Lecture 07 + 08 : Feb 11 & 12, 2025

$\mathbb{R}$ or $\mathbb{R}^1$: Set of real nos.

$\mathbb{R}^2$: Set of column vectors $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of two real components.

We denote

$\mathbb{R}^3$: Set of column vectors $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ of three real components.

$\vdots$

$\mathbb{R}^n$: Set of column vectors $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ of n real components.

In $\mathbb{R}^n$, we can add two vectors:

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \in \mathbb{R}^n$$

(addition of $n \times 1$ matrices).

and we can multiply all vectors by scalars:

$$c \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} cu_1 \\ \vdots \\ cu_n \end{pmatrix} \in \mathbb{R}^n, \quad c \in \mathbb{R}.$$

But we note that this property is also getting satisfied solutions of certain system of equations:

Note $Ax = b$ & say it has at least two sol^{n}s

then $Ax_1 = b$ & $Ax_2 = b$

$$\Rightarrow A(x_1 - x_2) = 0.$$

We have seen $x_1 + \alpha(x_1 - x_2)$ $\forall \alpha \in \mathbb{R}$

is a solution of $Ax = b$.

In fact consider the set

$$\{x : Ax = 0\}.$$

pick a y_1 from this set, then

$x_1 + y_1$ is a solution of $Ax = b$.

and you can get all the solutions of $Ax = b$

by knowing a particular sol^n. as

the solution — set $\{x : Ax = 0\} =: N(A)$

Note that this also has the property

$$x_1, x_2 \in N(A) \Rightarrow x_1 + x_2 \in N(A).$$

& $cx_1 \in N(A)$ for $c \in \mathbb{R}$.

— resemblance with $\mathbb{R}^n$.

It is clear we need to know the set

$$\{x : Ax = 0\}$$

and understand its properties!

But it seems that it is a part of a bigger universe!

Some more examples:

- $M_{m\times n}(\mathbb{R})$ — $m\times n$ matrices.

  $A+B$, $cA = ((ca_{ij}))$

- X — non empty set.

  $$\mathcal{F}(X, \mathbb{R}) := \{f \mid f : X \longrightarrow \mathbb{R}\}$$

  $$(f+g)(x) := f(x) + g(x)$$

  $$\alpha f(x) := \alpha f(x).$$

- polynomials with coefficient in $\mathbb{R}$.

- $\mathbb{C}$

- $\{f : \mathbb{R} \longrightarrow \mathbb{R} : f \text{ is cts}\}$. $f+g$, αf

- $\{f : \mathbb{R} \longrightarrow \mathbb{R} : f \text{ is differentiable}\}$

  $f+g$, αf.

- Sol^n of homogeneous differential equation with constant co. eff

$$y'' + y' + y = 0.$$

One can verify the following eight properties in all the examples above:

– we write for $\mathbb{R}^n$ to have a concrete example.

For any $x, y, z \in \mathbb{R}^n$, and $c_1, c_2 \in \mathbb{R}$,

i) $x + y = y + x$

ii) $x + (y+z) = (x+y) + z$

additive identity

iii) $x + 0 = 0 + x = x$

iv) $x + (-x) = 0$ additive inverse

additive group

identity elt for scalar mult.

v) $1 . x = x$

Scalar mult is associative

vi) $\alpha(\beta x) = \underbrace{(\alpha\beta)}_{\text{usual multiplication of } \mathbb{R}} x$

properties of scalar multiplication

vii) $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

– scalar multiplication is distributive over vector addition.

viii) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot y.$

multiplication by vector is distributive over scalar addition.

Definition : (Real vector space)

Let V be a non-empty set and let there be fns

$\underset{\sim}{+} : V \times V \longrightarrow V$ — called "vector addition"

$+(x_1, x_2) \in V$ — denoted as $x_1 \underset{\sim}{+} x_2$

$\underset{\sim}{+}$ is a symbol here

and

$\underset{\sim}{\cdot} : \mathbb{R} \times V \longrightarrow V$ — called "scalar multiplication"

$\underset{\sim}{\cdot}(\alpha, v) \in V$ — denoted as $\alpha \underset{\sim}{\cdot} v$.

$\underset{\sim}{\cdot}$ is symbol here!

satisfying properties (i) to (viii).

Then $(V, +, \cdot)$ is a vector space over $\mathbb{R}$ or real vector space.

Often while writing we say $V/\mathbb{R}$.

Note :

1) property (iii) should be understood as "$\exists$ a unique element in V, called "zero vector" & denoted by '0', such that $x \underset{\sim}{+} 0 = x \quad \forall x \in V$.

2) property (iv) should be understood as: for every $x \in V$, $\exists$ a unique element, denoted by $-x$, such that

$$x \underset{\sim}{+} (-x) = 0 .$$

3) In property (vi), (viii),

$\alpha, \beta \in \mathbb{R}$, $\alpha + \beta$, $\alpha\beta$ means the usual addition and multiplication that already exist in $\mathbb{R}$.

So rewriting all the condⁿ would look like:

i) $x \underset{\sim}{+} y = y \underset{\sim}{+} x$, $\forall x, y \in V$

ii) $x \underset{\sim}{+} (y \underset{\sim}{+} z) = (x \underset{\sim}{+} y) \underset{\sim}{+} z$.

iii) $\exists$ a unique element in V, called the 'zero-vector', & denoted by θ, such that

$$x \underset{\sim}{+} \theta = x \quad \forall x \in V$$

iv) For each $x \in V$, $\exists$ a unique element, known as 'additive inverse' & denoted by '$-x$' such that

$$x \underset{\sim}{+} (-x) = \theta$$

zero vector.

v) $1 \underset{\sim}{\cdot} x = x \quad \forall x \in V$

here 1 is the usual multiplicative identity in $\mathbb{R}$.

vi) $\alpha \underset{\sim}{\cdot} (\beta \underset{\sim}{\cdot} x) = (\alpha\beta) \underset{\sim}{\cdot} x \quad \forall x \in V, \alpha \in \mathbb{R}$.

multiplication of vector by a scalar (the 2nd operation)

usual multiplication in $\mathbb{R}$.

(vii) $\alpha \underset{\sim}{\cdot} (x \underset{\sim}{+} y) = \alpha \underset{\sim}{\cdot} x \underset{\sim}{+} \alpha \underset{\sim}{\cdot} y$. $\forall x, y \in V, \alpha \in \mathbb{R}$.

(viii) $(\alpha + \beta) \underset{\sim}{\cdot} x = \alpha \underset{\sim}{\cdot} x \underset{\sim}{+} \beta \underset{\sim}{\cdot} y$.

here the addition is the usual addition in $\mathbb{R}$.

Examples:

1) $(\mathbb{R}^n, +, \cdot)$

$\cdot$ → usual scalar multiplication by an element of $\mathbb{R}$ to a vector

$+$ → usual addition of column vector

2) $S = \{x \in \mathbb{R}^n : Ax = 0\}$

0 → $(m \times 1)$ zero vector in $\mathbb{R}^m$.

A → $m \times n$ - matrix

x → thought of as $n \times 1$ column vector.

$\boxed{\text{Null space of } A \text{ denoted by } N(A).}$

The usual addition and scalar multiplication in example 1, works here as well to make $(S, +, \cdot)$ is a real vector space.

Note $S_b = \{x \in \mathbb{R}^n : Ax = b\}$ $b \neq 0$ - $m \times 1$ vector in $\mathbb{R}^m$

is not a vector space

as $x_1, x_2 \in S_b$, we have $Ax_1 = b$, $Ax_2 = b$.

Then $A(x_1 + x_2) = Ax_1 + Ax_2 = b + b = 2b$.

So $x_1 + x_2 \notin S_b$.

So to be precise S_b is a not vector space over $\mathbb{R}$ w.r.to usual addition and scalar multiplication.

3) $M_{m\times n}(\mathbb{R}) := \{A : A$ is a $m\times n$ real matrix$\}$

$\boxed{\text{Matrices as vectors}}$

given $\alpha \in \mathbb{R}$,

define $\alpha \,\tilde{\cdot}\, A = ((\alpha a_{ij}))_{\substack{1\le i\le m \\ 1\le j\le n}}$.

$M_{m\times n}(\mathbb{R})$ is a vector space over $\mathbb{R}$ w.r.t. usual matrix addition and the scalar multiplication $\tilde{\cdot}$.

(One can think of

$$\alpha \,\tilde{\cdot}\, A = (\alpha I_m) \cdot A$$

↳ matrix multiplication we know.)

4) $\mathcal{F}(\mathbb{R},\mathbb{R}) := \{f \mid f : \mathbb{R} \to \mathbb{R}\}$

$\boxed{\text{functions as vectors}}$

Define $(f \,\tilde{+}\, g)(x) := f(x) + g(x)$

↳ usual addition in $\mathbb{R}$

$$(\alpha \,\tilde{\cdot}\, f)(x) := \alpha \, f(x)$$

↳ usual multiplication in $\mathbb{R}$

$(\mathcal{F}(\mathbb{R},\mathbb{R}), \tilde{+}, \tilde{\cdot})$ – forms a real vector space.

5) $C(\mathbb{R}) := \{f \mid f : \mathbb{R} \to \mathbb{R}$ cts$\}$

$\boxed{\text{cts. fns as vectors}}$

$(C(\mathbb{R}), \tilde{+}, \tilde{\cdot})$ is a vector space over $\mathbb{R}$.

same as in (4)

6) $\mathcal{D}(\mathbb{R}) := \{f \mid f : \mathbb{R} \to \mathbb{R}$ differentiable$\}$

$(\mathcal{D}(\mathbb{R}), \tilde{+}, \tilde{\cdot})$ is a real vector space.

as in (4)

$\boxed{\text{Diff'ble fns as vectors}}$

7) $S := \{ f \mid f : \mathbb{R} \longrightarrow \mathbb{R},$

$$\frac{d^2 f}{dx^2} + \frac{df}{dx} + f = 0 \}$$

$(S, \underset{\sim}{+}, \underset{\sim}{\cdot})$ - vector space $/\mathbb{R}$

same op as in (4)

Sol^{n}s to Diff. eq^{n} as vectors

8) $\mathcal{P}_2 := \{ p \mid p$ is a poly of degree $\leq 2 \}$.
(with real coefficients).

polynomials as vectors

$p_1, p_2 \in \mathcal{P}_2$

$p_1 = a + bx + cx^2$ $\qquad a, b, c, d, e, f \in \mathbb{R}$.

$p_2 = d + ex + fx^2$

$p_1 + p_2 := (a+d) + (b+e)x + (c+f)x^2 \in \mathcal{P}_2$

$\alpha \cdot p_1 = \alpha a + \alpha b x + \alpha c x^2 \in \mathcal{P}_2$ $\qquad \alpha \in \mathbb{R}$.

$(\mathcal{P}_2, +, \cdot)$ is a real vector space

Note the set $\{ p \mid p$ is a polynomial of degree $2 \}$ is not a vector space w.r.t the addition & multiplication defined above.

$$x^2 + (-x^2) = 0 \notin \mathcal{P}_2 .$$

The same ops would make $\mathcal{P}_n := \{ p \mid p$ polynomial s.t. $\deg p \leq n \}$ or $\mathcal{P} := \{ p \mid$ polynomial$\}$ — real vector space.

Exc: Verify that the examples above are real vector spaces.

A striking example:

$\mathbb{R}_+ := \{ x \in \mathbb{R} \mid x > 0 \}$

— not a vector space w.r.t. usual addition and multiplication of $\mathbb{R}$.

$0 \notin \mathbb{R}_+$, additive inverse is not in $\mathbb{R}_+$

Now we define a addition & scalar multiplication

$x \underset{\sim}{+} y := \underbrace{xy}$ ↳ usual multiplication in $\mathbb{R}$

$\alpha \underset{\sim}{\cdot} x := x^{\alpha}$ exponents of +ve reals.

i) $x \underset{\sim}{+} y := xy = yx = y \underset{\sim}{+} x$

ii) $x \underset{\sim}{+} (y \underset{\sim}{+} z) = x \underset{\sim}{+} (yz) = x(yz) = (xy)z$

$= (xy) \underset{\sim}{+} z$

$= (x \underset{\sim}{+} y) \underset{\sim}{+} z$

iii) $\theta = 1$, $\quad x \underset{\sim}{+} \theta = x \cdot 1 = x$.

iv) $-x = \frac{1}{x}$, $\quad x \underset{\sim}{+} (-x) = x \cdot \frac{1}{x} = 1 = \theta$.

v) $1.\ x = x^1 = x$

vi) $\alpha \underset{\sim}{\cdot} (\beta \underset{\sim}{\cdot} x) = \alpha \underset{\sim}{\cdot} (x^{\beta}) = (x^{\beta})^{\alpha} = x^{\beta\alpha}$

$= x^{\alpha\beta}$

$= (\alpha\beta) \underset{\sim}{\cdot} x$.

vii) $\alpha \tilde{\cdot} (x \tilde{+} y) = \alpha \tilde{\cdot} (xy) = (xy)^{\alpha}$

$= x^{\alpha} y^{\alpha} = x^{\alpha} \tilde{+} y^{\alpha} = \alpha \tilde{\cdot} x \tilde{+} \alpha \tilde{\cdot} y.$

viii) $(\alpha+\beta) \tilde{\cdot} x = x^{\alpha+\beta} = x^{\alpha} x^{\beta}$

$= x^{\alpha} \tilde{+} x^{\beta}$

$= \alpha \tilde{\cdot} x \tilde{+} \beta \tilde{\cdot} x.$

So $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ is a vector space over $\mathbb{R}$.

Showing that the name addition mult. came with usual operations but it need not be addition or multiplication in reality.

Now question is why over $\mathbb{R}$?

what is special?

$\mathbb{F}$ — field.

comes with two operations $\hat{+}$, $\hat{\cdot}$

$\hat{+} : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ - called addition

$\hat{\cdot} : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ - called multiplication

operations could be anything satisfying the following properties

such that

i) $\alpha \hat{+} \beta = \beta \hat{+} \alpha$ → Commutativity ← v) $\alpha \hat{\cdot} \beta = \beta \hat{\cdot} \alpha$

ii) $\alpha \hat{+} (\beta \hat{+} \gamma) = (\alpha \hat{+} \beta) \hat{+} \gamma$ → associativity ← vi) $\alpha \hat{\cdot} (\beta \hat{\cdot} \gamma) = (\alpha \hat{\cdot} \beta) \hat{\cdot} \gamma$

iii) $\exists\ 0$ - zero elt. (additive identity) → identity ← vii) $\exists\ 1$ - multiplicative identity

such that $\alpha \hat{+} 0 = \alpha$ | s.t $\alpha \hat{\cdot} 1 = \alpha$

iv) For each x, $\exists\ -x \in \mathbb{F}$ → inverse ← viii) For each $x \neq 0$, $\exists\ x^{-1} \in \mathbb{F}$ s.t

s.t $x + (-x) = 0$ (zero elt) | $x \hat{\cdot} x^{-1} = 1$ (mult. identity).

'mult.' is distributive over 'addition'

$\alpha \hat{\cdot} (\beta \hat{+} \gamma) = \alpha \hat{\cdot} \beta \hat{+} \alpha \hat{\cdot} \gamma.$

Examples of a field:

$(\mathbb{F}, \hat{+}, \hat{\cdot}) = (\mathbb{R}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ — all are with usual addition & multiplication

$\mathbb{Z}_2 = \{0,1\}$

$0+0=0$	$0\cdot 0 = 0$
$1+0=1$	$1\cdot 0 = 0$
$0+1=1$	$1\cdot 1 = 1.$
$1+1=0$	

$(\mathbb{Z}_2, +, \cdot)$ — field.

One can consider v. sp over a field $\mathbb{F}$.

Cond^n v) becomes $1 \in \mathbb{F} \rightarrow$ multiplicative identity

& $1 \,\tilde{\cdot}\, x = x \quad \forall x \in$

i – iv remains same

vi) becomes $\rightarrow$ field mult.

$\alpha \,\tilde{\cdot}\, (\beta \,\tilde{\cdot}\, x) = (\alpha \,\hat{\cdot}\, \beta) \,\tilde{\cdot}\, x.$

vii) remains as it is

viii) $(\underbrace{\alpha \,\hat{+}\, \beta}_{\text{field addition}}) \,\tilde{\cdot}\, x = \alpha \,\tilde{\cdot}\, x \,\tilde{+}\, \beta \,\tilde{\cdot}\, x.$

Remark: We can also consider complex vector spaces, (Analysis).

or vector spaces over $\mathbb{Q}$ (number theory)

or vector spaces over $\mathbb{Z}_2$ (Cryptography).

$\mathbb{Z}_2$: finite field; coding

Some properties of vector spaces:

For sake of simplicity I use $+$ instead of $\underset{\sim}{+}$ & $\cdot$ instead of $\underset{\sim}{\cdot}$

- **Cancellation law for vector addition.**

If $x, y, z \in V$ s.t. $x + z = y + z$, then $x = y$.

Pf. $x = x + 0 = x + \{z + (-z)\}$

$= (x+z) + (-z)$ — associativity of vector addition

$= (y+z) + (-z)$

$= y + \{z + (-z)\}$

$= y + 0$

$= y$.

Exc. 1. Uniqueness of additive identity
2. uniqueness of additive inverse

No need to assume.

Pf. of 1: $\exists\ \theta_1, \theta_2$ — additive identity

$\Rightarrow x + \theta_1 = x = x + \theta_2$

Cancellation law $\Rightarrow \theta_1 = \theta_2$.

Pf. of 2: $\exists\ x_1, x_2$ — additive inverse of x

then $x + x_1 = \theta = x + x_2$

Cancellation law $\Rightarrow x_1 = x_2$.

3. $0 \cdot x = \theta \quad \forall x \in V$

Pf. $0 + 0 = 0 \Rightarrow (0+0)\cdot x = 0 \cdot x$

$\overset{(viii)}{\Rightarrow} 0 \cdot x + 0 \cdot x = 0 \cdot x = 0 \cdot x + \theta$

Cancellation law $\Rightarrow 0 \cdot x = \theta$

4. $\alpha \cdot \theta = \theta \quad \forall \alpha \in \mathbb{R}$

Pf. $\theta + \theta = \theta \implies \alpha \cdot (\theta + \theta) = \alpha \cdot \theta$

iii) $\implies$ $\overset{(vii)}{\implies} \alpha \cdot \theta + \alpha \cdot \theta = \alpha \cdot \theta = \alpha \cdot \theta + \theta$

Cancellation law $\implies \alpha \cdot \theta = \theta$.

5. If $\alpha \cdot x = \theta$, then either $\alpha = 0$ or $x = \theta$

Pf. If $\alpha \neq 0$, then $\alpha^{-1} \in \mathbb{F}$,

so $\alpha^{-1} \cdot (\alpha \cdot x) = \alpha^{-1} \cdot \theta = \theta$

$\overset{(vi)}{\implies} (\alpha^{-1} \cdot \alpha) \cdot x = \theta$

$\implies 1 \cdot x = \theta$

$(v) \implies x = \theta$.

6. $-x = (-1) \cdot x$

Pf. $1 + (-1) = 0$ in Field.

$\implies \{1 + (-1)\} \cdot x = 0 \cdot x = \theta$

$\overset{(viii)}{\implies} 1 \cdot x + (-1) \cdot x = \theta$

$\overset{(v)}{\implies} x + (-1)\, x = \theta = x + (-x)$.

Cancellation law $\implies (-1)\, x = -x$.

Exc. 7. $(-\alpha) \cdot x = \alpha\, (-x) = -\alpha \cdot x$.

Remark:

Note for two matrices $AB = 0$ does not say either $A = 0$ or $B = 0$ $\left(\begin{array}{l} AB = A0 \\ \text{or } AB = 0 \cdot B \end{array} \right)$

– So cancellation law fails w.r. to the operation.

Subspace:

In the study of any algebraic structure, it is of interest to examine subsets that possesses same structure as the set under consideration.

Def^n: Let $V/_{\mathbb{F}}$ be a vector space.

A non-empty subset W of V is said to be a subspace of V if W is itself a vector space over $\mathbb{F}$ w.r.to the operations addition and scalar multiplication defined on V.

Ex: By def^n $\{0\}$, V are subspaces of $V/_{\mathbb{F}}$.

So how do we check a subset is a subspace or not!

— checking all the eight cond^n of the vector space.

BUT note

if $x, y \in W$, first we need both the operations to be defined on W itself.

So, $x, y \in W \Rightarrow x + y \in W$ ...... (a)

known as W is closed under addition.

$x \in W \Rightarrow \alpha . x \in W \quad \forall \alpha \in \mathbb{F}$ ---- (b)

known as W is closed under scalar multiplication

Now if

Cond^n. (a) happens

then i) & ii) are automatic as as the vectors were in V as well.

Cond^n. (b) happens

Then v) to (viii) are automatic as the vectors were in V as well.

Cond^n iii) becomes true

as for $x \in W$, $0 \cdot x \in W$

$\Rightarrow \theta \in W$.

Cond^n iv) becomes true

as for each $x \in W$, $(-1) \cdot x \in W$

$\Rightarrow -x \in W$.

So we have the following proposition:

Proposition:

A non-empty subset W of V is a subspace if and only if it is closed under addition and scalar multiplication.

Ex: Given an $m \times n$ matrix A

define null space $N(A)$ of A by

$$N(A) := \{ \underbrace{x}_{n\times 1} \in \mathbb{R}^n : Ax = \underbrace{0}_{m\times 1} \}$$

Clearly if $x_1, x_2 \in N(A)$

then $Ax_1 = 0 = Ax_2$

$$\Rightarrow A(x_1 + x_2) = Ax_1 + Ax_2 = 0$$

$$\Rightarrow x_1 + x_2 \in N(A).$$

& for $x_1 \in N(A)$

$$A(\alpha x_1) = \alpha A x_1 = 0.$$

$$\Rightarrow \alpha x_1 \in N(A) \quad \forall \alpha \in \mathbb{R}.$$

Thus $N(A)$ is a subspace of $\mathbb{R}^n / \mathbb{R}$ under usual addition and scalar multiplication

Remark:

1) This gives an easy way to check some space is a vector space or not if you can check it is a subspace of bigger space which you know as a vector space. You only need to check 2 condn. instead of 8.

2) Anyone who is following closely — would somehow notice that we have few vector spaces and others are subspaces of them, like the previous example.

Ex: $\mathcal{D}(\mathbb{R}) \subseteq C(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$.

Ex: $V = \mathbb{R}^2$

$$W = \{(x, 0) : x \in \mathbb{R}\}$$
$$= \{(x, mx) : m \in \mathbb{R}\}$$
$$= \{(0, y) : y \in \mathbb{R}\}$$

all lines passing through origin.

needed as $0 \in W$.

Ex: $W = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_1 + u_2 + u_3 = 0\}$

$(u_1, u_2, u_3) \in W$ & $(v_1, v_2, v_3) \in W$

then $(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3)$

$$= (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0.$$

$$\Rightarrow (u_1 + v_1, u_2 + v_2, u_3 + v_3) \in W.$$

& $\alpha(u_1 + u_2 + u_3) = 0$.

$$\Rightarrow \alpha(u_1, u_2, u_3) \in W.$$

— W is a subspace.

Qn:

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$$

is it a subspace of $\mathbb{R}^2$?

$$S_1 = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_1 \geq 0\}$$

is it a subspace of $\mathbb{R}^3$?

$$S_2 = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_1 + u_2 = 5\}$$

is it a subspace of $\mathbb{R}^3$?

Now we would like to create more subspaces of given vector space $/\mathbb{F}$.

Pick a vector $v \in V$ and fix it.

Let $W = \{\alpha v : \alpha \in \mathbb{F}\}$.

Note $\alpha_1 v + \alpha_2 v = (\alpha_1 + \alpha_2) \cdot v$.

& $\beta(\alpha v) = (\beta\alpha) v$.

& hence W is a subspace of $V/\mathbb{F}$.

This subspace is known as subspace generated by / spanned by the vector v, and denoted as $\text{span}\{v\}$.

Qn: why to stop at 1 vector?

Given $v_1, v_2, \dots v_k \in V$

- finitely many vector in V.

by a linear combination of vectors $v_1, v_2, \dots v_k$, we mean.

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$$

for some $\alpha_1, \alpha_2 \dots \alpha_k \in \mathbb{F}$.

Consider

$$\left\{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k : \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{F} \right\}$$

Note $(\alpha_1 v_1 + \alpha_2 v_2 \; \dots + \alpha_k v_k) +$

$(\beta_1 v_1 + \beta_2 v_2 \; \dots + \beta_k v_k)$

$$= (\alpha_1 + \beta_1) v_1 + \dots + (\alpha_k + \beta_k) v_k$$

& $\alpha \, (\alpha_1 v_1 + \dots + \alpha_k v_k)$

$$= (\alpha \alpha_1) v_1 + \dots + (\alpha \alpha_k) v_k.$$

So it is a subspace of $V/\mathbb{F}$.

- This subspace is said to be spanned by the vectors $\{v_1 \dots v_k\}$ & denoted by

span $\{v_1 \dots v_k\}$