Solution Set - 07

MA 1201 Spring Sem, 2025

Below, "*special solutions*" refer to a set of basis vectors for the null space of the matrix A associated with the system $Ax = b$. Since the number of free variables determines the dimension of the null space, one way to construct these special solutions is by setting one free variable to 1 at a time while assigning 0 to all other free variables.

1. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions.

Solution: Consider the system of equations:

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_1 + 4x_2 + 6x_3 = 1.$$

This system has no solution because the second equation is inconsistent with the first.

Now, if we change the right-hand side to zero, we obtain the homogeneous system:

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_1 + 4x_2 + 6x_3 = 0.$$

Since the second equation is simply a multiple of the first, the system has infinitely many solutions. The general solution can be expressed as:

$$\left\{ \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.$$

2. Reduce A and B to echelon form. Which variables are free?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the special solutions to $Ax = 0$ and $Bx = 0$. Find all solutions.

Solution: For A

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is the row ecelon form of A. From the row ecelon form of A we see x_3, x_4 are free variables. The solution space is $\{(-2x_3 - x_4, -x_3, x_3, x_4) : x_3, x_4 \in \mathbb{R}\}$.
For B

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_3 \to \frac{R_3 - R_1}{2}]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{R_2 - 3R_1}{-3}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

is the row echelon form of B. From the row echelon form of B we see x_3 is a free variable. The solution space is $\{(x_3, -2x_3, x_3) : x_3 \in \mathbb{R}\}$.

3. Find the echelon form $\mathcal{E}$, the free variables, and the special solutions:

$$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Find condition on b so that $Ax = b$ has a solution. Find the complete solution for one such b.

Solution: Consider the augmented matrix $(A|b)$,

$$(A|b) = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & b_1 \\ 0 & 2 & 0 & 6 & b_2 \end{array}\right] \xrightarrow{R_2 : R_2 - 2R_1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right] = (R|\bar{b})$$

where,

$$R = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is an echelon form of matrix A. Then the 1st, 3rd and 4th variables are free variables while 2nd variable is leading variable. If

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \text{ then } Rx = 0 \Rightarrow x_2 + 3x_4 = 0$$

Since, the special solutions are basis vectors of null spaces of A and null space of A is equal to null space of echelon form R. Therefore, the special solution of the system is given by,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The given system has a solution, only when $b_2 - 2b_1 = 0$. Consider,

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Clearly,

$$\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

is a particular solution of the system $Ax = b$. Hence, the complete solution of the system $Ax = b$ is given by,

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \text{ where } c_1, c_2, c_3 \in \mathbf{R}.$$

4. Carry out the same steps as in the previous problem to find the complete solution of $Mx = b$:

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Solution: Consider the augmented matrix $(M|b)$,

$$\left[\begin{array}{cc|c} 0 & 0 & b_1 \\ 1 & 2 & b_2 \\ 0 & 0 & b_3 \\ 3 & 6 & b_4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & 2 & b_2 \\ 0 & 0 & b_1 \\ 0 & 0 & b_3 \\ 3 & 6 & b_4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cc|c} 1 & 2 & b_2 \\ 3 & 6 & b_4 \\ 0 & 0 & b_3 \\ 0 & 0 & b_1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 2 & b_2 \\ 0 & 0 & b_4 - 3b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & b_1 \end{array}\right] = (R|\bar{b})$$

where,

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is an echelon form of matrix M. Then the 1st variables is free variables while 2nd variable is leading variable. If

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ then } Rx = 0 \Rightarrow x_1 + 2x_2 = 0$$

Since, the special solutions are basis vectors of null spaces of M and null space of M is equal to null space of echelon form R. Therefore, the special solution of the system is given by,

$$\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

The given system has a solution, only when $b_3 = b_1 = b_4 - 3b_2 = 0$. Consider,

$$b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

Clearly,

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is a particular solution of the system $Mx = b$. Hence, the complete solution of the system $Mx = b$ is given by,

$$c \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \text{ where } c \in \mathbf{R}.$$

5. Write the complete solutions to these systems:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Solution:

- The augmented matrix corresponding to the system can be written as

$$\begin{pmatrix} 1 & 2 & 2 & | & 1 \\ 2 & 4 & 5 & | & 4 \end{pmatrix}$$

Next, we convert this in row-reduced echelon form. Note that

$$\begin{pmatrix} 1 & 2 & 2 & | & 1 \\ 2 & 4 & 5 & | & 4 \end{pmatrix} \sim^{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 2 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix} \sim^{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 2 & 0 & | & -3 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

The complete solutions of this system can be then written as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \left\{ \begin{pmatrix} -3 - 2\alpha \\ \alpha \\ 2 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

- The augmented matrix corresponding to the system can be written as

$$\begin{pmatrix} 1 & 2 & 2 & | & 1 \\ 2 & 4 & 4 & | & 4 \end{pmatrix}$$

Next, we convert this in echelon form. Note that

$$\begin{pmatrix} 1 & 2 & 2 & | & 1 \\ 2 & 4 & 4 & | & 4 \end{pmatrix} \sim^{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 2 & | & 1 \\ 0 & 0 & 0 & | & 2 \end{pmatrix}$$

The last row indicates that the underlying system has no solution.

6. Describe the set of attainable right-hand sides b (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is a particular solution?

Solution:

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$

- The augmented matrix corresponding to the system can be written as

$$\begin{pmatrix} 1 & 0 & | & b_1 \\ 0 & 1 & | & b_2 \\ 2 & 3 & | & b_3 \end{pmatrix}$$

  Next, we convert this in row reduced echelon form. Note that

$$\begin{pmatrix} 1 & 0 & | & b_1 \\ 0 & 1 & | & b_2 \\ 2 & 3 & | & b_3 \end{pmatrix} \sim^{R_3 \to R_3 - 2R_1} \begin{pmatrix} 1 & 0 & | & b_1 \\ 0 & 1 & | & b_2 \\ 0 & 3 & | & b_3 - 2b_1 \end{pmatrix} \sim^{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 0 & | & b_1 \\ 0 & 1 & | & b_2 \\ 0 & 0 & | & b_3 - 2b_1 - 3b_2 \end{pmatrix}$$

  The last row indicates that the underlying system has a solution if and only if $b_3 - 2b_1 - 3b_2 = 0$.

- Let us take $b_1 = 1, b_2 = 0, b_3 = 2$ so that not all b_1, b_2, b_3 are zero. Now we find the solution of $Ax = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

  Now to we convert the corresponding augmented matrix to its row reduced echelon form.

$$\begin{pmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 0 \\ 2 & 3 & | & 2 \end{pmatrix} \sim^{R_3 \to R_3 - 2R_1} \begin{pmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 0 \\ 0 & 3 & | & 0 \end{pmatrix} \sim^{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$$

Therefore we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which implies that $u = 0, v = 0$.

So a particular solution is $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

7. Find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$\begin{aligned} u + v + 2w &= 2 \\ 2u + 3v - w &= 5 \\ 3u + 4v + w &= c \end{aligned}$$

Solution: Here the augumented matrix,

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & c \end{array}\right] \sim^{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 3 & 4 & 1 & c \end{array}\right] \sim^{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -5 & c-6 \end{array}\right]$$

$$\sim^{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right]$$

$$\sim^{R_1 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right]$$

So this system have solution if $c - 7 = 0 \Rightarrow c = 7$.

Now, From the second row: $v - 5w = 1 \quad \Rightarrow \quad v = 5w + 1$ and from the first row: $u + 7w = 1 \quad \Rightarrow$ $u = 1 - 7w$.

The complete solutions of this system when $c = 7$ can be then written as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \left\{ \begin{pmatrix} 1 - 7\alpha \\ 5\alpha + 1 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

8. Under what conditions on b_1 and b_2 (if any) does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Find two vectors in the nullspace of A, and the complete solution to $Ax = b$.

Solution: The augmented matrix is:
$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 7 & b_2 \end{array}\right] \sim^{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - 2b_1 \end{array}\right] \sim^{R_1 \to R_1 - 3R_2}$
$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 7b_1 - 3b_2 \\ 0 & 0 & 0 & 1 & b_2 - 2b_1 \end{array}\right]$
Thus for homogeneous system,

$$x_1 + 2x_2 = 0 \quad \Rightarrow \quad x_1 = -2x_2$$

Therefore, the solution to $Ax = 0$ is:

$$x_1 = -2x_2, \quad x_2 = x_2, \quad x_3 = x_3, \quad x_4 = 0$$

Thus, the general solution is:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$$

We can write this as a linear combination of two free variables x_2 and x_3:

$$x = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus, the two vectors in the nullspace of A are:

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and for any $b_1, b_2 \in \mathbb{R}$ $\quad x_1 + 2x_2 = 7b_1 - 3b_2, \quad x_4 = b_2 - 2b_1$ have solutions.
The complete solutions of this system can be then written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \left\{ \begin{pmatrix} -2\alpha + 7b_1 - 3b_2 \\ \alpha \\ \beta \\ b_2 - 2b_1 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

9. (a) Find the special solutions to $Ux = 0$. Reduce U to the row-reduced echelon form R and repeat:

$$Ux = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) If the right-hand side is changed from $(0, 0, 0)$ to $(a, b, 0)$, what are all solutions?

Solution:

(a) We convert U into its row reduced echelon form R as follows:

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim^{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

From $Rx = 0$ we have,

$$\begin{aligned} x_1 + 2x_2 - 2x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

Since x_2, x_4 are free variables, therefore assuming $x_2 = \alpha$ and $x_4 = \beta$ we get, $x_1 = 2\beta - 2\alpha$ and $x_3 = -2\beta$. Thus the solution set is

$\{(2\beta - 2\alpha, \alpha, -2\beta, \beta) : \alpha, \beta \in \mathbb{R}\} = \{\alpha(-2, 1, 0, 0) + \beta(2, 0, -2, 1) : \alpha, \beta \in \mathbb{R}\}$.

(b) We find the row reduced echelon form of the augmented matrix as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & | & a \\ 0 & 0 & 1 & 2 & | & b \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \sim^{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 3 & 4 & | & a - 3b \\ 0 & 0 & 1 & 2 & | & b \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus we have the new system of equatuons as follows:

$$\begin{aligned} x_1 + 2x_2 - 2x_4 &= a - 3b \\ x_3 + 2x_4 &= b \end{aligned}$$

Putting $x_2 = 0, x_4 = 0$ we have,

$x_1 = a - 3b, x_3 = b$.

Hence the set of all solutions is

$\{(a - 3b, 0, b, 0) + \alpha(-2, 1, 0, 0) + \beta(2, 0, -2, 1) : \alpha, \beta \in \mathbb{R}\}$.

10. Find a 2 by 3 system $Ax = b$ whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Find a 3 by 3 system with these solutions exactly when $b_1 + b_2 = b_3$.

Solution: We need to find a 2×3 system $Ax = b$ whose complete solution is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad w \in \mathbb{R}.$$

Since the homogeneous solution $x_h = w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ belongs to the null space of A, we require $A \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \mathbf{0}$.

A suitable choice is

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -4 \end{bmatrix}.$$

This ensures that the solution set of the homogeneous system $Ax = 0$ is

$$\left\{ w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \mid w \in \mathbb{R} \right\}.$$

To find b, we compute Ax_p for $x_p = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$:

$$b = A \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Thus, the required system is:

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

11. Write a 2 by 2 system $Ax = b$ with many solutions $Ax = 0$ but no particular solution. (Therefore the system has no solution.) Which b 's allow a particular solution ?

12. Find the row-reduced echelon forms R:

(a) The 3 by 4 matrix of all 1 s .

Solution: Given the 3×4 matrix of all 1 s:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Subtract the first row from the second and the third:

$$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is the row-reduced echelon form of A.

(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.

Solution: The matrix $a_{ij} = (-1)^{ij}$ looks like

(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

Solution:

(a)

(b)

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \sim_{R_3 \to R_3 - R_1, R_4 \to R_4 - R_2} \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim_{R_2 \to R_2 + R_1} \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim_{R_1 \to R_1 - \frac{1}{2}R_2} \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim_{R_1 \to -R_1, R_2 \to \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the row-reduced echelon form of the matrix defined by $a_{ij} = (-1)^{ij}$ is given as

$$R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c)

13. Find R for each of these (block) matrices, and the special solutions:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} A & A \end{bmatrix} \quad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

Solution: (A): We shall convert the matrix A into row-reduced echelon forms by using elementary row operatons,

(a) Interchange $R_1 \leftrightarrow R_3$, we get

$$A_1 = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Convert $R_1 \to \frac{1}{2}R_1$ and $R_2 \to \frac{1}{3}R_2$ and we get the matrix R which is in the row-reduced echelon form,

$$R = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

By the theorem proved in the lecture, equation $Ax = 0$ is equivalent tp $Rx = 0$. Thus we get the equations $x_1 + 2x_2 + 3x_3 = 0, x_3 = 0$, which is the same as $x_1 + 2x_2 = 0, x_3 = 0$. Consider $x_2 = \alpha$, then

$$N(A) = \{(-2\alpha, \alpha, 0) : \alpha \in \mathbb{R}\}.$$

This shows that $(-2, 1, 0)$ forms a basis for the null space of the matrix A.

Sol(B): The expanded form of the matrix B is,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 & 6 \end{bmatrix}$$

Row operations:

(a) Interchange $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Apply, $R_1 \to \frac{1}{2}R_1$ and $R_2 \to \frac{1}{3}R_2$, we get row-reduced rcheleon matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the above row-reduced one can see there the variable x_2, x_4, x_5, x_6 are free. Let $x_2 = \alpha, x_4 = \beta, x_5 = \gamma, x_6 = \delta$. Thus null space of the matrix B is,

$$N(A) = \{(x_1, x_2, x_3, x_4, x_5, x_6) : x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 + 3x_6 = 0, x_3 + x_6 = 0\}$$

Putting the free variables in the equations we get, $x_3 = -\delta$ and $x_1 = -2\alpha - \beta - 2\gamma$. Let $(x_1, x_2, x_3, x_4, x_5, x_6) \in N(A)$ then

$$v = (x_1, x_2, x_3, x_4, x_5, x_6) = (-2\alpha - \beta - 2\gamma, \alpha, -\delta, \beta, \gamma, \delta)$$

which we can write,

$$v = \alpha(-2, 1, 0, 0, 0, 0) + \beta(-1, 0, 0, 1, 0, 0) + \gamma(-2, 0, 0, 0, 1, 0) + \delta(0, 0, -1, 0, 0, 1)$$

Thus the vector, $v_1 = (-2, 1, 0, 0, 0, 0), v_2 = (-1, 0, 0, 1, 0, 0), v_3 = (-1, 0, 0, 0, 1, 0), v_4 = (0, 0, -1, 0, 0, 1)$ forms a basis for null space of A.

Sol(C):The expanded form of the matrix C is,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 2 & 4 & 6 & 0 & 0 & 0 \end{bmatrix}$$

Row operations:

(a) Interchange $R_1 \leftrightarrow R_3, R_4 \leftrightarrow R_6$

$$\begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Interchange $R_2 \leftrightarrow R_4$,

$$\begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Interchange $R_3 \leftrightarrow R_5$,

$$\begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) Apply $R_2 \to R_1 - R_2$

$$\begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 0 & -2 & -4 & -6 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) Apply $R_4 \to R_4 - R_3$

$$\begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 0 & -2 & -4 & -6 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(f) Apply $R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 2 & 4 & 6 & 2 & 4 & 6 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(g) Apply the row operatons, $R_1 \to \frac{1}{2}R_1, R_2 \to \frac{1}{3}R_2, R_3 \to \frac{-1}{2}R_3, R_4 \to \frac{1}{3}R_4$ we get the following row-reduced matrix R,

$$R = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The free variables are $x_2 = \alpha, x_5 = \beta$. From the matrix R we get the following equations,

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 + 3x_6 &= 0 \\ x_3 &= 0 \\ x_4 + 2x_5 + 3x_6 &= 0 \\ x_6 &= 0 \end{aligned}$$

Thus we get $x_4 = -2\beta$ and $x_1 = -2\alpha$. Thus, the null space is given by

$$N(A) = \{(-2\alpha, \alpha, 0, -2\beta, \beta, 0) : \alpha, \beta \in \mathbb{R}\}$$

Now, $(-2\alpha, \alpha, 0, -2\beta, \beta, 0) = \alpha(-2, 1, 0, 0, 0, 0) + \beta(0, 0, 0, -2, 1, 0)$ and $v_1 = (-2, 1, 0, 0, 0, 0), v_2 = (0, 0, 0, -2, 1, 0)$ forma bais for the null space of a block matrix C.