MA 1201 - Mathematics II

ODE - Week 01

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Basic Concepts

Let y be a function defined from I to $\mathbb{R}$, where I is a subset of $\mathbb{R}$ or $I = \mathbb{R}$, i.e., $x \mapsto y(x) \in \mathbb{R}, \quad \forall\, x \in I$. We usually take I to be an open interval.

Here x is the independent variable and $y(\cdot)$ is the dependent variable.

An ordinary differential equation is an equation containing the derivatives of an unknown function y, the unknown function y itself, and known functions of x including constants.

Notation. The n-th order derivative of y with respect to the independent variable x, $\frac{d^n}{dx^n} y(x)$ will be denoted by $y^{(n)}(x)$.

Explicit ODE

In other words, an ODE is a relation between the derivatives $y, y^{(1)}, \ldots, y^{(n)}$ and functions of x:

$$F(x, y, y^{(1)}, \ldots, y^{(n)}) = 0, \text{ for each } x \in I,$$

and for some map $F : I \times \mathbb{R}^{n+1} \to \mathbb{R}$.

Note that from the equation above, it does not follow that

$$y^{(n)}(x) = G(x, y, y^{(1)}, \ldots, y^{(n-1)}), \text{ for each } x \in I,$$

$G : I \times \mathbb{R}^n \to \mathbb{R}$ - this form is known as explicit form of an ODE. For example consider the differential equations:

1. $y'' - 2y' - 6 = 0$,
2. $\sin(y'')(y'')^3 - 2y' - 6 = 0$.

Observe that while in the first equation we can write y'' explicitly in terms of y, its lower order derivatives, and x, we cannot do so in the second equation.

In this course, we consider explicit ODEs only.

Examples

DE's occur naturally in physics, engineering and so on.

Can you think of some obvious examples? Since velocity and acceleration are derivatives, they often give rise to differential equations.

Example 1: A falling object.

A body of mass m falls under the force of gravity. The drag force due to air resistance is $c \cdot v^2$ where v is the velocity and c is a constant. Then

$$m\frac{dv}{dt} = mg - c \cdot v^2.$$

An ODE of first order.

Examples

Example 2: Radioactive decay.

A radioactive substance decomposes at a rate proportional to the amount present. Let $y(t)$ be the amount present at time t. Then

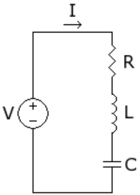
$$\frac{dy}{dt} = k \cdot y$$

where k is a physical constant whose value is found by experiments ($-k$ is called the decay constant). ODE of first order.

Examples

Example 3: Electrical circuits.

Consider a basic RLC circuit:
resistance of the resistor - R ohms,
inductance of the inductor - L henrys,
capacitance of the capacitor - C farads.
These are wired and connected to an electromotive force $V(t)$ volts.



Let $Q(t)$ (coulombs) be the total charge in the capacitor at time t.

$$I(t) = \frac{dQ}{dt} = \text{ current.}$$

By Kirchhoff's voltage law, $L\frac{dI}{dt} + RI + \frac{Q(t)}{C} = V(t)$, i.e.,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C} \cdot Q = V(t).$$

Can we solve it?

Given an equation, you would like to solve it.

Questions:

1. What is a solution?
2. Does an equation always have a solution? If so, how many?
3. Can the solutions be expressed in a 'nice form' (representation formula)? If not, how to get a feel for it?
4. How much can we proceed in a systematic manner?

order - first, second, ..., n^{th}, ...
linear or non-linear?

What is a solution?

Consider a n-th order ODE: $F(x, y, y', \ldots, y^{(n)}) = 0$.

Definition (Solution of a n-th order ODE)

An explicit solution of the n-th order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ on an interval $I = (\alpha, \beta)$ is a real-valued function ϕ defined on the interval (α, β) such that all n-derivatives of ϕ, i.e.,$\phi'(\cdot)$, $\phi''(\cdot), \cdots, \phi^{(n)}(\cdot)$ exist on the interval (α, β) satisfying

$$F(x, \phi(x), \phi'(x), \cdots, \phi^{(n)}(x)) = 0, \quad \forall\, \alpha < x < \beta.$$

Solution curve: If $\phi(\cdot)$ is a solution of the ODE on an interval (α, β), then the solution curve is given by the graph of ϕ in xy plane

$$\{(x, \phi(x)) \mid \alpha < x < \beta\}.$$

Examples

Example 4: Given an amount of a radioactive substance, say 1 gm, find the amount present at any later time.
The relevant ODE is

$$\frac{dy}{dx} = k \cdot y.$$

By inspection, $y(x) = ce^{kx}$, for an arbitrary constant c, is an explicit solution of the above ODE.
Now, initial amount given is 1 gm at time $x = 0$, i.e.,

$$y(0) = 1.$$

The initial condition determines $c = 1$. Hence

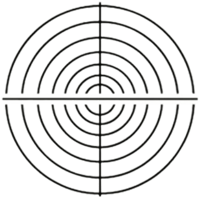
$$y(x) = e^{kx}$$

is a particular solution to the above ODE with the given initial condition.

Implicit solution of ODE

We say a relation $\psi(x, y) = 0$ is an implicit solution to the ODE $F(x, y, y^{(1)}, \dots, y^{(n)}) = 0$, on $I = (\alpha, \beta)$, if it defines at least one real function $\phi : I \to \mathbb{R}$ of x variable which is an explicit solution of the ODE on I.

Example.
The relation $\psi(x, y) := x^2 + y^2 - 25 = 0$ is an implicit solution of the ODE $yy' + x = 0$ in $(-5, 5)$. The given relation ψ defines two real valued functions $\phi_1(x) := \sqrt{25 - x^2}$ and $\phi_2(x) := -\sqrt{25 - x^2}$ in $(-5, 5)$ and both are explicit solutions of the given ODE.
In the image below, the upper semi-circles and lower semi-circles are solutions on appropriate domain.



Example ctd.

In the previous example, could we have just differentiated the relation $x^2 + y^2 - 25 = 0$ with respect to x to obtain $2x + 2yy' = 0$ and concluded that it is implicit solution?

The answer is a 'NO'! For instance, consider $x^2 + y^2 + 25 = 0$ - it satisfies the ODE but is not an implicit solution because $y(x) := \pm\sqrt{-25 - x^2}$ cannot be expressed as (even "locally") graph of a function.

A remark on implicit solution

Note that it is not always possible to have such ϕ from the ψ and even if we get the interval, it need not be the interval you started the differentiation with!

A way out is the implicit function theorem which helps us to see whether we have such ϕ from ψ on some interval. Here is a two variable version of it:

If $\psi(x, y)$ is a function that is continuously differentiable in an open interval of the point (x_0, y_0), and $\frac{\partial \psi}{\partial y}(x_0, y_0) \neq 0$, then there exists a unique differentiable function ϕ such that $y_0 = \phi(x_0)$ and $\psi(x, \phi(x)) = 0$ in an open interval around x_0.

In this course, when asked for an implicit solution, it is sufficient to provide ψ without demonstrating the existence of ϕ (unless you are asked to and it follows from a straightforward computation like the previous example). The existence of ϕ will be addressed in a later course on ODEs.

Examples

Example 5: Find the curve through the point $(1, 1)$ in the xy-plane having at each of its points, the slope $\frac{y}{x}$.
The relevant ODE is

$$\frac{dy}{dx} = \frac{y}{x}.$$

By inspection,

$$y(x) = cx$$

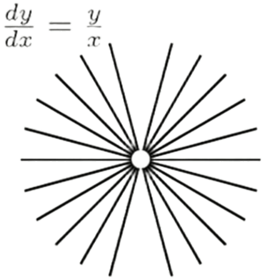
is its general solution for an arbitrary constant c; i.e., a family of straight lines.
However, we must mention that $y : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a solution but not $y : \mathbb{R} \to \mathbb{R}$. The differential equation is undefined on $x = 0$.

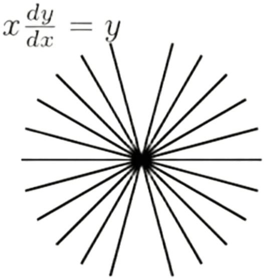
The initial condition given is $y(1) = 1$, which implies $c = 1$. Hence a particular solution for the above problem is $y(x) = x$ on $\mathbb{R} \setminus \{0\}$.

Examples ctd.

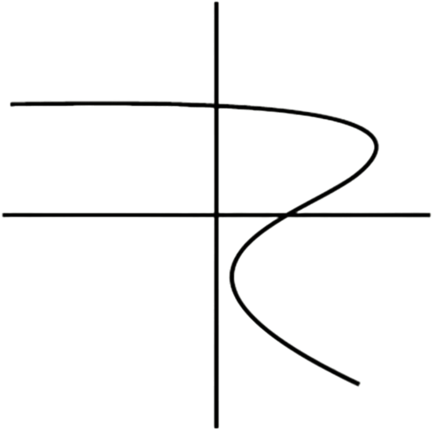
Instead of $\frac{dy}{dx} = \frac{y}{x}$, if we consider the ODE $x\frac{dy}{dx} = y$, then we can see $y : \mathbb{R} \to \mathbb{R}$ given by $y(x) = cx$ is a solution.

The solution curves:



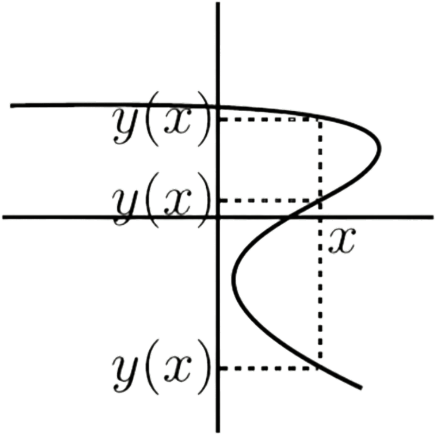


Can this graph be a solution curve?



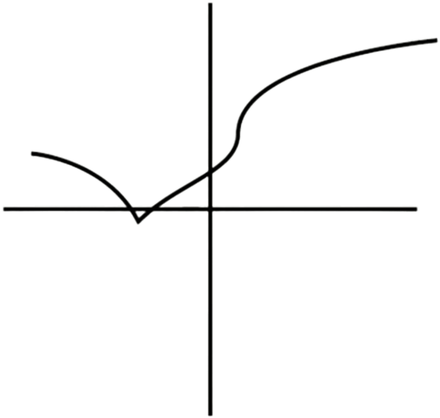
Not even a function!

To be a solution, it should be a function first!



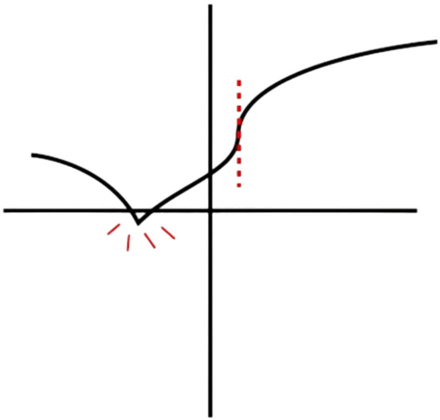
Another example

Let us consider another graph: can this graph be a solution curve?



Not even differentiable!

To be a solution, it should be differentiable upto the highest derivative involved in the ODE of which it is supposed to be a solution!

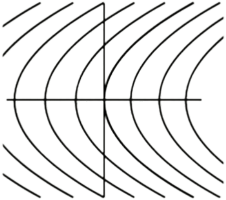


Some more examples

Let us consider the differential equation

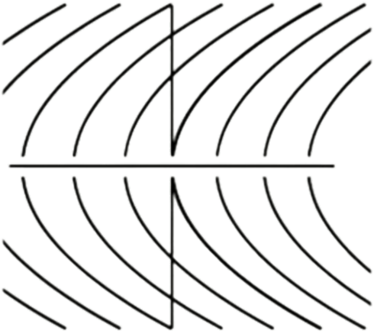
$$\frac{dy}{dx} = \frac{1}{y}.$$

Observe that $y^2 = 2(x + c)$ are implicit solutions. Let us see the graph:



Again not a function, on x-axis it can't be differentiable!

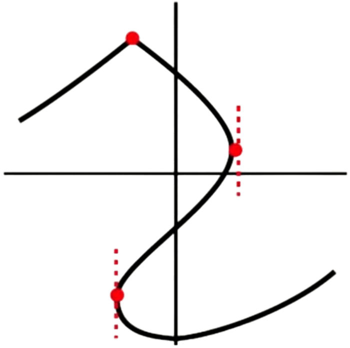
The way out



$y : (-c, \infty)$ given by $y(x) = \sqrt{2(x + c)}$ (the curves above x-axis) and
$y : (-c, \infty)$ given by $y(x) = -\sqrt{2(x + c)}$ (the curves below x-axis)are solutions.

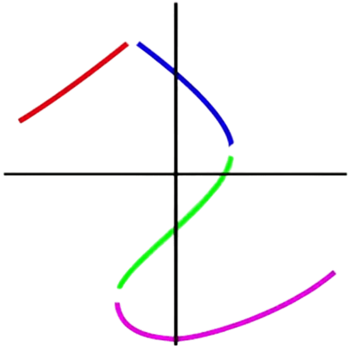
How to proceed?

Solve the equation by the methods we will describe later. Then look for the points where the solution defines a function and is differentiable upto the the highest order of the ODE we are solving and see whether the ODE is defined or not! Let us look at the example:



The cheat code

Delete the points where the function is not differentiable (upto the highest derivative in the ODE)!



Each coloured part represents a solution.

A vague (not so rigorous) explanation

The problem of having two values of y at a single x also gets eliminated by removing points where the derivative is undefined!

If from the figure in the previous slide, you can consider x and the function (say g of y), that is, $g(y) = x$ on some interval, then having two values of y would yield $g(y_1) = g(y_2)$ and by Rolle's theorem, there exists y_0 such that $g'(y_0) = 0$. By chain rule, if $y(x_0) = y_0$, then

$$y'(x_0) = \frac{1}{g'(y_0)}$$

would give y' undefined at x_0 - these points we have already deleted!

Having seen enough examples, let us now move towards the Classification of ODEs, and along the way solve a few of them.

Order of an ODE

Definition

The order of an ODE is n if the nth derivative of the unknown function y is the highest derivative of y in the equation.

Examples :

1. $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ (ODE, 2nd order)
2. $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$ (ODE, 4th order)

Non-examples:

1. $\frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} = v$ (PDE, 1st order)
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ (PDE, 2nd order)

Solving first order ODE's

Let's now start building up a systematic way of attacking ODE's. As we remarked at the very start, there is no general method to solve an arbitrary first order ODE:

$$y' = f(x, y).$$

We consider special forms of the above ODE to find solutions.

Separable ODE's

Definition (Separable ODE)

An ODE of the form
$$M(x) + N(y)y' = 0 \tag{1}$$
is called a separable ODE.

Let's first assume that $y(\cdot)$ is a solution. Let $\widetilde{M}(\cdot)$ and $\widetilde{N}(\cdot)$ be antiderivatives of $M(\cdot)$ and $N(\cdot)$, i.e.,
$$\widetilde{M}'(x) = M(x) \text{ and } \widetilde{N}'(z) = N(z) \tag{2}$$

Separable ODE's ctd.

Then, from the chain rule

$$\frac{d}{dx}\widetilde{N}\left(y(x)\right) = \widetilde{N}'\left(y(x)\right)y'(x) = N\left(y(x)\right)y'(x)$$

Then, (1) is equivalent to

$$\frac{d}{dx}\widetilde{N}\left(y(x)\right) = -\frac{d}{dx}\widetilde{M}(x)$$

Integrating both sides with respect to x gives

$$\widetilde{N}(y) + \widetilde{M}(x) = c, \tag{3}$$

where c is a constant.

Separable ODE's

Example: Solve the differential equation:

$$y' = -2xy.$$

Note that $y(x) = 0$ is a solution of the ODE.
Separating the variables, we get: for $y \neq 0$,

$$\frac{dy}{y} = -2x dx.$$

Integrating both sides, we get:

$$\ln|y| = -x^2 + c_1.$$

Thus, the solutions are

$$y(x) = ce^{-x^2}.$$

How do they look?

Separable ODE's

The solutions are

$$y(x) = ce^{-x^2}.$$

If we are given an initial condition

$$y(x_0) = y_0,$$

then we get:

$$c = y_0 e^{x_0^2}$$

and

$$y = y_0 e^{x_0^2 - x^2}.$$

Separable ODE's

Example: Find the solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2};\ y(0)=1.$$

Assume $y \neq 0$. Then,

$$\frac{1+2y^2}{y}dy = \cos x\ dx.$$

Integrating,

$$\ln|y| + y^2 = \sin x + c.$$

As $y(0)=1$, we get $c=1$. Hence a particular solution to the IVP is

$$\ln|y| + y^2 = \sin x + 1.$$

Note: $y \equiv 0$ is a solution to the DE but it is not a solution to the given IVP.

Initial Value Problem for first order ODE

Definition

Initial value problem (IVP) : A DE along with an initial condition is an IVP.

$$y' = f(x, y),\ y(x_0) = y_0.$$

A solution of the above Initial Value Problem for first order ODE is a real-valued function ϕ defined on an interval (α, β) containing x_0 such that $\phi'(\cdot)$, the derivative of ϕ, exists on the interval (α, β) satisfying

$$\phi'(x) = f(x, \phi(x)), \forall\, \alpha < x < \beta, \quad \phi(x_0) = y_0.$$

Separable ODE's

Example: Escape velocity.

A projectile of mass m moves in a direction perpendicular to the surface of the earth. Suppose v_0 is its initial velocity. We want to calculate the height the projectile reaches.

Neglect the force due to air resistance and other celestial bodies. If x denotes the height at time t and R denotes the radius of the earth, then the equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{mgR^2}{(R+x)^2};\ v(0) = v_0.$$

Separable ODE's

By chain rule,

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx}.$$

Thus,

$$v \cdot \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}.$$

This ODE is separable. Linear or non-linear? (NL)
Separating the variables and integrating, we get:

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c.$$

For $x = 0$, we get $\frac{v_0^2}{2} = gR + c$, hence, $c = \frac{v_0^2}{2} - gR$, and,

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}.$$

Separable ODE's

Suppose the body reaches the maximum height H. Then $v = 0$ at this height.

$$v_0^2 - 2gR + \frac{2gR^2}{(R+H)} = 0.$$

Thus,

$$v_0^2 = 2gR - \frac{2gR^2}{R+H} = 2gR\left(\frac{H}{R+H}\right).$$

The escape velocity is found by taking limit as $H \to \infty$. Thus,

$$v_e = \sqrt{2gR} \sim 11 \ \mathrm{km/sec}.$$

Homogeneous function

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called homogeneous if for some $d \in \mathbb{Z}$

$$f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$$

for all $t \neq 0$ and for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. The number d is called the degree of $f(x_1, \ldots, x_n)$.

Examples:
$f(x, y) = x^2 + xy + y^2$ is homogeneous of degree 2.
$f(x, y) = y + x\cos^2\left(\frac{y}{x}\right)$ is homogeneous of degree 1.

First Order Homogeneous ODE's

Definition

The first order ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is called homogeneous if M and N are homogeneous of the same degree.

Solving first order homogeneous ODE's

Consider

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where M and N are homogeneous of degree d. Put

$$y = xv.$$

Then,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v.$$

Substituting this in the given ODE, we get:

$$M(x,xv) + N(x,xv)\left(x\frac{dv}{dx} + v\right) = 0.$$

Thus,

$$x^d M(1,v) + x^d N(1,v)\left(x\frac{dv}{dx} + v\right) = 0.$$

Solving first order homogeneous ODE's Continued

$$x^d M(1, v) + x^d N(1, v)\left(x\frac{dv}{dx} + v\right) = 0.$$

Let $x \neq 0$. Then,

$$M(1, v) + N(1, v) \cdot v + N(1, v) \cdot x\frac{dv}{dx} = 0.$$

Thus,

$$\frac{dx}{x} + \frac{N(1, v)}{M(1, v) + N(1, v) \cdot v} dv = 0.$$

This is a separable equation.
NOTE: The above method can be applied to any ODE which takes the form

$$y' = f(\frac{y}{x}).$$

Remark

We will later use the term “homogeneous” in a different context to describe DE’s of the form $Ly = 0$ (as opposed to $Ly = b(x)$) for a differential operator L. If you recall, for a linear system, you used the term in this sense, in the part Linear Algebra.

Example

Example: Solve the ODE:

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0.$$

Put $y = vx$. Thus, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
Substituting this in the given ODE, we get:

$$(v^2x^2 - x^2)\left(v + x\frac{dv}{dx}\right) + 2x^2v = 0.$$

Thus, for $x \neq 0$,

$$(v^2 - 1)v + x(v^2 - 1)\frac{dv}{dx} + 2v = 0;$$

i.e.,

$$(v^3 + v) + x(v^2 - 1)\frac{dv}{dx} = 0.$$

Thus, we have the separable ODE:

$$\frac{v^2 - 1}{v(v^2 + 1)}dv + \frac{dx}{x} = 0.$$

Homogeneous ODE's

$$\frac{v^2 - 1}{v(v^2 + 1)} dv + \frac{dx}{x} = 0.$$

Integrating, we get:

$$\ln|x| + \int \left(\frac{2v}{v^2 + 1} - \frac{1}{v} \right) dv = 0.$$

Thus,

$$\ln|x| + \ln(v^2 + 1) - \ln|v| = c.$$

Hence,

$$\frac{x(v^2 + 1)}{v} = c,$$

or

$$y^2 + x^2 = cy,$$

which is

$$x^2 + \left(y - \frac{c}{2} \right)^2 = \frac{c^2}{4}.$$

Linear equations

Definition (Linear ODE)

The ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ is called linear if F when written as $\mathcal{L}(x, y, y', \ldots, y^{(n)}) = b(x)$, $\mathcal{L}$ becomes a linear function of the variables $y, y', \ldots, y^{(n)}$, meaning, for each $x \in I$, the map $\mathcal{L}_x : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$\mathcal{L}_x(x_1, x_2, \ldots, x_{n+1}) := \mathcal{L}(x, x_1, x_2, \ldots, x_{n+1})$$

is a linear map. So there exists scalars $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{R}$ (depends on x) such that

$$\mathcal{L}_x(x_1, x_2, \ldots, x_{n+1}) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{n+1} x_{n+1}.$$

Thus linear ODE of order n is of the form

$$a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_n(x) y = b(x)$$

where $a_0, a_1, \ldots, a_n, b$ are functions of x and $a_0(x) \neq 0$.

Check list : If the dependent variable is y, no products of y and/or its derivatives are there.

Operator equation $Ly = b$ in comparison to matrix equation $Ax = b$

Let

$$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$$
$$C^n(I) = \{f : I \to \mathbb{R} \mid f, f', f'', \ldots, f^{(n)} \text{ are continuous}\}.$$

Check: $C(I) C^2(I)$ are vector spaces with addition and scalar multiplication defined as:

$$(f + g)(x) = f(x) + g(x), \; x \in I,$$

$$(k \cdot f)(x) = kf(x), \; k \in \mathbb{R}, x \in I.$$

Solution space of linear ODE

Define

$$L : C^n(I) \to C(I)$$

by

$$L(f) = a_0 f^{(n)} + a_1 f^{(n-1)} + \ldots + a_n f,$$

that is,

$$L(f)(x) = a_0(x) f^{(n)}(x) + a_1(x) f^{(n-1)}(x) + \cdots + a_n(x) f(x), \text{ for each } x \in I.$$

Check that L is a linear transformation, i.e.,

$$L(cf + dg) = cL(f) + dL(g),$$

for all $c, d \in \mathbb{R}$ and for all $f, g \in C^2(I)$.

Observe that the null space $N(L)$ consists of solutions of the ODE $Ly = 0$, and by what we have learnt in linear algebra, (in solving the matrix equation $Ax = b$) every solution of ODE $Ly = b$ is of the form

$$y_0 + y_1,$$

where $y \in N(L)$ and y_1 is a particular solution of $L(y) = b$.

Examples

1. $y'' + 5y' + 6y = 0$ Ans. 2nd order, linear
2. $y^{(4)} + x^2 y^{(3)} + x^3 y' = xe^x$ Ans. 4th order, linear
3. $y'' + 5(y')^3 + 6y = 0$ Ans. 2nd order, non-linear.
4. $y'(t) = y^2(t)$ Ans. 1st order, non-linear.
5. $y'(t) = t\sin(y(t))$ Ans. 1st order, non-linear.

Warm up questions!

1. The ODE $e^x y' + 3y = x^2 y$ is linear & separable.
   Qn. TRUE OR FALSE?
   Ans. True
2. The ODE $yy' + 3x = 0$ is linear & separable.
   Qn. TRUE OR FALSE?
   Ans. False, non-linear and separable.
3. The ODE $2xyy' = y^2 - x^2$ is non-linear & Homogeneous.
   Qn. TRUE OR FALSE?
   Ans. True
4. The ODE $\frac{dy}{dx} = \frac{2yx + \cos x}{1 + x^2}$ is linear & separable & Homogeneous.
   Qn. TRUE OR FALSE?
   Ans. False, Linear but neither separable nor homogeneous form.

1st order Linear ODEs

Definition

An ODE of the form

$$\frac{dy}{dx} + p(x)y = r(x)$$

is called a 1st order linear ODE in standard form.

Here we assume $p(x)$ and $r(x)$ are continuous on an interval I.

In order to solve the linear ODE, we would like to write the linear equation above in form of a separable equation $\dfrac{du}{dx} = b(x)$.

Solving 1st order Linear ODEs

The idea is to multiply both sides of the differential equation by a function f so that

$$f(x)\frac{dy}{dx} + f(x)p(x)y = \frac{du}{dx},$$

and find f accordingly.

Note $\frac{d}{dx}(f(x)y(x)) = f(x)\frac{dy}{dx} + f'(x)y$ and so if we compare with the equation above and choose f such that

$$f'(x) = f(x)p(x),$$

then $u(x) = f(x)y(x)$ gives us the separable equation as desired where $b(x) = f(x)r(x)$.

Integrating factor

Observe that $\frac{df}{dx} = fp$ is another separable equation:

$$-p(x) + \frac{1}{f(x)}\frac{df}{dx} = 0,$$

provided $f \neq 0$. Solving for f, we have $\log|f| = \int p dx + c$ and hence $f = c_1 e^{\int p dx}$. Choose in particular $c_1 = 1$.

The function $e^{\int p dx}$ is called the integrating factor corresponding to the ODE: $\frac{dy}{dx} + p(x)y = r(x)$.

If $\tilde{p}$ is an antiderivative $\int p dx$, that is, $\tilde{p}'(x) = p(x)$, then

$$e^{\int p dx}\frac{dy}{dx} + e^{\int p dx}(py) = e^{\tilde{p}(x)}\frac{dy}{dx} + e^{\tilde{p}(x)}(\tilde{p}'(x)y(x)) = \frac{d}{dx}(e^{\tilde{p}(x)} \cdot y).$$

Solving via integrating factor

So multiplying the integrating factor $e^{\int p dx}$ both sides of the ODE: $\dfrac{dy}{dx} + p(x)y = r(x)$, we get

$$\frac{d}{dx}\left(e^{\int p dx} \cdot y\right) = e^{\int p dx} \cdot r$$

.

Thus taking $u = e^{\int p dx} \cdot y$, we have the separable equation

$$\frac{du}{dx} = e^{\int p dx} \cdot r$$

Thus the solution is

$$u = \int e^{\int p dx} \cdot r \; dx$$

and therefore we get the solution to the first order linear ODE:

$$y = e^{-\int p dx}\left(\int e^{\int p dx} \cdot r \; dx + c\right).$$

Example

Consider $y' + \left(\frac{2x+1}{x}\right) y = e^{-2x}$. Clearly $x \neq 0$. Let's take $x > 0$.
By our notation $\tilde{p}$ is antiderivative of p and hence,

$$\tilde{p}(x) = \int \frac{2x+1}{x} dx = \int (2 + \frac{1}{x}) dx = 2x + \log x.$$

So the integrating factor is

$$e^{\tilde{p}(x)} (= e^{\int p dx}) = \exp(2x + \log x) = e^{2x} e^{\log x} = xe^{2x}.$$

Using the I.F. in the ODE we get

$$xe^{2x} y' + e^{2x}(2x+1)y = x$$

or, equivalently,

$$(xe^{2x} y)' = x.$$

Thus, $xe^{2x} y - \frac{x^2}{2} = c$ is a family of solutions.

Example

Example: Solve the IVP: $t \neq 0$,

$$ty' + 2y = 4t^2;\ y(1) = 2.$$

The standard form of the given DE is:

$$y' + \frac{2y}{t} = 4t.$$

An integrating factor is

$$e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2.$$

Multiplying the given DE by t^2, we get:

$$t^2y' + 2yt = 4t^3;$$

i.e.,

$$\frac{d}{dt}(t^2y) = 4t^3.$$

Example continued

Integrating

$$\frac{d}{dt}(t^2 y) = 4t^3,$$

we get:

$$y(t) = t^2 + \frac{c}{t^2}.$$

The initial condition $y(1) = 2$ gives $c = 1$. Hence $y(t) = t^2 + \frac{1}{t^2}$ is a solution of the IVP.

Bernoulli's form

Definition

The first order DE

$$y' + p(x)y = q(x)y^n$$

is called Bernoulli's DE.

If $n = 0$ or 1, Bernoulli's DE is a linear differential equation.

Bernoulli DE $y' + p(x)y = q(x)y^n$

Let $n \geq 2$. Divide the above equation by y^n to get:

$$\frac{y'}{y^n} + \frac{p(x)}{y^{n-1}} = q(x).$$

Put

$$u(x) = \frac{1}{y^{n-1}}.$$

Then,

$$\frac{du}{dx} = \frac{1-n}{y^n}\frac{dy}{dx}.$$

Substituting this in the given DE, we get:

$$\frac{du}{dx} + (1-n)p(x)u(x) = (1-n)q(x).$$

This is a first order linear ODE. We solve it by finding an integrating factor.

Example

Example: Solve the DE

$$6x^2\frac{dy}{dx} - yx = 2y^4.$$

The equation in standard form is

$$\frac{dy}{dx} - \frac{y}{6x} = \frac{y^4}{3x^2},$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. This is a Bernoulli DE for the unknown function $y = y(x)$. If y is a non-trivial solution of the above ODE, there must be some open interval I on which y has no zeroes. On I,

$$\frac{1}{y^4}\frac{dy}{dx} - \frac{1}{6y^3x} = \frac{1}{3x^2}.$$

Put $u = \frac{1}{y^3}$. Then,

$$\frac{du}{dx} = -\frac{3}{y^4}\frac{dy}{dx}.$$

Example continued

Substituting in the given DE, we get:

$$-\frac{1}{3}\frac{du}{dx} - \frac{u}{6x} = \frac{1}{3x^2};$$

i.e.,

$$\frac{du}{dx} + \frac{u}{2x} = -\frac{1}{x^2}.$$

An integrating factor is:

$$e^{\int \frac{1}{2x}dx} = \sqrt{|x|}.$$

Thus,

$$\sqrt{|x|}\frac{du}{dx} + \frac{\sqrt{|x|}u}{2x} = -\frac{\sqrt{|x|}}{x^2}.$$

If $x > 0$, then

$$\frac{d}{dx}(u\sqrt{x}) = -x^{-\frac{3}{2}},$$

Example ctd.

which implies that

$$u\sqrt{x} = \frac{2}{\sqrt{x}} + c,$$

i.e.,

$$\frac{\sqrt{x}}{y^3} = \frac{2}{\sqrt{x}} + c.$$

Thus,

$$y^3 = \frac{x}{2 + c\sqrt{x}} \text{ for } x > 0.$$

Similarly, it can be checked that

$$y^3 = \frac{-x}{2 + c'\sqrt{-x}} \text{ for } x < 0.$$