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{R}$, their product, $\sqrt{2}$, is not in $\mathbb{Q}$.

(c) The set $\mathbb{R}_{\geq 0}$ of non-negative real numbers is a vector space over $\mathbb{R}$.

Solution: The given statement is **FALSE**. Since $1 \in \mathbb{R}_{\geq 0}$ has no additive inverse in $\mathbb{R}_{\geq 0}$, therefore $\mathbb{R}_{\geq 0}$ is not a vector space.

(d) The set $\mathbb{C}$ of complex numbers is a vector space over $\mathbb{R}$.

Solution: The given statement is **True**. Since, $\mathbb{C}$ is a field and $\mathbb{R} \subset \mathbb{C}$, then by axiom of fields, immediately follows that $\mathbb{C}$ is a vector space over $\mathbb{R}$.

(e) The set $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - 2x_3 = 0\}$ is a vector space $\mathbb{R}$.

Solution: The set $V = \{(2x, y, x) : x, y \in \mathbb{R}\}$. Clearly, all the axioms of the vector space are satisfied by this set with $(0, 0, 0)$ as the zero element and 1 as the identity element for scalar multiplication.

(f) The set $V = \{f : \mathbb{R} \to \mathbb{R} : f(t) = f(-t), \forall t \in \mathbb{R}\}$ (that is, the set of **even functions**) is a vector space $\mathbb{R}$.

Solution: The given statement is true. We first check that V is closed under addition and scalar multiplication. Let $f, g \in V, \alpha \in \mathbb{R}$. Now, $(f + \alpha g)(t) = f(t) + (\alpha g)(t) = f(t) + \alpha g(t) = f(-t) + \alpha g(-t) = f(-t) + (\alpha g)(-t) = (f + \alpha g)(-t)$. So, $f + \alpha g \in V$. Now, $V \subset W$, where W is the vector space of all real valued functions on $\mathbb{R}$. V inherits other vector space properties from W.

(g) The set $V = \{f : \mathbb{R} \to \mathbb{R} : f(t) = -f(-t), \forall t \in \mathbb{R}\}$ (that is, the set of **odd functions**) is a vector space $\mathbb{R}$.

Solution: The given statement is true. Let $f, g \in V$ and $a \in \mathbb{R}$. Then $(f + ag)(t) = f(t) + ag(t) = -f(-t) - ag(-t) = -(f + ag)(-t)$. Hence V is a vector space over $\mathbb{R}$.

(h) The set $\mathbb{R}^2$ with usual addition and new scalar multiplication defined by

$$\alpha(x_1, x_2) = \begin{cases} (0, 0), & \text{if } \alpha = 0; \\ (\alpha x_1, \frac{x_2}{\alpha}), & \text{if } \alpha \neq 0, \end{cases}$$

is a vector space over $\mathbb{R}$.

Solution: False. $(1+2)(x_1, x_2) = (3x_1, \frac{1}{3}x2)$ and $1(x_1, x_2) + 2(x_1, x_2) = (3x_1, \frac{3}{2}x_2)$.
So, $(1+2)(x_1, x_2) \neq 1(x_1, x_2) + 2(x_1, x_2)$ contradicts $\mathbb{R}^2$ with this scalar multiplication is not vector space, Since for a vectorspace V over $\mathbb{R}$ $(c+d)v = cv + dv \forall c, d \in \mathbb{R}, v \in V$.

(i) *The set $\mathbb{R}^3$ with usual addition and new scalar multiplication defined by

$$\alpha(x_1, x_2, x_3) = (\alpha x_1, x_2, x_3)$$

is a vector space over $\mathbb{R}$.

Solution: The new scalar multiplication on $\mathbb{R}^3$ is not a vector space. Any vector space has the property $(\alpha+\beta)(v) = \alpha v + \beta v$ where $\alpha, \beta \in \mathbb{R}$ and $v \in \mathbb{R}^3$. Let $v = (x_1, x_2, x_3)$ such that x_2 or x_3 is non zero. Then see that $(\alpha+\beta)(x_1, x_2, x_3) = ((\alpha+\beta)x_1, x_2, x_3)$ but $\alpha(x_1, x_2, x_3) + \beta(x_1, x_2, x_3) = ((\alpha+\beta)x_1, (\alpha+\beta)x_2, (\alpha+\beta)x_3$ and hence $(\alpha+\beta)(v) \neq \alpha v + \beta v$.

(j) The set $\mathbb{C}$ with usual addition and new scalar multiplication defined by $\alpha.x = \alpha^2 x$ is a vector space over $\mathbb{C}$.

Solution: False. The given scalar multiplication $\alpha \cdot x = \alpha^2 x$ fails the distributive property since:

$$(\alpha+\beta) \cdot x = (\alpha+\beta)^2 x = (\alpha^2 + 2\alpha\beta + \beta^2)x$$

$$\alpha \cdot x + \beta \cdot x = \alpha^2 x + \beta^2 x = (\alpha^2 + \beta^2)x$$

The extra $2\alpha\beta x$ term causes a mismatch, violating the vector space axioms. Hence, $\mathbb{C}$ is not a vector space under this operation.

(k) *The set $\mathbb{C}$ with usual addition and new scalar multiplication defined by $\alpha.x = (\text{Re } \alpha)x$ is a vector space over $\mathbb{C}$.

Solution: The given statement is **FALSE**. Note that, if $x = 1$, $\alpha = \beta = i$, then $\alpha.(\beta.x) = (\text{Re } \alpha)((\text{Re } \beta)x) = 0(0.1) = 0$, and, $(\alpha\beta).x = \text{Re } (\alpha\beta)x = \text{Re } (i^2)x = (-1).1 = -1$. Therefore $\alpha.(\beta.x) \neq (\alpha\beta).x$.

(l) The set $\mathbb{F}^2$ with usual addition and new scalar multiplication defined by $\alpha.(\beta, \gamma) = (\alpha\beta, 0)$ is a vector space over $\mathbb{F}$.

Solution: The given statement is **false**. Suppose take an element $(\beta, \gamma) \in \mathbb{F}^2$ such that $\gamma \neq 0$. Then

$$1 \cdot (\beta, \gamma) = (\beta, 0) \neq (\beta, \gamma)$$

Hence, by definition of vector space, w.r.t the given scalar multiplication, $\mathbb{F}^2$ is not forms a vector space.

(m) Let $V = \mathbb{R}_+ :=$ Set of all positive real numbers, with addition defined by

$$x + y := xy$$

and scalar multiplication defined by

$$\alpha.x := x^\alpha$$

is a vector space over $\mathbb{R}$.

Solution: The statement is true.

Axioms for addition:

- (Closed) For $x, y \in \mathbb{R}_+$, $x + y = xy \in \mathbb{R}_+$ as product of two positive real numbers is positive.
- (Commutative) $x + y = y + x$ as $\mathbb{R}$ is a field, so multiplication commutes.
- (Associative) $(x + y) + z = x + (y + z)$ as $\mathbb{R}$ is a field, so multiplication is associative.
- (Additive identity) For any $x \in \mathbb{R}_+$, $x + 1 = x.1 = x$ as 1 is the multiplicative identity of the field $\mathbb{R}$. So, $1 \in \mathbb{R}_+$ is the additive identity.
- (Additive inverse) Note that for any $x \in \mathbb{R}_+$, $y = \frac{1}{x} \in \mathbb{R}_+$ such that

$$x + y = xy = x\frac{1}{x} = 1.$$

  This shows that every element of $\mathbb{R}_+$ has an additive inverse.

Axioms for scalar multiplication:

- (Closed) For $x \in \mathbb{R}_+$, and $\alpha \in \mathbb{R}$, we have

$$\alpha.x = x^\alpha \in \mathbb{R}_+,$$

  as any power of a positive real number is positive.
- (Compatibility of scalar multiplication with field multiplication) Let $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}_+$. Then, using the definition of vector addition and scalar multiplication, we have

$$\alpha.(\beta.x) = \alpha.x^\beta = (x^\beta)^\alpha = x^{\alpha\beta} = (\alpha\beta).x$$

- (Identity element for scalar multiplication) For any $x \in \mathbb{R}_+$, we have

$$1.x = x^1 = x.$$

  So, $1 \in \mathbb{R}_+$ is the identity element for scalar multiplication.
- (Distribution of scalar multiplication over addition) Let $\alpha \in \mathbb{R}$ and let $x, y \in \mathbb{R}_+$. Then, using the definition of vector addition and scalar multiplication we have

$$\alpha.(x + y) = \alpha.(xy) = (xy)^\alpha = x^\alpha y^\alpha$$
$$\text{and } \alpha.x + \alpha.y = x^\alpha + y^\alpha = x^\alpha x^\beta$$

  This shows $\alpha.(x + y) = \alpha.x + \alpha.y$

- (Distributivity of scalar multiplication with respect to field addition) Let $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}_+$. Then, using the definition of vector addition and scalar multiplication

$$(\alpha+\beta).x = x^{\alpha+\beta} = x^{\alpha}x^{\beta}, \tag{1}$$
$$\text{and } \alpha.x + \alpha.y = x^{\alpha} + y^{\beta} = x^{\alpha}x^{\beta} \tag{2}$$

(n) Each of the sets $P := \{(x,y) \in \mathbb{R}^2 : y^2 = 4ax, a > 0\}, E := \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a, b > 0\}, H := \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a, b > 0\}$ is a vector space over $\mathbb{R}$.

Solution: P is not a vector space over $\mathbb{R}$. Note that $(1, 2\sqrt{a}) \in P$, but $(2, 4\sqrt{a}) \notin P$. Because $(4\sqrt{a})^2 = 16a \neq 8a = 4a2$. So P is not closed under scalar multiplication.

E is not a vector space because E does not contain the additive identity $(0,0)$.

H is not a vector space because E does not contain the additive identity $(0,0)$.

4. Prove that if $\mathbb{G}$ is a subfield of $\mathbb{F}$ (subset of $\mathbb{F}$ which is a field itself with respect to the addition and multiplication in $\mathbb{F}$), then $\mathbb{F}$ is a vector space over $\mathbb{G}$.

Solution: Since $\mathbb{F}$ is a field and $\mathbb{G} \subset \mathbb{F}$, then by axiom of fields, immediately follows that $\mathbb{F}$ is a vector space over $\mathbb{G}$.

5. Prove that $-(-x) = x$ for every $x \in V$.

Solution: Let us denote the additive inverse of x by y, i.e.,

$$y = -x. \tag{3}$$

By the definition of additive inverse, we have $x + (-x) = 0$. Using (3), we have $x + y = 0$. This shows that x is additive inverse of y, i.e., $x = -y$. Using (3), we get $x = -(-x)$.

6. Prove that if $\alpha \in \mathbb{F}$ and $x \in V$ such that $\alpha x = \theta$, then either $\alpha = 0$ or $x = \theta$. This shows that the singleton set $\{x\}$, for $x \neq \theta$, is lineraly indenpendent.

Solution: We first prove that for any $\lambda \in F, \lambda\theta = \theta$.

$\lambda\theta = \lambda(\theta + \theta)$ (θ is the additive identity)

$= \lambda\theta + \lambda\theta$ (by distributibity of scalar multiplication over addition)

Now adding an additive inverse of $\lambda\theta$ to both sides, we get $\lambda\theta = \theta$.

Suppose $\alpha(\in F) \neq 0$. Then there exists $\alpha^{-1} \in F$.

Now $\alpha x = \theta$. Multiplying bothsides by α^{-1}, we get, $\alpha^{-1}(\alpha x) = \alpha^{-1}\theta$. By distributivity of scalars, we have, $(\alpha^{-1}\alpha)x = \alpha\theta$. This implies $1x = \alpha\theta$. By vector space axioms, we have $1x = x$ and from what we proved above, we have $\alpha\theta = \theta$. So we have, $x = \theta$.

7. *Give examples of nonempty subset S of $\mathbb{R}^2$ such that

   (a) S is closed under addition and under taking additive inverses but not a subspace of $\mathbb{R}^2$.

   (b) S is closed under scalar multiplication but not a subspace of $\mathbb{R}^2$.

   This shows you are required to check closeness under both addition and scalar multiplication to check whether certain subset is a subspace.

   Solution:

   (a) Let $S = \{\mathbb{Z} \times 0\}$. See that S is closed under addition and inverse, but is not closed under scalar multiplication.

   (b) Let $S = (\mathbb{R} \times 0) \cup (0 \times \mathbb{R})$. See that S is closed under scalar multiplication but not closed under addtion.

8. Determine which of the following subsets S of the vector space V over $F = \mathbb{R}$ are subspaces.

   (a) $S = \{(x_1, x_2, x_3) : x_1 = x_2, x_3 = 2x_1\}$, $V = \mathbb{R}^3$.

   Solution: Given S is a subspsace of $\mathbb{R}^3$. Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S$ and $a \in \mathbb{R}$. We show $(x_1, x_2, x_3) + a(y_1, y_2, y_3) \in S$. See that $x_1 + ay_1 = x_2 + ay_2$ and $x_3 + ay_3 = 2(x_1 + ay_1)$. Hence S is a subspace.

   (b) *$S = \{(x_1, x_2, x_3) : x_1 = 0\}$, $V = \mathbb{R}^3$.

   Solution: Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S \subset \mathbb{R}^3$ then $x_1 = 0, y_1 = 0 \Rightarrow x_1 + y_1 = 0$, gives $(x_1 + y_1, x_2 + y_2, x_3 + y_3) \in S$ and as $cx_1 = 0$, so $c(x_1, x_2, x_3) \in S, \forall c \in \mathbb{R}$. So S is a subspace of $\mathbb{R}^3$.

   (c) *$S = \{(x_1, x_2, x_3) : x_1 = 1\}$, $V = \mathbb{R}^3$.

   Solution: If S is a subspace then $(0, 0, 0)$ must be in S. But here $(0, 0, 0)$ is not in S. So, S is not a subspace of V.

   (d) $S = \{(x_1, x_2, x_3) : x_2 x_3 = 0\}$, $V = \mathbb{R}^3$.

Solution: The set S is not a subspace of $\mathbb{R}^3$ because it contains the vectors $(1,1,0)$ and $(0,0,1)$, but their sum, $(1,1,1)$, is not in S.

(e) $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 0\}$, $V = \mathbb{R}^3$.

Solution: Note that $S = \{0\}$. By the solution of Q. 2 we have that, S is a vector space over $\mathbb{R}$. Since $S \subseteq \mathbb{R}^3$, therefore S is a vector subspace of $\mathbb{R}^3$.

(f) $S = \{(x_1, x_2, x_3) : 3x_1 - x_2 + x_3 = 0\}$, $V = \mathbb{R}^3$.

Solution: Let $\alpha = (x_1, x_2, x_3) \in S, \beta = (y_1, y_2, y_3) \in S$ and $c \in \mathbb{R}$. Then we do things,

$$3x_1 - x_2 + x_3 = 0 \tag{4}$$
$$3y_1 - y_2 + y_3 = 0 \tag{5}$$

Now, $c\alpha + \beta = (cx_1 + y_1, cx_2 + y_2, cx_2 + y_2)$

$$3(cx_1 + y_1) - (cx_2 + y_2) + cx_3 + y_3 = c(3x_1 - x_2 + x_3) + 3y_1 - y_2 + y_3 = c \cdot 0 + 0 = 0$$

Therefore, $c\alpha + \beta \in S$ for all $\alpha, \beta \in S$ and $c \in \mathbb{R}$, Hence, S is a subspace of $V = \mathbb{R}^3$.

(g) $S = \{(x_1, x_2, x_3) : x_1 + x_2 = 0\}$, $V = \mathbb{C}^3$.

Solution: To show S is a subspace, we need to check if the vector addition and scalar multiplication are closed or not.
Let $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in S$, then

$$x_1 + x_2 = 0 \tag{6}$$
$$y_1 + y_2 = 0. \tag{7}$$

Let $\alpha \in \mathbb{R}$.
To show: $X + Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in S$.
Note that $(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2)$ using associativity of addition over $\mathbb{R}$. Using (14) and (15), we get $(x_1 + y_1) + (x_2 + y_2) = 0 + 0 = 0$ as 0 is the additive identity. This shows that $X + Y \in S$.
To show: $\alpha X = (\alpha x_1 + \alpha x_2, \alpha x_3) \in S$.
Using the distributive property, we have

$$\alpha . x_1 + \alpha . x_2 = \alpha .(x_1 + x_2).$$

Again using (14), we get

$$\alpha . x_1 + \alpha . x_2 = \alpha . 0 = 0 \text{ using the property of additive identity.}$$

(h) $S = \{(x_1, x_2, x_3) : x_1 + x_2 \geq 0\}$, $V = \mathbb{C}^3$.

Solution: $\mathcal{S}$ is not a subspace of V. $(1,1,0) \in \mathcal{S}$, but $-5(1,1,0) = (-5,-5,0) \notin \mathcal{S}$. So $\mathcal{S}$ is not closed under scalar multiplication.

(i) $S =$ the set of all polynomials whose constant term is zero, $V = \mathcal{P}(\mathbb{R})$.

(j) $S =$ the set of all polynomials whose degree is equal to 2, $V = \mathcal{P}(\mathbb{R})$.

Solution: Here $x + x^2, x - x^2$ is in S but their sum which is $2x$ is not in S. Since any subspace of V is closed under addition, so S is not a subspace.

(k) $S =$ the set of all polynomials $f(x)$ such that $f'(1) = 0$, $V = \mathcal{P}(\mathbb{R})$.

Solution: The zero polynomial is not in S.

(l) All combinations of two given vectors $(1,1,0)$ and $(2,0,1)$.

Solution: Here $\alpha = (1,1,0), \beta = (2,0,1)$ and let $x, y \in S = \{c\alpha + d\beta : c, d \in \mathbb{R}\} \subseteq \mathbb{R}^3$ then $x = c_1\alpha + d_1\beta$ and $y = c_2\alpha + d_2\beta$. $x + y = (c_1 + c_2)\alpha + (d_1 + d_2)\beta \in S$ and $cx = cc_1\alpha + cd_1\beta \in S$ for any $c \in \mathbb{R}$. So S is a subspace of $\mathbb{R}^3$.

(m) $S = \{A \in M_{n\times n}(\mathbb{R}) : A^T = 2A\}$, $V = M_{n\times n}(\mathbb{R})$.

Solution: Let $A, B \in S$ and $\alpha \in \mathbb{R}$. See that $(A+\alpha B)^T = A^T + \alpha B^T = 2A + 2\alpha B = 2(A+\alpha B)$. Hence $A + \alpha B \in S$, so S is a vector subspace of $M_{n\times n}(\mathbb{R})$.

9. $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a subspace of the vector space $\mathbb{R}$ over $\mathbb{Q}$. (You can construct many such non-trivial subspaces of $\mathbb{R}$ over $\mathbb{Q}$)

Solution: Recall that the non-empty subset S of a vector space V over a field $\mathbb{F}$ is a subspace if and only if it satisfies,

$$\text{For}, x, y \in S \text{ and } a \in \mathbb{F}, \text{we have } ax + y \in S.$$

Let $a \in \mathbb{Q}$, $x = c + d\sqrt{2}$, $y = e + f\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$. Then, we have

$$ax + y = a(c + d\sqrt{2}) + (e + f\sqrt{2}) = ((ac + e) + (ad + f)\sqrt{2})$$

Now, both $ac + e, ad + f \in \mathbb{Q}$. Hence $\mathbb{Q}[\sqrt{2}]$ is a vector subspace of $\mathbb{R}$ over $\mathbb{Q}$.

10. *Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(For further thinking) What about union of more than two subspaces?

Solution: ($\implies$): Let $W_1 \cup W_2$ be a subspace of V. Assume, if possible, that $W_1 \subsetneq W_2$ and $W_2 \subsetneq W_1$.
That means there exists $x \in W_1$ such that $x \notin W_2$, and $y \in W_2$ such that $y \notin W_1$. Then, $x, y \in W_1 \cup W_2$ and hence $x + y \in W_1 \cup W_2$ by the assumption, which implies either $x + y \in W_1$ or $x + y \in W_2$. If $x + y \in W_1$, then $y = (x + y) - x \in W_1$, which is a contradiction. Similarly, if $x + y \in W_2$, then $x = (x + y) - y \in W_2$ is not possible.

($\Longleftarrow$): Assume $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. In both cases $W_1 \cup W_2$ is either W_1 or W_2. Hence $W_1 \cup W_2$ is a subspace of V.

11. Describe the column space and the nullspace of the following matrices:

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution: We know that the column space $C(A)$ of an $m \times n$ matrix A is the collection of all linear combinations of the columns of A, and that $C(A)$ is a subspace of $\mathbb{R}^m$.

On the other hand, the null space $N(A)$ of an $m \times n$ matrix A is the collection of all vectors $v \in \mathbb{R}^n$ such that $Av = 0$, and that $N(A)$ is a subspace of $\mathbb{R}^n$.

- Given matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

  We have

$$\begin{aligned} C(A) &= \left\{ s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\} \\ &= \left\{ (s - t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\} \\ &= \left\{ \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\}. \end{aligned}$$

  Clearly, $C(A)$ is collection of all scalar multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, that is, $C(A)$ is the x-axis in $\mathbb{R}^2$.

  On the other hand, for $v = \begin{bmatrix} x \\ y \end{bmatrix}$, we have $Av = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x - y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

  Comparing the entries of the last two columns we get $x - y = 0 \iff x = y$.

  So, $N(A) = \{(x, x) : x \in \mathbb{R}\} = \{x(1, 1) : x \in \mathbb{R}\}$.

  Thus, $N(A)$ is the collection of all scalar multiples of $(1, 1)$, that is, $N(A)$ is the straight line in $\mathbb{R}^2$ passing through $(1, 1)$ and $(0, 0)$.

- Given matrix $B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

  Observe that the first two columns are scalar multiple of each other, as we have $(0,2) = 2(0,1)$ and $(0,1) = \frac{1}{2}(0,2)$. So these two columns together are linearly dependent columns, and while taking linear combinations, we can drop any one of these two.

  Thus, $C(B) = C(D)$ where $D = \begin{bmatrix} 0 & 3 \\ 1 & 3 \end{bmatrix}$.

  One can easily check that the matrix D is invertible, which implies that $C(D) = \mathbb{R}^2$, and hence $C(B) = \mathbb{R}^2$.

  On the other hand, for $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have

$$Bv = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 3z \\ x + 2y + 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

  The above system of linear equations is same as $z = 0$ and $x + 2y = 0$.

  Thus,

$$\begin{aligned} N(B) &= \{(x,y,z) \in \mathbb{R}^3 : x + 2y = 0, z = 0\} \\ &= \{(x,y,0) \in \mathbb{R}^3 : x + 2y = 0\} \\ &= \{(-2y, y, 0) : y \in \mathbb{R}\} \\ &= \{y(-2,1,0) : y \in \mathbb{R}\}. \end{aligned}$$

  Hence, the null space $N(B)$ of B is the straight line in $\mathbb{R}^3$ which lies in the xy-plane and passes through $(0,0,0)$ and $(-2,1,0)$.

- Given matrix $C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$.

  Here observe that each column is the scalar multiple of every other column. So, while taking linear combinations, we can choose any one of these.

  Thus, the column space of C is $C(C) = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$.

  That is, the column space of the matrix C is the straight line in $\mathbb{R}^2$ passing through $(0,0)$ and $(0,1)$.

  On the other hand, for $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have

$$Cv = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 0 \\ x + 2y + 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The above system of linear equations is same as $x + 2y + 3z = 0$.

Thus,

$$\begin{aligned} N(A) &= \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\} \\ &= \{(-2y - 3z, y, z) \in \mathbb{R}^3 : y, z \in \mathbb{R}\} \\ &= \{y(-2, 1, 0) + z(-3, 0, 1) : y, z \in \mathbb{R}\}. \end{aligned}$$

Hence, the null space $N(C)$ of the matrix C is the plane in $\mathbb{R}^3$ passing through $(0, 0, 0)$, $(-2, 1, 0)$ and $(-3, 0, 1)$.

12. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form

(a) a plane;

(b) a line;

(c) a point;

(d) a subspace;

(e) the nullspace of A;

(f) the column space of A.

Solution: First check that A is matrix of rank 2. Hence the solution space will have one free variable and two dependent variable. Hence the solution space is $1-$dimensional vector space generated by the vector $(2, -1, -1)$.

13. *Write an example of a 2 by 2 system $Ax = b$ with many solutions for $Ax = 0$ but no solution $Ax = b$. (Therefore the system has no solution.)

Solution: Take the 2×2 matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$.

14. The columns of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) the column space of A. Give an example where the column spaces of A and AB are not equal.

Solution:

(a) Let A be an $m \times n$ matrix and B be an $n \times l$ matrix. We want to show that

$$C(AB) \subseteq C(A).$$

Let $b \in C(AB)$. Then, by definition, there exists $x \in \mathbb{R}^l$ such that

$$b = (AB)x.$$

Thus, if we write $v = Bx$ then we have $b = Av$ which implies that $b \in C(A)$.

Hence, $C(AB) \subseteq C(A)$.

(b) To find an example where the column spaces of A and AB are not equal, consider the following matrices A and B:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

so that

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that the column space $C(A)$ of A is the x-axis in $\mathbb{R}^2$, whereas the column space $C(AB)$ of AB is the trivial subspace $\{0\}$ of $\mathbb{R}^2$. Thus, in this particular example, the column space of AB is a proper subspace of the column space of A.

15. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $C(A)$ form a subspace.

(b) If $C(A)$ contains only the zero vector, then A is the zero matrix.

Solution: The statement is **True**. If A is non zero, then there exist at least one non zero coloumn, this non zero column vector should be inside $C(A)$, a contradiction. Hence, A is zero matrix.

(c) The column space of $2A$ equals the column space of A.

Solution: The statement is **True**. Let α be a column vector of A, since $C(A)$ is a vector space, so $2\alpha \in C(A)$, since α is arbitrary column vector, so $C(2A) \subset C(A)$. Now, let β be a column vector of A, Now we can write:

$$\beta = \frac{1}{2} \cdot 2\beta$$

Hence, $\beta \in C(2A)$. Therefore, $C(A) \subset C(2A)$. Hence, $C(A) = C(2A)$.

(d) The column space of $A - I$ equals the column space of A.

Solution: The statement is **False**. Take, $A = 0$, then $A - I = -I \neq 0$. Therefore, $C(A) = \{0\}$ but $C(A) \neq \{0\}$.

16. Prove that if $a = 0, d = 0$, or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

Solution: Let $v_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} b \\ d \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} c \\ e \\ f \end{bmatrix}$.

- **Case I:** Let $a = 0$.
  Then we have scalars $1, 0, 0$ such that

$$1. \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} b \\ d \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} c \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \left(\because \quad a = 0 \right)$$

  Thus, v_1, v_2 and v_3 are dependent.

  Case II: Let $d = 0$ and $a \neq 0$. Then we have scalars $b,\ -a,\ 0$, such that

$$b. \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + (-a) \cdot \begin{bmatrix} b \\ d \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} c \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

  Since not all $b,\ -a,\ 0$ are zero ($-a \neq 0$), we get that v_1, v_2 and v_3 are linearly dependent.

  Case III: Let $f = 0,\ d \neq 0,\ a \neq 0$.

  Then, we have $\frac{-(dc-be)}{da},\ \frac{-e}{d},\ 1$, such that

$$\frac{-(dc - be)}{da} \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \left(\frac{-e}{d} \right) \begin{bmatrix} b \\ d \\ 0 \end{bmatrix} + 1 \begin{bmatrix} c \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

  Hence, v_1, v_2 and v_3 are linearly dependent.

  Thus, if any of a, d and f is zero, then the set $\{v_1, v_2, v_3\}$ is linearly dependent.

- Now Let $a,\ d$ and f are all non zero and $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

We have already seen that any triangular matrix is invertible if and only if all the diagonal entries are non-zero. So, U is invertible, which implies

$$Ux = 0 \iff x = U^{-1}0 \iff x = 0.$$

Hence, $\nexists$ any $x = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \neq 0$ for some $p,\ q,\ r\ \in \mathbb{R}$ such that $Ux = 0$.

Thus, we conclude that U has linearly independent columns.

If a, d, f are all nonzero, show that the only solution to $Ux = 0$ is $x = 0$. Then U has independent columns.

17. Prove that columns of an upper triangular matrix are linearly independent if and only if all of its diagonal entries (PIVOTS) are non-zero.

18. *Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1 v_1 + \cdots + c_4 v_4 = 0$ or $Ac = 0$. The v's go in the columns of A.

Solution: Given, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

- Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ for some $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbb{R}$.

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ \alpha_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ \alpha_3 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The above system implies

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \tag{8}$$
$$\alpha_2 + \alpha_3 = 0, \tag{9}$$
$$\alpha_3 = 0. \tag{10}$$

Putting value of α_3 in (12), we get

$$\alpha_2 = 0.$$

Putting $\alpha_2 = \alpha_3 = 0$ in (11), we get

$$\alpha_1 = 0.$$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0.$

Hence, $\nexists\ \alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ and at least one of the α_i's is non zero.
$\therefore v_1, v_2$ and v_3 are linearly independent.

- Now, consider $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$.

Since the matrix A has 3 rows, the row rank of the matrix A is less than or equal to 3.

Also, we know that the column rank of a matrix is equal to the row rank.

Thus, the column rank of the matrix A is less than or equal to 3.

Hence, four columns of the matrix must be linearly dependent.

That is, the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ are linearly dependent.

19. (a) Under what conditions on the scalar ξ are the vectors $(\xi, 1, 0), (1, \xi, 1)$ and $(0, 1, \xi)$ in $\mathbb{R}^3/\mathbb{R}$ are linearly dependent?

(b) What is the answer to (a) for $\mathbb{Q}^3/\mathbb{Q}$ (in place of $\mathbb{R}^3/\mathbb{R}$)?

Solution: (a) Let, $v_1 = \begin{bmatrix} \xi \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ \xi \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ \xi \end{bmatrix}$.

Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ for some $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbb{R}$.

$$\Rightarrow \begin{bmatrix} \alpha_1 \xi \\ \alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ \alpha_2 \xi \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_3 \\ \alpha_3 \xi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The above system implies

$$\alpha_1 \xi + \alpha_2 = 0, \tag{11}$$

$$\alpha_1 + \alpha_2 \xi + \alpha_3 = 0, \tag{12}$$

$$\alpha_2 + \alpha_3 \xi = 0. \tag{13}$$

Applying the Gaussian elimination it can be shown that the above system has a solution if and only if $\xi = 0, \sqrt{2}$ or $-\sqrt{2}$. Thus v_1, v_2, v_3 are linearly dependent in $\mathbb{R}^3/\mathbb{R}$ if and only if $\xi = 0, \sqrt{2}, -\sqrt{2}$.

(b) Since $\sqrt{(2)} \notin \mathbb{Q}$, therefore v_1, v_2, v_3 are linearly dependent over $\mathbb{Q}^3/\mathbb{Q}$ if and only if $\xi = 0$.

20. Find all possible values for a for which the vector $(3, 3, a)$ is in the span of the vectors $(1, -1, 1)$ and $(1, 2, -3)$.

Solution: Let us consider the following relation:

$$(3, 3, a) = x(1 - 1, 1) + y(1, 2, -3)$$

Therefore, we have

$$x + y = 3 \tag{14}$$
$$-x + 2y = 3 \tag{15}$$
$$x - 3y = a \tag{16}$$

From, equation (14),(15), we get $x = 1, y = 2$. finally, put the values of x, y, in equation (16), we get $a = -5$.

21. *If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, and $v_3 = w_1 - w_2$ are dependent. Find a combination of the v's that gives zero.

Solution: Note that $v_1 + v_3 = w_1 - w_3 = v_2$. That is, $v_1 - v_2 + v_3 = 0$, which shows that the vectors v_1, v_2, v_3 are linearly dependent.

22. If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3, v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are independent. [Hint: $c_1v_1 + c_2v_2 + c_3v_3 = 0$ in terms of the w's. Find and solve equations for the c's.]

Solution: Let $c_1v_1 + c_2v_2 + c_3v_3 = 0$ for some $c_1, c_2, c_3 \in \mathbb{R}$. Then expanding v_i's we get $(c_2 + c_3)w_1 + (c_1 + c_3)w_2 + (c_1 + c_2)w_3 = 0$. Then the w_i's being linearly independent gives us three equations

$$c_2 + c_3 = 0$$
$$c_1 + c_2 = 0$$
$$c_1 + c_2 = 0$$

Solving the above equation gives us $c_1 = c_2 = c_3 = 0$. Hence v_1, v_2, v_3 are linearly independent vectors.

23. Find a basis for the column space (in $\mathbb{R}^2$) and nullspace (in $\mathbb{R}^5$) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

Solution: Note that U is a rank-2 matrix, hence the column space is whole R^2. We can take the usual basis $\{(1,0),(0,1)\}$ for the column space. Null space of U is same as the solution of the following system of equation.

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 + x_4 &= 0 \end{aligned}$$

Solving the above system of equation tells us that $\{(1,0,0,0,-1),(0,0,1,0,-1),(0,1,0,-1,0)\}$ is a basis of the solution space and hence they are basis of the null space of U.

24. *Find a basis for the subspace $W = \{(x_1,x_2,x_3,x_4) : x_1 - 3x_2 + x_3 = 0\}$ of $\mathbb{R}^4$.

Solution: Here $W = \{(x_1,x_2,x_3,x_4) : x_1 - 3x_2 + x_3 = 0\}$. $\Rightarrow W = \{(3x_2 - x_3, x_2, x_3, x_4) : x_4, x_2, x_3 \in \mathbb{R}\}$
$\Rightarrow W = \{x_2(3,1,0,0) + x_3(-1,0,1,0) + x_4(0,0,0,1) : x_4, x_2, x_3 \in \mathbb{R}\}$
Now check that $B = \{(3,1,0,0),(-1,0,1,0),(0,0,0,1)\} \subset \mathbb{R}^4$ is linearly independent over $\mathbb{R}$, which gives us B is a basis of W.
As $c_1(3,1,0,0) + c_2(-1,0,1,0) + c_3(0,0,0,1) = (0,0,0,0)$
$\Rightarrow (3c_1 - c_2, c_1, c_2, c_3) = (0,0,0,0)$
$\Rightarrow c_3 = 0, c_2 = 0, c_1 = 0.$

25. *Let $V = M_n(\mathbb{R})$, the vector space of all $n \times n$ real matrices and S denote the subset of V of all symmetric matrices, that is, $S = \{A \in M_n(\mathbb{R}) : A^T = A\}$.
    (a) Prove that S is a subspace of V.
    (b) Find a basis for V and S.

    Answer the same set of questions when S denote the subset of all skew symmetric matrices ($A^T = -A$).

Solution:

(a) Let $A, B \in S$, $c \in \mathbb{R}$. Then $(cA+B)^T = (cA)^T + B^T = cA^T + B^T = cA + B$, thus $A + B \in S$. This shows that S is a subspace of V.

(b)
- **Basis for** V: Consider the set $C = \{E_{ij} \in M_n(\mathbb{R}) \mid 1 \le i,j \le n\}$ where, E_{ij} is the matrix $n \times n$ whose (i,j)-th entry is 1 and 0 elsewhere. We shall prove that the set C forms a basis for V.
  Span: Let $A = (a_{ij}) \in M_n(\mathbb{R})$, then $A = \sum_{i,j}^{n} a_{ij}E_{ij}$. Because the kl entry of LHS is,

$$\left(\sum_{i,j}^{n} a_{ij}E_{ij}\right)_{kl} = \sum_{i,j}^{n} a_{ij}(E_{ij})_{kl} = a_{kl}$$

Remember, $(E_{ij})_{kl} = 1$ if $i = k, j = l$, otherwise 0.

Linearly independent: Consider,

$$\sum_{i,j} c_{ij} E_{ij} = 0$$

then, for any $k, l \in \{1, 2, \cdots, n\}$, $\left(\sum_{i,j} c_{ij} E_{ij}\right)_{kl} = c_{kl} = 0$. Thus, $\sum_{i,j} c_{ij} E_{ij} = 0$, implies $c_{ij} = 0$ for all $i, j \in \{1, 2, \cdots, n\}$.

- The subspace S consists of all symmetric matrices, i.e., matrices satisfying $A^T = A$. To construct a basis for S, we consider the structure of a symmetric matrix:

$$A = (a_{ij}) \quad \text{with} \quad a_{ij} = a_{ji}.$$

A basis for $D = \{E_{ii} \in M_n(\mathbb{R}) \mid 1 \leq i \leq n\} \cup \{E_{ij} + E_{ji} \in M_n(\mathbb{R}) \mid 1 \leq i < j \leq n\}$. Let $A \in S$ be a symmetric matrix, then

$$A = \sum_{i=1}^{n} a_{ii} E_{ii} + \sum_{1 \leq i < j \leq n} a_{ij} (E_{ij} + E_{ji})$$

Which shows the Span$D = S$. Now assume, $\sum_{i=1}^{n} a_{ii} E_{ii} + \sum_{1 \leq i < j \leq n} a_{ij} (E_{ij} + E_{ji}) = 0$, then

$$\sum_{i=1}^{n} a_{ii} E_{ii} + \sum_{1 \leq i < j \leq n} a_{ij} (E_{ij}) + \sum_{1 \leq i < j \leq n} a_{ij} (E_{ji}) = 0$$

which shows that diagonal entries $a_{ii} = 0$ and for $1 \leq i < j \leq n$, $a_{ij} = 0$. For $i > j$, $a_{ij} = a_{ji} = 0$ and we are done.