Problem Set - 5

MA 1201 Spring Sem, 2025

1. Given an example of two 2 by 2 matrices B and C such that $B \neq C$ but $AB = AC$, where $A = \begin{bmatrix} 1 & 5 \\ 3 & 15 \end{bmatrix}$.
2. If the inverse of A^2 is B, show that the inverse of A is AB. (Thus A is invertible whenever A^2 is invertible.)
3. Find three 2 by 2 matrices, other than I and $-I$, that are their own inverses: $A^2 = I$.
4. Give examples of 2 by 2 matrices matrices A and B such that
   (a) $A + B$ is not invertible although A and B are invertible.
   (b) $A + B$ is invertible although A and B are not invertible.
   (c) All of A, B, and $A + B$ are invertible.
5. Let A and B be n by n matrices such that all of A, B, and $A + B$ are invertible. In this case, show that $C = A^{-1} + B^{-1}$ is also invertible, and find a formula for C^{-1}.
6. Under what conditions on their entries are A and B invertible?
$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}.$$
7. **(Remarkable)** Let A and B be n by n matrices. Prove that $I - BA$ is invertible if and only if $I - AB$ is invertible. [Hint: One can make use of the identity $B(I - AB) = (I - BA)B$.]
8. Invert these matrices A by the Gauss-Jordan method starting with $[A\ I]$:
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$
9. True or false (with a counterexample if false and a reason if true):
   (a) A 4 by 4 matrix with a row of zeros is not invertible.
   (b) A matrix with 1s down the main diagonal is invertible.

End of problem sheet