1. *Describe the intersection of the three planes $u+v+w+z=6$ and $u+w+z=4$ and $u+w=2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u=-1$ is included? Find a fourth equation that leaves us with no solution.

Solution:

Part 1: We need to solve for $(u,v,w,z) \in \mathbb{R}^4$ satisfying the following equations,

$$u+v+w+z=6, \tag{1}$$
$$u+w+z=4, \tag{2}$$
$$\text{and} \qquad u+w=2. \tag{3}$$

We employ the method of elimination. We replace Equation (1) by the one obtained by subtracting Equation (2) from Equation (1). After that we replace Equation (2) by the one obtained by subtracting Equation (3) from Equation (2). We know that linear equations (1) to (3) has exactly the same set of solutions as that of the following set of equations:

$$\begin{aligned} v &= 2, \\ z &= 2, \\ \text{and} \quad u+w &= 2. \end{aligned}$$

Rearranging (permuting) the above equations to have the diagonal form, we obtain

$$\begin{aligned} u+w &= 2, \\ v &= 2, \\ \text{and} \quad z &= 2. \end{aligned}$$

Thus the intersection of the three planes is the set S described by

$$\begin{aligned} S &= \{(u,v,w,z) \in \mathbb{R}^4 : \ v=2, \ z=2, \ \text{and } u+w=2\} \\ &= \{(u,2,w,2) \in \mathbb{R}^4 : \ u+w=2\} \\ &= \{(t,2,2-t,2) : t \in \mathbb{R}\} \\ &= \{(0,2,2,2) + t(1,0,-1,0) : \ t \in \mathbb{R}\}. \end{aligned}$$

Thus the set S is the line passing through the point $(0,2,2,2)$ along the vector $(1,0,-1,0)$.

Part 2: Along with the three planes, let us also consider the fourth plane $u=-1$. That is, we now should solve the system of equations given by

$$u=-1, \tag{4}$$
$$u+v+w+z=6, \tag{5}$$
$$u+w+z=4, \tag{6}$$
$$\text{and} \quad u+w=2. \tag{7}$$

Question 1 continues on the next page...

Using the argument as before, by applying Gaussian elimination and permutation steps, this is equivalent to solving the system of equations

$$\begin{aligned} u &= -1, \\ u + w &= 2, \\ v &= 2, \\ \text{and} \quad z &= 2. \end{aligned}$$

We subtract the first equation from the second equation above to obtain

$$u = -1, \; w = 3, \; v = 2, \text{ and } z = 2.$$

Thus the set of points $(u, v, w, z) \in \mathbb{R}^4$ satisfying equations (4) to (7) is the singleton set $\{(-1, 2, 2, 3)\}$.

<u>Part 3</u>: We now find a fourth equation which along with equations (1) to (3) has no solution. There are infinitely many choices for such a equation, we only need to find one such equation that is inconsistent with any of the following equations

$$u + w = 2, \; v = 2, \text{ or } \quad z = 2.$$

For example, $u + w = 0$ is one such equation. Such a system has no solution as $u + w$ cannot be simultaneously 0 and 2.

2. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$
$$x - y = 2$$
$$y = 1.$$

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

Solution:

Part 1: The first line $x + 2y = 2$ passes through the points $(0, 1)$ and $(2, 0)$. The second line $x - y = 2$ passes through the points $(0, -2)$ and $(2, 0)$. And the line $y = 1$ passes through the point $(0, 1)$ and is parallel to the $x-$axis. The lines are sketched in the figure below.

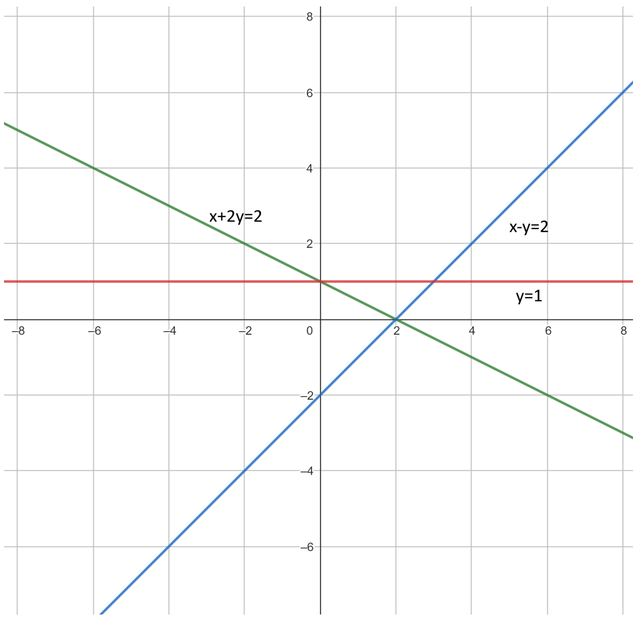

Figure 1: The figure consists of the three lines.

We note that the lines $x + 2y = 2$ and $x - y = 2$ intersects at the point $(2, 0)$ but this point does not lie on the line $y = 1$. Hence the system of the equations has no solution.

Part 2: We now consider the system of equations given by

$$x + 2y = 0,$$
$$x - y = 0,$$
$$\text{and} \quad y = 0.$$

We note that $(0, 0)$ is an obvious solution. To see that this is the only solution, note that the third equation forces $y = 0$ and putting it into any any of the two equations $x + 2y = 0$ or $x - y = 0$, we get $x = 0$. This implies that $(0, 0)$ is the only solution.

Question 2 continues on the next page. . .

Part 3: We now want to find $a, b, c \in \mathbb{R}$ such that the following system has a solution

$$x + 2y = a, \tag{8}$$
$$x - y = b, \tag{9}$$
$$\text{and} \quad y = c. \tag{10}$$

We again add twice the second equation to the first equation to obtain,

$$\begin{aligned} 3x &= a + 2b, \\ x - y &= b, \\ \text{and} \qquad y &= c. \end{aligned}$$

This implies that $x = \frac{a+2b}{3}$ and $y = c$ but we must also have $x - y = b$. This provides us the necessary condition $a - b - 3c = 0$ for the solution to exist. Hence the system of equations (8)-(10) has a solution for all $a, b, c \in \mathbb{R}$ satisfying $a - b - 3c = 0$ and the solution is the singleton set $\{(\frac{a+2b}{3}, c)\}$.

3. Explain why the system

$$\begin{aligned} u+v+w&=2 \\ u+2v+3w&=1 \\ v+2w&=0 \end{aligned}$$

has no solution by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions, and what is one of the solutions?

Solution:

Part 1: We are given the system of linear equations:

$$u+v+w=2 \tag{11}$$
$$u+2v+3w=1 \tag{12}$$
$$v+2w=0. \tag{13}$$

The linear combination eq. (11) + eq. (13) - eq. (12) results into $0 = 1$. This proves the singularity of the system.

Part 2: Subtracting eq. (11) from eq. (12) and then replacing eq. (12) with it, we get the following equivalent system of equations:

$$u+v+w=2 \tag{14}$$
$$v+2w=-1 \tag{15}$$
$$v+2w=0. \tag{16}$$

The above system will become consistent if we replace the right side of eq. (16) by -1. After doing this replacement, we can subtract eq. (15) from eq. (16) and then replace eq. (16) with it, to get the following equivalent system of equations:

$$u+v+w=2 \tag{17}$$
$$v+2w=-1 \tag{18}$$
$$0=0. \tag{19}$$

From eq. (18) we get $v = -2w - 1$. Now putting v equals $-2w - 1$ in eq. (17), we obtain $u = 3 + w$.

Hence the solution set for the above system of equations is $\{(3 + w, -2w - 1, w) : w \in \mathbb{R}\}$.

Now, for a particular solution choose $w = -1$, then $u = 3 + (-1) = 2$, and $v = -2(-1) - 1 = 1$. Thus, one of the possible solutions of the new system made of equations eq. (11), eq. (12) and eq. (13) (with -1 in place of 0 in the right hand side of eq. (13)) is $(2, 1, -1)$.

4. Under what condition on $y_1,\, y_2,\, y_3 \in \mathbb{R}$ do the points $(0, y_1), (1, y_2)$, and $(2, y_3)$ lie on a straight line?

Solution:

Consider a general form of the line in the plane $\mathbb{R}^2$:

$$ax + by = c, \quad a, b, c \in \mathbb{R}. \tag{20}$$

Now the three points $(0, y_1), (1, y_2)$, and $(2, y_3)$ lie on a same straight line if and only if they satisfy eq. (20) simultaneously. So, putting these points one by one in the eq. (20) we get the following system of equations in the unknowns a, b, c:

$$\begin{aligned} by_1 &= c \\ a + by_2 &= c \\ 2a + by_3 &= c, \end{aligned}$$

which is equivalent to

$$by_1 - c = 0 \tag{21}$$
$$a + by_2 - c = 0 \tag{22}$$
$$2a + by_3 - c = 0. \tag{23}$$

Now this is a homogeneous system. To find that straight line we need a nontrivial solution of the system made of eq. (21), eq. (22) and eq. (23).

Now interchanging eq. (22) with eq. (21), we get the equivalent system:

$$a + by_2 - c = 0 \tag{24}$$
$$by_1 - c = 0 \tag{25}$$
$$2a + by_3 - c = 0. \tag{26}$$

Now subtracting twice of eq. (24) from eq. (26) and then replacing eq. (26) with it, we get the following equivalent system of equations:

$$a + by_2 - c = 0 \tag{27}$$
$$by_1 - c = 0 \tag{28}$$
$$b(y_3 - 2y_2) + c = 0. \tag{29}$$

Now, adding eq. (28) with eq. (29) and then replacing eq. (29) with it, then we get the following equivalent system of equations:

$$a + by_2 - c = 0 \tag{30}$$
$$by_1 - c = 0 \tag{31}$$
$$b(y_3 - 2y_2 + y_1) + 0 = 0. \tag{32}$$

Note that from eq. (32) we have $b(y_3 - 2y_2 + y_1) = 0$.

Question 4 continues on the next page...

There are two possibilities: either $b = 0$ or $b \neq 0$.

If $b = 0$ then from eq. (31), we get $c = by_1 \implies c = 0$. And then from eq. (30), we get $a = -by_2 + c \implies a = 0$. But, that just says that the line $ax + by = c$ doesn't exist.

So, let us take $b \neq 0$ and fix it. In this case, we get $y_3 - 2y_2 + y_1 = 0$. Fix also y_1 and y_2 arbitrarily.

eq. (31) gives $c = by_1$. Wit that eq. (30) gives $a = c - by_2 = b(y_1 - y_2)$.

Thus we see that we get non-trivial value of a, b, c if and only if $y_3 - 2y_2 + y_1 = 0 \iff 2y_2 = y_1 + y_3$.

Therefore, the three points $(0, y_1), (1, y_2)$, and $(2, y_3)$ lie on a straight line exactly when $y_1, y_2, y_3 \in \mathbb{R}$ satisfy the relation $\boldsymbol{2y_2 = y_1 + y_3}$.

5. These equations are certain to have the solution $x = y = 0$. For which values of d is there a whole line of solutions?

$$dx + 2y = 0$$
$$2x + dy = 0.$$

Solution:

To determine the values of d for which the following system of linear equations has a whole line of solutions, we start by analyzing the system:

$$dx + 2y = 0 \tag{33}$$
$$2x + dy = 0. \tag{34}$$

We will consider different cases based on the value of d:

Case 1 : Let $d = 2$. Substituting this into (33) and (34), the system becomes

$$2x + 2y = 0 \tag{35}$$
$$2x + 2y = 0. \tag{36}$$

Subtracting (35) from (36) and then replacing it with (36), we get the simplified system:

$$2x + 2y = 0 \tag{37}$$
$$0 = 0. \tag{38}$$

The solution set to (37) and (38) is $\{(x, y) \in \mathbb{R}^2 : x = -y\}$, which is a line in $\mathbb{R}^2$. Since this is also the solution set for the system (35) and (36), the original system (33) and (34) will indeed have a full line of solutions when $d = 2$.

Case 2 : If we substitute $d = -2$ in (33) and (34), we obtain the system:

$$-2x + 2y = 0 \tag{39}$$
$$2x - 2y = 0. \tag{40}$$

Adding (39) and (40), and then replacing it with (40), the system becomes:

$$-2x + 2y = 0 \tag{41}$$
$$0 = 0. \tag{42}$$

Since the solution set to the system (41) and (42) is also a line in $\mathbb{R}^2$, namely $\{(x, y) \in \mathbb{R}^2 : x = y\}$, it follows from a similar argument as in Case 1 that the system (33) and (34) has a full line of solutions when $d = -2$.

Case 3 : Now, let us consider the case where $d = 0$. Substituting $d = 0$ into (33) and (34), we obtain the system:

$$0x + 2y = 0 \tag{43}$$
$$2x + 0y = 0. \tag{44}$$

Question 5 continues on the next page...

By interchanging (43) and (44), we get the simplified system:

$$2x + 0y = 0 \tag{45}$$
$$0x + 2y = 0. \tag{46}$$

Clearly, the solution to the system (45) and (46) is $x = 0$, $y = 0$. Consequently, the solution to the system (33) and (34) when $d = 0$, is $x = 0$, $y = 0$, which represents a single point $(0, 0)$ in $\mathbb{R}^2$.

Case 4 : Finally, we consider the case where $d \in \mathbb{R}$ and $d \neq 0, 2, -2$. Subtracting $(\frac{2}{d})$ times of (33) from (34) and then replacing it with (34), we get the new system:

$$dx + 2y = 0 \tag{47}$$
$$\left(\frac{d^2 - 4}{d}\right) y = 0. \tag{48}$$

Since $d \neq 2, -2$, we have

$$\frac{d^2 - 4}{d} \neq 0.$$

Implementing this fact in (48), we obtain $y = 0$. Substituting $y = 0$ in (47) and noting that $d \neq 0$, we get $x = 0$. Thus, when $d \in \mathbb{R}$ and $d \neq 0, 2, -2$, the solution to the system (33) and (34) is $(x, y) = (0, 0)$, a single point in $\mathbb{R}^2$.

In conclusion, the system of linear equations (33) and (34) will have an entire line of solutions if and only if $d = 2$ or $d = -2$.

6. What multiple of equation 1 should be subtracted from equation 2 ?

$$2x - 4y = 6$$
$$-x + 5y = 0.$$

After this elimination step, solve the triangular system. If the right-hand side changes to $(-6, 0)$, what is the new solution?

Solution:

Since the problem consists of two parts, we will address each one separately. Consider the system of linear equations:

$$2x - 4y = 6 \tag{49}$$
$$-x + 5y = 0. \tag{50}$$

Part 1 : Subtracting $(-\frac{1}{2})$ times of (49) from (50) and then replacing (50) with it, we obtain the triangular system:

$$2x - 4y = 6 \tag{51}$$
$$3y = 3. \tag{52}$$

From (52), we have $y = 1$. Plugging in $y = 1$ in (51), we obtain $x = 5$. Therefore, the solution to the triangular system (51) and (52) is $x = 5$ and $y = 1$.

Part 2 : If we replace the right-hand sides of (49) and (50) with -6 and 0, respectively, we obtain the modified system:

$$2x - 4y = -6 \tag{53}$$
$$-x + 5y = 0. \tag{54}$$

Once again, we perform the same operation as in Part 1. Subtracting $(-\frac{1}{2})$ times of (53) from (54) and then replacing it with (54), yields:

$$2x - 4y = -6 \tag{55}$$
$$3y = -3. \tag{56}$$

Solving (56), we get $y = -1$. Substituting $y = -1$ into (55), we obtain $x = -5$. Therefore, the solution to the modified system (53) and (54) is $x = -5$ and $y = -1$.

7. *For which numbers d does elimination break down (i) permanently, and (ii) temporarily ?

$$dx + 3y = -3$$
$$4x + 6y = 6.$$

Solve for x and y after fixing the second breakdown by a row exchange.

Solution:

Assuming that d is an arbitrary real number, we attempt to apply elimination to the given system of equations:

$$dx + 3y = -3, \tag{57}$$
$$4x + 6y = 6. \tag{58}$$

For $d \neq 0$, the pivot element in the first equation is nonzero. To eliminate x from the second equation, we subtract $\frac{4}{d}$ times (57) from (58), yielding the transformed system:

$$dx + 3y = -3, \tag{59}$$
$$\left(6 - \frac{12}{d}\right) y = \left(6 + \frac{12}{d}\right). \tag{60}$$

Elimination fails if the coefficient of y in the second equation becomes zero, i.e.,

$$6 - \frac{12}{d} = 0.$$

Solving for d:

$$6 = \frac{12}{d} \quad \Rightarrow \quad d = 2.$$

(i) **Permanent breakdown:** A permanent breakdown occurs if elimination leads to an inconsistent equation. Substituting $d = 2$ into (60), we obtain:

$$0y = 6 + \frac{12}{2} = 6 + 6 = 12,$$

which is a contradiction. Thus, elimination permanently breaks down for $d = 2$.

(ii) **Temporary breakdown:** This situation occurs if and only if $d = 0$. In this case, the given system becomes

$$3y = -3 \tag{61}$$
$$4x + 6y = 6. \tag{62}$$

Exchanging eq. (61) and eq. (62), we get

$$4x + 6y = 6 \tag{63}$$

$$3y = -3. \tag{64}$$

If x and y are real numbers satisfying the above system, then we have $y = -1$ from eq. (64), and substituting it back into eq. (63) gives us that $x = 3$. Also, note that $x = 3$ and $y = -1$ indeed solve the given system of equations with $d = 0$. Thus, we conclude that the solution of the given system of equations with $d = 0$ is $\mathbf{x = 3}$ and $\mathbf{y = -1}$.

8. Apply elimination (circle the pivots) and back-substitution to solve

$$2x - 3y = 3$$
$$4x - 5y + z = 7$$
$$2x - y - 3z = 5.$$

List the three row operations: Subtract __ times row __ from row __ .

Solution:

We express the given system of equations in the form of a matrix equation as follows.

$$\begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \tag{65}$$

We apply elimination technique to solve this equation, which consists of the following row operations, while circling the pivots in each step.

Step 1: Replace row 2 in eq. (65) by "the subtraction of twice of row 1 from row 2". Then, we get

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}. \tag{66}$$

Step 2: Replace row 3 in eq. (66) by "the subtraction of row 1 from row 3". This gives

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}. \tag{67}$$

Step 3: Replace row 3 in eq. (67) by "the subtraction of twice of row 2 from row 3" to get

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}. \tag{68}$$

By performing the matrix multiplication, eq. (68) yields

$$\begin{bmatrix} 2x - 3y \\ y + z \\ -5z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}. \tag{69}$$

Note that eq. (69) (by equating the entries of the matrices on either side of the equality sign) gives the following system of equations, which has exactly the same solutions as that of the given system of equations.

$$2x - 3y = 3 \tag{70}$$
$$y + z = 1 \tag{71}$$
$$-5z = 0 \tag{72}$$

eq. (72) gives $z = 0$. Plugging $z = 0$ in eq. (71) gives $y = 1$. By substituting $y = 1$ in eq. (70), we get $x = 3$.

Hence, $(x, y, z) = (3, 1, 0)$ is the only solution of the given system of equations.

9. *Apply elimination to the system

$$u+v+w=-2$$
$$3u+3v-w=6$$
$$u-v+w=-1.$$

When a zero arises in the pivot position, exchange that equation for the one below it and proceed. What coefficient of v in the third equation, in place of the present -1, would make it impossible to proceed, and force elimination to break down?

Solution:

The given system is:

$$u+v+w=-2 \tag{73}$$
$$3u+3v-w=6 \tag{74}$$
$$u-v+w=-1. \tag{75}$$

Part 1: Subtracting three times of eq. (73) from eq. (74), we get

$$(3u+3v-w)-(3u+3v+3w)=6-(-6) \implies -4w=12.$$

Next, subtracting eq. (73) from eq. (75), we get

$$(u-v+w)-(u+v+w)=-1-(-2) \implies -2v=1.$$

In this way, we get a new system of equations as shown below:

$$u+v+w=-2 \tag{76}$$
$$-4w=12 \tag{77}$$
$$-2v=1. \tag{78}$$

Notice that zero appears in the pivot position in eq. (77). By exchanging eq. (77) and eq. (78), we get the following system of equations:

$$u+v+w=-2$$
$$-2v=1 \tag{79}$$
$$-4w=12. \tag{80}$$

Simplifying eq. (79) and eq. (80), we get $v=-\frac{1}{2}$ and $w=-3$. Substitute these values of v and w in eq. (73) to find u:

$$u+(-\frac{1}{2})+(-3)=-2 \implies u=\frac{3}{2}$$

Thus, the solution of the system is:

$$\mathbf{u}=\mathbf{\frac{3}{2}}, \mathbf{v}=-\mathbf{\frac{1}{2}} \text{ and } \mathbf{w}=-\mathbf{3}.$$

Question 9 continues on the next page...

Part 2: What coefficient of v in the third equation, in place of the present -1, would make it impossible to proceed?

Replace the coefficient of v in the third equation, that is, in eq. (75) by 1. Then, we get

$$u + v + w = -1. \tag{81}$$

Now, if we start the elimination as earlier, that is, subtracting eq. (73) from eq. (81), we get

$$(u + v + w) - (u + v + w) = -1 - (-2) \implies 0 = 1,$$

which is a contradiction.

Thus, if the coefficient of v in the third equation were to be 1, elimination would break down permanently.

10. *Suppose A commutes with every 2 by 2 matrix ($AB = BA$), and in particular

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ commutes with } B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Show that $a = d$ and $b = c = 0$.

If $AB = BA$ for all matrices B, then A is a multiple of the identity.

Solution:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, commute with every 2 by 2 matrix B, that is, $AB = BA$. Then, in particular

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ commutes with } B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We shall show that $a = d$ and $b = c = 0$.

Let us begin with computing AB_1 and B_1A. We have

$$AB_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \text{ and } B_1A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Since A commutes with B_1, that is, $AB_1 = B_1A$, by equating the entries in the two matrices, we get $b = c = 0$.

Next, we compute AB_2 and B_2A. We have

$$AB_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \text{ and } B_2A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}.$$

As A commutes with B_2, that is, $AB_2 = B_2A$, we get $a = d$.

Thus, $a = d$ and $b = c = 0$.

In other words,

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

that is, A is a multiple of the identity matrix I.

We cannot say anything extra for the real number a. This is because every multiple of the identity matrix commutes with every matrix B. In fact,

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap & aq \\ ar & as \end{bmatrix} \text{ and } \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ap & aq \\ ar & as \end{bmatrix}.$$

Conclusion: $AB = BA$ for all matrices B if and only if A is a multiple of the identity.

11. *Which of the following matrices are guaranteed to equal $(A+B)^2$?

$$A^2 + 2AB + B^2, \quad A(A+B) + B(A+B), \quad (A+B)(B+A), \quad A^2 + AB + BA + B^2.$$

Solution:

1. For any matrix A and B, we will show the following:
   (a) $(A+B)^2 = (A+B)(B+A)$.
   (b) $(A+B)^2 = A(A+B) + B(A+B)$.
   (c) $(A+B)^2 = A^2 + AB + BA + B^2$.
2. We also give an example of matrices A and B for which

$$(A+B)^2 \neq A^2 + 2AB + B^2.$$

Part 1: Recall that the matrix addition is commutative, that is, $A+B = B+A$. Therefore,

$$(A+B)^2 = (A+B)(A+B) = (A+B)(B+A).$$

This proves (a). We will prove (b) and (c) together. Since the matrix multiplication is distributive over addition, we get

$$\begin{aligned}(A+B)^2 = (A+B)(A+B) &= A(A+B) + B(A+B) \\ &= A^2 + AB + BA + B^2.\end{aligned}$$

Part 2: Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $AB \neq BA$. In (c), we have already shown that $(A+B)^2 = A^2 + AB + BA + B^2$. Thus, $(A+B)^2 \neq A^2 + 2AB + B^2$.

12. *By trial and error find examples of 2 by 2 matrices such that

(i) $A^2 = -I$, A having only real entries.

(ii) $B^2 = 0$, although $B \neq 0$.

(iii) $CD = -DC$, but $CD \neq 0$.

(iv) $EF = 0$, although no entries of E or F are zero.

Solution:

(i) Let us consider the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then, we have

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

(ii) Consider the matrix $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Clearly, $B \neq 0$. But

$$B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

(iii) Take $C = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then,

$$\begin{aligned} CD &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \\ \text{and} \quad DC &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Clearly, $CD \neq 0$ but $CD = -DC$.

(iv) Consider the matrices $E = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $F = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. Clearly, none of the entries of E and F is zero, but

$$EF = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

13. Find the powers A^2, A^3 (A^2 times A), and B^2, B^3, C^2, C^3. What are A^k, B^k, and C^k?

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } C = AB = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

Solution:

(1) We are given that $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. We need to find A^2, A^3 and A^k for higher positive integers k. Now,

$$A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

So, we have $A^2 = A$.

Now, $A^3 = A^2 A = AA = A^2 = A$.

Similarly, for any higher positive integer k, we get $A^k = A$.

(2) We are given that $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Now,

$$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus, B^2 is the identity matrix.

Now, $B^3 = B^2 B = IB = B$.

To find B^k for any higher positive integers k, consider the following two cases.

Case I: If k is even, that is, $k = 2m$ for some positive integer m, then we can write

$$B^k = B^{2m} = B^2 B^2 \dots B^2 \ (m\text{-times}).$$

Using $B^2 = I$, we get $B^k = I$.

Case II: If k is odd, then $k-1$ is even and case I implies that $B^{k-1} = I$. Thus,

$$B^k = B^{k-1} B = IB = B.$$

We conclude that

$$B^k = \begin{cases} B & \text{if } k \text{ is odd} \\ I & \text{if } k \text{ is even.} \end{cases}$$

(3) Consider the matrix $C = AB$. We have

$$C = AB = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

Then,

$$C^2 = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

Question 13 continues on the next page...

that is, C^2 is the zero matrix.

Now,

$$C^3 = C^2C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, C^3 is also the zero matrix.

In general, for any $k \geq 4$,

$$C^k = C^2C^{k-2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} C^{k-2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, we conclude that C^k is the zero matrix for all $k \geq 2$.

14. Which three matrices E, F, G put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad GFEA = U.$$

Solution:

We have matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}.$$

We shall transform the matrix A into triangular form. For that, we need to eliminate $(2,1)^{th}$, $(3,1)^{th}$ and $(3,2)^{th}$ entries from the matrix.

To eliminate $(2,1)^{th}$ entry, we replace row 2 with the result of "subtracting 4 times of row 1 from row 2", and we get

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_1 = E_{21}\,A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}.$$

Now, to eliminate $(3,1)^{th}$ entry from matrix A_1, we replace row 3 by "the sum of row 3 and twice of row 1", and we get

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = E_{31}\,A_1 = E_{31}\,E_{21}\,A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}.$$

We now eliminate $(3,2)^{th}$ entry of matrix A_2. To do so, we replace row 3 with the result of "subtracting twice of row 2 from row 3". This gives

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad U = E_{32}\,A_2 = E_{32}\,E_{31}\,E_{21}\,A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

Conclusion: We have the following elimination matrices

$$E = E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = E_{31} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad G = E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

15. *The following 4 by 4 matrix needs which elimination matrices E_{21} and E_{32} and E_{43}?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Solution:

Recall that by $(i,j)^{th}$ entry of a matrix we mean the entry in the intersection of the i^{th}-row and the j^{th}-column.

Our goal is to transform matrix A into an upper triangular matrix.

To begin with, from the first column of matrix A we only need to eliminate $(2,1)^{th}$ entry as $(3,1)^{th}$ and $(4,1)^{th}$ entries are already zero. For that we can replace row 2 of matrix A by "the addition of row 1 and twice of row 2", and thus

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_1 = E_{21}\,A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Next, we need to eliminate $(3,2)^{th}$ entry from the matrix A_1 as its $(4,2)^{th}$ entry is already zero. For the same, we can replace row 3 of matrix A_1 by "the addition of row 2 and thrice of row 3", and thus

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = E_{32}\,A_1 = E_{32}\,E_{21}\,A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Finally, we are left with eliminating the $(4,3)^{th}$ entry from the matrix A_2. To do this, we can replace row 4 of matrix A_2 by "the addition of row 3 and four times of row 3", and thus

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \text{and} \quad U = E_{43}\,A_2 = E_{43}\,E_{32}\,E_{21}\,A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Conclusion: We have the following elimination matrices

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{32} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$