PH-567: Non-Linear Dynamics

The Duffing Oscillator: From Simple Harmonics to Complex Chaos

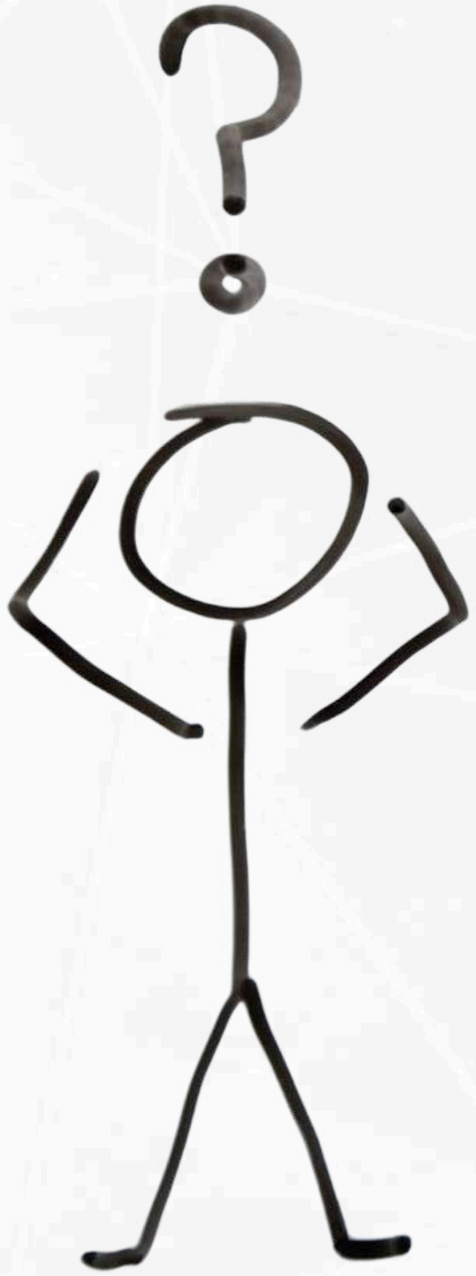
Soumik Sahoo

Onkar Dagade

Vasudev Dubey

Outline

1. Simple spring-mass system
2. Introducing non-linearity
3. Duffing Oscillator
4. Simulations
5. Analogous analogy
6. Wait a minute...



Why this topic?

References

Dynamics of a Duffing Oscililator System

Jongoh (Andy) Jeong
Professor Anita Raja
Ph235 Physics Simulations
29 April 2019

Abstract
The chaotic behavior in motion is not unusual to observe in practice in common nonlinear
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Survey of Regular and Chaotic Phenomena in the Forced Duffing Oscillator

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(*Received* 11 *March* 1991)

Abstract—The periodically forced Duffing oscillator

$$\ddot{x} + k\dot{x} + x^3 = B\cos t$$

exhibits a wide variety of interesting phenomena which are fundamental to the behavior of nonlinear dynamical systems, such as regular and chaotic motions, coexisting attractors, regular and fractal basin boundaries, and local and global bifurcations. Analog and digital simulation experiments have provided a survey of the most significant types of behavior; these experiments are essential to any complete understanding, but the experiments alone are not sufficient, and careful interpretation in terms of the geometric theory of dynamical systems is required. The results of the author's survey, begun over 25 yr ago, are here brought together to give a reasonably complete view of the behavior of this important and prototypical dynamical system.

Analogue Electrical Circuit for Simulation of the Duffing-Holmes Equation

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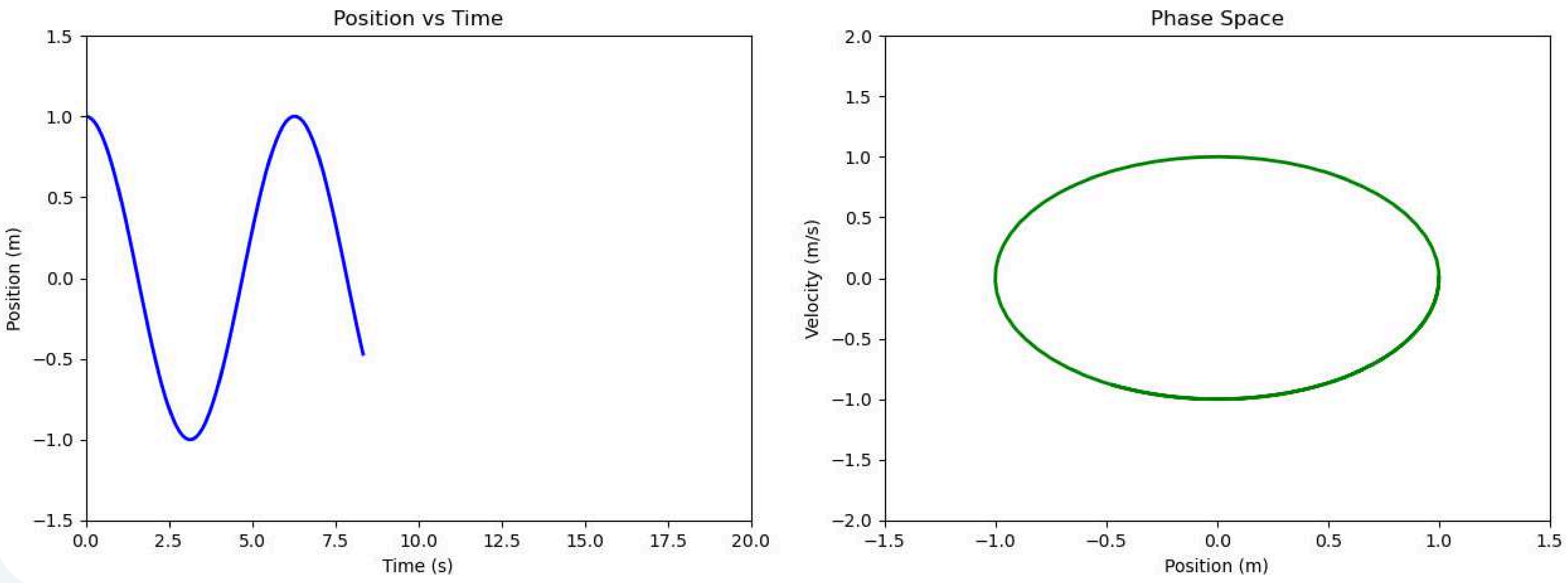
ceived: 15.10.2007 **Revised:** 31.01.2008 **Published online:** 02.06.2008

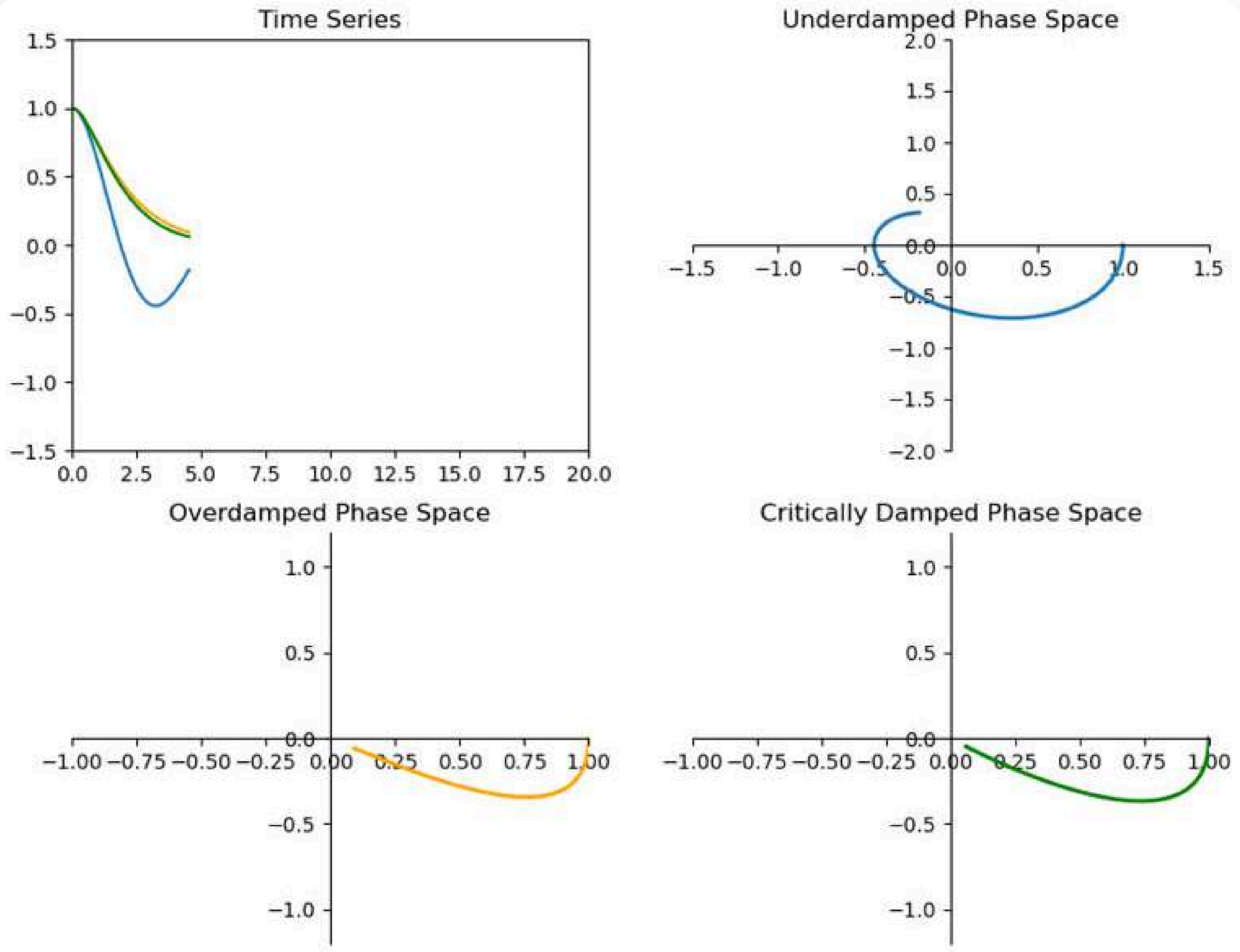
stract. We describe an extremely simple second order analogue electrical circuit for
ulating the two-well Duffing-Holmes mathematical oscillator. Numerical results and
logue electrical simulations are illustrated with the snapshots of chaotic waveforms,
h the phase portraits (the Lissajous figures) and with the stroboscopic maps (the
ncaré sections).

ywords: nonlinear dynamics, chaos, electrical circuits.

Spring-mass system



Damping + Spring–mass system



$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k_1 x = 0$$

$$\frac{d^2x}{dt^2} + 2\zeta\frac{dx}{dt} + \omega_0^2 x = 0$$

- **Underdamping** $(\zeta < 1)$

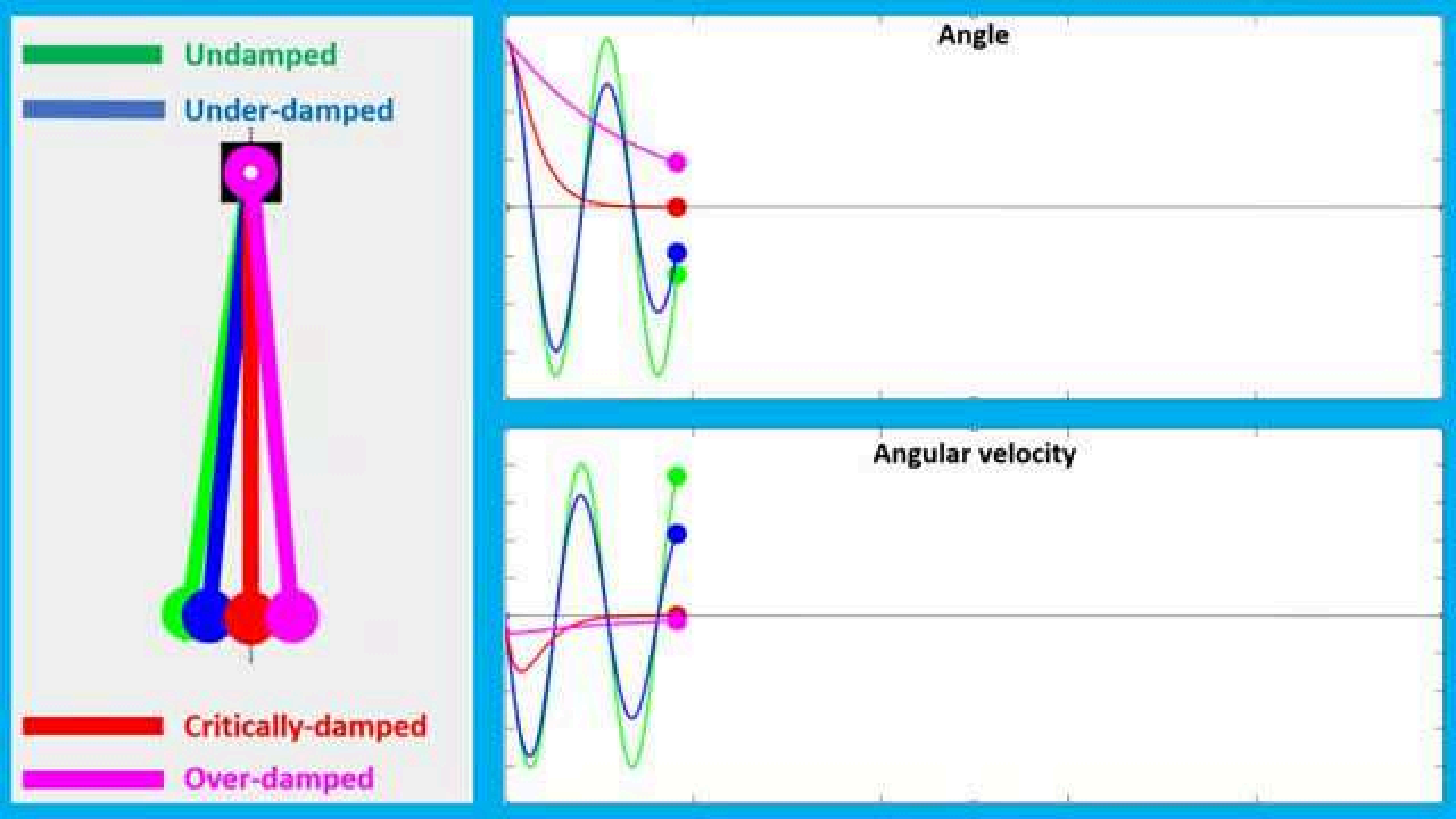
$$x(t) = Ae^{-\zeta\omega_0 t}(cos(\omega t + \phi))$$

- **Critically-damping**

$$x(t) = (at + b)e^{-\zeta\omega_0 t}$$

- **Overdamping**

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$


Undamped
Under-damped
Critically-damped
Over-damped
Angle
Angular velocity

Force + Damping + Spring-Mass system

$$\frac{d^2x}{dt^2} + 2\zeta\frac{dx}{dt} + \omega_0^2 x = f(t) = fcos(\omega t)$$

$$x(t) = x_{homo}(t) + Acos(\omega t - \delta)$$

$$\delta = tan^{-1}\left(\frac{2\zeta\omega}{\omega_0^2 - \omega^2}\right)$$

- Underdamping $(\zeta < 1)$

$$x(t) = Ae^{-\zeta\omega_0 t}(cos(\omega t + \phi))$$

- Critically-damping

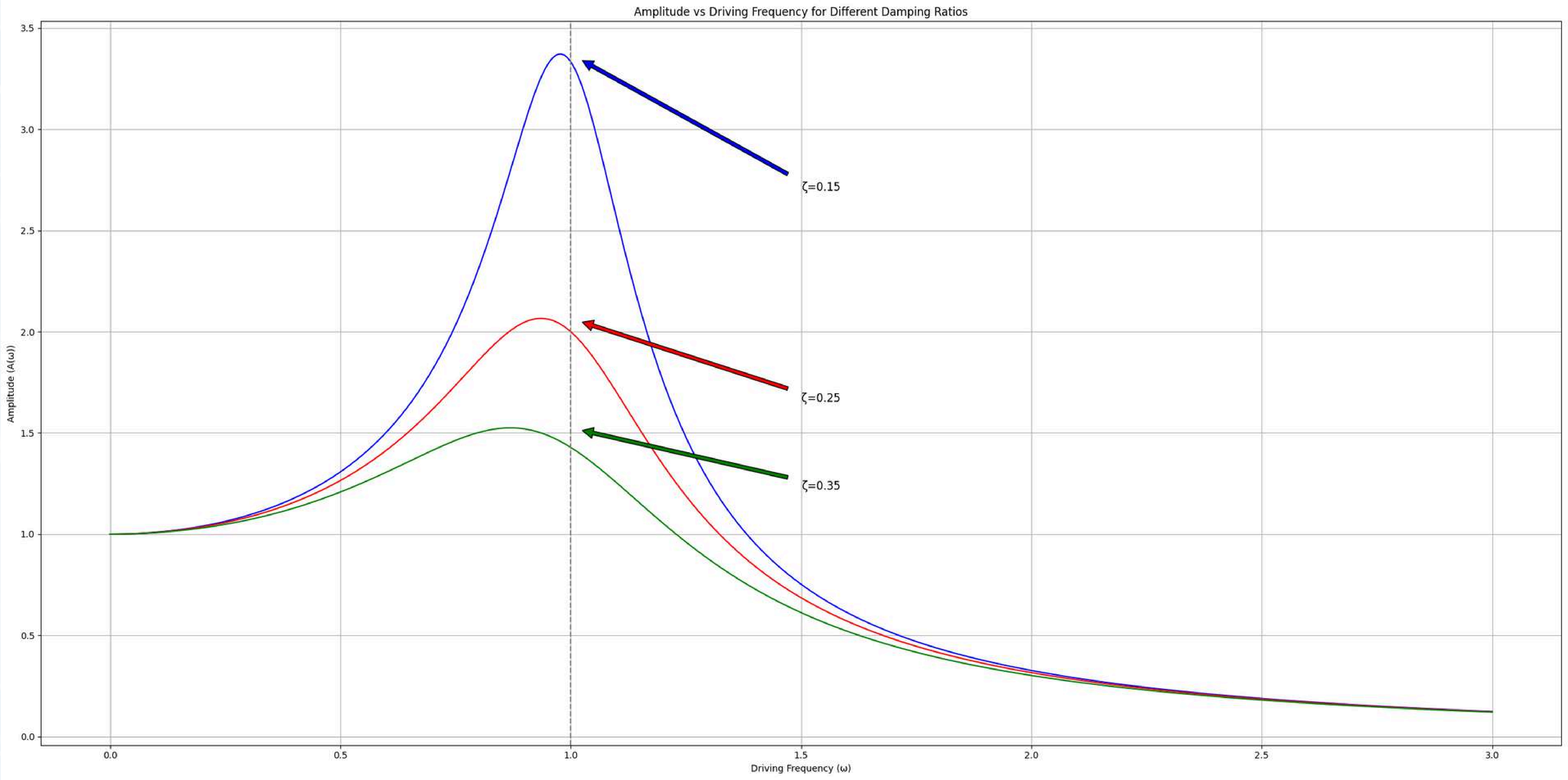
$$x(t) = (at + b)e^{-\zeta\omega_0 t}$$

- Overdamping

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

$$A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\zeta\omega_0\omega)^2}}$$

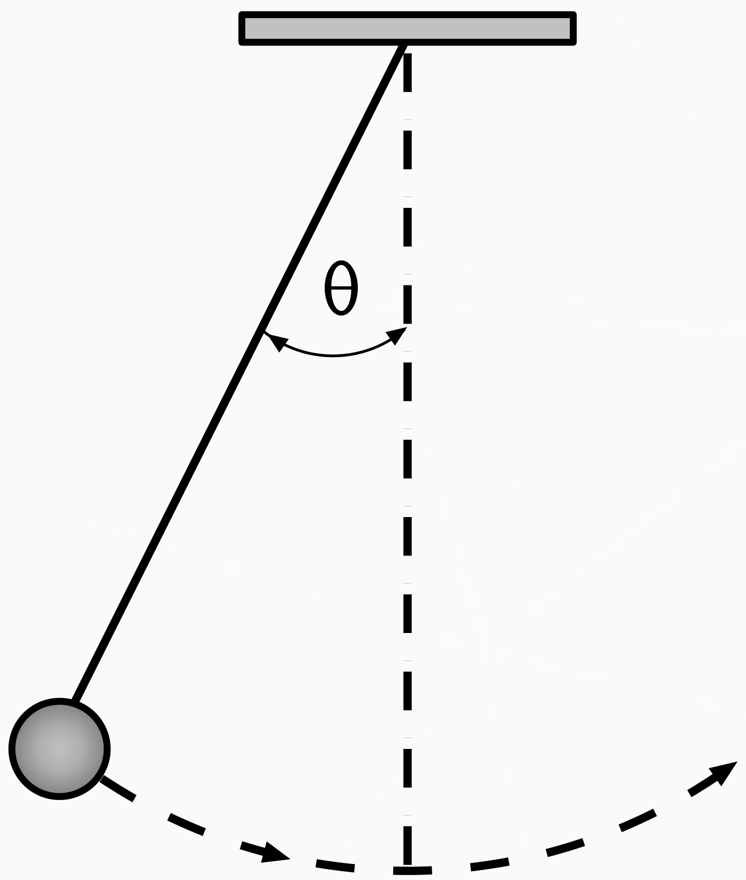
Force + Damping + Spring–Mass system



DUFFING OSCILLATOR

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} = -\frac{g}{l}sin\theta$$

$$sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots$$



For simple case we assume for small θ and sin(θ) ~ θ.
But if we add a non linear term θ then ...

Duffing Oscillator

- **Non-autonomous differential equation**

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma cos(\omega t)$$

- δ = damping coefficient,
- α= linear stiffness coefficient,
- β = nonlinearity in the restoring force,
- Y = amplitude of the periodic driving force,
- ω = angular frequency of the periodic driving force

- Interestingly, the special case with no forcing,

$$\dot{x} = y \qquad \dot{y} = -\alpha x - \beta x^3$$

- Does it mean, energy will be conserved?

- can be integrated by quadratures. Differentiating and plugging in gives

$$\ddot{x} = \dot{y} = -\alpha x - \beta x^3 \Rightarrow \ddot{x}\dot{x} + \alpha \dot{x} x + \beta \dot{x} x^3 = 0$$

- But this can be written as

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2+\alpha\frac{1}{2}x^2+\beta\frac{1}{4}x^4\right)=0$$

$$\rightarrow \text{let } (\dots) = h$$

$$\dot{x}^2 = 2h-\alpha x^2-\beta\frac{1}{2}x^4$$

$$\frac{dx}{dt}=\sqrt{2h-\alpha x^2-\beta\frac{1}{2}x^4} \qquad t=\int dt=\int\frac{dx}{\sqrt{2h-\alpha x^2-\beta\frac{1}{2}x^4}}$$

- Note that the invariant of motion satisfies

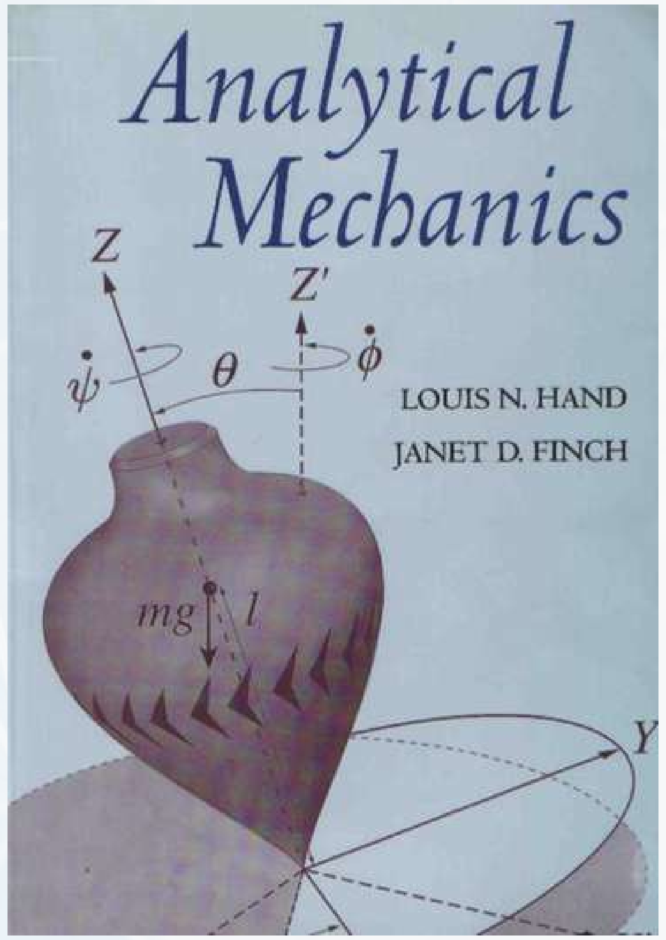
$$\frac{\partial h}{\partial \dot{x}} = \frac{\partial h}{\partial y} = \dot{x}$$

LOOKING SO FAMILIAR :)

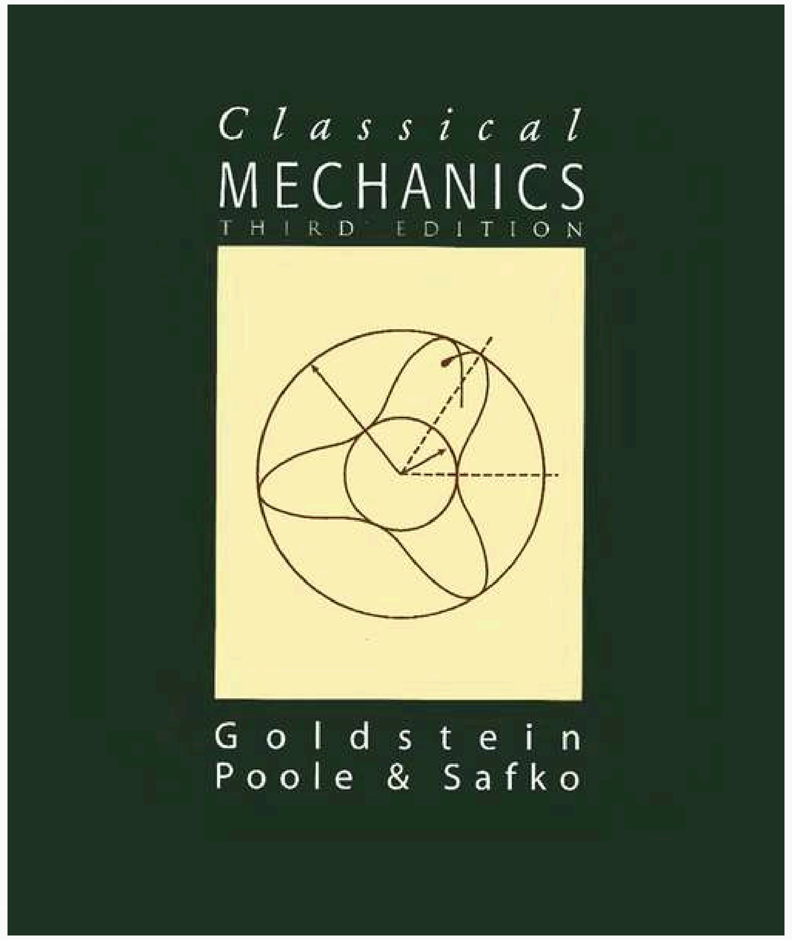
$$\frac{\partial h}{\partial x} = \alpha x + \beta x^3 = -\dot{y}$$

- So the equations of the Duffing oscillator are given by the Hamiltonian system !!!!!!

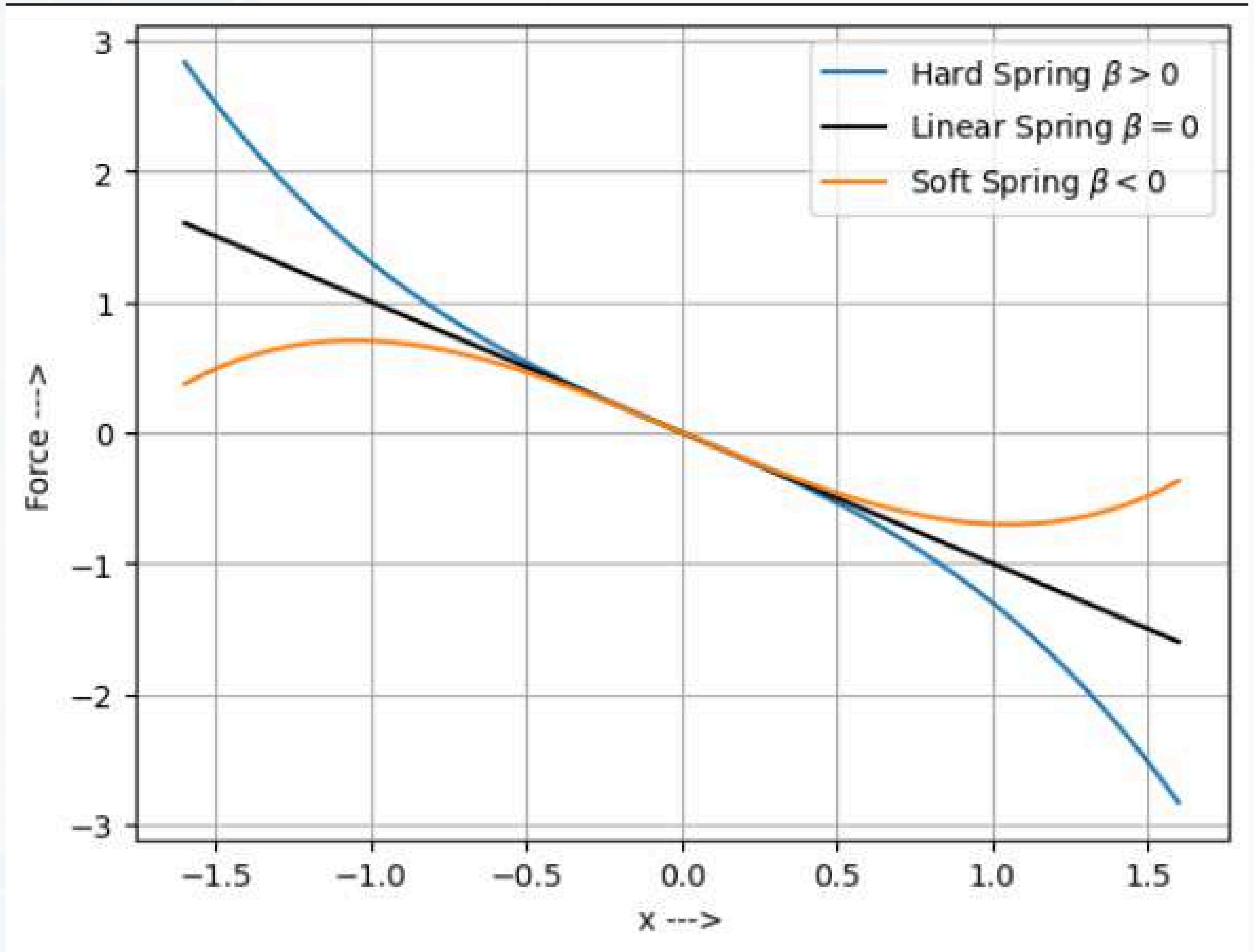
- Now everyone knows what to do :)
- If not, below are the references:



ChatGPT



Duffing Oscillator



$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma cos(\omega t)$$

- For no damping and zero external force case, and
- For β>0, can be interpreted as a forced oscillator with spring restoring force:

$$F = -\alpha x - \beta x^3$$

Frequency Response

- In the case when there was no anharmonicity (x^3 term), the frequency response was

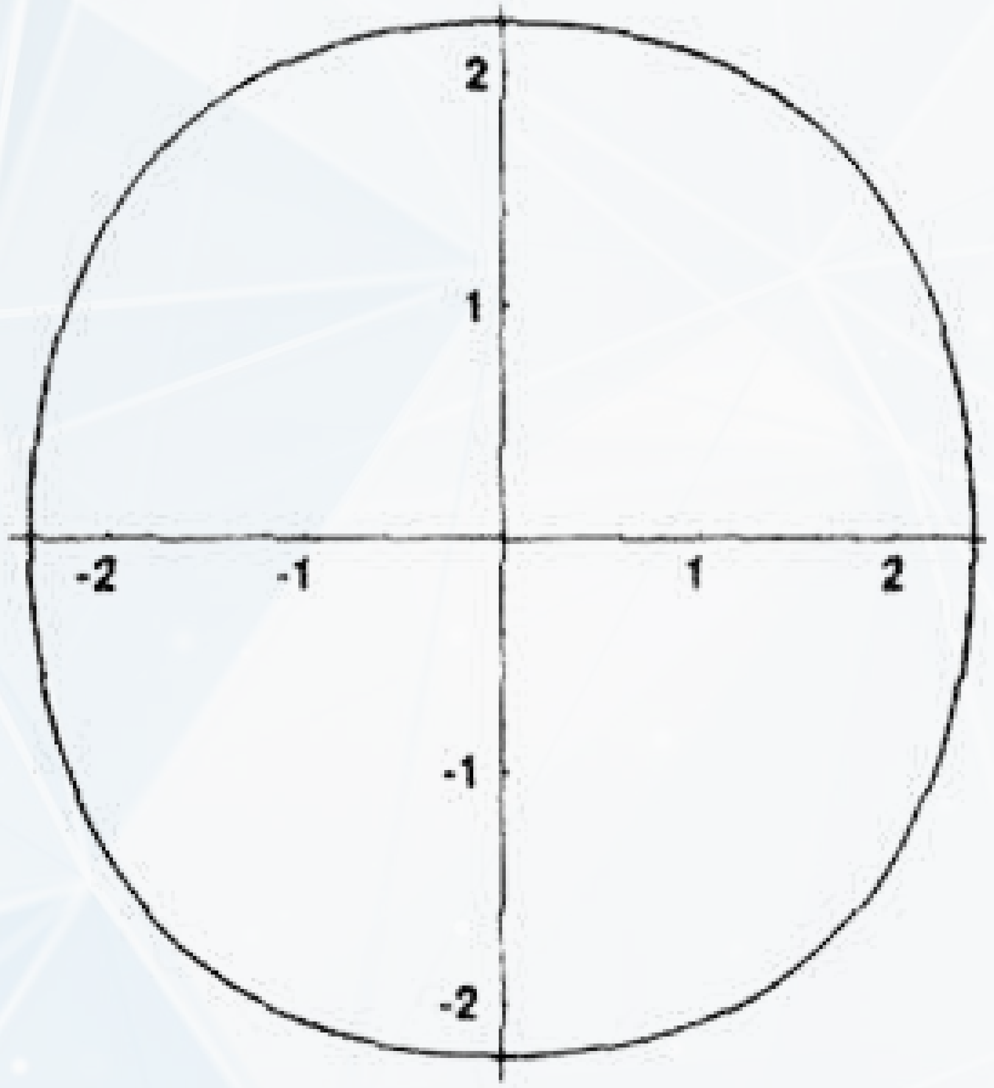
$$A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\zeta\omega_0\omega)^2}}$$

- After adding non-linearity (taking damping ~ 0),

$$\ddot{x} + 2\zeta\dot{x} + x + \beta x^3 = \gamma cos(\omega t)$$

Harmonic Analysis

- Without non-linear term, we would get a perfect circle.



- Taking β=0.1, Y=1, ω=1 (resonant frequency).
- Introduction of non-linear term resulted in distorted circle.
- We can assume a periodic solution and expand it as a Fourier series. Using symmetry of the equation;

$$x(t) = \sum_{n=1,3,5,\ldots} A_n cos(n\omega t)$$

Frequency Response

```python
import numpy as np
from scipy.optimize import fsolve

# Define parameters
omega = 1.0  # Angular frequency
f = 1.0      # External force amplitude
epsilon = 0.1 # Small nonlinearity parameter

# Define the equations for A1 and A3
def equations(vars):
    A1, A3 = vars
    eq1 = (1 - omega**2) * A1 + (3/4) * epsilon * A1**3 - f
    eq2 = (1 - 9 * omega**2) * A3 + (1/4) * epsilon * A1**3
    return [eq1, eq2]

# Initial guess for A1 and A3
initial_guess = [1.78, 0.1]

# Solve the system of equations
solution = fsolve(equations, initial_guess)
A1, A3 = solution

# Output the solution
print(f"A1 = {A1}")
print(f"A3 = {A3}")
```

✓ 0.0s

```
A1 = 2.371262202993375
A3 = 0.041666666666666664
```

Harmonic Analysis

- By numerically analysis, we found that

$$A_1 \approx 2.37126$$
$$A_3 \approx 0.04166$$

- So, with good approximation, we can write

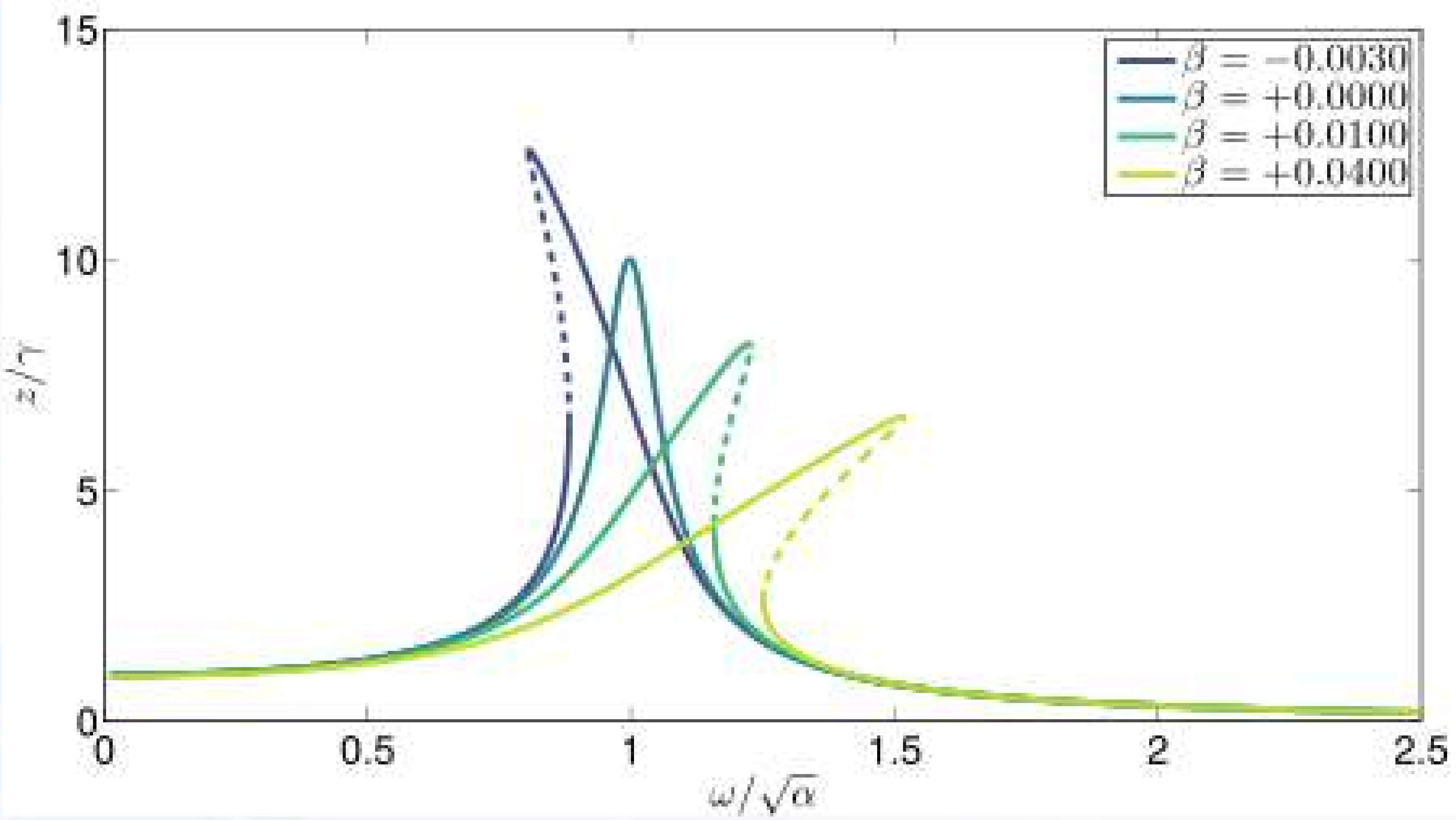
$$x(t) = A_1 cos(\omega t) + A_3 cos(3\omega t)$$

- Putting in the differential equation and equating different coefficients:

$$(1-\omega^2)A_1 + \frac{3}{4}\beta A_1^3 = \gamma$$
$$(1-9\omega^2)A_3 + \frac{1}{4}\beta A_1^3 = 0$$

Harmonic Analysis

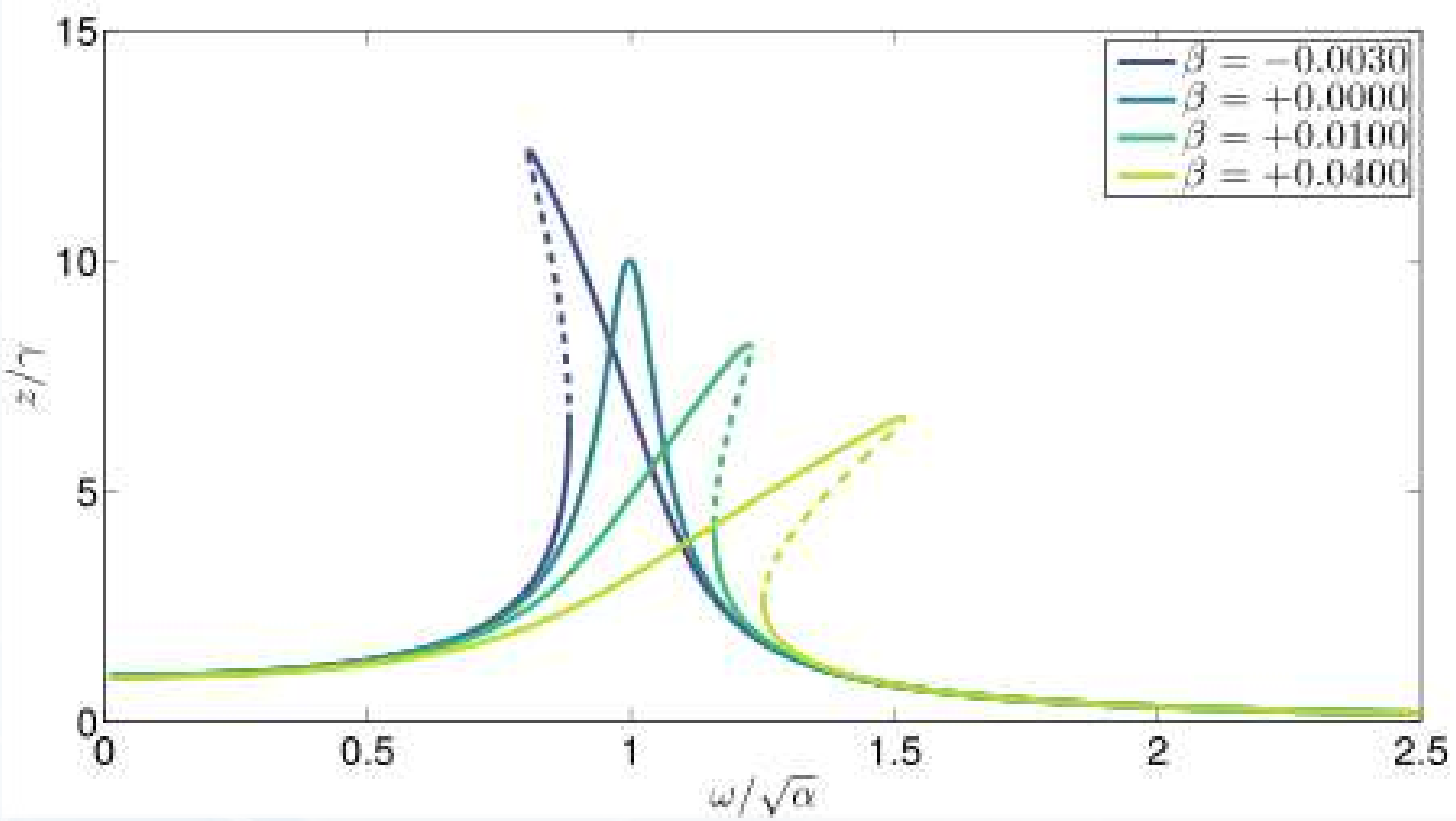
- For given frequency and force amplitude, we can solve these equations.



- The cubic term here suggests the possibility of multiple co-existing solutions for a given frequency of external force.

Duffing Oscillator

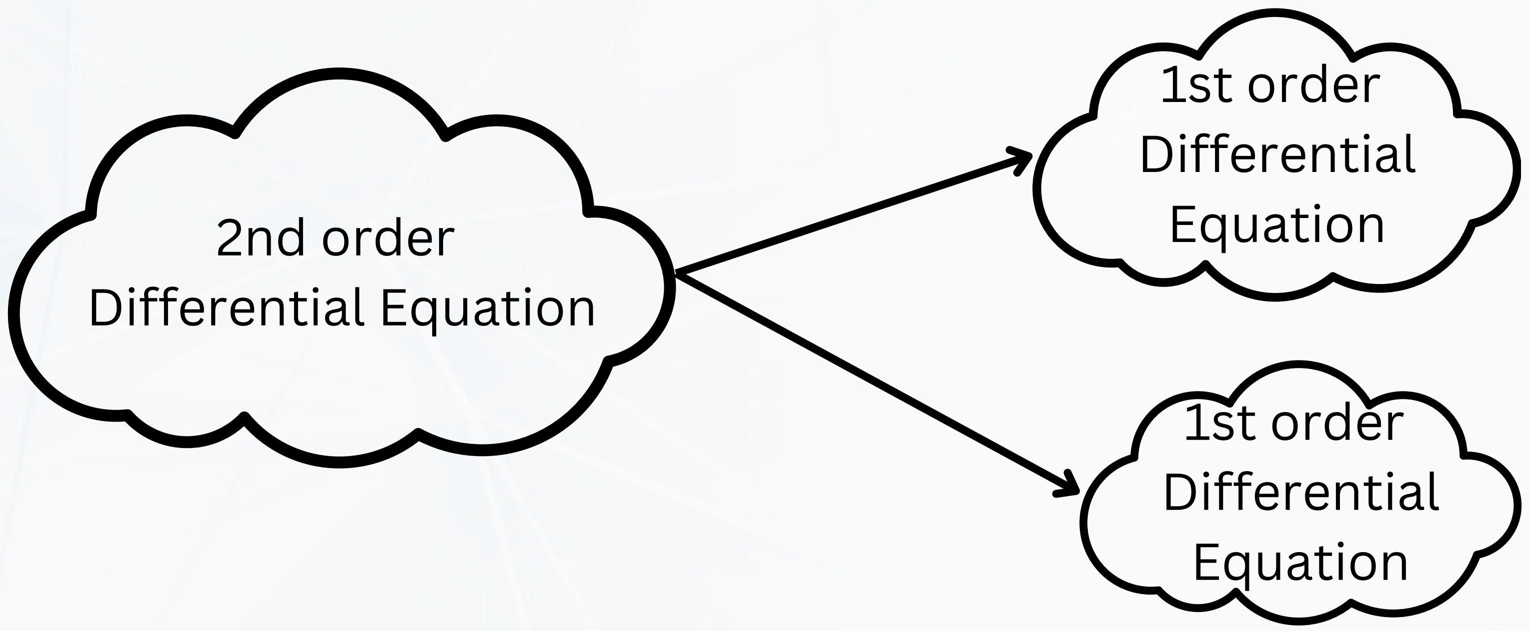
- For hard spring, the effective restoring force increases with increasing displacement, so it’s peak will shift towards higher frequency.



- For soft spring, the peak will shift towards lower frequencies.

Duffing Oscillator

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma cos(\omega t)$$



$$\dot{x} = y \qquad \dot{y} = -\alpha x - \beta x^3 - \delta y + \gamma cos(\omega t)$$

Simple case ($\gamma = 0$)

- The fixed points of the set of coupled differential equation is given by

$$\dot{x} = 0, \dot{y} = 0$$

$$\dot{x} = y = 0 \qquad \dot{y} = -x(\alpha + \beta x^2) = 0$$

$$X_{fixed} = \{0, \sqrt{-\alpha/\beta}, -\sqrt{-\alpha/\beta}\}$$

$$(X_{fixed}, Y_{fixed}) \in \{(0,0), (\sqrt{-\alpha/\beta}, 0), (-\sqrt{-\alpha/\beta}, 0)\}$$

Stability Analysis

- Analysis of the stability of the fixed points can be point by linearizing the equations.
- The equations will be written as:

$$\dot{x} = y$$

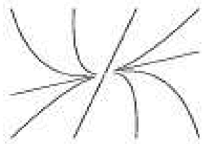
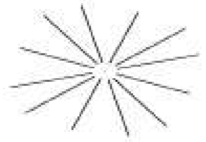
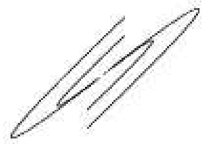
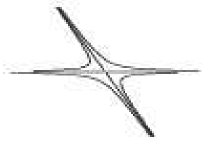
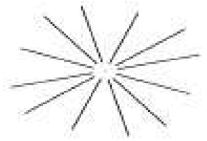
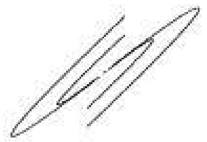
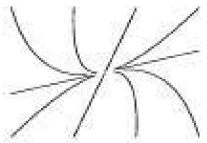
$$\dot{y} = -\alpha x - \beta x^3 - \delta y$$

$$\ddot{y} = -(\alpha + 3\beta x^2)\dot{x} - \delta\dot{y}$$

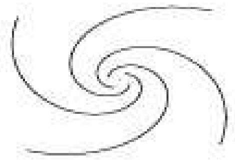
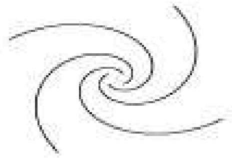
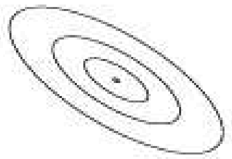
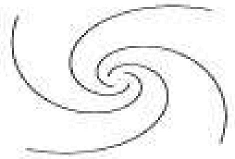
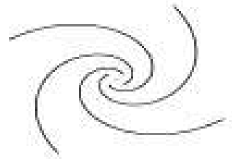
- Writing as the matrix form and analysing the stability

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha - \beta x^2 & -\delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Real eigenvalues λ and μ

$0 < \lambda < \mu$		Unstable node
$0 < \lambda = \mu$, A diagonalizable		Degenerate unstable node
$0 < \lambda = \mu$, A non-diagonalizable		Unstable node
$\lambda < 0 < \mu$		Saddle
$\lambda = \mu < 0$, A diagonalizable		Degenerate stable node
$\lambda = \mu < 0$, A non-diagonalizable		Stable node
$\mu < \lambda < 0$		Stable node

Complex eigenvalues ($\alpha \pm i\omega$, $\omega \neq 0$)

$\alpha > 0$	 ou 	Unstable spiral
$\alpha = 0$		Center
$\alpha < 0$	 ou 	Stable spiral

Stability check at (0,0)

$$\begin{vmatrix} 0-\lambda & 1 \\ -\alpha & -\delta-\lambda \end{vmatrix} = \lambda(\lambda+\delta)+\alpha = \lambda^2+\lambda\delta+\alpha = 0$$

$$\lambda_{(0,0)\pm} = \frac{1}{2}\left(-\delta \pm \sqrt{\delta^2-4\alpha}\right)$$

$$\text{case-1} : \alpha < 0 \implies \lambda_1 \le 0, \lambda_2 \ge 0.$$

So it's saddle.

$$\text{case-2} : \Delta < 0 \implies \Re(\lambda) \le 0.$$

So it is a focus and stable.

Stability Check at ($\pm\sqrt{-\alpha/\beta}$, 0)

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\alpha-\beta x^2 & -\delta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\begin{vmatrix}0-\lambda & 1\\ -\alpha(1\pm\sqrt{\frac{-\alpha}{\beta}}) & -\delta-\lambda\end{vmatrix} = \lambda(\lambda+\delta)+\alpha(1\pm\sqrt{\frac{-\alpha}{\beta}}) = \lambda^2+\lambda\delta+\alpha(1\pm\sqrt{\frac{-\alpha}{\beta}}) = 0$$

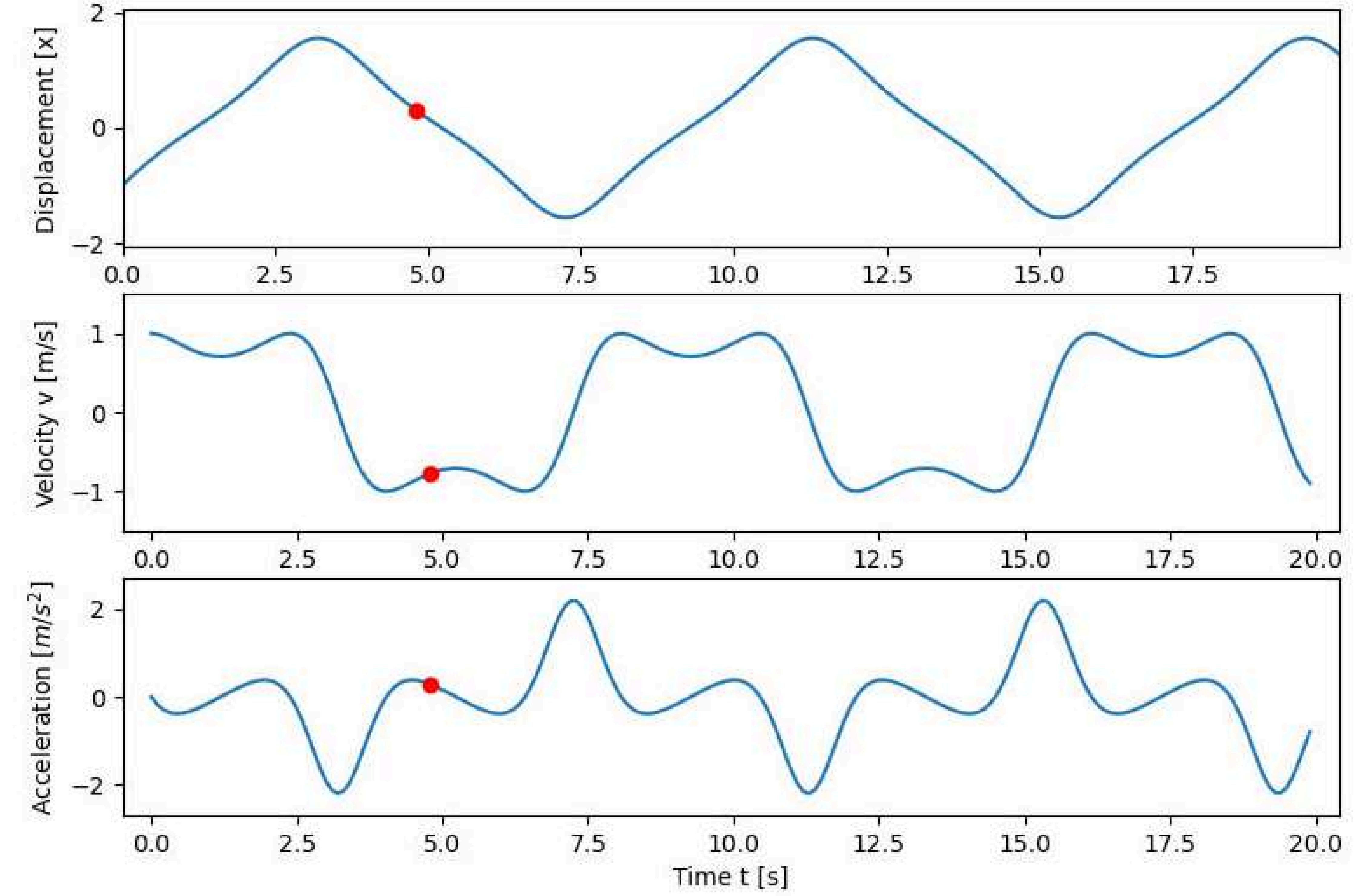
Duffing Oscillator

$$\lambda_{(\pm\sqrt{-\alpha/\beta},0)} = \frac{1}{2}\left(-\delta \pm \sqrt{\delta^2 - 4\alpha(1 \pm \sqrt{\frac{-\alpha}{\beta}})}\right)$$

Similarly You can check stablities here for different cases!!

Simulations

Free Motion (Stable Equilibrium)



Initial conditions:

x = -1, v = 1

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0$$

$$\gamma = 0$$

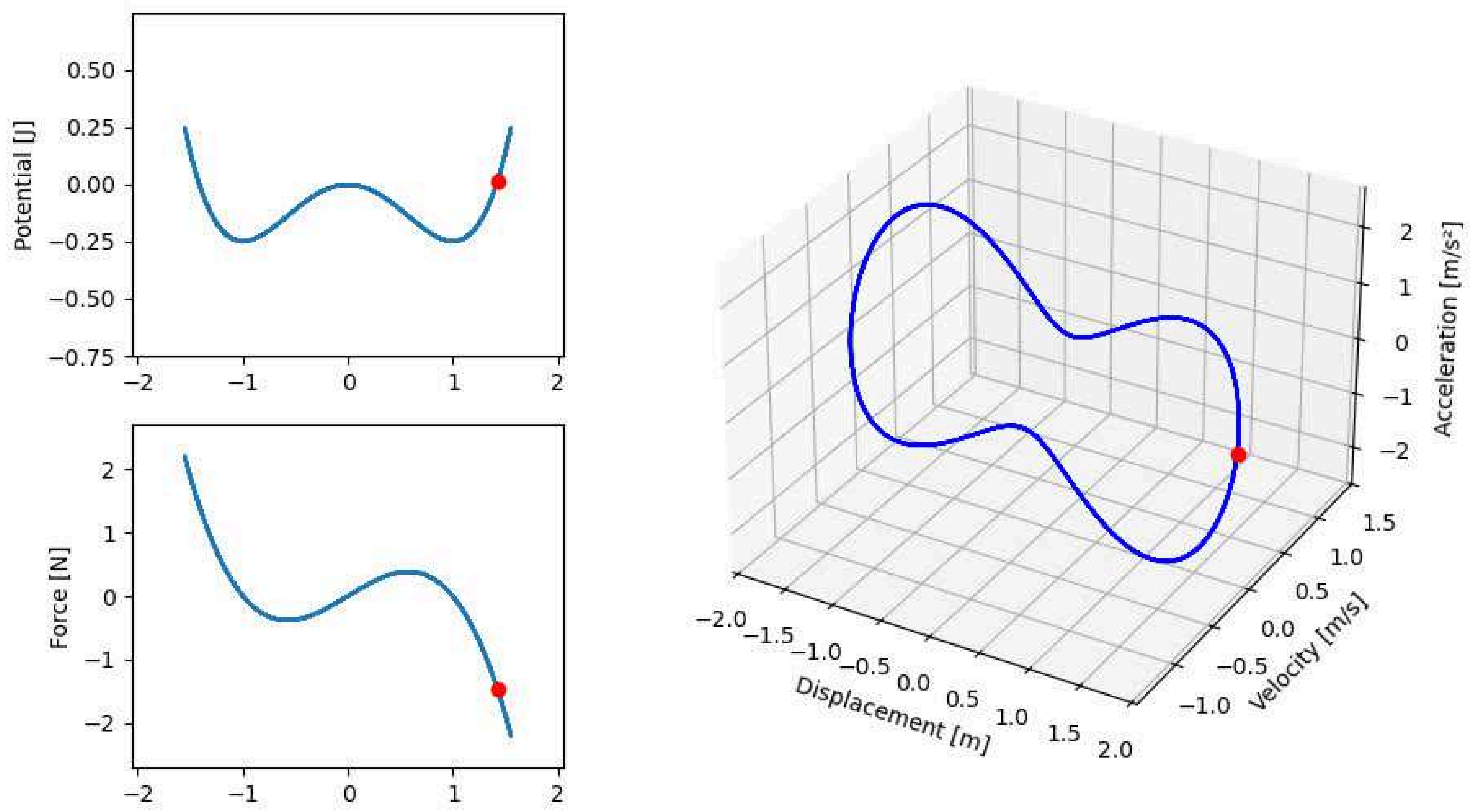
$$\omega = 0$$

$$m = 1$$

Time series

Simulations

Free Motion (Stable Equilibrium)



Initial conditions:

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$$\delta = 0$$

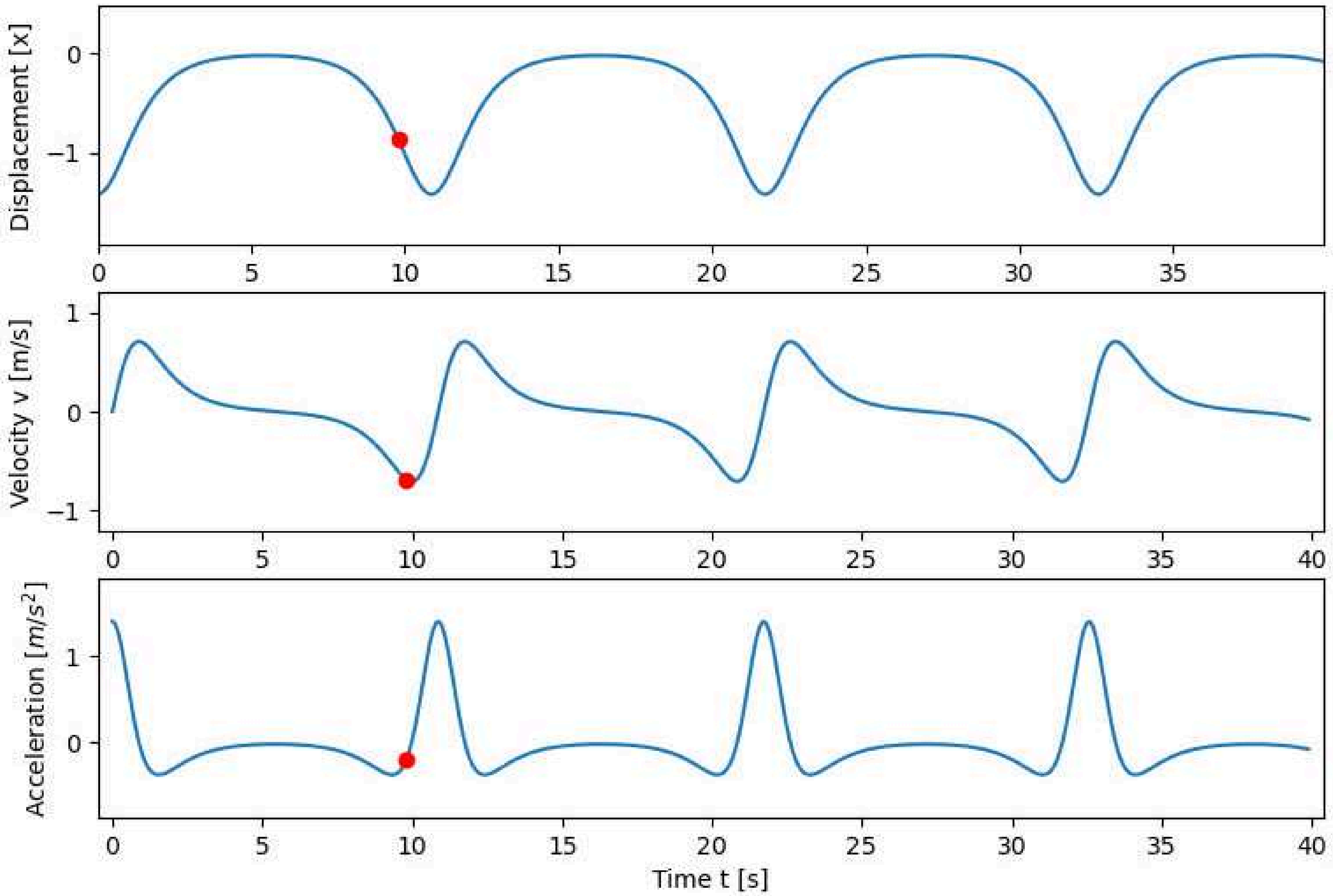
$$\gamma = 0$$

$$\omega = 0$$

$$m = 1$$

PE and Force vs displacement
and
Phase diagram

Free Motion (Unstable Equilibrium)



Initial conditions:

x = -1.414, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0$$

$$\gamma = 0$$

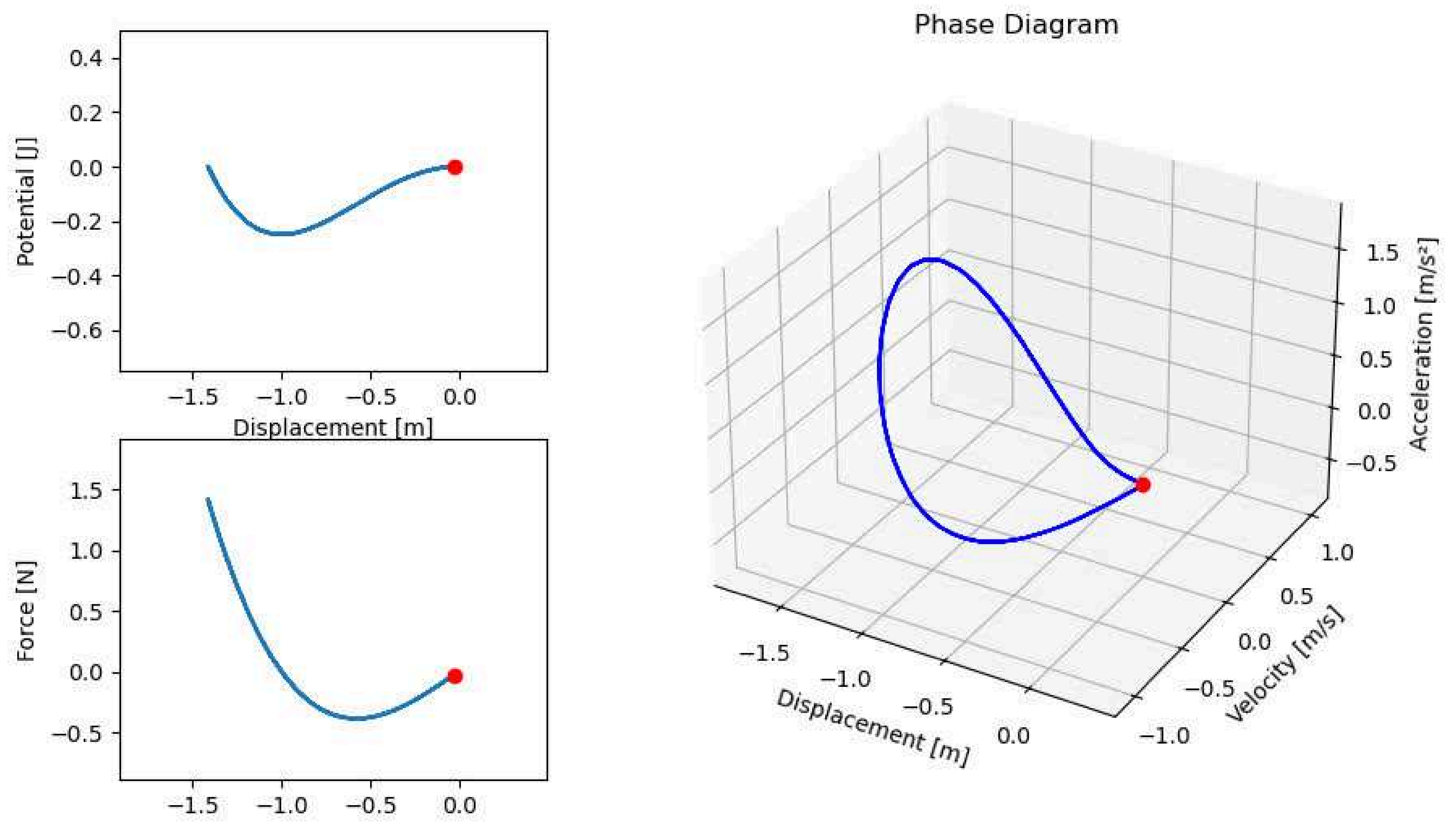
$$\omega = 0$$

$$m = 1$$

Time series

Simulations

Free Motion (Unstable Equilibrium)



Initial conditions:
x = -1.414, v = 0

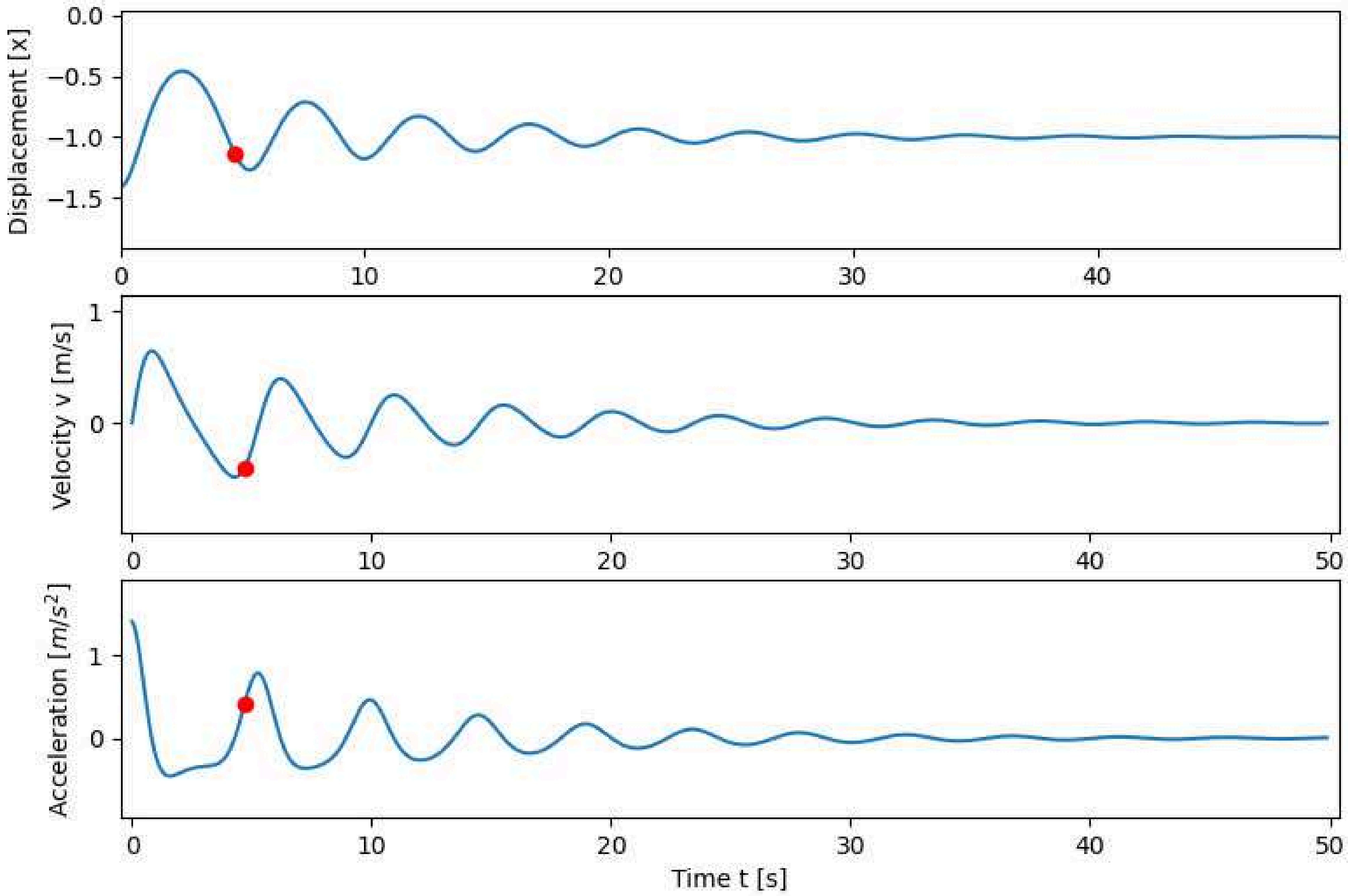
Parameter values:

$$\alpha = -1$$
$$\beta = 1$$
$$\delta = 0$$
$$\gamma = 0$$
$$\omega = 0$$
$$m = 1$$

PE and Force vs displacement and Phase diagram

Simulations

Damped Motion



Initial conditions:

x=-1.414, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.2$$

$$\gamma = 0$$

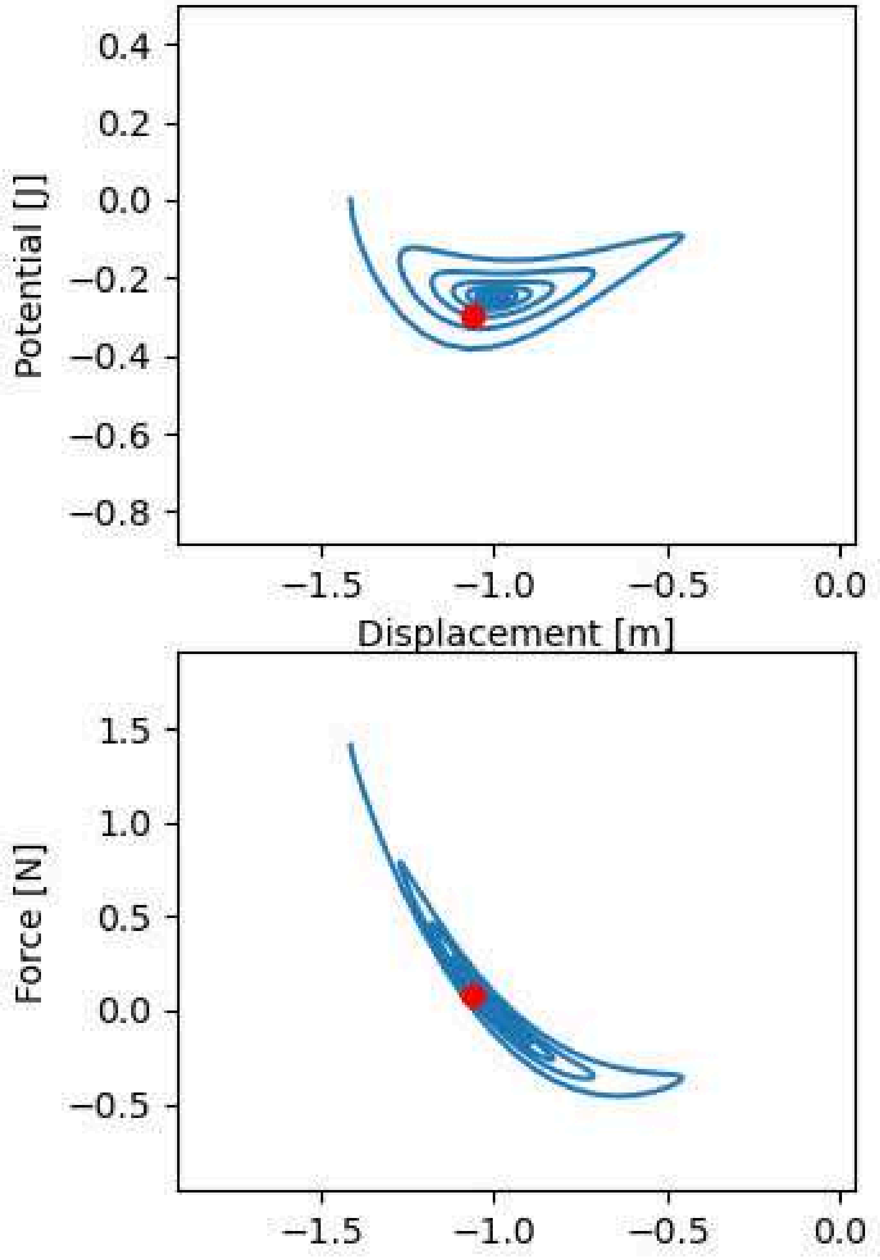
$$\omega = 0$$

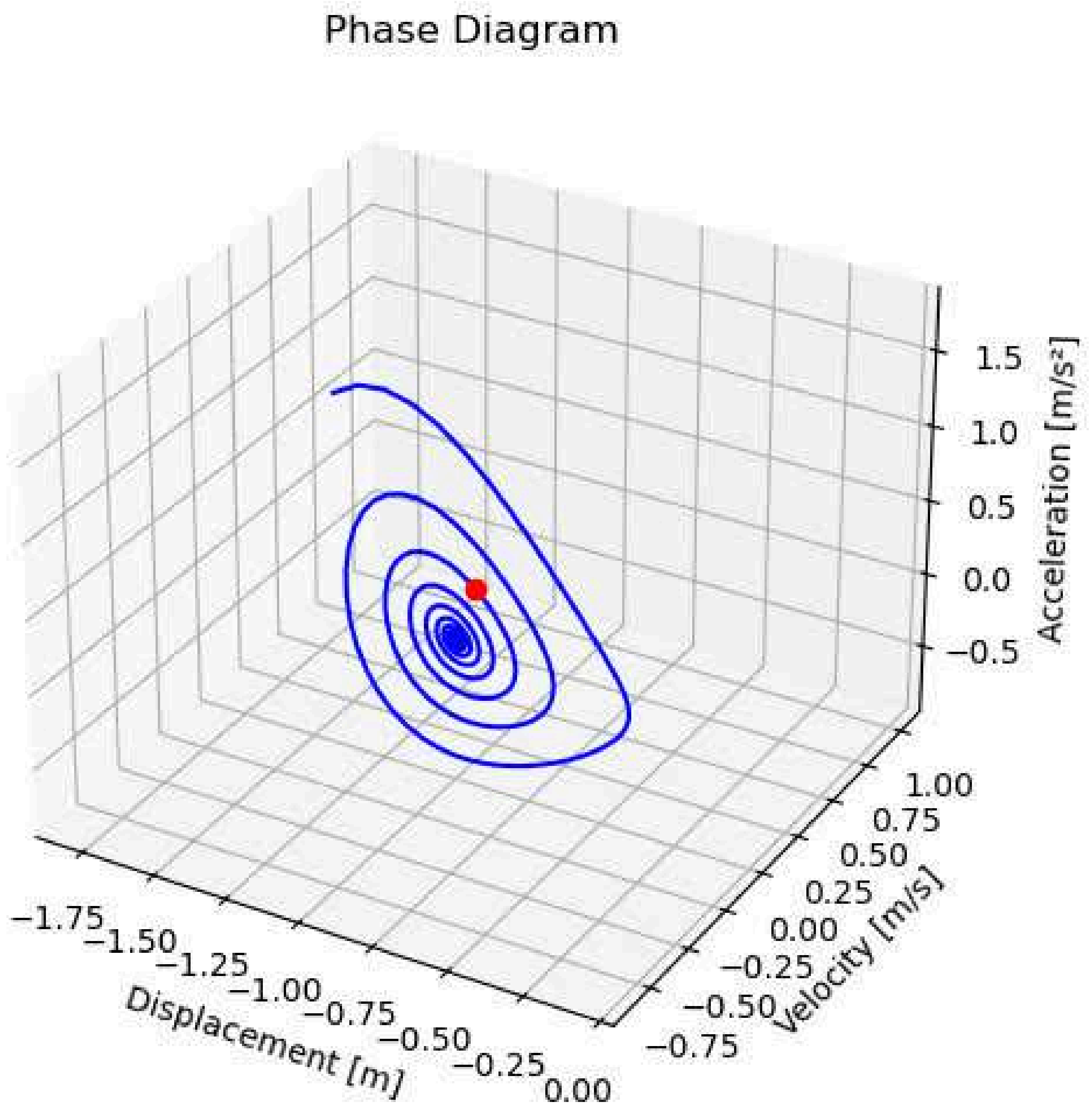
$$m = 1$$

Time series

Simulations

Damped Motion





Initial conditions:
x=-1.414, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.2$$

$$\gamma = 0$$

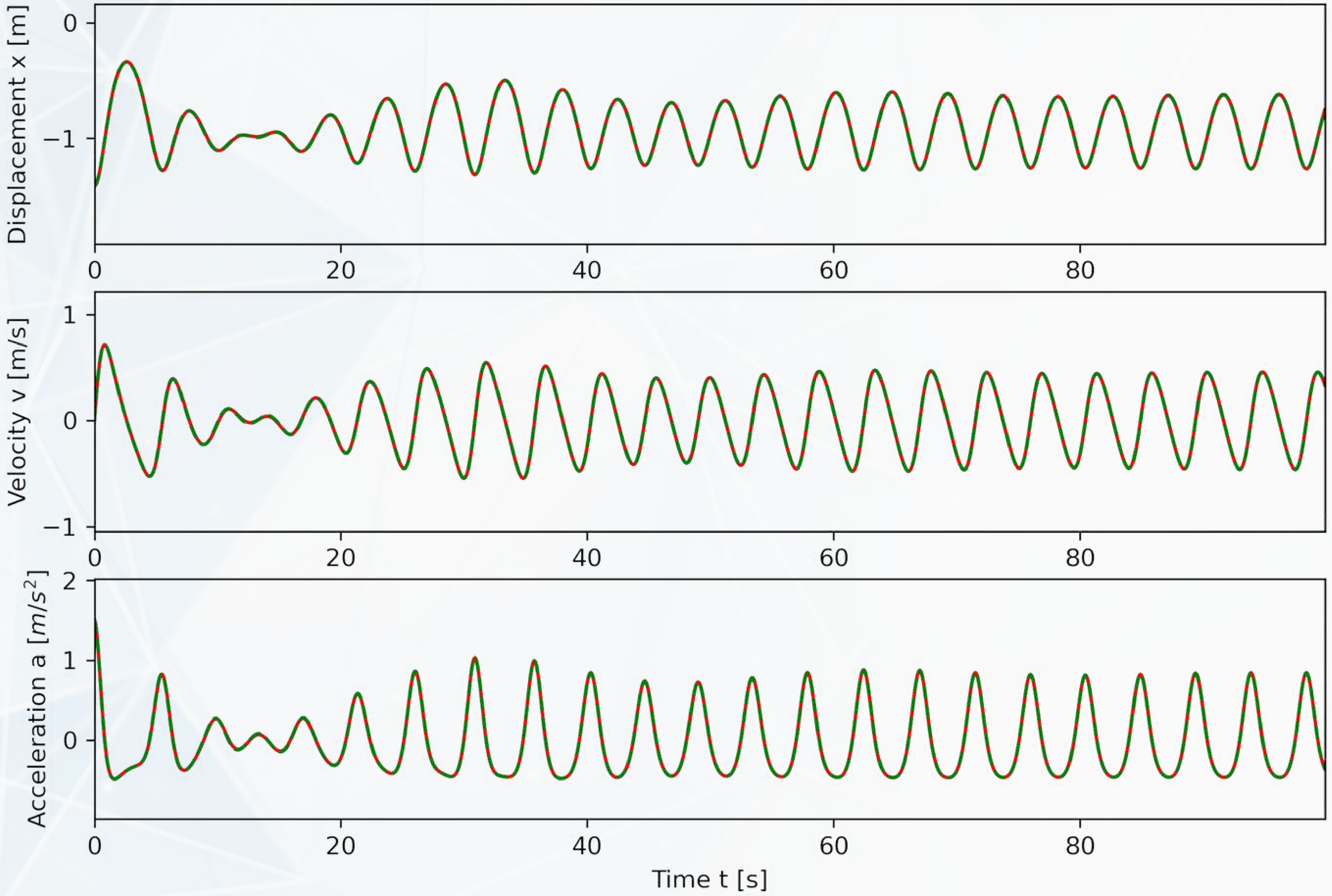
$$\omega = 0$$

$$m = 1$$

PE and Force vs displacement and Phase diagram

Simulations

Forced Damped Motion



Initial conditions:

x = -1.414, v = 0

x = -1.413, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

$$\gamma = 0.1$$

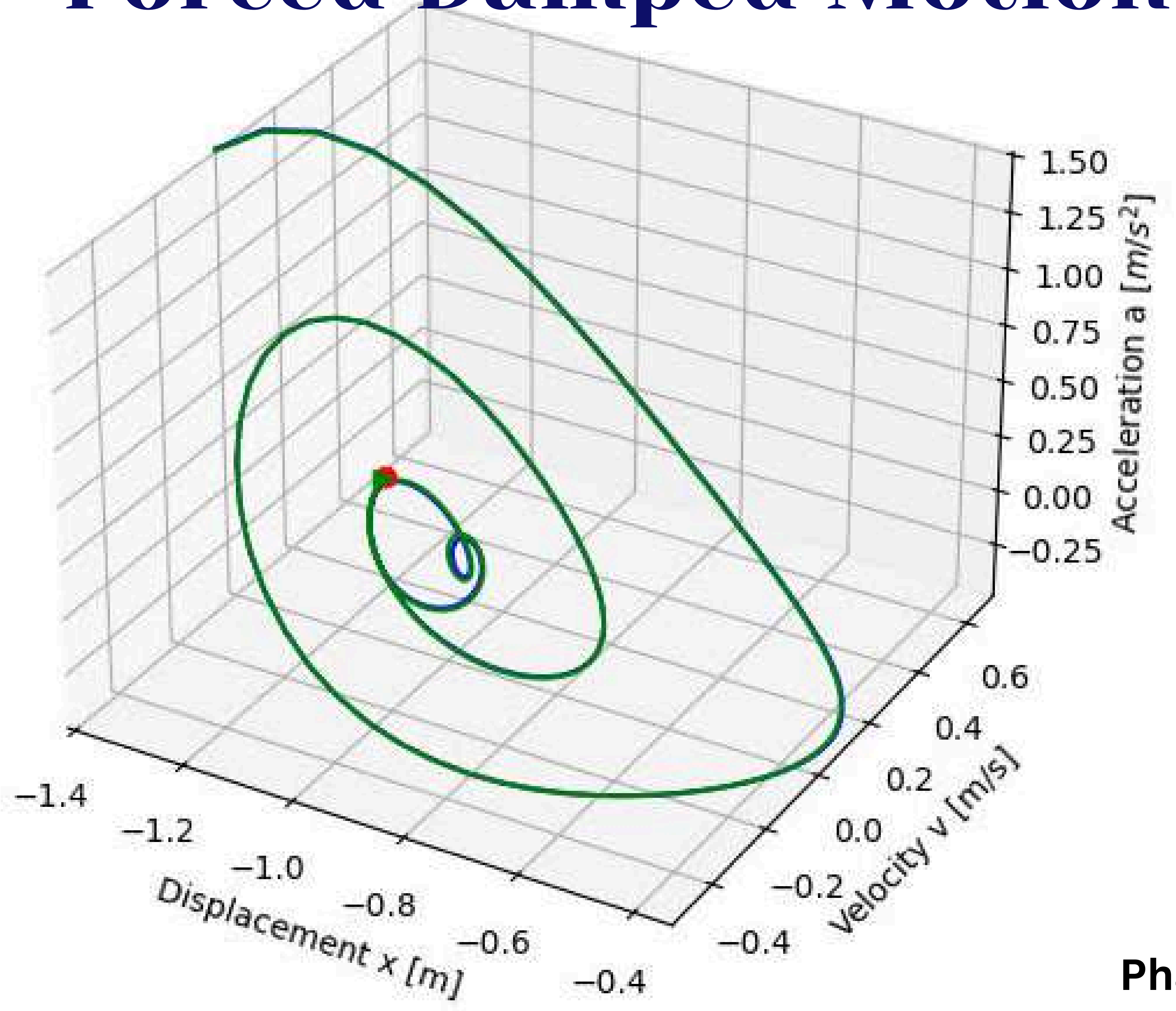
$$\omega = 1.4$$

$$m = 1$$

Time series

Simulations

Forced Damped Motion



Phase diagram

Initial conditions:

x = -1.414, v = 0

x = -1.413, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

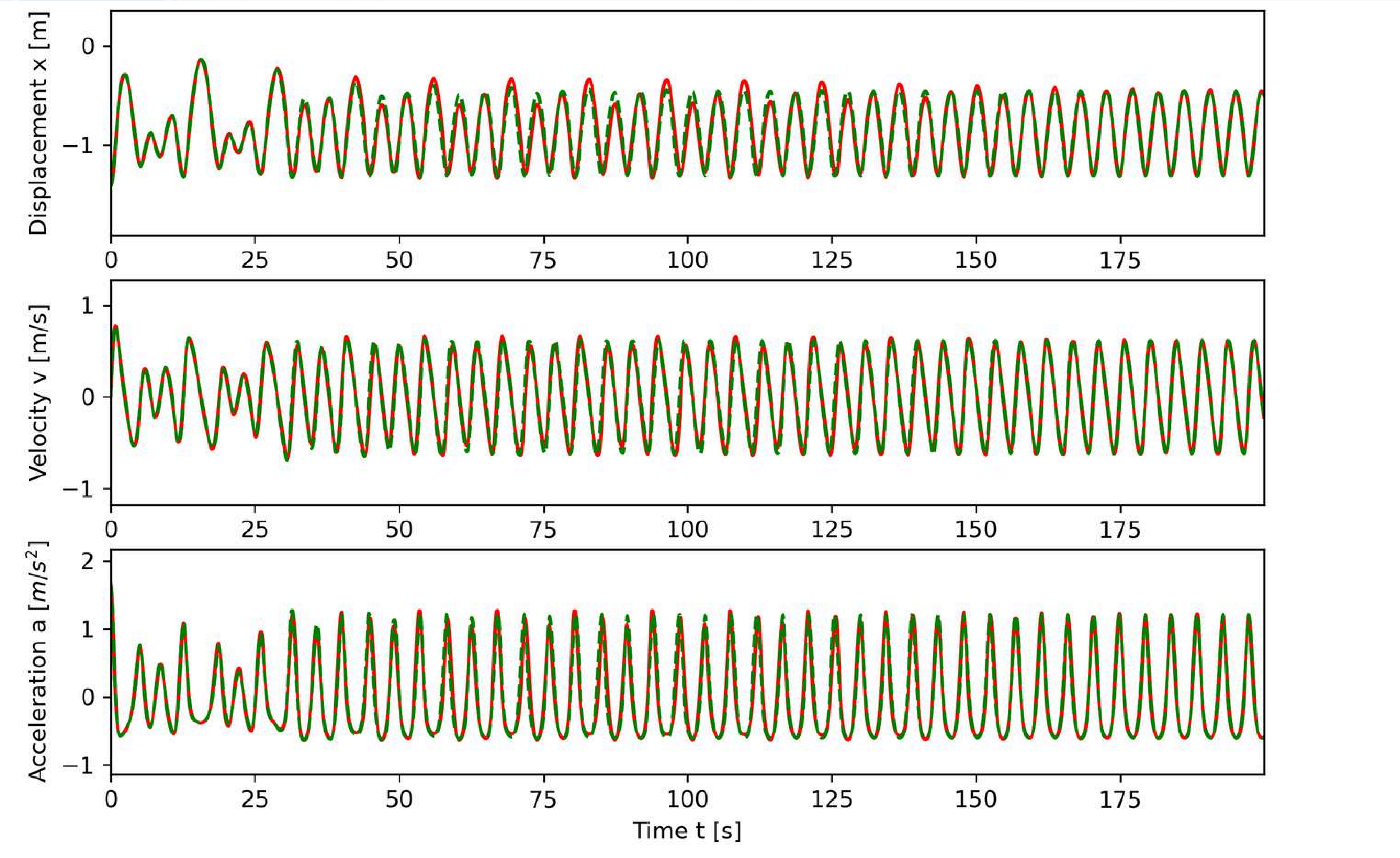
$$\gamma = 0.1$$

$$\omega = 1.4$$

$$m = 1$$

Simulations

Forced Damped Motion



Initial conditions:

x = -1.414, v = 0

x = -1.413, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

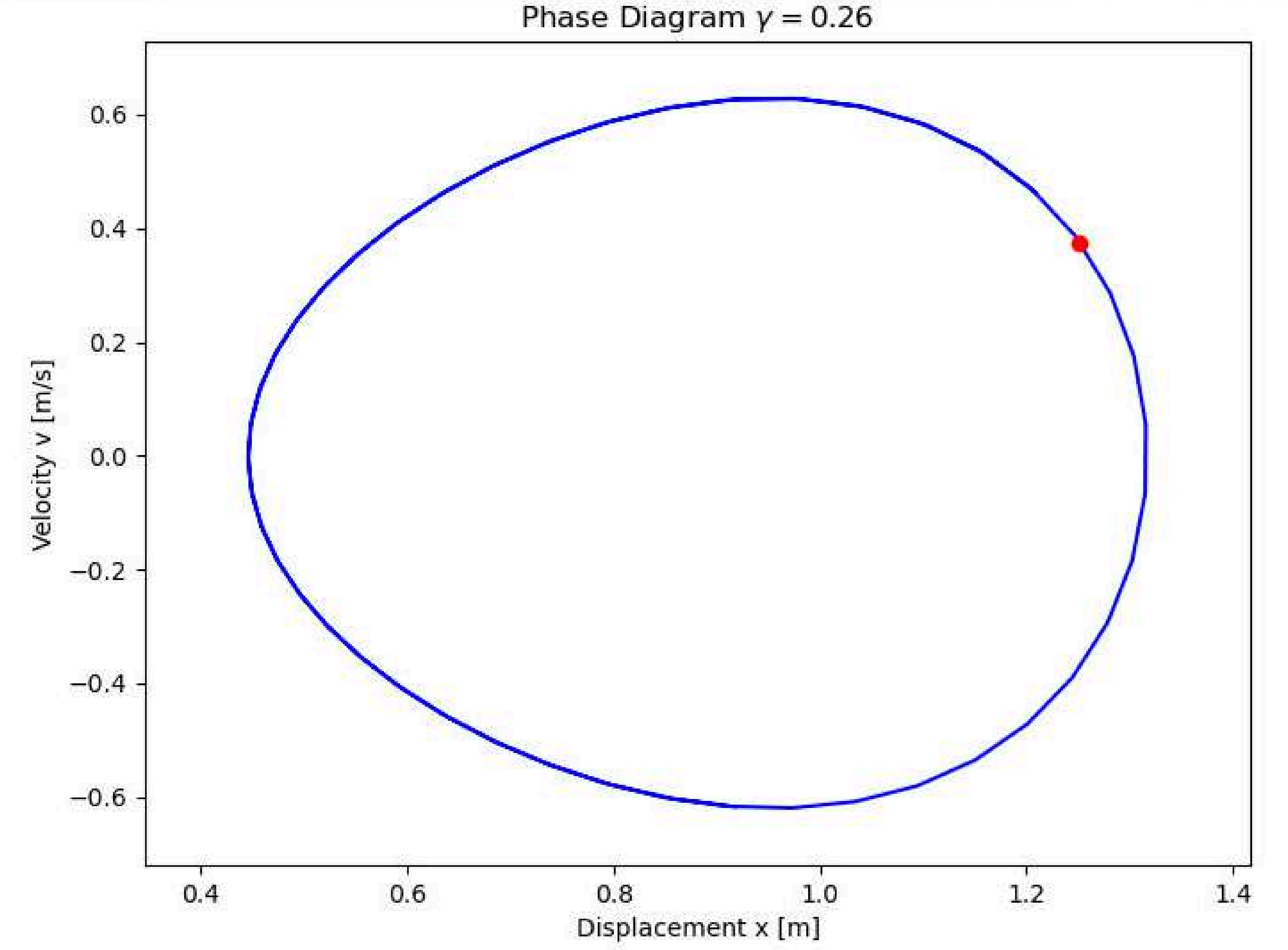
$$\gamma = 0.26$$

$$\omega = 1.4$$

$$m = 1$$

Simulations

Forced Damped Motion



Initial conditions:

x = -1.414, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

$$\gamma = 0.26$$

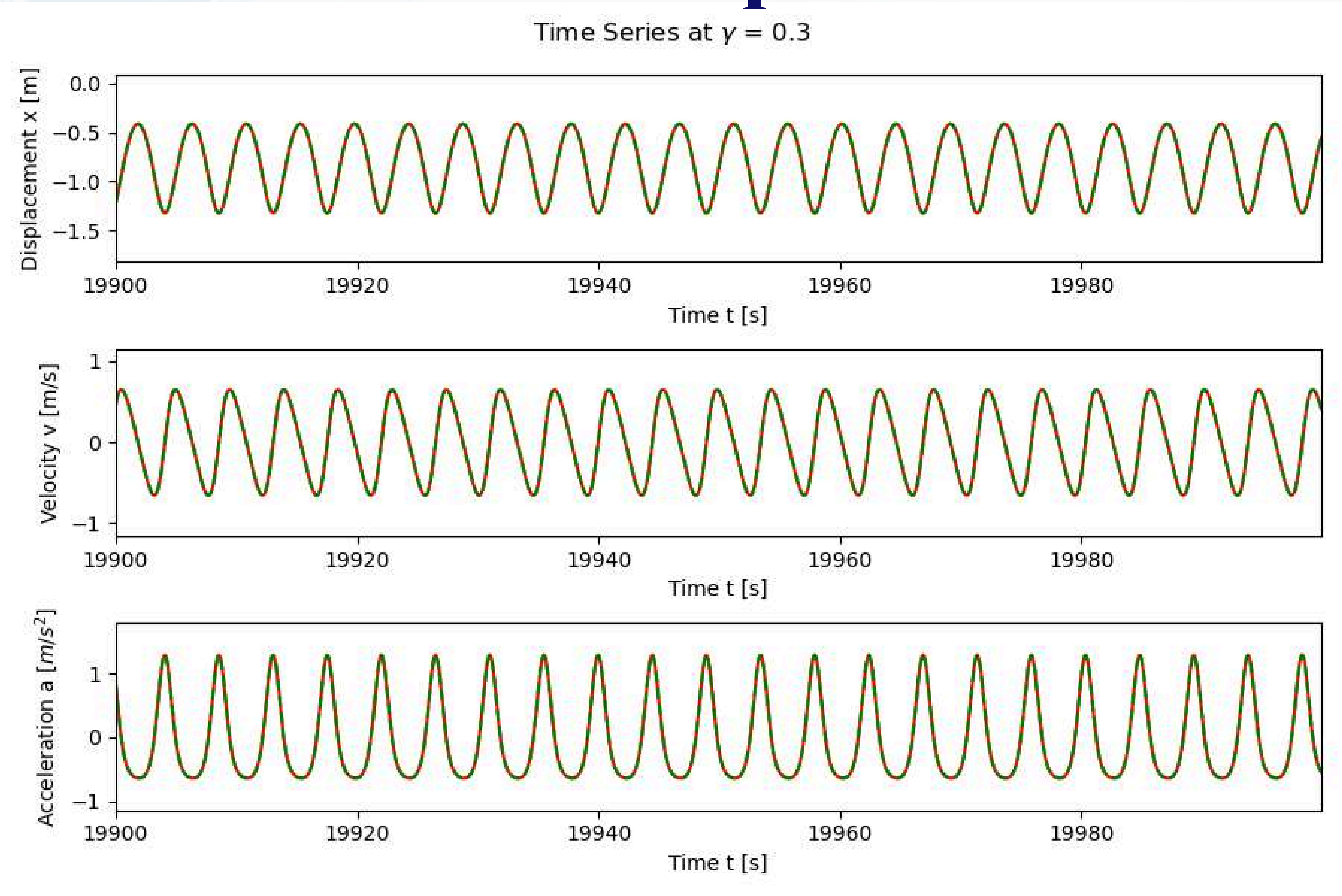
$$\omega = 1.4$$

$$m = 1$$

Phase diagram

Simulations

Forced Damped Motion



Initial conditions:

x = -1.414, v = 0

x = -1.413, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

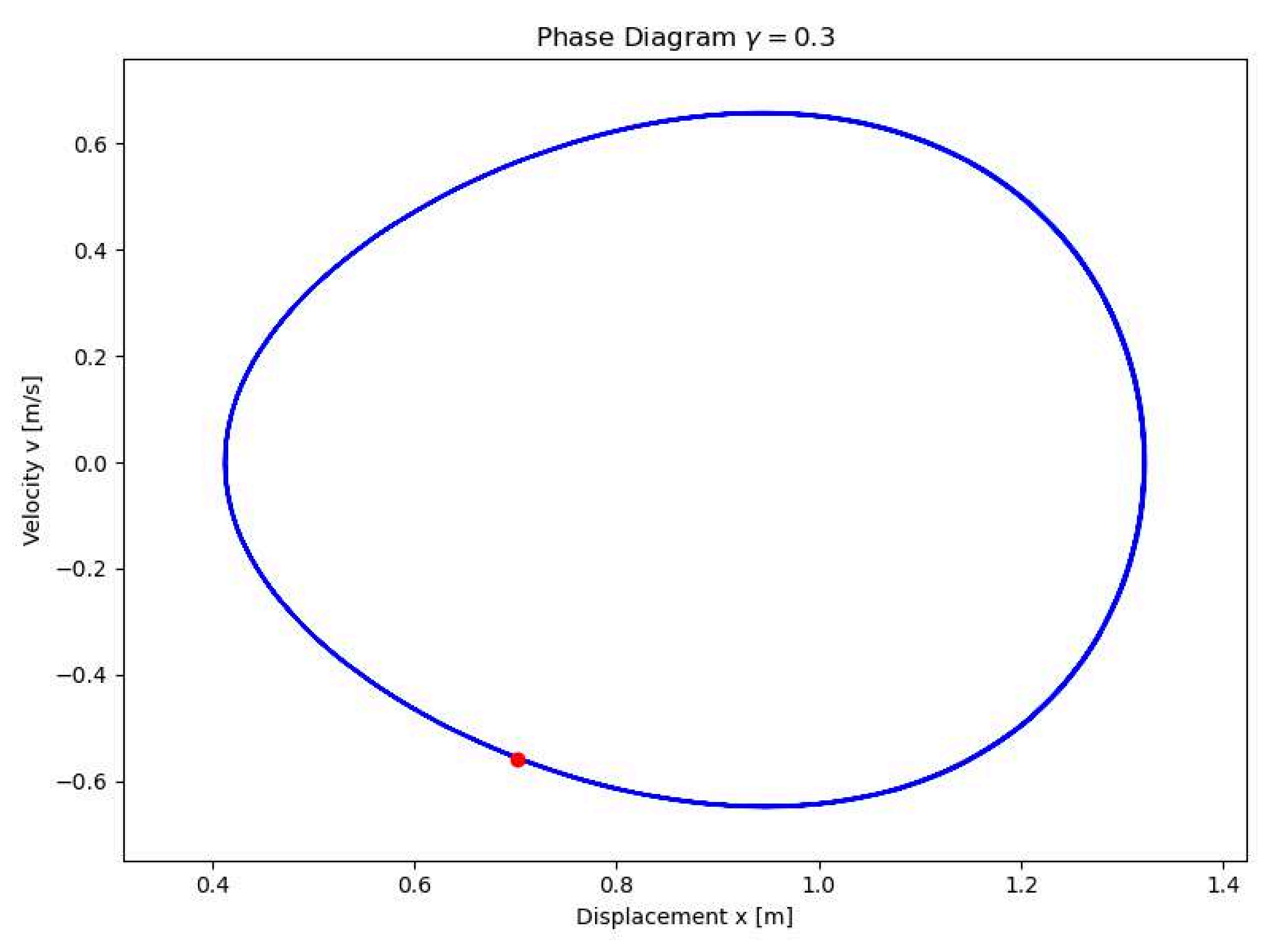
$$\gamma = 0.30$$

$$\omega = 1.4$$

$$m = 1$$

Simulations

Forced Damped Motion



Initial conditions:

x = -1.414, v = 0

x = -1.413, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

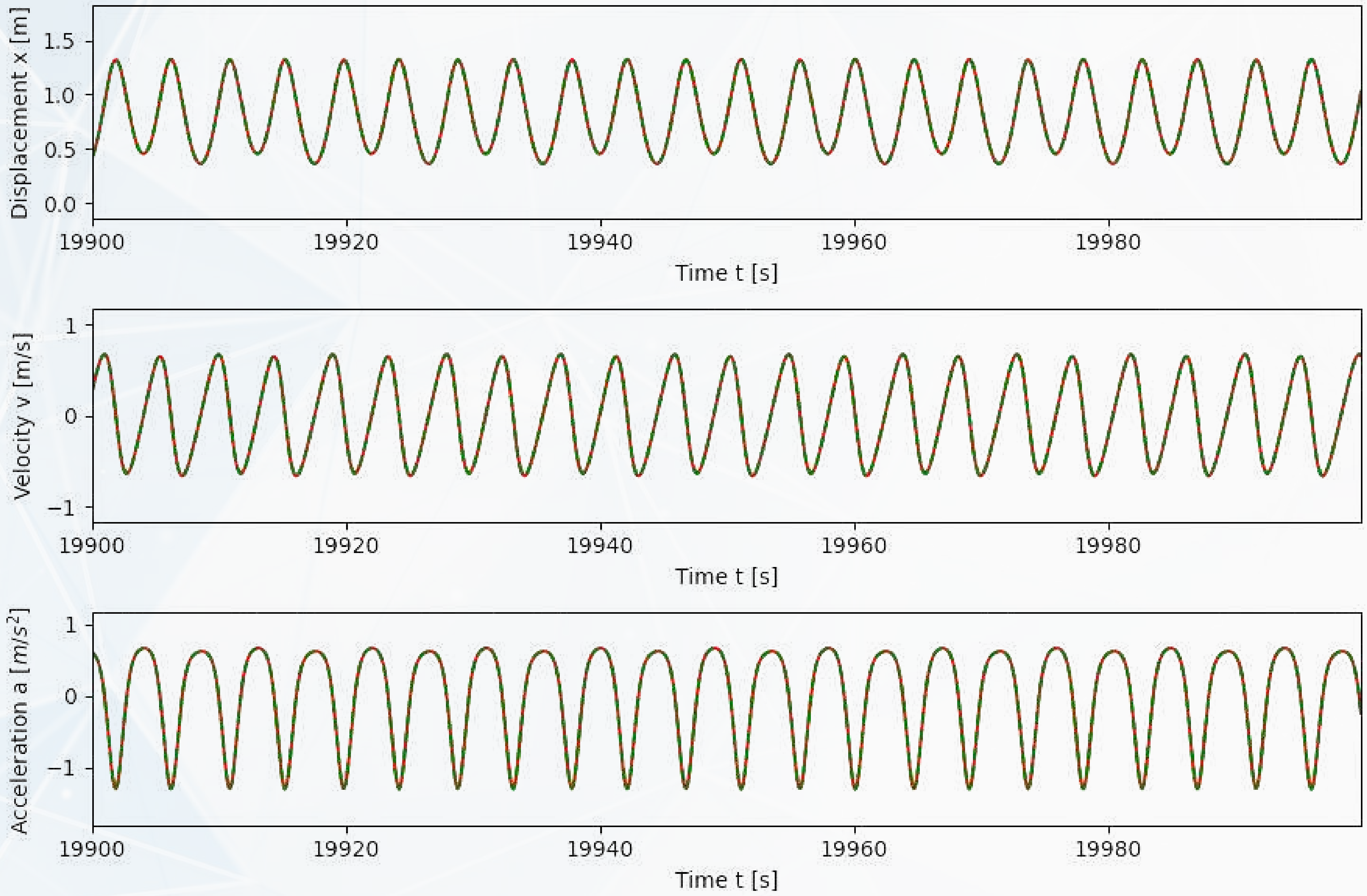
$$\gamma = 0.30$$

$$\omega = 1.4$$

$$m = 1$$

Simulations

Forced Damped Motion



Initial conditions:

x = -1.414, v = 0

x = -1.413, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

$$\gamma = 0.31$$

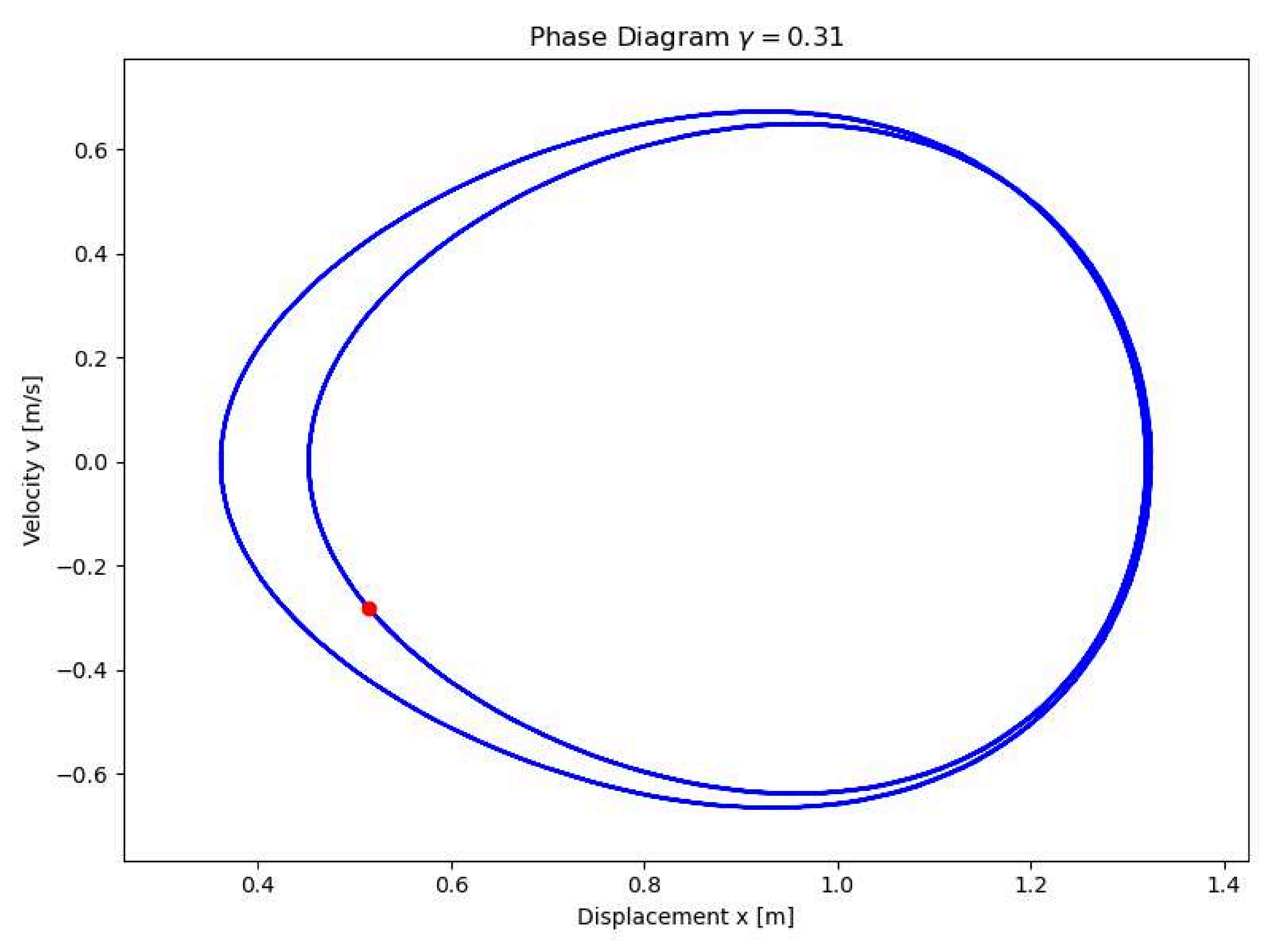
$$\omega = 1.4$$

$$m = 1$$

Time Series

Simulations

Forced Damped Motion



Initial conditions:
x = -1.414, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

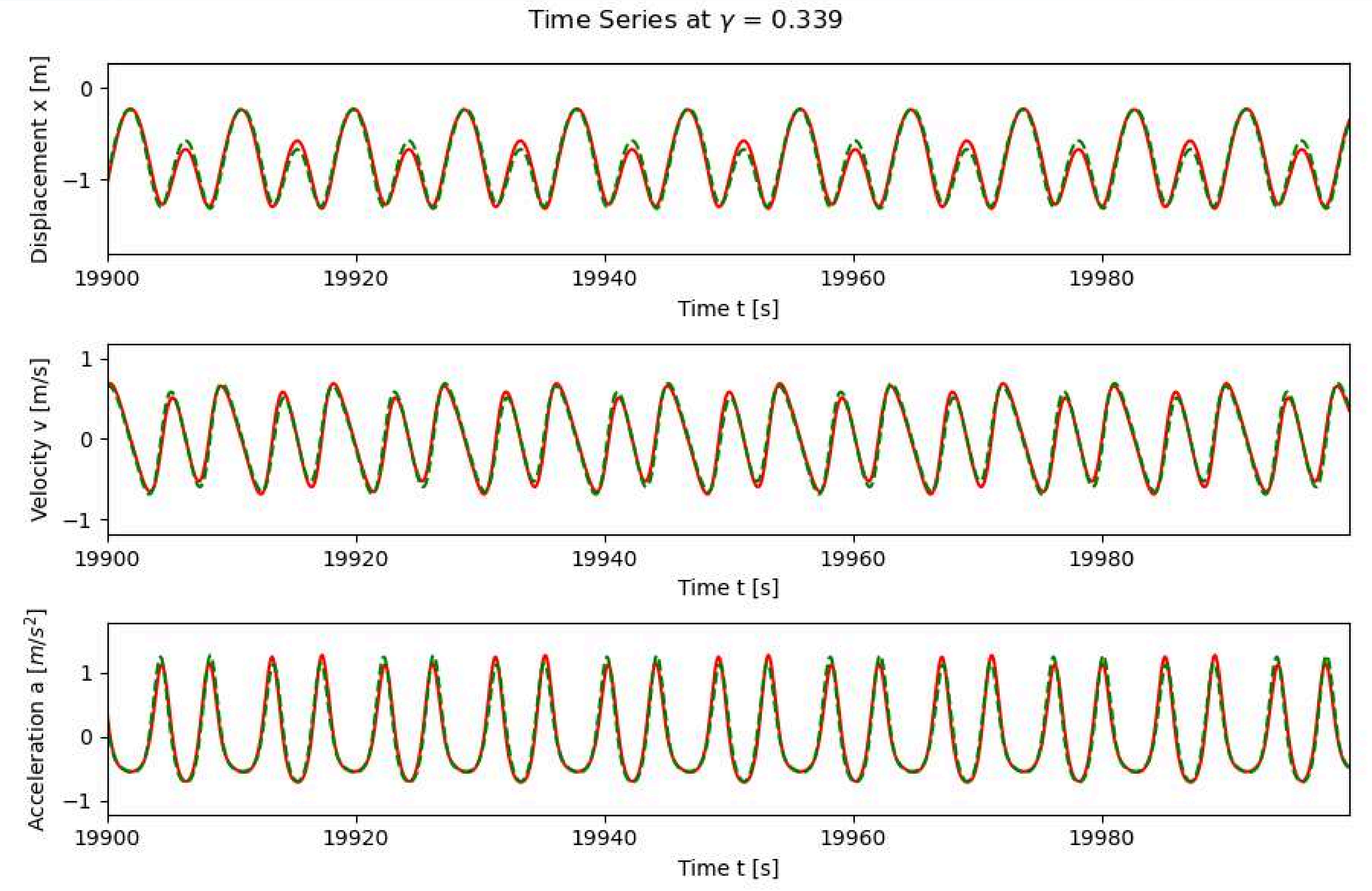
$$\gamma = 0.31$$

$$\omega = 1.4$$

$$m = 1$$

Simulations

Forced Damped Motion



Initial conditions:

x = -1.414, v = 0

x = -1.413, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

$$\delta = 0.1$$

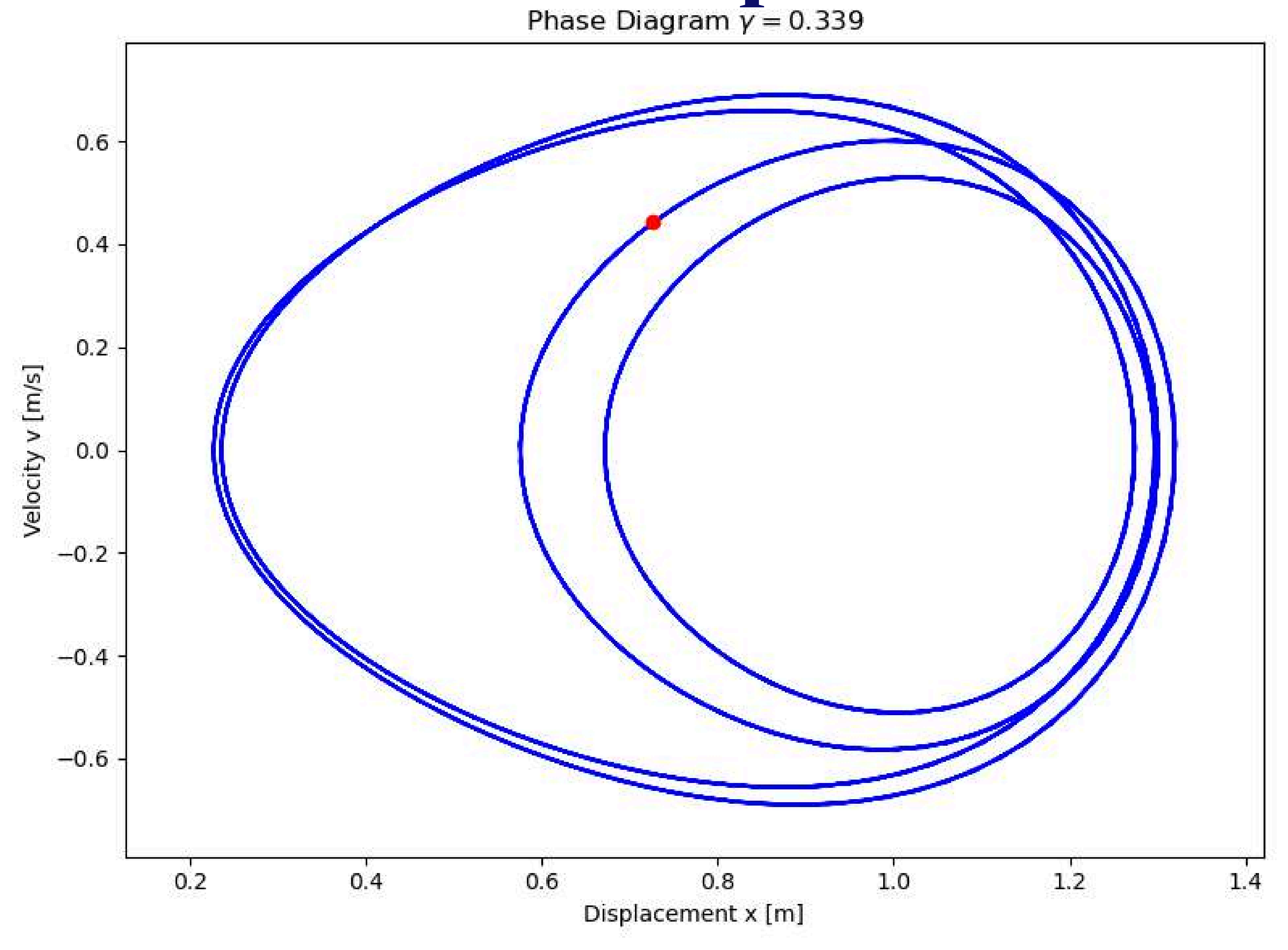
$$\gamma = 0.339$$

$$\omega = 1.4$$

$$m = 1$$

Simulations

Forced Damped Motion



Initial conditions:
x = -1.414, v = 0

Parameter values:

$$\alpha = -1$$

$$\beta = 1$$

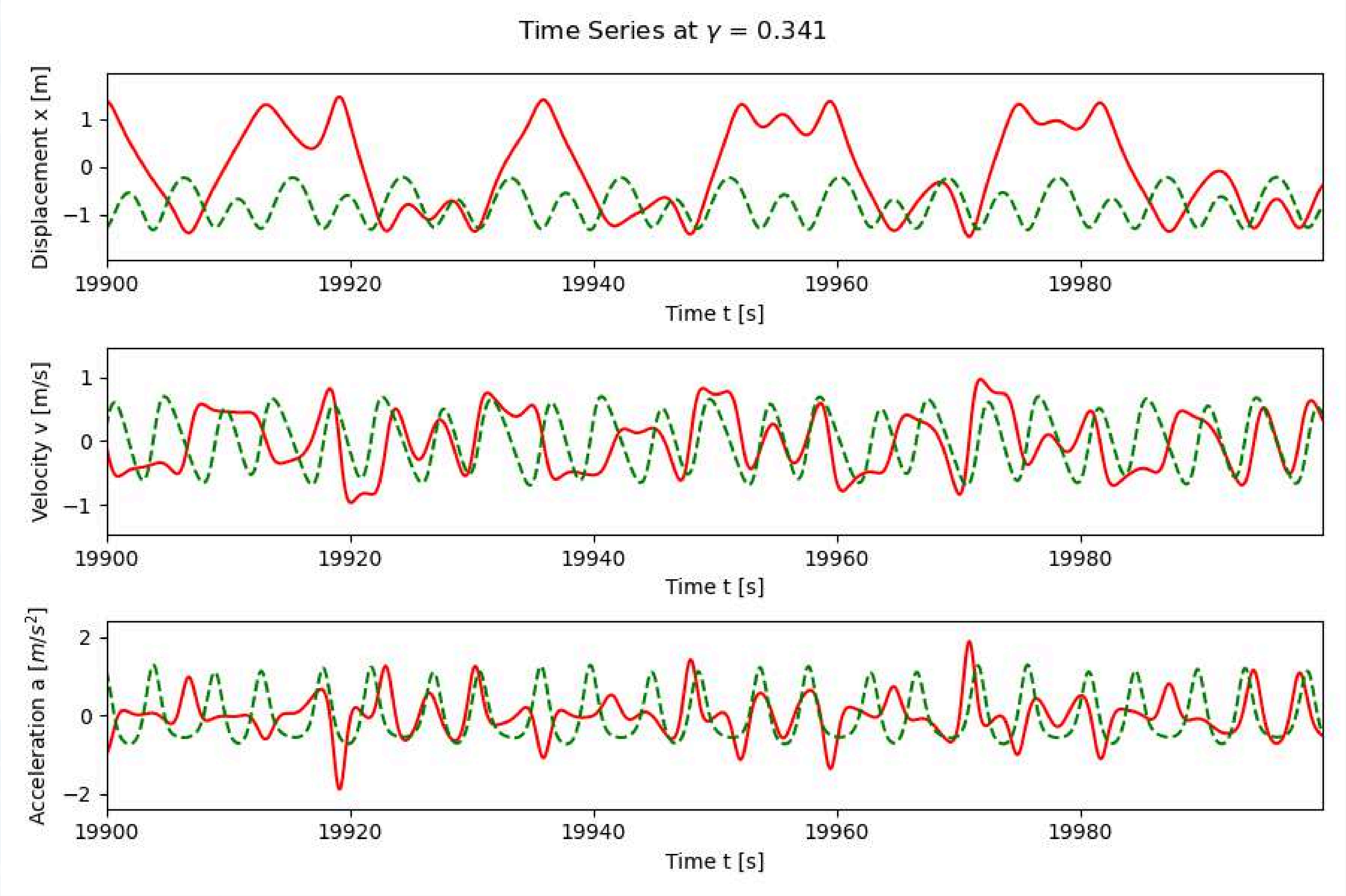
$$\delta = 0.1$$

$$\gamma = 0.339$$

$$\omega = 1.4$$

$$m = 1$$

Forced Damped Motion



$$\alpha = -1$$

$$\beta = 1$$

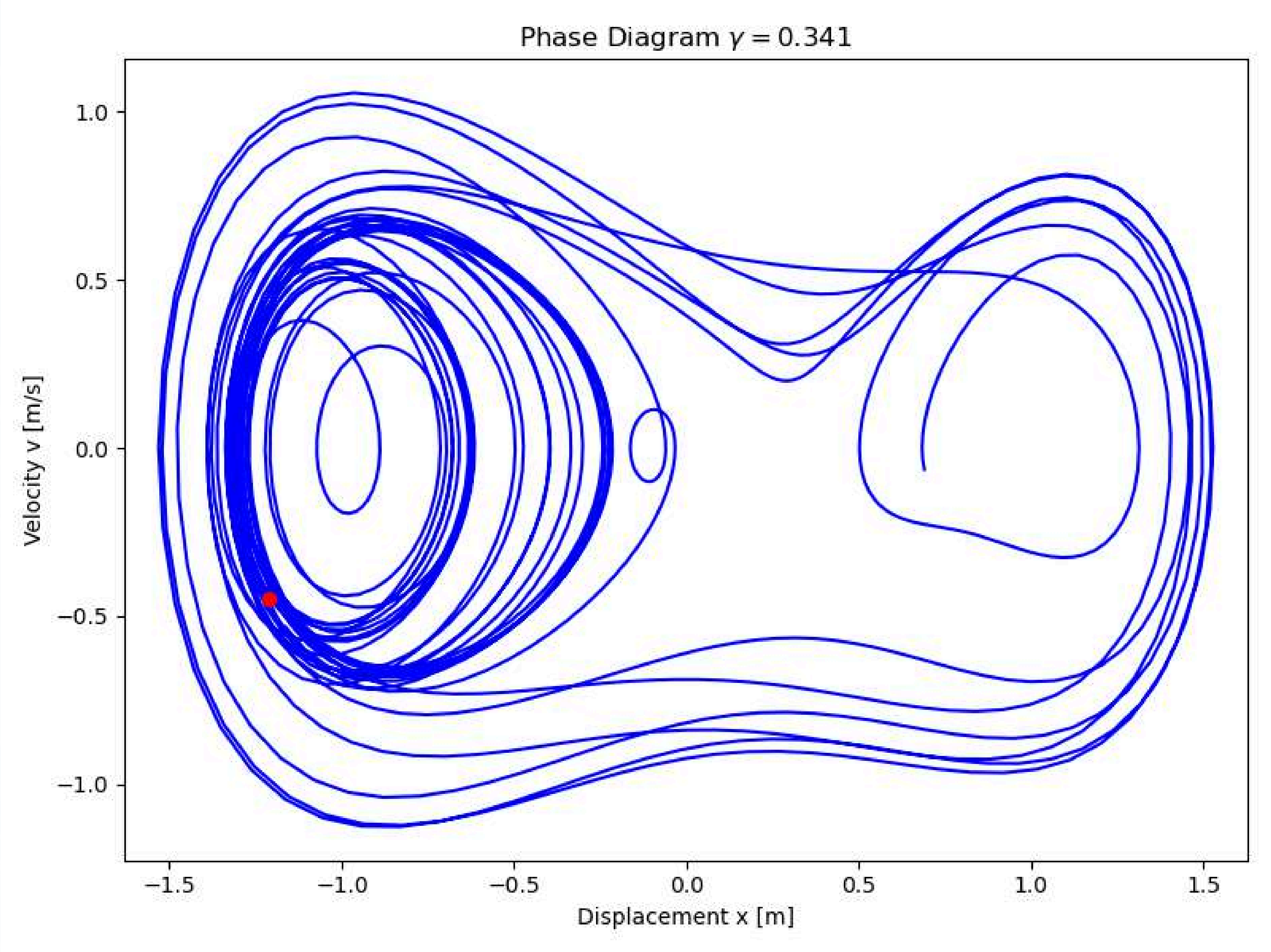
$$\delta = 0.1$$

$$\gamma = 0.341$$

$$\omega = 1.4$$

$$m = 1$$

Forced Damped Motion



$$\alpha = -1$$

$$\beta = 1$$

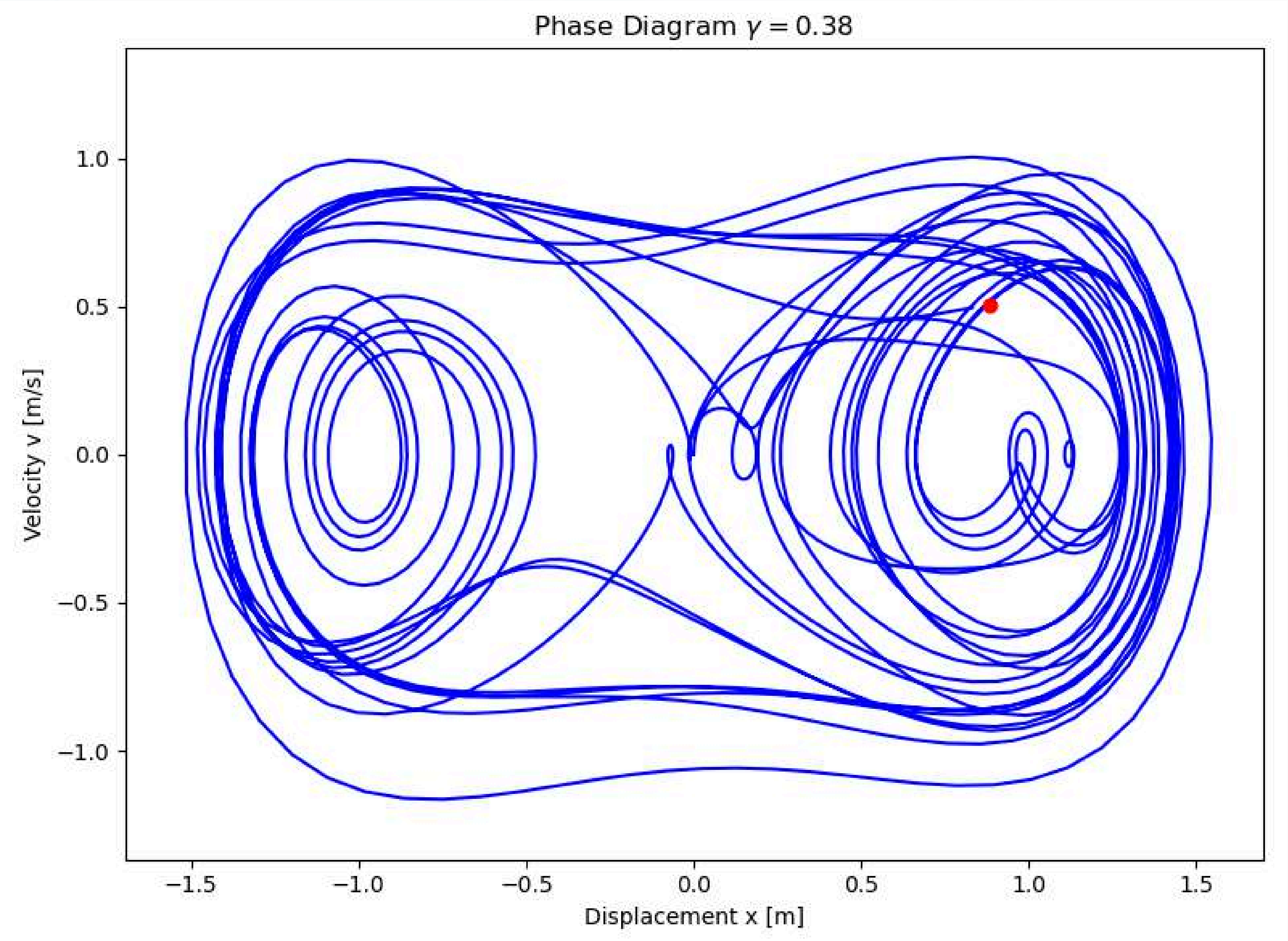
$$\delta = 0.1$$

$$\gamma = 0.341$$

$$\omega = 1.4$$

$$m = 1$$

Period Doubling and ... CHAOS

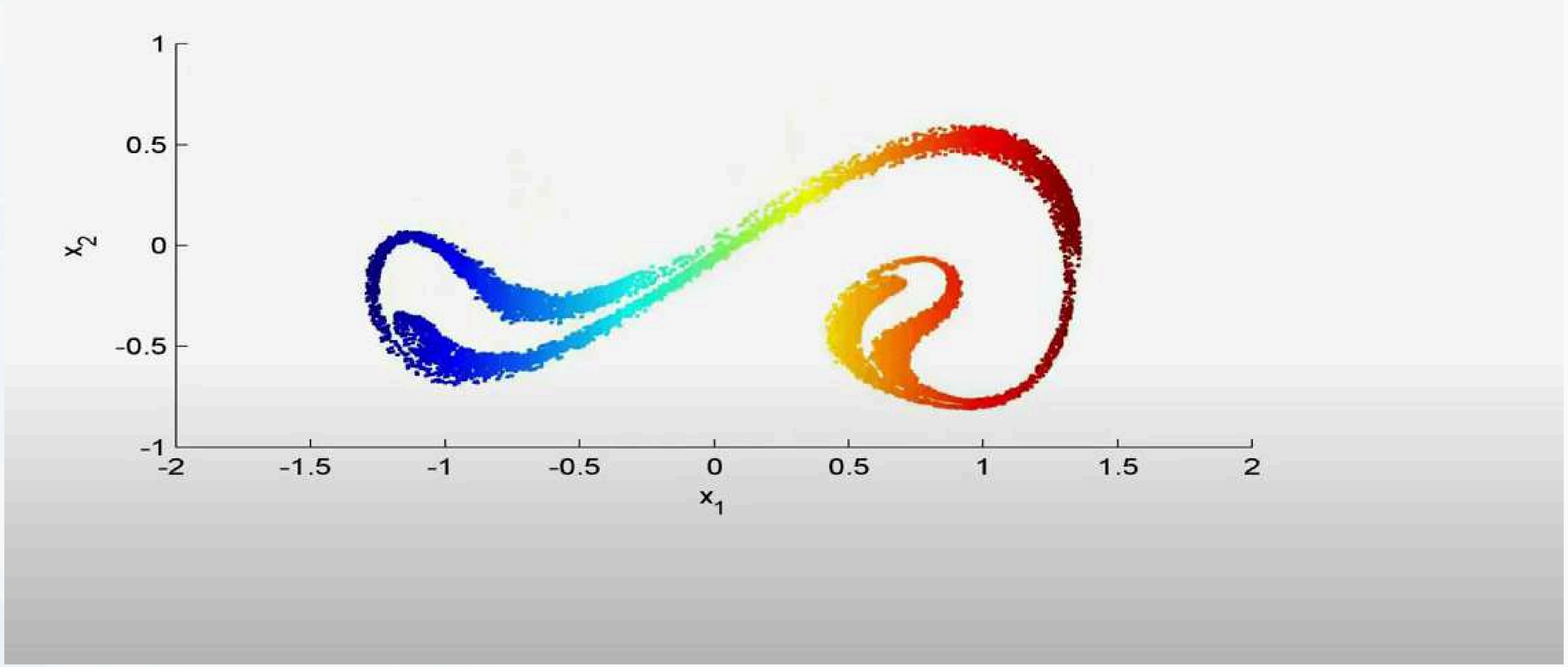


Period Doubling and ... CHAOS

How the Poincaré section will look like?

Let's see (*Enjoy the music!*)

Period Doubling and ... CHAOS



Analogous Circuit

The non-linear equation we mentioned so far in this session has a analogous electrical circuit which show same kind of phenomena.

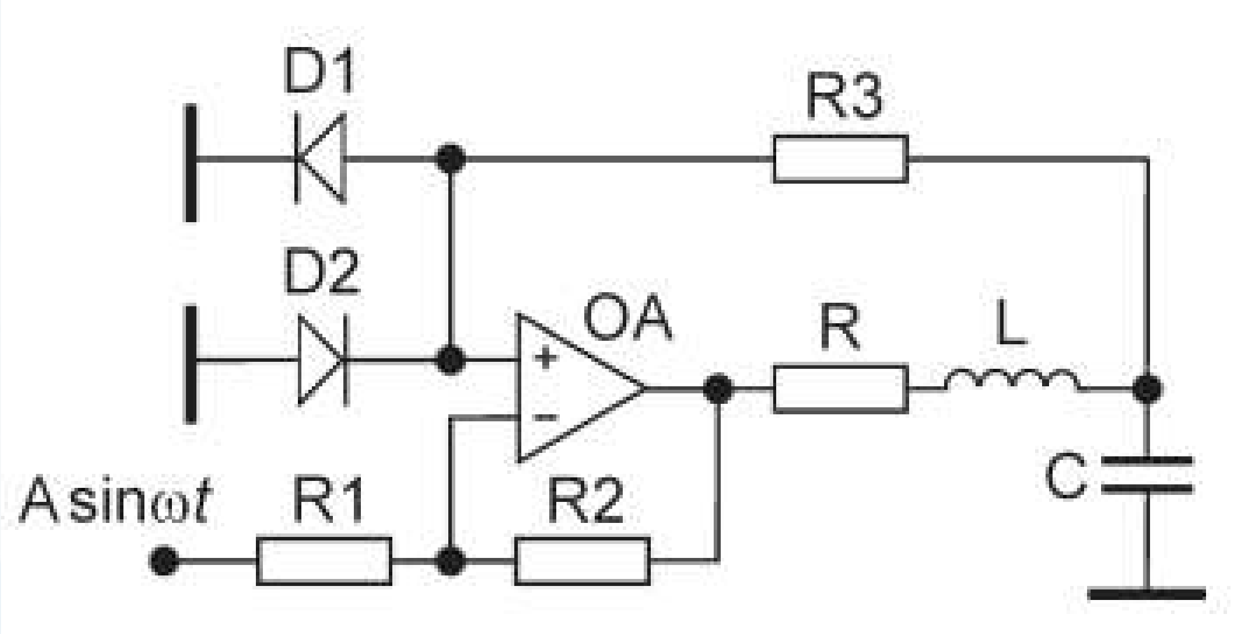
If we put $\alpha = -1$ and $\beta = 1$ then we get a equation
Famously known as Duffing-Holmes equation

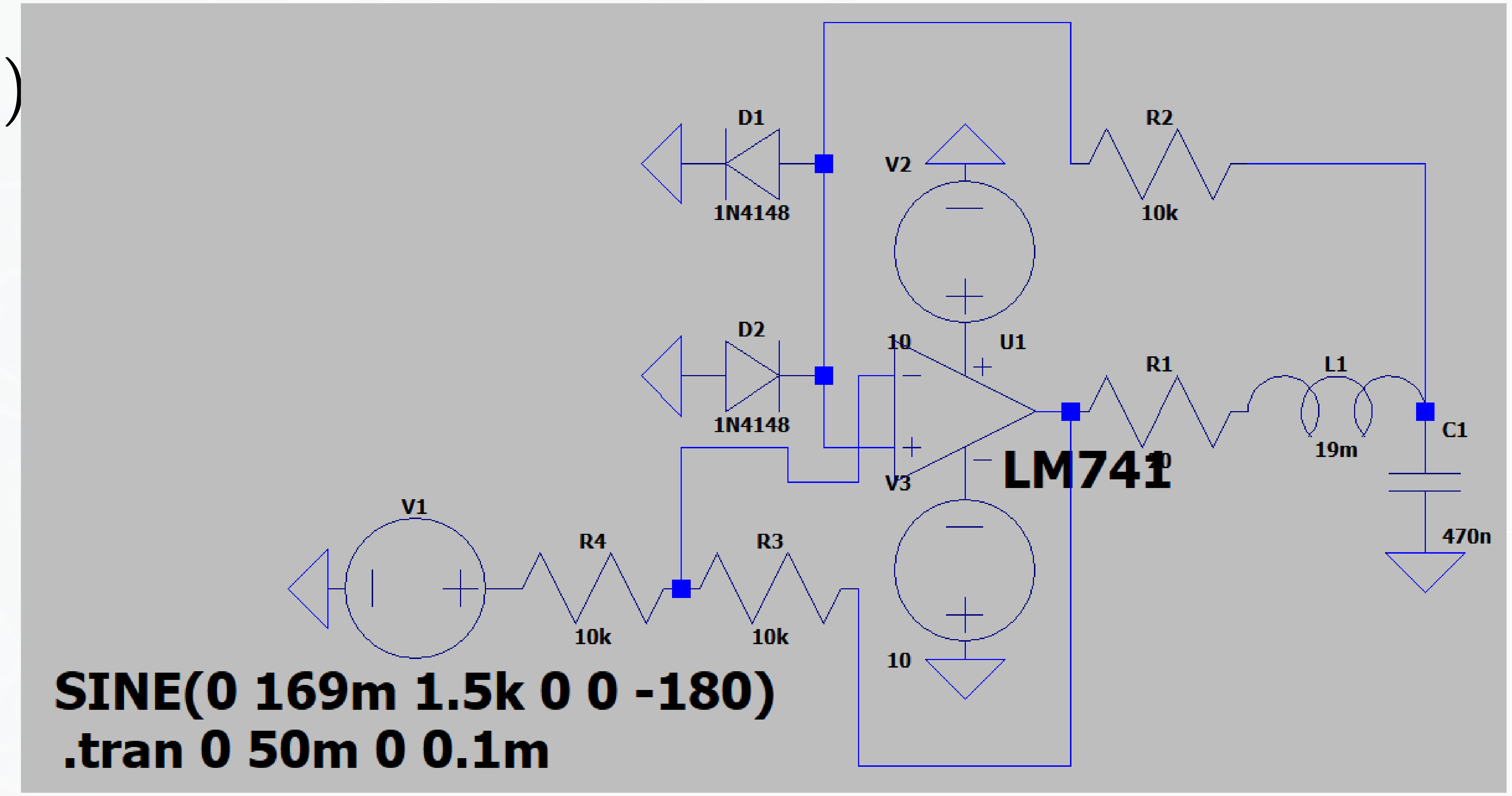
And we can make a analogous circuit to see the dynamics of the system.

Analogous Circuit

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} - x + x^3 = \gamma cos(\omega t)$$

with: $\alpha = -1, \beta = 1$





Analogous Circuit

$$C\frac{dV_C}{dt} = I_L, L\frac{dI_L}{dt} = F_E(V_C) - I_L R + A\sin(\omega t - \pi)$$

V_C is the voltage across capacitor C, and I_L is the current through inductor L.

Assuming $R_3 \gg \rho = \sqrt{\frac{L}{C}}$, we can ignore the phase π in $A\sin(\omega t - \pi)$.

The function $F_E(V_C)$ is approximated as:

$$F_E(V_C) = \begin{cases} -(V_C + kV^*), & V_C < -V^*, \\ (k-1)V_C, & -V^* \leq V_C \leq V^*, \\ -(V_C - kV^*), & V_C > V^* \end{cases}$$

Analogous Circuit

$k = \dfrac{R_2}{R_1} + 1$ is the gain of the amplifying stage.

V^* is the voltage drop across an opened diode (for silicon diodes V^* $\approx 0.5\,V$ at $0.1\,\mathrm{mA}$).Choose $k = 2$ by setting $R_2 = R_1$.

assume $R_{d0} \gg R_3 \gg R_{d1}$, where R_{d0} and R_{d1} are the resistances of the diode in the closed and opened states, respectively.

Introducing dimensionless variables and parameters:

Analogous Circuit

$$x = \frac{V_C}{2V^*}, \quad y = \frac{\rho I_L}{2V^*}, \quad \frac{t}{\sqrt{LC}} \to t, \quad \omega\sqrt{LC} \to \omega,$$

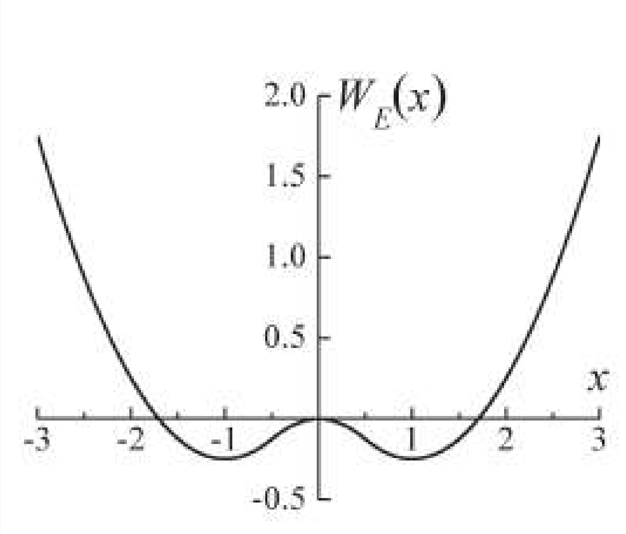
$$a = \frac{A}{2V^*}, \quad b = \frac{R}{\rho}, \quad \rho = \sqrt{\frac{L}{C}},$$

equations convenient for analysis and numerical simulation are obtained:

$$\dot{x} = y,$$
$$\dot{y} = F_E(x) - by + a\sin\omega t$$

Analogous Circuit

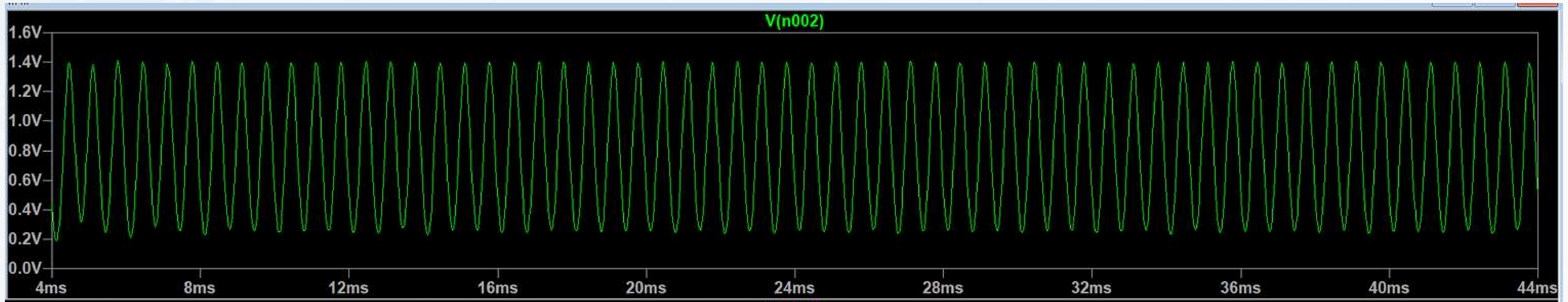
$$F_E(x) = \begin{cases} -(x+1), & x < -0.5, \\ x, & -0.5 \le x \le 0.5, \\ -(x-1), & x > 0.5. \end{cases}$$



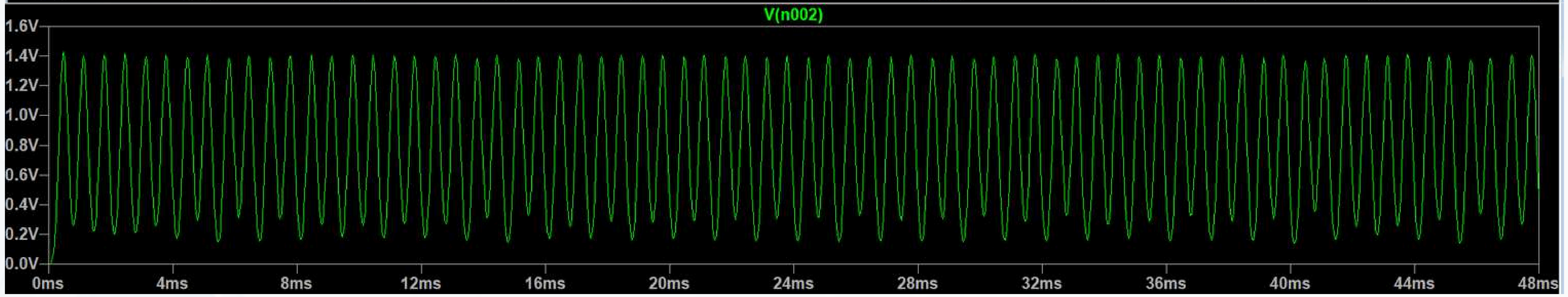
It mimics like the actual potential but not exact

$$W_E(x) = -\int F_E(x)\,dx = \frac{1}{2}\begin{cases} (x+1)^2 - 0.5, & x < -0.5, \\ -x^2, & -0.5 \le x \le 0.5, \\ (x-1)^2 - 0.5, & x > 0.5. \end{cases}$$

Analogous Circuit

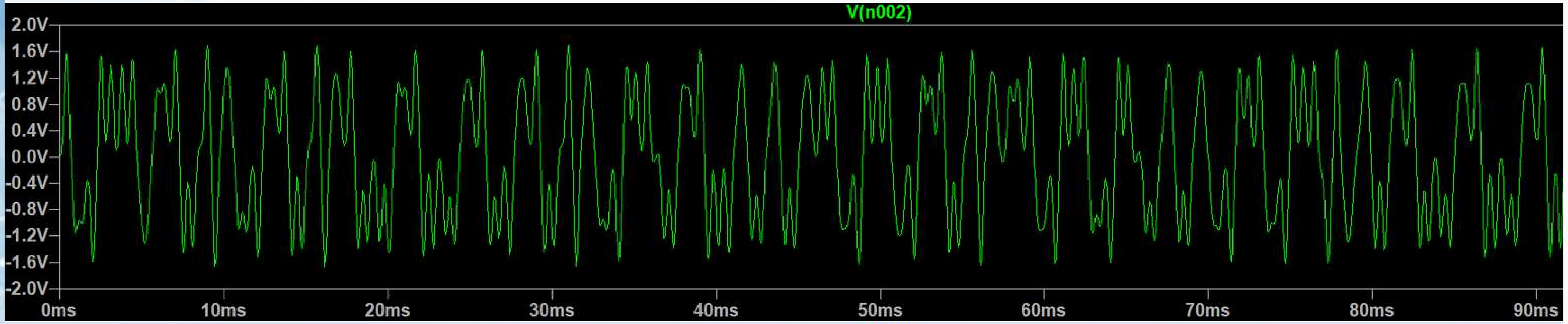


Period-1

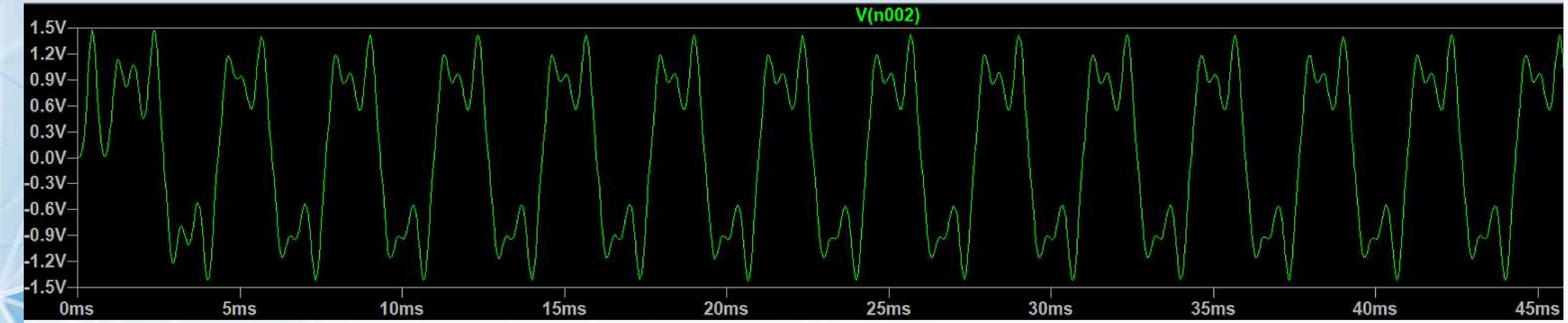


Period-2

Analogous Circuit

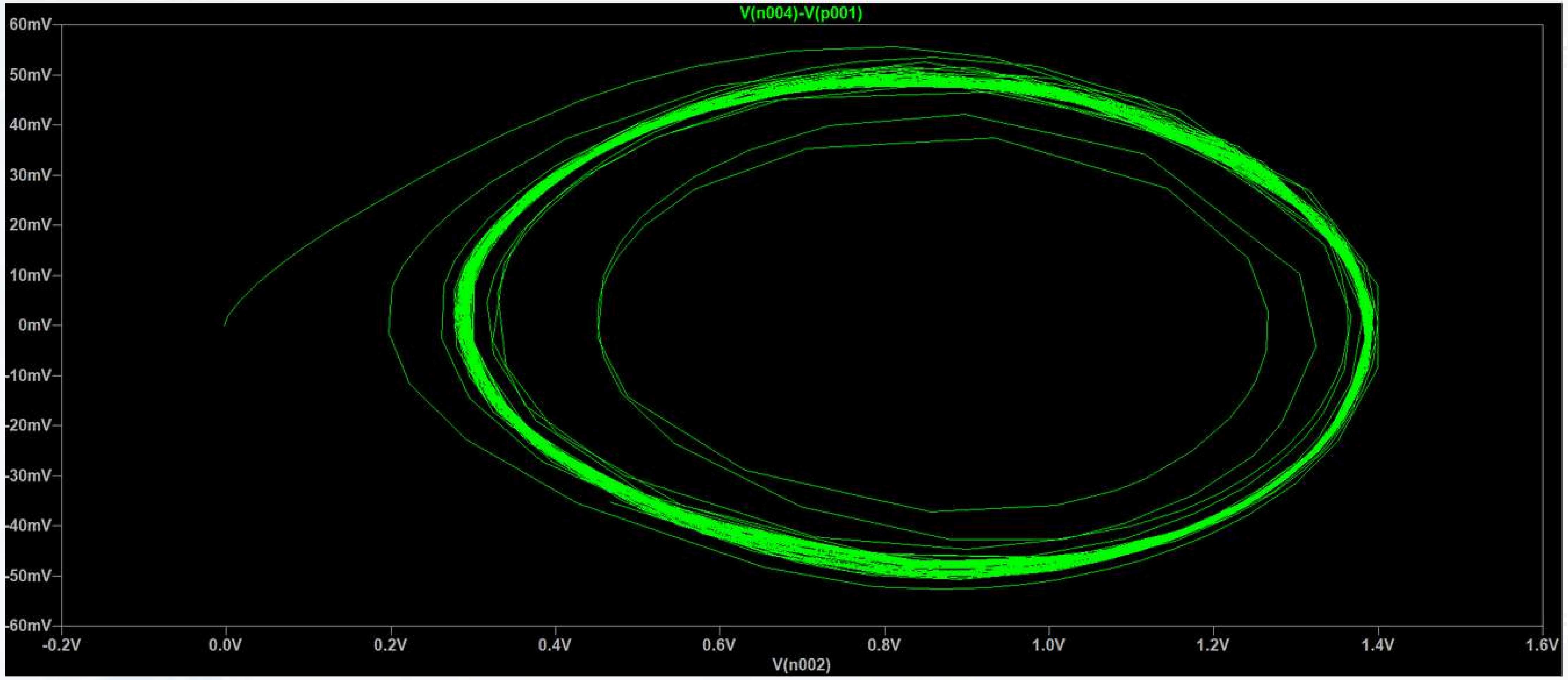


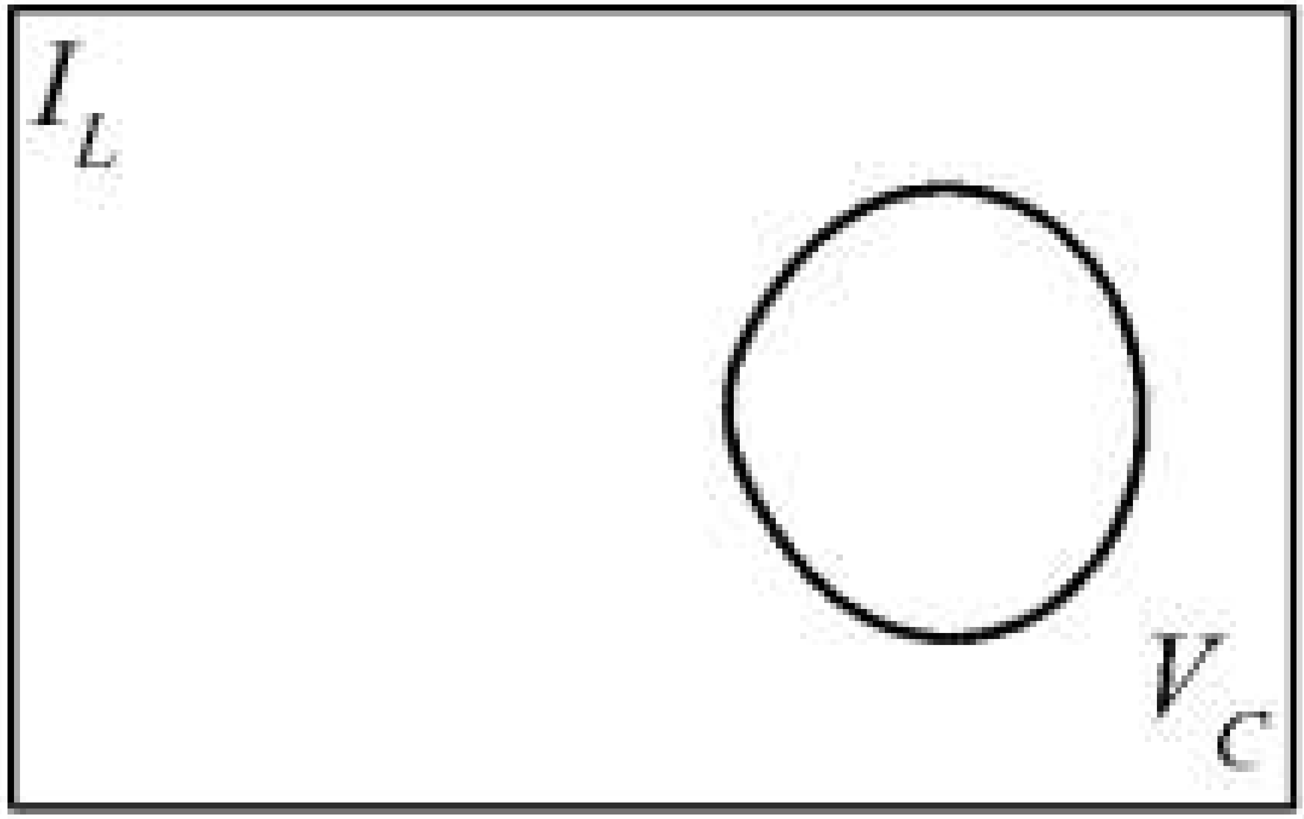
Chaos



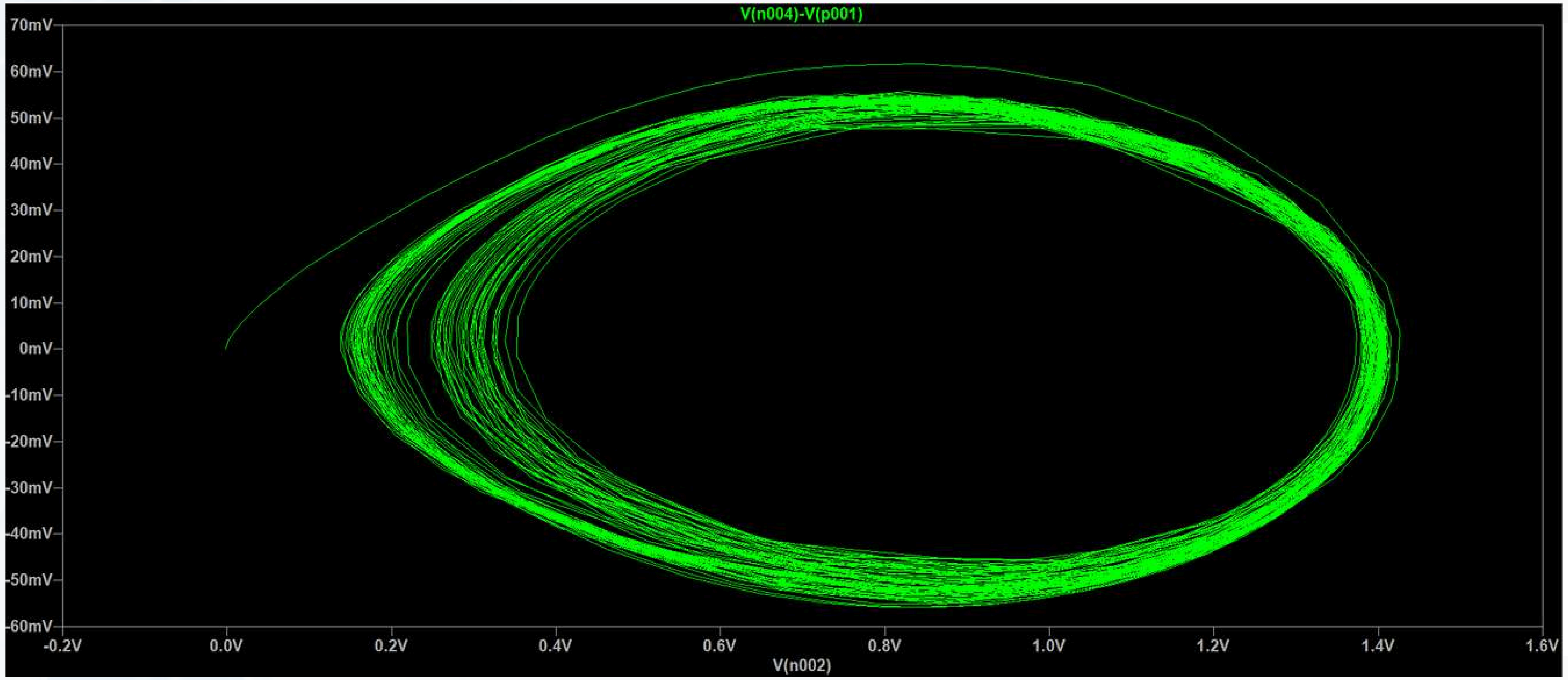
Period-5

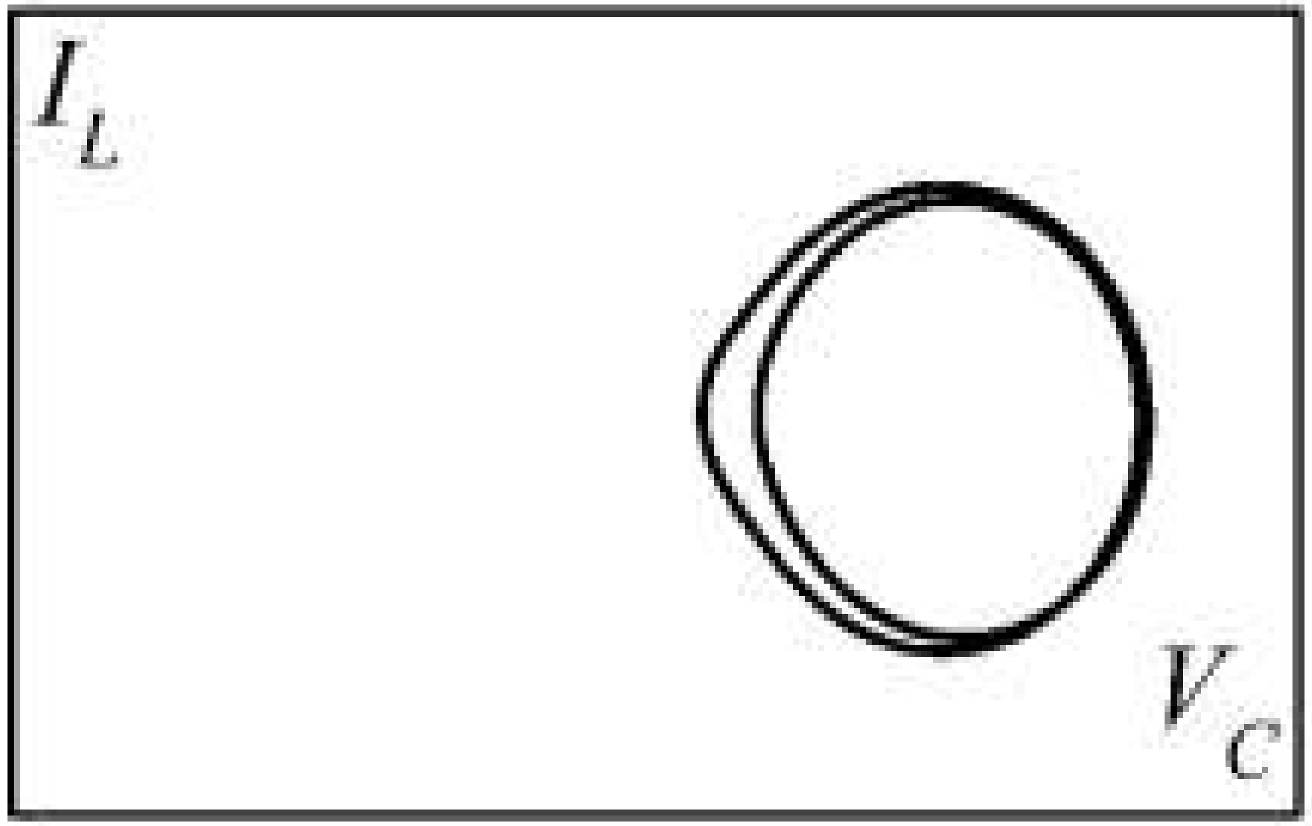
iL vs Vc (Period-1)at Vc=120m



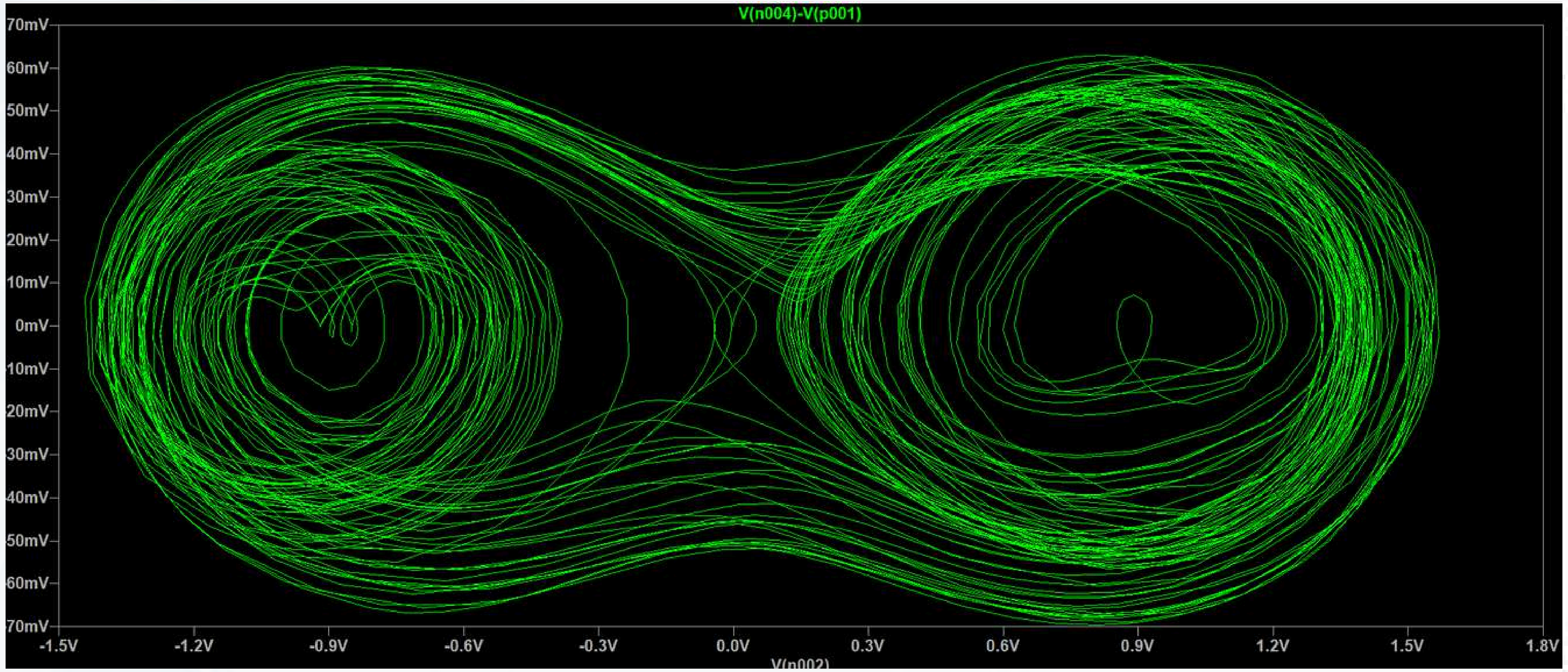


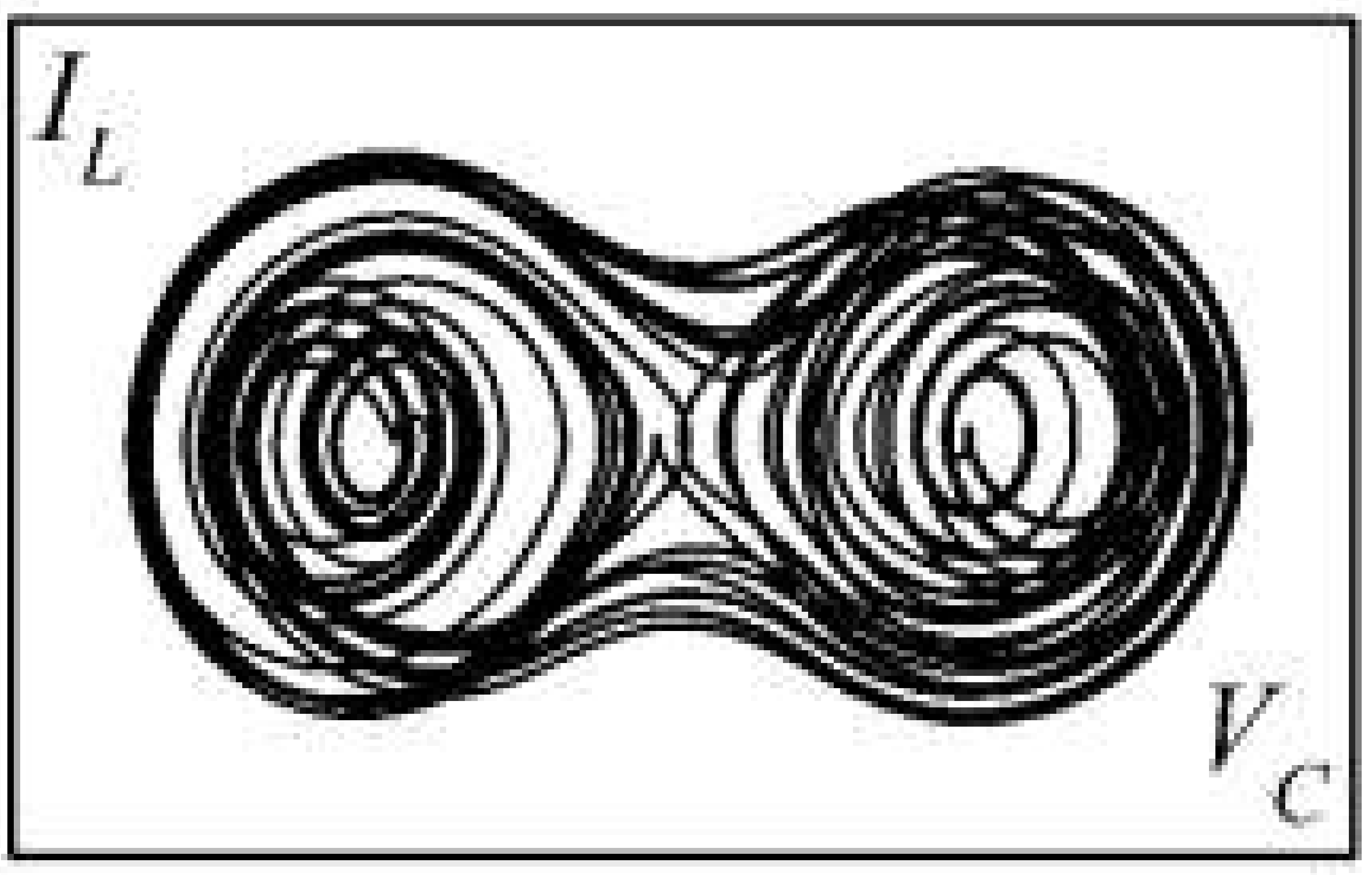
iL vs Vc (Period-2 Vc=142mv)





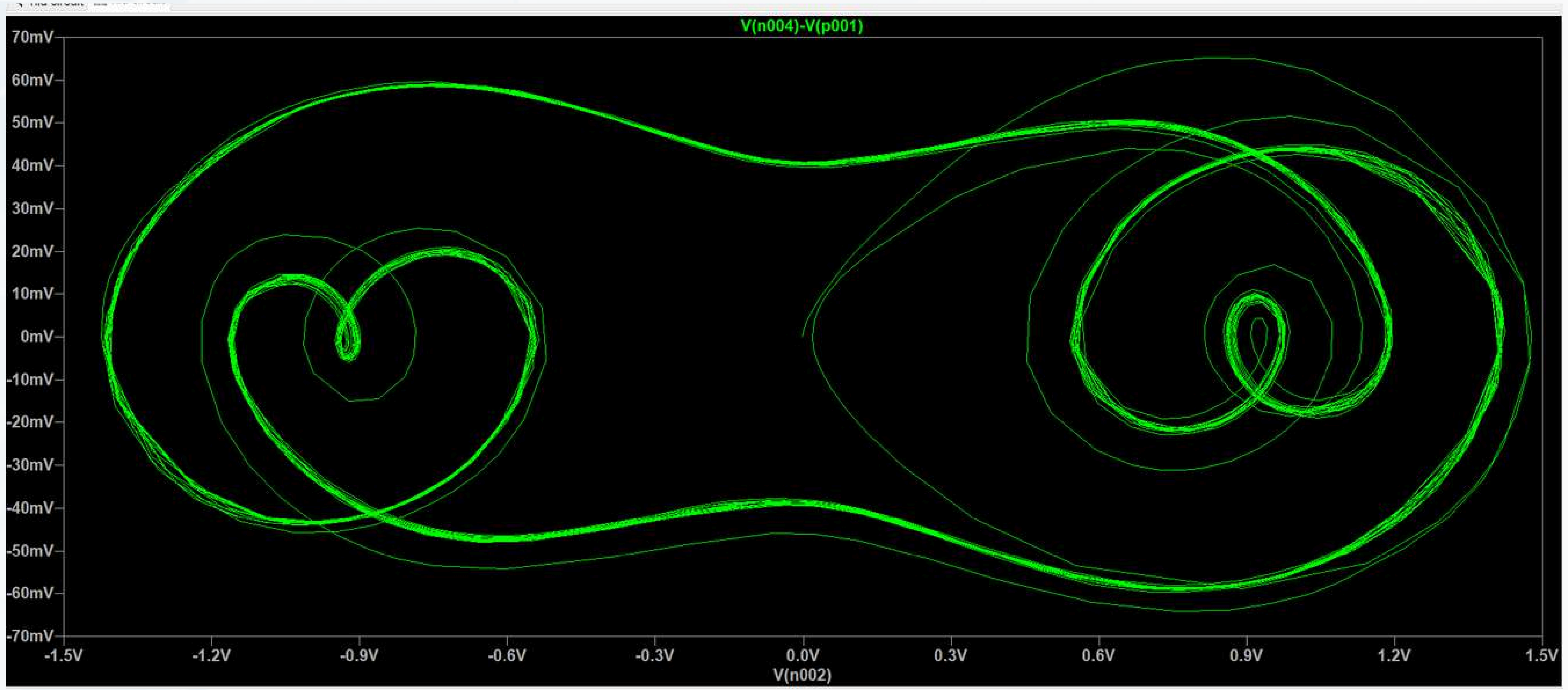
iL vs Vc (Chaos at 143mv)

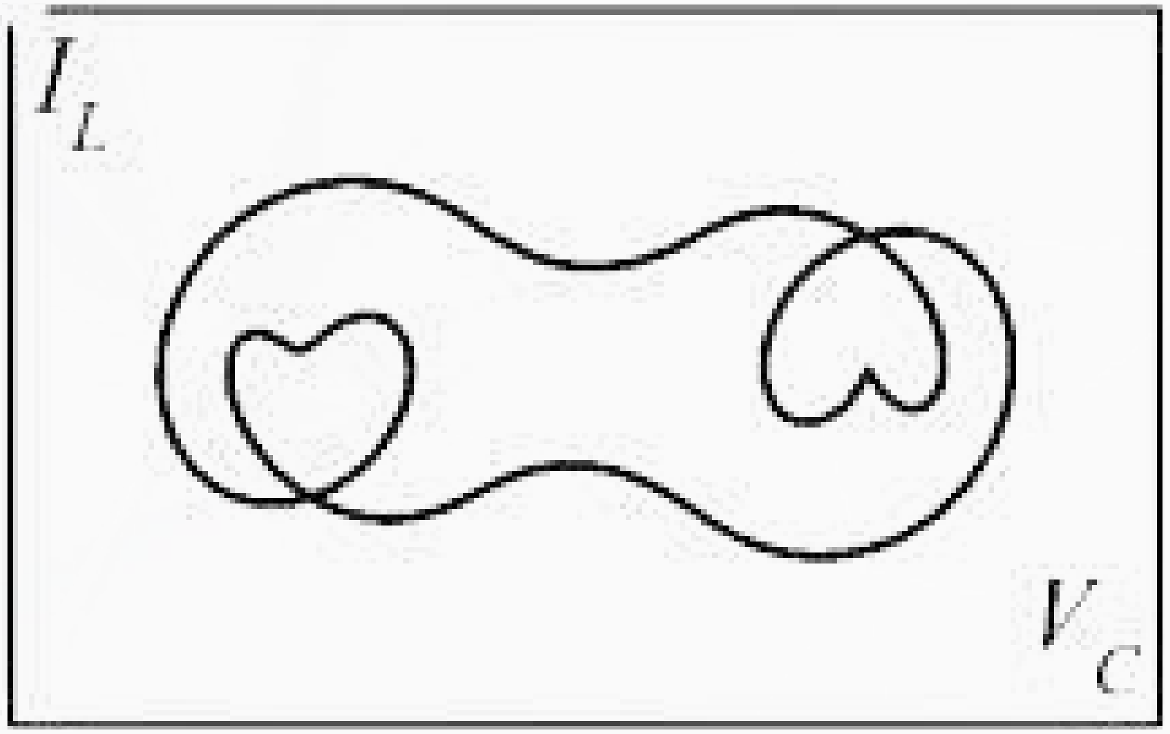




Analogous analogy

iL vs Vc (Period-5 at 169mv)





Thank You

and Good Night

Wait A minute...

In case I don’t see ou on 31st ...