Personal Memoranda

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To be renewed on :

Credit Cards :

Type : No. : Due Date :

My Best

Teacher :

Friend :

Subject :

Picnic Spot :

Food :

Sport Player :

Film Star :

Important Contacts :

Daddy Office : Friend :

School Teacher : Friend :

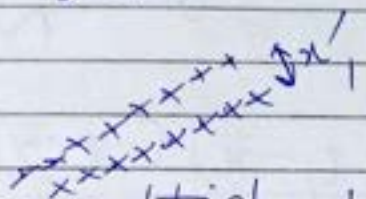
Classes Teacher : Friend :

(1)

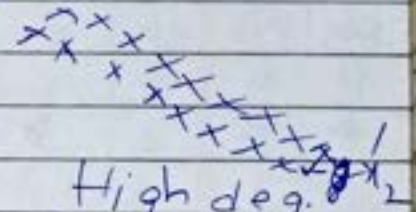
Perfect +ve correlation
$r = +1$

Perfect -ve correlation
$r = -1$

(2)

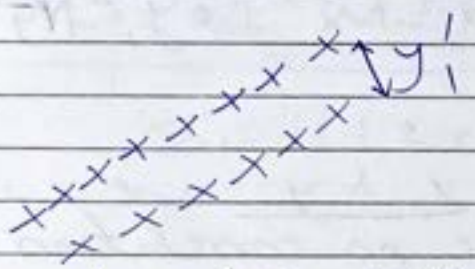


High deg. +ve correlation
$r = +ve$



High deg. -ve correlation
$r = -ve$

(3)



Low deg. +ve correlation
$r = +ve$

Low deg. -ve correlation
$r = -ve$

$x_1', x_2' < y_1', y_2'$

(4)

No correlation
$r = 0$

Karl Pearson's coeff. of correlation

$r = \frac{cov(x,y)}{\sigma_x \times \sigma_y}$

where $cov(x,y) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{N}$

$$\sigma_x = \sqrt{\frac{\Sigma(x_i-\bar{x})^2}{N}} \quad \text{- S.d. of } x$$

$$\sigma_y = \text{s.d. of } y = \sqrt{\frac{\Sigma(y_i-\bar{y})^2}{N}}$$

$$r = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\Sigma(x_i-\bar{x})^2}\sqrt{\Sigma(y_i-\bar{y})^2}}$$

Also $\boxed{r = \dfrac{\Sigma x_i y_i - N\bar{x}\bar{y}}{\sqrt{\Sigma x_i^2 - N\bar{x}^2}\sqrt{\Sigma y_i^2 - N\bar{y}^2}} \quad (*)}$

Note ① $-1 \le r \le 1$

② If x & y are independent, then there is no correlation b/w x & y

③ r is independent of change of correlation and change of scale.

Q. Calculate r for the following data.

X	Y
23	18
27	22
28	23
29	24
30	25

X	Y
31	26
33	28
35	29
36	30
39	32

$$r = \frac{\Sigma uv - N\bar{u}\bar{v}}{\sqrt{\Sigma u^2 - N\bar{u}^2}\sqrt{\Sigma v^2 - N\bar{v}^2}} \quad (*)$$

Sol:

$u = X - 30$	$v = Y - 25$	u^2	v^2	uv
-7	-7	49	49	49
-3	-3	9	9	9
-2	-2	4	4	4
-1	-1	1	1	1
0	0	0	0	0
1	1	1	1	1
3	3	9	9	9
5	4	25	16	20
6	5	36	25	30
9	7	81	49	63
$\Sigma u = 11$	$\Sigma v = 7$	$\Sigma u^2 = 215$	$\Sigma v^2 = 163$	$\Sigma uv = 186$

$$\bar{u} = \frac{\Sigma u}{N} = 1.1$$

10×1.21

$$\bar{v} = \frac{\Sigma v}{N} = 0.7$$

0.49

$$r = \frac{186 - (10 \times 0.77)}{\sqrt{215 - 12.1}\sqrt{163 - 4.9}}$$

178.3

14.244 × 12.574

= 0.995

Spearman's Rank correlation coeff.

$$R = 1 - \left(\frac{6 \sum d_i^2}{n^3 - n}\right)$$

$\sum d_i$ = diff. betn two ranks.

$-1 \le R \le 1$

Student no.	Rank in Eng. (R_1)	Rank in Math (R_2)
1	1	2
2	3	1
3	7	4
4	5	5
5	4	6
6	6	9
7	2	7
8	10	8
9	9	10
10	8	2

$d^2 = (R_1 - R_2)^2$

4
4
9
0
4
9
25
4
1
36

$\sum d_i = \cancel{-2} \cancel{-1} \cancel{-2} + 0 \cancel{-3} \cancel{-5} \cancel{+2} \cancel{+1} + 6$

$= 0$

$\sum d^2 = 96$

$$R = 1 - \left(\frac{6 \times 96}{990}\right)$$

$1 - 0.582$

$= 0.418 \simeq 0.42$

Obsv. $\sum d_i = 0$ ALWAYS

~~Only~~

PSOT

$$R = 1 - \frac{6\Sigma d^2}{n^2 - n}$$

For equal Ranks

$$R = 1 - \frac{6\left[\Sigma d^2 + \frac{1}{12}(m_1^2 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \dots\right]}{n^3 - n}$$

where m_1 = no. of times item is repeated $n = 10$

X	R_1	Y	R_2	$d = R_1 - R_2$	d^2
32	3	40	5	-2	4
55	9	30	3.5	5.5	30.25
49	7.5	70	9	-1.5	2.25
60	10	20	1	9	81
43	5.5	30	3.5	2	4
37	4	50	7	3	9
45	5.5	72	10	-4.5	20.25
49	7.5	60	8	-0.5	0.25
10	1	45	6	-5	25
20	2	25	2	0	0

$m_1 = 2$ (for 43)
$m_2 = 2$ (for 49)
$m_3 = 2$ (for 30)

$$1 - 6\left[176 + \frac{1}{12}(4-2) + \frac{1}{12}(8-2) + \frac{1}{12}(16-2) + \dots\right] / 990$$

$$= 1 - 6\left[176 + \frac{1}{6} + \frac{1}{2} + \frac{7}{6}\right] / 990$$

$$= \frac{1 - 6(176 + 3/2)}{990} = -0.075$$

Q. $r = 0.4$
$cov(x,y) = 16$
$\sigma_y^2 = 25$ find σ_x

Soln.

$$r = \frac{cov(x,y)}{\sigma_x \times \sigma_y} \Rightarrow \sigma_x = 0.8$$

(2) $R = 0.143$

$\Sigma d^2 = 48$ find n

Sol^n $R = 1 - \frac{6\,\Sigma d^2}{n^3 - n}$

$\Rightarrow n^3 - n - 336 = 0$

$\Rightarrow n = 7$

$0.143 = 1 - \left(\frac{6 \times 48}{n^3 - n}\right)$

$(0.143)(n^3 - n) = n^3 - n - 288$

$(-0.857)(n^3 - n) = -288$

$n^3 - n - 336.056 = 0$

$n = 7$

Find r for $N = 10$, $\Sigma x = 140$, $\Sigma y = 150$

$\Sigma (x-10)^2 = 180$, $\Sigma (y-15)^2 = 215$

$\Sigma (x-10)(y-15) = 60$

$u = x - a$

$v = y - b$

Sol^n Let $\boxed{\begin{array}{l} u = x - 10 \\ v = y - 15 \end{array}}$

$\therefore \Sigma u^2 = 180$

$\Sigma v^2 = 215$

$\Sigma uv = 60$

$\Sigma u = \Sigma (x - 10)$

$= \Sigma x - \Sigma 10$

$= 140 - 10 \times 10$

$= 40$

$\Sigma v = \Sigma (y - 15)$

$= \Sigma y - \Sigma 15$

$= 150 - (15 \times 10)$

$= 0$

$$\therefore \bar{u} = \frac{\Sigma u}{N} = \frac{40}{10} = 4$$

$$\bar{v} = \frac{\Sigma v}{N} = 0$$

$$r_{uv} = \frac{60 - 10(4)(0)}{\sqrt{180 - 160}\sqrt{215 - 0}}$$

$$= \frac{60}{}$$

$$= 0.915$$

Q. Ten competitors in a musical contest ranked. Divis which pair of judges has the nearest approach to the common likings in music.

(R_1) (x)	(R_2) (y)	(R_3) (z)	$d_{xy} = R_1 - R_2$	$d_{xz} = R_1 - R_3$	$d_{yz} = R_2 - R_3$
1	3	6	-2	-3	-5
6	5	4	1	1	2
5	8	9	-3	-1	-4
10	4	8	6	-4	2
3	7	1	-4	6	2
2	10	2	-8	8	0
4	2	3	2	-1	1
9	1	10	8	-9	-1
7	6	5	1	1	2
8	9	7	-1	2	1

d_{xy}^2	d_{yz}^2	d_{xz}^2
4	9	25
1	1	4
9	1	16
36	16	4
16	36	4
64	64	0
4	1	1
64	81	1
1	1	4
1	4	1
$\Sigma d_{xy}^2 = 200$	$\Sigma d_{yz}^2 = 214$	$\Sigma d_{xz}^2 = 60$

$$R_{xy} = 1 - \frac{6\Sigma d_{xy}^2}{n^3 - n} = 1 - \frac{6(200)}{990} = -0.212$$

$$R_{yz} = 1 - \frac{6\Sigma d_{yz}^2}{n^3 - n} = 1 - \frac{6(214)}{990} = -0.08$$

$$R_{xz} = 1 - \frac{6\Sigma d_{xz}^2}{n^3 - n} = 1 - \frac{6(60)}{990} = 0.636$$

3rd Judge

Q. A sample of 25 pairs of (x,y) values gives the following results.

$\Sigma x = 127$

$\Sigma y = 100$

$\Sigma x^2 = 760$

$\Sigma y^2 = 449$

$\Sigma xy = 500$

(8,14) & (8,6) ✗

(8,12) & (6,8)

Later on, it was found that 2 pair of value were taken as

Find correct co-relational coeff.

Soln

correct Σx = wrong Σx − (8+8) + (8+12)

correct Σy = wrong Σy − (14+6) + (12+8)

correct Σx^2 = wrong Σx^2 − (8^2+8^2) + (8^2+12^2)

correct Σy^2 = wrong Σy^2 − (14^2+6^2) + (12^2+8^2)

correct Σxy = wrong Σxy − ((8×14) + (8×6)) + (8×12) + (12×8)

correct $\bar{x} = \frac{\Sigma x}{n}$ =

correct $\bar{y} = \frac{\Sigma y}{n}$ =

127 − 16 + 14 = 125 = correct Σx

100 − 20 + 20 = 100 = correct Σy

760 − 128 + 100 = 732 = correct Σx^2

= 732

425 = correct Σy^2

484 = correct Σxy

∴ correct $\bar{x} = \frac{125}{25} = 5$

correct $\bar{y} = \frac{100}{25} = 4$

$$r = \frac{\Sigma xy - n\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - n\bar{x}^2}\sqrt{\Sigma y^2 - n\bar{y}^2}}$$

$$= \frac{484 - (25 \times 20)}{\sqrt{732 - (25 \times 25)}\sqrt{425 - (25 \times 16)}}$$

$= -0.309$

Regression lines:

Reg. line of y on x

$y - \bar{y} = b_{yx}(x - \bar{x})$

where $b_{yx} = r\frac{\sigma_y}{\sigma_x}$ (Reg. co-eff.)

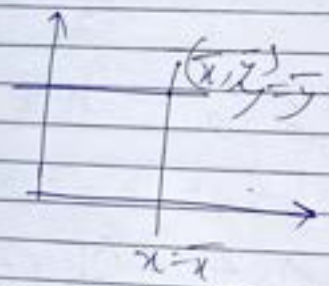
Reg. line of x on y

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

where $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$

b_{yx} & b_{xy} are the slopes of the regression lines of y on x and x on y respectively

Remarks: ① If $r = 0$, then $x = \bar{x}$ & $y = \bar{y}$



If $r = \pm 1$, then both lines are ||el to each other.

② $$b_{yx} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$$

$$b_{xy} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma y^2 - n\bar{y}^2}$$

$$r = \frac{\Sigma xy - n\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - n\bar{x}^2}\sqrt{\Sigma y^2 - n\bar{y}^2}}$$

$= -0.309$

③ $r = \pm\sqrt{b_{xy} b_{yx}}$

④ If $b_{yx} > 1$ then $b_{xy} < 1$

⑤ $$\frac{b_{xy} + b_{yx}}{2} \geq r$$

⑥ If θ is the $\angle$le b/w 2 regression lines, then

$$\tan\theta = \left(\frac{1 - r^2}{r}\right)\frac{\sigma_x \times \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

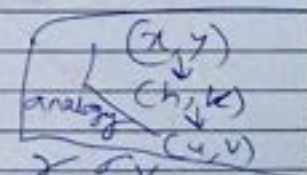
If $r = 0 \Rightarrow \tan\theta = \infty \Rightarrow \theta = \pi/2$

If $r = \pm 1 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

⑦ Regression coeff are independent of change of origin but NOT change of scale.

i.e. If $u = \dfrac{X - a}{h}$, $v = \dfrac{Y - b}{k}$

then $\sigma_x = h\sigma_u$
$\sigma_y = k\sigma_v$



$$\therefore b_{yx} = \frac{r\sigma_y}{\sigma_x} = \frac{k}{h} r \frac{\sigma_v}{\sigma_u}$$

$$= \frac{k}{h} b_{vu}$$

Similarly $b_{xy} = \dfrac{h}{k} b_{uv}$

Example A panel of two judges A & B ~~judges~~ aded the following performances.

Perf No	A(X)	$\bar{X}$	$\bar{Y}$	B(Y)	u=y-$\bar{y}$	v=x-$\bar{x}$
1	36			35	3	2
2	32			33	-1	0
3	34	33	33	31	1	-2
4	31			30	-2	-3
5	32			34	-1	1
6	32			32	-1	-1
7	34			36	1	3
~~8~~	~~38~~			~~32~~		
	$\Sigma x = 231$			$\Sigma y = 231$	$\Sigma u = 0$	$\Sigma v = 0$

n=7

If B was present, how many marks he'd have awarded?

~~Perf No~~ A, $v-\bar{v} = b_{vu}(u-\bar{u})$

$$b_{vu} = \frac{\Sigma uv - n\bar{u}\bar{v}}{\Sigma u^2 - n\bar{u}^2}$$

$\Sigma u^2 = 18$

$\bar{u} = \frac{0}{7} = 0$

$\bar{v} = \frac{0}{7} = 0$

$\bar{x} = \frac{231}{7} = 33$

$\bar{y} = \frac{231}{7} = 33$

~~$r = b_{vu}$~~ $\Sigma uv = 6 - 2 + 6 - 1 + 1 + 3$

$= 13$

$$b_{vu} = \frac{13 - 7 \times 0}{18 - (7 \times 0)} = 0.722$$

$\bar{v} - \bar{v} = b_{vu}(u - \bar{u})$

$\Rightarrow v = 0.72u \quad (\because \bar{u} = \bar{v} = 0)$

$\Rightarrow y - \bar{y} = (0.72)(x - \bar{x})$

$\Rightarrow y = 33 + 0.72(x - 33)$

when $x = 38$

$y = 36.6$

$y \approx 37$

	Maths	Eng
Mean	80	50
S.D	15	10

$r = 0.4$

Estimate marks of student in Maths who scored 60 in English and marks of student in English who scored 70 in Maths.

Sol^n: Maths (X) Eng (Y)

$Y - \bar{Y} = b_{yx}(X - \bar{X})$

$x = 70, y = ?$

$X - \bar{X} = b_{xy}(Y - \bar{Y})$

$y = 60, x = ?$

$b_{yx} = r\frac{\sigma_y}{\sigma_x}$ $b_{xy} = r\frac{\sigma_x}{\sigma_y}$

$\bar{x} = 80$ $\bar{y} = 50$

$\sigma_x = 15$ $\sigma_y = 10$

$b_{yx} = 0.4 \times \frac{10}{15} = 0.267$

$b_{xy} = 0.4 \times \frac{15}{10} = 0.6$

i) $Y - 50 = 0.267(X - 80)$

ii) $X - 80 = 0.6(Y - 50)$

i) $\overset{y}{\cancel{60}} - 50 = 0.267(\overset{70}{\cancel{x}} - 80)$

ii) $\overset{x}{\cancel{70}} - 80 = 0.6(\overset{60}{\cancel{y}} - 50)$

$x = 86$

$\cancel{X - 80 = 9.733 = 89.733}$

$y = 50 - 2.6 = 47.33$

Given $6y = 5x + 90$ — (1)

$15x = 8y + 130$ — (2)

$\sigma_x^2 = 16$

Find (1) $\bar{x}$ & $\bar{y}$

(2) r

(3) σ_y^2

Sol^n:

$\cancel{(6y - 5x = 90) \times 8}$

$\cancel{15x - 8y = 130}$

$\cancel{\Rightarrow 18y - 15x = 270}$

$(6y - 5x = 90) \times 3$

$15x - 8y = 130$

$10y = 400$

$y = 40$

$\therefore \bar{x} = 30$

$\cancel{10y = 300}$

$\cancel{y = 30}$

$\cancel{\Rightarrow 18y - 15x = 90}$

$\cancel{x = 18}$

(1) $\Rightarrow y = \frac{5}{6}x + \frac{90}{6} \Rightarrow b_{yx} = \frac{5}{6}$

(2) $\Rightarrow x = \frac{8}{15}x + \frac{130}{15}$

$r = \sqrt{b_{xy}\, b_{yx}}$

$= \frac{2}{3}$ — Ans 2

$b_{xy} = r\frac{\sigma_x}{\sigma_y} \Rightarrow \sigma_y^2 = 25$ Ans. (3)

If $\sigma_x = \sigma_y = 1$ and $\theta = \tan^{-1}(3)$ find r.

Sol^n

$$3 = \left(\frac{1-r^2}{r}\right)\frac{\sigma_x^2}{2\sigma^2}$$

$$3 = \left(\frac{1-r^2}{r}\right)\left(\frac{1}{2}\right)$$

$$\left(\frac{1-r^2}{r}\right) = 6$$

$$r^2 - 1 + 6r = 0$$

$$r = \frac{-6 \pm \sqrt{36+4}}{2}$$

$$= -3 \pm \sqrt{10}$$

$\because r > 0$

We take $-3+\sqrt{10}$ Ans

If the line of regression of $\bar{x} = 5$, $\bar{y} = 10$, is $\parallel^{el}$ to the line $20y = 9x + 40$ find y for $x = 30$.

Sol^n: $(x - \bar{x}) = b_{xy}(y - \bar{y})$

$(y - \bar{y}) = b_{yx}(x - \bar{x})$

$9x - 20y + 40 = 0$ slope $= \frac{9}{20} = b_{yx}$

$\because \parallel^{el}$

$$(y - 10) = \frac{9}{20}(30 - 5)$$

$$= 11.25$$

$y = 21.25$ Ans

Q. If line of regression made by is $\tan\theta = 0.6$, $\sigma_y = 2\sigma_x$. Find r.

$$0.6 = \left(\frac{1-r^2}{r}\right)\frac{2\sigma_x^2}{\sigma_x^2}$$

$$\frac{3}{5} = \left(\frac{1-r^2}{r}\right)\frac{2}{5}$$

$$3r = 2 - 2r^2$$

$$\frac{3}{5} = \left(\frac{1-r^2}{r}\right)\frac{2}{5}$$

$$3r = 2 - 2r^2$$

$$2r^2 + 3r - 2 = 0$$

Given: $b_{yx} = 0.6$

$\sigma_y = 2\sigma_x$ $\quad 2r^2 = 1$

$$\Rightarrow r\frac{\sigma_y}{\sigma_x} = 0.6$$

$$\Rightarrow r\frac{2\sigma_x}{\sigma_x} = 0.6 \Rightarrow r = 0.3$$ Ans

$P(A|B) = \frac{n(A \cap B)}{n(B)}$ A: card drawn is king

$P(A) = 4/52$

B: Red card

26 red cards

$n(B) = 26$

$n(A \cap B) = 2$ (Red kings)

$\therefore P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{26}$

Q. Data on the leadership of certain magazine shows that proportion of male leader under 35 = 0.4

over 35 = 0.2

If proportion of leaders under 35 = 0.7, find the proportion of subscribers that are females over 35 years. Also find the probability that randomly selected male subscriber is under 35 years of age.

Sol^n A: Male reader

B: Reader is over 35 years of age

$P(A \cap B) = 0.2$

$P(A \cap \bar{B}) = 0.4$

$P(\bar{B}) = 0.7$

$\Rightarrow P(B) = 0.30$ (Given)

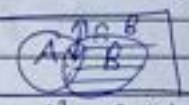
Proportion of subscribers that are female

(i) P(Female reader is over 35)

$= P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$= 0.3 - 0.2$

$= 0.10$



(ii) $P\left(\frac{\bar{B}}{A}\right) = \frac{P(A \cap \bar{B})}{P(A)}$ $(\because P(A \cap B) = P(\frac{B}{A}) P(A))$

Also $P(A \cap B) = P\left(\frac{A}{B}\right) P(B)$

But $P(A) = P(A \cap B) + P(A \cap \bar{B})$

$\therefore P(\bar{B}/A) = \frac{0.40}{0.60} =$

Q. From a city population, probability of selecting

(i) a male or a smoker is $\frac{7}{10}$.

(ii) a female smoker is $\frac{2}{5}$ &

(iii) a male, if smoker is already selected is $\frac{2}{7}$.

Find the probability of selecting: (1) non-smoker

(2) Male

(3) Smoker, if a male is first selected

Sol^n Given ~~A made~~ $P(A \cup B) = \frac{7}{10}$

$P(A \cap B) = \frac{2}{5}$

(1) $P\left(\frac{A}{B}\right) = \frac{2}{3}$

$P(\bar{B}) = 1 - P(B)$

$= 1 - \frac{P(A \cap B)}{P(A/B)}$

$= 1 - \frac{2/5}{2/3}$

$= 1 - \frac{3}{5} = \frac{2}{5}$

$\Rightarrow P(B) = \frac{3}{5}$ — (*)

(2) $P(A) = P(A \cap B) + P(A \cup B) - P(B)$ (Addition theorem)

$= \cancel{\frac{7}{10} + \frac{2}{5} - \frac{3}{5}}$

$\frac{2}{5} + \frac{7}{10} - \frac{3}{5}$

$= \frac{7}{10} - \frac{1}{5} = \frac{7-2}{10} = \frac{1}{2}$

(I) $P(B/A) = \frac{P(A \cap B)}{P(A)}$

$= \frac{\frac{2}{5}}{1/2} = \frac{4}{5}$

Baye's Theorem

If $E_1, E_2, \ldots E_n$ are mutually disjoint events, $[E_i \cap E_j = \phi \ \forall i \neq j]$ $P(E_i) \neq 0$

then for any arbitrary event A

$A \subseteq \bigcup_{i=1} E_i$

and $P(A) > 0$

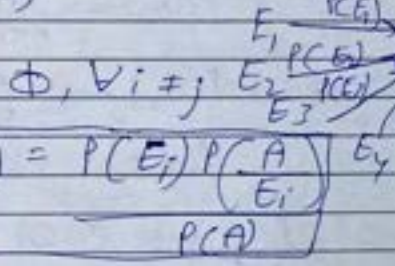
we have

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \times P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)} \Big\} P(A)$$

$$= \frac{P(E_i) \times P\left(\frac{A}{E_i}\right)}{P(A)}$$

$A \subseteq \bigcup E_i$

$E_i \cap E_j = \phi, \forall i \neq j$



then $\boxed{P(E_i/A) = \frac{P(E_i) P\left(\frac{A}{E_i}\right)}{P(A)}}$

Q. A product is produced in three factories X, Y & Z. It is already known that X produces price as many times as factory Y and, factory Y & Z produces same no. of items. It's also known that, 3% of items produced by X and Z are defective, 5% of items by

Y are defective. An item is selected at random.

i) Probability that the item is defective.
ii) If an item is already found to be defective, what is the probability that it's produced by factory X, Y or Z?

Ans: Items produced by $Y = n$.
$\therefore$ items " " $X = 3n$.
$\therefore$ Items " " $Z = n$.

Let E_i be an event that item is produced by X, Y, Z respectively.
Let A = item is defective.

$$P(E_1) = \frac{3n}{n+3n+n} = \frac{3}{5}$$

$$P(E_2) = \frac{n}{5n} = \frac{1}{5}$$

$$P(E_3) = \frac{n}{5n} = \frac{1}{5}$$

$$P\left(\frac{A}{E_1}\right) = 0.03$$

$$P\left(\frac{A}{E_2}\right) = 0.05$$

$$P\left(\frac{A}{E_3}\right) = 0.03$$

(i) $P(A) = \sum_{i=1}^{3} P(E_i) \times P\left(\frac{A}{E_i}\right)$

$$= \left(\frac{3}{5} \times 0.03\right) + \left(0.2 \times 0.05\right) + \left(0.2 \times 0.03\right)$$

$$= \underline{0.034}$$

(ii) $$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\, P\left(\frac{A}{E_1}\right)}{\sum P(E_i)\, P\left(\frac{A}{E_i}\right) = P(A)}$$

$$= \frac{0.6 \times 0.03}{0.034} = \frac{0.018}{0.034} = \frac{9}{17}$$

$$= \underline{0.529}$$

(iii) $$P\left(\frac{E_2}{A}\right) = \frac{0.2 \times 0.05}{0.034} = \frac{1}{3.4} = \frac{5}{17}$$

$$= \underline{0.294}$$

and,

$$P\left(\frac{E_3}{A}\right) = \frac{3}{17} = \underline{0.1765}$$

Q. Pre-candidates for the position of principal, whose chances of getting appointed are in the proportion 4:2:3 respectively. Probability that X gets selected, who would introduce ~~4~~ core education in the college is 0.3. Probability that Y gets selected in the same condition are respectively 0.5 and 0.8.

Q(a) Probability that there'll be co-education in the college.

Q(b) If already there is co-education in the college, then Mr. Z is the principal?

Sol^n Let E_i be the event Mr. X gets selected

A is the event there's co-ed^n in the college.

~~Event E_i | Prior prob. $P(E_i)$ | conditional prob. $P(A/E_i)$ | Joint prob. $P(A \cap E_i)$~~

Event E_i	Prior prob $P(E_i)$	conditional prob $P(A/E_i)$	Joint prob $P(A \cap E_i)$	Posterior prob $P(E_i/A)$
E_1	4/9	0.3	* 2/15	** 0.16
E_2	2/9	0.5	* 1/9	0.27 **
E_3	1/3	0.8	* 4/15	0.522 **
			$P(A) = 0.511$	

$$\frac{2/15}{2/15 + \frac{1}{9} + \frac{4}{15}} = \frac{2/15}{\frac{23}{45}} = \frac{\frac{2}{15}}{\frac{23}{45}}$$

Q. A factory produces certain type of output by daily mar. M_1: 3000 units M_2: 2500 M_3: 4500. Past experience shows that 1% of the output by M_1 is defective, 1.2% by M_2 is defective and 2% by M_3 is defective. Item is drawn at random and found to be defective. What is the probability that:

E_i = Machine M_1, M_2, M_3 produced the machine

A = The product is defective.

Event E_i (1)	Prior prob $P(E_i)$ (2)	Conditional prob $P(A/E_i)$	Joint prob $P(A \cap E_i)$	Posterior prob $P(E_i/A)$
E_1	0.3	$\frac{1}{100}$	0.003	1/5
E_2	0.25	$\frac{1.2}{100}$	0.003	1/5
E_3	0.45	$\frac{2}{100}$	0.009	3/5

$P(A) = 0.015$

Q. There are two bags I & II

I → n white, 2 black balls.

II → 2 white, n black balls.

One bag is selected at random and 2 balls are drawn from it. If both the balls drawn are white, and the probability that bag I

was used is $\frac{6}{7}$, find the value of n.

Sol^n: Let E_1 = bag 1 is selected
and E_2 = bag 2 is selected
and A = both balls drawn are white.

Then, $P\left(\frac{A}{E_1}\right) = \cancel{\frac{6}{7}}$?

$P(E_1) = \frac{1}{2}$

$P(E_2) = \frac{1}{2}$

Bayes' theorem

$$P(E_i/A) = \frac{P(E_i)\,P\left(\frac{A}{E_i}\right)}{P(A)}$$

$\cancel{P(E_1) = \frac{1}{2}}$ But $P\left(\frac{E_1}{A}\right) = \frac{6}{7}$

$\cancel{P(E_2) = \frac{1}{2}}$

$$= \frac{P(E_1)\,P\left(\frac{A}{E_1}\right)}{P(E_1)\,P\left(\frac{A}{E_1}\right) + P(E_2)\,P\left(\frac{A}{E_2}\right)}$$

Here $P\left(\frac{A}{E_1}\right) = \frac{{}^nC_2}{{}^{n+2}C_2} = \frac{\frac{n!}{\cancel{2!}(n-2)!}}{\frac{(n+2)!}{n!\,\cancel{2!}}}$

$= \frac{n! \times n!}{(n-2)!\,(n+2)!} = \frac{n(n-1) \times n!}{(n+2)!}$

$= \frac{n(n-1)}{(n+1)(n+2)}$

$\cancel{= \frac{6}{7}}$

$\Rightarrow 7(n^2 - n) = 6(n^2 + 3n + 2)$

$\Rightarrow n^2 - 25n - 12 = 0$

$P\left(\frac{A}{E_2}\right) = \frac{{}^2C_2}{{}^{n+2}C_2} = \frac{1}{\frac{(n+2)!}{n!\,2!}}$

$= \frac{n!\,2!}{(n+2)!} = \frac{2}{(n+1)(n+2)}$

$\therefore P\left(\frac{E_1}{A}\right) = \frac{\frac{1}{\cancel{2}}\,\frac{n(n-1)}{(n+1)(n+2)}}{\frac{n(n-1)}{\cancel{2}(n+1)(n+2)} + 2}$

$= \frac{n(n-1)}{n(n-1) + 2} = \frac{6}{7}$

$\therefore n > 0$

$\boxed{n = 4}$

Ans

$$\left[\because \; (n^2-n) = 6(n^2-n)+12 \right.$$

$$\Rightarrow \; n^2-n-12=0$$

$$(n-4)(n+3)=0$$

$$\left. \underline{n=4, \; \cancel{-3}} \; (\because n>0) \right]$$

Q. Probability of X, Y, Z becoming managers are $\frac{4}{9}$, $\frac{2}{9}$ and $\frac{1}{3}$ respectively. Probability that bonus will be introduced if X, Y, Z becomes manager are $\frac{3}{10}$, $\frac{1}{2}$ and $\frac{4}{5}$ respectively.

- What is the probability that bonus will be introduced?
- If bonus scheme has already been introduced, what is the probability that manager introduced was X?

Soln.

E_1 : X becomes manager
E_2 : Y " "
E_3 : Z " "

Q. The contents of an urn 1, 2 and 3 are as follows:

I → 1W, 2B, 3R

II → 2W, 1B, 1R

III → 4W, 5B, 3R

W → White balls
B → Black balls
R → Red balls

One urn is chosen at random, and 2 balls drawn. They happen to be W and R. What is the probability that the balls come from urn I, II or III?

Soln. Let E_1, E_2 & E_3 be event that balls are chosen from urn I, II or III.

$$\therefore P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$\therefore P\left(\frac{A}{E_1}\right) = \frac{1 \times 3}{{}^6C_2} \quad P\left(\frac{A}{E_2}\right) = \frac{4 \times 3}{{}^{12}C_2} \quad P\left(\frac{A}{E_3}\right) = \frac{2 \times 1}{{}^4C_2}$$

$P\left(\frac{A}{E_1}\right) = \frac{1}{5}$

$P\left(\frac{A}{E_3}\right) = \frac{12}{{}^{12}C_2} = \frac{12}{11 \times 6} = \frac{2}{11}$

$P\left(\frac{A}{E_2}\right) = \frac{1}{3}$

Q. In answering a question of Multiple Choice Test, a student either knows the answer or he guesses. p be the probability that he knows the answer, $1-p$ be probability he guesses. Assume that the student who guesses the answer will be correct answer, with probability $\frac{1}{5}$.

If S is the no. of alternatives, what is the conditional prob. that student knew the answer to question given that he answered it correctly.

Sol^n E_1: student knows right answer.
E_2: student guesses " "
A: answer is correct
$P(E_1) = p$

$P(E_2) = 1-p$ $\quad \therefore P(E_1) + P(E_2) = 1$

$P\left(\frac{A}{E_1}\right) = 1$

$P\left(\frac{A}{E_2}\right) = \frac{1}{5}$

$\therefore P\left(\frac{E_1}{A}\right) = \frac{p \times 1}{p + (1-p)\frac{1}{5}} = \frac{5p}{4p+1}$

$\because P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right)}$

Two-dim. random variable

Let X, Y be two random var defined as same sample space S, then the function (X,Y) which assigns point in R^2 i.e. $R \times R$ i.e. $\{(x,y) \mid x \in R, y \in R\}$.

$S = \{(HHH), (HHT), (HTH), (THH), (HTT), (THT), (TTH), (TTT)\}$

R.V X = no of Tails

$X = 0$
$X = 1$
$X = 2$
$X = 3$

$(X,Y) \in R^2 = R \times R$

$X \in R$ $\quad = \{(x,y) \mid x \in R, x \in R\}$

When the function (X,Y), which assigns Cartesian product of 2 sets is called 2-dimensional

2-dimensional or joint probability mass function (PMF).

Defn: If (X,Y) is 2-D discrete random variable, with each possible outcome (x_i, y_j) with probabilities P_{ij} satisfying the condition $P_{ij} \geq 0$ for every i,j

1. $P(x_i, y_j) \geq 0, \forall i,j$

2. $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(x_i, y_j) = 1$

Then, P is called joint probability mass function of (X,Y) (jt pmf) and its probability distribution is given by

$X \backslash Y$	x_1	x_2	x_3	y_1	y_2	y_3 TOTAL
y_1	P_{11}	P_{21}	P_3	P_1		
y_2	P_{12}	P_{22}	P_{32}	P_2		
y_3	P_{13}	P_{23}	P_{33}	P_3		
	P_1'	P_2	P_3'	1		

↑ Joint prob. distribution of function (X,Y)

Marginal probability distribution

If X is a random variable taking values $x_1, x_2, x_3, \ldots x_n$ with probabilities $p_1, p_2, \ldots p_n$, then the marginal ~~prob distribution~~ of X is given by

$P(X) = p_1, p_2, \ldots p_n$

where $p_i' = \sum_{j=1}^{n} p_{ij}$

Q. Joint conditional probability is given by:

$X = x_i$, given that $Y = y_i$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(X=x_i, Y=y_i)}{P(Y=y_i)}$$

$$P(X=x_i / Y=y_i) = \frac{P(X=x_i, Y=y_i)}{P(Y=y_i)}$$

Q. Joint prob distribution of $P(X=x, Y=y) = \frac{x+3y}{24}$, $x, y = 1, 2$.

Find pmf and marginal prob. distribution of X & Y.

$Y \backslash X$	1	2	Total
1	$\frac{4}{24} = \frac{1}{6}$	$5/24$	$9/24$
2	$\frac{7}{24}$	$\frac{8}{24} = \frac{1}{3}$	$15/24$
	$11/24$	$13/24$	1

Marginal prob. distribution of

X	1	2	Total
$P(X)$	$11/24$	$13/24$	1

Q. Joint probability distribution of $X_1 = x_1$, $X_2 = x_2$

$$P(X_1 = x_1, X_2 = x_2) = \frac{1}{27}(x_1 + 2x_2)$$

$$x_1, x_2 = 0, 1, 2$$

Find pmf of X_1 & X_2 in terms of x_1 & x_2.

Soln

Y/X	x_1	x_2	x_3	Total
y_1	P_{11}	P_{21}	P_{31}	$P_1 = \sum_{i=1}^{3} P_{i1}$
y_2	P_{12}	P_{22}	P_{32}	$P_2 = \sum_{i=1}^{3} P_{i2}$
y_3	P_{13}	P_{23}	P_{33}	$P_3 = \sum_{i=1}^{3} P_{i3}$
	$P_1' =$			1

X: x_1, x_2, x_3

$P(X)$: P_1, P_2, P_3

$$P(X_1 = x_1) = \sum_{x_2=0}^{2} P(x_1, x_2)$$

$$= \frac{1}{27}[x_1 + (x_1+2) + (x_1+4)]$$

$$= \frac{x_1 + 2}{9}$$

Marginal pmf for X_1

X_1: 0 1 2

$P(X_1)$: $\frac{6}{27}$ $\frac{9}{27}$ $\frac{12}{27}$

Marginal pmf for X_2

$$P(X_2 = x_2) = \sum_{x_1=0}^{2} P(x_1, x_2)$$

$$= \sum_{x_1=0}^{2} \frac{1}{27}(x_1 + 2x_2)$$

$$= \frac{1}{27}[(2x_2) + (1+2x_2) + (2+2x_2)]$$

$$= \frac{1}{9}[2x_2 + 1]$$

X_2: 0 1 2

$P(X_2)$: $1/9$ $3/9$ $5/9$

Q. Three balls are drawn at random without replacement, from a box containing 2 white 4 black & 3 red.

X = no. of white balls

Y = no. of red ball.

1. Joint Pd. of (X, Y)
2. $P(X \le 1)$, $P(X \le 1, Y \le 1)$, $P(Y \le 2 / X \le 1)$, $P(X+Y \le 2)$
3. M org. prob. distribution of X.
4. M org. " " of Y.
5. cond. prob dist. of X given Y=1.
6. " " " " Y " X=2
7. X, Y indep

Soln

Y/X	0	1	2	3	Total
0	$\frac{1}{21}$	$\frac{4}{21}$	$1/7$	$\frac{1}{84}$	$\frac{37}{84}$
1	$1/7$	$\frac{2}{7}$	$1/4$	0	$\frac{12}{84}$
2	$1/21$	$1/28$	0	0	$\frac{7}{84}$
Total	$\frac{7}{21}$	$\frac{17}{28}$	$\frac{3}{14}$	$\frac{1}{84}$	1

$P(X=0, Y=0) = \frac{{}^4C_2}{{}^9C_2} = \frac{1}{21}$

$P(A) = \frac{n(A)}{n(S)}$

$$P(X=1, Y=0) = \frac{{}^2C_1 \times {}^4C_2}{{}^9C_3} = \frac{2 \times 6}{84} = \frac{1}{7}$$

$$P(X \le 1) = P(X=0) + P(X=1) = \frac{35}{84} + \frac{42}{84} = \frac{77}{84}$$

$$P(X \le 1, Y \le 2) = P(X=0, Y \le 2) + P(X=1, Y \le 2)$$
$$= \left(\frac{1}{21} + \frac{3}{14} + \frac{1}{7}\right) + \left(\frac{1}{7} + \frac{2}{7} + \frac{1}{14}\right)$$
$$= \frac{19}{21}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(Y \le 2 / X \le 1) = \frac{19/21}{77/84} = \frac{76}{77}$$

$$P(X+Y \le 2) = P(X=0, Y \le 2) + P(X=1, Y=1) + P(X=2, Y=0)$$
$$= \frac{17}{42} + \frac{3}{7} + \frac{1}{21} = \frac{27}{42}$$

(2)

X:	0	1	2
f(x):	3/84	42/84	7/84

Y:	0	1	2	3
P(Y):	5/21	15/28	3/14	1/84

$$P(X/Y=1)$$
$$= P(X=0/Y=1) + P(X=1/Y=1) + P(X=2/Y=1)$$
$$= \frac{P(X=0, Y=1)}{P(Y=1)} + \frac{P(X=1, Y=1)}{P(Y=1)} + \frac{P(X=2, Y=1)}{P(Y=1)}$$
$$= \frac{3/14}{15/28} + \frac{2/7}{15/28} + \frac{1/28}{15/28}$$

cond prob. of Y given X=2

$$\frac{P(X=2, Y=0)}{P(X=2)} \quad \frac{P(X=2, Y=1)}{P(X=2)} \quad \frac{P(X=2, Y=2)}{P(X=2)} \quad \frac{P(X=2, Y=3)}{P(X=2)}$$

$$P(0,1) = 3/14$$
$$P(0) = 35/84$$
$$P(1) = 42/84$$
$$P(0)P(1) = \frac{35}{84} \times \frac{15}{28} \neq \frac{3}{14}$$

Let (X,Y)

(1) $f_{XY}(x,y) \Rightarrow$ pdf

(i) $f_{XY}(x,y) \ge 0$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

(iii) $\int_c^d \int_a^b f_{XY}(x,y)\,dx\,dy = P(a \le x \le b, c \le y \le d)$

(2) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

(3) $f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$, $\int_{-\infty}^{\infty} f_y(y)\,dy = 1$

$$f_{xy}(x,y) = \begin{cases} 15e^{-3x-5y}, & x \geq 0,\ y > 0 \\ 0, & o.w \end{cases}$$

Find (a)

(1) $p(1<x<2,\ 0.2<y<0.3)$

(2) $p(x<2,\ y>0.2)$

(b) Marginal dist^n of x & y

Sol^n (a) (1) $p(1<x<2,\ 0.2<y<0.3)$

$$= \int_{0.2}^{0.3}\int_{1}^{2} 15e^{-3x-5y}\,dx\,dy$$

$$= 15\left[\int_{0.2}^{0.3} e^{-5y}\,dy \int_{1}^{2} e^{-3x}\,dx\right]$$

$$= 15(0.029 \times 0.016)$$

$$= \underline{0.0684}$$

(2) $P(x<2, y>0.2) = \int_{0.2}^{\infty}\int_{0}^{2} 15e^{-3x-5y}\,dx\,dy$

$$= \underline{0.367}$$

(3) $f_x(x) = 15e^{-3x}\int_{y=0}^{\infty} e^{-5y}\,dy = 3e^{-3x}$

$$f_y(y) = 15e^{-5y}\int_{x=0}^{\infty} e^{-3x}\,dx$$

$$= \underline{5e^{-8x}}$$

<u>Given</u>: $f_{xy}(x,y) = \begin{cases} cx(x-y), & 0<x<2,\ -x<y<x \\ 0, & o.w \end{cases}$

Find

(1) evaluate c

(2) find $f_x(x)$

(3) $f_{y|x}(y|x)$

(4) $f_y(y)$

Sol^n we have $\int_{x=0}^{2}\int_{y=-x}^{x} cx(x-y)\,dx\,dy = 1$

$$\Rightarrow 2c\int_{x=0}^{2} x\left[xy\right]_{0}^{x}\,dx = 1$$

$$\Rightarrow 2c\int_{0}^{2} x^3\,dx = 1$$

$$\Rightarrow c = \frac{1}{2\times 4} = \frac{1}{8}$$

(2) $f_x(x) = \int_{y=-x}^{x} \frac{1}{8}x(x-y)\,dy$

$$= \frac{2}{8}x\int_{0}^{x} x\,dy = \frac{1}{4}x\left[\frac{x^2}{2}\right]_{0}^{x}$$

(3) $f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$, $\int_{-\infty}^{\infty} f_y(y)\,dy = 1$

$$f_{xy}(x,y) = \begin{cases} 15e^{-3x-5y}, & x \geq 0, y > 0 \\ 0, & \text{o.w} \end{cases}$$

Find (a)

(1) $p(1<x<2, 0.2<y<0.3)$

(2) $p(x<2, y>0.2)$

(b) Marginal dist^n of x & y

Sol^n (a) (1) $p(1<x<2, 0.2<y<0.3)$

$$= \int_{0.2}^{0.3}\int_{1}^{2} 15e^{-3x-5y}\,dx\,dy$$

$$= 15\left[\int_{0.2}^{0.3} e^{-5y}\,dy \times \int_{1}^{2} e^{-3x}\,dx\right]$$

$$= 15(0.029 \times 0.016)$$

$$= \underline{0.0684}$$

(2) $P(x<2, y>0.2) = \int_{0.2}^{\infty}\int_{0}^{2} 15e^{-3x-5y}\,dx\,dy$

$$= \underline{0.367}$$

(3) $f_x(x) = 15e^{-3x}\int_{y=0}^{\infty} e^{-5y}\,dy = 3e^{-3x}$

$$f_y(y) = 15e^{-5y}\int_{x=0}^{\infty} e^{-3x}\,dx$$

$$= \underline{5e^{-8x}}$$

Given: $f_{xy}(x,y) = \begin{cases} cx(x-y), & 0<x<2, -x<y<x \\ 0, & \text{o.w} \end{cases}$

Find

(1) evaluate c

(2) find $f_x(x)$

(3) $f_{y|x}(y|x)$

(4) $f_y(y)$

Sol^n we have $\int_{x=0}^{2}\int_{y=-x}^{x} cx(x-y)\,dx\,dy = 1$

$$\Rightarrow 2c\int_{x=0}^{2} x\left[xy\right]_{0}^{x}\,dx = 1$$

$$\Rightarrow 2c\int_{0}^{2} x^3\,dx = 1$$

$$\Rightarrow c = \frac{1}{2\times 4} = \frac{1}{8}$$

(2) $f_x(x) = \int_{y=-x}^{x} \frac{1}{8}x(x-y)\,dy$

$$= \frac{2}{8}x\int_{0}^{x} x\,dy = \frac{1}{4}x\left[\frac{x^2}{2}\right]_{0}^{x}$$

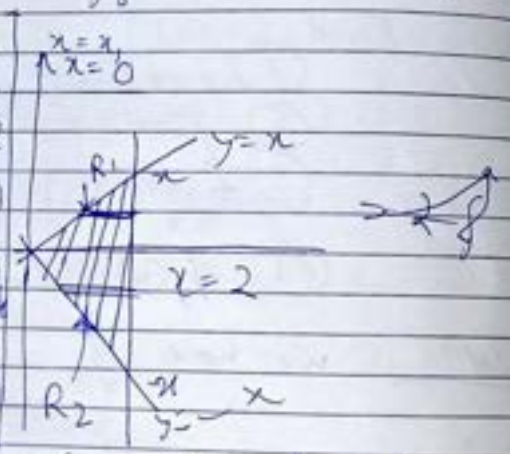
$= \frac{x^2}{2}[x]$

$= \frac{x^3}{4}, \; 0<x<2$

(4) $f_1(y) = \int_{y}^{2} \frac{1}{8} x(x-y)\,dx, \quad -x<y<x$

$= \int_{x=y}^{2} \frac{x}{8}(x-y)\,dx$, $0<y<2$ (in R_1)

$= \int_{x=y}^{2} \frac{1}{8} x(x-y)\,dx$, $-2<y<0$ (in R_2)



* The distribution func of discrete random var X.

Definition Let X be a discrete ~~continuous~~ random variable taking values $x_1, x_2 \ldots x_n$.

$x_1 < x_2 < x_3 \ldots < x_n$

$F(x_1) = P(x_1)$

$F(x_n) = \sum_{i=1}^{n} P(x_i)$

~~It is a function which takes such that~~ F is called CDF $x_i; F(x_i) \to CPD$

e.g. X: 1, 2, 3, 4

s.t. $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$

Consider a f^n & Find pmf (probability mass function)

Sol^n Let $P(X=3) = k \Rightarrow$

X:	1	2	3	4
$P(X=x_i)$:	k	$\frac{k}{3}$	k	$\frac{k}{5}$

$\therefore \sum P(x_i) = 1 \Rightarrow k + \frac{k}{3} + k + \frac{k}{5} = 1$

$30k + 20k + 60k + 12k = 60$

$k = \frac{30}{61}$

X:	1	2	3	4
$P(X=x_i)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

$\to$ pmf

find pmf & cdf

Sol^n Let $P(X=3) = k$.

Discrete : pmf p dist^n

Continuous : pdf

cdf

Let $P(X=3) = k$

$P(X=1) = k/3$

$P(X=2) = k/3$

$P(X=4) = k/5$

$\therefore \sum P(X = x_i) = 1 \Rightarrow k = \frac{30}{61}$

X:	1	2	3	4
$P(X=x_i)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

X_i is a random variable

P_i's are pmf

And $(X_i, P(X_i))$ are probability distribution function.

$F(X=x_i)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

$P(1 \le X \le 3) = P(X=2)$
$= \frac{10}{61}$

$P(X \le 3) = \frac{55}{61}$

pdf for a continuous RV X:

Definition: A continuous function $f_{XY}(x,y)$

$y = f(x)$

(i) $f(x)$ is integrable
(ii) $f(x) \ge 0$

Discrete	Continuous
(1) $- P(X=x_i) \ge 0$	$- P(X=x_i) \ge 0$
(2) $- \Sigma P(X=x_i) = 1$	$- \int_a^b f(x)\,dx = 1,\ x \in [a,b]$
(3) $- P(a<X<b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$	$- \int_\alpha^\beta f(x)\,dx = P(\alpha \le X \le \beta),\ a < \alpha < \beta < b$

then f is called probability density function of a continuous random variable X.

Example: A continuous random variable has following pdf.

$f(x) = kx^2,\ 0 \le x \le 2$

Find k & find
(i) $P(0.2 \le X \le 0.5)$
(ii) $P(X \ge \frac{3}{4})$ given that $X \ge \frac{1}{2}$.

Soln: $\because f(x)$ is a pdf,

$\cancel{\Sigma f(x)}$

$\int_0^2 f(x)\,dx = 1$

$\Rightarrow k\left[\frac{x^3}{3}\right]_0^2 = 1$

$\Rightarrow \frac{8}{3}k = 1$

$\boxed{k = \frac{3}{8}}$

Now,
(i) $P(0.2 \le X \le 0.5)$
$= \int_{0.2}^{0.5} kx^2\,dx$
$= k\left[\frac{x^3}{3}\right]_{0.2}^{0.5} = \underline{0.0146}$

Let A: $X \ge 1/2$
B: $X \ge 3/4$

$P(A) = \int_{1/2}^{2} f(x)\,dx = \int_{1/2}^{2} \frac{3}{8}x^2\,dx = \frac{\left[x^3\right]_{1/2}^{2}}{8}$

$= \underline{0.984}$

$$P(B) = \int_{3/4}^{2} \frac{3}{8} x^2 dx = \frac{[x^3]_{3/4}^{2}}{8}$$

$$= \underline{0.947}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{P(B)}{P(A)} = \frac{0.947}{0.984} = \underline{0.96}$$

Q. Find the value of k such that the following will be the pdf. Also find $P(x \le 1.5)$, where $f(x)$ is,

$$f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k, & 1 \le x \le 2 \\ k(3-x), & 2 \le x \le 3 \end{cases}$$

Soln: $\sum f(x) = 1$

$kx + k + k(3-x)$

[$\because$ $f(x)$ ~~should~~ be a pdf, it must've $\int_0^3 f(x)\,dx = 1$]

$$\Rightarrow \int_0^1 kx\,dx + \int_1^2 k\,dx + \int_2^3 k(3-x) = 1$$

$$\Rightarrow k \cdot \frac{1}{2} + k + 3k - k\left(\frac{x^2}{2}\right)_2^3$$

$$\Rightarrow \frac{3k}{2} + 3k - \frac{5}{2}k = \underline{\underline{2k}} = 1$$

$$\Rightarrow k = \underline{\frac{1}{2}}$$

$$P(X \le 1.5) = \int_0^{1.5} f(x)\,dx = \int_0^1 kx\,dx + \int_1^{1.5} k\,dx$$

$$= \frac{k}{2} + \frac{k}{2} = k = \frac{1}{2}$$

- A continuous standard variable (cd)

$f(x) = A + Bx, \quad 0 \le x \le 1$

Mean $= 1/3$. Find A & B.

Soln: we have,

$$\int_0^1 f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 (A + Bx)\,dx = 1$$

$$\Rightarrow \left[Ax + \frac{Bx^2}{2}\right]_0^1 = 1$$

$$\Rightarrow A + \frac{B}{2} = 1$$

$$\Rightarrow 2A + B = 2 \quad \text{—} (1)$$

$$\text{Mean} = \int_0^1 x f(x)\,dx = \frac{1}{3} \text{ (Given)}$$

$$\Rightarrow \left[\frac{Ax^2}{2} + \frac{Bx^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\Rightarrow \frac{A}{2} + \frac{B}{3} = \frac{1}{3}$$

$$\Rightarrow 3A + 2B = 2 \quad \text{—} (2)$$

$$4A + 2B = 4$$

$$\underline{A = 2} \quad \underline{B = -2}$$

Q. Find k if

$$f(x) = \begin{cases} kx^2(1 - x^3), & 0 \le x \le 1 \\ 0, & \text{o.w} \end{cases}$$

is a cdf

Also, find (1) $P(0 \le x \le 1/2)$

(2) mean

(3) var

Soln

$$\int_0^1 kx^2(1 - x^3) = 1 \quad \Rightarrow k \cdot \frac{1}{6} = 1$$

$$k = 6$$

① $P(0 \le x \le 1/2) = \int_0^{1/2} 6x^2(1-x^3)\,dx$

$\Rightarrow \cancel{k = 0.039}$

$= \underline{0.2344}$

② Mean $= \int_0^1 x f(x)\,dx$

$= 6\int_0^1 x^3(1-x^3)\,dx$

$= \underline{0.643}$

③ $E(X^2) = \int_0^1 x^2 f(x)\,dx$

$= 6\int_0^1 x^4(1-x^3)\,dx = \frac{9}{20}$

$var(X) = E(X^2) - (E(X))^2 = \frac{9}{20} - \left(\frac{9}{14}\right)^2$

$= \underline{0.037}$

*Binomial distribution

A Random variable X follows a binomial distribution with parameters n & p.

if $P(X=x)$ ${}^nC_x p^x q^{n-x}$

where n = no. of trials

p = prob. of success

q = prob. of failure

s.t $p+q=1$

Prop: ① If X_1 & X_2, (n_1, p_1) & (n_2, p_2) Parameters

$X_1 \rightarrow (n_1, p_1)$

$X_2 \rightarrow (n_2, p_2)$

then $X_1 + X_2 \sim B(n_1+n_2, p)$

where $p = p_1 = p_2$

It follows a binomial distribution with parameters --- (--- =

② Mean $= E(X)$

$= np$

③ $var(X) = \boxed{npq}$

e.g. If X is binomially distributed with

$E(X) = 2 = np$

& $var(X) = 4/3 = npq$ } Given

Find $P(X)$ & $\underbrace{P(\text{atleast 1 success})}_{P(\ \ \ \

$\Rightarrow 2q = \frac{4}{3}$

$\Rightarrow q = \frac{2}{3}$

$\therefore p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

$np = 2$ $\quad \therefore P(X=x) = {}^nC_x p^x q^{n-x}$

$\Rightarrow n \times \frac{1}{3} = 2$ $\quad = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$

$\Rightarrow \boxed{n = 6}$

Given

x	0	1	2	3	4	5	6
$P(X=x)$	${}^6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^6$	${}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5$	${}^6C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$	${}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3$	${}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2$	${}^6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1$	${}^6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0$

x	0	1	2	3	4	5	6
$P(X=x)$	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

$P(\text{at least 1 success}) = 1 - P(\text{no success})$

$= 1 - \frac{64}{729} = \frac{665}{729} = 0.9$

Q. Prob. that man aged 60 will live upto 70 $= 0.65$ $(P(A) = 0.65)$

~~what~~ Prob (atleast 7 out of 10 such men will live upto 70) $= ?$ $(P(B) = ?)$

Sol^n X = no. of men who'll live upto 70

$p = 0.65$

$q = 0.35$

$P(X=x) = {}^nC_x p^x q^{n-x}$

$= {}^{10}C_x (0.65)^x (0.35)^{10-x}$

We want $P(X \geq 7) = \sum_{x=7}^{10} P(x)$

$= P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= \sum_{x=7}^{10} {}^{10}C_x (0.65)^x (0.35)^{10-x}$

$= 0.51$

Q. All Find p if $n = 6$ & $9P(X=4) = P(X=2)$ (use $q = 1-p$)

Ans. $p = 1/4$ (DIY) $\boxed{9 \times ({}^6C_4 p^4 q^2) = ({}^6C_2 p^2 q^4)}$ $\Rightarrow 9p^2 = 1 - p$

Q. The ratio of probability of 3 indep. successes to prob. of 2 indep. successes in 5 independent trials to the prob. of 2 successes in 5 independent trials is 1/4. What is the prob. of 4 successes in 6 independent trials.

Sol^n If $x = 3$, $n = 5$,

$P(X=x) = {}^5C_3 p^3 q^2$

If $x = 2$, $n = 5$,

$P(X=x) = {}^5C_2 p^2 q^3$

Given that $\frac{P(X=3)}{P(X=2)} = \frac{1}{4}$

$\Rightarrow \frac{{}^5C_3 p^3 q^2}{{}^5C_2 p^2 q^3} = \frac{1}{4} \Rightarrow \frac{p}{q} = \frac{1}{4}$

$\Rightarrow 4p = q = 1-p$

$\Rightarrow \boxed{p = \frac{1}{5}}$

If $x = 4,\ n = 6$

$P(X=4) = {}^6C_4 (p^4)(q^2)$

$= {}^6C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^2$

$= \frac{15 \times 16}{5^6}$

$= \boxed{\frac{48}{3125}}$

Q. A citem consists of n components, each functions with probability p. Total citem will function effectively if atleast half of its ~~components~~ components are functioning. For what value of p a 5 component citem more likely to operate effectively than a 3 component system?

Sol: $P(X=x) = {}^nC_x p^x q^{n-x}$

mean $= np$

variance $= npq$

$P(X=x) = {}^nC_x p^x q^{n-x}$

$P(5 \text{ comp sys}) > P(3 \text{ comp sys})$

$\frac{5}{2} = 2.5$ $\qquad$ $\frac{3}{2} = 1.5$

$(3, 4, 5)$ $\qquad$ $(2, 3)$

$X =$ No. of components

$P(5 \text{ comp. system functioning})$

$= P(X = 3, 4, 5)$

$= P(X=3) + P(X=4) + P(X=5)$

$= \sum_{x=3}^{5} {}^5C_3 p^x q^{5-x}$

& $P(3 \text{ comp. system functioning})$

$= P(X = 2, 3) = \sum_{x=2}^{3} {}^3C_x p^x q^{3-x}$

5-component system will work more effectively than a 3-component system if

$P(X = 3, 4, 5) > P(X = 2, 3)$

i.e. ${}^5C_3 p^3 q^2 + {}^5C_4 p^4 q + {}^5C_5 p^5 q^0 > {}^3C_2 p^2 q + {}^3C_3 p^3 q$

i.e. $10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3 \cancel{(1-p)}$

$\Rightarrow 10p^3(p^2 + 1 - 2p) + 5p^4 - 5p^5 + p^5 - 3p^2 + 3p^3 - p^3 > 0$

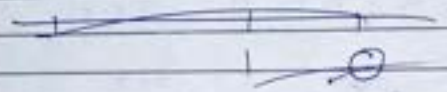
$\Rightarrow 10p^5 + 10p^3 - 20p^4 + 5p^4 - 4p^5 - 3p^2 + 2p^3 > 0$

$\Rightarrow 6p^5 + 12p^3 - 15p^4 - 3p^2 > 0$

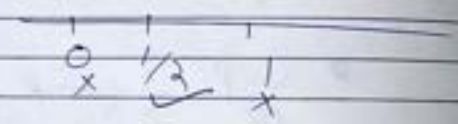
$\Rightarrow 2p^5 + 4p^3 - 5p^4 - p^2 > 0$

i.e. $p^2(2p^3 + p^2 - 5p^2 - 1) > 0$

$$p^2(p-1)^2(2p-1) > 0$$



i.e.



$x \to$ Atr. ind.

i.e. $\cancel{p > 1/2}$ $\sum_{x=3}^{5} {}^5C_x p^x q^{5-x} > \sum_{x=2}^{3} {}^3C_x p^x q^{3-x}$

i.e. ~~A 5 component system will work more effectively than 3~~

Q. 7 die are thrown 729 times. How many times do you expect atleast 4 die to show 3 or 5?

Soln.

1st □ □ □ □ □ □ □

2nd

3rd

729th

Soln. $N = 729$

$n = 7$

$X =$ No. of Dice showing 3 or 5

$p = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$

$P(X=x) = {}^nC_x p^x q^{n-x}$

$= {}^7C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{7-x}$

We want

$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6) + P(X=7)$

$= \frac{379}{3^7}$

Exp. $= NP(X \geq 4)$

$= 729 \times \frac{379}{3^7} = \frac{3^6}{3^7} \times 379$

$= \frac{379}{3}$

$= \underline{126}$

Q. Let X, Y be two binomial variant with parameter $n_1 = 6$, $p_1 = 1/2$, $n_2 = 4$, $p_2 = 1/2$ respectively. Evaluate:

① $P(X+Y=3)$

② $P(X+Y \geq 3)$

Soln. Let $Z = X+Y$ which is a binomial variant with parameters

$n = n_1 + n_2 = 10$ & $p = 1/2$

i.e. ① $P(Z=3) = {}^{10}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^7 =$

(2) $P(z \geq 3) = 1 - P(z < 3)$

$= 1 - [P(z=1) + P(z=2) + P(z=0)]$

$= 1 - [{}^{10}C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^9 + {}^{10}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8 + {}^{10}C_0 \left(\frac{1}{2}\right)^{10}]$

Q. 3 fair coins are tossed.

Let X = No of heads

$p = 1/2$, $N = 3000$

$q = 1/2$

$n = 3$

$P(X=x) = {}^nC_x p^x q^{n-x}$

$= {}^3C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x}$

X:	0	1	2	3
P(X=x):	1/8	3/8	3/8	1/8
f = Np:	$3000 \times \frac{1}{8}$ = 375	1125	1125	375

mean $= np = 1.5$

$SD = \sqrt{var}$

$= \sqrt{npq} = \sqrt{0.75} = 0.866$

$\bar{x} = \frac{0 + 1125 + 2250 + 1125}{375 + 1125 + 1125 + 375}$

$= 1.5$

Q. 7 coins are tossed.

Fitting of Binomial Distⁿ:

X:	0	1	2	3
	$\frac{3}{8}$			

$2^7 = 128$ times

No. of heads ~~and~~ obtained are as follows:

$(7-1)! = 720 \times 2^7$

No. of heads:	0	1	2	3	4	5	6	7
Freq.:	7	6	19	35	30	23	7	1

TOTAL (128)

Fit a binomial distribution if:

Case-1 : Coins are unbiased

Case-2 : Nature of the coins are not known.

Case-1 : $p = 1/2$, $q = \frac{1}{2}$, $n = 7$, $N = 128$

$P(X=x) = {}^nC_x p^x q^{n-x} = {}^7C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}$

X:	0	1	2	3	4	5	6	7
P(X=x):	$\left(\frac{1}{2}\right)^7$	$\left(\frac{7}{2}\right)^7$	$\left(\frac{21}{2}\right)^7$	$\left(\frac{35}{2}\right)^7$	$\left(\frac{35}{2}\right)^7$	$\left(\frac{21}{2}\right)^7$	$\left(\frac{7}{2}\right)^7$	$\left(\frac{1}{2}\right)^7$
Exp.	1	7	21	35	35	21	7	1

Now $\bar{x} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{\cancel{512}}{\cancel{128}} = \cancel{4.0}$ 3.98

$\therefore np = 3.38$

$\Rightarrow p = \frac{3.38}{n} = \frac{3.38}{7} = 0.48$

$\therefore P(X=x) = {}^7C_x (0.48)^x (0.52)^{7-x}$

Poisson Distribution

$n \to \infty$
$p \to 0$
but $m = np$ is finite

A random variable X follows a Poisson distribution with parameter m.
If $P(X=x) = \frac{e^{-m} m^x}{x!}$

mean = var = m

NOTE: If $x_1, x_2, \ldots x_n$ are n poisson variants, with parameters $m_1, m_2, \ldots$
Then, $y = x_1 + x_2 + \ldots + x_n$
is also poisson variant with parameter
$m = m_1 + m_2 + \ldots + m_n$
But, the variants such as
$y = a_1x_1 + a_2x_2 + \ldots + a_nx_n$
or $y = x_1 + x_2 + x_3 + \ldots + x_n$
are not poisson variants, all constants!
$\therefore$ mean $= m_1a_1 + m_2a_2 + \ldots$
var. $=$

Q. If mean of the Poisson distribution is 4, find $P(m-2\sigma < x < m+2\sigma)$

Sol^n: Given: mean = var = 4 = m
$\sigma = S.D = \sqrt{4} = 2$
$= \sqrt{var}$
(S.D > 0)

$\therefore P(m-2\sigma < x < m+2\sigma)$
$= P(0 < x < 8) = P(X = 1, 2, \ldots 7)$

but $P(X=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-4} 4^x}{x!}$

$\therefore P(0 < x < 8) = e^{-4}\left(\frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \ldots \frac{4^7}{7!}\right)$

$= e^{-4}(4 + 8 + \ldots) = 0.93$

Q. A car hire firm has

The no. of demands for a car on each day follows Poisson distribution
(NOTE: If p = small : Poisson,
p = large : Binomial)
Find the prob that
i) Neither car is used
ii) Same demand is refused

Sol^n. Let x = no. of demands

$m = 1.5$

$$P(X=x) = \frac{e^{-m} m^x}{x!}$$

$$= \frac{e^{-1.5} (1.5)^x}{x!}$$

① $P(X=0) = \frac{e^{-1.5} (1.5)^0}{0!} = e^{-1.5}$

② $P(X>2) = 1 - P(X \le 2) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$$= 1 - \left(e^{-1.5} + (e^{-1.5} \times 1.5) + \left(\frac{e^{-1.5}}{2} \times (1.5)^2\right)\right)$$

$= 0.19$

Q. If x_1, x_2, x_3 are the Poisson variants ~~of~~ and parameters $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ resply.

find ① $P[(x_1 + x_2 + x_3) \ge 3]$

② $P[(x_1 = 1) / x_1 + x_2 + x_3 = 3]$

Sol^n ① $z = x_1 + x_2 + x_3$ is a Poisson variant

$m_1 + m_2 + m_3 = 6$

$$P(z=x) = \frac{e^{-m} m^z}{z!} = \frac{e^{-6} 6^x}{z!}$$

$\therefore P(z \ge 3) = 1 - P(z < 3)$

$= 1 - [P(z=0) + P(z=1) + P(z=2)]$

$= 0.938$

② $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$P[(X_1 = 1) / X_1 + X_2 + X_3 = 3]$$

$$= \frac{P((X_1 = 1) \cap (X_1 + X_2 + X_3) = 3)}{P(X_1 + X_2 + X_3 = 3)}$$

$$= \frac{P[(X_1 = 1) \cap (X_2 + X_3 = 2)]}{P(X_1 + X_2 + X_3 = 3)}$$

$$= \frac{\left(\frac{e^{-1} \times 1}{1!}\right)\left(\frac{e^{-5} 5^2}{2!}\right)}{\left(\frac{e^{-6} 6^3}{3!}\right)}$$

$$= \frac{25}{72}$$

Q. An insurance company found that only 0.01% of the population meets with an accident each year. If 1000 policy holders were selected, what's the probability that no more than two of it's clients meet with an accident next year?
[P(success) = 0.01% = p]

Soln.

$p = 0.0001$

$n = 1000$

$np = 10^3 \times 10^{-4} = \underline{0.1}$

$P(X=x) = \frac{(e^{-m} m^x)}{x!}$

$\therefore m = 0.1$

$\therefore P(X=x) = \frac{e^{-0.1} (0.1)^x}{x!}$

we want $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \underline{0.9998}$

Q. Using Poisson distribution, find approximate value of : ${}^{300}C_2 (0.02)^2 (0.98)^{298}$

${}^{300}C_2 (0.02)^2 (0.98)^{298} + {}^{300}C_3 (0.02)^3 (0.98)^{297}$

$q = 0.98$

$p = 0.02 \quad q = 0.98$

$m = np = 6, \quad x = 2, 3$

$\therefore P(X=x) = \frac{e^{-m} m^x}{x!}$

$= \frac{e^{-6} 6^x}{x!}$

$\therefore P(X=2) + P(X=3) = \underline{0.1338}$

Q. Out of 1000 taxi drivers, find approximate no. of drivers with
(i) no accidents
(ii) more than 3 accidents in a year.

Soln. Let X = no. of accidents

$m = 3 \qquad N = 1000$

(i) $P(X=x) = \frac{e^{-m} m^x}{x!}$

$= \frac{e^{-3} 3^x}{x!}$

$\rightarrow P(X=0) = \frac{e^{-3} 3^0}{0!} = e^{-3} = \underline{0.0498}$

$\therefore$ Exp. no of drivers with no accidents $= 1000 \times 0.0498 = \underline{49.8}$

(ii) $P(X \geq 3) = 1 - [P(X=1) + P(X=2) + P(X=3)]$

$= 1 - \left[\frac{e^{-3}3}{} + \frac{e^{-3}3^2}{2} + \frac{e^{-3}3^3}{6}\right]$

$= 1 - [(0.0498) \times (3 + 4.5 + 4.5)]$

$= 1 - [0.0498 \times (12)]$

$= \cancel{0.402}$ $\underline{0.353}$

$\therefore$ Expected no. of drivers with more than 3 accidents $= 1000 \times 0.10$
$= \underline{\underline{353}}$

Q. A certain ~~of~~ factory turning out small chokes, any blade defective there's a small chance 1 out of 500 for any blade to be defective. Blades are supplied in a packet of 10 use Poisson distribution to calculate approx no. of packets containing
(i) no defective
(ii) one defective
(iii) two defective blades
in a consignment of 10,000 packets.

Soln: $n = 10$
$m = 10,000$ $p = 1/500$
$M = np = 0.02$

$P(X=x) = \frac{e^{-M} M^x}{x!}$

$= \frac{e^{-0.02} (0.02)^x}{x!}$

$P(X=0) = e^{-0.02} = 0.9802$ ~~defective~~
No. of packets ~~with no defective~~
$\Rightarrow 10,000 \times 0.9802$
$= \underline{9802}$

$\therefore$ No. of packets with 1 defective blade $\Rightarrow N = 10$
i.e. $P(X=1) = \frac{e^{-0.02} (0.02)^1}{1} \times 10000$

$= \underline{0.0196 \times 10^4 = 196}$

$P(X=2) = 0.9802 \times \cancel{100} (0.02)^2 \times 10000$

$= \cancel{0.9802 \times 2}$ $\cancel{1.9604 \times 10^4}$ $\underline{980}$

Q. Fit a Poisson distribution to the following data:

x_i: No. of deaths:	0	1	2	3	4	Total
f_i: Freq.	123	59	14	3	1	200

~~Total~~ For a Poisson

$$m = \text{mean} = \frac{\sum x_i f_i}{\sum f_i}$$

$$= \frac{0 + \cancel{123}\,59 + 28 + 9 + 4}{200}$$

$$= 0.5$$

$$P(X=x) = \frac{e^{-0.5}(0.5)^x}{x!}$$

$N = 200$

$P(X=0) = 0.6065$

$P(X=1) = 0.3033$

$P(X=2) = 0.07\cancel{6}58$

$P(X=3) = 0.0126$

$P(X=4) = 1.579 \times 10^{-3} = 0.001579$

Normal Distribution

A continuous random variable follows a normal distribution if ~~a normal distribution~~ it's pdf is given by

$$f(x) = \frac{e^{-1/2\left(\frac{x-m}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}, \quad -\infty < x < \infty, \quad -\infty < m < \infty$$

m (mean) & σ^2 (variance)

$P(X_i) = 1$

$\sum p_i = 1$

$$P(a < x < b) = \int_a^b f(x)\,dx$$

Remark: If X is a normal variate with parameter m and σ then

$Z = \frac{X - m}{\sigma}$ is also a normal variate with mean 0 and standard deviation 1.

~~If~~ Z is called standard normal variate, if X follows a normal ~~variation~~ distribution, $\boxed{X \sim N(m, \sigma) \text{ then } Z \sim N(0,1)}$.

Mean & variance of normal distribution

1. mean = mode = median = m

2. If x_1, x_2, x_n are normal variates, ~~they~~ with mean ~~and~~ $m_1, m_2, \ldots, m_n$ and variate $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, then $Y = a_1x_1 + a_2x_2 + \ldots + a_nx_n$

$$m = a_1m_1 + a_2m_2 + \ldots + a_nm_n$$

$$\& \ \sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots + a_n^2\sigma_n^2$$

3. ~~Quartiles~~

$Q_1 \quad Q_2 \quad Q_3$

$Q_1 = m - \frac{2\sigma}{3} \quad Q_3 = m + \frac{2\sigma}{3}$

$$Q.D. = \frac{Q_3 - Q_1}{2} = \frac{2\sigma}{3}$$

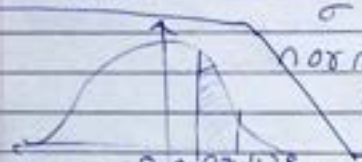
4. If ... P.T.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

4. If $X \sim N(m, \sigma)$

then $Z = \frac{X-m}{\sigma} \sim N(0,1)$ is std normal variate (SNV) is also a normal variate.

Q.



Find $P(0.87 < z < 1.28)$

Sol^n $P(0.87 < z < 1.28) = P(0 < z < 1.28) - P(0 < z < 0.87)$

$= 0.3997 - 0.3078$

$= \underline{0.0919}$

Q. 1. Normal distribution is $m = 50$, $\sigma = 15$

Find: (i) Q_1, Q_3 (ii) Q.D.

2 If $Q_1 = 36$, $Q_3 = 44$, find mean, std.

1.96 * $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$, $P(a \le x \le b) = ?$

$P(a \le x \le b) = \int_a^b f(x)\,dx$

$\frac{z = 0}{1}$ $\frac{x-m}{\sigma}$

Q. Find

(i) $P(-c < z < c) = 0.95$ $m = 120$, $\sigma = 10$

(ii) $P(|z| > c) = 0.01$

(iii) $P(X > c) = 0.02$

(iv) $P(X < c) = 0.05$

Given $m = 120$, $\sigma = 10$

Sol^n (i) value of z corresponding to 0.475, ~~ke cor~~

Reason: $\because -c < z < c$

$\Rightarrow 2P(\ )$ it'll be

Also, $1 - 0.95 = 0.05$ $\frac{0.05}{2} = 0.025$

so, $\underline{0.05}$

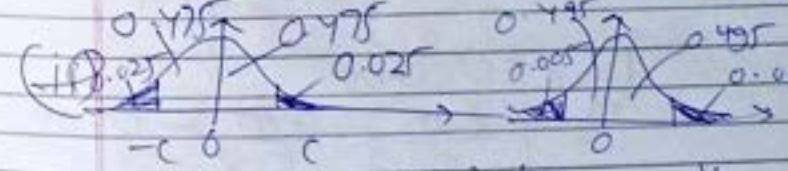
Note: ~~$z < c \Rightarrow c < 0$~~ $z = 1.96$

subtract from 0.5

$\underline{0.495}$ ~~(0 to c)~~ $(0.5 - 0.05)$

(print error mdf.)

$\underline{2.575}$ $\underline{c = 2.575}$



If z is a S.N.V then $z = \frac{X-m}{\sigma}$

(ii) $z = \frac{X-120}{10}$

check out

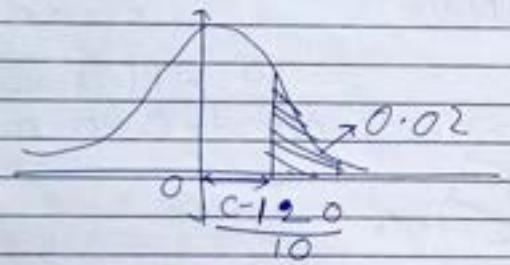
Find c if

(1) $P(X>c) = 0.02$
(2) $P(X<c) = 0.05$

Given $m=120$, $\sigma=10$

$$P\left(\frac{x-m}{\sigma} > \frac{c-m}{\sigma}\right) = 0.02$$

$$\Rightarrow P\left(z > \frac{c-120}{10}\right) = 0.02$$



$$P\left(0<z<\frac{c-120}{10}\right) = 0.5-0.02 = 0.48$$

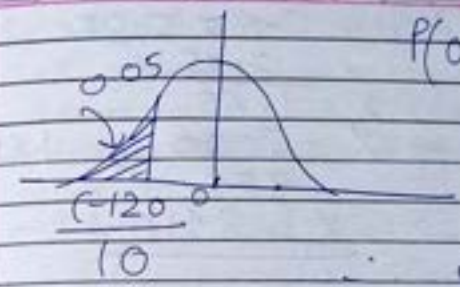
$$\therefore \frac{c-120}{10} = 2.06$$

$$\Rightarrow \boxed{c = 140.6}$$

$$P(X<c) = 0.05$$

$$\Rightarrow P\left(\frac{x-m}{\sigma} < \frac{c-m}{\sigma}\right) = 0.05$$

$$\Rightarrow P\left(z < \frac{c-120}{10}\right) = 0.05$$



$$P\left(0<z<\frac{c-120}{10}\right)$$

$$= 0.5 - 0.05$$

$$= 0.45$$

$$\therefore \frac{c-120}{10} = -1.65$$

$$c = 10.35$$

Q. Monthly salary X is normal distributed with mean (μ) ₹3000 & std. dev^n (σ) ₹250. What should be the minimum salary of a worker so that he belongs to top 5% workers?

Sol^n Let X = ~~monthly~~ monthly salary of a worker.

$m = 3000$ $\sigma = 250$

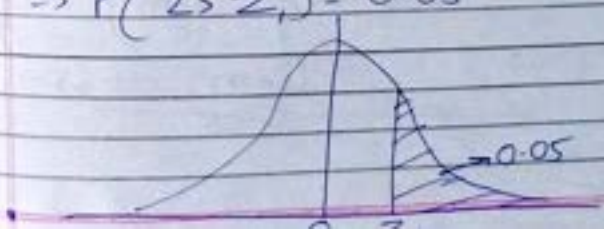
Let X_1 be minimum salary of top 5% workers

$$Z = \frac{X-m}{\sigma} = \frac{X-300}{250}$$

$$P(X > X_1) = 5\%$$

$$\Rightarrow P\left(\frac{X-1}{\sigma}\right) > \frac{X_1-1}{\sigma} = 5\% = 0.05$$

$$\Rightarrow P(z > z_1) = 0.05$$



$P(0<z<z_1) = 0.5 - 0.05$
$= 0.45$
$\therefore z = 1.65$
$\Rightarrow \frac{X_1 - 3000}{250} = 1.65$
$\Rightarrow X_1 = 3412$

Q ~~Th~~ Diameters of normally distributed (an tops) with standard deviation 0.05 cm.
At what mean diameter of the machine should be set so that not more than 5% of the tops can have diameter ~~at th~~ exceeding 3cm?

Soln

~~X = 1~~
$\sigma = 0.05$
$P(X > 3) = 5\%$
$\Rightarrow P\left(\frac{X-m}{\sigma} > \frac{3-m}{\sigma}\right) = 0.05$
$\Rightarrow P\left(z > \frac{z-m}{\sigma}\right) = 0.05$

$\therefore P\left(0 < z < \frac{3-m}{\sigma}\right) = 0.5 - 0.05$
$= 0.45$
$\Rightarrow \frac{3-m}{0.05} = 1.65 \Rightarrow m = 2.92$

Q ~~If~~ X, Marks obtained by 1000 students are found to be normally distributed, ~~the~~ with mean 70 & std. deviation ~~5~~ 5. ~~Calculate~~ (estimate) the no. of students whose marks will be (i) b/w 60 & 75 and those (ii) greater than or equal to 75.

Soln: X = marks of student
(i) $P(60 < X < 75) =$ 818
No. of students = NP
N = 1000
(ii) $P(X \geq 75)$ 159

N = 1000, ~~f.N~~
$z = \frac{x-m}{\sigma} = \frac{x-70}{5}$

(i) $X = 60, \; z = -2$
$X = 75, \; z = 1$
$P(60 \leq X \leq 75) = P(-2 < z < 1)$
$= P(0 < z \leq 2) + P(0 < z \leq 1)$
$= 0.4772 + .3413$
$= .8185$
$np = 1000 \times .8185$
$= 818$

(ii) $P(X \geq 75) = P(z \geq 1)$
$= 0.5 - P(0 < z \leq 1)$
$= 0.5 - .3413$
$= 0.1587$
$= 0.159$

Q. X_1 & X_2 are normal events with mean

$m_1 = 30$

$m_2 = 25$

$\sigma_1^2 = 16$

$\sigma_2^2 = 12$

$Y = 3x_1 - 2x_2$

Find probability of ...

$P(60 \le Y \le 80)$

REPEAT

Remark: If $x_1, x_2 \ldots x_n$ are normal variates, $m_1, m_2, \ldots m_n$, $\sigma_1^2, \sigma_2^2, \sigma_n^2$ (written back pages)

$Y = 3x_1 - 2x_2 = 3x_1 + (-2)x_2$

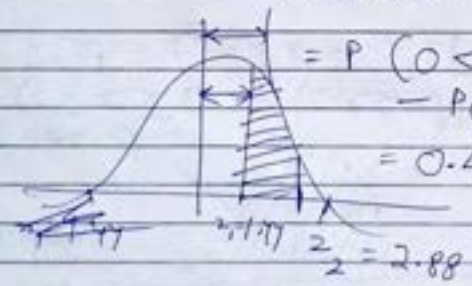
~~Find P(60 ≤~~

Solⁿ: Y is a normal variate with mean $m = (3 \times 30) + (-2) \times 25 = 40$

& $\sigma^2 = \Sigma a_i^2 \sigma_i^2 = (9 \times 16) + (4 \times 12) = 192$

$\therefore P(60 \le Y \le 80) = P\left(\frac{60-40}{\sqrt{192}} \le z \le \frac{80-40}{\sqrt{192}}\right)$

$= P(1.44 \le z \le 2.88)$



$= P(0 < z < 2.88) - P(0 < z < 1.44)$

$= 0.498 - 0.4251$

$= 0.0729$

Q. 2 independent random variables X & Y are distributed normally with mean and standard deviation (52, 3) & (50, 2) resply. Find the probability that randomly pair of values of X & Y will differ by 1.7 or more.

Solⁿ: i.e. $P(|X-Y| \ge 1.7)$

let $X - Y = 0$

$= P(|0| \ge 1.7)$

mean of $U = X$

$m_1 = 52$, $\sigma_1 = 3 \Rightarrow \sigma_1^2 = 9$

$m_2 = 50$, $\sigma_2 = 2 \Rightarrow \sigma_2^2 = 4$

$\hookrightarrow m = 9 + 4 = 13$

$\therefore P(|0| \ge 1.7) = 1 - P(|0| < 1.7)$

$= 1 - P[-1.7 < |0| < 1.7]$

OR $1 - P[-1.7 < U < 1.7]$

$= 1 - P\left(\frac{-1.7 - 2}{\sqrt{13}} < z < \frac{1.7-2}{\sqrt{13}}\right)$

$= 1 - P\left(-1.026 < z < \frac{-0.3}{\sqrt{13}}\right)$

$= 1 - \left\{P\left(0 < z < \frac{-0.3}{\sqrt{13}}\right) - P\left(0 < z < -1.026\right)\right\}$

~~$= 1 - \{$~~ $= -1.026 < \frac{U-2}{\sqrt{13}} < -0.08$

$= 0.6734$

Q. Incomes of a group of 10,000 persons are normally distributed ~~with~~ $\mu = 520$, $\sigma = 60$. Find

i) No. of persons having income b/w 400 & 550. $0.6687 \times 10000 = 6687$

ii) lowest income of richest 500.

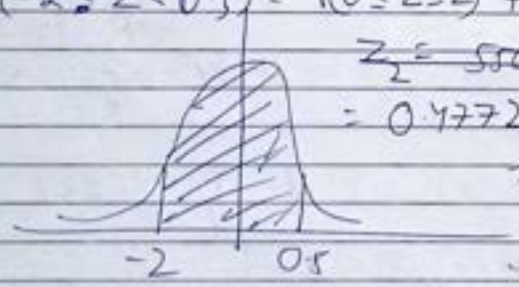
Sol: Income of the person

$z_1 = \frac{400-520}{60} = -2$ $\quad z_2 = \frac{550-520}{60} = 0.5$

$\therefore P(-2 \le z < 0.5) = P(0 \le z \le 2) + P(0 \le z \le 0.5)$

$= 0.4772 + 0.1915$

$= 0.6687$



$\therefore Np = 6687$

(ii) Let x_1 be the lowest income of richest 500.

$P(x > x_1) = 5\% = 0.05$ Rev. problem

$\Rightarrow$ Area $P(0 \le z \le ?) = 0.5 - 0.05 = 0.45$

From table, $? = 1.645$ at $p = 0.45$

$\therefore$ Req. value of $z = 1.645$

But $z = \frac{x - 520}{60}$, $\therefore 1.645 = \frac{x-520}{60}$

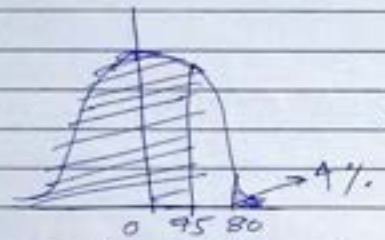
$x = 618.7$

Thus lowest income of richest 500 = ₹ 618.7

Testing hypothesis: $P(x > x_1) = 0.05$

Q. Find the mean and standard deviation of marks in an exam where 58% of candidates obtained marks below 75, and rest b/w 75 & 80.

Soln



X = marks.

$P(0 \le X \le 75) = 8\% = 0.08$

$\Rightarrow P\left(0 \le z \le \frac{75-m}{\sigma}\right) = 0.08$

$\Rightarrow \frac{75-m}{\sigma} = 0.2$

$\Rightarrow 0.2\sigma + m = 75$

$\frac{75-m}{\sigma}$

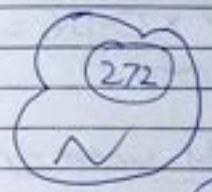
$P(75 \le X \le 80) = 0.92$

$\Rightarrow P\left(\frac{75-m}{\sigma} \le z \le \frac{80-m}{\sigma}\right) = 0.92$

$\Rightarrow P\left(z \le \frac{80-m}{\sigma}\right) - P\left(z \le \frac{75-m}{\sigma}\right) = 0.92$

Q. A large no. of automotive batteries have an avg. life of 24 months. If 34% of them have an average life of 22 - 26 months and 272 of them last longer than 29 months, how many total batteries were tested using normal distribution?

Soln



Given $m = 24$.

X = Avg. life of batteries

$\sigma = ?$

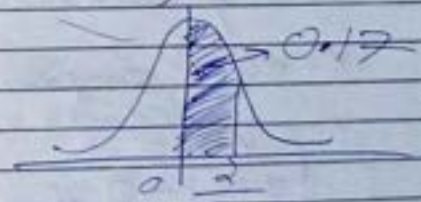
Given $P(22 \le X \le 26) = 0.34$

$\Rightarrow P\left(\frac{22-24}{\sigma} \le \frac{X-24}{\sigma} \le \frac{26-24}{\sigma}\right) = 0.34$

$\Rightarrow P\left(\frac{-2}{\sigma} \le z \le \frac{2}{\sigma}\right) = 0.34$

$\Rightarrow 2P\left(0 \le z \le \frac{2}{\sigma}\right) = 0.34$

$\therefore P\left(0 \le z \le \frac{2}{\sigma}\right) = 0.17$

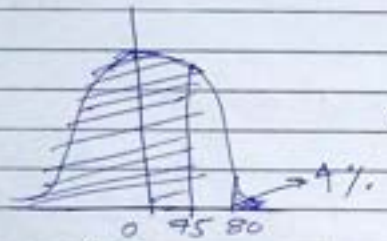


$\therefore \frac{2}{\sigma} = 0.44$

$\boxed{\sigma = \frac{2}{0.44} = 4.54}$

Q. Find the mean and standard deviation of marks in an exam where 58% of candidates obtained marks below 75, and rest b/w 75 & 80.

Soln



X = marks.

$P(0 \le X \le 75) = 8\% = 0.08$

$\Rightarrow P\left(0 \le Z \le \frac{75-m}{\sigma}\right) = 0.08$

$\Rightarrow \frac{75-m}{\sigma} = 0.2$

$\Rightarrow 0.2\sigma + m = 75$

$\frac{75-m}{\sigma}$

$P(75 \le X \le 80) = 0.92$

$\Rightarrow P\left(\frac{75-m}{\sigma} \le Z \le \frac{80-m}{\sigma}\right) = 0.92$

$\Rightarrow P\left(Z \le \frac{80-m}{\sigma}\right) - P\left(Z \le \frac{75-m}{\sigma}\right) = 0.92$

Q. A large no. of automotive batteries have an avg. life of 24 months. If 34% of them have an average life of 22 - 26 months and 272 of them last longer than 29 months, how many total batteries were tested using normal distribution?

Soln

Given $m = 24$

X = Avg. life of batteries

$\sigma = ?$

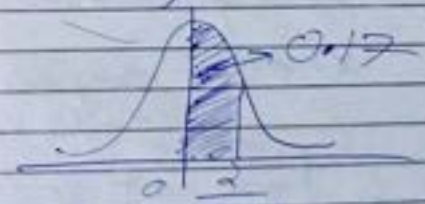
Given $P(22 \le X \le 26) = 0.34$

$\Rightarrow P\left(\frac{22-24}{\sigma} \le \frac{X-24}{\sigma} \le \frac{26-24}{\sigma}\right) = 0.34$

$\Rightarrow P\left(\frac{-2}{\sigma} \le Z \le \frac{2}{\sigma}\right) = 0.34$

$\Rightarrow 2P\left(0 \le Z \le \frac{2}{\sigma}\right) = 0.34$

$\therefore P\left(0 \le Z \le \frac{2}{\sigma}\right) = 0.17$



$\therefore \frac{2}{\sigma} = 0.44$

$\sigma = \frac{2}{} = 4.54$

Exp. no of batteries having life > 29
$= NP(X>29) = 272$

having life > 29

$\Rightarrow NP\left(z > \frac{29-24}{4.54}\right) = 272$

$\Rightarrow N \times P(z > 1.1) = 272$

$\Rightarrow N \times 0.136 = 272$

$\therefore N = \frac{272}{0.136} = 2000$

Here, 2000 batteries were tested.

Q. The probability that an electronic component will fail less than 1200 hours of continuous use is 0.25. Using normal approximation ~~x = no of~~ through normal distribution, find the probability that among 200 such components fewer than 45 will fail in less than 1200 hours of continuous use.

Soln: X = no of components failing in 1200 of cont. use

$p = 0.25$

$q = 1 - 0.25 = 0.75$

$n = 200$

Req prob $= P(X < 45)$

$m = np = 50$

$\sigma = \sqrt{npq} = 6.1237$

$P(X \le 44)$

Standard rule add. 5

Req. prob $= P(X \le 44)$

$= P(X \le 44.5)$

(∵ not discrete, but no. of components)

$= P\left(z \le \frac{44.5-50}{6.1237}\right)$

$= P(z \le -0.898)$

$= 0.5 - P(0 \le z \le 0.898)$

$= 0.5 - 0.3133$ (from table)

$= 0.1867$

Q. Using normal distribution find the probability of getting 55 heads in a toss of 100 fair coins.

Soln: $n = 100$

$p = 1/2$

$q = 1/2$

$m = np = 50$

$\sigma = \sqrt{npq} = 5$

$P(X=55) = P(55 \le X \le 55)$

$= P(54.5 \le X \le 55.5)$

$= P(0.9 \le z \le 1.0)$

$= P(0 \le z \le 1.1) - P(0 \le z \le 0.9)$

NORMAL DISTRIBUTION

Q. In a competitive exam, top 15% of students will get grade A, while bottom 20% will be declared fail. The marks are normally distributed, with mean % of marks 75 and $\sigma = 10$. Determine the lowest % of marks received grade A and lowest % of marks that passes.

Soln.

$m = 75$

$\sigma = 10$

~~Let~~ X = marks

Let X_1 = Lowest marks to get grade A.

$P(X > X_1) = 0.15$ (given)

$\Rightarrow P\left(\frac{X-m}{\sigma} > \frac{X_1-m}{\sigma}\right) = 0.15$

i.e. $P\left(Z > \frac{X_1-75}{10}\right) = 0.15$

$\therefore P\left(0 \le Z \le \frac{X_1-75}{10}\right) = 0.5 - 0.15 = 0.35$

$\therefore \frac{X_1-75}{10} = 1.04$

$\Rightarrow x_1 = 85.4$

Let x_2 = lowest marks that passes

$P(X < x_2) = 0.20$ (given)

$\Rightarrow P\left(\frac{X-m}{\sigma} < \frac{x_2-m}{\sigma}\right) = 0.20$

i.e. $P\left(z < \frac{x_2 - 75}{10}\right) = 0.2$

$\therefore \cancel{P\left(\frac{x_2 - 75}{10}\right) = 0.85}$

$\cancel{x_2 - 75 = 8.5}$

$x_2 = 8.5$

Q The actual amount of coffee which a filling machine puts into 6 ounce jars having normal distribution with standard deviation $\sigma = 0.05$.

If only 3% of the jars are certain less than 6 ounce of coffee, what must be the mean fill of the jars?

Soln $\sigma = 0.05$

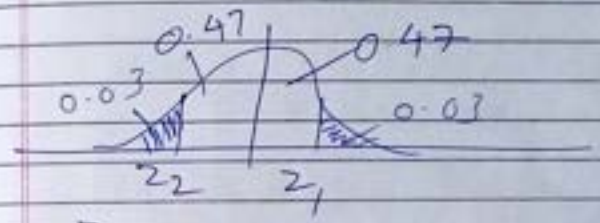
$m = ?$

X = Amount of coffee

$P(X < 6) = 0.03$ (given)

Ans : $\boxed{m = 6.09}$

Let $z = \frac{X-m}{\sigma}$ $\therefore z = \frac{6-m}{0.05}$



~~But 0.2 area left of m here~~

~~$z = -0.525$ (Reason: $-0.5 + -0.025$)~~

$\therefore$ ~~35 m~~ $P(z \leq 3)$?

$P\left(\frac{6-m}{0.05}\right) = P(z) = 0.03$

z_1 for 0.47 $z_1 = 1.88$

$\therefore z_2 = -1.88$

$\frac{6-m}{0.05} = -1.88$

$\therefore m = 6 + (0.05 \times 1.88)$

$= 6.094$ ounce Ans

Normal Distribution

A continuous random variable follows normal distribution if ~~a normal distribution~~ it's pdf is given by

m (mean) & σ^2 (variance)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2(x-m)^2/\sigma^2}, \quad -\infty < x < \infty, \quad -\infty < m < \infty$$

$$P(X=x) = 1$$

$$\sum p_i = 1$$

$$P(a<x<b) = \int_a^b f(x)\,dx$$

Remark: If X is a normal variate, with parameter m and σ, then

$Z = \frac{X-m}{\sigma}$ is also a normal variate with mean 0 and standard deviation 1.

Z is called standard normal variate if X follows a normal ~~variation~~ distribution, $\boxed{X \sim N(m, \sigma)}$

$\boxed{\text{then } Z \sim N(0,1).}$

Mean & variance of normal distribution

1. mean = mode = median = m

2. If $x_1, x_2, \ldots x_n$ are normal variates, ~~they~~ with mean ~~and~~ $m_1, m_2, \ldots, m_n$ and variate $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, then $Y = a_1x_1 + a_2x_2 + \ldots + a_nx_n$

$$m = a_1m_1 + a_2m_2 + \ldots + a_nm_n$$

$$\& \quad \sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots + a_n^2\sigma_n^2$$

3. Quartiles

$Q_1 \quad Q_2 \quad Q_3$

$Q_1 = m - \frac{2\sigma}{3} \quad Q_3 = m + \frac{2\sigma}{3}$

$Q.D. = \frac{Q_3 - Q_1}{2} = \frac{2\sigma}{3}$

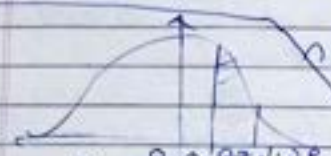
4. If . . . P.T.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

4. If $X \sim N(m, \sigma)$

then $Z = \frac{X-m}{\sigma} \sim N(0,1)$ is std normal variate (SNV) is also a normal variate.

Q.



Find $P(0.87 < z < 1.28)$

Soln: $P(0.87 < z < 1.28) = P(0 < z < 1.28) - P(0 < z < 0.87)$

$= 0.3997 - 0.3078$

$= \underline{0.0919}$

Q. 1. Normal distribution is $m = 50$, $\sigma = 15$

Find: (i) Q_1, Q_2 (ii) Q.D.

2 If $Q_1 = 36$, $Q_3 = 44$, find mean, std

~~1.96~~ * $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$, $P(a \le x \le b) = ?$

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

$z = \frac{0}{1}$ & $\frac{x-m}{\sigma}$

Q Find

(i) $P(-c < z < c) = 0.95$ $m = 120$, $\sigma = 10$

(ii) $P(|z| > c) = 0.01$

(iii) $P(X > c) = 0.02$

(iv) $P(X < c) = 0.05$

Given $m = 120$, $\sigma = 10$

Soln (i) value of z corresponding to $\underline{0.475}$. ~~we can~~

Reason: $-c < z < c$

$\Rightarrow 2P(\ )$ ~~to~~ it'll be

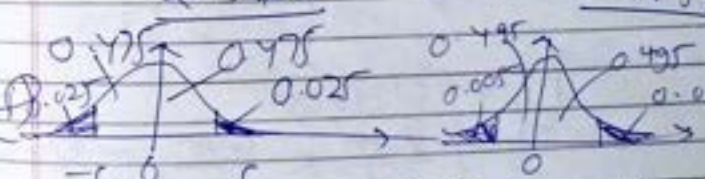
Also, $1 - 0.95 = 0.05$ $\frac{0.05}{2} = 0.025$

so, $\underline{0.05}$

NOTE: ~~$z < c \Rightarrow c < 0$~~ $z = 1.96$

$\underline{0.475}$ ~~(1 - 0.05)~~ (0.5 - 0.05) (print error diff.)

$\underline{2.5\cancel{75}}$ $\underline{c = 2.5\cancel{75}}$



If z is a S.N.V then $z = \frac{X-m}{\sigma}$

(ii) $z = \frac{X-120}{10}$

check out

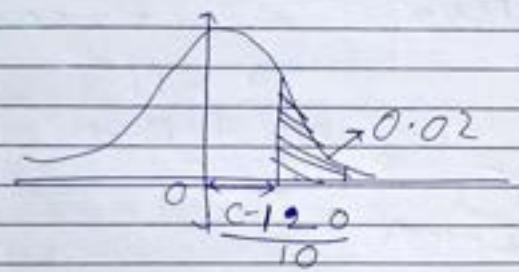
Find c if

(1) $P(X>c) = 0.02$
(2) $P(X<c) = 0.05$

Given $m = 120$, $\sigma = 10$

$$P\left(\frac{x-m}{\sigma} > \frac{c-m}{\sigma}\right) = 0.02$$

$$\Rightarrow P\left(z > \frac{c-120}{10}\right) = 0.02$$



$$P\left(0 < z < \frac{c-120}{10}\right) = 0.5 - 0.02 = 0.48$$

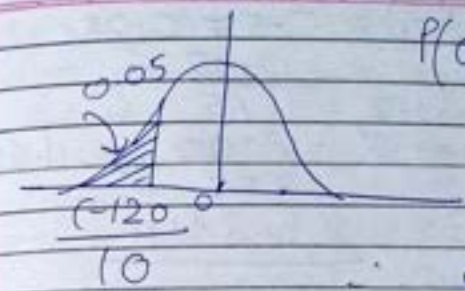
$$\therefore \frac{c-120}{10} = 2.06$$

$$\Rightarrow \boxed{c = 140.6}$$

$P(X<c) = 0.05$

$$\Rightarrow P\left(\frac{x-m}{\sigma} < \frac{c-m}{\sigma}\right) = 0.05$$

$$\Rightarrow P\left(z < \frac{c-120}{10}\right) = 0.05$$



$$P\left(0 < z < \frac{c-120}{10}\right) = 0.5 - 0.05 = 0.45$$

$$\therefore \frac{c-120}{10} = -1.65$$

$$c = 10.35$$

Q. Monthly salary X is normal distributed with mean (μ) ₹3000 & std. dev^n (σ) ₹250. What should be the minimum salary of a worker so that he belongs to top 5% workers?

Sol^n: Let X = monthly salary of a worker.

$m = 3000$ $\sigma = 250$

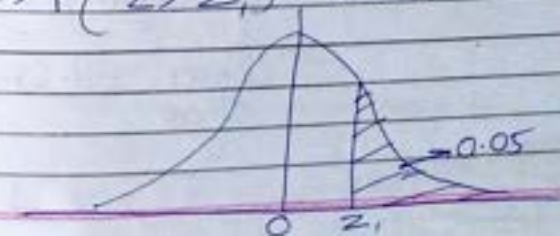
Let x_1 be minimum salary of top 5% workers

$$z = \frac{x-m}{\sigma} = \frac{x-300}{250}$$

$P(X > x_1) = 5\%$

$$\Rightarrow P\left(\frac{x-1}{\sigma}\right) > \frac{x-1}{\sigma} = 5\% = 0.05$$

$$\Rightarrow P(z > z_1) = 0.05$$



$P(0<z<z) = 0.5 - 0.05$

$= 0.45$

$\therefore z = 1.65$

$\Rightarrow \frac{X_1 - 3000}{250} = 1.65$

$\Rightarrow X_1 = 3412$

Q. ~~Th~~ Diameter of normally distributed (an tops) with standard deviation 0.05 cm.

At what mean diameter of the machine should be set so that not more than 5% of the tops can have diameter ~~at th~~ exceeding 3 cm?

Sol

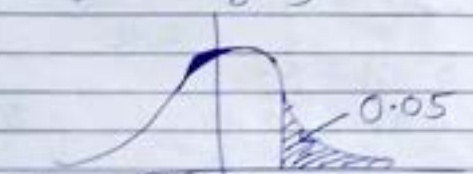
~~X = 1~~

$\sigma = 0.05$

$P(X > 3) = 5\%$

$\Rightarrow P\left(\frac{X-m}{\sigma} > \frac{3-m}{\sigma}\right) = 0.05$

$\Rightarrow P\left(z > \frac{3-m}{\sigma}\right) = 0.05$



$\therefore P\left(0 < z < \frac{3-m}{\sigma}\right) = 0.5 - 0.05$

$= 0.45$

$\Rightarrow \frac{3-m}{0.05} = 1.65 \Rightarrow m = 2.918$

Q. ~~If~~ X, Marks obtained by 1000 students are found to be normally distributed, ~~it~~ with mean 70 & std. deviation ~~is~~ 5. ~~Calculate~~ estimate the no. of students whose marks will be (i) b/w 60 & 75 and those (ii) greater than or equal to 75

Sol^n: X = marks of student

(i) $P(60 < X < 75) =$ 818

No. of students = NP

N = 1000

(ii) $P(X \geq 75)$ 159

N = 1000

$z = \frac{x-m}{\sigma} = \frac{x-70}{5}$

(i) $X = 60, \; z = -2$

$X = 75, \; z = 1$

$P(60 \leq X \leq 75) = P(-2 < z < 1)$

$= P(0 \leq z \leq 2) + P(0 \leq z \leq 1)$

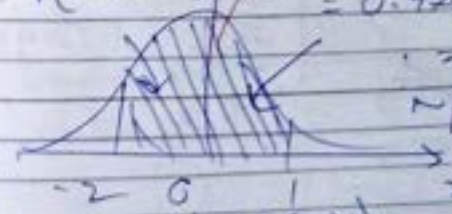
$= 0.4772 + .3413$

$= .8185$

$np = 1000 \times .8185$

$= 818$

$\therefore$ Nearest $= 818$



(ii) $P(X \geq 75) = P(z \geq 1)$

$= 0.5 - P(0 \leq z \leq 1)$

$= 0.5 - .3413$

$= 0.1587$

≈ 0.159

Q. X_1 & X_2 are normal events with mean

$m_1 = 30$

$m_2 = 25$

$\sigma_1^2 = 16$

$\sigma_2^2 = 12$

$Y = 3x_1 - 2x_2$

Find probability of

$P(60 \le Y \le 80)$

REPEAT

Remark: If $x_1, x_2 \ldots x_n$ are normal variates, $m_1, m_2, \ldots m_n$, $\sigma_1^2, \sigma_2^2, \sigma_n^2$ (written back pages)

$Y = 3x_1 - 2x_2 = 3x_1 + (-2)x_2$

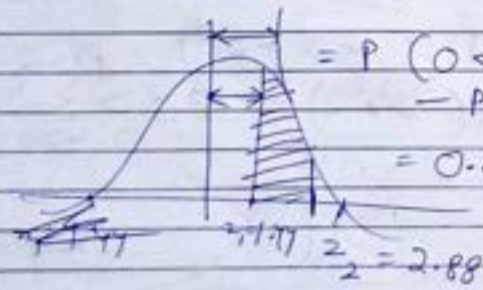
~~Find P(60 ≤~~

Solⁿ: Y is a normal variate with mean $m = 3 \times 30 + (-2) \times 25 = 40$

& $\sigma^2 = \Sigma a_i^2 \sigma_i^2 = (9 \times 16) + (4 \times 12) = 192$

$\therefore P(60 \le Y \le 80) = P\left(\frac{60-40}{\sqrt{192}} \le z \le \frac{80-40}{\sqrt{192}}\right)$

$= P(1.44 \le z \le 2.88)$



$= P(0 < z < 2.88) - P(0 < z < 1.44)$

$= 0.498 - 0.4251$

$= 0.0729$

Q. 2 independent random variables X & Y are distributed normally with mean and standard deviation (52, 3) & (50, 2) resply. Find the probability that randomly pair of values of X & Y will differ by 1.7 or more.

Solⁿ: i.e. $P(|X-Y| \ge 1.7)$

let $X - Y = 0$

$= P(|0| \ge 1.7)$

mean of $U = X$

$m_1 = 52$, $\sigma_1 = 3 \Rightarrow \sigma_1^2 = 9$

$m_2 = 50$, $\sigma_2 = 2 \Rightarrow \sigma_2^2 = 4$

$\hookrightarrow m = 9 + 4 = 13$

$\therefore P(|0| \ge 1.7) = 1 - P(|0| < 1.7)$

$= 1 - P[-1.7 < |0| < 1.7]$

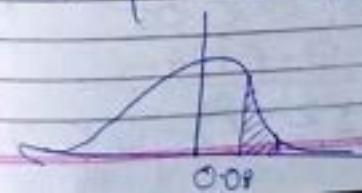
OR $1 - P[-1.7 < U < 1.7]$

$= 1 - P\left(\frac{-1.7 - 2}{\sqrt{13}} < z < \frac{1.7 - 2}{\sqrt{13}}\right)$

$= 1 - P\left(-1.026 < z < \frac{-0.3}{\sqrt{13}}\right)$

$= 1 - \left\{P\left(0 < z < \frac{-0.3}{\sqrt{13}}\right) - P(0 < z < -1.026)\right\}$

$= 1 - \{ = -1.026 < \frac{U-2}{\sqrt{13}} < -0.08 \}$



$= 0.6234$

Q. Incomes of a group of 10,000 persons are normally distributed ~~defn~~ $\mu = 520$, $\sigma = 60$. Find

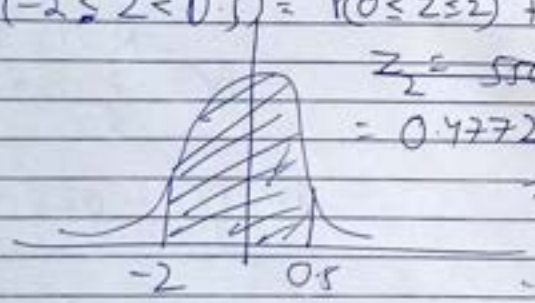
i) No. of persons having income b/w 400 & 550. $0.6687 \times 10000 = 6687$

ii) lowest income of richest 500.

Sol: Income of the person

$z_1 = \frac{400-520}{60} = -2$ $\quad z_2 = \frac{550-520}{60} = 0.5$

$\therefore P(-2 \le z \le 0.5) = P(0 \le z \le 2) + P(0 \le z \le 0.5)$



~~$z_2 = 550-520$~~

$= 0.4772 + 0.1915$

$= 0.6687$

$\therefore Np = 6687$

(ii) Let x_1 be the lowest income of richest 500.

$P(x > x_1) = 5\% = 0.05$ Rev problem

~~$\Rightarrow z = 0.0199$~~

$\Rightarrow$ ~~Area~~ $P(0 \le z \le ?) = 0.5 - 0.05 = 0.45$

From table, $? = 1.645$ at 0.45

$\therefore$ Req value of $z = 1.645$

But $z = \frac{x-520}{60}$, $\therefore 1.645 = \frac{x-520}{60}$

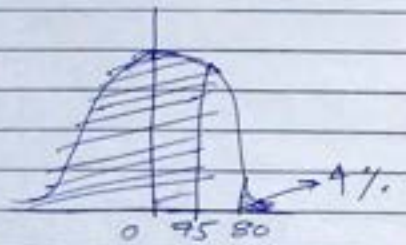
$x = 618.7$

Thus lowest income of richest 500 = ₹ 618.7

~~Testing hypothesis~~: $P(x > x_1) = 0.05$

Q. Find the mean and standard deviation of marks in an exam where 5~~0~~8% of ~~m~~ candidates obtained marks below 75, and rest b/w 75 & 80.

Soln



X = marks.

$P(0 \le X \le 75) = 8\% = 0.08$

$\Rightarrow P\left(0 \le z \le \frac{75-m}{\sigma}\right) = 0.08$

$\Rightarrow \frac{75-m}{\sigma} = 0.2$

$\Rightarrow 0.2\sigma + m = 75$

$\frac{75-m}{\sigma}$

$P(75 \le X \le 80) = 0.92$

$\Rightarrow P\left(\frac{75-m}{\sigma} \le z \le \frac{80-m}{\sigma}\right) = 0.92$

$\Rightarrow P\left(z \le \frac{80-m}{\sigma}\right) - P\left(z \le \frac{75-m}{\sigma}\right) = 0.92$

Q. A large no. of automotive batteries have an avg. life of 24 months. If 3~~0~~4% of them have an average life of 22 - 26 months and 272 of them last longer than 29 months, how many total batteries were tested using normal distribution?

Soln

Given $m = 24$

X = Avg. life of batteries

$\sigma = ?$

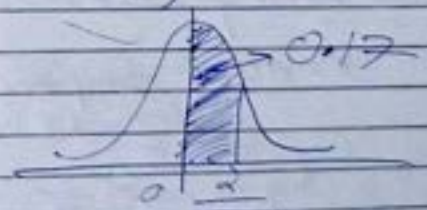
Given $P(22 \le X \le 26) = 0.34$

$\Rightarrow P\left(\frac{22-24}{\sigma} \le \frac{X-24}{\sigma} \le \frac{26-24}{\sigma}\right) = 0.34$

$\Rightarrow P\left(\frac{-2}{\sigma} \le z \le \frac{2}{\sigma}\right) = 0.34$

$\Rightarrow 2P\left(0 \le z \le \frac{2}{\sigma}\right) = 0.34$

$\therefore P\left(0 \le z \le \frac{2}{\sigma}\right) = 0.17$



$\therefore \frac{2}{\sigma} = 0.44$

$\boxed{\sigma = \frac{2}{0.44} = 4.54}$

Exp. no of batteries having life > 29 $= NP(X>29) = 272$

and having life > 29

$\Rightarrow NP\left(z > \frac{29-24}{4.54}\right) = 272$

$\Rightarrow N \times P(z > 1.1) = 272$

$\Rightarrow N \times 0.136 = 272$

$\therefore N = \frac{272}{0.136} = 2000$

Here, 2000 batteries were tested.

Q. The probability that an electronic component will fail less than 1200 hours of continuous use is 0.25. Using normal approximation ~~X = no of~~ through normal distribution, find the probability that among 200 such components fewer than 45 will fail in less than 1200 hours of continuous use.

Soln. X = no. of components failing in 1200 hrs of cont. use

$p = 0.25$

$q = 1 - 0.25 = 0.75$

$n = 200$

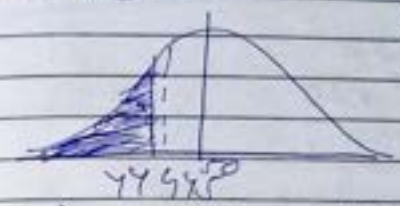
Req. prob. $= P(X < 45)$

$m = np = 50$

$\sigma = \sqrt{npq} = 6.1237$

$P(X \le 44)$

Standard rule add .5



Req. prob. $= P(X \le 44)$

$= P(X \le 44.5)$

($\because$ not discrete, but no. of components)

$= P\left(z \le \frac{44.5-50}{6.1237}\right)$

$= P(z \le -0.898)$

$= 0.5 - P(0 \le z \le 0.898)$

$= 0.5 - 0.3133$ (from table)

$= 0.1867$

Q. Using normal distribution, find the probability of getting 55 heads in a toss of 100 fair coins.

Soln. $n = 100$

$p = 1/2$

$q = 1/2$

$m = np = 50$

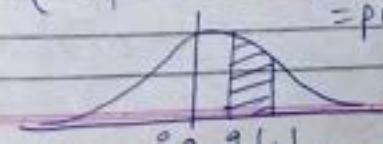
$\sigma = \sqrt{npq} = 5$

$P(X=55) = P(55 \le X \le 55)$

$= P(54.5 \le X \le 55.5)$

$= P(0.9 \le z \le 1.0)$

$= P(0 \le z \le 1.1) - P(0 \le z \le 0.9)$



$= 0.3643 - 0.3159$

$= 0.0484$

Exponential distribution

$\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & o.w \end{cases}$$

Note: $\int_0^\infty f(x)\,dx = 1$

Mean $= E(X) = \frac{1}{\lambda}$

$$Var = E(X^2) - [E(X)]^2$$
$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$
$$= \frac{1}{\lambda^2}$$

Note $P(X > k) = \int_k^\infty f(x)\,dx$

$$= e^{-\lambda k}$$

$$\text{Mean} = \frac{1}{\lambda} = 40{,}000$$

$$\therefore \lambda = \frac{1}{40{,}000}$$

$$\therefore f(x) = \frac{1}{40{,}000} e^{-x/40000}$$

$$P(X \geq 20{,}000) = \int_{20{,}000}^\infty f(x)\,dx$$
$$= e^{-\frac{1}{40000} \times 20000}$$
$$= e^{-0.5} = 0.606$$

$$P(X \leq 30000) = \int_0^{30000} f(x)\,dx$$
$$= \left[\frac{e^{-\frac{1}{40000} x}}{-\frac{1}{40{,}000}}\right]_0^{30000}$$
$$= \left[e^{-0.75} - 1\right] = 1 - e^{-0.75}$$

Q The amount of time a watch will run without having to be wed is a random variable having a exponential distribution with mean 120 days, find probability that such watch will (1) have to be reset within 24 hours after reset, (2) it will run for 180 days

Ans $\lambda e^{-\lambda x}$

$$E(x) = 120$$
$$\lambda = \frac{1}{120}$$

$$\frac{1}{120} e^{-1/120 \times x} = \lambda e^{-\lambda x}$$

(1) $$P(X \leq 24) = \int_0^{24} f(x)\,dx$$
$$= \frac{1}{120} \int_0^{24} e^{-x/120}\,dx$$
$$= \frac{1}{120} \left(\frac{e^{-x/120}}{-1/120}\right)_0^{24/120}$$

$= -(e^{-0.2} - 1)$

$= \underline{0.1812}$

(2) $P(X \leq 180)$

Memory-less

$$P\left(\frac{X > s+t}{X > s}\right) = P(X > t) \ \forall \ s,t > 0$$

Q The length of a shower on a tropical island has an exponential distribution with parameter 2. Time being measured in minutes, what is the probability that a shower will last more than 3 minutes?
If the shower has already lasted for 2 minutes what is the probability that it will last for atleast 1 more minute?

Ans X = length of shower
$\lambda = 2$

$$P(X > 3) = \int_3^\infty f(x)\,dx = e^{-2\times 3} = e^{-6}$$
$$= \underline{0.0025}$$

Given that
$X > 2$

To find $P(X > 2+1 / X > 2) = P(X > 3 / X > 2)$
$= P(X > 1)$
$= e^{-2\times 1} = e^{-2}$
$= \underline{0.1353}$

DISCRETE UNIFORM DISTRIBUTION

Discrete random variable X is said to have uniform distribution if its pmf is given by

$$P(X = x) = \frac{1}{n}, \quad x = 1, 2, \ldots n$$

$$\text{mean} = E(X) = \frac{n+1}{2}$$

$$\text{Variance}(X) = E(X^2) - [E(X)]^2$$
$$= \frac{n(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$
$$= \frac{n^2-1}{12}$$

Q Roll a six-faced fair dice, Suppose X denotes number appearing on the top of a dice. Find

1. P(even no. = X)
2. P(X<3)
3. E(X) = ?
4. Var(X) = ?

Sol^n $P(X=x) = \frac{1}{6}$

1. $P(X=2,4,6) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}$
$= \underline{0.5}$

2. $P(X<3) = \frac{1}{6}+\frac{1}{6} = \underline{0.33}$

3. $\frac{n+1}{2}$ $\frac{6+1}{2} = \underline{3.5}$

4. $E(X^2) - (E(X))^2$
$= \frac{n^2-1}{12} = \underline{\frac{35}{12}}$

Continuous uniform distribution

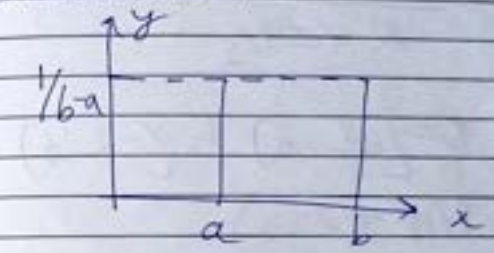
A continuous random variable X is said to follow a uniform distribution with parameters in the interval [a, b] if its pdf is given by $f(x) = \begin{cases} k, & x \in [a,b] \\ 0, & \text{o.w.} \end{cases}$

$\int_a^b f(x)\,dx = 1$

$\Rightarrow k = \frac{1}{b-a}$

$\therefore f(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{o.w.} \end{cases}$

This Distribution is also known as "rectangular distribution".



mean = $E(X) = \frac{b+a}{2}$

$var(X) = \frac{(b-a)^2}{12}$

Note that, for any sub-interval (c, d), where $a \le c \le d \le b$ then

$P(c \le X \le d) = \int_c^d f(x)\,dx = \int_c^d \frac{1}{b-a}\,dx$

$= \boxed{\frac{d-c}{b-a}}$

Q. If X is uniformly distributed $-2.5 \le x \le 2$ find ① $P(X \le 1)$ ② $P(|X-1| \ge 1/2)$

(or) ① $\therefore x<1 \Rightarrow x \in (-2,1) = (c,d)$

$\therefore P(x<1) = \frac{1-(-2)}{2-(-2)} = \frac{3}{4}$

② $|x-1| \geq 1/2$

$\Rightarrow -\frac{1}{2} \leq 1-x \leq \frac{1}{2}$

$\Rightarrow x \geq 3/2$

& $x \leq 1/2$

$\therefore x \in [\frac{3}{2}, 2]$ & $(-2, 1/2]$

$\therefore P(|x-1| \geq 1/2)$

$= P(-2 \leq x \leq 1/2) + P(\frac{3}{2} \leq x \leq 2)$

$= \frac{1}{5} + \frac{1}{2}$

$= \frac{2}{5} + 2$

$= \frac{1/2+2}{4} + \frac{2-3/2}{4}$

$= \frac{5}{8} + 1/8$

$= \frac{3}{4}$

Q In a group of 7 there are 30 participants, a question is given to all 30 and time allowed is 30 sec.
Find the probability of participant answering within 6 sec.

(or) $f(x) = \frac{1}{25}$

$P(x<6) = \int_0^6 f(x)\,dx$ $[0,6]$

$= \frac{6}{25}$

$\therefore$ No. of participants answering within 6 sec

$= 30 \times \frac{6}{25}$

$= \frac{180}{25} = 7.2 \simeq 7$ Ans

Q In uniform distribution, suppose a random number n is taken from 690 to 850. Find the prob that $n \geq 790$

(or) $\int_{690}^{850} f(x)\,dx = \frac{1}{850-690} = \frac{1}{160}$

$P(x>790)$ & $P(x<850)$ combine

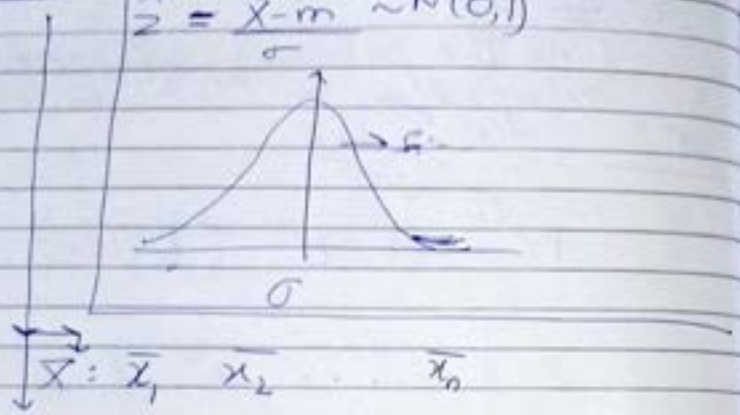
$\Rightarrow P(790 < x < 850)$

$= \frac{850-790}{850-690}$

Large sampling : $(n \geq 30)$

Thr^m : $X \sim N(m, \sigma)$

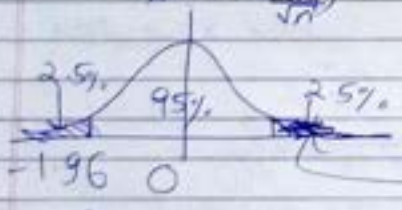
$$z = \frac{X - m}{\sigma} \sim N(0,1)$$



If $\bar{X}$ is the mean of each sample of size n drawn from the population with mean μ and standard deviation σ, then:

- $\bar{X}$ is normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$.
- $z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$ is snv with mean 0.

$\bar{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$



⇒ for an snv curve, 95% area under the curve lies b/w ± 1.96.

99%
2.575

99.73% area lies b/w ± 3.

The levels marked by the probabilities ~~0.05, 0.01 & 0.0027~~ (1.96, 2.58, 3), which decide the significance of the events are known as level of significance (α), and the corresponding regions to level of significance are called 'critical regions'.

$P(|z| < 1.96) = 0.95 \simeq P(|z| > 1.96) = 0.05$

$P(|z| < 2.58) = 0.99$

$P(|z| < 3) = 0.9973$

- The limits within which we expect z to lie with specified probabilities are called confidence limits and the region within the confidence limits ~~and the region within the con~~ are called as confidence level.

- INTERVAL ESTIMATION

We know that at 5% LOS, $|z| < 1.96$.

Now, $z = \frac{\bar{X} - \mu}{(\sigma/\sqrt{n})} < 1.96$

$\Rightarrow |\bar{X} - \mu| < \frac{1.96 \times \sigma}{\sqrt{n}}$

$\Rightarrow \mu \in \left(\bar{X} - \frac{1.96\sigma}{\sqrt{n}}, \bar{X} + \frac{1.96\sigma}{\sqrt{n}}\right)$

$\Rightarrow \frac{-1.96 \times \sigma}{\sqrt{n}} < \bar{X} - \mu < \frac{1.96 \times \sigma}{\sqrt{n}}$

Eg 1 Measurements of the weights of a random sample of 200 ball bearings showed a mean of 0.824 newtons and standard deviation 0.042 newtons. Find 95% confidence limits for the mean weight of all ball bearings.

$n = 200$

Sol^n $\bar{x} = 0.824$ newtons

$s = 0.042$ "

$\sigma_{\bar{x}} = s/\sqrt{n} = 0.003$

$\therefore$ at 5% los,

$\mu \in (\bar{x} - 1.96\sigma_{\bar{x}}, \bar{x} + 1.96\sigma_{\bar{x}})$

$\therefore \mu \in (0.81812, 0.8$[illegible]$)$

Eg 2 A random sample of size 65 was drawn in the process of estimating mean annual income of 50 families. Mean and sd of sample ~~used~~ were found to be 4730 & 765 resp. Find confidence for population mean.

Sol^n $n = 65$, $N = 950$, $\bar{x} = 4730$, $s = 765$

$\frac{n}{N} = 0.068 > 0.05$

find $\sqrt{\frac{N-n}{N-1}} = 0.932$

$\therefore \sigma_{\bar{x}} = \frac{s}{\sqrt{n}} \times \sqrt{\frac{N-n}{N-1}}$

$= \cancel{91.62}\ 88.43$

$\therefore \mu \in (4550.295, 6909.705)$

Testing hypothesis:

Statistical method used to determine if there's enough evidence in a sample data to draw conclusions about a population.

H_0 : Null hypothesis

H_a : Alternative hypothesis

Type-1 error: If the hypothesis is true and it's getting rejected then it's called Type-1 error. Type-2 False & getting accepted.

Q. A random sample of 50 items.

$\bar{x} = 6.2$

$s^2 = 10.24$

Can it be regarded as drawn from a normal population with mean 5.4 ($\mu = 5.4$) and 5% LOS?

Sol^n

Null hypothesis H_0 : $\mu = 5.4$

Alternate hypothesis H_a : $\mu \neq 5.4$

$|z| = \left|\frac{\bar{x} - \mu}{s/\sqrt{n}}\right| = \left|\frac{6.2 - 5.4}{3.2/5\sqrt{2}}\right| = \frac{\sqrt{2}}{4}$

$= 1.77$

at 5% LOS

$Z_\alpha = 1.96$

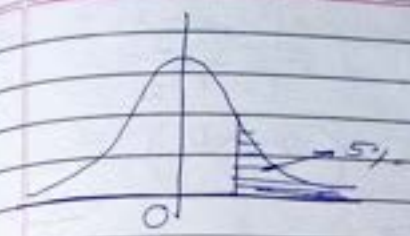
H_0 is accepted
i.e. the sample is drawn from the normal population.

Q. A ~~tie~~ tyre company claims that the lives of tyres have mean 42000 km with standard deviation 4000 km. A change in the production process is believed to result in better product. A test sample of 81 new tyres has a mean life of 42500 km. Test at 5% LOS that the new product is significantly better than the old one.

Sol^n $\mu = 42000$
$\sigma = 4000$
$n = 81$
$\bar{x} = 42,500$
$H_0: \mu = 42000$
$H_a: \mu > 42000$

$|z|_{cal} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ (no modulus, since one-tailed test)

$= 1.125$



$Z_{cal} < Z_\alpha$

At 5% LOS
$Z_\alpha = 1.65$

Thus,
there is no improvement in the product.

Q. If distribution of difference b/w the means; if $\bar{x}_1$ & $\bar{x}_2$ denote the means of the samples drawn from the first and second population respectively having means μ_1 & μ_2 and standard deviation σ_1 & σ_2 and the sample sizes are n_1 & n_2, then we can prove that the distribution of diff. b/w the means $\bar{x}_1 - \bar{x}_2$ is normally distributed with mean $\mu_1 - \mu_2$.

$\bar{X} \in N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

μ, σ

$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \in N(0,1)$

It is normally distributed with mean

$\mu_1 - \mu_2$ & $s = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

i.e. $Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$ ~ SNV

NOTE: If the samples are drawn from the same population so that σ known.

(1) $\sigma_1 = \sigma_2 = \sigma$ $(\mu_1 = \mu_2)$

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = SE.$$

(2) If $\sigma_1 = \sigma_2 = \sigma$ (μ unknown).
then $SE = \sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}}$

"unbiased estimate of population std. variate"

deg of freedom

for $n < 30$

(1) If σ is known
then $Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

(2) If σ - unknown
(a) then $t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$ where $s = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n-1}}$

(b) $t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$, where $s = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$

Sol^n: $n = 10$
$\bar{x} = 40$
$s = 10$

$\hat{\sigma}_{\bar{x}} = \frac{s}{\sqrt{n-1}} = 3.33$

at 1% LOS & $n-1 = 9$ dof

$t_\alpha = 3.25$

$\therefore \mu \in (\bar{x} - 3.25\sigma$

one-tail test
left-tail: z_α & t_α -ve (either smallest or largest α)
right-tail: z_α & t_α +ve (")

- Testing the hypothesis for the population mean μ

Q. A soap manufacturing company distributes soaps to retail stores. Before the advertisement, the mean sale of the soap was 140 dozen. After the advertisement, a sample of 26 shops was taken, and the mean sale was found to be 147 dozen. Can you consider the advertisement effective?

$t_{cal} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = 2.19$

at 5% LOS & $n-1 = 25$ dof

$t_\alpha = 1.708$

$\left(\because \begin{array}{l} n = 26 \\ \mu = 140 \\ \bar{x} = 147 \\ s = 16 \end{array} \right)$

$H_0 : \mu = 140$ ✓

$H_a : \mu > 140$ (one tailed test)

~~t_{cal}~~ $= x$

Since $t_{cal} > t_\alpha$

We Reject H_0.

Q. 9 items of a sample had following values:

45
47
50
52
48
47
49
53
51

Does the mean of 9 items differ significantly from the population mean 47.5?

Ans. $\bar{x} = 49.11 = \frac{442}{9}$

$s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{(45-49.11)^2 + (47-49.11)^2 + \ldots}{9}}$

$= \sqrt{5.603} = 2.37$

$H_0 : \mu = 47.5$

$H_a : \mu \neq 47.5$

$|t|_{cal} = \left| \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \right| = 1.84$

at 5% LOS & $n-1 = 8$ dof

$t_\alpha = 2.306$

$t_{cal} < t_\alpha$

We accept H_0.

- Therefore the mean of 9 items doesn't differ significantly from the population mean

Testing the hypothesis for difference b/w the means

Case-1 Independent sample

- If $n_1 + n_2 - 2$ = SAMPLE SIZE is small

OR $(n_1 + n_2 - 2) < 30$

Then, the unbiased estimate of common population standard deviation is given by

$$S_p = \sqrt{\frac{\Sigma(X_{i1}-\bar{X}_1)^2 + \Sigma(X_{i2}-\bar{X}_2)^2}{n_1+n_2-2}}$$

If S_1 & S_2 are unbiased estimates of population standard deviations pop-1 & pop-2 then s_1 & s_2 are given by

$$s_1 = \sqrt{\frac{\Sigma(X_{i1}-\bar{X}_1)^2}{n_1-1}}$$

$$s_2 = \sqrt{\frac{\Sigma(X_{i2}-\bar{X}_2)^2}{n_2-1}}$$

then $S_p = \sqrt{\frac{S_1^2(n_1-1)+S_2^2(n_2-1)}{n_1+n_2-2}}$

If σ_1 & σ_2 are standard deviation of the two samples s_1 & s_2

i.e.

$$\sigma_1 = \sqrt{\frac{\Sigma(X_{i1}-\bar{X}_1)^2}{n_1}}$$

$$\sigma_2 = \sqrt{\frac{\Sigma(X_{i2}-\bar{X}_2)^2}{n_2}}$$

then $S_p = \sqrt{\frac{\sigma_1^2 n_1 + \sigma_2^2 n_2}{n_1+n_2-2}}$

$$\therefore S.E. = S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

& $t = \frac{\bar{X}_1-\bar{X}_2}{S.E.}$ under the condition $\mu_1=\mu_2$

Q. If two independent samples of sizes 15 & 8 have means and populations standard deviation 980 & 1012 reply

$n_1 = 15$ $\bar{x}_1 = 980$

$\sigma_1 = 75$ $n_2 = 8$ $\bar{x}_2 = 1012$

$\sigma_2 = 80$

Test $\mu_1 = \mu_2$ at 5% LOS.

Sol: $H_0 = \mu_2$

$H_a : \mu_1 \neq \mu_2$

$$S.E = \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} =$$

$$|Z_{cal}| = \left|\frac{\bar{X}_1-\bar{X}_2}{S.E}\right| = 0.92$$

at 5% LOS ~~& $n_1+n_2-2 = 21$~~

$Z_\alpha = 1.96$

$Z_{cal} < Z_\alpha$

accept $H_0 : \mu_1 = \mu_2$

$n_1 = 3 = n_2 = 7$

Q. $\bar{X}_1 = 196.42$

$\Sigma(X_{i1}-\bar{X}_1)^2 = 26.94$

$\bar{X}_2 = 198.82$

$\Sigma(X_{i2}-\bar{X}_2)^2 = 18.73$

Sol: $H_0 : \mu_1 = \mu_2$

$H_2 = \mu_1 \neq \mu_2$

$$S_p = \sqrt{\frac{\Sigma(x_{1i}-\bar{x})^2 + \Sigma(x_{2}-\bar{x})^2}{n_1+n_2-2}}$$

$= 1.81$

$$S.E = S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 0.91$$

$$|t_{cal}| = \left|\frac{\bar{x}_1-\bar{x}_2}{S.E}\right| = 2.64$$

at 5% LOS and $n_1+n_2-2 = 14$ dof

$t_\alpha = 2.145$

$\therefore t_{cal} > t_\alpha$

H_0 is rejected.

with an unbiased standard deviation 2.2, ~~which took~~

Q. $n_1 = 6$ $n_2 = 6$

$\bar{x}_1 = 15.4$ $S_1 = 2.2$

$\bar{x}_2 = 11.2$ $S_2 = 2.6$

The difference in the doses have no effect.

0.5 mg

1.5 mg

Ans. $H_0: \mu_1 = \mu_2$ (difference in the doses have no effect)

$H_a: \mu_1 \neq \mu_2$

$$S_p = \sqrt{\frac{S_1^2(n_1-1) + S_2^2(n_2-1)}{n_1+n_2-2}}$$

$= \sqrt{5.8} = 2.41$

$$S.E = S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 1.39$$

$$|t_{cal}| = \left|\frac{12.8}{1.39}\right| = 9.21$$

$$\left|\frac{15.4-11.2}{1.39}\right| = 3.02$$

5% LOS & $n_1+n_2-2 = 10$ dif $t_\alpha = 2.228$

$\therefore t_\alpha$ calc.

$t_{cal} > t_\alpha$

$\therefore$ no H_0 rejected

Q. The height of 6 salox (in inches) are 63, 65, 68, 69, 71, 72

and height of 10 sldiers 61 62 65 66 69 69 70 71 72 73

X_1	X_2	$(X_1-\bar{X}_1)^2$	$(X_2-\bar{X}_2)^2$
63	61		
	62		
65	65		
68	66		
69	69		
71	69		
72	70		
$\Sigma X_1 =$	71		
	73		
	$\Sigma X_2 =$		

$\Sigma(X_1-\bar{X}_1)^2 = 60$ $\Sigma(X_2-\bar{X}_2)^2$

Discuss the soldiers on an avg are taller than sailors

$\bar{X}_1 = \frac{\Sigma X_1}{6} = 68$

$\bar{X}_2 = \frac{\Sigma X_2}{10} = 67.8$

$H_0: \mu_1 = \mu_2$

$H_0: \mu_2 > \mu_1$

$S_p = \sqrt{\frac{\Sigma(X_1-\bar{X}_1)^2 + \Sigma(X_2-\bar{X}_2)^2}{n_1+n_2-2}}$

$= 3.9$

$S_E = S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 2.014$ LOS 5%

$|t_{cal}| = \frac{\bar{X}_1 - \bar{X}_2}{S.E} =$ ~~(n1+n2)-2 df~~ ~~=10~~

$t_\alpha = 2.145$ & 5% LOS at 14 dof

$t_{cal} = \frac{\bar{X}_1 - \bar{X}_2}{S.E} = \frac{68-67.8}{2.014} = 0.099$

(iii) LOS = 5%

(iv) Critical value (obtained t_α)

(v) Decision: Since the computed value $|t| = 0.099$ is smaller than the table value $t_\alpha = 2.145$, the hypothesis is accepted.

∴ The means are equal i.e. the suggestion that the soldiers on average are taller than sailors can't be accepted.

Q. Ten school boys were given a test in Statistics and their scores were recorded. They're given a month special coaching and a second test was given to them in the same subject at the end of the coaching period. Test if the marks given below give evidence to the fact that the students are benefited by coaching

Marks Test I:	70	68	56	75	30	90	68	75	56	58
Marks Test II:	68	70	52	73	75	76	80	92	54	55

T_1:

T_2:

x_i: $T_2 - T_1$:

$\bar{x} = \frac{\Sigma x_i}{n} = 0.1$

$H_0: \mu = 0$ (coaching wasn't effective)

$H_1: \mu \neq 0$

$\Sigma (x_i - \bar{x})^2 : \Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n}$

$= 679 - \frac{0.01}{10} = 678.999$

$s^2 = \frac{\Sigma (x_i - \bar{x})^2}{n} = 67.90$

(ii) Calc^n of test statistic : Since the sample size is small, we use students distribution.

$t_{cal} = \left| \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \right|$, $n = 10$ $\quad t_{cal} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$

$|t| = 0.036$ $\quad$ where $s = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{n}}$

$t_\alpha = 2.262$

(iv) at 5% LOS & $n - 1 = 10 - 1 = 9$ dof

$t_{cal} < t_\alpha$ we accept H_0

Students are not benefited by the ~~test~~ coaching

χ^2 distribution (chi square)

$$\chi^2 = \frac{\Sigma (O - E)^2}{E}$$

where O = observed frequencies

E = expected frequencies

Calculate expected frequencies on certain predictions.

1) A coin / die is unbiased
2) There's no assoc^n b/w the attributes
3) Accidents occur evenly on all days
4) Errors occur evenly on all ~~data~~ pages
5) Events occur in the given ratio.
6) Events occur acc. to the given distribution
(eg. Binomial, Poisson or normal)
This is called "testing goodness of fit."
7) ~~If~~ we compare the calculated value of χ^2 with the table value for the given degree of freedom at specified LOS.

Condition for χ^2 test:

1) N = total no of observations must be sufficiently large.
2) Frequency of every cell should be greater than 5. If freq ≤ 5, then it's combined with the neighbouring

freq. so that combined freq. > 5 and the degree of freedom are reduced accordingly.

χ^2_{cal} = Yate's correction

In a 2×2 table, degree of freedom is

$$dof = (r-1)(c-1) = 1$$

But, if any ~~cell~~ freq. ≤ 5, we've to use pooling method.

e.g.

2+7	6+8
7	8

If this will result in 0 dof, which is not admissible.
Therefore, we apply Yate's correction formula as

$$\chi^2 = \Sigma \frac{(|O-E|-.5)^2}{E}$$

Type 1

Independence of attributes

Investigate the association b/w the darkness of eye color in father and son from the following data.

Color of father's eyes

Color of son's eyes	Dark	Not dark	Total
Dark	48	90	138
not Dark	80	782	862
Total	128	872	1000

Soln: H_0: There's no assoc^n b/w the darkness of eye color in father & son.

H_a: There's an association.

$$\frac{x}{128} = \frac{138}{1000} \Rightarrow x = 17.664 = 18$$

O	E	~~Observe~~ $(O-E)^2$	$(O-E)^2/E$
48	18	900	50
80	110	900	8.182
90	120	900	7.5
782	752	900	1.197

$$\chi^2_{cal} = \frac{\Sigma(O-E)^2}{E}$$

$$= 58.182 + 8.007$$

$$= 66.88$$

At 5% LOS & $(r-1)(c-1) = (2-1)(2-1) = 1$ dof

$\chi^2_{table} = 3.841$

$\therefore \chi^2_{cal} > \chi^2_{table}$

we reject H_0.

Thus, there is an association b/w father and son.

Age	Party A	B	C	Total
20-35	25	20	25	70
35-50	20	25	35	80
>50	25	25	30	80
Total	70	70	90	230

Q. Following table gives the result of opinion poll of party A, B, C. Test whether age & choice of the party are independent at 5% LOS.

Soln

$x = \frac{70}{230}$

O	E	$(O-E)^2/E$
25	21	0.762
20	24	0.667
25	25	0
20	21	0.048
25	24	0.0417
25	25	0
25	28	0.324
25	32	0.281
30	30	0

$\Sigma (O-E)^2/E = \chi^2 = 2.121$

H_0: No association b/w ~~inoculation~~ inoculation & disease.

H_a: Is an association.

$\therefore \chi^2_{cal} = 2.11$

at 5% LOS & $(r-1)(c-1) = 4$ df

$\chi^2_{table} = 9.488$

$\therefore \chi^2_{cal} < \chi^2_{table}$

Accept H_0

Q. Following table gives no. of accidents during a week. Find whether the accidents are uniformly distributed over a week.

DAY	Sunday	Mon	Tue	Wed	Thu	Fri	Sat	TOT.
NO. OF ACCIDENTS	~~9~~	~~7~~	~~12~~	~~10~~	~~14~~	~~84~~	~~84~~	
	13	15	9	7	12	10	14	84

Accidents occur uniformly during a week

H_0

H_a: Acci don't occur uniformly weekly

Soln. If H_0 ~~is not~~ is true 12 accidents.

$\chi^2 = \frac{\Sigma (O-E)^2}{E} = \frac{(13-12)^2}{12} + \frac{(15-12)^2}{12} +$

$$\frac{(7-12)^2}{12} + \frac{(7-12)^2}{12} + 0 + \frac{4}{12} + \frac{4}{12}$$

$$= \frac{10}{12} + \frac{9}{12} + \frac{25}{12} + \frac{8}{12}$$

$$= \frac{28}{12} = 2.333$$

we accept H_0.

Q Theory predicts that the proportion of beans in 4 groups A, B, C, D should be 9:3:3:1. In an expt among 1600 beans, the no. in the 4 groups were 882, 313, 287 & 118. Thus experimental result support the theory.

H_0: Proportion of the beans in 4 gps = 9:3:3:1 ($p = \frac{9}{16}$)

H_a: Proportion is not given as in the ratio

$r = \frac{9}{16} \times 1600 = 900$

∴ The sum of the proportion is 9+3+3+1 = 16. The no of ~~free~~ beans in the 4 groups will be 900.

Similarly.. No of beans group B = 300

No of beans group C = 300

No of beans group D = 100

$$\chi^2 = \frac{\Sigma(O-E)^2}{E}$$

$$= \frac{(882-900)^2}{900} + \frac{(313-300)^2}{300} + \frac{(287-300)^2}{300} + \frac{(118-100)^2}{100}$$

$$= 4.73$$

at 5% LOS & $n-1 = 4-1 = 3$ dof

$\chi^2_{table} = 7.81$

$\therefore \chi^2_{cal} < \chi^2_{table}$

accept H_0.

Q Figures given below are having same mean and standard deviation, find total freq. as in A

(a) 1 12 66 220 495 792 924 792 495 220 66 12 1

(b) 2 15 66 210 484 799 943 799 484 210 66 15 2

Apply χ^2 goodness of fit test

O	E
1 } 13	2 } 17
12	15
66	66
220	210
495	484
792	799
924	947
792	799
495	484
220	210
66	66
12 } 13	15 } 17
1	2

Appl^n χ^2_{test} of goodness of fit.

Sol^n H_0: If it is good.
H_a: If it's not good.

$(O-E)^2/E$
0.94
0
0.476
0.25
0.061
0.383
0.061
0.25
0.476
0
0.94

(i) ~~Null~~ H_0: The fit is good.
H_a: The fit is not good.
(ii) Calc^n of $\chi^2 = 3.84$
LOS $= 0.05 = \alpha$
(iii) dof $= 13 - 2 - 3 = 8$
(iv) Critical value: For 8 dof at 5% LOS, table value of $\chi^2 = 15.5$
(v) Decision: $\because 3.84 < 15.5$
(Accepted): Fit is good.

$\chi^2 = 3.84 = \frac{(O-E)^2}{E}$

expl^n: Originally there were 17 classes. Since they're reduced by 2 by grouping twice, so dof gets reduced by 2.
Also, because in χ^2,
- mean same
- std. deviation same

Also, 3 constraints are introduced like mean, standard deviation and frequency of original data. Therefore, dof gets reduced by 3. At 5% LOS, & $13-2-3 = 8$ dof

$\chi^2_{table} = 15.51$
$\because \chi^2_{calc} < \chi^2_{table}$
we accept H_0.
- fit is good. Ans

Q. The following mistakes per page were observed in a book.

No. of mistakes per page	No of pages
0	211
1	90
2	19
3	5
4	0

Fit a Poisson distribution and test the goodness of fit

Sol^n H_0: fit is good
H_a: fit is not good.

Sol^n: $P(X=x) = \frac{e^{-m} m^x}{x!}$

mean, $m = \frac{\Sigma f_i x_i}{\Sigma f_i}$

$= \frac{143}{325} = \cancel{1.43}\ 0.44$

$P(X=x) = \frac{e^{-0.44} \times (0.44)^x}{x!}$

Exp. Freq $= N \times P(X=x) = 325 \times P(X=x)$

x	O	E	$(O-E)^2/E$
0	211	209	0.019
1	90	92	0.043
2	19	20	
3	5	3	0
4	0	1	

24 24

dof $= 5 - 2 = 1$

at 5% LOS

& 1 dof

$\chi^2_{table} = 3.84$

$\chi^2_{cal} < \chi^2_{table}$

Accept H_0

Hypothesis concerning several proportions

	Shipment A	Shipment B	Shipment C	Total
Defective	5 (6)	8 (7)	9 (9)	22
Non-defective	35 (34)	42 (43)	51 (51)	128
Total	40 (40)	50 (50)	60 (60)	150

Test whether the proportion of defective items is same in the three shipments at 5% LOS.

Sol^n: H_0: Proportion is same. $(P_1 = P_2 = P_3)$

H_A: Proportion isn't same $(P_1 \neq P_2 \neq P_3)$

$\cancel{\chi^2 = 0.361}$

$A \rightarrow \frac{40 \times 21}{150} = 5.6$ remn. 34.4 non defective

$B \rightarrow \frac{50 \times 21}{150} = 7$ remn. 43 non defective

$C \rightarrow \frac{60 \times 21}{150} = 8.4$ & remn. 51.6 non defective

$\chi^2 = 0.0309$ LOS $\alpha = 0.05$

dof $= (2-1)(3-1) = 2$

Critical val: Table val of $\chi^2 = 5.991$

(v) Decision: Calc $\chi^2 = 0.0309$ < table val. χ^2

Accepted: Proportion of defective same in all shipments.

OPTIMIZATION

Q. Consider a LPP

$$z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

(objective fun^n)

Subject to $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$ — (1)

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$

and so on...

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$

(constraints) — (2)

$x_1, x_2 \dots x_n \ge 0$ — (3) Non-negativity conditions

(I) Any set of values of $x_1, x_2, \dots x_n$ which satisfies the constraints in (2) is called solution of LPP.

(II) Any solution which satisfies the non-negativity restrictions is called feasible solution.

(III) Any feasible solution which maximizes/minimizes the obj. fn is called optimal feasible solution of LPP.

Q. If the constraints of LPP are $\le$ type, ~~then~~ i.e.

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i,$$

$i = 1, 2, \dots m$

then, we add non-negative variables S_i so that inequality gets changed to equality, i.e.

$$\sum_{j=1}^{n} a_{ij} x_j + S_i = b_i$$

These S_i – slack variables

→ If we've

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i$$

then $\sum_{j=1}^{n} (a_{ij} x_j - S_i) = b_i$

where S_i – surplus variables

Canonical form of LPP

The canonical form is given by

$$\text{Max } Z = \sum_{i=1}^{n} c_i x_i$$

subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i$

$x_j \ge 0$

Characteristics of canonical form

Q. If the objective function is of minimization type, then we must

have ≥ inequality, and if it's of maximization type, then we multiply ≤ inequality.

(2)- If the constraint is of equality then $a_1x_1 + a_2x_2 = b$ → express it in the form of inequality

then, $a_1x_1 + a_2x_2 \le b$,

& $a_1x_1 + a_2x_2 \ge b$.

(3) All decision variables should be restricted ($x_i \ge 0$) if

Suppose x_i is, unrestricted, then

$x_i = x_i' - x_i''$

where $x_i', x_i'' \ge 0$

(cts. of canonical form)

(4) Standard form of LPP.

If the standard form express objective function in terms of, we express the objective fn as well as the constraints in the form of slack variables.

i.e $\max Z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0s_1 + 0s_2 + \dots + 0s_m$

Subject to

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 + 0s_2 + \dots + 0s_m = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + 0s_1 + s_2 + \dots + 0s_m = b_2$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + 0s_1 + 0s_2 + \dots + s_m = b_m$

& $x_1, x_2 \dots x_n, s_1, s_2 \dots s_m \ge 0$

Characteristics of abje

1- All the constraints are expressed in the form of equations with the help of slack variables.

2- RHS of the constraints are non-ve.

3- obj. fn must be of maximization type.

4- All decision and slack variables are non-ve.

Basic solution :- A solution obtained by setting any (m+n) variables out of m+n variables to zero and solving for these 'm' variables is called basic solution.

These 'm' variables are called basic variable.

Therefore, the no. of basic solutions = ${}^{m+n}C_m$

Basic feasible solution (BFS) -

The basic solutions which satisfies the non-negativity solution is called

basic feasible solution.

Non-Degenerate basic visible solution:

If in BFS, the value of all basic variables are positive, then it is called non-degenerate BFS.

Degenerate BFS: If in BFS, the value of atleast 1 basic variable is zero, then it's called degenerate BFS.

Max $Z = x_1 + 3x_2 + 3x_3$

Sub to $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$

No. of Non-basic variable = No. of variables − No. of constraints $= 3 - 2 = 1$

Simplex

Solⁿ	No. of Basic solⁿ	No. of Basic variables	Basic variables	Eqⁿ's and values of Basic var	Is the solⁿ feasible	Is the solⁿ degenerate	Value of Z	Is the solⁿ optimal
1.	$x_3 = 0$	x_1, x_2	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	Yes	No	5	Yes
2	$x_1 = 0$	x_3	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_2 = 2, x_3 = 1$	Yes	No	4	No
3	$x_2 = 0$	x_1	x_2, x_3	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = 1, x_3 = 1$	No	No	-	-

Q. Solve the foll. LPP using simplex method.

Max $Z = 3x_1 + 12x_2$

sub to $3x_1 + 2x_2 \le 18$

$0 \le x_1 \le 4$

$0 \le x_2 \le 6$

Solⁿ: Standard form of LPP is

max $Z - 3x_1 - 2x_2 = 0$

Sub. to $3x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 18$

$x_1 + 0x_2 + 0s_1 + s_2 + 0s_3 = 4$

$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 6$

$x_1, x_2, s_1, s_2, s_3 \ge 0$

No. of initial basic feasible solⁿ is

$x_1 = 0 \quad x_2 = 0$

$s_1 = 18 \quad s_2 = 6$

$s_2 = 4$

$> 2 : 0$

~~Z = 0~~

~~Iteration no. | Basic variables | Coeff of x_1 x_2 s_1 s_2 s_3~~

$Z = 0$

Iteration no.	Basic variables	Coeff of x_1	x_2	s_1	s_2	s_3	RHS	Ratio
0	Z	-3	-2	0	0	0	0	
x_1 enters	s_1	3	2	1	0	0	18	18/3 = 6
s_2 leaves	s_2	1	0	0	1	0	4	4/1 = 4
	s_3	0	1	0	0	1	6	6/0 = ∞
	Z	0	-2	0	3	0	12	
x_2 enters	s_1	0	2	1	-3	0	6	6/2 = 3 → leaves
s_1 leaves	x_1	1	0	0	1	0	4	4/0 = ∞
	s_3	0	1	0	0	1	6	6/1 = 6
	Z	0	0	1	0	0	18	
	x_2	0	1	1/2	-1/2	0	3	
	x_1	1	0	0	0	0	4	
	s_2	0	0	-1/2	3/2	1	3	

All entries in Z row are non-negative, the current solution is the feasible solution.

$R_1 + 3R_2$

$R_2 - 3R_1$

$x_1 = 4$

$x_2 = 3$

$s_1 = 0$

$s_2 = 0$

$s_3 = 3$

$Z_{max} = 18$

Practice qs.

1) Tests made on breaking strength of 10 pcs of metal wire gave the foll. result → 578, 572, 570, 568, 572, 570, 570, 572, 596 & 584 in kgs.

Test if the breaking strength of the metal wire can be assumed to be 577 kg?

X	$d_i = x_i - 580$	$d_i^2 = (x_i - 580)^2$
578		
572		
570		
568		
572		
570		
572		
596		
584		
Total	-48	864

Q29) $\bar{X} = 108.75$

$\mu = 108.75$

$(x_i - \bar{x}) = 843.75$ and $n = 16$

(alc. std. dev.)

$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n} = \frac{843.75}{16} = 52.73$

Step 1 Null hypothesis $H_0 = \mu = 108.75$

Alternative hypothesis $H_1 : \mu \neq 108.75$

Step 2: Calc^n of test statistic

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n-1}} = \frac{108.75 - 108.75}{\sqrt{52.74}/\sqrt{16-1}}$$

$$= \frac{-5}{1.875}$$

$$= -2.67$$

$$\therefore |t| = |-2.67| = 2.67$$

Step 3: LOS = 5%

i.e. $\alpha = 0.05$

Step 4 Critical val ⇒ The value of t_α for 5% LOS & DOF $V = 16-1 = 15$ from the table is 2.13 (%-pt. distribution table)

Step 5 $\because |t| = 2.67 > t_\alpha$

Rejected.

We CAN'T say the population mean is 108.75.

Ans.

Max $Z = 107x_1 + x_2 + 2x_3$

sub to $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5$

$3x_1 - x_2 - x_3 \le 0$

$x_1, x_2, x_3, x_4 \ge 0$

Sol^n Std. form is

Max $Z - 107x_1 - x_2 - 2x_3 = 0$

Sub to $\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = \frac{7}{3}$

$16x_1 + \frac{1}{2}x_2 - 6x_3 + s_1 = 5$

$3x_1 - x_2 - x_3 + s_2 = 0$

$x_1, x_2, x_3, x_4, s_1, s_2 \ge 0$

Let $x_1 = x_2 = x_3 = 0$

$\therefore x_4 = \frac{7}{3}, s_1 = 5, s_2 = 0$

Step	Basic var	Coeff of x_1	x_2	x_3	x_4	s_1	s_2	RHS sol^n	Ratio
0	Z	-107	-1	-2	0	0	0	0	
1 selected level	x_4	14/3	1/3	-2	1	0	0	7/3	2/4 = 1/2
	s_1	16	1/2	-6	0	1	0	5	5/16
	s_2	3*	-1	-1	0	0	1	0	0/3 = 0
2 1 per tem	~~Z~~	~~0~~	~~-11~~	~~-107/3~~	~~-111/3~~	~~0~~	~~0~~	~~107/3~~	~~0~~
	Z	0	-107/3	-111/3	0	0	107/3	0	~~-12~~

Big-M method

Max $Z = c_1x_1 + c_2x_2 + c_3x_3$

Sub to $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ge b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2$

d

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le b_3$

$x_1, x_2, x_3 \ge 0$

$\max Z = c_1x_1 + c_2x_2 + c_3x_3$

Sub to $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - S_1 + A_1 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + S_2 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + S_3 = b_3$

$x_1, x_2, x_3, S_1, S_2, S_3, A_1 \ge 0$

∴ we get

$\max Z = (c_1 + Ma_{11})x_1 + (c_2 + Ma_{12})x_2 + (c_3 + Ma_{13})x_3 - MS_1 - Mb_1$

Q. Use penalty method or Big-M-method to solve the following LPP

$\max Z = 6x_1 + 4x_2$

Sub to $2x_1 + 3x_2 \le 30$

$3x_1 + 2x_2 \le 24$

$x_1 + x_2 \ge 3$

$x_1, x_2 \ge 0$

Is the solution unique? If not, find another solution.

Sol^n

$\max Z = 6x_1 + 4x_2 - MA_3$

Sub to $2x_1 + 3x_2 + S_1 = 30$

$3x_1 + 2x_2 + S_2 = 24$

$x_1 + x_2 - S_3 + A_3 = 3$

$x_1, x_2, S_1, S_2, S_3, A_3 \ge 0$

we get $Z = (6+M)x_1 + (4+M)x_2 - MS_3 - 3M$

i.e. $Z - (6+M)x_1 - (4+M)x_2 + MS_3 + 3M = 0$

~~Let Initial~~

Let $x_1 = x_2 = 0 = S_3$

∴ $S_1 = 30$, $S_2 = 24$

$A_3 = 3$

~~Iter~~

Iter. no.	Basic var.	x_1	x_2	S_1	S_2	S_3	A_3	RHS sol^n	Ratio
0	Z	-6-M	-4-M	0	0	M	0	-3M	
x_1 enters	S_1	2	3	1	0	0	0	30	5
	S_2	3	2	0	1	0	0	24	8
	A_3	1	1	0	0	-1	1	3	3 →
1	Z	0	2	0	0	-6		18	
S_2 enters S_1 leaves	S_1	0	1	1	0	2		24	12
	S_2	0	-1	0	1	3		15	5 →
	x_1	1	1	0	0	-1	1	3	-3
	Z	0	0	0	2	0		48	
	S_1	0	5/3	01	-2/3	0		14	
	S_2	0	-1/3	0	1/3	1		5	-15
	x_1	1	2/3	0	1/3	0		8	12

$R_1 + (6+M) R_4$

$R_3 - 2R_4$

$R_2 - 3R_4$

$x_1 = 8,\ x_2 = 0,\ S_1 = 14,\ S_2 = 0$

$S_3 = 5$

$Z_{max} = 48$

The non-basic variable x_2 has value 0 in Z-row.
Therefore, this LPP has alternate solution.

Final Answer: Alternate optimal sol^n is

$x_1 = \frac{12}{5},\ x_2 = \frac{42}{5},\ S_1 = 0,\ S_2 = 0$

and $S_3 = 39/5$

Q Using Penalty or Big M-method, solve the following LPP

Min $Z = x_1 + 2x_2 + x_3$

Sub. $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \le 1$

$\frac{3}{2}x_1 + 2x_2 + x_3 \ge 8$

$x_1, x_2, x_3 \ge 0$

Sol^n Std. form is

Max $Z' = -Z = -x_1 - 2x_2 - x_3 - MA_1$

Subt. to $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + S_1 = 1$ — (1)

$\frac{3}{2}x_1 + 2x_2 + x_3 - S_2 + A_1 = 8$ — (2)

$x_1, x_2, x_3, S_1, S_2, A_1 \ge 0$

mult. (2) by M & add it to Z'

$\therefore$ Max $Z' = \left(-1 + \frac{3}{2}M\right)x_1 + (-2+2M)x_2 + (-1+M)x_3 - MS_2 - 8M$

i.e. $Z' + \left(1 - \frac{3}{2}M\right) + (2-2M)x_2 + (1-M)x_3 + MS_2 + 8M$

initial BFS: $x_1 = 0 = x_2 = x_3 = S_2$

$\therefore S_1 = 1,\ A_1 = 8$

Iter No	Basic variables	coeff. of x_1	x_2	x_3	S_1	S_2	A_1	RHS sol^n	Ratio
0	Z'	$1-\frac{3}{2}M$	$2-2M$	$1-M$	0	M	0	$-8M$	
1st	S_1	1	1/2	1/2	1	0	0	1	2
1st	A_1	3/2	2	1	0	-1	1	8	4
1	Z'		0	$-4M$	$-4+4M$	M	0	$-4+4M$	
	x_2	2	1	1	2	0	0	2	
	A_1	-5/2	0	-1	-4	-1	1	4	

Since all the entries in 2 row are non-ve, but artical variable A_1 remains in the basis with non-zero value 4 the solution is called pseudo-basic optimal solution.

$R_1 - (2-2M)R_2$

$R_3 - 2R_2$

$\left(1 - \frac{3}{2}M\right) - (2-2M)2$

$= 1 - \frac{3}{2}M - 4 + 4M$

$= -3 + \frac{5}{2}M$

DUALITY:

Q. Write the dual of the given LPP

$$\text{Max } Z = 2x_1 + x_2 + x_3$$

Sub to $x_1 + x_2 + x_3 \geq 6$

$3x_1 - 2x_2 + 3x_3 = 3$

$-4x_1 + x_3 \leq 10$

$x_1, x_3 \geq 0$ x_2 - unrestricted

Sol: Let $x_2 = x_2' - x_2''$

then Max $Z = 2x_1 + (x_2' - x_2'') + x_3$

subject to $-x_1 - (x_2' - x_2'') - x_3 \leq -6$

$3x_1 - 2(x_2' - x_2'') + 3x_3 \leq 3$ (*)

$3x_1 - 2x_2 + 3x_3 \leq 3$

$3x_1 - 2x_2 + 3x_3 \geq 3$

i.e. $-3x_1 + 2x_2 - 3x_3 \leq -3$

(*) $-3x_1 + 2(x_2' - x_2'') - 3x_3 \leq -3$

$-4x_1 + x_3 \leq 10$

$x_1, x_2', x_2'', x_3 \geq 0$

Let $y_1, y_1', y_2', y_2'', y_3$

Dual LPP becomes

Min $W = -6y_1 + 3y_2' - 3y_2'' + 10y_3$

Sub to $-y_1 + 3y_2' - 3y_2'' - 4y_3 \geq 2$

$-y_1 - 2y_2' + 2y_2'' + 0y_3 \geq 1$

$y_1 + 2y_2' - 2y_2'' + 0y_3 \geq -1$

$-y_1 + 3y_2' - 3y_2'' + y_3 \geq 1$

when $y_1, y_2', y_2'', y_3 \geq 0$

i.e. Min $W = -6y_1 + 3y_2' - 3y_2'' + 10y_3$

Sub. to $-y_1 + 3y_2' - 3y_2'' - 4y_3 \geq 2$

$-y_1 - 2y_2' + 2y_2'' + 10y_3 = 1$

$-y_1 + 3y_2'' - 3y_3'' + y_3 \geq 1$

Now let $y_2 = y_2' - y_2''$

Min $W = -6y_1 + 3y_2 + 10y_3$

Sub to $-y_1 + 3y_2 - 4y_3 \geq 2$

$-y_1 - 2y_2 + 0y_3 = 1$

$-y_1 + 3y_2 + y_3 \geq 1$

$y_1, y_3 \geq 0$ y_2 - unrestricted

Q. $\text{Max } Z = x_1 + 3x_2 - 2x_3 + 5x_4$

subject to $3x_1 - x_2 + x_3 - 4x_4 = 6$

$5x_1 + 3x_2 - x_3 - 2x_4 = 4$

$x_1, x_2 \geq 0,\ x_3, x_4$ unrestricted

Soln Canonical form

Let $x_3 = x_3' - x_3''$

$x_4 = x_4' - x_4''$

$\text{Max } Z = x_1 + 3x_2 - 2x_3' + 2x_3'' + 5x_4' - 5x_4''$

Sub. to $3x_1 - x_2 + x_3' - x_3'' - 4x_4' + 4x_4'' \leq 6$

$-3x_1 + x_2 - x_3' + x_3'' + 4x_4' - 4x_4'' \leq -6$

$5x_1 + 3x_2 - x_3' + x_3'' - 2x_4' + 2x_4'' \leq 4$

$-5x_1 - 3x_2 + x_3' - x_3'' + 2x_4' - 2x_4'' \leq -4$

$x_1, x_2, x_3', x_3'', x_4', x_4'' \geq 0$

Let y_1, y_2, y_3, y_4 be dual variables

Dual LPP is

$\text{Min } w = 6y_1 - 6y_2 + 4y_3 - 4y_4$

Subject to $3y_1 - 3y_2 + 5y_3 - 5y_4 \geq 1$

$-y_1 + y_2 + 3y_3 - 3y_4 \geq 3$

$-y_1 + y_2 - y_3 + y_4 \geq -2$

$4y_1 - 4y_2 - 2y_3 + 2y_4 \geq 5$

$4y_1 - 4y_2 + 2y_3 - 2y_4 \geq -5$

$y_1, y_2, y_3, y_4 \geq 0$

∴ 2nd LPP becomes

$\text{min } w = 6y_1 - 6y_2 + 4y_3 - 4y_4$

Sub to

$3y_1 - 3y_2 + 5y_3 - 5y_4 \geq 1$

$-y_1 + y_2 + 3y_3 - 3y_4 \geq 3$

$-y_1 + y_2 + y_3 - y_4 = 2$

$-4y_1 + 4y_2 - 2y_3 + 2y_4 = 5$

Let $y_1 - y_2 = w_1$, $y_3 - y_4 = w_2$

∴ w_1, w_2 are unrestricted.

∴ Dual LPP is

$\text{Min } w = 6w_1 + 4w_2$

Subject to

$3w_1 + 5w_2 \geq 1$

$-w_1 + 3w_2 \geq 3$

$-w_1 + w_2 = 2$

$-4w_1 + 2w_2 = 5$

w_1, w_2 are unrestricted

No. of variables in primal is same as no. of constraints in primal, and

- The variable which is unrestricted corresponding constraints in dual is equality and vice-versa
- In primal the constraints which is equality, corresponding variables in dual is unrestricted.

Non-linear programming problem

We shall consider 3 types of non-linear problems:

- Quadratic programming problem with no constraints.
- Quadratic Programming problem with linear equality constraints
- Quadratic Programming problem with linear inequality constraints

$$z = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2$$
$$a_{12}x_1x_2 + a_{13}x_1x_3 + \cdots + a_{1n}x_1x_n +$$
$$\cdots$$

put $\frac{\partial f}{\partial x_i} = 0, \; 1 \le i \le n$

& find $X_0(x_1, x_2, \ldots x_n)$

Define a Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

If (i) All the principal minors of X_0 matrix at X_0 positive then

$$\frac{\partial^2 f}{\partial x^2} = r \qquad rt - s^2 > 0, \; r > 0 \Big\} \text{ minima}$$

$$\frac{\partial^2 f}{\partial x \partial y} = s \qquad rt - s^2 > 0, \; r < 0$$

$$\frac{\partial^2 f}{\partial y^2} = t$$

(ii) the principal minor determinants $D_1, D_3, \ldots$ are -ve, and $D_2, D_4 \ldots$ are +ve TO SUM UP,

(i) If all $D_1, D_2, \ldots$ +ve → minima

(ii) $D_1, D_3 \ldots$ are -ve, & $D_2, D_4, \ldots$ are +ve } maxima

(iii) If any random pt is indefinite, then X_0 is a saddle point.

Eg. Optimize $Z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 8x_2 - 10x_3$

Sol^n Let us take $Z = f(x_1, x_2, x_3)$

$$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 2x_1 - 6 = 0 \Rightarrow \underline{x_1 = 3}$$

$$\frac{\partial f}{\partial x_2} = 0 \Rightarrow 2x_2 - 8 = 0 \Rightarrow \underline{x_2 = 4}$$

$$\frac{\partial f}{\partial x_3} = 0 \Rightarrow 2x_3 - 10 = 0 \Rightarrow \underline{x_3 = 5}$$

$\therefore X_0(3,4,5)$ is a stationary point.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(iii) $D_1 = |[2]| = 2$

$D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

$D_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$

Since D_1, D_2, D_3 all are +ve, X_0 is a point of minima and therefore Z_{min} is

Let $f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$

$Z_{min} =$

$\frac{\partial f}{\partial x_1} = 0 \Rightarrow x_1 = \frac{1}{2}$

$\frac{\partial f}{\partial x_2} = 0 \Rightarrow x_2 = \frac{2}{3}$

$\frac{\partial f}{\partial x_3} = 0 \Rightarrow x_3 = \frac{4}{3}$

$\therefore X_0(1/2, 2/3, 4/3)$ is a stationary pt

Consider $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$

NLPP with one equality constraint

Consider following NLPP

$Z = f(x_1, x_2, x_3 \ldots x_n)$

Subj. to $g(x_1, x_2 \ldots x_n) = b$

i.e. $g(x_1, x_2 \ldots x_n) - b = 0$

$= h(x_1, x_2, \ldots x_n)$

Construct a new function called Lagrangian funct^n using Lagrangian multipliers i.e.

$L(x_1, x_2 \ldots x_n, \lambda) = f(x_1, x_2 \ldots x_n) - \lambda h(x_1, x_2 \ldots x_n)$ — (1)

The necessary conditions for maximal minima, subject to the constraint $h(x_1, x_2, \ldots x_n)$ are

$\frac{\partial L}{\partial x_1} = 0$

$\frac{\partial L}{\partial x_2} = 0$

$\vdots$

$\frac{\partial L}{\partial x_n} = 0$

$\frac{\partial L}{\partial \lambda} = 0$ — (2)

From (1) we get $\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}$

$\frac{\partial L}{\partial \lambda} = -h$

Using (2), we get following (n+1) conditions

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\vdots$$

$$h(x_1, x_2 \ldots x_n) = 0$$

Solving these $(n+1)$ conditions, we find $x_1, x_2 \ldots x_n$ and λ, and find a point of maximal minima.

Now find value of the following determinant of order $(n+1)$ at x_0.

$$\Delta_{n+1} = \begin{vmatrix} 0 & & & & \\ \frac{\partial h}{\partial x_1} & & & & \\ \frac{\partial h}{\partial x_2} & - & - & - & - \\ \vdots & & & & \\ \frac{\partial h}{\partial x_n} & & & & \end{vmatrix}$$

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & & & \end{vmatrix}$$

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_n} & \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_n} & \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} & \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n^2} & \end{vmatrix}$$

- If the signs of the principal minors $n \geq 1$ $\Delta_3, \Delta_4, \Delta_5 \ldots$ are alternatively +ve and −ve, the point x_0 is a point of maxima.
- If all the principal minors $\Delta_3, \Delta_4, \Delta_5 \ldots \Delta_{n+1}$ are −ve, then the point x_0 is a point of minima.

Q. Use the method of Lagrange multiplier to solve the following NLPP.

$$Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

subject to $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 \geq 0$

Sol^n $L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) - \lambda h(x_1, x_2, x_3)$

$= 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda(x_1 + x_2 + x_3 - 20)$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 + 10 - \lambda = 0 \quad \text{—} (1)$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 + 8 - \lambda = 0 \quad \text{—} (2)$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 6x_3 + 6 - \lambda = 0 \quad \text{—} (3)$$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 + x_2 + x_3 = 20$ — (4)

Multiply (1) by (3), (2) by 6 and (3) by 2

and use (4)

$12(x_1 + x_2 + x_3) + 90 - 11\lambda = 0$

But $x_1 + x_2 + x_3 = 20$

$\Rightarrow \lambda = \frac{330}{11} = 30$

$\therefore$ from (1) (2) & (3),

$x_1 = 5,\ x_2 = 11,\ x_3 = 4$

$\therefore$ $x_0(5, 11, 4)$ is a stationary point.

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} \qquad C_2 - C_4 \Rightarrow C_2 - C_4$$

$$\therefore \Delta_4 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -6 & -6 & 6 \end{vmatrix}$$

$$= 1\left(\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 2 \\ 1 & -6 & -6 \end{vmatrix}\right)$$

$$= -1\left(12 - 4(-8)\right)$$

$$= -44$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$\therefore$ Δ_3 & Δ_4 are both -ve, x_0 is a point of minima.

Therefore,

$Z_{min} = 281$.

Q. Simplex

$Z = 4x_1 + 10x_2$

Subj. to $2x_1 + x_2 \le 50$

$2x_1 + 5x_2 \le 100$

$2x_1 + 3x_2 \le 90$

$x_1, x_2 \le 0$

Sol^n: $Z - 4x_1 - 10x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

$2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 50$

$2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 100$

$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 90$

Iter. no	Basic var.	Coeff. of x_1	x_2	s_1	s_2	s_3	RHS sol^n	Ratio
0	Z	-4	-10	0	0	0	0	
s_1 leaves	s_1	2	1	1	0	0	50	50/1=50
x_2 enters	s_2	2	5	0	1	0	100	100/5=20
	s_3	2	3	0	0	1	90	90/3=30
1	Z	0	0	0	2	0	200	
	s_1	8/5	0	1	1	-1/5	0	30
	x_2	2/5	1	0	0	1/5	0	20
	s_3	4/5	0	0	0	-3/5	1	30

$\therefore$ $x_1 = 0,\ x_2 = 20,\ Z_{max} = 200$

Q Maximise $z = x_1 + 3x_2$

subject to $x_1 + 2x_2 \le 10$

$0 \le x \le 5$

$0 \le x_2 \le 4$

Sol:

$z - x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

$x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 10$

$x_1 + 0x_2 + 0s_1 + s_2 + 0s_3 = 5$

$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 4$

$z = 0 \quad s_1 = 10 \quad s_2 = 5 \quad s_3 = 4$

Kuhn-Tucker method

$\text{Max } z = 2x_1^2 - 7x_2^2 + 12x_1x_2$

subject to $2x_1 + 5x_2 \le 98$

$x_1, x_2 \ge 0$

Sol^n Let $f(x_1, x_2) = 2x_1^2 - 7x_2^2 + 12x_1x_2$

and $h(x_1, x_2) = 2x_1 + 5x_2 - 98$

Now Kuhn-Tucker conditions are

$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$

$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$

$h(x_1, x_2) \le 0$

$\lambda(x_1, x_2) = 0, \lambda \ge 0$

we get

$4x_1 + 12x_2 - 2\lambda = 0$ — (1)

$-14x_2 + 12x_1 - 5\lambda = 0$ — (2)

$2x_1 + 5x_2 - 98 \le 0$ — (3)

$\lambda(2x_1 + 5x_2 - 98) = 0$ — (4)

$x_1, x_2, \lambda \ge 0$ — (5)

from (4) either $\lambda = 0$ or $2x_1 + 5x_2 - 98 = 0$

Case (i) : If $\lambda = 0$, then from (1) & (2)

$4x_1 + 12x_2 = 0$

& $-14x_2 + 12x_1 = 0$

$\Rightarrow x_1 = 0 = x_2$

$\therefore z = 0$

Hence, for $\lambda = 0$, feasible solution is not obtained.

Therefore, we reject these values.

Case (ii) : If $\lambda \ne 0$

then $2x_1 + 5x_2 - 98 = 0$ — (6)

Mult (1) by 5 & eq^n (2) by 2 & subtract.

$\Rightarrow 20x_1 + 60x_2 + 28x_2 - 24x_1 = 0$

$\Rightarrow -x_1 + 22x_2 = 0$

i.e. $x_1 = 22x_2$

put $x_1 = 22x_2$ in (6)

$\Rightarrow x_1 = 44$ & $x_2 = 2$

$\therefore$ from (1) $\lambda = 100 > 0$

Therefore, these values will satisfy all the necessary Kuhn-Tucker conditions and therefore the optimal solution is $x_1 = 44$, $x_2 = 2$ &

$z_{max} = 4900$

Q. Max $z = 8x_1 + 10x_2 - x_1^2 - x_2^2$
subject to $3x_1 + 2x_2 \le 6$.
$x_1, x_2 \ge 0$

Sol^n. Let $f(x_1, x_2) = 8x_1 + 10x_2 - x_1^2 - x_2^2$
and $h(x_1, x_2) = 3x_1 + 2x_2 - 6$

Now, Kuhn-Tucker conditions are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$h(x_1, x_2) \le 0$$

$$\lambda h(x_1, x_2) = 0, \quad \lambda \ge 0$$

$\therefore$ we get

~~$4x_1 + 12x_2$~~

~~$8 + 2x_1 =$~~ $8 - 2x_1 = 0$ — (1)

$8 - 2x_1 - 3\lambda = 0$ — (1)

$10 - 2x_2 - 4\lambda x_2 = 0$ — (2)

$3x_1 + 2x_2 - 6 \le 0$ — (3)

$\lambda(3x_1 + 2x_2 - 6) = 0$ — (4)

$x_1, x_2, \lambda \ge 0$ — (5)

from (4) either $\lambda = 0$ or $3x_1 + 2x_2 - 6 = 0$

case (i): If $\lambda = 0$ then from (1) & (2)
$x_1 = 4, \quad x_2 = 5$
$\therefore z = 82 - 16 - 25$
$= 41$

But eq^n (3) is not satisfied.
$\therefore$ These values are not accepted

case (ii) If $\lambda \ne 0$
then $3x_1 + 2x_2 - 6 = 0$ — (6)
mult (1) by 2 & (2) by 3 & subtract
$x_1 = 4/13$
$x_2 = 33/13$
$\therefore$ from (1) $\lambda = \frac{32}{13} > 0$

Again, 7.81 satisfies eq^n (3)
$\therefore$ optimal sol^n is satisfied.
$x_1 = \frac{4}{13}, \quad x_2 = \frac{33}{10}$
$z_{max} = \frac{277}{13}$.

Q4. A scooter mechanic can accomodate 5 scooters at a time, 4 in the queue and 1 being repaired.

Sol^n (1) $P_0 = 0.048 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1 - 3/2}{1-(3/2)^6}$

(2) $P_N = P_0 \rho^N = 0.3652$
$\therefore$ 36.5% of the customers will be turned away.

(3) Eff arrival rate $= \lambda_{eff} = \lambda(1-P_N)$
$= 6(1-0.3652) = 3.8088$