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Solving a system of fluid-induced fault slip

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1 Introduction

The action of injecting fluid into the subsurface level of Earth is often followed by a number of earthquake events in the area. Figure 1 shows a 3D picture of injection of fluid through a borehole. Now, the fluid goes to the subsurface level and starts to move in radially outward direction in a region where there is already a damage zone in the rock. The host rock has a low permeability (k_0) and high elastic modulus (G_0) compared to the fault damage zone (k and G, respectively).

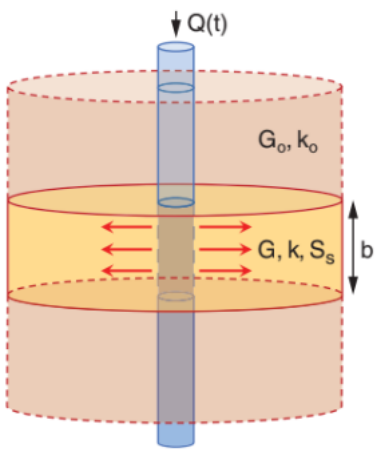

Figure 1: Fluid is injected through a borehole with a rate Q(t). The fault damage zone (in yellow) has a width of b and elastic modulus G. The host rock has an elastic modulus $G_0 >> G$. [Figure: *P. Bhattacharya and R. C. Viesca (2019)*]

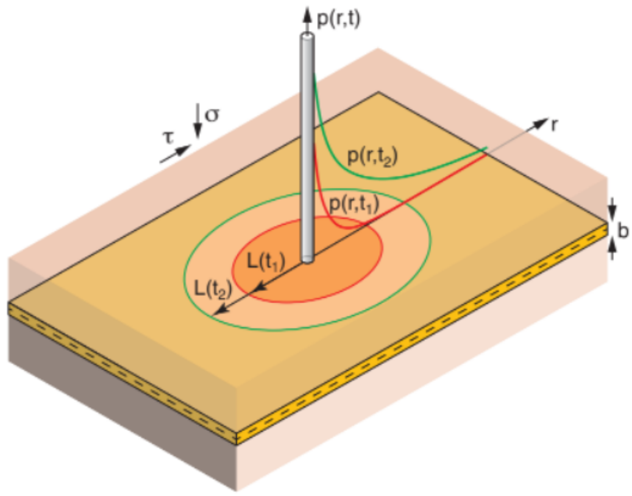

Figure 2: The pore pressure (p(r,t)) reduces the effective normal stress. The elliptical rupture grows radially outward the direction of r. [Figure: *P. Bhattacharya and R. C. Viesca (2019)*]

In figure 2, the picture is shown in a cylindrical geometry (r, θ, ϕ). The pore pressure is a purely radial function, which decreases with distance from the borehole centre and increases with time. This pore-pressure term effectively reduces the normal stress($\bar{\sigma}$).

$$\bar{\sigma} = \sigma - p(\bar{r}, t) \tag{1}$$

As a result this reduction in stress decreases the frictional resistance to sliding. This resistance term given as a product of frictional coefficient (f) and $\bar{\sigma}$. Apart from the frictional resistance, there is a background shear traction, τ, which arises in the boundary between damage zone and host rock. This helps the growth of an elliptical shaped rupture along the major axis (r), parallel to the slip direction. The force balance equation of the problem is given by,

$$f[\sigma - p(r,t)] - \tau = -\frac{G_o}{2\pi}\int_0^{L(t)} \frac{\partial}{\partial s}\delta(s,t)\left[\frac{1}{s-r}E\left\{k\left(\frac{r}{s}\right)\right\} + \frac{1}{s+r}F\left\{k\left(\frac{r}{s}\right)\right\}\right]ds \tag{2}$$

In equation 2, the L(t) is the rupture length, which increases with time; δ (r,t) is the slip profile; E and F are the complete elliptic integrals of respectively 2nd and 1st kinds; $k(U) = 2\sqrt{U}/(1+U)$;

s is another coordinate system along the same direction as r. We cannot take s and r as the same systems as that will give an infinite coefficient term for $\frac{\partial \delta}{\partial r}$ at s= r.

Now, in this problem, the boundary value, i.e. L(t) is a function of t. However, the growth of the rupture follows a diffusion equation and we can relate the the rupture length and time using a dimensionless constant λ.

$$L(t) = 4\lambda\sqrt{\alpha t}$$

We further substitute s and r to non-denationalise the equation.

$$\bar{s} = 2s/L - 1, \bar{r} = 2r/L - 1$$

The pore-pressure distribution p(r,t) is given by,

$$p(r,t) = P_* E_1\left(\frac{r^2}{4\alpha t}\right) \tag{3}$$

E1 is the exponential integral function. Now, substituting all these in equation (2) leads us to,

$$\begin{aligned}&\left(1-\frac{\tau}{f\sigma}\right)\frac{\sigma}{P_*} - E_1\left[\lambda^2(\bar{r}+1)^2\right] = \\ &-\frac{1}{2\pi}\int_{-1}^{1}\frac{d\bar{\delta}}{d\bar{s}}\left[\frac{1}{\bar{s}-\bar{r}}E\left\{k\left(\frac{\bar{r}+1}{\bar{s}+1}\right)\right\} + \frac{1}{\bar{s}+\bar{r}+2}F\left\{k\left(\frac{\bar{r}+1}{\bar{s}+1}\right)\right\}\right]d\bar{s}\end{aligned} \tag{4}$$

Now, if we can solve the right hand side of the equation, then we will get a value of λ for a given value of T ($T = \left(1-\frac{\tau}{f\sigma}\right)\frac{\sigma}{P_*}$; in practical scenarios, if we can fit the diffusion equation to get the value of λ, we can solve T. Please note that T is taken constant for this part, however, f can also be a function of r and t as the slipping progresses.

Now, we proceed to find the $\frac{d\delta}{ds}$ term to solve the equation. For an axi-symmetric non-singular crack, we take solution to be in the form $d\bar{\delta}/d\bar{s} = \sqrt{1-\bar{s}^2}\phi(s)$

We use the Gauss-Chebyshev quadrature of Type II to get the solution.

$$\begin{aligned}I(\bar{r}_i) &= \int_{-1}^{1}\frac{d\bar{\delta}}{d\bar{s}}\left[\frac{1}{\bar{s}-\bar{r}}E\left\{k\left(\frac{\bar{r}+1}{\bar{s}+1}\right)\right\} + \frac{1}{\bar{s}+\bar{r}+2}F\left\{k\left(\frac{\bar{r}+1}{\bar{s}+1}\right)\right\}\right]d\bar{s} \\ &= \sum_{j=1}^{n}\frac{\pi}{n+1}\left(1-\bar{s}_j^2\right)\left[\frac{1}{\overline{s_j}-\overline{r_i}}E\left\{k\left(\frac{\overline{r_i}+1}{\overline{s_j}+1}\right)\right\} + \frac{1}{s_j+\overline{r_i}+2}F\left\{k\left(\frac{\overline{r_i}+1}{\overline{s_j}+1}\right)\right\}\right]\phi_j \\ &= \sum_{j=1}^{n} w_{ij}K_{ij}\phi_j\end{aligned} \tag{5}$$

2 Gauss Chebyshev Quadrature

In numerical analysis, a quadrature rule is an approximation of a definite integral of a function taken as a weighted sum of function values at specified points within the domain of integration. In case of rectangular quadrature, we take the quadrature points at equal intervals and take the weights after solving a system of linear equations for known polynomials. However, Gaussian quadratures give ourselves the freedom to choose not only the weighting coefficients, but also the location of the abscissas at which the function is to be evaluated. Thus we have twice the degrees of freedom compared to rectangular quadratures.

In Gaussian quadrature, we can arrange the choice of weights and abscissas to make the integral exact for a class of integrands polynomials times some known function W(x). For a given weight function, W(x) and a given order of quadrature (n), we can find a set of weights w_j and abscissas x_j such that,

$$\int_a^b W(x)f(x)dx \approx \sum_{j=0}^{n-1} w_j f\left(x_j\right) \tag{6}$$

The approximation gives exact solution if f(x) is in polynomial form.

Chebyshev calculated the values of w_j and x_j when the weight function is in the form

$$W(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

and α and β are $\pm 1/2$ each.

In our case, we have a non-singular crack which corresponds to the case when both α and β are $+1/2$. The formula of weights and abscissas for the equation are given by the following set of equations,

$$\begin{aligned} \bar{r}_i &= \cos\left(\frac{\pi}{2}\frac{2i-1}{n+1}\right) \\ \bar{s}_j &= \cos\left(\pi\frac{j}{n+1}\right) \\ w_{ij} &= \frac{\pi}{n+1}\left(1-\bar{s}_j^2\right) \end{aligned} \tag{7}$$

Here r_i are the points along which we calculate the slip gradients, i=1,2,3...n+1 for a chosen degree n.
s_j are the abscissas at each r_i and j=1,2,3...n.
w_j are the weight corresponding to each s_j

3 Solving the equation using n-dimensional Newton-Raphson

Taking all terms of equation 4 at one side gives us the following force balance equation to solve,

$$F_i = \left(1 - \frac{\tau_\infty}{f\sigma}\right)\frac{\sigma}{P_*} - E_1\left[\lambda^2\left(\bar{r}_i+1\right)^2\right] + \sum_j \frac{1}{2\pi} W_{ij} K_{ij} \phi_j \tag{8}$$

Here we have n+1 number of equations (i=1,2,3...n+1) and n+1 variables (λ, ϕ_j) to solve (j=1,2,3..n). The set of equations are not linear, because the pore-pressure term (p(t)) is given in exponential integral form. As a result, we cannot solve the equation using any matrix inversion method, instead we go for n-dimensional Newton-Raphson method.

In traditional Newton-Raphson method for single variable x and a function f(x), we initiate the process with a guess of x=x_0, reasonably close to the actual solution, and continue the iteration as follows,

$$x_{k+1} = x_k - \frac{f\left(x_k\right)}{f'\left(x_k\right)}$$

Now, in case of n-dimensional Newton-Raphson, let us take the function F as,

$$F_i(x_1, x_2, ...x_n) = 0, i = 1, 2, 3..n$$

Let $\mathbf{x}$ denote the entire vector of values x_i and $\mathbf{F}$ denote the entire vector of functions F_i. In the neighborhood of x, each of the functions F_i can be expanded in Taylor series,

$$F_i(\mathbf{x} + \delta\mathbf{x}) = F_i(\mathbf{x}) + \sum_{j=0}^{N-1} \frac{\partial F_i}{\partial x_j} \delta x_j + O\left(\delta\mathbf{x}^2\right)$$

The matrix of partial derivatives is called the *Jacobian Matrix*, $\mathbf{J}$,

$$J_{ij} \equiv \frac{\partial F_i}{\partial x_j}.$$

$$\mathbf{F}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \mathbf{J} \cdot \delta\mathbf{x} + O\left(\delta\mathbf{x}^2\right)$$

Now, neglecting the higher order terms and setting $\mathbf{F}(\mathbf{x} + \delta\mathbf{x})$ as zero, we obtain

$$\delta\mathbf{x} = -\mathbf{F} \cdot \mathbf{J}^{-1}$$

the J_1 term is calculated by using numpy library for LU decomposition method. Finally we get the next iteration vector $\mathbf{x}$ by using,

$$\mathbf{x}_{\text{new}} = \mathbf{x}_{\text{old}} + \delta\mathbf{x}$$

In our problem the $J_{i,j}$ matrix is given by,

$$\begin{aligned} J_{1...(n+1),1} &= \frac{\partial F_i}{\partial \lambda} = \frac{2}{\lambda} \exp\left[-\lambda^2 (\bar{r}_i + 1)^2\right] \\ J_{1...(n+1),2...n} &= \frac{\partial F_i}{\partial \phi_j} = \frac{1}{2\pi} W_{ij} K_{ij} \end{aligned} \tag{9}$$

We now solve the system of equations for $[\lambda, \phi_j]$ for different values of T ($\left(1 - \frac{\tau_\infty}{f\sigma}\right) \frac{\sigma}{P_*}$).

4 Results

Continuity of the solution

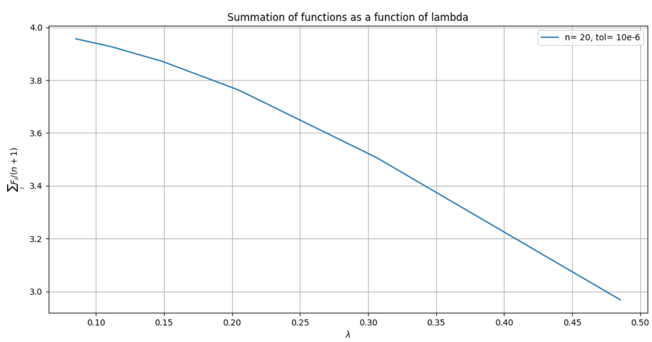

Figure 3: Normalised summation of F_i with changing λ

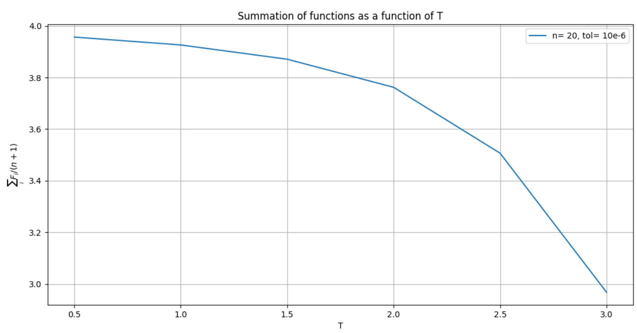

Figure 4: Normalised summation of F_i with changing T

- **Discussion:**

The plot being continuous shows that there is no discontinuity in the taken range. However, we cannot draw any further conclusion unless we can get a minima in the plot.

Relation between T, λ and slip gradient (δ_r)

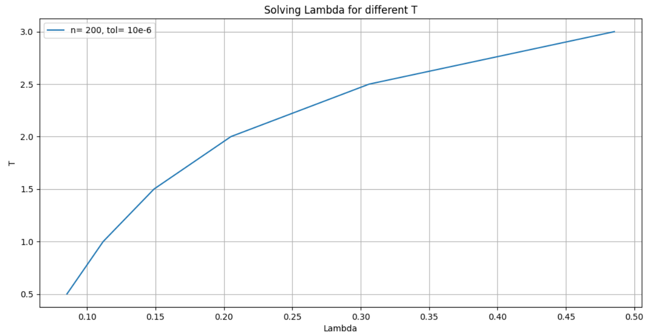

Figure 5: Change of λ with T

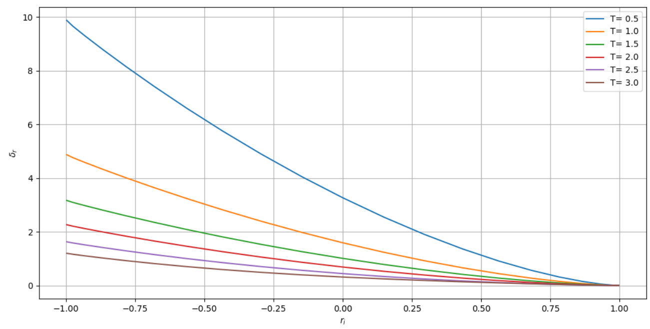

Figure 6: Slip profile as a function of coordinate

- **Discussion:**

As expected, an increase in T essentially means a decrease in frictional coefficient (f), so the rupture length (L(t)) increases more with time compared to higher friction. So, the constant of the diffusion equation between L(t) and t, λ, also increases.

In the second plot (Figure 4), the coordinate is from -1 to 1, where -1 corresponds to borehole centre and 1 corresponds to rupture length. As T increases, frictional coefficient decreases and slipping increases as a result of that.

References

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