Chiral Tunnelling and the Klein Paradox in Graphene

PH 436 Presentation

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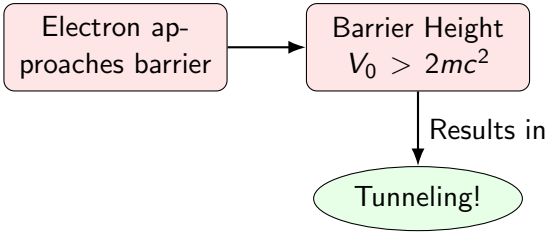
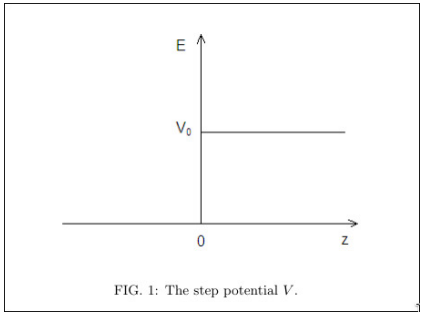
Ref: Katsnelson, M., Novoselov, K., Geim, A. (2006)

The Klein Paradox

What is the paradox?

It's a counterintuitive phenomenon that ultimately comes from relativistic quantum mechanics.





Barrier Scattering Schematic

The paradox

- Transmission probability increases with barrier height.
- Reflection coefficient can exceed 1.

The Governing Law: Dirac Equation

Since we are effectively dealing with relativistic particles we need the Dirac equation.

Free Dirac Equation

$$(i\gamma_\mu \partial^\mu - m)\psi = 0$$

Terminology

- ψ are four-component object called spinors.
- γ_μ is a 4-vector where μ goes from 0 to 3

$$\gamma = \left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \right)$$

- ∂^μ yet another 4-vector

$$\partial^\mu = (\partial_t \ , -\nabla)$$

Minimal Coupling

Free Dirac equation to be modified to include potential and still maintain Lorentz invariance

Change $p^\mu \to p^\mu - A^\mu$.
$A^\mu = (\phi, A)$ where ϕ is scalar potential and A is vector potential.
$p^\mu \to i\partial_\mu$
$p^\mu - A^\mu \to i\partial_\mu - A_\mu$

Dirac Equation with Potential

$$\left[\gamma^\mu\left(i\partial_\mu - A_\mu\right) - m\right]\psi = 0$$

Dirac Equation for Electrostatic Potential

A purely electrostatic field
$A^\mu = (V, 0, 0, 0)$
$\gamma^\mu A_\mu = \gamma^0 V$
Which makes our **final Dirac equation that we will finally apply as**

Final Dirac Equation

$$\Big[i\gamma^\mu \partial_\mu - V\gamma^0 - m\Big]\psi = 0$$

Ansatz for Dirac Equation

We pick an ansatz for solving our step potential coming only from z-direction

$$\psi = u \exp(-ip^{\mu} x_{\mu})$$

Where

$$u = \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}$$

$p^{\mu} = (E \quad 0 \quad 0 \quad p_z)$
$x_{\mu} = (t \quad 0 \quad 0 \quad -z)$
$p^{\mu} x_{\mu} = Et - zp_z$
Ideally should pick most general case but just increases algebraic complexity and no extra insight offered. u is simplest factor possible for its internal degrees of freedom.

Applying Dirac on Ansatz

We get

$$\Big[i\gamma^\mu\partial_\mu - V\gamma^0 - m\Big]\, u\exp(-ip^\mu x_\mu) = 0$$

Applying it and simplifying it gives us

$$\Big[(E-V)^2 - (p_z^2 + m^2)\Big]\, u = 0$$

Since relation true for any general u the operator evaluates to 0.

Energy Momentum Relation

$$(E-V)^2 = p_z^2 + m^2$$

Positive and Negative Energy States

We can plug u into Energy Momentum relation to get

$$\frac{\alpha}{\beta} = \frac{E - V - p}{m}$$

$$u = N \begin{pmatrix} 1 \\ 0 \\ \frac{m}{E-V-p} \\ 0 \end{pmatrix}$$

Note that depending upon the sign of $E - V$ we will have a different normalization factor N

Positive and Negative Energy States

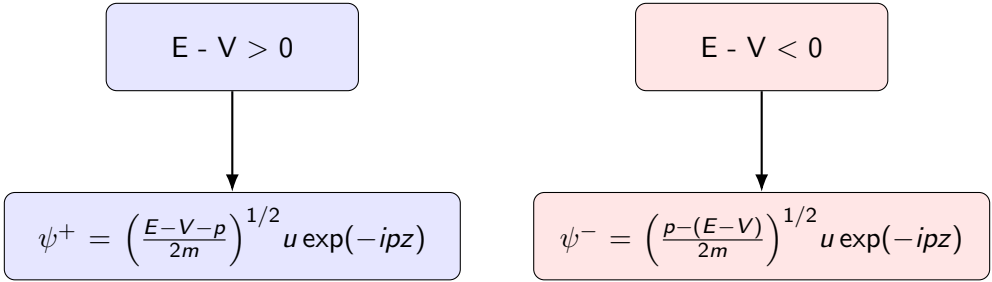
We can plug u into Energy Momentum relation to get

$$\frac{\alpha}{\beta} = \frac{E - V - p}{m}$$

$$u_{norm} = N \begin{pmatrix} 1 \\ 0 \\ \frac{m}{E-V-p} \\ 0 \end{pmatrix} = Nu$$

Note that depending upon the sign of $E - V$ we will have a different normalization factor N

Positive and Negative Energy States



Wavefunctions in Different Regions

In **Region-1** wavefunction will be superposition of incident and reflected wave.

$$\psi_I = a\psi_i + b\psi_r$$

$$\implies \psi_I = au(E,p)\exp(ipz) + bu(E,-p)\exp(-ipz)$$

In **Region-2** you will have cases depending upon the height of potential barrier.

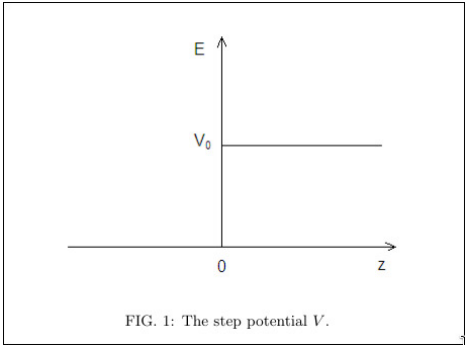


FIG. 1: The step potential V.

Wavefunctions in Different Region

Case 1: $V_0 < E - m$ In this case you will use ψ_+ in region-2 as $E - V > 0$.

Case 2: $E - m < V_0 < E + m$ In this case $p = \sqrt{(E - V_0)^2 - m}$ will be imaginary and there will be exponential decay.

Case 3: $V_0 > E + m$ This is the paradox region so we will analyze it a bit further. Here we will implement ψ_-
In Case3 we will have to implement continuity relation at the boundary between z<0 and z>0.

For Case3 ψ_{II} =cv(E-V_0, q)$exp(iqz)$

Continuity Relation

$$\psi_I\Big|_{z=0} = \psi_{II}\Big|_{z=0}$$

$$a\begin{pmatrix} 1 \\ 0 \\ \frac{m}{E-p} \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ \frac{m}{E+p} \\ 0 \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \\ \frac{m}{E-V_0-q} \\ 0 \end{pmatrix}$$

Solving this gives us

$$a + b = c$$

$$a\frac{m}{E-p} + b\frac{m}{E-p} = c\frac{m}{E-V_0-q}$$

Probability Density Current

And solving the 2 linear equations gives us

$$\frac{b}{a} = \frac{(E+p)(p-V_0-q)}{(E-p)(p+V_0+q)}$$
$$\frac{c}{a} = \frac{2p(E-V_0-q)}{(E-p)(p+V_0+q)}$$

In Dirac's formula current density is famously given by

$$j^i = \psi^* \gamma^0 \gamma^1 \psi$$

Probability Density Current

Calculating density current for each incident, reflected and transmitted wave gives us

$$j^i = a^* a \frac{m^2 - (E-p)^2}{(E-p)^2}$$

$$j_r = b^* b \frac{m^2 - (E+p)^2}{(E+p)^2}$$

$$j_t = c^* c \frac{m^2 - (E-V_0-q)^2}{(E-V_0-q)^2}$$

Reflection and Transmission Coefficients

Now for the reflection and transmission coefficients we get

$$R = \frac{|j_r|}{|j_i|} = \frac{E+p}{E-p}\left(\frac{p-V_0-q}{p+V_0+q}\right)^2$$

$$T = \frac{|j_t|}{|j_i|} = \frac{2p}{E-p}\frac{(E-V_0-q)^2-m^2}{(p+V_0+q)^2}$$

A simple addition shows that $\mathbf{1+T=R}$!!!!

This is the first form of Klein's Paradox where Reflection coefficient is greater than 1. Although this can be quickly fixed.

Pauli's Fix

Pauli pointed out that we assume a implicit +q group velocity when we wrote the solution in z>0 .This is the wrong intuition we carried over from non-relativistic QM where $q^2 = \frac{2m}{\hbar^2}(E - V_0)$ and for E> V_0 we take e^{iqz} for right moving waves. This assumption breaks in relativistic QM because

$$v_g = \frac{dE}{dq} = \frac{q}{E - V_0}$$

When $E < V_0$ in the klein regime the group velocity is opposite to that of q. So to get a right moving wave we choose $q < 0$. In all the formulas we have to replace q by -q

Corrected Coefficients

Replacing q by -q and redoing the calculations we get the following transmission coefficients:

$$R = \frac{E+p}{E-p}\frac{(p-V_0+q)^2}{(p+V_0-q)^2}$$

$$T = \frac{2p}{1-p}\frac{m^2-(E-V_0+q)^2}{(p+V_0-q)^2}$$

R + T = 1 which is as expected.

Klein's Paradox

Now take the limit as $V_0 \to \infty$ then

$$R = \frac{E - p}{E + p}$$

$$T = \frac{2p}{E + p}$$

Which means that even at high potential barrier there is still a non-zero chance of transmitting through.

Tight-binding Hamiltonian (A Slightly different Approach)

- Graphene: two-site basis (A,B). Nearest-neighbour hopping amplitude $-t$.
- Real-space tight-binding Hamiltonian (nearest neighbours only):

$$\hat{H} = -t \sum_{\langle ij \rangle} (\hat{a}_i^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{a}_i),$$

  where $i \in A$, $j \in B$, and $\langle ij \rangle$ denotes nearest neighbours.
- Re-expressimg sum by picking each A-site and summing over its 3 neighbour displacements $\{\boldsymbol{\delta}\}$:

$$\sum_{\langle ij \rangle} (\hat{a}_i^\dagger \hat{b}_j + \text{H.c.}) = \sum_{i \in A} \sum_{\delta} (\hat{a}_i^\dagger \hat{b}_{i+\delta} + \hat{b}_{i+\delta}^\dagger \hat{a}_i).$$

Fourier transform to momentum space

- Use Bloch expansions (number of A sites $= N/2$):

$$\hat{a}_i^\dagger = \frac{1}{\sqrt{N/2}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_i}\, \hat{a}_{\mathbf{k}}^\dagger, \quad \hat{b}_{i+\delta} = \frac{1}{\sqrt{N/2}} \sum_{\mathbf{k}'} e^{-i\mathbf{k}'\cdot(\mathbf{r}_i+\delta)}\, \hat{b}_{\mathbf{k}'}.$$

- Substitute into $\hat{H}$ and sum over $i \in A$:

$$\sum_{i\in A} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_i} = \frac{N}{2}\, \delta_{\mathbf{k},\mathbf{k}'}.$$

- This collapses the $\mathbf{k}'$ sum and yields a block-diagonal Hamiltonian:

$$\hat{H} = -t \sum_{\mathbf{k}} \sum_{\delta} \left(e^{-i\mathbf{k}\cdot\delta}\, \hat{a}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} + \text{H.c.} \right).$$

Bloch Hamiltonian and $\Delta_{\mathbf{k}}$

- Collect operators into a two-component spinor:

$$\Psi_{\mathbf{k}} = \begin{pmatrix} \hat{a}_{\mathbf{k}} \\ \hat{b}_{\mathbf{k}} \end{pmatrix}, \qquad \hat{H} = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^{\dagger}\, h(\mathbf{k})\, \Psi_{\mathbf{k}}.$$

- Define

$$\Delta_{\mathbf{k}} \equiv \sum_{\delta} e^{i\mathbf{k}\cdot\delta},$$

  where $\delta \in \{\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_3\}$ are the three nearest-neighbour vectors.

- The 2×2 Bloch Hamiltonian is

$$h(\mathbf{k}) = -t \begin{pmatrix} 0 & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^{*} & 0 \end{pmatrix}.$$

Explicit form of $\Delta_{\mathbf{k}}$ and band energies

- Using the standard geometry (lattice constant a) one can show (algebra below):

$$\Delta_{\mathbf{k}} = e^{-ik_x a}\Big[1 + 2e^{i\frac{3}{2}k_x a}\cos\Big(\frac{\sqrt{3}}{2}k_y a\Big)\Big].$$

- The eigenvalues of $h(\mathbf{k})$ are

$$E_{\pm}(\mathbf{k}) = \pm t\sqrt{\Delta_{\mathbf{k}}\Delta_{\mathbf{k}}^{*}} = \pm t\,|\Delta_{\mathbf{k}}|.$$

- Expanding $|\Delta_{\mathbf{k}}|^2$ yields the familiar band formula

$$E_{\pm}(\mathbf{k}) = \pm t\sqrt{1 + 4\cos\Big(\tfrac{3}{2}k_x a\Big)\cos\Big(\tfrac{\sqrt{3}}{2}k_y a\Big) + 4\cos^2\Big(\tfrac{\sqrt{3}}{2}k_y a\Big)}.$$

Derivation detail (expand $\Delta_{\mathbf{k}}$) – algebra

$$\begin{aligned}\Delta_{\mathbf{k}} &= e^{i\mathbf{k}\cdot\delta_1} + e^{i\mathbf{k}\cdot\delta_2} + e^{i\mathbf{k}\cdot\delta_3} \\ &= e^{-ik_x a}\left[1 + e^{i\frac{3}{2}k_x a}e^{i\frac{\sqrt{3}}{2}k_y a} + e^{i\frac{3}{2}k_x a}e^{-i\frac{\sqrt{3}}{2}k_y a}\right] \\ &= e^{-ik_x a}\left[1 + 2e^{i\frac{3}{2}k_x a}\cos\left(\frac{\sqrt{3}}{2}k_y a\right)\right].\end{aligned}$$

Then

$$|\Delta_{\mathbf{k}}|^2 = 1 + 4\cos\left(\frac{3}{2}k_x a\right)\cos\left(\frac{\sqrt{3}}{2}k_y a\right) + 4\cos^2\left(\frac{\sqrt{3}}{2}k_y a\right).$$

Dirac Quasiparticles in Graphene (contd.)

- The hamiltonian of an electron in a monolayer graphene is given by:

$$\hat{H}0 = -i\hbar v_F \sigma \cdot \nabla$$

- The Hamiltonian matrix when expanded around Dirac point K' comes out as

$$H = \hbar v_F \begin{pmatrix} 0 & q_x - iq_y \\ q_x + iq_y & 0 \end{pmatrix}$$

- The eigen-values are

$$E = \pm \hbar v_F |q|$$

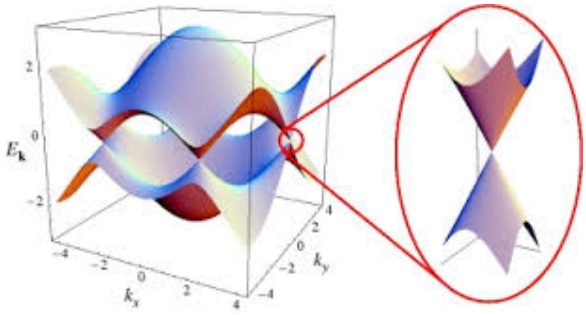
- and the eigen vectors

$$\psi_{\pm} = e^{i(q_x x + q_y y)} \begin{pmatrix} 1 \\ \pm e^{i\phi} \end{pmatrix}$$

- where

$$\phi = \tan^{-1}\left(\frac{q_y}{q_x}\right)$$

Visualising the Band Structure



Dirac Points (K and K')

- These are special locations in momentum space (k-space).
- The conduction and valence bands **touch** here.
- This "touching point" is the key to Graphene's unique electronic properties.

Dirac Quasiparticles in Graphene

- The solution of graphene like dirac spinors contains **2 components** i.e. contribution from the 2 atoms on a lattice point, which is analogous to the spin and is here referred to as '**pseudo-spin**'
- In the Dirac framework, **chirality** is defined by an operator that projects spin along the momentum axis.
  - When the spin aligns with the momentum, the eigenvalue is $+1$, representing **electrons**.
  - If it is opposite to the momentum, the eigenvalue is -1, corresponding to **positrons**.

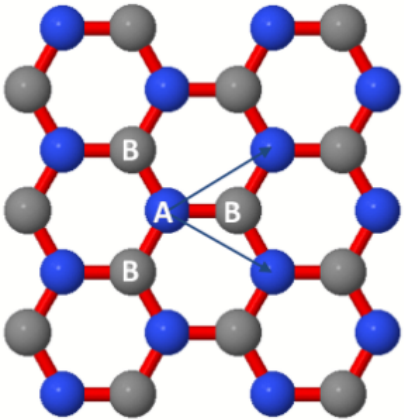
Understanding the Pre-requisites - Pseudo-spin

Remember the A-B sublattice that we studied in graphene. Due to this we wrote the wavefunction as

$$\psi(r) = c_A(r)\,|A\rangle + c_B(r)\,|B\rangle$$

We can also write this as

$$\psi = \begin{pmatrix} c_A \\ c_B \end{pmatrix}$$



Understanding the Pre-requisites - Pseudo-spin (Contd.)

The two-component description for graphene is very similar to the one by spinor wavefunctions in QED, but the 'spin' index for graphene indicates the sublattice occupancy rather than the real spin of electrons and is usually referred to as 'pseudo-spin' σ

Now, in the presence of a potential, it can change only if the scatterer couples differently to the two sublattices (A/B). We assume that the potential is effectively constant within the lattice dimensions, hence

Pseudospin effectively remains conserved

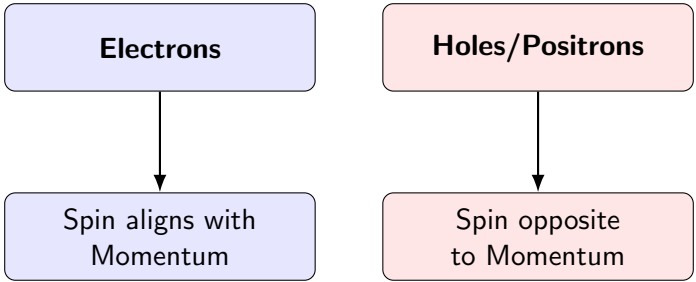
Understanding the Pre-requisites - Chirality

In QED, for dirac fermions, we define another **conserved** quantity 'Chirality', which is given as:

$$\chi = sgn(E)\frac{\langle\sigma|\sigma\rangle \cdot \vec{p}}{|\vec{p}|}$$

Since for electrons (in conduction band) E>0 and for holes (in valence band) E<0, we have:

Chirality Operator logic:



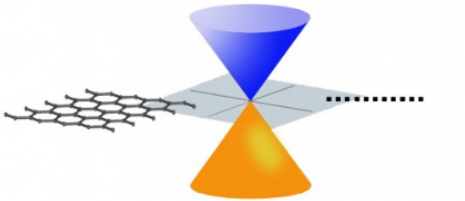
Reformulating the Paradox for Graphene

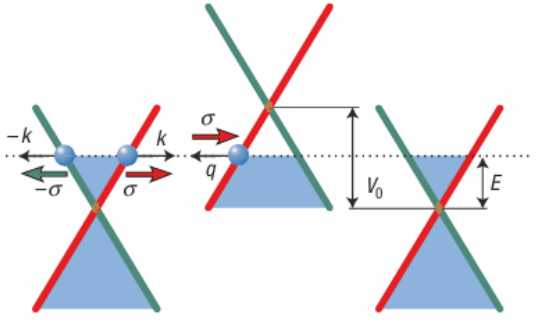
Let's set up a potential barrier of width D:

$$V(x) = \begin{cases} V_0 & 0 < x < D \\ 0 & \text{otherwise} \end{cases}$$

- **Inversion:** The barrier inverts charge carriers. Underneath the barrier, electrons turn into holes (positrons).
- **Massless Advantage:** Since Dirac fermions in graphene are massless ($m = 0$), we don't need the massive electric field usually required in vacuum QED.

Quasiparticle Spectrum: Single Layer





Visualizing how quasiparticles transition across the potential barrier in single-layer graphene

Why Reflection is Forbidden

State	k direction	v_g direction	Pseudospin	Allowed?
Incoming electron	$+k\hat{x}$ (right)	$+v_F\hat{x}$ (right)	$+\hat{x}$	Yes
Reflected electron	$-k\hat{x}$ (left)	$-v_F\hat{x}$ (left)	$-\hat{x}$	No
Transmitted hole	$-q\hat{x}$ (left)	$+v_F\hat{x}$ (right)	$+\hat{x}$	Yes

Table: Momentum, group velocity, and pseudospin directions for Klein tunneling at normal incidence in single-layer graphene.

Let's get Mathematical

We assume that the incident electron wave propagates at an angle ϕ with respect to the x axis and then try the components of the Dirac spinor ψ_1 and ψ_2 for the hamiltonian $H = H_0 + V(x)$ in the following form:

$$\psi_1(x,y) = \begin{cases} e^{ik_x x} + re^{-ik_x x} & e^{ik_y y}, x < 0, \\ ae^{iq_x x} + be^{-iq_x x} & e^{ik_y y}, 0 < x < D, \\ te^{ik_x x + ik_y y}, & x > D, \end{cases}$$

$$\psi_2(x,y) = \begin{cases} s(e^{ik_x x + i\phi} - re^{-ik_x x - i\phi})e^{ik_y y} & x < 0, \\ s'(ae^{iq_x x + i\theta} - be^{-iq_x x - i\theta})e^{ik_y y} & 0 < x < D, \\ ste^{ik_x x + ik_y y + i\phi} & x > D, \end{cases}$$

Let's get Mathematical (Contd.)

$$k_F = 2\pi/\lambda \quad k_x = k_F \sin\phi \quad k_y = k_F \cos\phi$$

$$q_x = \sqrt{\frac{(E - V_0)^2}{\hbar^2 v_F^2} - k_y^2} \quad \theta = \tan^{-1}(k_y/q_x)$$

$$s = \mathrm{sgn}(E) \quad s' = \mathrm{sgn}(E - V_0)$$

- coefficients are a, b, t, r.
- t and r are transmission and reflection coefficients.

The Magic of High Barriers

Matching boundary conditions gives us the reflection coefficient r:

$$r = \frac{2ie^{i\phi}\sin(q_x D)(\sin\phi - ss'\sin\theta)}{ss'\left(e^{-iq_x D}\cos(\phi+\theta) + e^{iq_x D}\cos(\phi-\theta)\right) - 2i\sin(q_x D)}$$

If the barrier is very strong ($|V_0| \gg |E|$), the formula simplifies:

$$|T|^2 = 1 - |r|^2 \qquad T = \frac{\cos^2\phi}{1 - \cos^2(q_x D)\sin^2\phi}$$

Key Observations

1. **Resonance:** When $q_x D = \pi N$, the barrier becomes transparent ($T = 1$).
2. **Klein Paradox:** At normal incidence ($\phi = 0$), transmission is **always 1**.
3. **No Backscattering:** Right-moving e^- turns into left-moving e^+ perfectly.

Calculating Reflection & Transmission coefficients

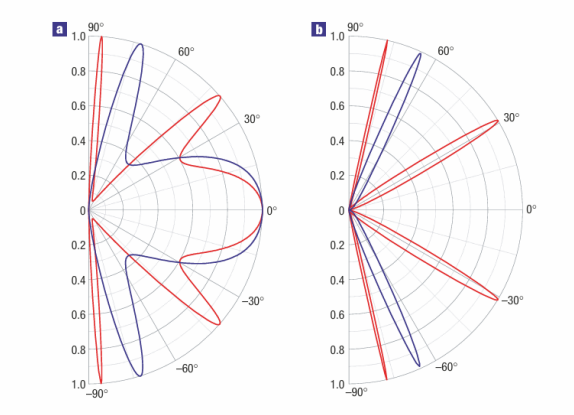


Figure: Transmission probability T through a 100-nm-wide barrier as a function of the incident angle ϕ for single- **(a)** and bi-layer **(b)** graphene.

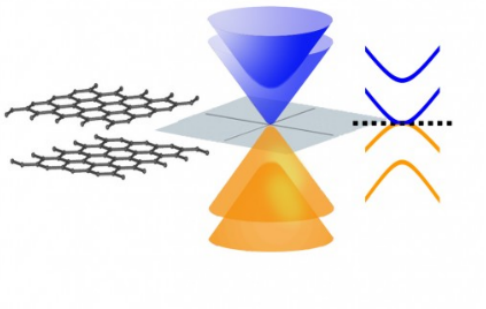
Bilayer Graphene

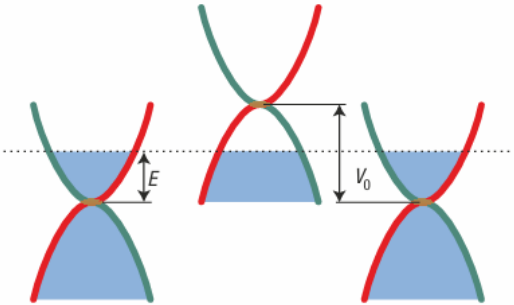
What happens if we add a second layer?

- **Spectrum:** Becomes parabolic (like standard non-relativistic electrons).
- **Mass:** Quasiparticles acquire mass.
- **Solutions:** There are now **four** solutions:
  - 2 Propagating waves.
  - 2 Evanescent (decaying) waves.

This extra complexity changes the tunneling behavior.

Quasiparticle Spectrum: Bilayer





Note the difference in band curvature (parabolic) compared to single layer.

Chiral Tunnelling in Bilayer Graphene

- Charge carriers in bilayer graphene have a **parabolic energy spectrum**
- **Massive quasiparticles** with a finite density of states at zero energy, similar to conventional non-relativistic electrons
- They are **chiral** and described by spinor wavefuntions
- Hamiltonian :

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\begin{pmatrix} 0 & (k_x - ik_y)^2 \\ (k_x + ik_y)^2 & 0 \end{pmatrix}$$

- There are **four possible solutions** for a given energy $E = \pm\frac{\hbar^2 k_F^2}{2m}$.
- Two of them correspond to propagating waves and the other two to evanescent waves.

Tunnelling in Bilayer Graphene

For constant potential V_i, eigenstates of the hamiltonian are :

$$\psi_1(x,y) = (a_i e^{ik_{ix}x} + b_i e^{-ik_{ix}x} + c_i e^{\kappa_{ix}x} + d_i e^{-\kappa_{ix}x}) e^{ik_y y}$$

$$\psi_2(x,y) = s_i(a_i e^{ik_{ix}x+2i\phi_i} + b_i e^{-ik_{ix}x-2i\phi_i} - c_i h_i e^{\kappa_{ix}x} - \frac{d_i}{h_i} e^{-\kappa_{ix}x}) e^{ik_y y}$$

where

$$s_i = \mathrm{sgn}(V_i - E) \qquad \hbar k_{ix} = \sqrt{2m|E - V_i|}\cos\phi_i$$

$$\hbar k_y = \sqrt{2m|E - V_i|}\sin\phi_i \qquad \kappa_{ix} = \sqrt{k_{ix}^2 + 2k_y^2}$$

$$h_i = \frac{\sqrt{1 + \sin^2\phi_i} - \sin\phi_i}{2}$$

- $d_1 = 0$ for $x < 0$ and $b_3 = c_3 = 0$ for $x > D$

Tunneling at Normal Incidence

Let's look at an electron beam hitting the barrier head-on ($\phi = 0$).

Low Barrier ($V_0 < E$):

- Chirality doesn't really show up.
- Behaves like standard Schrödinger electrons.

High Barrier ($V_0 > E$):

- Electrons outside $\rightarrow$ Holes inside.
- **Surprise:** Massive chiral fermions are **perfectly reflected** at normal incidence!

Bilayer Transmission Coefficient

Transmission coefficient :

$$t = \frac{4ik_1k_2}{(k_2 + ik_1)^2 e^{-k_2 D} - (k_2 - ik_1)^2 e^{k_2 D}}$$

- $D \longrightarrow \infty$?
- completely reflect a normal beam

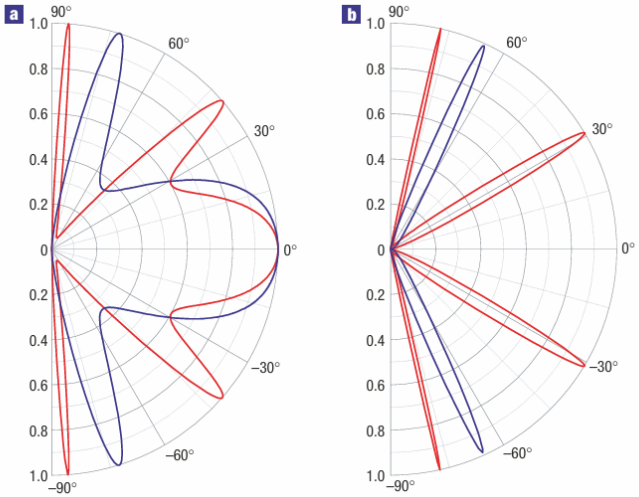
For a single $p - n$ junction with $V_0 \gg E$

$$T = \frac{E}{V_0} \sin^2(2\phi)$$

$$\implies T = 0 \text{ for } \phi = 0$$

Contrast with single-layer graphene, where normally incident electrons are always perfectly transmitted.

Visual Proof: Monolayer vs Bilayer



Why the Difference? Symmetry!

It all comes down to **Charge-Conjugation Symmetry**.

Monolayer (Spin 1/2 like)

Wavefunction matches perfectly.

$\downarrow$

$\mathbf{T = 1}$

Bilayer (Spin 1 like)

Wavevector k transforms to ik.
(Propagating $\rightarrow$ Evanescent)

$\downarrow$

$\mathbf{T = 0}$

The 'Anti-Klein' Difference

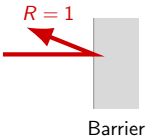
Single-Layer Graphene

"The Ghost"

- **Phase:** π (Spin 1/2-like)
- **Physics:** Spin mismatch forbids reflection.
- **Result: Klein Tunneling**

Bilayer Graphene

"The Wall"



- **Phase:** 2π (Spin 1-like)
- **Physics:** 360° rotation allows reflection.
- **Result: Anti-Klein Tunneling**

References

1. Katsnelson, M., Novoselov, K., Geim, A. *Chiral tunnelling and the Klein paradox in graphene*. Nature Phys 2, 620-625 (2006).
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3. Franz Utermohlen. *Tight-Binding Model for Graphene* (Course Notes, 2018).
4. A. H. Castro Neto et al. *The electronic properties of graphene*. Rev. Mod. Phys. 81, 109 (2009).