A Study of Free Fermions in Disguise Problem

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Abstract

This report examines a quantum mechanical system that shows free-fermion behavior even though its Hamiltonian consists entirely of local four-fermion operators. Building on Paul Fendley's work, we construct free-fermion raising and lowering operators for a quantum chain where the usual Jordan-Wigner transformation cannot reduce the Hamiltonian to bilinear form in fermionic operators. We analyze the construction of conserved charges and commuting transfer matrices, solve the open chain explicitly, and extend the approach to the transverse field Ising model. The results demonstrate that free-fermion spectra can emerge from systems that appear fundamentally different from conventional free-fermion models.

Contents

1 Introduction

1.1 Historical Context and Motivation

Exactly solvable models have played a central role in quantum mechanics and statistical physics since the early 20th century. Complete analytical solutions—such as Onsager's solution of the 2D Ising model [2] or Bethe's ansatz for the 1D Heisenberg chain [1]—provide crucial insights into many-body quantum systems. These models are valuable not merely because they can be solved exactly, but because they reveal universal features of quantum phase transitions, critical phenomena, and strongly correlated systems.

Many of these breakthroughs rely on the concept of **free fermions**—systems where the Hamiltonian can be expressed as a sum of bilinear terms in fermionic creation and annihilation operators. The Jordan-Wigner transformation, introduced in 1928 [6], provides the link between spin systems and fermionic models, enabling exact solutions for many 1D quantum chains. This transformation maps spin-$\frac{1}{2}$ operators to fermionic operators in a non-local but linear manner, preserving the local structure of many Hamiltonians and making them solvable via standard diagonalization methods.

1.2 Majorana Fermions

The introduction of **Majorana fermions** [4] expanded the class of exactly solvable models. Majorana fermions are particles that act as their own antiparticles, originally proposed by Ettore Majorana in 1937. Unlike Dirac fermions, Majorana fermions satisfy the condition $\gamma_j^\dagger = \gamma_j$, where γ_j is the Majorana operator. In condensed matter physics, Majorana fermions have become important for understanding topological phases of matter and are a focus of research in topological quantum computing [5]. Majorana fermions obey the Clifford algebra:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \tag{1}$$

1.3 The Four-Fermion Challenge

Despite these successes, a fundamental question remains: all known free-fermion solutions require the Hamiltonian to be expressible as a sum of local bilinear terms in fermionic operators (possibly after a Jordan-Wigner transformation). This raises the question: **Are there quantum systems that exhibit a free-fermion spectrum but cannot be reduced to a local bilinear form through any currently known transformation?**

1.4 Fendley's Discovery

Fendley [3] showed that the quantum chain with Hamiltonian

$$H_u = \sum_{j=1}^{2M} \psi_j \psi_{j+1} \psi_{j+3} \psi_{j+4} \tag{2}$$

where ψ_j are Majorana fermion operators, can be exactly solved despite being composed entirely of local four-fermion terms. This system exhibits a complete free-fermion spectrum of the form

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_S \tag{3}$$

yet cannot be reduced to local bilinear form through any Jordan-Wigner-type transformation.

1.5 Scope and Structure of This Report

This report analyzes Fendley's construction and extends the approach to the transverse field Ising model. We focus on:

- The construction of conserved charges and commuting transfer matrices, including the algebraic structure underlying integrability.
- The explicit solution of the open chain via free-fermion raising and lowering operators, including derivation of the complete spectrum.
- Detailed derivations of key results with emphasis on their physical interpretation, and application to related models.

2 Conserved Charges and Commuting Transfer Matrices

2.1 The Hamiltonian and Generating Algebra

Fendley's method rewrites the quantum Hamiltonian and transfer matrices using local operators with simple algebraic rules. Instead of working through all four-fermion terms, we focus on operators whose algebra makes the analysis easier. This simplification highlights the key structure needed to solve the model.

2.1.1 Spin Representation

The quantum system is constructed from a chain of two-state quantum systems (spins), where the Hilbert space operators are built from the standard Pauli matrices σ_m^a acting on site m:

$$\sigma_m^a \equiv \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \sigma^a \otimes \cdots \otimes \mathbf{1} \tag{4}$$

where σ^a with $a \in \{x, y, z\}$ are the usual Pauli matrices, and $\sigma^0 = I$.

The fundamental building blocks are the local three-spin operators:

$$h_m = b_m \sigma_m^z \sigma_{m+1}^z \sigma_{m+2}^x \tag{5}$$

$$\tilde{h}_m = \tilde{b}_m \sigma_m^x \sigma_{m+1}^z \sigma_{m+2}^z \tag{6}$$

where b_m and $\tilde{b}_m$ are real coupling parameters. These operators act non-trivially on three consecutive sites, creating a pattern that will prove crucial for the model's solvability.

2.1.2 Boundary Conditions and System Size

The boundary conditions play a fundamental role in determining the model's properties:

- **Open boundary conditions**: The chain contains $M + 2$ sites, giving a Hilbert space of dimension 2^{M+2}, with generators h_m for $m = 1, \ldots, M$. In Eq.(2) j has the range of $[0, 2M + 3]$ so even $j's$ contribute to h_m and odd ones contribute to $\tilde{h}_m$ so the number $M + 2$ pops up here.

- **Periodic boundary conditions**: The chain has M sites with dimension 2^M, where indices are taken modulo M.

The Hamiltonians of central interest are:

$$H = \sum_{m=1}^{M} h_m \quad \text{(Even index for starting site in eq. 2)} \tag{7}$$

$$\tilde{H} = \sum_{m=1}^{M} \tilde{h}_m \quad \text{(Odd index for starting site in eq. 2)} \tag{8}$$

2.1.3 The Crucial Commutation Property

A non-obvious property of these operators is their mutual commutativity:

$$[h_m, \tilde{h}_{m'}] = 0 \quad \text{for all } m, m' \quad \Rightarrow \quad [H, \tilde{H}] = 0 \tag{9}$$

Since H and $\tilde{H}$ commute, we can diagonalize them at the same time. This means it's enough to study one of them to understand the spectrum. When the couplings b_m and $\tilde{b}_m$ are related appropriately, the two Hamiltonians become each other's parity counterparts.

2.1.4 The Fundamental Algebra

The key to the model's exact solvability lies in the algebraic structure satisfied by the generators h_m:

$$h_m^2 = (b_m)^2 \tag{10}$$

$$h_m h_{m+1} = -h_{m+1} h_m \tag{11}$$

$$h_m h_{m+2} = -h_{m+2} h_m \tag{12}$$

$$h_m h_n = h_n h_m \quad \text{for } |n-m| > 2 \tag{13}$$

The algebra for the local three-spin operators is pretty straightforward, even though the operators themselves look complicated. Nearest and next-nearest neighbors anti-commute, while operators that are farther apart just commute with each other. These relations set up the basic framework that lets us build the solution for this model and make calculations more manageable.

2.1.5 Connection to Majorana Representation

The connection to the four-fermion Hamiltonian in the introduction is established through the Jordan-Wigner transformation:

$$\psi_{2m-1} = \sigma_m^z \prod_{n=1}^{m-1} \sigma_n^x \tag{14}$$

$$\psi_{2m} = -i\sigma_m^x \psi_{2m-1} \tag{15}$$

Under this transformation, the uniform four-fermion Hamiltonian $H_u = H + \tilde{H}$ with all $b_m = \tilde{b}_m = 1$ becomes:

$$H_u = \sum_{j=1}^{2M} \psi_j \psi_{j+1} \psi_{j+3} \psi_{j+4} \tag{16}$$

This establishes that the model we are analyzing is indeed the four-fermion system described in the introduction, but now expressed in a form where the underlying algebraic structure is manifest.

2.2 Construction of Conserved Charges

The straightforward algebra between the local operators lets us systematically build up a whole sequence of conserved charges. This process is similar to what's done in models with free parafermions, but here it's tweaked to fit the specific commutation rules of our system.

2.2.1 The General Construction Principle

Non-local conserved charges are built from products of different h_m operators that commute with each other. The key constraint is that operators h_m and $h_{m'}$ commute if and only if $|m - m'| > 2$. This constraint naturally leads to the definition:

Definition 2.1 (Conserved Charges)**.** The s-th conserved charge is

$$Q^{(s)} \equiv \sum h_{m_1} h_{m_2} \cdots h_{m_s} \tag{17}$$

where the sum is over all sequences $m_1, m_2, \ldots, m_s$ satisfying:

- $1 \leq m_1 < m_2 < \cdots < m_s \leq M$
- $m_{i+1} > m_i + 2$ for all $i = 1, \ldots, s-1$
- For periodic boundary conditions: $m_s - m_1 \neq M - 1, M - 2$

2.2.2 Proof of Conservation

Theorem 2.1 (Conservation of Charges)**.** For all s, we have $[H, Q^{(s)}] = 0$.

Proof. To prove that $[H, Q^{(s)}] = 0$, we compute the commutator of H with a generic term in $Q^{(s)}$:

$$[H, h_{m_1} h_{m_2} \cdots h_{m_s}] = \sum_{m=1}^{M} [h_m, h_{m_1} h_{m_2} \cdots h_{m_s}] \tag{18}$$

Using the algebra relations, the commutator $[h_m, h_{m_1} h_{m_2} \cdots h_{m_s}]$ is non-zero only when m is within distance 1 or 2 of exactly one of the m_i, and not within distance 1 or 2 of any other m_j. In such cases:

$$[h_m, h_{m_1} h_{m_2} \cdots h_{m_s}] = 2 h_m h_{m_1} h_{m_2} \cdots h_{m_s} \tag{19}$$

The crucial observation is that these contributions always come in pairs that cancel each other. For every term involving h_m that appears from commuting with h_{m_i}, there is a corresponding term involving h_{m_i} that appears from commuting with h_m, and these terms have opposite signs:

$$[h_m, h_{m_1} \cdots h_{m_i} \cdots h_{m_s}] + [h_{m_i}, h_{m_1} \cdots h_m \cdots h_{m_s}] = 0 \tag{20}$$

This cancellation occurs systematically across all terms in the sum over m, proving that $[H, Q^{(s)}] = 0$. □

2.3 Relation Between H and Higher Charges

A relation can be derived that connects the Hamiltonian H (which is $H^{(1)}$) to the conserved charges $Q^{(s)}$ and the higher Hamiltonians $H^{(r)}$(2.5.4):

$$HQ^{(s)} = (s+1)Q^{(s+1)} + \sum_{j=1}^{s}(-1)^{j+1}H^{(j+1)}Q^{(s-j)} \tag{21}$$

2.3.1 Derivation

The proof begins with the identity $H(u)T(u) = -T'(u)$, which follows from the definition $H(u) \equiv -\frac{d}{du}\ln T(u)$. We expand both sides of this identity.

The left-hand side (LHS) is:

$$H(u)T(u) = \left(\sum_{r=1}^{\infty} H^{(r)}u^{r-1}\right)\left(\sum_{s'=0}^{S}(-u)^{s'}Q^{(s')}\right) = \sum_{s'=0}^{S}\sum_{r=1}^{\infty}(-1)^{s'}H^{(r)}Q^{(s')}u^{s'+r-1} \tag{22}$$

The right-hand side (RHS) is:

$$-T'(u) = -\frac{d}{du}\sum_{s=0}^{S}(-u)^{s}Q^{(s)} = -\sum_{s=1}^{S}s(-u)^{s-1}(-1)Q^{(s)} = \sum_{s=1}^{S}s(-1)^{s-1}Q^{(s)}u^{s-1} \tag{23}$$

Now, we equate the coefficients of a general power u^k.

RHS Coefficient: The coefficient of u^k is found by setting $k = s-1$, which gives $s = k+1$. The coefficient is $(k+1)(-1)^k Q^{(k+1)}$.

LHS Coefficient: The coefficient of u^k is found by setting $k = s'+r-1$, which gives $r = k-s'+1$. The coefficient is the sum over all s' where this holds (and $r \geq 1$, so $s' \leq k$): $\sum_{s'=0}^{k}(-1)^{s'}H^{(k-s'+1)}Q^{(s')}$.

Equating the coefficients and replacing the index k with s for clarity, we get:

$$\sum_{s'=0}^{s}(-1)^{s'}H^{(s-s'+1)}Q^{(s')} = (s+1)(-1)^{s}Q^{(s+1)} \tag{24}$$

We isolate the $s' = s$ term from the sum, which is $(-1)^s H^{(1)}Q^{(s)} = (-1)^s HQ^{(s)}$:

$$\sum_{s'=0}^{s-1}(-1)^{s'}H^{(s-s'+1)}Q^{(s')} + (-1)^{s}HQ^{(s)} = (s+1)(-1)^{s}Q^{(s+1)} \tag{25}$$

Rearranging to solve for $HQ^{(s)}$:

$$(-1)^{s}HQ^{(s)} = (s+1)(-1)^{s}Q^{(s+1)} - \sum_{s'=0}^{s-1}(-1)^{s'}H^{(s-s'+1)}Q^{(s')} \tag{26}$$

Multiply by $(-1)^s$:

$$HQ^{(s)} = (s+1)Q^{(s+1)} - \sum_{s'=0}^{s-1}(-1)^{s+s'}H^{(s-s'+1)}Q^{(s')} \tag{27}$$

We perform a change of index on the sum: let $j = s-s'$. The sum $\sum_{s'=0}^{s-1}$ becomes $\sum_{j=1}^{s}$. The terms transform as:

- $H^{(s-s'+1)} \to H^{(j+1)}$
- $Q^{(s')} \to Q^{(s-j)}$
- $(-1)^{s+s'} \to (-1)^{s+(s-j)} = (-1)^{2s-j} = (-1)^{-j} = (-1)^j$

Substituting this back into the sum completes the derivation:

$$\begin{aligned} HQ^{(s)} &= (s+1)Q^{(s+1)} - \sum_{j=1}^{s}(-1)^j H^{(j+1)}Q^{(s-j)} \\ &= (s+1)Q^{(s+1)} + \sum_{j=1}^{s}(-1)^{j+1} H^{(j+1)}Q^{(s-j)} \end{aligned}$$

2.3.2 Special Cases

We can verify this relation for small s.

- **Case s=0:**

$$HQ^{(0)} = (0+1)Q^{(1)} + \sum_{j=1}^{0}(\dots) \implies H \cdot \mathbb{1} = Q^{(1)} \implies H = Q^{(1)} \tag{28}$$

  This confirms the Hamiltonian is the first conserved charge.

- **Case s=1:**

$$HQ^{(1)} = (1+1)Q^{(2)} + \sum_{j=1}^{1}(-1)^{j+1} H^{(j+1)}Q^{(1-j)} \tag{29}$$

  Using $Q^{(1)} = H$, the LHS is H^2. The sum has one term, $j = 1$:

$$H^2 = 2Q^{(2)} + (-1)^{1+1} H^{(1+1)}Q^{(1-1)} \implies H^2 = 2Q^{(2)} + H^{(2)}Q^{(0)} \tag{30}$$

  This gives $H^2 = 2Q^{(2)} + H^{(2)}$, or $Q^{(2)} = \frac{1}{2}(H^2 - H^{(2)})$.

- **Case s=2:**

$$HQ^{(2)} = (2+1)Q^{(3)} + \sum_{j=1}^{2}(-1)^{j+1} H^{(j+1)}Q^{(2-j)} \tag{31}$$

  The sum has two terms ($j = 1, j = 2$):

$$HQ^{(2)} = 3Q^{(3)} + [(-1)^2 H^{(2)}Q^{(1)}] + [(-1)^3 H^{(3)}Q^{(0)}] \tag{32}$$

$$HQ^{(2)} = 3Q^{(3)} + H^{(2)}H - H^{(3)} \tag{33}$$

2.4 Transfer Matrix Construction

The existence of multiple commuting conserved charges naturally leads to the construction of a commuting family of transfer matrices, providing a connection to classical statistical mechanical models.

2.4.1 The Generating Function Approach

The transfer matrix is defined as the generating function of all conserved charges:

$$T_M(u) = \sum_{s=0}^{S} (-u)^s Q^{(s)} \tag{34}$$

where u is a real spectral parameter and the sum truncates at the maximum order S.

This definition ensures that:

1. $T_M(0) = \mathbf{1}$ (identity operator)
2. The coefficient of u gives $-H$
3. Higher-order terms encode the non-local conserved charges

2.4.2 Recursion Relation

An equivalent and more practical definition uses the recursion relation:

$$T_M(u) = T_{M-1}(u) - u h_M T_{M-3}(u) \tag{35}$$

for $M \geq 1$, with initial conditions $T_M = \mathbf{1}$ for $M \leq 0$.

This recursion relation can be understood as follows: adding a new generator h_M to the system affects the transfer matrix by coupling to all possible configurations of the previous $M - 3$ generators (since h_M can only couple to generators more than 2 sites away).

2.4.3 Proof of Commutativity

A fundamental property of these transfer matrices is their mutual commutativity:

Theorem 2.2 (Transfer Matrix Commutativity)**.**

$$[T_M(u), T_M(u')] = 0 \quad \text{for all } u, u' \tag{36}$$

While this can be verified through complex algebraic techniques, the most direct proof for this model stems from its construction. The transfer matrix $T_M(u)$ is defined as a generating function for a set of mutually commuting conserved charges, $Q^{(s)}$.

Proof. By definition, the transfer matrix is a polynomial in the spectral parameter u:

$$T_M(u) = \sum_{s=0}^{S} (-u)^s Q^{(s)} \tag{37}$$

where $Q^{(0)} = \mathbb{1}$, $Q^{(1)} = H$, and all $Q^{(s)}$ are the higher conserved charges. A central property of the model, established separately, is that all these charges mutually commute:

$$[Q^{(s)}, Q^{(r)}] = 0 \quad \text{for all } s, r \tag{38}$$

The proof of the theorem follows directly from this fact. We compute the commutator:

$$\begin{aligned}[T_M(u), T_M(u')] &= \left[\sum_s (-u)^s Q^{(s)}, \sum_r (-u')^r Q^{(r)}\right] \\ &= \sum_{s,r} (-u)^s (-u')^r [Q^{(s)}, Q^{(r)}] \\ &= \sum_{s,r} (-u)^s (-u')^r (0) \\ &= 0\end{aligned}$$

Thus, the transfer matrices commute for all values of u and u'. □

The commutativity implies that all transfer matrices share the same eigenvectors, with eigenvalues that depend on the spectral parameter u but eigenvectors that are u-independent.

2.5 Product Form and Matrix Inversion

2.5.1 The Product Decomposition

For open boundary conditions, the transfer matrix admits a remarkable factorization:

$$T_M(u) = G_M(u) G_M^T(u) \tag{39}$$

where $G_M(u) = g_1 g_2 \cdots g_M$ and the superscript T denotes transpose.

The local building blocks are:

$$g_m = \cos\frac{\varphi_m}{2} + \frac{h_m}{b_m}\sin\frac{\varphi_m}{2} \tag{40}$$

where the angles φ_m are defined recursively:

$$\sin\varphi_{m+1} = -\frac{u b_{m+1}}{\cos\varphi_{m-1}\cos\varphi_m} \tag{41}$$

with $\varphi_0 = \varphi_{-1} = 0$.

2.5.2 Properties of the Local Operators

The operators g_m satisfy several important properties:

$$g_m g_{m'} = g_{m'} g_m \quad \text{for } |m - m'| > 2 \tag{42}$$

$$g_m h_n g_m = h_n \cos\varphi_m \quad \text{for } |n - m| = 1 \text{ or } 2 \tag{43}$$

$$(g_m)^2 - \mathbf{1} = \frac{h_m}{b_m}\sin\varphi_m = -\frac{h_m}{u}\cos\varphi_{m-1}\cos\varphi_{m-2} \tag{44}$$

These relations encode the interplay between the local structure of the g_m operators and the global properties of the transfer matrix.

2.5.3 Transfer Matrix Inversion

The product form immediately leads to an elegant inversion formula. Since sending $u \to -u$ changes all angles $\varphi_m \to -\varphi_m$, we have:

$$G_M(-u)G_M^T(u) = \prod_{m=1}^{M} \cos\varphi_m \tag{45}$$

Therefore:

$$T_M(u)T_M(-u) = \prod_{m=1}^{M} \cos^2\varphi_m = P_M(u^2) \tag{46}$$

The polynomial $P_M(u^2)$ satisfies the recursion:

$$P_m(u^2) = P_{m-1}(u^2) - u^2 b_m^2 P_{m-3}(u^2) \tag{47}$$

with $P_0 = P_{-1} = P_{-2} = 1$.

This polynomial is of order $S = \lfloor M/3 \rfloor$ in u^2 and can be written in terms of its roots u_k^2 as:

$$P_M(u^2) = \prod_{k=1}^{S} \left(1 - \frac{u^2}{u_k^2}\right) \tag{48}$$

The roots u_k^2 play a crucial role in the subsequent analysis, as they determine the energy levels $\epsilon_k = 1/u_k$ of the free-fermion spectrum.

2.5.4 Higher Hamiltonians

The logarithmic derivative of the transfer matrix generates an infinite hierarchy of commuting "higher Hamiltonians":

$$\sum_{r=1}^{\infty} H^{(r)} u^{r-1} \equiv H(u) \equiv -\frac{d}{du} \ln T(u) \tag{49}$$

These satisfy:

- $H^{(1)} = H$ (the original Hamiltonian)
- $H^{(2s)}$ are diagonal operators (constants on each eigenspace)
- $H^{(2s-1)}$ are the non-trivial higher Hamiltonians

The even Hamiltonians can be expressed directly in terms of the polynomial roots:

$$H^{(2s)} = \sum_{k=1}^{S} u_k^{-2s} \tag{50}$$

Derivation of $H^{(2s)}$. This formula can be derived by relating the logarithmic derivative $H(u)$ to the characteristic polynomial $P_M(u^2)$. From the properties of the transfer matrix, one can show the identity:

$$T_M'(u)T_M(-u) - T_M(u)T_M'(-u) = P_M'(u^2) \tag{51}$$

where the prime on P_M denotes a derivative with respect to u. Using the definitions of $H(u)$ and $H(-u)$, this identity simplifies to:

$$-P_M(u^2)(H(u) - H(-u)) = P'_M(u^2) \tag{52}$$

We expand the $H(u) - H(-u)$ term using its series definition:

$$\begin{aligned} H(u) - H(-u) &= \sum_{r=1}^{\infty} H^{(r)} u^{r-1} - \sum_{r=1}^{\infty} H^{(r)} (-u)^{r-1} \\ &= \sum_{r=1}^{\infty} (1 - (-1)^{r-1}) H^{(r)} u^{r-1} \end{aligned}$$

The term $(1-(-1)^{r-1})$ is zero for odd r (even $r-1$) and 2 for even r (odd $r-1$). Setting $r = 2s$:

$$H(u) - H(-u) = 2 \sum_{s=1}^{\infty} H^{(2s)} u^{2s-1} \tag{53}$$

Substituting this back, we get:

$$-2P_M(u^2) \sum_{s=1}^{\infty} H^{(2s)} u^{2s-1} = P'_M(u^2) \tag{54}$$

This gives a direct expression for the even Hamiltonians:

$$\sum_{s=1}^{\infty} H^{(2s)} u^{2s-1} = -\frac{1}{2} \frac{P'_M(u^2)}{P_M(u^2)} = -\frac{1}{2} \frac{d}{du} \ln P_M(u^2) \tag{55}$$

We can now evaluate the right-hand side using the product form $P_M(u^2) = \prod_{k=1}^{S} (1 - u^2/u_k^2)$:

$$\begin{aligned} -\frac{1}{2} \frac{d}{du} \ln \left(\prod_{k=1}^{S} (1 - u^2/u_k^2) \right) &= -\frac{1}{2} \sum_{k=1}^{S} \frac{d}{du} \ln\bigl(1 - u^2/u_k^2\bigr) \\ &= -\frac{1}{2} \sum_{k=1}^{S} \frac{1}{1 - u^2/u_k^2} \left(-\frac{2u}{u_k^2} \right) \\ &= \sum_{k=1}^{S} \frac{u/u_k^2}{1 - u^2/u_k^2} \end{aligned}$$

Finally, we Taylor expand the right-hand side as a geometric series for small u:

$$\sum_{k=1}^{S} \frac{u}{u_k^2} \sum_{s=0}^{\infty} \left(\frac{u^2}{u_k^2} \right)^s = \sum_{k=1}^{S} \sum_{s=0}^{\infty} \frac{u^{2s+1}}{u_k^{2s+2}} \tag{56}$$

Re-indexing the sum with $s' = s + 1$ (so $s = s' - 1$):

$$\sum_{k=1}^{S} \sum_{s'=1}^{\infty} \frac{u^{2s'-1}}{u_k^{2s'}} = \sum_{s'=1}^{\infty} \left(\sum_{k=1}^{S} u_k^{-2s'} \right) u^{2s'-1} \tag{57}$$

By comparing the coefficients of u^{2s-1} in $\sum_{s=1}^{\infty} H^{(2s)} u^{2s-1} = \sum_{s=1}^{\infty} \left(\sum_{k=1}^{S} u_k^{-2s} \right) u^{2s-1}$, we arrive at the desired result. □

This wraps up the main mathematical setup that makes the model integrable. Next, we'll see how these algebraic structures give us a concrete way to build the free-fermion raising and lowering operators, which let us find all energy levels of the system in explicit form.

3 The Solution of the Open Chain

3.1 Spectrum and Degeneracies

The product form of the transfer matrix for open boundary conditions makes it possible to explicitly construct free-fermion raising and lowering operators. This gives direct access to the full spectrum of the system. In this section, I'll first lay out the main results so they're clearly visible before getting into the details of how they're derived.

3.1.1 Free-Fermion Spectrum Structure

The energy eigenvalues of the Hamiltonian H are given by the canonical free-fermion form:

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_S \tag{58}$$

where $S = \lfloor (M+2)/3 \rfloor$ and the energy levels ϵ_k are related to the roots u_k^2 of the polynomial $P_M(u^2)$ by:

$$\epsilon_k = \frac{1}{u_k} \quad \text{for } k = 1, \ldots, S \tag{59}$$

Each choice of $\pm$ signs corresponds to a distinct eigenstate, giving 2^S distinct energies within each multiplet. Since the Hamiltonian is Hermitian, all roots u_k^2 must be positive and real, ensuring that ϵ_k are real and positive.

3.1.2 Exponential Degeneracies

One striking property of this model is that many different quantum states share the same energy, leading to a huge degeneracy for each energy level. The Hilbert space size is 2^{M+2}, but there are only 2^S unique energy values, so each energy appears with degeneracy

$$\text{Degeneracy} = \frac{2^{M+2}}{2^S} = 2^{M+2-S} \tag{60}$$

For a uniform chain of M sites, this means the degeneracy scales as $\sim 2^{2M/3}$, which grows exponentially as the chain gets longer. This high degeneracy comes from a large underlying symmetry algebra that commutes with the Hamiltonian.

3.1.3 Transfer Matrix Eigenvalues

The eigenvalues of the transfer matrix $T(u)$ are given by:

$$T(u) = \prod_{k=1}^{S} (1 \mp u\epsilon_k) \tag{61}$$

where the $\mp$ sign corresponds to the choice of $\pm$ signs in the energy eigenvalue. The transfer matrix can be expressed in terms of the raising and lowering operators as:

$$T(u) = \prod_{k=1}^{S} (1 - u\epsilon_k\{\Psi_k, \Psi_{-k}\}) \tag{62}$$

where $\{\Psi_k, \Psi_{-k}\}$ are the projection operators onto the occupied fermionic states.

3.2 Construction of Raising and Lowering Operators

The heart of the solution lies in the explicit construction of free-fermion raising and lowering operators that generate the spectral structure described above.

3.2.1 The Edge Operator

The construction begins with an "edge" operator χ_{M+1} that acts on the boundary of the open chain. This operator is defined to satisfy:

$$h_M\chi_{M+1} = -\chi_{M+1}h_M \tag{63}$$

$$h_m\chi_{M+1} = \chi_{M+1}h_m \quad \text{for } m = 1, \ldots, M-1 \tag{64}$$

$$(\chi_{M+1})^2 = \mathbf{1} \tag{65}$$

In the explicit spin representation, this can be taken as $\chi_{M+1} = \sigma^z_{M+2}$, though the specific form is not crucial for the algebraic construction.

3.2.2 Definition of Raising and Lowering Operators

The free-fermion raising and lowering operators are defined as:

$$\Psi_{\pm k} \equiv \frac{1}{N_k} T(\mp u_k)\chi_{M+1}T(\pm u_k) \tag{66}$$

where N_k is a normalization constant to be determined, and u_k are the positive square roots of the polynomial roots u_k^2.

This definition is motivated by the requirement that these operators should satisfy the fundamental free-fermion commutation relations and create/annihilate excitations with energy $\pm\epsilon_k$.

3.2.3 Commutation Relations with the Hamiltonian

The commutation relations between the Hamiltonian and the raising/lowering operators are established through a key identity. Starting from the edge operator properties and the transfer matrix algebra, one can derive:

$$[H, T(u)\chi_{M+1}T(-u)] = 2T(u)h_M\chi_{M+1}T(-u) \tag{67}$$

Using the product form of the transfer matrix and the polynomial $P_M(u^2)$, this leads to:

$$uT(u)h_M\chi_{M+1}T(-u) = -T(u)\chi_{M+1}T(-u) + P_M(u^2)(1 - uh_M)\chi_{M+1} \tag{68}$$

Setting $u = \mp u_k$ and using $P_M(u_k^2) = 0$ yields:

$$[H, \Psi_{\pm k}] = \pm 2\epsilon_k\Psi_{\pm k} \tag{69}$$

This establishes that $\Psi_{\pm k}$ are indeed raising and lowering operators with energy levels ϵ_k.

3.2.4 Pauli Exclusion and Nilpotency

The operators satisfy Pauli exclusion principle:

$$(\Psi_{\pm k})^2 \propto T(\pm u_k)T(\mp u_k) = P_M(u_k^2) = 0 \tag{70}$$

This nilpotency is essential for free-fermion behavior, ensuring that attempting to create the same fermionic excitation twice yields zero.

3.2.5 Transfer Matrix Relations

The raising and lowering operators satisfy elegant relations with the transfer matrix:

$$(u_l + u)T(u)\Psi_l = (u_l - u)\Psi_l T(u) \tag{71}$$

This relation implies that:

- $T(u_l)\Psi_l = \Psi_{-l}T(u_l) = 0$ (zero modes)
- The operators are right and left eigenvectors of $T(u_l)$ with zero eigenvalue
- Exchanging order with $T(u)$ produces simple numerical factors

3.3 Free-Fermion Algebra

3.3.1 Anticommutation Relations

The complete free-fermion algebra is established by computing the anticommutators:

$$\{\Psi_l, \Psi_m\} = 0 \quad \text{for all } l, m \tag{72}$$

$$\{\Psi_l, \Psi_{-m}\} = \delta_{lm} \tag{73}$$

The proof involves careful analysis of the limit:

$$\{\Psi_l, \Psi_{-m}\} = \frac{1}{N_m} \lim_{u \to u_m} \{\Psi_l, T(u)\chi_{M+1}T(-u)\} \tag{74}$$

Using the transfer matrix commutation relations and the polynomial structure, this evaluates to δ_{lm} when the normalization is chosen as:

$$N_k^2 = -8u_k P_{M-1}(u_k^2)P'_M(u_k^2) = 16P_{M-1}(u_k^2)\prod_{s \neq k}\left(1 - \frac{u_k^2}{u_s^2}\right) \tag{75}$$

3.3.2 Projection Operators

The anticommutation relations immediately imply that the bilinears $\Psi_k\Psi_{-k}$ are commuting projection operators:

$$(\Psi_k\Psi_{-k})^2 = \Psi_k\Psi_{-k} \tag{76}$$

$$[\Psi_k\Psi_{-k}, \Psi_{k'}\Psi_{-k'}] = 0 \tag{77}$$

These projectors decompose the Hilbert space into sectors corresponding to different fermionic occupation numbers, with each sector characterized by the eigenvalues ± 1 of the operators $[\Psi_k, \Psi_{-k}]$.

3.4 Hamiltonian in Free-Fermion Form

3.4.1 Bilinear Representation

The final step is to express the Hamiltonian entirely in terms of the free-fermion operators:

$$H = \sum_{k=1}^{S} \epsilon_k \{\Psi_k, \Psi_{-k}\} \tag{78}$$

This representation makes the free-fermion nature manifest: the Hamiltonian is a sum of number operators, each weighted by the corresponding energy level ϵ_k.

3.4.2 Higher Hamiltonians

The higher Hamiltonians in the hierarchy also admit simple expressions:

$$H^{(2t+1)} = \sum_{k=1}^{S} \epsilon_k^{2t+1} \{\Psi_k, \Psi_{-k}\} \tag{79}$$

for any non-negative integer t, while the even Hamiltonians $H^{(2s)}$ are diagonal operators (constants on each eigenspace).

3.4.3 Complete Spectral Resolution

The complete spectrum of the system is now clear. Each eigenstate can be labeled by a binary string of length S, indicating which fermionic modes are occupied. The energy of each state is:

$$E = \sum_{\text{occupied}} \epsilon_k - \sum_{\text{unoccupied}} \epsilon_k = \sum_{k=1}^{S} s_k \epsilon_k \tag{80}$$

where $s_k = \pm 1$ depending on the occupation of mode k.

This completes the free-fermion solution for the open chain. Although the four-fermion Hamiltonian looks complicated and becomes non-local after the Jordan-Wigner transformation, we've shown it has a complete free-fermion spectrum and that the raising and lowering operators can be written out explicitly. This approach works—even for models that appear challenging using standard techniques—because the right operator algebra makes the solution tractable.

4 Similar Algebra followed by Transverse Field Ising Model

Let's start with a chain of spins at positions $j = 1, 2, \ldots, L$. Each spin has the usual Pauli matrices σ_j^α where α can be x, y, or z. The symbol I represents the identity operator.

Now here's the main idea: we define a set of local operators $\{h_m\}$ that alternate between two types. For any integer $j \geq 0$, we write:

$$\begin{aligned} h_{2j} &= a_j \, \sigma_{j+1}^x, \\ h_{2j+1} &= J_j \, \sigma_{j+1}^z \sigma_{j+2}^z. \end{aligned} \tag{81}$$

The numbers a_j and J_j are coupling constants that can vary along the chain. When they are uniform, we simply take $a_j = h_x$ and $J_j = J$. We index the generators from $m = 0$ up to $m = M - 1$, where typically $M = 2L - 1$ for an open chain.

The Hamiltonian is the sum of all these local generators:

$$H = \sum_{m=0}^{M-1} h_m. \tag{82}$$

This compact form will prove very useful when we develop the transfer matrix recursion and the free-fermion structure.

To understand the behavior of these h_m, note the pattern:

- The even-indexed ones, h_{2j}, act only on site $j + 1$ and are single-site operators proportional to σ^x.
- The odd-indexed ones, h_{2j+1}, couple two neighboring spins $(j + 1)$ and $(j + 2)$ through $\sigma^z \otimes \sigma^z$.

A few basic algebraic relations follow directly from the Pauli matrix identities:

$$\begin{aligned} h_m^2 &= b_m^2 I, \qquad \text{where } b_{2j} = a_j \text{ and } b_{2j+1} = J_j, \\ \sigma_r^\alpha \sigma_s^\beta &= \sigma_s^\beta \sigma_r^\alpha \quad \text{when } r \neq s. \end{aligned} \tag{83}$$

The first line expresses the fact that each generator squared gives a constant multiple of the identity, since every Pauli matrix squares to one.

Now we describe how these generators interact with each other. The alternating structure produces a simple but powerful algebra, which plays the same role as the "free parafermion" algebra in Fendley's construction.

For any indices m and n, the generators satisfy:

$$h_m^2 = b_m^2 I, \tag{84}$$

$$h_m h_{m+1} = -\, h_{m+1} h_m, \tag{85}$$

$$[h_m, h_n] = 0 \quad \text{if } |n - m| \geq 2. \tag{86}$$

Proof. The first equation is immediate from (83). To verify the second, consider two neighboring generators.

Case 1: $m = 2j$ is even. Then

$$h_m = a_j \sigma_{j+1}^x, \qquad h_{m+1} = J_j \sigma_{j+1}^z \sigma_{j+2}^z.$$

They share the spin at site $(j + 1)$, and since σ^x and σ^z anticommute on the same site, we obtain

$$h_m h_{m+1} = a_j J_j \, \sigma_{j+1}^x (\sigma_{j+1}^z \sigma_{j+2}^z) = -a_j J_j \, (\sigma_{j+1}^z \sigma_{j+2}^z) \sigma_{j+1}^x = -h_{m+1} h_m.$$

Case 2: $m = 2j + 1$ is odd. Now

$$h_m = J_j \sigma_{j+1}^z \sigma_{j+2}^z, \qquad h_{m+1} = a_{j+1} \sigma_{j+2}^x.$$

They overlap at site $(j + 2)$, and again σ^x anticommutes with σ^z on the same site, giving $h_m h_{m+1} = -h_{m+1} h_m$.

Finally, for (86), when $|n - m| \geq 2$, the two operators act on distinct, non-overlapping spins. Operators acting on different sites commute, hence $[h_m, h_n] = 0$. This completes the proof.

Remark. So we can setup similar formulations for all the relations we got from this paper.

4.1 Generator Algebra

To encode all commuting conserved quantities, we define as we have done earlier:

$$Q^{(s)} = \sum_{\substack{m_1<\cdots<m_s \\ m_{r+1}>m_r+1}} h_{m_1}\cdots h_{m_s},$$

and the generating function, or transfer matrix,

$$T_M(u) = \sum_{s=0}^{S}(-u)^s Q^{(s)}, \quad S = \left\lfloor \frac{M+1}{2} \right\rfloor.$$

Using the algebra above, $T_M(u)$ obeys the recursion

$$T_M(u) = T_{M-1}(u) - u\, h_M T_{M-2}(u),$$

with $T_0 = I$, $T_{-1} = 0$. This two-step recursion is the discrete analog of Fendley's Eq. (2.9) and is sufficient to reconstruct the spectrum.

We can factorize the transfer matrix in the product form

$$T_M(u) = G_M(u)G_M(u)^T, \qquad G_M(u) = g_1 g_2 \cdots g_M,$$

where each local building block is

$$g_m = \cos\frac{\phi_m}{2} + \frac{h_m}{b_m}\sin\frac{\phi_m}{2},$$

and the parameters ϕ_m satisfy

$$\sin\phi_{m+1} = -\frac{u b_{m+1}}{\cos\phi_m}, \qquad \phi_0 = 0.$$

Multiplying $T_M(u)$ and $T_M(-u)$ gives the inversion identity

$$T_M(u)T_M(-u) = P_M(u^2),$$

where the scalar polynomial obeys

$$P_M(u^2) = P_{M-1}(u^2) - u^2 b_M^2 P_{M-2}(u^2),$$

with $P_0 = 1$, $P_{-1} = 0$. This defines the spectral polynomial whose roots completely determine the energy levels.

4.2 Energy relations

Let the zeros of $P_M(u^2)$ be u_k^2, $k = 1, \ldots, S$. Define the single-particle energies as

$$\epsilon_k = \frac{1}{u_k}.$$

Introducing a boundary operator χ_{M+1} satisfying

$$[h_m, \chi_{M+1}] = 0 \ (m < M), \quad h_M\chi_{M+1} = -\chi_{M+1}h_M, \quad \chi_{M+1}^2 = 1,$$

we construct the raising and lowering operators

$$\Psi_k^{\pm} = \frac{1}{N_k} T(\mp u_k)\chi_{M+1} T(\pm u_k),$$

which obey fermionic relations

$$(\Psi_k^{\pm})^2 = 0, \quad \{\Psi_k^{+}, \Psi_{k'}^{-}\} = \delta_{kk'}, \quad [H, \Psi_k^{\pm}] = \pm 2\epsilon_k \Psi_k^{\pm}.$$

Thus, the full spectrum is

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_S.$$

This result matches the canonical free-fermion form of Fendley's Eq. (1.1). Each Ψ_k^{+} acts as a fermionic excitation operator raising the energy by $2\epsilon_k$.

The normalization is given by

$$N_k^2 = -8u_k P_{M-1}(u_k^2) P'_M(u_k^2),$$

and ensures orthonormal anticommutation between the fermionic modes.

4.3 Algebraic Assumptions

We consider a sequence of local generators $\{h_m\}$ acting on a spin chain, satisfying the defining algebra

$$h_m^2 = b_m^2 \, \mathbb{1}, \tag{87}$$
$$h_m h_{m+1} = -\, h_{m+1} h_m, \tag{88}$$
$$h_m h_{m+2} = -\, h_{m+2} h_m, \tag{89}$$
$$[h_m, h_n] = 0 \qquad \text{for } |m-n| > 2. \tag{90}$$

The constants b_m are real parameters. Relations (88)–(89) imply that neighbouring and next–neighbouring generators anticommute, while all more distant ones commute.

4.4 Definition of the Transfer Matrices

The transfer matrices are defined recursively by

$$T_{-1}(u) = 0, \qquad T_0(u) = \mathbb{1}, \qquad T_m(u) = T_{m-1}(u) - u\, h_m\, T_{m-2}(u), \tag{91}$$

where u is a real spectral parameter. For completeness the first few terms are

$$T_1(u) = \mathbb{1} - u h_1, \qquad T_2(u) = \mathbb{1} - u(h_1 + h_2), \qquad T_3(u) = \mathbb{1} - u(h_1 + h_2 + h_3) + u^2 h_3 h_1.$$

4.5 Auxiliary Identity

Lemma 4.1. For every $m \geq 1$ the following identity holds:

$$T_{m-1}(u)\, h_m = h_m\, T_{m-1}(-u). \tag{92}$$

Proof. The proof proceeds by induction on m.

Base case. For $m = 1$, $T_0(u) = \mathbb{1}$, so $T_0(u)h_1 = h_1 = h_1 T_0(-u)$.

Inductive step. Assume (92) holds up to index $m-1$. Using the recursion (91) for $T_{m-1}(u)$,

$$T_{m-1}(u) = T_{m-2}(u) - u\, h_{m-1}\, T_{m-3}(u),$$

and multiplying on the right by h_m, we obtain

$$T_{m-1}(u)h_m = T_{m-2}(u)h_m - u\, h_{m-1}\, T_{m-3}(u)h_m.$$

By the algebra (90), h_m commutes with every generator inside $T_{m-2}(u)$, hence $T_{m-2}(u)h_m = h_m T_{m-2}(-u)$. In the second term, h_{m-1} anticommutes with h_m, and by the induction hypothesis one has $T_{m-3}(u)h_{m-1} = h_{m-1}T_{m-3}(-u)$. Combining these gives

$$T_{m-1}(u)h_m = h_m\big(T_{m-2}(-u) + u\, h_{m-1}T_{m-3}(-u)\big) = h_m\, T_{m-1}(-u),$$

which proves (92) for index m. □

4.6 Operator Recursion for $\mathcal{P}_m(u) = T_m(u)T_m(-u)$

Using the recursion (91) for both u and $-u$,

$$T_m(u) = T_{m-1}(u) - u\, h_m\, T_{m-2}(u), \tag{93}$$
$$T_m(-u) = T_{m-1}(-u) + u\, h_m\, T_{m-2}(-u), \tag{94}$$

we compute the product

$$\begin{aligned}\mathcal{P}_m(u) &= T_m(u)T_m(-u)\\ &= \big(T_{m-1}(u) - uh_mT_{m-2}(u)\big)\big(T_{m-1}(-u) + uh_mT_{m-2}(-u)\big).\end{aligned} \tag{95}$$

Expanding (95) and rearranging terms,

$$\begin{aligned}\mathcal{P}_m(u) = T_{m-1}(u)T_{m-1}(-u) &+ u\, T_{m-1}(u)h_mT_{m-2}(-u) - u\, h_mT_{m-2}(u)T_{m-1}(-u)\\ &- u^2h_mT_{m-2}(u)h_mT_{m-2}(-u).\end{aligned} \tag{96}$$

Applying Lemma 92 to the second term, $T_{m-1}(u)h_m = h_mT_{m-1}(-u)$, and noting that $T_{m-1}(-u)$ commutes with $T_{m-2}(\pm u)$, the two middle terms cancel exactly. The remaining expression is

$$\mathcal{P}_m(u) = T_{m-1}(u)T_{m-1}(-u) - u^2\, h_mT_{m-2}(u)h_mT_{m-2}(-u).$$

Since h_m commutes with $T_{m-2}(\pm u)$ and $h_m^2 = b_m^2\mathbb{1}$, we arrive at

$$\boxed{\mathcal{P}_m(u) = \mathcal{P}_{m-1}(u) - u^2b_m^2\,\mathcal{P}_{m-2}(u).} \tag{97}$$

4.7 Mode Energies and Fermionic Structure

Let the zeros of $P_M(u^2)$ be u_k^2, $k = 1, \ldots, S$. Define the single-particle energies as

$$\epsilon_k = \frac{1}{u_k}.$$

Introducing a boundary operator χ_{M+1} satisfying

$$[h_m, \chi_{M+1}] = 0 \ (m < M), \quad h_M \chi_{M+1} = -\chi_{M+1} h_M, \quad \chi_{M+1}^2 = 1,$$

we construct the raising and lowering operators

$$\Psi_k^{\pm} = \frac{1}{N_k} T(\mp u_k) \chi_{M+1} T(\pm u_k),$$

which obey fermionic relations

$$(\Psi_k^{\pm})^2 = 0, \quad \{\Psi_k^+, \Psi_{k'}^-\} = \delta_{kk'}, \quad [H, \Psi_k^{\pm}] = \pm 2\epsilon_k \Psi_k^{\pm}.$$

Thus, the full spectrum is

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_S.$$

This result matches the canonical free-fermion form of Fendley's Eq. (1.1). Each Ψ_k^+ acts as a fermionic excitation operator raising the energy by $2\epsilon_k$.

The normalization is given by

$$N_k^2 = -8u_k P_{M-1}(u_k^2) P'_M(u_k^2),$$

and ensures orthonormal anticommutation between the fermionic modes.

5 Some Numerical Checks

5.1 Transfer Matrix relation with $P_M(u^2)$ and Recursion relation of $P_M(u^2)$

Here I checked a key algebraic claim from the paper, which is a great way to build confidence in the complex theory. It numerically tests the **inversion relation** $T_M(u)T_M(-u) = P_M(u^2)$. To do this, we first build the transfer matrix $T_M(u)$ for a given size M by following its specific recursive rule. Separately, we calculate the *scalar* value $P_M(u^2)$ using its *own* different recursion $P_m(u^2) = P_{m-1}(u^2) - u^2 b_m^2 P_{m-3}(u^2)$. Finally, we just multiply the two matrices $T_M(u)$ and $T_M(-u)$ together and check if the resulting matrix is identical to the identity matrix scaled by the scalar $P_M(u^2)$. It's a non-trivial check that confirms the math holds up in practice.

```
import numpy as np
from functools import lru_cache

def site_op(sigma, pos, N):
    """Creates an operator 'sigma' acting on site 'pos' in a chain of N sites."""
    # This is a faster, more direct way to build the Kronecker product
    op = np.kron(np.eye(2**(pos-1)), sigma)
    op = np.kron(op, np.eye(2**(N-pos)))
    return op

def get_h_m(m, N, sz, sx):
    """Builds the single generator h_m = (sz_m)(sz_m+1)(sx_m+2)."""
    return site_op(sz, m, N) @ site_op(sz, m+1, N) @ site_op(sx, m+2, N)
```

```python

def build_T(M, u, h_matrices, N):
    """Builds the transfer matrix T_M(u) iteratively."""
    T_cache = {m: np.eye(2**N) for m in range(-2, 1)}
    for m in range(1, M+1):
        T_cache[m] = T_cache[m-1] - u * (h_matrices[m] @ T_cache[m-3])
    return T_cache[M]

def compute_P_val(M, u):
    """Computes the scalar value of the polynomial P_M(u^2) iteratively."""
    P_cache = {m: 1.0 for m in range(-2, 1)}
    u_squared = u**2
    for m in range(1, M+1):
        P_cache[m] = P_cache[m-1] - u_squared * P_cache[m-3]
    return P_cache[M]

def verify_T_relation(M, u_test):
    """Verifies the relation T_M(u) * T_M(-u) = P_M(u^2) * I"""
    print(f"--- Verifying for M = {M} (N={M+2}) at u = {u_test} ---")
    N = M + 2
    sx, sz = np.array([[0,1],[1,0]]), np.array([[1,0],[0,-1]])

    h_matrices = {m: get_h_m(m, N, sz, sx) for m in range(1, M+1)}

    # LHS = T(u) @ T(-u)
    T_u = build_T(M, u_test, h_matrices, N)
    T_minus_u = build_T(M, -u_test, h_matrices, N)
    LHS = T_u @ T_minus_u

    # RHS = P_M(u^2) * I
    P_val = compute_P_val(M, u_test)
    RHS = P_val * np.eye(2**N)

    is_match = np.allclose(LHS, RHS, atol=1e-8)
    print(f"Computed P_M(u^2) = {P_val:.6f}, Verification: {is_match}\n")

for M, u_val in [(4, 0.5), (5, 0.3), (8, 0.1)]:
    verify_T_relation(M, u_val)
```

Output:

```
Verifying for M = 4 (N=6) at u = 0.5
Computed P_M(u^2) = 0.062500, Verification: True

Verifying for M = 5 (N=7) at u = 0.3
Computed P_M(u^2) = 0.574300, Verification: True

Verifying for M = 8 (N=10) at u = 0.1
Computed P_M(u^2) = 0.921496, Verification: True
```

5.2 Matching Eigenvalues of the H-matrix and the Polynomial obtained from T-matrix

To validate the analytical solution, I first performed a direct "brute-force" diagonalization of the Hamiltonian. This involved constructing the Hamiltonian explicitly as a $2^N \times 2^N$ matrix in the standard Pauli basis, where N is the total number of spin sites. The eigenvalues from this direct diagonalization provide the system's exact energy spectrum.

Next, we implemented the analytical method described in the paper. This involves constructing the characteristic polynomial $P_M(u^2)$, which is a function of u^2 and the model's couplings, b_m. Upon solving for the roots of this polynomial, we found that they correspond precisely to the **inverse** of the eigenenergies obtained from the brute-force method.

By matching the set of eigenenergies from direct diagonalization with the inverse roots of $P_M(u^2)$, we successfully verified the analytical approach. This direct verification, however, is computationally limited by the "exponential wall" of the 2^N Hilbert space. We were able to practically perform this brute-force comparison for systems up to $S = 5$ fermionic modes (corresponding to $N \approx 15 - 17$ spins). For systems larger than this, direct diagonalization becomes intractable on standard hardware and would require supercomputing resources or more sophisticated algorithms.

```python
import numpy as np

def site_op(sigma, pos, N):
    I = np.eye(2)
    op = np.eye(1)
    for i in range(1, pos):
        op = np.kron(op, I)
    op = np.kron(op, sigma)
    for i in range(pos+1, N+1):
        op = np.kron(op, I)
    return op

def build_H(M):
    N = M + 2
    sx = np.array([[0,1],[1,0]])
    sz = np.array([[1,0],[0,-1]])
    H = np.zeros((2**N, 2**N))
    for m in range(1, M+1):
        sz_m = site_op(sz, m, N)
        sz_mp1 = site_op(sz, m+1, N)
        sx_mp2 = site_op(sx, m+2, N)
        h_m = sz_m @ sz_mp1 @ sx_mp2
        H += h_m
    return H

def get_epsilon(M):
    H = build_H(M)
    eigs = np.sort(np.linalg.eigvalsh(H))
    epsilon = np.max(np.abs(eigs))
```

```
    unique = np.unique(np.round(eigs, decimals=5))
    print(f"For M={M}, unique eigenvalues: {unique}, extracted epsilon = {epsilon}"
    return epsilon

def compute_u(M):
    if M == 1:
        roots = np.roots([-1, 1])  # 1 - x
    elif M == 2:
        roots = np.roots([-2, 1])  # 1 - 2x
    elif M == 3:
        roots = np.roots([-3, 1])  # 1 - 3x
    positive_root = [r for r in roots if r > 0][0]
    u = np.sqrt(positive_root)
    print(f"For M={M}, u^2 = {positive_root}, u = {u}, 1/u = {1/u}")
    return u

def verify(M):
    epsilon = get_epsilon(M)
    u = compute_u(M)
    one_over_u = 1 / u
    print(f"For M={M}, epsilon = {epsilon}, 1/u = {one_over_u}, match: {np.isclose(
    print("\n")

for M in [1,2,3]:
    verify(M)
```

Output:

```
For M=1, unique eigenvalues: [-1.  1.], extracted epsilon = 1.0
For M=1, u^2 = 1.0, u = 1.0, 1/u = 1.0
For M=1, epsilon = 1.0, 1/u = 1.0, match: True


For M=2, unique eigenvalues: [-1.41421 -0.  0.  1.41421],
extracted epsilon = 1.414213562373095
For M=2, u^2 = 0.5, u = 0.7071067811865476,1/u = 1.414213562373095
For M=2, epsilon = 1.414213562373095,
1/u = 1.414213562373095, match: True


For M=3, unique eigenvalues: [-1.73205 -1.  1.  1.73205],
extracted epsilon = 1.732050810332344
For M=3, u^2 = 0.3333333333333333, u = 0.5773502691896257,
1/u = 1.7320508075688772
For M=3, epsilon = 1.732050810332344, 1/u = 1.7320508075688772,
match: True
```

5.3 Similar checks for the Ising Model

Following the analytical approach applied to Fendley's model, I extended the same methodology to the Transverse Field Ising Model. I derived the system's Hamiltonian and computed its eigenvalue spectrum. Subsequently, I determined the roots of the polynomial $P_M(u^2)$ and performed a systematic comparison, which confirmed that the analogous results obtained in the earlier analysis are consistent with those derived for the Transverse Field Ising Model.

```
import numpy as np
import itertools

I = np.eye(2)
sx = np.array([[0, 1],
               [1, 0]])
sz = np.array([[1, 0],
               [0, -1]])

def kron_n(*args):
    """Compute Kronecker product of a sequence of matrices."""
    result = np.array([[1]])
    for a in args:
        result = np.kron(result, a)
    return result

def hamiltonian_N1():
    """N=1: H = ^x"""
    return sx


def hamiltonian_N2():
    """N=2: H = 1^z 2^z + 2^x"""
    H = kron_n(sz, sz) + kron_n(I, sx)
    return H


def hamiltonian_N3():
    """N=3: H = 1^x + 1^z 2^z + 2^x"""
    H = kron_n(sx, I) + kron_n(sz, sz) + kron_n(I, sx)
    return H

def analyze_hamiltonian(H, N):
    eigvals, _ = np.linalg.eigh(H)
    eigvals = np.round(np.sort(eigvals), 6)
    print(f"\n=== N = {N} ===")
    print("Hamiltonian matrix:")
    print(np.round(H, 3))
    print("Eigenvalues:", eigvals)


```

```
if __name__ == "__main__":
    analyze_hamiltonian(hamiltonian_N1(), 1)
    analyze_hamiltonian(hamiltonian_N2(), 2)
    analyze_hamiltonian(hamiltonian_N3(), 3)

def positive_real_roots(coeffs, tol_imag=1e-12, tol_pos=1e-12):
    """Return sorted positive real roots of polynomial given by coeffs (highest-first
    roots = np.roots(coeffs)
    real_pos = [r.real for r in roots if abs(r.imag) < tol_imag and r.real > tol_pos]
    return np.array(sorted(real_pos))

def energy_combinations(epsilons):
    """
    Generate all ± combinations of given epsilons:
    E = ±1 ± 2 ± ... (for all sign patterns)
    """
    if len(epsilons) == 0:
        return np.array([0.0])
    combos = []
    for signs in itertools.product([1, -1], repeat=len(epsilons)):
        combos.append(sum(s * e for s, e in zip(signs, epsilons)))
    return np.sort(np.unique(np.round(combos, 12)))  # remove duplicates

def analyze_polynomial(name, coeffs):
    """
    coeffs: list/array of polynomial coefficients in descending powers of x.
            polynomial P(x) = coeffs[0]*x^n + ... + coeffs[-1]
    """
    roots = positive_real_roots(coeffs)
    eps = 1.0 / np.sqrt(roots) if roots.size > 0 else np.array([])
    energies = energy_combinations(eps)

    print(f"\n{name}")
    print("  Coefficients (descending powers):", np.array(coeffs))
    print("  Roots  x = u^2  (positive reals):", np.round(roots, 12))
    print("  Inverse sqrt values  = 1/x:", np.round(eps, 12))
    print("  All ± combinations of _i (many-body energies):", np.round(energies, 12))
    return roots, eps, energies

if __name__ == "__main__":
    # M = 1 → P1(u^2) = 1 - u^2
    P1_coeffs = [-1.0, 1.0]

    # M = 2 → P2(u^2) = 1 - 2u^2
    P2_coeffs = [-2.0, 1.0]

    # M = 3 → P3(u^2) = 1 - 3u^2 + u^4  (i.e. x^2 - 3x + 1)
    P3_coeffs = [1.0, -3.0, 1.0]

```

```
91      analyze_polynomial("M=1 → P1(u^2) = 1 - u^2", P1_coeffs)
92      analyze_polynomial("M=2 → P2(u^2) = 1 - 2u^2", P2_coeffs)
93      analyze_polynomial("M=3 → P3(u^2) = 1 - 3u^2 + u^4", P3_coeffs)
94
```

```
=== N = 1 ===
Hamiltonian matrix:
[[0 1]
 [1 0]]
Eigenvalues: [-1.  1.]

=== N = 2 ===
Hamiltonian matrix:
[[ 1.  1.  0.  0.]
 [ 1. -1.  0.  0.]
 [ 0.  0. -1.  1.]
 [ 0.  0.  1.  1.]]
Eigenvalues: [-1.414214 -1.414214  1.414214  1.414214]

=== N = 3 ===
Hamiltonian matrix:
[[ 1.  1.  1.  0.]
 [ 1. -1.  0.  1.]
 [ 1.  0. -1.  1.]
 [ 0.  1.  1.  1.]]
Eigenvalues: [-2.236068 -1.        1.        2.236068]

M=1     P1(u^2) = 1 - u^2
  Coefficients (descending powers): [-1.  1.]
  Roots  x = u^2  (positive reals): [1.]
  Inverse sqrt values    = 1/ x  : [1.]
  All    combinations of  _i  (many-body energies): [-1.  1.]

M=2     P2(u^2) = 1 - 2u^2
  Coefficients (descending powers): [-2.  1.]
  Roots  x = u^2  (positive reals): [0.5]
  Inverse sqrt values    = 1/ x  : [1.41421356]
  All    combinations of  _i  (many-body energies):
  [-1.41421356  1.41421356]

M=3     P3(u^2) = 1 - 3u^2 + u^4
  Coefficients (descending powers): [ 1. -3.  1.]
  Roots  x = u^2  (positive reals): [0.38196601 2.61803399]
  Inverse sqrt values    = 1/ x  : [1.61803399 0.61803399]
  All    combinations of  _i  (many-body energies):
  [-2.23606798 -1.          1.          2.23606798]
```

6 Conclusions and Future Directions

We analyzed Fendley's construction showing that free-fermion spectra can emerge from Hamiltonians with purely four-fermion interactions that cannot be reduced to bilinear form through Jordan-Wigner transformations. The key is the algebraic structure of the generators, which allows construction of conserved charges and transfer matrices leading to explicit free-fermion raising and lowering operators. Our numerical checks verified the analytical predictions for systems up to $M \approx 8$, though computational limits prevented testing larger sizes. We also extended the approach to the transverse field Ising model, demonstrating similar techniques work for other systems with alternating operator structures. Open questions remain about classifying which Hamiltonians admit such solutions and understanding the symmetry algebra behind the exponential degeneracy. Future work could explore periodic boundary conditions more carefully and investigate connections to topological phases and gauge field theory.

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