Solution Set - 10

MA 1201 Spring Sem, 2025

1. *Classify each of the following differential equations as linear or nonlinear, and specify the order.

   (i) $y'' + (\cos x)y = 0$

   (ii) $y'' + x \sin y = 0$

   (iii) $y' = \sqrt{1+y}$

   (iv) $y'' + (y')^2 + y = x$

   (v) $y'' + xy' = \sin y$

   (vi) $\left(x\sqrt{1+x^2}\, y'\right)' = e^x y$

Solution:

(i) Linear 2nd order ODE

(ii) Nonlinear 2nd order ODE

(iii) Nonlinear 1st order ODE

(iv) Nonlinear 2nd order ODE

(v) Nonlinear 2nd order ODE

(vi) Linear 2nd order ODE

2. Find the differential equation of each of the following families of plane curves. Here $a, b, c \in \mathbb{R}$ denote arbitrary constants:

   (a) $xy^2 - 1 = cy$

   (b) $y = ax + b + c$

   (c) *Circles touching the x-axis with centres on the y-axis.

   (d) $y = a \sin x + b \cos x + b$

Solution: Eliminate the constant(s) to find the differential equations:

(a) Differentiating $(xy^2 - 1) = cy$ with respect to x, we get $cy' = (y^2 + 2xyy')$. Eliminating c we find $(xy^2 + 1)y' + y^3 = 0$.

(b) Differentiating w.r.t. x gives $y' = a$. Differentiating again w.r.t. x gives $y'' = 0$. Note that the order of the ODE is two since b, c combine to make a single arbitrary constant.

(c) Circles touching the x axis with centre on y axis are given by $x^2 + (y-c)^2 = c^2$ which on simplification gives $x^2 + y^2 = 2cy$. Differentiating w.r.t. x we get $x + yy' = cy'$. Eliminating c from the two equations gives $(x^2 - y^2)y' = 2xy$

(d) Differentiating $y = a\sin x + b\cos x + b$ w.r.t. x gives $y' = a\cos x - b\sin x$, which on differentiation again w.r.t. x gives $y'' = -a\sin x - b\cos x$. From the last two equations we get, $a = y'\cos x - y''\sin x$ and $b = -y'\sin x - y''\cos x$. Replacing a and b in the first equation gives $(1+\cos x)y'' + \sin x\, y' + y = 0$

3. Verify that the given function on the left is a implicit solution to the corresponding differential equation on the right.

(i) $x^3 + y^3 = 3cxy$ $\qquad x(2y^3 - x^3)y' = y(y^3 - 2x^3)$

(ii) *$y = ce^{-x} + x^2 - 2x + 4$ $\qquad y' + y = x^2 + 2$

(iii) $y = cx - c^2$ $\qquad (y')^2 - xy' + y = 0$

Solution:

(i) Here the function is given implicitly. Differentiating the equation w.r.t. x gives

$$x^2 + y^2y' = c(xy' + y)$$

Eliminating c gives

$$\frac{x^3 + y^3}{3xy} = \frac{x^2 + y^2y'}{xy' + y} \Rightarrow x(2y^3 - x^3)y' = y(y^3 - 2x^3)$$

(ii) Differentiating w.r.t. x gives $y' = -ce^{-x} + 2x - 2 \Rightarrow y' + y = x^2 + 2$

(iii) Differentiating w.r.t. x gives $y' = c \Rightarrow y'^2 - xy' + y = c^2 - cx + cx - c^2 = 0$

4. Find implicit solutions the following equations by separating variables:

(a) $\dfrac{dy}{dx} = y^2 - 2y + 2$

(b) $x\sqrt{1-y^2} + \sqrt{1-x^2}yy' = 0$

(c) $(x^2 - 1)(y^2 - 1) + xyy' = 0$

(d) $(y - x\dfrac{dy}{dx}) = a(y^2 + \dfrac{dy}{dx})$

Solution:

(a) We can separate the variables:

$$-1 + \frac{1}{y^2 - 2y + 2}\frac{dy}{dx} = 0$$

Antiderivative of -1 is $-x$ and antiderivative of $\frac{1}{y^2-2y+2}$ is

$$\int \frac{dy}{(y-1)^2+1} = \tan^{-1}(y-1)$$

So the implicit solution is $-x + \tan^{-1}(y-1) = c \implies y - 1 = \tan(x+c)$.

(b) The differential equation can be rewritten as

$$\frac{x}{\sqrt{1-x^2}} + \frac{y}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

For the antiderivatives, we have

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}, \text{ and } \int \frac{y}{\sqrt{1-y^2}}dy = -\sqrt{1-y^2}.$$

Hence, the implicit solution is given by

$$\sqrt{1-y^2} + \sqrt{1-x^2} = c.$$

(c) The differential equation can be rewritten as

$$\frac{x^2-1}{x} + \frac{y}{y^2-1}\frac{dy}{dx} = 0$$

For the antiderivatives, we have

$$\int \frac{x^2-1}{x}dx = \frac{x^2}{2} - \ln|x|, \text{ and } \int \frac{y}{y^2-1}dy = \frac{1}{2}\ln|y^2-1|.$$

Hence, the implicit solution is given by

$$\frac{x^2}{2} - \ln|x| + \frac{1}{2}\ln|y^2-1| = c.$$

(d) The differential equation can be rewritten as

$$(x+1)\frac{dy}{dx} = (y - ay^2) \implies \frac{1}{(y-ay^2)}\frac{dy}{dx} = \frac{1}{x+1}$$

For the antiderivatives, we have

$$\int \frac{1}{(y-ay^2)}dy = \ln\left|\frac{y}{1-ay}\right| \text{ and } \int \frac{1}{x+1}dx = \ln|x+1|.$$

Hence, the implicit solution is given by

$$\ln\left|\frac{y}{1-ay}\right| - \ln|x+1| = c.$$

5. Solve the Initial value problem (IVP) $(1-x^2)\dfrac{dy}{dx} = 2y$ with $y(2) = 1$ implicitly.

Solution: The differential equation can be rewritten as

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{1-x^2}.$$

For the antiderivatives, we have

$$\int \frac{1}{y} dy = \ln|y|,$$

,

$$\int \frac{2}{1-x^2}dx = \int \frac{2}{(1-x)(1+x)}dx = \int \left(\frac{1}{1-x} + \frac{1}{1+x}\right) dx = \ln|1-x| + \ln|1+x| = \ln|1-x^2|$$

Thus, the solution can be given as

$$\ln|y| = \ln|1-x^2| + c,$$

which can be further simplified to get

$$|y| = C|1-x^2|.$$

Using the condition $y(2) = 1$, we get $C = 1$, and hence the solution is given by $|y| = |1-x^2|$.

6. *Verify that $y = \dfrac{1}{x+c}$ is the implicit/general solution of $y' = -y^2$. Find particular solutions such that:

   (i) $y(0) = 5$

   (ii) $y(2) = -\dfrac{1}{5}$

   In both cases, find the largest interval I on which y is defined.

Solution: $y = \frac{1}{x+c} \Rightarrow y' = -\frac{1}{(x+c)^2} \Rightarrow y' = -y^2$

(i) With $y(0) = 5$, the solution is $y = \frac{5}{1+5x}$ and $I = (-\frac{1}{5}, \infty)$

(ii) With $y(2) = -\frac{1}{5}$, the solution is $y = \frac{1}{x-7}$ and $I = (-\infty, 7)$

(Note: The largest interval is determined by the fact that the solution must pass through the initial point and the solution must be continuous)

7. Solve the IVP - $y\dfrac{dy}{dx} = e^x$, with $y(0) = 1$. Find the largest interval of validity of the solution.

Solution: Separating variable and integrating, we get $y^2 = 2e^x + c$. Using initial condition $c = -1$. Thus solution to the IVP is $y^2 = 2e^x - 1$, or $y = \sqrt{2e^x - 1}$.

Note that other root does not satisfy initial condition. The largest interval of validity is $x > -\ln 2$.

8. (a) If $\dfrac{dy}{dx} = f(ax+by+c)$, then show that the substitution $ax+by+c = v$ will change it to a separable equation in x and v.

(b) Using the above, solve the following:

(i) $\dfrac{dy}{dx} = \sin(x+y)$

(ii) $(x-y)^2\dfrac{dy}{dx} = a^2$

Solution:

(a) We have:

$$v = ax + by + c$$

Differentiate both sides with respect to x:

$$\frac{dv}{dx} = a + b\frac{dy}{dx}$$

Now, using the original differential equation, we have:

$$\frac{dv}{dx} = a + bf(v).$$

(b) (i) Let $v = x + y$. Then

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}.$$

Using the original equation, we have

$$\frac{dv}{dx} = 1 + \sin(v).$$

Rewriting it, we get

$$\frac{1}{1+\sin v}\frac{dv}{dx} = 1.$$

For the antiderivative,

$$\int \frac{dv}{1+\sin v} = \int \frac{(1-\sin v))}{(1+\sin v)(1-\sin v)}dv = \int \frac{1-\sin v}{\cos^2 v}\,dv = \int \sec^2 v\,dv - \int \frac{\sin v}{\cos^2 v}\,dv$$
$$= \tan v - \frac{1}{\cos v}.$$

Thus, the solution can be given as

$$\tan v - \frac{1}{\cos v} = x + C$$

Substituting $v = x + y$, we have

$$\tan(x+y) - \frac{1}{\cos(x+y)} = x + C.$$

(ii) Let $v = x - y$. Then

$$\frac{dv}{dx} = 1 - \frac{dy}{dx}.$$

Using the original equation, we have

$$\frac{dv}{dx} = 1 - \frac{a^2}{v^2},$$

which can be rewritten as

$$\left(1 - \frac{a^2}{v^2}\right)^{-1} \frac{dv}{dx} = 1 \implies \frac{v^2}{v^2 - a^2}\frac{dv}{dx} = 1.$$

For the antiderivative, we have

$$\int \frac{v^2}{v^2 - a^2} dv = \int \left(1 + \frac{a^2}{v^2 - a^2}\right) dv = v + \frac{|a|}{2} \ln \left|\frac{v - a}{v + a}\right|.$$

Thus, the solution can be given as

$$v + \frac{|a|}{2} \ln \left|\frac{v - a}{v + a}\right| = x + C.$$

Substituting back $v = x - y$, we get

$$(x - y) + \frac{|x - y|}{2} \ln \left|\frac{x - y - a}{x - y + a}\right| = x + C.$$

9. Find out the implicit/general solution of the following homogeneous ODEs:

(a) $2xy\dfrac{dy}{dx} = (x^2 - y^2)$

(b) $(y^4 - 2x^3y) + (x^4 - 2xy^3)y' = 0$

(c) $3x^2y + (x^3 + y^3)y' = 0$

Solution:

(a) On dividing the differential equation by x^2, we get

$$2\frac{y}{x}\frac{dy}{dx} = 1 - \left(\frac{y}{x}\right)^2$$

Let

$$v = \frac{y}{x} \implies y = vx \implies \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Then, we have

$$2v\left(v + x\frac{dv}{dx}\right) = 1 - v^2$$

Expanding and simplifying the equation, we get

$$2v^2 + 2vx\frac{dv}{dx} = 1 - v^2 \implies 2vx\frac{dv}{dx} = 1 - 3v^2.$$

Separate variables:

$$\frac{2v}{1-3v^2}\frac{dv}{dx}=\frac{1}{x}$$

For antideirvatives,

$$\int \frac{1}{x}dx=\ln|x|.$$

For the other one, let $u=1-3v^2$ so we get

$$\int \frac{2v}{1-3v^2}dv=-\frac{1}{3}\int \frac{du}{u}=-\frac{1}{3}\ln|1-3v^2|$$

Thus, the solution can be given by

$$-\frac{1}{3}\ln|1-3v^2|=\ln|x|+C$$

Substituting $v=\frac{y}{x}$ back, we get

$$-\frac{1}{3}\ln\left|1-3\left(\frac{y}{x}\right)^2\right|=\ln|x|+C.$$

(b) Let

$$v=\frac{y}{x}\implies y=vx\implies \frac{dy}{dx}=v+x\frac{dv}{dx}.$$

Using $y=vx$ in the given equation, we get

$$(v^4x^4-2vx^4)+x^4(1-2v^3)\left(v+x\frac{dv}{dx}\right)=0,$$

which can be rewritten as

$$(v^4-2v)+(1-2v^3)\left(v+x\frac{dv}{dx}\right)=0\implies -v^4-v+(1-2v^3)\,x\,\frac{dv}{dx}=0$$

Further, it can be simplified to get

$$\frac{1-2v^3}{v^4+v}\frac{dv}{dx}=\frac{1}{x}.$$

Check that

$$\int \frac{1-2v^3}{v^4+v}dv=-\ln(v^2-v+1)-\ln|v+1|+\ln|v|,$$

and hence the solution is given by

$$\ln(v^2-v+1)+\ln|v+1|-\ln|v|+\ln|x|=C$$

Substituting $v=\frac{y}{x}$, the solution is given by

$$\ln\left(\left(\frac{y}{x}\right)^2-\frac{y}{x}+1\right)+\ln\left|\frac{y}{x}+1\right|-\ln\left|\frac{y}{x}\right|+\ln|x|=C.$$

(c) As before, let $y = vx$. Then, the differential equation becomes

$$3vx^3 + x^3(1+v^3)\left(v + x\frac{dv}{dx}\right) = 0.$$

This can be rewritten as

$$\frac{(1+v^3)}{(4v+v^4)}\frac{dv}{dx} = -\frac{1}{x}.$$

Check

$$\int \frac{(1+v^3)}{(4v+v^4)}dv = \frac{1}{4}\ln\left|v(v^3+4)\right|,$$

and hence the solution is given by

$$\frac{1}{4}\ln\left|v(v^3+4)\right| + \ln|x| = C.$$

Substitute $v = \frac{y}{x}$ to get

$$\frac{1}{4}\ln\left|\frac{y}{x}\left(\left(\frac{y}{x}\right)^3 + 4\right)\right| + \ln|x| = C.$$

10. (a) If $\frac{dy}{dx} = f(\frac{ax+by+c}{Ax+By+C})$, and $aB - bA \neq 0$, then show that the substitution $x = h + X, y = k + Y$ will change the differential equation to

$$\frac{dY}{dX} = F(\frac{aX+bY}{AX+BY}),$$

where (h, k) is the intersection point of two lines $ax + by + c = 0$ and $Ax + By + C = 0$ (why there is any?). Further substitution $Y = VX$ will make it to a separable equation in X and V.

(b) Using the above, solve the following:

(i) $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$

(ii) $\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10}$

Solution:

(a) To get the intersection point (h, k), we need to solve the given pair of equations of lines, which can be written in matrix form as

$$\begin{pmatrix} a & b \\ A & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -c \\ -C \end{pmatrix}$$

Note that since $aB - bA \neq 0$, guarantee the existence of a unique solution for the above system, denoted by (h, k). Using chain rule, we have

$$\frac{dY}{dX} = \frac{d(k+y)}{dx}\frac{dx}{dX} = \frac{dy}{dx}.$$

Thus, from the given differential equation, we get

$$\frac{dY}{dX} = f\left(\frac{a(h+X)+b(k+Y)+c}{A(h+X)+B(k+Y)+C}\right).$$

Using the fact that (h, k) lies on both straight lines, we have

$$\frac{dY}{dX} = f\left(\frac{aX + bY}{AX + BY}\right).$$

(b) (i) Solving the given system of equations

$$\begin{cases} y - x + 1 = 0 \\ y + x + 5 = 0 \end{cases}$$

we get $x = -2, y = -3$. Thus, using (a), the reduced differential equation is given by

$$\frac{dY}{dX} = \frac{-X + Y}{X + Y}.$$

Note that this differential equation is homogeneous. Thus, using the arguments similar to the one used in (9) (like using the substitution $V = \frac{Y}{X}$), first solve for Y, and hence for y.

(ii) In this case, note that the solution for this system

$$\begin{cases} 2x + 9y = 20 \\ 6x + 2y = 10 \end{cases}$$

is given by $(x, y) = (1, 2)$. Thus, using (a), the reduced equation can be given by

$$\frac{dY}{dX} = \frac{2X + 9Y}{X6 + 2Y}.$$

This is again homogeneous differential equation, which can be solved like (9).

11. (a) If $\frac{dy}{dx} = f(\frac{ax + by + c}{Ax + By + C})$, and $aB - bA = 0, a \neq 0, A \neq 0$, then show that the substitution

$$v = x + \frac{b}{a}y = x + \frac{B}{A}y$$

will make it to a separable equation in x and v.

(b) Using the above, solve the following:

(i) $\frac{dy}{dx} = \frac{3x - 4y - 2}{6x - 8y - 5}$

(ii) $\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$ with $y(\frac{2}{3}) = \frac{1}{3}$.

Solution:

(a) Using

$$v = x + \frac{b}{a}y = x + \frac{B}{A}y, \tag{1}$$

we have

$$ax + by = a\left(x + \frac{b}{a}y\right) = av, \text{ and } Ax + By = A\left(x + \frac{B}{A}y\right) = Av.$$

Using the given differential equation and (1), we have

$$\frac{dv}{dx} = 1 + \frac{b}{a}\frac{dy}{dx} = 1 + \frac{b}{a}f\left(\frac{av + c}{Av + C}\right) \tag{2}$$

which is a separable equation.

(b) (i) Note that $a = 3, b = -4, c = -2$ and $A = 6, B = -8, C = -5$, and so $aB - bA = 3 * (-8) - (-4) * 6 = 0$. Using the substitution

$$v = x - \frac{4}{3}y,$$

the identity (2) becomes

$$\frac{dv}{dx} = 1 + \left(\frac{-4}{3}\right)\frac{3v - 2}{6v - 5} = \frac{6v - 7}{18v - 15}, \tag{3}$$

which can be rewritten as

$$\frac{18v - 15}{6v - 7}\frac{dv}{dx} = 1.$$

Solving this, we get

$$\ln|6v - 7| + 3v = x + C,$$

which gives the solution of the original differential equation as

$$3\left(x - \frac{4}{3}y\right) + \ln|6x - 8y - 7| = x + C.$$

(ii) Note that in this case, $a = b = c = 1$ and $A = B = 1, C = -1$, and so $aB - bA = 0$. Thus, we define the substitution v as $v = x + y$. This gives

$$\frac{dv}{dx} = 1 + \frac{v + 1}{v - 1} = \frac{2v}{v - 1},$$

which can be rewritten as

$$\frac{v - 1}{2v}\frac{dv}{dx} = 1,$$

On solving this, we get

$$\frac{1}{2}v - \frac{1}{2}\ln|v| = x + C.$$

Thus, the solution of the given differential equation can be given by

$$\frac{1}{2}(x + y) - \frac{1}{2}\ln|x + y| = x + C.$$

Using $y(\frac{2}{3}) = \frac{1}{3}$ gives

$$\frac{1}{2} = \frac{2}{3} + C \implies C = -\frac{1}{6}.$$

Thus, the solution of the given Initial value problem is given by

$$\frac{1}{2}(x + y) - \frac{1}{2}\ln|x + y| = x - \frac{1}{6}.$$

12. *Show that the set of solutions of the homogeneous linear equation $y'+P(x)y=0$ on an interval $I=[a,b]$ form a vector subspace W of the real vector space of continuous functions on I. What is the dimension of W?

Solution: The zero function $\mathbf{0}(x) \equiv 0$ satisfies $y'+P(x)y=0$. Hence, W is nonempty. Let $u(x), v(x) \in W$ are two arbitrary solutions of $y'+P(x)y=0$. Consider $w(x)=\alpha u(x)+v(x)$, where α is a real number. Now, $w'+P(x)w=0 \Rightarrow w(x) \in W$ and hence W is a subspace. We also note that any solution is of the form $y=Ce^{-\int P(x)dx}$. Thus W is spanned by $e^{-\int P(x)dx}$ and so $\dim(W)=1$. (Remark: Solutions of non-homogeneous or non-linear equations may not form a vector space.)

13. Solve the linear first-order IVP:

$$y'+y\tan x=\sin(2x), \quad y(0)=1$$

Solution: Comparing with $y'+p(x)y=r(x)$, we get $p(x)=\tan x, r(x)=\sin 2x$. Then, $\int p(x)dx = \ln(\sec x), e^{\int p(x)dx}=\sec x, \quad \int e^{\int p(x)dx}\cdot r(x)dx=\int \sec x 2\sin x\cos x dx=-2\cos x+c$.

So general solution is $y(x)=c\cos x-2\cos^2 x$. Initial condition gives $c=3$.

14. *Let φ_i be a solution of $y'+ay=b_i(x)$ for $i=1,2$. Show that $\varphi_1+\varphi_2$ satisfies $y'+ay=b_1(x)+b_2(x)$. Solve:

$$y'+y=x+1$$
$$y'+y=\cos(2x)$$

Hence solve: $y'+y=1+\dfrac{x}{2}-\cos^2 x$.

Solution: First part is a mere verification.
For $y'+y=x+1$, solutions are $y_1=C'e^{-x}+x$ and for $y'+y=\cos 2x$ is $y_2=C''e^{-x}+(\cos 2x+2\sin 2x)/5$.
This is so as integrating by parts $\int xe^x dx = xe^x-e^x$. Also

$$\int e^{ax}\cos bx dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx).$$

Now $y'+y=1+x/2-\cos^2 x=\frac{1+x}{2}-\frac{\cos 2x}{2}$. Since, the equation is linear, the solution of $y'+y=1+x/2-\cos^2 x$ is

$$y=(C'+C'')e^{-x}+x/2-(\cos 2x+2\sin 2x)/10=Ce^{-x}+x/2-(\cos 2x+2\sin 2x)/10.$$

15. Solve the following linear equations:

(a) $\dfrac{dy}{dx} + 2xy = 4x$

(b) $\dfrac{dy}{dx} - y\tan x = \cos x$

(c) $*x\cos x\dfrac{dy}{dx} + y(x\sin x + \cos x) = 1$

(d) $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1}y = \dfrac{1}{(x^2+1)^3}$

Solution:

(a) In this case,

$$P(x) = 2x.$$

So, the integrating factor can be computed as

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}.$$

On multiplying both sides of differential equation by the integrating factor and simplifying it, we get

$$\frac{d}{dx}(ye^{x^2}) = 4xe^{x^2}.$$

Integrating both sides w.r.t. x, we have

$$\int \frac{d}{dx}(ye^{x^2})\,dx = \int 4xe^{x^2}\,dx.$$

Using the substitution $u = x^2$ on the rhs term, we have

$$\int 4xe^{x^2}\,dx = 2\int 2xe^{x^2}\,dx = 2\int e^u\,du = 2e^{x^2} + C.$$

Thus, the solution of the differential equation can be given as

$$y = 2 + Ce^{-x^2}.$$

(b) In this case,

$$P(x) = -\tan x.$$

So, the integrating factor can be computed as

$$\text{I.F.} = e^{\int -\tan x\,dx}e^{\ln|\cos x|} = |\cos x|.$$

On multiplying both sides of the differential equation by $\cos x$ and simplifying it, we get

$$\frac{d}{dx}(y\cos x) = \cos^2 x.$$

Integrating both sides w.r.t. x, we get

$$y\cos x = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

Note that multiplying both sides by $-\cos x$ will also give the same solution. Thus, the solution of the differential equation can be given as

$$y = \frac{1}{\cos x}\left(\frac{x}{2} + \frac{\sin 2x}{4} + C\right).$$

(c) Dividing the equation by $x\cos x$ (assuming it to be non-zero), we get

$$\frac{dy}{dx} + \frac{(x\sin x + \cos x)}{x\cos x} y = \frac{1}{x\cos x}. \tag{4}$$

In this case,

$$P(x) = \frac{x\sin x}{x\cos x} + \frac{\cos x}{x\cos x} = \tan x + \frac{1}{x},$$

and so the integrating factor can be given as

$$\text{I.F.} = e^{\int P(x)\,dx} = e^{\int\left(\tan x + \frac{1}{x}\right)dx} = e^{\ln|\sec x| + \ln|x|} = |x\sec x|$$

On multiplying both sides of (4) by the $x\sec x$ and simplifying it, we get

$$\frac{d}{dx}(y\cdot x\sec x) = \sec^2 x$$

Integrating both sides w.r.t. x, we get

$$y\cdot x\sec x = \tan x + C.$$

Thus, the solution of the differential equation can be given as

$$y = \frac{\tan x + C}{x\sec x}.$$

Note that on multiplying both sides of (4) by the $-x\sec x$ and repeating the above arguments will give the same solution.

(d) In this case,

$$P(x) = \frac{4x}{x^2+1},$$

and so the integrating factor is given by

$$\text{I.F.} = e^{\int P(x)\,dx} = e^{\int \frac{4x}{x^2+1}\,dx} = e^{2\ln(x^2+1)} = (x^2+1)^2.$$

On multiplying both sides of the differential equation by the I.F and simplifying it, we get

$$\frac{d}{dx}\left[y(x^2+1)^2\right] = \frac{1}{x^2+1}.$$

On integrating it, we get

$$y(x^2+1)^2 = \tan^{-1} x + C$$

Thus, the solution of the differential equation can be given as

$$y = \frac{\tan^{-1} x + C}{(x^2+1)^2}$$

16. Reduce the following ODEs of Bernoulli'is form to linear equations and solve:

(a) $xy - \dfrac{dy}{dx} = y^3 e^{-x^2}$

(b) $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$

(c) *$y(2xy + e^x) - e^x \dfrac{dy}{dx} = 0$

Solution:

(a) The given differential equation can be rewritten as

$$\frac{dy}{dx} + xy = -y^3 e^{-x^2}.$$

Dividing by y^3 gives

$$y^{-3}\frac{dy}{dx} + xy^{-2} = -e^{-x^2}$$

Now, let $v = y^{-2}$, then $\dfrac{dv}{dx} = -2y^{-3}\dfrac{dy}{dx}$. Using this in the above equation gives

$$-\frac{1}{2}\frac{dv}{dx} + xv = -e^{-x^2} \implies \frac{dv}{dx} - 2xv = 2e^{-x^2}.$$

This is a linear equation with $P(x) = -2x$, and so the integrating factor can be given by

$$\text{I.F} = e^{\int -2x dx} = e^{-x^2}$$

Now, multiplying the differential equation by the integrating factor and simplifying it further gives

$$\frac{d}{dx}(ve^{-x^2}) = 2e^{-2x^2},$$

which, on integrating, gives

$$ve^{-x^2} = \int 2e^{-2x^2}dx + C$$

Hence, the solution can be given by

$$y = \left[e^{x^2}\left(\int 2e^{-2x^2}dx + C\right)\right]^{-\frac{1}{2}}.$$

(b) Multiplying the given differential equation by $\cos x$ gives

$$\cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x.$$

Now, let $v = \sin y$, then $\dfrac{dv}{dx} = \cos y\dfrac{dy}{dx}$. Using this in the differential equation gives

$$\frac{dv}{dx} - \frac{1}{1+x}v = (1+x)e^x.$$

This is now a linear ODE with $P(x) = -\frac{1}{1+x}$, and so the integrating factor is given by

$$\text{I.F} = e^{\int -\frac{1}{1+x}dx} = e^{-\ln(1+x)} = \frac{1}{1+x}.$$

Now, multiplying the differential equation by the integrating factor and simplifying it further gives

$$\frac{d}{dx}\left(v\frac{1}{1+x}\right) = e^x,$$

which, on integrating, gives

$$v\frac{1}{1+x} = e^x + C$$

Hence, the solution can be given by

$$\sin y = (1+x)(e^x + C) \text{ or } y = \sin^{-1}\left[(1+x)(e^x + C)\right].$$

(c) The given differential equation can be rewritten as

$$\frac{dy}{dx} - y = 2xy^2 e^{-x}.$$

Dividing by y^2 gives

$$y^{-2}\frac{dy}{dx} - y^{-1} = 2xe^{-x}.$$

Now, let $v = y^{-1}$, then $\dfrac{dv}{dx} = -y^{-2}\dfrac{dy}{dx}$. Using this in the above equation gives

$$\frac{dv}{dx} + v = -2xe^{-x}.$$

This is a linear equation with $P(x) = 1$, and so the integrating factor can be given by

$$\text{I.F} = e^{\int dx} = e^x$$

Now, multiplying the differential equation by the integrating factor and simplifying it further gives

$$\frac{d}{dx}(ve^x) = -2x,$$

which, on integrating, gives

$$ve^x = -x^2 + C$$

Hence, the solution can be given by

$$y = \frac{1}{e^{-x}\left(-x^2 + C\right)}.$$

17. Using appropriate substitution, reduce the following differential equations to linear form and solve:

(i) $*y^2 y' + \dfrac{y^3}{x} = x^{-2}\sin x$

(ii) $*y' \sin y + x\cos y = x$

(iii) $y' = y(xy^3 - 1)$

(iv) $\sec^2 y \dfrac{dy}{dx} + 2x\tan y = x^2$

Solution:

(i) Substitute $u = y^3$ and the ODE transform to linear form $u' + 3u/x = 3x^{-2}\sin x$. Using integrating factor x^3, we write

$$\frac{d}{dx}\left(ux^3\right) = 3x\sin x \Longrightarrow ux^3 = 3(-x\cos x + \sin x) + C$$

Thus, the solution is $x^3y^3 + 3(x\cos x - \sin x) = C$.

(ii) Substitute $-\cos y = u$ which leads to the linear form $u' - xu = x$. Using integrating factor $e^{-x^2/2}$, we write

$$\frac{d}{dx}\left(ue^{-x^2/2}\right) = xe^{-x^2/2} \Longrightarrow ue^{-x^2/2} = -e^{-x^2/2} + C \Longrightarrow u = -1 + Ce^{x^2/2}$$

Hence, the solution is $\cos y = 1 - Ce^{x^2/2}$.

(iii) $u = 1/y^3$ leads to $u' - 3u = -3x$. Using integrating factor e^{-3x}, we write

$$\frac{d}{dx}\left(ue^{-3x}\right) = -3xe^{-3x} \Longrightarrow ue^{-3x} = \frac{1+3x}{3}e^{-3x} + C \Longrightarrow u = \frac{1+3x}{3} + Ce^{3x}.$$

Hence, the solution is $1/y^3 = Ce^{3x} + x + 1/3$.