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1. Laplace Transform } ISE
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1. **Def^n:** ~~If~~ f(t) is a function of t, $0 \le t < \infty$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$
$$= \phi(s) = \bar{f}(s)$$

If $f(t) = \begin{cases} \cos t, & 0 \le t \le \pi \\ \sin t, & t > \pi. \end{cases}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$
$$= \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$$

Apply $\begin{cases} \int e^{ax}\cos bx\, dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \\ \int e^{ax}\sin bx\, dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \end{cases}$

(Don't write spf. a & b)

$$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^\pi$$
$$+ \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_\pi^\infty$$
$$= \left(\frac{e^{-s\pi}}{s^2+1}(-s(-1) + 0) - \frac{1}{s^2+1}(-s)\right)$$

$(\log 0 = -\infty)$

$$+\left(0 - \frac{e^{-\pi s}}{s^2+1}(-1)(-1)\right)$$

$$= \frac{s e^{-\pi s}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}$$ Ans

Laplace of standard functions

(1) $L[e^{at}] = \frac{1}{s-a}$ → If $a=0$ $\therefore L[1] = \frac{1}{s}$

(2) $L[e^{-at}] = \frac{1}{s+a}$

(3) $L[\sin at] = \frac{a}{s^2+a^2}$

(4) $L[\cos at] = \frac{s}{s^2+a^2}$

(5) $L[\sinh at] = \frac{a}{s^2-a^2}$

(6) $L[\cosh at] = \frac{s}{s^2-a^2}$

(7) $L[t^n] = \frac{\Gamma n+1}{s^{n+1}} = \frac{n!}{s^{n+1}}$

Linearity prop.

$L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)]$

ex: $L[t^2 - e^{-2t} + \cosh^2 3t + \sin^3 t]$

$= \cancel{L(t^2)}\ L(t^2) - L(e^{-2t}) + L(\cosh^2 3t) + L(\sin^3 t)$

$$\frac{2!}{s^3} - \frac{1}{s+2} + L\left(\frac{e^{3t}+e^{-3t}}{2}\right)^2$$

$$+ L\left(\frac{e^{3x} - e^{-3x}}{2}\right)$$

$$= \frac{2}{s^3} - \frac{1}{s+2} + L\left(\frac{e^{6t}+e^{-6t}+2}{2}\right)$$

$$+ \left(+ \frac{3}{s^2+9}\right)$$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2s} + \frac{1}{2}\frac{s}{s^2-36} + \frac{3}{s^2+9}$$

Let $x = \cos t + i\sin t$

$x + \frac{1}{x} = 2\cos t$

$x - \frac{1}{x} = 2i\sin t$ } DMT

$L[\sin^5 t]$

$\therefore \left(x - \frac{1}{x}\right)^5 = (2i\sin t)^5 = 2^5 i \sin^5 t$

$$\sin^5 t = \frac{1}{32i}\left(x^5 - \frac{1}{x^5} - 5x^3 + 10x - 10\left(\frac{1}{x}\right) + \frac{5}{x^3}\right)$$

$$= \frac{1}{32i}\left(\left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)\right)$$

$$\frac{1}{32i}\left(\cos 5t + i\sin 5t - (\cos 5t - i\sin 5t)\right) + 10(\cos t + i\sin t$$

$$- 5(\cos 3t + i\sin 3t - (\cos 3t - i\sin 3t)$$

$\therefore \sin^5 t = \frac{1}{2^4}\left(\sin 5t - 5\sin 3t + 10\sin t\right)$

$\therefore L(\sin^5 t) = \frac{1}{2^4}\left[\frac{5}{s^2+25} - \frac{5\times 3}{s^2+9} + \frac{10}{s^2+1}\right]$

$= \frac{120}{(s^2+1)(s^2+9)(s^2+25)}$

$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \phi(s)$

$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

(1) $L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$

(2) Change of scale:

$$\boxed{\begin{array}{l} \text{If } L[f(t)] = \phi(s) \\ \text{then } L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right) \end{array}}$$

e.g. $L(\cos 3t) = \frac{s}{s^2+9}$

$f(t) = \cos t$ ~~But~~ By property

$L[f(t)] = \frac{s}{s^2+1}$

$= \phi(s)$

$\frac{1}{3}\phi\left(\frac{s}{3}\right) = \frac{1}{3}\frac{s/3}{(s/3)^2+1}$

$= \frac{s}{s^2+9}$

(3) First shifting prop.

$$\boxed{\begin{aligned} &\text{If } L[f(t)] = \phi(s) \\ &\text{then } L\{e^{at} f(t)\} = \phi(s-a) \\ &\text{or } L[e^{-at} f(t)] = \phi(s+a) \end{aligned}}$$

e.g. $L[\cos 4t] = \dfrac{s}{s^2+16} = \phi(s)$

$\therefore L[e^{3t} \cos 4t] = \phi(s-3)$

$= \dfrac{s-3}{(s-3)^2+16}$

Q. $\boxed{\text{S.T. } L\left[\sinh\left(\frac{t}{2}\right) \sin \frac{\sqrt{3}}{2} t\right] = \frac{\sqrt{3}}{2} \frac{s}{s^4+s^2+1}}$

$\sinh(t/2) = \dfrac{e^{t/2} - e^{-t/2}}{2}$

Sol^n: $L\left[\left(\dfrac{e^{t/2} - e^{-t/2}}{2}\right) \sin \dfrac{\sqrt{3}}{2} t\right]$

$\phi =$

$= \dfrac{1}{2}\left\{L\left[e^{t/2} \sin \dfrac{\sqrt{3}}{2} t\right] - L\left[e^{-t/2} \sin \dfrac{\sqrt{3}}{2} t\right]\right\}$

Now $L\left[\sin \dfrac{\sqrt{3}}{2} t\right] = \dfrac{\frac{\sqrt{3}}{2}}{s^2 + \frac{3}{4}} = \phi(s)$ — (1)

$\therefore L\left[e^{t/2} \sin \dfrac{\sqrt{3}}{2} t\right]$

$= \phi\left(s - \dfrac{1}{2}\right)$

$= \dfrac{\sqrt{3}/2}{\left(s-\frac{1}{2}\right)^2 + \frac{3}{4}} = \dfrac{\sqrt{3}/2}{s^2 - s + 1}$ — (2)

$\cancel{\frac{s^2+1}{4}} - s + 3/4 \quad s^2$

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$$L\left(e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right) = \phi\left(s+\frac{1}{2}\right)$$

$$= \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{\sqrt{3}/2}{s^2+s+1} \quad (3)$$

Adding (2) & (3) eqⁿs,

(from (1), (2) & (3))

$$\frac{\sqrt{3}}{2}\left(\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right)$$

$$\boxed{L[\sinh(t/2)\sin\sqrt{3}/2\, t] = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\frac{1}{s^2-s+1} - \frac{\sqrt{3}}{2}\frac{1}{s^2+s+1}\right)}$$

$$= \frac{\sqrt{3}}{4}\left(\frac{2s}{(s^2+1)^2-s^2}\right) \qquad (a-b)(a+b) = a^2-b^2$$

$$= \frac{\sqrt{3}}{2}\left(\frac{1}{s^4-s^2+2s^2+1}\right) = \frac{\sqrt{3}}{2}\left(\frac{1}{s^4+s^2+1}\right)$$

(SECOND SHIFTING PROPERTY)

If $L[f(t)] = \phi(s)$

and $g(t) = \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$

then $L[g(t)] = e^{-as}\phi(s)$

e.g. $g(t) = \begin{cases} \cos(t-2\pi/3) & \text{for } t > 2\pi/3 \\ 0 & \text{for } t < 2\pi/3 \end{cases}$

$L[g(t)] = ?$

$f(t) = \cos t, \quad a = 2\pi/3$

$$\therefore L[g(t)] = e^{-2\pi/3 \, s} \, \phi(s)$$

$$= e^{-2\pi/3 s} \cdot \frac{s}{s^2+1}$$

→ class notes
→ Practice prblms

Effect of multiplication by t

- $$\boxed{\begin{aligned} &\text{If } L[f(t)] = \phi(s) \\ &\text{then } L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\phi(s)] \\ &n=1 : L[t f(t)] = -\frac{d}{ds}(\phi(s)) = -\phi'(s) \\ &n=2 : L[t^2 f(t)] = \phi''(s) \end{aligned}}$$

- Find $L[t e^{3t} \sqrt{1+\sin t}]$.

$\underline{\text{Sol}^n}$: $n=1$

$$L(\sqrt{1+\sin t}) = L\left(\sqrt{\sin^2 t/2 + \cos^2 t/2 + 2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$1 = \sin^2 t/2 + \cos^2 t/2$$

$$= L(\sin t/2 + \cos t/2)$$

$$= L(\sin t/2) + L(\cos t/2)$$

$$= \frac{\frac{1}{2}}{s^2+\frac{1}{4}} + \frac{s}{s^2+\frac{1}{4}} = \phi_1(s)$$

don't, for shortcut

$$\frac{2s+1}{2s^2+\frac{1}{2}}$$

$$\therefore L\left[t\sqrt{1+\sin t}\right] = -\phi_1'(s)$$

$$= -\left[\frac{1}{2}\left(\frac{-1}{(s^2+1/4)^2}\times 2s\right) + \frac{(s^2+1/4) - s(2s)}{(s^2+1/4)^2}\right]$$

$$= \frac{s}{(s^2+1/4)^2} - \frac{(1/4 - s^2)}{(s^2+1/4)^2} = \phi_2(s)$$

$$\therefore L\left[e^{3t}\,t\sqrt{1+\sin t}\right] = \phi_2(s-3)$$

$$= \frac{s-3}{((s-3)^2+1/4)^2} - \frac{(1/4-(s-3)^2)}{((s-3)^2+1/4)^2}$$

Q. Evaluate $\int_0^\infty e^{-3t}\,t^2\cos t\,dt$

Sol^n $= \int_0^\infty t^2\left(e^{-3t}\cos t\right)dt$

$\therefore f(t) = \left(e^{-3t}\cos t\right)$

Rough

~~Data Structures~~

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$\rightarrow$ $L[t^2\cos t] = \int_0^\infty e^{-st}(t^2\cos t)\,dt$

$= \Phi(s)$

now $L(\cos t) = \dfrac{s}{s^2+1}$

$\therefore L[t^2\cos t] = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+1}\right)$

$= \dfrac{d}{ds}\left(\dfrac{(s^2+1) - s\times 2s}{(s^2+1)^2}\right)$

$= \dfrac{d}{ds}\left(\dfrac{1-s^2}{(s^2+1)^2}\right)$

$= \dfrac{(s^2+1)^2(-2s) - (1-s^2)2(s^2+1)\times 2s}{(s^2+1)^4}$

$L[t^2\cos t] = \dfrac{(s^2+1)(-2s) - 4s(1-s^2)}{(s^2+1)^3}$

$= \int_0^\infty e^{-st}\,t^2\cos t\,dt$

Put $s = 3$

$$\therefore \int_0^\infty e^{-3t} t^2 \cos t \, dt = \frac{36}{1000}$$

effect of division by t.

If $L[f(t)] = \Phi(s)$

then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \Phi(s)\, ds$

Find $L\left[\frac{e^{-2t} \cosh t \sin 2t}{t}\right]$

Sol^n: $L[e^{-2t} \cosh t \sin 2t]$

$$= L\left[e^{-2t}\left(\frac{e^t + e^{-t}}{2}\right) \sin 2t\right]$$

$$= \frac{1}{2} L\left[(e^{-t} + e^{-3t}) \sin 2t\right]$$

$$= \frac{1}{2}\left[L(e^{-t} \sin 2t) + L(e^{-3t} \sin 2t)\right]$$

$$\because L(\sin 2t) = \frac{2}{s^2+4}$$

$$\therefore L(e^{-t} \sin 2t) = \frac{2}{(s+1)^2+4}$$

$$\& \ L(e^{-3t} \sin 2t) = \frac{2}{(s+3)^2+4}$$

$$\therefore L(e^{-2t} \cosh t \sin 2t) = \frac{1}{2}\left[\frac{2}{(s+1)^2+4} + \frac{2}{(s+3)^2+4}\right]$$

$$= \Phi(s)$$

$$\therefore L\left(\frac{e^{-2t}\cosh t \sin 2t}{t}\right) = \int_s^{\infty} \phi(s)\, ds$$

$$= \int_s^{\infty} \left(\frac{1}{(s+1)^2+4} + \frac{1}{(s+3)^2+4}\right) ds$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s+1}{2}\right) + \frac{\pi}{2} - \tan^{-1}\left(\frac{s+3}{2}\right)\right]$$

Q. Find $L\left[\frac{\sin^2 t}{t}\right]$

Hence P.T $\int_0^{\infty} e^{-t}\frac{\sin^2 t}{t}\, dt = \frac{1}{4}\log_e 5$

Sol: $L\left[\frac{\sin^2 t}{t}\right]$; $f(t) = \sin^2 t$

$$L[\sin^2 t] = \phi(s)$$

$$= L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+4)}$$

$$L(\sin^2 t) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$\therefore L\left(\frac{\sin^2 t}{t}\right) = \frac{1}{2}\int_s^{\infty}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\cancel{\frac{1}{2}}\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left(\log s^2 - \log(s^2+4)\right]_0^\infty$$

$$= \frac{1}{4}\left[\log \frac{s^2}{s^2+4}\right]_s^\infty$$

(dividing N^r & d^r)

$$= \frac{1}{4}\left(\log \frac{1}{1+\frac{4}{s^2}}\right)_s^\infty \quad \cancel{= \frac{1}{4}\left(\log 1 - \log\left(\frac{s^2}{s^2+4}\right)\right)}$$

$$= \frac{1}{4}\left(\log 1 - \log\left(\frac{s^2}{s^2+4}\right)\right)$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

$$\therefore L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right).$$

$$= \int_0^\infty e^{-st}\frac{\sin^2 t}{t}\,dt$$

Put $s=1$

we get $\int_s^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log_e 5$

Ex. Find $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$ (using Laplace only, even if NOT mentioned)

Sol^n: $L(\cos 6t - \cos 4t) = \frac{s}{s^2+36} - \frac{s}{s^2+16}$

$\therefore L\left(\frac{\cos 6t - \cos 4t}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right) ds$

[using division by t prop.]

$= \left[\frac{1}{2}\log(s^2+36) - \frac{1}{2}\log(s^2+16)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{1+\frac{36}{s^2}}{1+\frac{16}{s^2}}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$

$= \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$

$= \int_0^\infty e^{-st}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt$

Put $s = 0$

we get $\int_0^\infty \frac{(\cos 6t - \cos 4t)}{t}\,dt = \frac{1}{2}\log\frac{4}{9}$

$= \frac{1}{2}\log\left(\frac{2}{3}\right)^2 = \boxed{\log(2/3)}$ Ans

(7) **Laplace of derivative property**

If $L[f(t)] = \phi(s)$

then $L[f^{n}(t)]$

$= s^n \phi(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \dots - s^{n-n} f^{n-1}(0)$

$n=1$: $L[f'(t)] = s\phi(s) - f(0)$.

$n=2$: $L[f''(t)] = s^2\phi(s) - sf(0) - f'(0)$.

$n=3$: $L[f'''(t)] = s^3\phi(s) - s^2 f(0) - sf'(0) - f''(0)$

Exam. Using Laplace of $\cos(at)$ find laplace of $\sin(at)$.

Sol:

Let $f(t) = \cos at$

$L[f(t)] = \dfrac{s}{s^2+a^2}$

but $f'(t) = -a.\sin at$

$\therefore L[\sin at] = -\dfrac{1}{a} L[f'(t)]$.

$= \dfrac{-1}{a}[s\phi(s) - f(0)]$

$= \dfrac{-1}{a}\left[\dfrac{s^2}{s^2+a^2} - 1\right]$

$= \dfrac{-1}{a}\left(\dfrac{-a^2}{s^2+a^2}\right)$

$= \dfrac{a}{s^2+a^2}$

(8) **Laplace of integral property**

If $L[f(t)] = \phi(s)$.

Then $L\left[\int_0^t f(u)\,du\right] = \dfrac{\phi(s)}{s}$.

(1) Find $L[\operatorname{erf}_c \sqrt{t}]$.

Sol^n. $\operatorname{erf}\sqrt{t} = \dfrac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\,du$ } definition of erf. — (1)

$\operatorname{erf}_c \sqrt{t} = 1 - \operatorname{erf}\sqrt{t}$

$\therefore L[\operatorname{erf}_c \sqrt{t}] = L(1) - L(\operatorname{erf}\sqrt{t})$

$= \dfrac{1}{s} - L(\operatorname{erf}\sqrt{t})$ — (2)

In (1) put $u^2 = v$

$\therefore 2u\,du = dv$.

$\therefore \operatorname{erf}\sqrt{t} = \dfrac{2}{\sqrt{\pi}} \int_{v=0}^{t} e^{-v} \dfrac{dv}{2\sqrt{v}}$

$= \dfrac{1}{\sqrt{\pi}} \int_{v=0}^{t} e^{-v} v^{-1/2}\,dv$

$n - \frac{1}{2}$

$\therefore L[\operatorname{erf}\sqrt{t}] = \dfrac{1}{\sqrt{\pi}} L\left[\int_0^t e^{-v} v^{-1/2}\,dv\right]$

but $L(v^{-1/2}) = \dfrac{\Gamma 1/2}{s^{1/2}} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$

$\therefore L(e^{-v} v^{-1/2}) = \dfrac{\sqrt{\pi}}{\sqrt{s+1}}$ (using first shifting property)

$\therefore L[\operatorname{erf}_c \sqrt{t}] = \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{\sqrt{\pi}}{\sqrt{s+1} \cdot s} = \dfrac{1}{s\sqrt{s+1}}$

$$\therefore L[\text{erfc}\sqrt{t}] = \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$$ from (2)

(1) Find $L\left[\int_0^t u e^{-3u} \cos^2 2u \, du\right]$.

Sol^n: $L(\cos^2 2u) = L\left(\frac{1+\cos 4u}{2}\right)$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

$$\therefore L(u\cos^2 2u) = -\frac{d}{ds}\,\frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

(By Effect of multiplication by t property)

$$= \frac{-1}{2}\left(-\frac{1}{s^2} + \frac{(s^2+16) - 2s^2}{(s^2+16)^2}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2} + \frac{16-s^2}{(16+s^2)^2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2-16}{(s^2+16)^2}\right].$$

$$\therefore L[e^{-3u}(u\cos^2 2u)] = \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{(s+3)^2 - 16}{((s+3)^2+16)^2}\right] = \phi(s)$$

$$\therefore L\left[\int_0^t \cancel{f} e^{-3u}(u\cos^2 2u)\, du\right]$$

$$= \frac{\phi(s)}{s}$$

Find $L\left[\cosh t \underbrace{\left(\int_0^t e^{u}\cosh u\, du\right)}_{=f(t)}\right]$.

$= L[\cosh t\, f(t)]$.

$= L\left[\left(\frac{e^{t}+e^{-t}}{2}\right) f(t)\right]$.

$= \frac{1}{2}\left[L\left(e^{t} f(t)\right) + L\left(e^{-t} f(t)\right)\right]$.

(first shifting -ve)
(after finding $L(f(t))$)

$L\left(\int_0^t e^{u}\cosh u\, du\right)$

Inverse Laplace transform

If $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$, then $f(t)$ is called inverse laplace of $\phi(s)$, written as $f(t) = L^{-1}[\phi(s)]$.

Std. formulae :

① $L[e^{at}] = \frac{1}{s-a}$

$\therefore \boxed{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$

② $L[e^{-at}] = \frac{1}{s+a}$

$\therefore \boxed{L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}}$

③ $L[\sin at] = \frac{a}{s^2+a^2}$

$\therefore \boxed{L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at}$

④ $L(\cos at) = \frac{s}{s^2+a^2}$

$\therefore \boxed{L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at}$

⑤ $L(\sinh at) = \frac{a}{s^2-a^2}$

$\therefore \boxed{L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at}$

⑥ $L(\cosh at) = \frac{s}{s^2-a^2}$

$\therefore \boxed{L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at}$

⑦ $L(t^n) = \frac{\Gamma n}{s^n}$

$\therefore \boxed{L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\Gamma n}}$

1) Linearity prop.

$$L(af(t)+bg(t)) = aL(f(t))+bL(g(t))$$

$$\therefore \boxed{L^{-1}[a\phi_1(s)+b\phi_2(s)] = aL^{-1}[\phi_1(s)] + bL^{-1}(\phi_2(s))}$$

Q. $\phi(s) = \dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{s^2-9} + \dfrac{1}{s+1} + \dfrac{5}{s^4}$.

$\therefore L^{-1}[\phi(s)]$

$= \dfrac{2}{4}L^{-1}\left(\dfrac{s}{s^2+\frac{25}{4}}\right) \cancel{+ 4L^{-1}(s-} - \dfrac{5}{4}L^{-1}\left(\dfrac{1}{s^2+\frac{25}{4}}\right)$

$+ 4L^{-1}\left(\dfrac{s}{s^2-9}\right) - 18L^{-1}\left(\dfrac{1}{s^2-9}\right)$

$+ L^{-1}\left(\dfrac{1}{s+1}\right) + 5L^{-1}\left(\dfrac{1}{s^4}\right)$

$= \dfrac{1}{2}L^{-1}\left(\dfrac{s}{s^2+\left(\frac{5}{2}\right)^2}\right) - \dfrac{5}{4}L^{-1}\left(\dfrac{1}{s^2+(5/2)^2}\right)$

$+ 4L^{-1}\left(\dfrac{s}{s^2-3^2}\right) - 18L^{-1}\left(\dfrac{1}{s^2-3^2}\right)$

$+ L^{-1}\left(\dfrac{1}{1+s}\right) \cancel{+18L^{-1}\left(\dfrac{1}{s^2-3^2}\right)} \; 5L^{-1}(s^{-4})$

$= \dfrac{1}{2}\cos\dfrac{5t}{2} - \dfrac{5}{4}\times\dfrac{1}{5/2}\sin\dfrac{5t}{2} + 4\cosh 3t$

$- 18\times\dfrac{1}{3}\sinh 3t + e^{-t} \cancel{-18} + \dfrac{5.t^3}{\Gamma 4}$

$$= \boxed{\dfrac{1}{2}\cos\dfrac{5t}{2} - \dfrac{1}{2}\sin\dfrac{5t}{2} + 4\cosh 3t - 6\sinh 3t + e^{-t} + \dfrac{5}{6}t^3}$$

Ans

(2) First shifting property

L: If $L[f(t)] = \phi(s)$
then $L[e^{at} f(t)] = \phi(s-a)$
$L[e^{-at} f(t)] = \phi(s+a)$

L^{-1}: If $L^{-1}[\phi(s)] = f(t)$
then $L^{-1}[\phi(s-a)] = e^{at} f(t) = e^{at} L^{-1}[\phi(s)]$

& $L^{-1}[\phi(s+a)] = e^{-at} f(t) = e^{-at} L^{-1}[\phi(s)]$

$$L^{-1}\left[\frac{s+4}{(s+4)^2+9}\right]$$
$$= L^{-1}[\phi(s+4)]$$
$$= e^{-4t} L^{-1}\left[\frac{s}{s^2+9}\right]$$
$$= \underline{e^{-4t} \cos 3t}$$

(1) $\dfrac{s^2}{(s-1)^3}$

(2) $\dfrac{s}{s^4+s^2+1}$

(3) $\dfrac{s^2+16s-24}{s^4+20s^2+64}$

Sol^n: $\phi(s) = \dfrac{s^2}{(s-1)^3}$

$$= \frac{(s-1)^2 + 2s - 1}{(s-1)^3}$$

$$= \frac{1}{s-1} + 2\frac{s}{(s-1)^3} - \frac{1}{(s-1)^3}$$

$$= \frac{(s-1)^2 + 2(s-1) + 1}{(s-1)^3}$$

$$= \frac{1}{s-1} + 2\frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$\therefore L^{-1}[\phi(s)] = L^{-1}\left(\frac{1}{s-1}\right) + 2L^{-1}\left(\frac{1}{(s-1)^2}\right) + L^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$= e^t + 2e^t L^{-1}\left(\frac{1}{s^2}\right) + e^t L^{-1}\left(\frac{1}{s^3}\right)$$

$$= e^t + 2e^t \cdot \frac{t}{\sqrt{2}} + e^t \cdot \frac{t^2}{\lfloor 3}$$

Sdn. Ans.

$$= e^t + te^t + \frac{e^t}{2} t^2$$

NOTE: Can also apply

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \qquad \frac{1}{(x-a)^3} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

and apply LCM then.

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(2) $\phi(s) = \frac{s}{s^4 + s^2 + 1}$

$$= \frac{s}{s^2(s^2) + 1^2} = \frac{s}{(s^2+1)^2 - s^2}$$

$$= \frac{s}{(s^2+1-s)(s^2+1+s)}$$

M-1 $= \frac{1}{2}\frac{(s^2+1+s) - (s^2+1-s)}{(s^2+1-s)\ (s^2+1+s)} = \frac{1}{2}\left(\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right)$

$$= \frac{1}{2}\left(\frac{1}{(s-1)^2+s} - \frac{1}{(s+1)^2 - s}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$\therefore$ ~~Laplace~~ $L^{-1}[\phi(s)] = \frac{1}{2} L^{-1}\left(\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$

$$= \frac{1}{2}\left(L^{-1}\left(\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right) - L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)\right)$$

$\cancel{= \frac{1}{2}\left(\frac{1}{\frac{\sqrt{3}}{2}} \sin\frac{\sqrt{3}}{2}\right.}$ | $\cancel{= e^{t/2} L^{-1}\phi(s)}$

$= \frac{e^{t/2}}{2}\left(\frac{1}{(\sqrt{3}/2)} \sin\frac{\sqrt{3}}{2}t\right) - \frac{e^{-t/2}}{2}\left(\frac{1}{(\sqrt{3}/2)}\sin\frac{\sqrt{3}}{2}t\right)$

(using first shifting theorem)

$\therefore$ Laplace inverse of given function,

$L^{-1}[\phi(s)] = \frac{2}{2} \times \frac{1}{\sqrt{3}} \times \left(e^{t/2} - e^{-t/2}\right) \sin\frac{\sqrt{3}}{2}t$

$$\Rightarrow \boxed{L^{-1}[\phi(s)] = \frac{2}{\sqrt{3}} \sinh\frac{t}{2} \sin\frac{\sqrt{3}}{2}t} \text{ Ans}$$

Q3) $\phi(s) = \frac{s^2+16s-24}{s^4+20s^2+64}$

$= \frac{s^2+16s-24}{(s^2+16)(s^2+4)}$

$$= \frac{As+B}{s^2+16} + \frac{Cs+D}{s^2+4}$$

$$= \frac{(As+B)(s^2+4) + (Cs+D)(s^2+16)}{(s^2+16)(s^2+4)}$$

$$= s^2+16s-24 = s^3(A+C) + s^2(B+D) + s(4A+16C) + (4B+16D)$$

$A+C=0$ — (1)

$B+D=1$ — (2)

$4A+16C=16$

$A+4C=4$ — (3)

$4B+16D=-24$

$B+4D=-6$ — (4)

Solving (1) & (3), & (2) & (4),

$A = \frac{-4}{3}$

$B = 10/3$

$C = \frac{4}{3}$

$D = \cancel{\frac{-7}{4}} \; \frac{-7}{3}$

$$\therefore \phi(s) = \frac{-4}{3}\frac{s}{s^2+16} + \frac{10/3}{s^2+16} + \frac{4}{3}\frac{s}{s^2+4} - \frac{\frac{7}{3}}{s^2+4}$$

$$\boxed{\therefore L^{-1}[\phi(s)] = \frac{-4}{3}\cos 4t + \frac{10}{3}\frac{1}{4}\sin 4t + \frac{4}{3}\cos 2t - \frac{7}{3}\frac{1}{2}\sin 2t}$$ Ans

Second Shifting property

By ~~first~~ second Shifting property of Laplace transform

if $L[f(t)] = \phi(s)$

and $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

then $L[g(t)] = e^{-as}\phi(s)$

$\therefore$ If $f(t) = L^{-1}[\phi(s)]$

then $L^{-1}[e^{-as}\phi(s)] = g(t)$

$= \begin{cases} f(t-a) & \text{for } t>a \\ 0 & \text{for } t<a \end{cases}$

Q. $L^{-1}\left[\dfrac{e^{-2s}}{s^2+8s+25}\right] = ?$

Sol^n Let $\phi(s) = \dfrac{1}{s^2+8s+25}$

$L^{-1}[\phi(s)] = L^{-1}\left[\dfrac{1}{s^2+8s+25}\right]$

$= L^{-1}\left[\dfrac{1}{(s+4)^2+\underset{(=3^2)}{9}}\right]$

$= e^{-4t} L^{-1}\left[\dfrac{1}{s^2+9}\right]$

$= \dfrac{e^{-4t}}{3}\sin 3t = f(t)$

$\therefore L^{-1}[e^{-2s}\phi(s)]$

$= \begin{cases} f(t-2) & \text{for } t>2 \\ 0, & t<2 \end{cases}$

$= \begin{cases} \dfrac{e^{-4(t-2)}}{3}\sin 3(t-2) & \text{for } t>2 \\ 0 & \text{for } t<2 \end{cases}$

Inverse laplace transform of derivative property

If $L[f(t)] = \phi(s)$
then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\phi(s)]$

$\therefore$ If $f(t) = L^{-1}[\phi(s)]$
then $L^{-1}[\phi^n(s)] = (-1)^n t^n f(t)$
$= (-1)^n t^n L^{-1}[\phi(s)]$

$n=1$
$\therefore L^{-1}[\phi'(s)] = -t\, L^{-1}[\phi(s)]$
$\therefore \boxed{L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi'(s)]}$

(1) $\tan^{-1}\left(\frac{2}{s^2}\right)$
(2) $\log\left(\sqrt{1+\frac{a^2}{s^2}}\right)$
(3) $2\tanh^{-1} s$
(4) $\log\left(\frac{s^2+1}{s(s+1)}\right)$

Sdn's (1) $\frac{d}{dx}\left(\tan^{-1} ax\right)$
$= \frac{1}{1+a^2x^2}\, a = \frac{a}{1+a^2x^2}$

$\phi(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

$\phi'(s) = \frac{1}{1+\frac{4}{s^4}} \times \frac{-4}{s^3}$

$= \frac{-4}{\frac{1}{s^4}(s^4+4)s^3} = \frac{-4s}{(s^2)^2+2^2}$

$$\therefore L^{-1}[\phi'(s)] = -4L^{-1}\left[\frac{s}{s^4+4}\right]$$

$$= -4L^{-1}\left[\frac{s}{(s^2)^2+(2i)^2}\right]$$ won't work because still will result in Deg. (N.S)

$$= -4L^{-1}\left[\frac{s}{(s^2+2)^2-4s^2}\right]$$

$$= -L^{-1}\left[\frac{4s}{(s^2+2-2s)(s^2+2+2s)}\right]$$

$$= -L^{-1}\left[\frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}\right]$$

$$= -L^{-1}\left[\frac{1}{(s-1)^2+1^2} - \frac{1}{(s+1)^2+1^2}\right]$$

$$= -\left[e^{t}L^{-1}\left[\frac{1}{s^2+1}\right] - e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]\right]$$

$$= -\left[e^{t}\sin t - e^{-t}\sin t\right]$$

$$= -2\sinh t \sin t \qquad \left[\because \sinh t = \frac{e^{t}-e^{-t}}{2}\right]$$

$$\therefore L^{-1}[\phi(s)] = \frac{-1}{t}L^{-1}[\phi'(s)]$$

$$= \frac{2\sin t \sinh t}{t}$$ Ans.

(2) $\phi(s) = \log\left(\sqrt{1+\frac{a^2}{s^2}}\right)$

$\phi(s) = \frac{1}{2}\log(s^2+a^2) - \frac{1}{\cancel{2}}\log s^{\cancel{2}}$

$= \log\sqrt{s^2+a^2} - \log s$

$\therefore \phi'(s) = \frac{1}{\cancel{2}}\left(\frac{1}{s^2+a^2}\times \cancel{2}s\right) - \frac{1}{s}$

$= \frac{s}{s^2+a^2} - \frac{1}{s}$

$= \frac{\cancel{s^2} - (\cancel{s^2}+a^2)}{s(s^2+a^2)}$ } ~~not required~~ cuz can easily get Laplace by above step

$\therefore L^{-1}[\phi'(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] - L^{-1}\left[\frac{1}{s}\right]$

$= \cos at - 1$

$\therefore L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi_1'(s)]$

$= \frac{-1}{t}(\cos at - 1)$

(3) $\phi(s) = 2\tanh^{-1}(s)$

$= 2\left(\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)\right)$

$= \log(1+s) - \log(1-s)$

$\phi'(s) = \frac{1}{1+s} - \frac{1(-1)}{1-s}$

$= \frac{1}{1+s} + \frac{1}{1-s} = \frac{1}{s+1} - \frac{1}{s-1}$

$L^{-1}[\phi'(s)] = e^{-t} - e^{t}$

$\therefore L^{-1}[\phi(s)] = \frac{-1}{t}(e^{-t} - e^{t}) = \frac{2\sinh t}{t}$

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(4) $\phi(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$

$= \log(s^2+1) - \log(s) - \log(s+1)$

$\phi'(s) = \left(\frac{1}{s^2+1} \times 2s\right) - \frac{1}{s} - \frac{1}{s+1}$

$\therefore L^{-1}[\phi'(s)] = \underline{-1 - e^{-t}}$

Convolution Thr^m

If $L^{-1}[\phi_1(s)] = f_1(u)$
& $L^{-1}[\phi_2(s)] = f_2(u)$

then $L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t f_1(u) f_2(t-u)\,du$

(1) $\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$

(2) $\frac{(s+3)^2}{(s^2+6s+5)^2}$

(3) $\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

Sol^n (1) $\frac{s(s+1)}{(s^2+1)(s^2+2s+2)}$

$\Rightarrow$ ~~s^2+s~~ $\phi_1(s) = \frac{s+1}{s^2+2s+2}$

$\phi_2(s) = \frac{s}{s^2+1}$

$$L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s+1}{(s+1)^2+1^2}\right]$$

$$= e^{-u}\cos u = f_1(u)$$ (using first shifting prop.)

$$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos u = f_2(u)$$

$\therefore$ By convolution theorem

$$L^{-1}[\phi_1(s)\,\phi_2(s)] = \int_0^t e^{-u}\cos u\,\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}\cdot 2\cos u\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}(\cos t + \cos(2u-t))\,du$$

$$= \frac{1}{2}\left[\cos t\,\Big[-e^{-u}\Big]_0^t + \int_0^t e^{-u}\cos(2u-t)\,du\right]$$

$$= \frac{1}{2}\left[\left[-e^{-t}+1\right]\cos t + \left[\frac{e^{-u}}{1+4}\left(-\cos(2u-t)+2\sin(2u-t)\right)\right]_0^t\right]$$

$$= \frac{1}{2}\left\{\cos t(1-e^{-t}) + \frac{e^{-t}}{5}(-\cos t + 2\sin t) - \frac{1}{5}(-\cos t - 2\sin t)\right\}$$

(2) $\dfrac{s+3}{s^2+6s+5} \cdot \dfrac{s+3}{s^2+6s+5}$

$\Rightarrow \phi_1(s) = \dfrac{s+3}{s^2+6s+5} = \phi_2(s)$ (using first shifting prop.)

$$L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s+3}{(s+3)^2-2^2}\right] = e^{-3u}\cosh 2u = f_1(u)$$

$$L^{-1}[\Phi_2(s)] = L^{-1}\left[\frac{s+3}{(s+3)^2-2^2}\right] = e^{-3u}\cosh 2u = f_2(u)$$

$\therefore$ By convolution thr^m,

$$\underline{L^{-1}[\Phi_1(s).\Phi_2(s)]} = \int_0^t (e^{-3u})^2 (\cosh 2u)^2\, du$$

$$= \int_0^t e^{-6u}\cosh^2 2u\, du$$

$$= \int_0^t e^{-6u}\left(\frac{e^{2u}+e^{-2u}}{2}\right)^2 du$$

$$= \frac{1}{4}\int_0^t e^{-6u}\left(e^{4u}+e^{-4u}+2\right) du$$

$$= \frac{1}{4}\int_0^t$$

$$= \int_0^t (e^{-3u}\cosh 2u)(e^{-3(t-u)}\cosh 2(t-u))\, du$$

$$= \frac{1}{2}\int_0^t (e^{-3u}\cosh 2u)(e^{-3t+3u}(2\cosh 2(t-u))\, du$$

$$= \frac{1}{2}\int_0^t e^{-3t}\cdot 2\cosh 2u \cosh 2(t-u)\, du$$

$$= \frac{1}{2}\int_0^t e^{-3t}\cdot(\cosh(2t) + \cosh(4u-2t))\, du$$

$$= \frac{e^{-3t}}{2}\int_0^t \cosh 2t + \cosh(4u-2t)\, du$$

$$= \frac{e^{-3t}}{2}(\cosh 2t)t + \frac{e^{-3t}}{2}\int_0^t \cosh(4u-2t)\, du$$

$$= \frac{e^{-3t}}{2}\left[t\cosh 2t + \left(\frac{\sinh(4u-2t)}{4}\right)_0^t\right]$$

$$= \frac{e^{-3t}}{2}\left[t\cosh 2t + \frac{1}{4}(\sinh 2t + \sinh 2t)\right]$$

Ans.

(3) Let $\phi_1(s) = \frac{s^2+2s+3}{s^2+2s+2} = 1 + \frac{1}{s^2+2s+2}$

$\phi_2(s) = \frac{1}{s^2+2s+5}$

$\Rightarrow \phi_1(s) = 1 + \frac{1}{(s+1)^2+1^2}$,

$\phi_2(s) = \frac{1}{(s+1)^2+2^2}$

$L^{-1}[\phi_1(s)] = L^{-1}\left[1 + \frac{1}{(s+1)^2+1^2}\right]$

$= \ldots$ M.D. Lengthy, won't work

Easier method Let $\phi(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

$= \frac{(s+1)^2+2}{((s+1)^2+1^2)((s+1)^2+2^2)}$

$L^{-1}[\phi(s)] = e^{-t}L^{-1}\left[\frac{s^2+2}{(s^2+1)(s^2+4)}\right]$

$= e^{-t}\left[L^{-1}\left(\frac{s^2}{(s^2+1)(s^2+4)}\right) + 2L^{-1}\left(\frac{1}{(s^2+1)(s^2+4)}\right)\right]$

Partial fraction

For 1st ~~case~~ $\phi_1(s) = \frac{s}{s^2+1^2}$ | ~~$\phi_1(s) =$~~

$\phi_2(s) = \frac{s}{s^2+2^2}$

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$\therefore \quad L^{-1}[\phi(s)] = \cos u = f_1(u)$

$L^{-1}[\phi_2(s)] = \cos 2t = f_2(u)$

$\therefore L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t \cos u \cos 2(t-u)\, du$

$= \frac{1}{2}\int_0^t 2\cos u \cos 2(t-u)\, du$

$= \frac{1}{2}\int_0^t \left(\cos(u+2t-2u) + \cos(3u-2t)\right) du$

$= \frac{1}{2}\left\{\left[-\sin(2t-u)\right]_0^t + \frac{1}{3}\left[\sin(3u-2t)\right]_0^t\right\}$

$= \frac{1}{2}\left\{-\sin t + \sin 2t + \frac{1}{3}(\sin t + \sin 2t)\right\}$

$\frac{2}{3}\left\{L^{-1}\left(\frac{1}{s^2+1^2} - \frac{1}{s^2+2^2}\right)\right\}$

$= \frac{2}{3}\left\{\frac{1}{1}\sin t - \frac{1}{2}\sin 2t\right\}$

<u>Ans.</u> $e^{-t}\left(\frac{1}{2}\left(-\frac{2}{3}\sin t + \frac{4}{3}\sin 2t\right) + \frac{2}{3}\left(\sin t - \frac{\sin 2t}{2}\right)\right)$

$= e^{-t}\left(\frac{-\sin t}{3} + \frac{2}{3}\sin 2t + \frac{2}{3}\sin t - \frac{1}{3}\sin 2t\right)$

$= \frac{e^{-t}}{3}\left(\sin 2t + \sin t\right)$ Ans.

$L\left(\frac{e^{2y}+e^{-2y}}{2}\right)$ Rough

$\frac{1}{2}\left(\frac{1}{s-2} + \frac{1}{s+2}\right)$

---

(1) Periodic function

If $f(t)$ is periodic fn^x

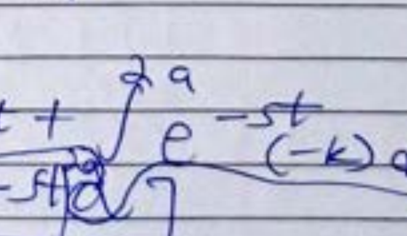
i.e. $f(t+a) = f(t)$

then $L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt$

Q. Find Laplace of a square wave function

$$f(t) = \begin{cases} k, & 0<t<a \\ -k, & a<t<2a \end{cases}$$

where $f(t) = f(t+2a)$



Soln. $L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$

$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} k\,dt + \int_a^{2a} e^{-st}(-k)\,dt\right]$

$= \frac{k}{1-e^{-2as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$

$$= \frac{k}{1-e^{-2as}}\left(\left(\frac{e^{-as}-1}{-s}\right) - \left(\frac{e^{-2as} - e^{-as}}{-s}\right)\right)$$

$$= \frac{k}{s(1-e^{-2as})}\left[1-e^{-as} - (e^{-as} - e^{-2as})\right]$$

$$= \frac{k}{s(1-e^{-2as})}\left[1+e^{-2as}-2e^{-as}\right]$$

$$= \frac{k}{s(1-e^{-2as})}(1-e^{-as})^2$$

$$= \frac{k\,\cancel{(1-e^{-as})}(1-e^{-as})}{s(1+e^{-as})\cancel{(1-e^{-as})}}$$

$$= \frac{k(1-e^{-as})}{s(1+e^{-as})}$$

$f(t) = |\sin\omega t|, t\geq 0$

Find Laplace transform of full wave rectification of $|\sin\omega t|, t\geq 0$

Solⁿ $f(t+\pi/\omega) = \left|\sin\omega\left(t+\frac{\pi}{\omega}\right)\right|$

$$= |\sin(\omega t+\pi)|$$

$$= |-\sin\omega t|$$

$$= |\sin\omega t| = f(t)$$

$$L[f(t)] = \frac{1}{1-e^{-\frac{\pi}{\omega}s}} \int_0^{\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi}{\omega}s}} \int_0^{\pi/\omega} e^{-st} \sin\omega t\, dt$$

$$\left[\because |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}\right.$$

$$\therefore |\sin\omega t| = \begin{cases} \sin\omega t, & \sin\omega t \geq 0 \ (\text{if } 0 \leq \omega t \leq \pi) \\ -\sin\omega t, & \sin\omega t < 0 \ (\text{if } \omega t > \pi) \end{cases}$$

$$\left. = \begin{cases} \sin\omega t & \text{if } 0 \leq t \leq \pi/\omega \\ -\sin\omega t & \text{if } t > \pi/\omega \end{cases}\right]$$

$$= \frac{1}{1-e^{-\frac{\pi}{\omega}s}} \left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{\cancel{e^{-st}}}{(1-e^{-\frac{\pi}{\omega}s})(s^2+\omega^2)}\left[0 + e^{-\pi/\omega \, s}\omega - (-0 - e^0 \omega)\right]$$

$$= \frac{\cancel{e^{-st}}\ \omega}{(1-e^{-(\pi/\omega)s})(s^2+\omega^2)} \cancel{[2\omega]} \left[1+e^{-\pi/\omega \, s}\right]$$

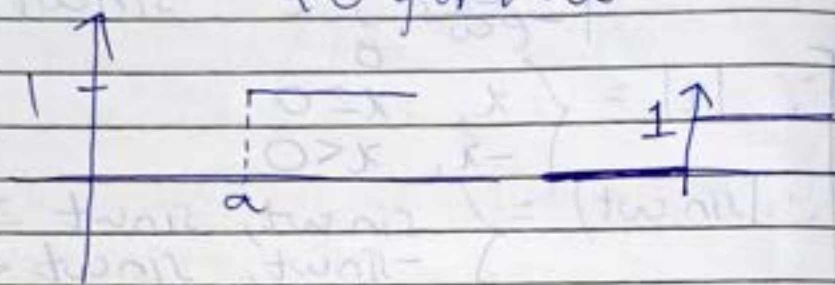
$\therefore$ N & D by $e^{-\pi s/2\omega}$

$$= \frac{\omega}{s^2+\omega^2} \cdot \frac{e^{\pi s/2\omega} + e^{-\pi s/2\omega}}{e^{\pi s/2\omega} - e^{-\frac{\pi s}{2\omega}}}$$

$$= \frac{\omega}{s^2+\omega^2} \coth\frac{\pi s}{2\omega}$$

Heaviside's unit step function

$$H(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$



If $a=0$, then

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

(1) $L[H(t-a)] = \frac{e^{-as}}{s}$

$\therefore L^{-1}\left[\frac{e^{-as}}{s}\right] = H(t-a)$

(2) $L[f(t-a) \cdot H(t-a)] = e^{-a}L[f(t)]$

(3) $L[f(t)\, H(t-a)] = e^{-as} L[f(t+a)]$

Revise

(1) $L^{-1}\left[\frac{e^{-as}}{s}\right] = H(t-a)$

(2) $L[f(t-a)H(t-a)] = e^{-as}L[f(t)]$

(3) $L[f(t)\, H(t-a)] = e^{-as} L[f(t+a)]$

(4) $f(t) = \begin{cases} f_1(t) & 0 \leq t \leq a \\ f_2(t), & a < t \leq b \\ f_3(t), & t > b \end{cases}$

then $f(t) = f_1(t)[H(t) - H(t-a)]$

$+f_2(t)[H(t-a)-H(t-b)]$
$+f_3(t)[H(t-b)]$

Q. Find $L[\sin t(H(t-\frac{\pi}{2})-H(t-\frac{3\pi}{2}))]$.

$= L[\sin t\, H(t-\pi/2)] - L[\sin t\, H(t-\frac{3\pi}{2})]$

$= e^{-\pi/2 s} L[\sin(t+\frac{\pi}{2})] - e^{-\frac{3\pi s}{2}} L[\sin(t+\frac{3\pi}{2})]$

$= e^{-\frac{\pi}{2}s} L(\cos t) - e^{-(3\pi/2)s} L[-\cos t]$

$= e^{-\pi/2} \frac{s}{s^2+1} + e^{-3\pi/2} L\left(\frac{s}{s^2+1}\right)$ Ans.

Q. Express the following function in terms of unit step fn and hence find its Laplace transform.

$f(t) = \begin{cases} t^2 & \text{for } 0<t<1 \\ 4t & \text{for } t>1. \end{cases}$

Soln $f(t) = t^2[H(t)-H(t-1)] + 4t\,H(t-1)$

$= t^2 H(t) + (4t-t^2) H(t-1)$

$L[f(t)] = L[t^2 H(t)] + L[(4t-t^2)H(t-1)]$

$= e^{-0s} L[t^2] + e^{-s} L[4(t+1)-(t+1)^2]$

$= \frac{\Gamma 2+1}{s^3} + e^{-s} L(4t+4-t^2-1-2t]$

$= \frac{2}{s^3} + e^{-s}\left(\frac{4}{s^2}+\frac{4}{s}-\frac{2}{s^3}-\frac{1}{s}-\frac{2}{s^2}\right)$

$= \frac{2}{s^3} + e^{-s}\left(\frac{2}{s^2}-\frac{2}{s^3}+\frac{3}{s}\right)$ Ans.

Q. $f(t) = \begin{cases} t-1 & \text{if } 1<t<2 \\ 3-t & \text{if } 2<t<3 \end{cases}$

Soln. $f(t) = (t-1)[H(t-1) - H(t-2)]$
$+ (3-t)[H(t-2) - H(t-3)]$

$= H(t-2)\{(3-t) - (t-1)\}$
$+ H(t-1)(t-1) - (3-t)H(t-3)$

$L[f(t)] = L[(-2t+4)H(t-2)]$
$+ L[H(t-1)(t-1)]$
$+ L[(t-3)H(t-3)]$

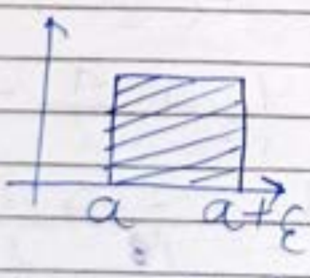
$= e^{-s}L[(t+1)-1] + e^{-2s}L(\ldots)$
$+ e^{-3s}L[t+3-3]$

$= e^{-s}L(t) - 2e^{-2s}L(t)$
$+ e^{-3s}L(t)$

$= e^{-s}\frac{1}{s^2} - \frac{2e^{-2s}}{s^2} + e^{-3s}\frac{1}{s^2}$ Ans.

Dirac-Delta function

Consider $F(t) = \begin{cases} 1/\varepsilon & \text{if } a \le t \le a+\varepsilon \\ 0 & \text{O.W} \end{cases}$



then $\lim_{\varepsilon \to 0} F(t) = \begin{cases} \infty & \text{if } t \ldots \\ 0 & \text{O.W} \end{cases}$

$= \delta(t-a)$

and is called Dirac-Delta function (impulse function)

$L(\delta(t-a)) = e^{-as}$

$\therefore L^{-1}[e^{-as}] = \delta(t-a)$

If $a=0$

$L^{-1}[1] = \delta(t)$

$L[f(t)\,\delta(t-a)] = e^{-as} f(a)$

(1) Find $L[t H(t-a) + t^2 \delta(t-a)]$

$= L[tH(t-a)] + L[t^2 \delta(t-a)]$

$= e^{-as} L[t+a] + e^{-as} a^2$

$= e^{-as}\left(\frac{1}{s} + \frac{a}{s}\right) + e^{-as} a^2$

(2) Evaluate $\int_0^\infty t e^{2t} \sin 3t\, \delta(t-2)\, dt$

$= \int_0^\infty e^{-(-2t)} t \sin 3t\, \delta(t-2)\, dt$

Now, $f(t) = t \sin 3t$

$\therefore L[f(t)\,\delta(t-2)] = e^{-2s}(2\sin 6)$

$= \int_0^\infty e^{-st} f(t)\,\delta(t-2)\, dt$

but $L[f(t)]$

$= \int_0^\infty e^{-st} f(t)\, dt$ At $s=-2$, $a=$

$\int_0^\infty e^{(-2t)} (t \sin 3t)\, \delta(t-2) = e^{4}(2\sin 6)$

Solving differential equations using Laplace transform

$L[f'(t)] = s\phi(s) - f(0)$

$= s\bar{y} - y(0)$

$y = f(x)$ $L[f''(t)] = s^2\phi(s) - sf(0) - f'(0)$

$= s^2\bar{y} - sy(0) - y'(0)$

$L[f'''(t)] = s^3\phi(s) - s^2 f(0) - s f'(0) - f$
$= s^3\bar{y} - s^2 y(0) - s y'(0) - y$

(1) Solve $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = 3te^{-t}$

Given: $y(0) = 4$ and $y'(0) = 2$

Sol^n. $y'' + 2y' + y = 3te^{-t}$

$y = y(t)$

$D^2 + 2D + 1 = 3te^{-t}$ (earlier - learnt)

We have

$L(y'') + 2L(y') + L(y) = L(3te^{-t})$

$\Rightarrow [s^2\bar{y} - s y(0) - y'(0)] + 2[s\bar{y} - y(0)] + \bar{y} = \ldots$

$y(0) = 4$

but $y'(0) = 2$

$\Rightarrow [s^2\bar{y} - 4s - 2] + 2[s\bar{y} - 4] + \bar{y} = \frac{3}{(s+1)^2}$

$\Rightarrow \bar{y}(s^2 + 2s + 1) - 4s - 10 = \frac{3}{(s+1)^2}$

$\therefore \bar{y} = \frac{(4s+10)}{(s+1)^2} + \frac{3}{(s+1)^2(s+1)^2}$

$= \frac{4s+10}{(s+1)^2} + \frac{3}{(s+1)^4}$

Let $\frac{4s+10}{(s+1)^2} = \frac{a}{s+1} + \frac{b}{(s+1)^2}$

$\Rightarrow \frac{4s+10}{(s+1)^2} = \frac{a(s+1) + b}{(s+1)^2}$

$\Rightarrow 4s + 10 = sa + (a+b)$

$\Rightarrow a = 4$

$\& \; a+b = 10 \Rightarrow b = 6$

$\therefore L[y] = \bar{y} = \frac{4}{s+1} + \frac{6}{(s+1)^2} - \frac{3}{(s+1)^4}$

$\therefore y = L^{-1}\left[\frac{4}{s+1} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4}\right]$

$y = 4e^{-t} + 6e^{-t}L^{-1}\left(\frac{1}{s^2}\right) + 3e^{-t}L^{-1}\left(\frac{1}{s^4}\right)$

$y(t) = 4e^{-t} + 6e^{-t}t + 3e^{-t}\frac{t^3}{6 \cdot 2}$

$= 4e^{-t} + 6e^{-t}t + e^{-t}\frac{t^3}{2}$

Q. Solve $y'' + y = t$. Given that $y(\pi) = 0$ & $y'(0) = 1$.

Sol^n. Let $y(0) = \alpha$.

$\Rightarrow L(y'') + L(y) = L(t)$

$\Rightarrow [s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = \frac{1}{s^2}$

But $y(0) = \alpha$ & $y'(0) = 1$

$\Rightarrow [s^2\bar{y} - s\alpha - 1] + \bar{y} = \frac{1}{s^2}$

$\Rightarrow \bar{y}(s^2+1) = \frac{1}{s^2} + s\alpha + 1$

$\Rightarrow \bar{y} = \frac{1}{s^2(s^2+1)} + \frac{s\alpha+1}{s^2+1}$

$\Rightarrow y = L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] + L^{-1}\left[\frac{s\alpha+1}{s^2+1}\right]$

$= L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] + L^{-1}\left(\frac{s\alpha+1}{s^2+1}\right)$

$$= L^{-1}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) + \alpha L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= t - \sin t + \alpha \cos t + \sin t$$

$$\Rightarrow \therefore \underline{y(t) = t + \alpha \cos t}$$

But $y(\pi) = 0$

$$\Rightarrow \pi + \alpha(-1) = 0$$

$$\Rightarrow \pi = \alpha$$

$$y'(t) = 1 - \alpha \sin t = 0$$

$$\Rightarrow \sin t = \frac{1}{\alpha}$$

extra step not req.

$$\underline{y(t) = t + \pi \cos t}$$

- Solve $y'' + 2y' + 5y = 8\sin t + 4\cos t$. Given that $y(0) = 1$, $y\left(\frac{\pi}{4}\right) = \sqrt{2}$.

$\underline{\text{Sol}^n}$. let $y'(0) = \alpha$

We've $y'' + 2y' + 5y = 8\sin t$

$$\Rightarrow L(y'') + 2L(y') + 5L(y) = L(8\sin t)$$

$$\Rightarrow [s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{8}{s^2+1} + \frac{4s}{s^2+1} = \frac{4s+8}{(s^2+1)}$$

$$\Rightarrow \bar{y} = \frac{4s+8}{(s^2+1)(s^2+2s+5)} + \frac{(\alpha + s + 2)}{(s^2+2s+5)}$$

$$= \frac{2(2s+4)}{(s^2+1)(s^2+2s+5)} + \frac{(\alpha+1) + (s+1)}{(s+1)^2+4}$$

$$= 2L^{-1}\left(\frac{1}{s^2+1} - \frac{1}{(s+1)^2+2^2}\right) + \frac{\alpha+1}{(s+1)^2+2^2} + \frac{s+1}{(s+1)^2+2^2}$$

$$2L^{-1}\left(\frac{1}{s^2+1}\right) = 2\sin t \quad \text{--- (1)}$$

$$2L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right) = \frac{2}{2}e^{-t}\sin 2t \quad (2)$$

$$(\alpha+1)L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right) = (\alpha+1)\left(\frac{1}{2}e^{-t}\sin 2t\right) \quad (3)$$

$$L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) = e^{-t}\cos 2t \quad (4)$$

(1) − (2) + (3) + (4)

we get

$$y(t) = 2\sin t - e^{-t}\sin 2t + \frac{(\alpha+1)}{2}e^{-t}\sin 2t + e^{-t}\cos 2t$$

But $y\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$\Rightarrow \frac{2}{\sqrt{2}} - e^{-\pi/4} + \frac{(\alpha+1)}{2}e^{-\pi/4} + e^{-\pi/4}(0) = \sqrt{2}$$

$$\Rightarrow \alpha+1 = 2$$

$$\underline{\alpha = 1}$$

$$\therefore y(t) = 2\sin t \cancel{- e^{-t}\sin 2t} \cancel{+ e^{-t}\sin 2t} + e^{-t}\cos 2t$$

$$= \underline{2\sin t + e^{-t}\cos 2t}$$

Q. $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t.\quad y(0) = 1$

Soln. $y' + 2y + \int_0^t y\,dt = \sin t$

$$L(y') + 2L(y) + \underbrace{L\left[\int_0^t y\,dt\right]}_{\frac{\bar{y}}{s}} = L(\sin t)$$

$$\Rightarrow s\bar{y} - y(0) + 2(\bar{y}) + \frac{\bar{y}}{s} = \frac{s}{s^2+1}$$

IV

$$s\bar{y} - 1 + 2\bar{y} + \frac{\bar{y}}{s} = \frac{s}{s^2+1}$$

$$\frac{(s+2+1)}{s}\bar{y} - 1 = \frac{s}{s^2+1}$$

$$\Rightarrow (s^2+2s+1)\bar{y} - s = \frac{s^2}{s^2+1}$$

$$\bar{y} = \frac{s^2+s^3+s}{(s^2+1)(s+1)^2}$$

$$= \frac{s^2(s+1)+s}{(s+1)^2(s^2+1)}$$

$$\text{OR} \quad \frac{s^2+s(s^2+1)}{(s^2+1)(s^2+2s+1)}$$

Fourier series:

$f(x)$ - periodic

Dirichlet's conditions

(1) If $f(x)$ is defined in the interval $[a, a+2l]$ so that $f(x) = f(x+2l)$

(2) $f(x)$ is continuous function in $[a, a+2l]$

or has finite number of discontinuities in the interval $[a, a+2l]$

(3) $f(x)$ has no maxima and minima or has finite no. of maxima and minima in the interval $(a, a+2l)$.

Then, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$ ——(1)

where $a_0 = \frac{1}{l}\int_a^{a+2l} f(x)\,dx$

$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{n\pi x}{l}\,dx$ ——(2)

eqⁿ (1) is called Fourier series in the interval $(a, a+2l)$ and coeff. in eq. (2) are called fourier coeff. of $f(x)$ in the interval $[a, a+2l]$

In $(0, 2\pi)$:

Ex: $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 \le x \le 2\pi$

Deduce that

(1) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

(2) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

(3) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

(4) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

Solⁿ:

(1) $$a_0 = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\int_0^{2\pi}(\pi^2+x^2-2\pi x)\,dx\right]$$

$$= \frac{1}{4\pi}\left[\pi^2 x+\frac{x^3}{3}-\pi x^2\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left\{\pi^2(2\pi)+\frac{8\pi^3}{3}-4\pi^3\right\}$$

$$= \frac{1}{2}\left\{\pi^2+\frac{4\pi^2}{3}-2\pi^2\right\}$$

$$a_0 = \frac{1}{2}\left(\frac{\pi^2}{3}\right) = \frac{\pi^2}{6} \quad (1)$$

(2) $$f(x) = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{\sin nx}{n}\right)-(-2)(\pi-x)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{-2}{4\pi}\ldots$$

$$= \frac{-1}{2\pi n}(-\pi-\pi\ldots$$

$$a_m = \frac{1}{n^2}$$

(3) $$b_m = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - (-2)(\pi - x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right)\right]$$

$$\boxed{b = 0}$$

$$\therefore \left(\frac{\pi - x}{2}\right)^2 = \frac{a_0}{2} + \sum_{n=1}^{a}\left[a_n \cos nx + a_n \sin nx\right]$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx \quad \text{— (2)}$$

(4) $\frac{\pi^2}{12} + \frac{\pi^2}{6}$ {i.e. (1) + (2)}

$$= \frac{\pi^2}{4}$$

$$\Rightarrow \frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad (3) \text{ H.P.}$$

By Parseval's identity

(1) $$\frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}(a_n^2 + b_n^2)$$ (full-Fourier)

$$\Rightarrow \frac{1}{16\pi}\int_0^{2\pi}(\pi - x)^4\,dx = \frac{\pi^4}{72} + \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{1}{16\pi}\left[\frac{(\pi-x)^5}{-5}\right]_0^{2\pi} = \frac{\pi^4}{72} + \sum_1^{\infty}\frac{1}{x^4}$$

Q. $f(x) = e^{-x}$ $0 \le x \le 2\pi$
Deduce value of $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$.

Series for cosech π.

Soln. $a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$

$= \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi}$

$= \frac{-1}{\pi}\left[e^{-2\pi}-1\right]$

$a_0 = \frac{(1-e^{-2\pi})}{\pi}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$

$= \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right]_0^{2\pi}$

$= \frac{-1}{\pi(n^2+1)}\left[e^{-x}\cos nx\right]_0^{2\pi}$

$= \frac{-1}{\pi(n^2+1)}\left[e^{-2\pi}-1\right]$

$a_n = \frac{(1-e^{-2\pi})}{\pi(n^2+1)}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$

$= \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}(-\sin nx - n\cos nx)\right]_0^{2\pi}$

$= \frac{-n}{\pi(n^2+1)}\left[e^{-x}\cos nx\right]_0^{2\pi}$

$$= \frac{-n}{\pi(n^2+1)}\left(e^{-2\pi}-1\right)$$

$$\therefore b_n = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}$$

$$\therefore e^{-x} = \frac{(1-e^{-2\pi})}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+1}$$

$$+ \frac{(1-e^{-2\pi})}{\pi}\sum_{n=1}^{x}\frac{n}{n^2+1}\sin nx$$

$$\boxed{\text{Put } x=\pi}$$

$$\Rightarrow e^{-\pi} = \frac{(1-e^{-2\pi})}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$= \cancel{\frac{1-e^{-2\pi}}{2\pi}} - \cancel{\frac{(1-e^{-2\pi})}{2\pi}} + \frac{(1-e^{-2\pi})}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\Rightarrow \frac{e^{-\pi}\pi}{(1-e^{-2\pi})} = \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\Rightarrow \frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=2}\frac{(-1)^n}{n^2+1} \quad (1)$$

$$\Rightarrow \frac{\pi}{2\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)} = \sum_{2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\Rightarrow \frac{\pi}{2}\frac{1}{\sinh\pi} = \sum_{2}$$

Ex.: $(-\pi, \pi)$ $f(x) = \begin{cases} 0, & \pi \le x < 0 \\ \sin x, & 0 \le x \le \pi. \end{cases}$

Solⁿ $f(-x) = \begin{cases} 0, & -\pi \le -x < 0. \\ \sin(-x), & 0 \le -x \le \pi. \end{cases}$

$= \begin{cases} 0, & 0 < x \le \pi \\ \sin(-x), & -\pi \le x \le 0 \end{cases}$

$\Rightarrow f$ is neither even nor odd.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{-1}{\pi}[-1-1]$$

$$a_0 = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\{\sin(n+1)x - \sin(n-1)x\}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right]_0^{\pi}, n \neq 1$$

$$= \frac{1}{2\pi}\left[\left(\frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1}\right) - \left(\frac{1}{n-1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{\left((-1)^{n-1}-1\right)}{n-1} - \frac{\left((-1)^{n+1}-1\right)}{n+1}\right] \quad (n \neq 1)$$

$$= \frac{1}{2\pi} \times \frac{2}{n^2-1}\left[(-1)^{n+1}-1\right]$$

$$= \begin{cases} -2 & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases}$$

$$\therefore a_n = \begin{cases} \dfrac{-2}{\pi(n^2-1)} & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases}$$

$$a_1 = \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{4\pi}\left[1-1\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left\{\cos(n-1)x - \cos(n+1)x\right\} dx$$

$$= \frac{1}{2\pi}\Bigg[\underbrace{\frac{\sin(n-1)x}{n-1}}_{=0} - \underbrace{\frac{\sin(n+1)x}{n+1}}_{=0}\Bigg]_0^{\pi}$$

$$\therefore b_n = 0 \quad \forall \; (n \neq 1)$$

$$b_1 = \frac{1}{2\pi}\int_0^{\pi} 2\sin^2 x \, dx$$

$$\therefore b_1 = \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\therefore \boxed{b_1 = 1/2}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$= \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k)^2-1} + \frac{1}{2}\sin x \quad (*)$$

Deduce that

(1) $\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$

(2) $\frac{1}{4}(\pi-2) = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots\dots$

Ans (1) $f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} + \frac{1}{2}\sin x \quad (*)$

Put $x = 0$ in $(*)$

$f(0) = 0$

$$\Rightarrow \frac{1}{2} = \sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$$

(2) put $x = \frac{\pi}{2}$ in $(*)$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)} + \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)}$$

$$\Rightarrow \frac{1}{2}-\frac{\pi}{4} = \sum_{1}^{\infty} \frac{(-1)^k}{(2k+1)(2k-1)} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \cdots \quad (3)$$

Q. Write Fourier series for

$$f(x) = \begin{cases} x+\pi/2, & -\pi<x<0 \\ \frac{\pi}{2}-x, & 0<x<\pi \end{cases}$$

Deduce (1) $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

(2) $\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$

Solⁿ: $$f(-x) = \begin{cases} \pi/2 - x, & 0<x<\pi \\ x+\frac{\pi}{2}, & -\pi<x<0 \end{cases}$$

$= f(x)$

$\therefore$ f is even functⁿ $\Rightarrow b_n = 0$

(directly write this down. no need to write separately)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)dx}_{\text{even}}$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)dx$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2}-\frac{\pi^2}{2}\right] = 0$$

$\therefore \boxed{a_0 = 0}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\underbrace{\left(\frac{\pi}{2}-x\right)}_{=u}\underbrace{\cos nx}_{=v}\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}-x\right)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$\cancel{= -\frac{2}{\pi n^2}[\cos nx]_0^{\pi}}$$

$$= \frac{-2}{\pi n^2}[\cos nx]_0^{\pi}$$

$$= \frac{-2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\hookrightarrow = \begin{cases} 0 & \text{if } n = 2k \\ -2 & \text{if } n = 2k+1 \end{cases}$$

$$\therefore a_n = \begin{cases} 0 & \text{if } n = 2k \\ \frac{4}{\pi n^2} & \text{if } n = 2k-1 \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}$$

Put $x = 0$

$$\Rightarrow f(0) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

f is discontinuous at $x = 0$.

$$\therefore f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x \to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\lim_{x\to 0}\left(x + \frac{\pi}{2}\right) + \lim_{x\to 0}\left(\frac{\pi}{2} - x\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{\pi}{2}$$

Common ~~errors~~ mistakes:

1) Finding ~~of~~ individually b_0 =

2) Not using property of ~~splitting~~ even odd fn^x

3) $f(x) = \{ \ldots$ they don't write

$$\therefore \frac{\pi}{2} = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

By Parseval's Id.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)^2 dx = \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$$

$$\Rightarrow \frac{2}{\pi}\left[\frac{\left(\frac{\pi}{2}-x\right)^3}{-3}\right]_0^{\pi} = \frac{16}{\pi^2}\sum_1^{\infty}\frac{1}{(2k-1)^4}$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{16}{\pi^2}\sum_1^{\infty}\frac{1}{(2k-1)^4}$$

$$\Rightarrow \sum_1^{\infty}\frac{1}{(2k-1)^4} = \frac{\pi^4}{96}$$

Q. $f(x) = \begin{cases} -\pi/4, & -\pi \le x \le 0 \\ \frac{\pi}{4}, & 0 \le x \le \pi \end{cases}$

Deduce

(1) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

(2) $\frac{\sqrt{3}\pi}{6} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \cdots$

Soln. f is odd $\Rightarrow a_0 = a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4}\sin nx\,dx$$

$$= \frac{1}{2}\left[-\cos nx\right]_0^{\pi}$$

$$= \frac{-1}{2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{if } n = 2k \\ -2 & \text{if } n = 2k-1 \end{cases}$$

$$b_n = \begin{cases} 0 & \text{if } n = 2k \\ \frac{1}{n} & \text{if } n = 2k-1 \end{cases}$$

$$\therefore f(x) = \sum b_n \sin nx$$

$$= \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{(2k-1)}$$

1/1/23

** Fourier series in the interval $(0, 2l)$

Q. Find the Fourier expansion for $f(x) = 2x - x^2$, $0 \le x \le 3$. Here $2l = 3$. $\therefore l = 3/2$.

Soln. $\because$ Period is $3 = 2l$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\,dx$$

$$= \frac{2}{3}\int_0^3 x^2 - \frac{x^3}{3}\Big]_0^3 = \frac{2}{3}(0)$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

Applying successive integration formula;

$$\int uv\,dx = u\int v - u'\iint v$$

$$= \frac{2}{3}\Big\{ (2x - x^2)\left(\sin\frac{2\pi n x}{3} \cdot \frac{3}{2n\pi}\right)$$

$$- (2 - 2x)\left(\frac{-\cos 2n\pi x}{3} \times \frac{9}{4\pi^2 n^2}\right)$$

$$+ (-2)\left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\frac{8n^2\pi^3}{27}}\right)\Big\}_0^3$$

$$= \frac{2}{3}\frac{9}{4\pi^2 n^2}\left[(2 - 2x)\cos\left(\frac{2\pi n x}{3}\right)\right]_0^3$$

$$= \frac{3}{2n^2\pi^2}\left[-4 - 2\right]$$

$$\boxed{a_n = \frac{-9}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\Big[\ \dots\ \Big]_0^3$$

(applying successive integration)

$$= \frac{2}{3}\Bigg[(2x-x^2)\left(-\cos\frac{2n\pi x}{3}\right)\times\frac{3}{2n\pi} - (2-2x)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2\pi \times n}{3}}\times\frac{3}{2n\pi}\right)$$

$$+ (-2)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2 x}{9}}\times\frac{3}{2n\pi}\right)\Bigg]_0^3$$

$$= \frac{2}{3}\Bigg[(2x+x^2)\left(\cos\frac{2n\pi x}{3}\right)\times\frac{3}{2n\pi} - (2-2x)\frac{9}{4\pi^2n^2}\sin\left(\frac{2n\pi x}{3}\right) + 2\times\frac{27}{8n^2\pi^2}\cos\frac{2n\pi x}{3}\Bigg]_0^3$$

$$= \frac{2}{3}\Bigg[\left(\frac{9}{2n\pi}\ \cancel{\cos} + \frac{27}{4n^2\pi^2}\ \cancel{\cos}\right)\cos 2n\pi - (-4)\frac{9}{4\pi^2n^2}\sin(2n\pi)\Bigg]$$

$$- \frac{9}{2\pi^2n^2}\sin(0) + \frac{27}{4n^2\pi^2}$$

$$= \frac{2}{3}\left[\left(\frac{9}{2n\pi} + \frac{27}{4n^2\pi^2}\right) + \frac{9}{\pi^2 n^2} \times 0 + \frac{27}{}\right]$$

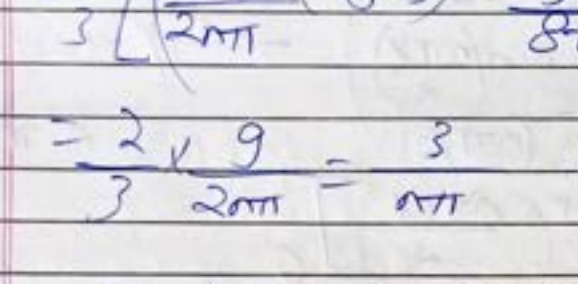
$$\frac{2}{3}\left[\left(\frac{-3}{2n\pi}(6-9) - \frac{54}{8\pi^3 n^3}\right) - \left(0 - \frac{54}{8\pi^3 n^3}\right)\right]$$

$$= \frac{2}{3} \times \frac{9}{2n\pi} = \frac{3}{n\pi} \qquad b_n = \frac{3}{n\pi}$$

25/8/23 - Find Fourier series for F.S. in $(-l, l)$.

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

Solⁿ:



$$f(-x) = \begin{cases} 0, & 1 < x < 2 \\ 1-x, & 0 < x < 1 \\ 1+x, & -1 < x < 0 \\ 0, & -2 < x < -1 \end{cases}$$

$$= f(x) \Rightarrow b_n = 0$$

$\therefore$ Even function

$$a_0 = \frac{1}{2}\int_{-2}^{2} \underbrace{f(x)}_{\text{even}}\, dx = \int_0^1 (1-x)\, dx$$

$$= \int_0^2 f(x)\, dx = \left(x - \frac{x^2}{2}\right)_0^1 \qquad a_0 = \frac{1}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2}{2}\int_{0}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 \underbrace{(1-x)}_{=4}\cos\frac{n\pi x}{2}\,dx$$

$$= \left[(1-x)\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)}\right) - (-1)\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{n^2\pi^2/4}\right)\right]_0^1$$

$$= \frac{-4}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^1$$

$$= \frac{-4}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$= \frac{1}{4} \times \frac{-4}{\pi^2}\sum_{n=1}^{\infty} \frac{\left(\cos\frac{n\pi}{2} - 1\right)}{n^2}\cos\frac{n\pi x}{2}$$

Q. Find fourier series for $f(x) = e^{-x}$ in $(-a, a)$.

Solⁿ. (Req.) (ans.) $a_0 = \frac{e^a - e^{-a}}{a} = \frac{2}{a}\sinh a$

$$a_n = \frac{2a\,(-1)^n}{(a^2+n^2\pi^2)}\sinh a$$

$$b_n = \frac{2n\pi\,(-1)^n}{(a^2+n^2\pi^2)}\sinh a$$

$$\left(\sinh a = \frac{e^a - e^{-a}}{2}\right)$$

Actual self-solⁿ $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$ $\quad l = a$

e^{-x} is not even/odd.

$$a_0 = \frac{1}{a}\int_{-a}^{a} e^{-x}\,dx$$

$$= \frac{1}{a}(e^{a} - e^{-a})$$

$$\because \frac{e^{a} - e^{-a}}{2} = \sinh a \quad \text{——} (1)$$

$$\therefore a_0 = \frac{2}{a}\sinh a$$

$$a_n = \frac{1}{a}\int_{-a}^{a} e^{-x}\cos\frac{n\pi x}{a}\,dx$$

formula $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

$$b_n = \frac{1}{a}\int_{-a}^{a} e^{-x}\sin\frac{n\pi x}{a}\,dx$$

formula $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$\therefore a_n = \frac{1}{a\left(1+\frac{n^2\pi^2}{a^2}\right)}\left(e^{-x}\left(-\cos\frac{n\pi x}{a} + \frac{n\pi}{a}\sin\frac{n\pi x}{a}\right)\right)_{-a}^{a}$$

$$= \frac{1}{a\left(1+\frac{n^2\pi^2}{a^2}\right)}\left[e^{-a}\left(-\cos\frac{n\pi a}{a} + \frac{n\pi}{2}\sin\frac{n\pi a}{a}\right) - e^{a}\left(-\cos n\pi(-1) - \frac{n\pi}{2}\sin(-n\pi)\right)\right]$$

$$= \frac{a^{2}}{a(a^2+n^2\pi^2)}\left((e^{a} - e^{-a})\cos n\pi + (-e^{a} - e^{-a})\times 0\right)$$

$\because \frac{e^{a} - e^{-a}}{2} = \sinh a \quad \therefore$ Above expression

$$= \frac{2a}{a^2+n^2\pi^2}(-1)^n\sinh a$$

H.Done

$$b_n = \left[\frac{e^{-x}}{1+\frac{n^2\pi^2}{a^2}}\left(-\sin\frac{n\pi x}{a} - \frac{n\pi}{a}\cos\frac{n\pi x}{a}\right.\right.$$

$$= \frac{e^{-a}\cdot a^2}{a^2+n^2\pi^2}\left(0-\frac{n\pi}{a}\cos n\pi\right) - \left(e^{a}\right)\left(0-\frac{n\pi}{a}\right.$$

$$= \frac{e^{a}-e^{-a}}{a^2+n^2\pi^2}\,\frac{n\pi}{a}\cos n\pi \cdot a^2$$

$$\therefore b_n = \frac{2n\pi a(-1)^n \sinh a}{a(a^2+n^2\pi^2)} \quad (\text{from } (1))$$

$$\boxed{f(x) = \frac{\sinh a}{a} + 2a\sinh a \sum \frac{(-1)^n}{a^2+n^2\pi^2}\cos\frac{n\pi x}{a} + 2\pi \sinh a \sum \frac{(-1)\ldots}{a^2+n^2\ldots}\sin\ldots}$$

31/8/23 **Half-range Fourier series**

(HRCS)

Type I: Half-range Fourier cosine series for the function $f(x)$ in the interval $(0,l)$, is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

Type-II:

Half-range Fourier sine series for the function $f(x)$ in (HRSS) the interval $(0,l)$, is given by

$$f(x) = \sum_{n=1} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Parseval's Id:

$$\frac{2}{l}\int_0^l f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$\therefore$ For Type II,

$$\frac{2}{l}\int_0^l f^2(x)\,dx = \sum_{n=1}^{\infty} b_n^2$$

$\rightarrow$ $f(x) = x$ in $(0, 2)$.

Deduce:

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

Soln.

$$a_0 = \frac{2}{2}\int_0^2 x\,dx$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\cancel{x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)} \rightarrow 0 - \cancel{\frac{x}{x}}\,1\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore a_n = \begin{cases} 0 & \text{if } n = 2k \\ \dfrac{-8}{n^2\pi^2} & \text{if } n = 2k-1 \end{cases}$$

$$\therefore x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$= 1 - \frac{8}{\pi^2}\sum_{k=1}^{\infty} \frac{\cos\left((2k-1)\frac{\pi x}{2}\right)}{(2k-1)^2}$$

By Parseval's identity

$$\frac{2}{2}\int_0^2 x^2\,dx = 2 + \frac{64}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$$

$$= \underbrace{\left(\frac{8}{3}-2\right)\frac{\pi^4}{64}} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$$

$$= \frac{\pi^4}{96}$$

Q. Find half-range cosine series for $f(x) = \sin x,\ 0 \le x \le \pi$.
Deduce that

(1) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots = \frac{1}{2}$

(2) $\frac{\pi^2 - 8}{16} = \frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \dots$ (Parseval's)

(3) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Soln. $a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx$

$$\boxed{= \frac{4}{\pi}}$$

$a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$

$= \frac{1}{\pi}\int_0^{\pi} 2\cos nx\sin x\,dx$

$= \frac{-4}{\pi(n^2-1)}$ for $n = 2k\ (n \ne 1)$.

$\therefore a_1 = 0$

$\therefore \sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{(4k^2-1)}$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos 2kx}{(2k-1)(2k+1)} \quad (*)$$

Put $x=0$ in $(*)$

$$\Rightarrow 0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots \quad (1)$$

From $(*)$ By putting $x = \pi/2$

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)(2k+1)}$$

$$\Rightarrow 1 = \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{2}{(2k-1)(2k+1)}\right)$$

$$\Rightarrow 1 = \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

$$\Rightarrow \left(1-\frac{2}{\pi}\right)\frac{\pi}{2} = \left(\frac{1}{1}-\frac{1}{3}\right) - \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) - \left(\frac{1}{7}-\frac{1}{9}\right) + \cdots$$

$$\Rightarrow \frac{\pi-2}{2} = 1 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} \cdots$$

$$\Rightarrow \frac{\pi}{2} = 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \quad (2)$$

Q. Prove that: $(0<x<\pi) \to$ for

$$\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} \cdots\right]$$

Sol^n: Let $f(x) = e^{ax} - e^{-ax}$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Here $l = \pi$

$$\therefore b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi (e^{ax}-e^{-ax})\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^\pi e^{ax}\sin nx\,dx - \int_0^\pi e^{-ax}\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{e^{ax}}{a^2+n^2}\right)\left(\underbrace{\cancel{a\sin nx}}_{=0} - n\cos nx\right)\Bigg]_0^\pi - \left(\frac{e^{-ax}}{a^2+n^2}\left(\underbrace{\cancel{-a\sin nx}}_{=0} - n\cos nx\right)\right)_0^\pi\right]$$

$$= \frac{-2n}{\pi(a^2+n^2)}\left[\left(e^{ax}\cos nx\right)_0^\pi - \left(e^{-ax}\cos nx\right)\right]$$

$$b_n = \frac{-2n(-1)^n}{\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$\therefore f(x) = e^{ax} - e^{-ax} = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}\sin nx}{(a^2+n^2)}$$

$$= \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+2^2} + \cdots\right]$$

Complex form of F.S.

$(a, a+2l)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

~~where~~

Q. Find complex form of Fourier series for function $f(x)$, where

[note: $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$

where $C_n = \frac{1}{2l}\int_a^{a+2l} f(x) e^{\frac{-in\pi x}{l}} dx$]

$$f(x) = \begin{cases} a & \text{if } 0<x<l \\ -a & \text{if } l<x<2l \end{cases}$$

Sol^n: $C_n = \frac{1}{2l}\left[\int_0^{2l} f(x) e^{\frac{-in\pi x}{l}} dx\right]$

$$= \frac{a}{2l}\left[\int_0^{l} e^{\frac{-in\pi x}{l}} dx - \int_l^{2l} e^{\frac{-in\pi x}{l}} dx\right]$$

$$= \frac{a}{2l}\left[\left(\frac{e^{-in\pi x/l}}{-\frac{in\pi}{l}}\right)_0^l - \left(\frac{e^{-in\pi x/l}}{\left(-\frac{in\pi}{l}\right)}\right)_l^{2l}\right]$$

$(n \neq 0)$

$$= \frac{a}{2l}\left(\frac{-l}{in\pi}\right)\left[\left(e^{-in\pi}-1\right) - \left(e^{-i2n\pi} - e^{-in\pi}\right)\right]$$

$$= \frac{ai}{2n\pi}\left[\left((-1)^n - 1\right) - \left(1 - (-1)^n\right)\right]$$

$$\therefore C_n = \frac{ai}{n\pi}\left((-1)^n - 1\right), \quad n \neq 0$$

$$C_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{a}{2l}\left[\int_0^l dx - \int_l^{2l} dx\right]$$

$$= \frac{a}{2l}\left[l - (l)\right] = \underline{\underline{0}}$$

Now $C_0 = \frac{a_0}{2} = 0$

$$\Rightarrow \underline{\underline{a_0 = 0}}$$

$$C_n + C_{-n} = a_n$$

$$= \frac{ai}{n\pi}\left[(-1)^n - 1\right] + \left(\frac{-ai}{n\pi}\right)\left[(-1)^n\right]$$

$$= \underline{0}$$

$$C_n - C_{-n} = -ib_n$$

$$\Rightarrow \frac{2ai}{n\pi}\left((-1)^n - 1\right) = -ib_n$$

$$\therefore b_n = \frac{2a}{n\pi}\left[1 - (-1)^n\right]$$

Q. If $-5 < x < 5$

$$f(x) = \cosh 2x + \sinh 2x$$

$$= \left(\frac{e^{2x} + e^{-2x}}{2}\right) + \left(\frac{e^{2x} - e^{-2x}}{2}\right)$$

$$= e^{2x}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/5}$$

$$C_n = \frac{1}{10}\int_{-5}^{5} e^{2x} e^{-\frac{in\pi x}{5}} dx$$

$$= \frac{1}{10}\int_{-5}^{5} e^{x\left(2 - \frac{in\pi}{5}\right)} dx$$

$$= \frac{1}{10}\int_{-5}^{5} e^{x\left(\frac{10-in\pi}{5}\right)} dx$$

$$= \frac{1}{10}\left[\frac{e^{x\left(\frac{10-in\pi}{5}\right)}}{\left(\frac{10-in\pi}{5}\right)}\right]_{-5}^{5}$$

$$= \frac{1}{2(10-in\pi)}\left[e^{(10-in\pi)} - e^{-(10-in\pi)}\right]$$

$$= \frac{10+in\pi}{2(100+n^2\pi^2)}\left[e^{10}\cdot\underbrace{e^{-in\pi}}_{=(-1)^n} - e^{-10}\cdot\underbrace{e^{in\pi}}_{=(-1)^n}\right]$$

$$= \frac{(-1)^n(10+in\pi)}{(100+n^2\pi^2)}\left[\frac{e^{10}-e^{-10}}{2}\right] = \frac{(-1)^n(10+in\pi)}{(100+n^2\pi^2)} \times \sinh 10$$

$\therefore$ from (*)

$$f(x) = \sinh 10 \sum_{n=-\infty}^{\infty} \frac{(-1)^n(10+in\pi)}{(100+n^2\pi^2)} e^{\frac{in\pi x}{5}}$$

Q. Find the complex form of Fourier series for $f(x) = \begin{cases} 0, & 0<x<l \\ a, & l<x<2l \end{cases}$

$$C_n = \frac{ai}{2n\pi}\left(1-(-1)^n\right)$$

$$C_0 = a/2$$

$$f(x) = \frac{ai}{2\pi}\sum_{\substack{-\infty \\ (n\neq 0)}}^{\infty} \frac{1-(-1)^n}{n} e^{in\pi x/l}$$

$$+ a/2$$

FOURIER INTEGRAL

If $f(x)$ satisfies Dirichlet's conditions in the finite interval $(-l, l)$ and $f(x)$ is integrable in the interval st $(-\infty, \infty)$, then, Fourier integral rep^n of $f(x)$ is given by ~~$f(x) = \frac{1}{\pi}\int^{\infty}_{\omega=0}$~~

$$\boxed{f(x)=\frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\cos\omega(s-x)\,d\omega\,ds}$$

Fourier cosine integral for the function $f(x)$ is given by ~~$f(x)=$~~

$$\boxed{f(x)=\frac{2}{\pi}\int_{\omega=0}^{\infty}\cos\omega x\left[\int_{s=0}^{\infty} f(s)\cos\omega s\,ds\right]d\omega}$$

If f is even

$$\boxed{f(x)=\frac{2}{\pi}\int_{\omega=0}^{\infty}\sin\omega x\left[\int_{s=0}^{\infty} f(s)\sin\omega s\,ds\right]d\omega}$$

If f is odd

$$f(x)=\begin{cases}1 & \text{for } |x|<1 \quad (-1<x<1)\\ 0 & \text{for } |x|>1 \quad (x>1,\ x<-1)\end{cases}$$

Evaluate $\int_0^{\infty}\frac{\sin\omega}{\omega}\cos\omega x\,d\omega$

Sol^n:

f is even fn $(\because f(-x)=f(x))$

$$=\frac{2}{\pi}\int_{\omega=0}^{\infty}\cos\omega x\left[\int_{s=0}^{1} 1\cos\omega s\,ds\right]d\omega$$

$$= \frac{2}{\pi}\int_{\omega=0}^{\infty} \cos\omega x \left[\frac{\sin\omega s}{\omega}\right]_0^1 d\omega$$

$$f(x) = \frac{2}{\pi}\int_{\omega=0}^{\infty} \cos\omega x \, \frac{\sin\omega}{\omega} \cdot d\omega \; //$$

$$\therefore \int_{\omega=0}^{\infty} \cos\omega x \, \frac{\sin\omega}{\omega}\, d\omega = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2} & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \\ =\pi/4 & \text{for } |x|=1 \end{cases}$$

$$\left\{ \begin{aligned} \therefore f(1) &= \frac{1}{2}\left[\lim_{x\to 1-} f(x) + \lim_{x\to +} f(x)\right] \\ &= \frac{1}{2}(1+0) = \frac{1}{2} = f(-1) \end{aligned} \right\}$$

HW Q. Find Fourier integral repⁿ for

$$f(x) = \begin{cases} 1-x^2 & \text{for } |x| \le 1 \\ 0 & \text{for } |x|>1. \end{cases}$$

HW Q. Find Fourier sine integral repⁿ for

$$f(x) = \frac{e^{-ax}}{x}.$$

Solⁿ. Fourier sine repⁿ

$$f(x) = \frac{2}{\pi}\int_{\omega=0}^{\infty} \sin\omega x \left[\int_{s=0}^{\infty} f(s)\sin\omega s\, ds\right] d\omega$$

$$f(x) = \frac{2}{\pi}\int_{\omega=0}^{\infty} \sin\omega x \underbrace{\left[\int_{s=0}^{\infty} \frac{e^{-as}}{s}\sin\omega s\, ds\right]}_{=I(\omega)} d\omega$$

(s won't come in answer)

$$I(\omega) = \int_0^{\infty} \frac{e^{-as}}{s}\sin\omega s\, ds \quad \text{——} (*)$$

$$\frac{dI}{d\omega} = \int_{s=0}^{\infty} \frac{\partial}{\partial\omega}\left(\frac{e^{-as}}{s}\sin\omega s\right) ds$$

$$= \int_{0}^{\infty} \frac{e^{-as}}{\cancel{s}}\, \cancel{s}\cos\omega s \, ds$$

$$= \left\{\frac{e^{-as}}{a^2+\omega^2}\left[-a\cos\omega s + \omega\sin\omega s\right]\right\}_0^{\infty}$$

$$\frac{dI}{d\omega} = \frac{a}{a^2+\omega^2}$$

$$dI = \frac{a}{a^2+\omega^2} d\omega$$

$$I = \int \frac{a}{a^2+\omega^2} d\omega$$

$$I(\omega) = \frac{1}{a}\tan^{-1}\frac{\omega}{a} + c$$

Put $\omega = 0$

$$\therefore I(0) = \tan^{-1} 0 + c$$

$$= 0 = 0 + c$$

$\therefore$ from $(*)$

$$\underline{c = 0}$$

$$\therefore I(\omega) = \tan^{-1}\frac{\omega}{a}$$

$$\therefore f(x) = \frac{2}{\pi}\int_{\omega=0}^{\infty} \sin\omega x \tan^{-1}\frac{\omega}{a}\, d\omega$$

Ans.

(can also be done through Laplace)

Date
Page

Fourier transform

(1) $F(f(t)) = F(s)$

$$= \boxed{\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt} \text{ — FT}$$

Inverse Fourier transform

(2) $f(x) = \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds}$ — IFT

But,

(3) Fourier cosine transform of $F(t)$ is given by

$$F_c[F(t)] = F_c(s) = \int_0^{\infty} f(t) \cos st\, dt \text{ — FCT}$$

(4) Inverse fourier cosine transform of $f(t)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(s) \cos sx\, ds \text{ — IFCT}$$

(5) Fourier sine transform of $f(t)$ is given by

$$F_s[F(t)] = F_s(s) = \int_0^{\infty} f(t) \sin st\, dt \text{ — FST}$$

(6) Inverse Fourier sine transform of $f(t)$ is given by

$$\cancel{F_s[F(t)] = F_s(s)}\ f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(s) \sin sx\, ds \text{ — IFST}$$

(1) $F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$

where $F(s) = F[f(t)]$

(2) $F[f(t)\cos at] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(3) $F_s[f(t)\sin at] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

(4) $F_c(f(t)\sin at] = \frac{1}{2}[F_s(s-a) - F_c(s+a)]$

(5) $F_c[f(t)\cos at] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

Eg. 1 $f(x) = \begin{cases} 1 & \text{for } |x|<1 \ (-1<x<1) \\ 0 & \text{for } |x|>1 \ (x>1, x<-1) \end{cases}$

evaluate $\int_0^\infty \frac{\sin x}{x} dx$

Solⁿ. $F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$= \int_0^1 1 e^{isx} dx$

$= \left[\frac{e^{isx}}{is}\right]^1 = \frac{1}{is}[e^{is} - e^{-is}]$

$= \frac{2}{is}\left(\frac{e^{is} - e^{-is}}{2i}\right) = \frac{2}{s}\sin s$

$\therefore$

$F(s) = \begin{cases} \frac{2\sin s}{s} & \text{if } s \neq 0 \\ 2 & \text{if } s = 0 \end{cases} \quad \left(\because \lim_{x\to 0} \frac{\sin x}{x} = \right.$

Now

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) e^{-isx} ds$

$= \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}\sin s}{s} e^{-isx} ds$

Put $x = 0$

$$\therefore f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(s)}{s}\, ds$$

$$1 = \frac{1}{\pi}\, 2 \int_{0}^{\infty} \frac{\sin s}{s}\, ds$$

$$\Rightarrow \frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin x}{x}\, ds \quad \text{Ans}$$

Q. $f(x) = \begin{cases} 1-x^2 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

Evaluate
$$\int_{0}^{\infty} \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$$

Sol^n: $F(s) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$

$$= \int_{-1}^{1} \underbrace{(1-x^2)}_{=u} \underbrace{e^{isx}}_{=v}\, dx$$

$$= \Bigg[\underbrace{(1-x^2)\left(\frac{e^{isx}}{is}\right)}_{\to 0} - (-2x)\left(\frac{e^{isx}}{i^2 s^2}\right) + (-2)\left(\frac{e^{isx}}{i^3 s^3}\right) \Bigg]_{-1}^{1}$$

$$= \left(-\frac{2e^{is}}{s^2} + \frac{2}{i}\frac{e^{is}}{s^3}\right) - \left(\frac{2e^{-is}}{s^2} + \frac{2e^{-is}}{i s^3}\right)$$

$$= \frac{-4}{s^2}\left(\frac{e^{is} + e^{-is}}{2}\right) + \frac{4}{s^3}\left(\frac{e^{is} - e^{-is}}{2i}\right)$$

$$= \frac{-4}{s^2}\cos s + \frac{4}{s^3}\sin s$$

$$F(s) = \frac{-4}{s^3}(s\cos s - \sin s)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)e^{-isx}\,ds$$

Put $x = 1/2$

$$\therefore f\left(\frac{1}{2}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4(s\cos s - \sin s)}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)ds$$

$$\underbrace{f(1/2)}_{=3/4} = \frac{-4}{2\pi}\left[2\int_0^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos\frac{s}{2}\,ds + 0\right]$$

$$\Rightarrow 3/4 \times \frac{-\pi}{4} = \int_0^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos\frac{s}{2}\,ds$$

→ Find

$f(x) = e^{-|x|}$

Show that $\int_0^{\infty} \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$

Soln: $F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$

$$= \int_0^{\infty} e^{-x}\sin sx\,dx$$

$$= \frac{e^{-x}}{s^2+1}\left[-\cancel{\sin sx} - s\cos sx\right]_0^{\infty}$$

$$F_s(s) = \frac{s}{s^2+1}$$

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, ds$$

$$\therefore e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+1} \sin sx \, ds$$

Replacing x by m

$$\Rightarrow e^{-m} \frac{\pi}{2} = \int_0^\infty \frac{s}{s^2+1} \sin ms \, ds$$

26/9/23

Vector Algebra

Let $\bar{a}, \bar{b}, \bar{c}$ be any three vectors then Scalar Triple Product (STP) of them is given by-

$[\vec{a} \times \vec{b}] \cdot \vec{c}$, which can also be denoted by box pdt.

(1) $\boxed{(\vec{a} \times \vec{b}) \cdot \vec{c} = [\bar{a}\ \bar{b}\ \bar{c}]}$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties

(1) $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a}$
$= (\vec{c} \times \vec{a}) \cdot \vec{b}$

(2) $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$

(3) $[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{a}\ \bar{c}\ \bar{b}]$
$= -[\bar{c}\ \bar{b}\ \bar{a}]$
$= -[\bar{b}\ \bar{a}\ \bar{c}]$

(4) $[\bar{a}\ \bar{a}\ \bar{b}] = 0$
($\because$ 2 rows/columns are same/proportional $\Rightarrow |\cdot| = 0$)

(5) If $\bar{a}, \bar{b}$ & $\bar{c}$ are coplanar vectors, then $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

(6) Vol. of tetrahedron
$= \frac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}]$.

(7) Vol. of parallelopiped $= [\bar{a}\ \bar{b}\ \bar{c}]$

Q. Show that
$$(\bar{p}+\bar{q})\cdot[(\bar{q}+\bar{r})\times(\bar{r}+\bar{p})] = 2[\bar{p}\ \bar{q}\ \bar{r}].$$

Sol: LHS $= \bar{p}\cdot[(\bar{q}+\bar{r})\times(\bar{r}+\bar{p})] + \bar{q}\cdot[(\bar{q}+\bar{r})\times(\bar{r}+\bar{p})]$

$= \bar{p}\cdot[\cancel{\bar{r}\times(\bar{q}+} \ \bar{q}\times(\bar{r}+\bar{p}) + \bar{r}\times(\bar{r}+\bar{p})]$
$+ \bar{q}\cdot[\bar{q}\times(\bar{r}+\bar{p}) + \bar{r}\times(\bar{r}+\bar{p})]$

$= \bar{p}\cdot[(\bar{q}\times\bar{r}) + (\bar{q}\times\bar{p}) + \cancel{(\bar{r}\times\bar{r})}^{0} + (\bar{r}\times\bar{p})]$
$+ \bar{q}\cdot[(\bar{q}\times\bar{r}) + (\bar{q}\times\bar{p}) + \cancel{(\bar{r}\times\bar{r})}^{0} + (\bar{r}\times\bar{p})]$

$= (\bar{p}\ \bar{q}\ \bar{r}) + 0 + 0 + 0 + 0 + [\bar{q}\ \bar{r}\ \bar{p}]$

$= [\bar{p}\ \bar{q}\ \bar{r}] + [\bar{p}\ \bar{q}\ \bar{r}]$
$= 2[\bar{p}\ \bar{q}\ \bar{r}] =$ RHS.

Vector Triple Product (VTP)

If $\bar{a}, \bar{b}, \bar{c}$ are any 3 vectors, then vector Triple product of 3 vector is
(2) given by $\boxed{(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\,\bar{b} - (\bar{b}\cdot\bar{c})\,\bar{a}}$
or $\boxed{\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\,\bar{b} - (\bar{a}\cdot\bar{b})\,\bar{c}}$

Eg If $\bar{a}, \bar{b}$ & $\bar{c}$ are coplanar vectors, then $(\bar{a}\times\bar{b})$, $(\bar{b}\times\bar{c})$, $(\bar{c}\times\bar{a})$ are also coplanar. (Remember as a property)

Solⁿ. Given $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

TPT: $[\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}] = 0$

$$LHS = [(\bar{a}\times\bar{b})\times(\bar{b}\times\bar{c})]\cdot(\bar{c}\times\bar{a})$$

Let $\bar{b}\times\bar{c} = \bar{m}$

$$= [(\bar{a}\times\bar{b})\times\bar{m}]\cdot(\bar{c}\times\bar{a})$$

$$= [(\bar{a}\cdot\bar{m})\bar{b} - (\bar{b}\cdot\bar{m})\bar{a}]\cdot(\bar{c}\times\bar{a})$$

$$= [(\bar{a}\cdot(\bar{b}\times\bar{c}))\bar{b} - \underbrace{(\bar{b}\cdot(\bar{b}\times\bar{c}))}_{=0}\bar{a}]\cdot(\bar{c}\times\bar{a})$$

$$= \underbrace{([\bar{a}\ \bar{b}\ \bar{c}]\bar{b})}_{=0}\cdot(\bar{c}\times\bar{a})$$

$$= \underline{0}.$$

Q. If vectors $\bar{u}, \bar{v}$ & $\bar{w}$ are non-coplanar, then show that $(\bar{u}\times\bar{v})$, $(\bar{v}\times\bar{w})$, $(\bar{w}\times\bar{u})$ are also non-coplanar. Hence obtain scalars l, m, n such that $\bar{u} = l(\bar{v}\times\bar{w}) + m(\bar{w}\times\bar{u}) + n(\bar{u}\times\bar{v})$

Solⁿ. $$\bar{u}\cdot\bar{u} = l\,\bar{u}\cdot(\bar{v}\times\bar{w}) + m\,\cancel{\bar{u}\cdot(\bar{w}\times\bar{u})}_{=0} + n\,\cancel{\bar{u}\cdot(\bar{u}\times\bar{v})}_{=0}$$

$$\therefore\ l = \frac{\bar{u}\cdot\bar{u}}{[\bar{u}\ \bar{v}\ \bar{w}]}$$

$$m = \frac{\bar{u}\cdot\bar{v}}{[\bar{u}\ \bar{v}\ \bar{w}]}$$

$$n = \frac{\bar{u}\cdot\bar{w}}{[\bar{u}\ \bar{v}\ \bar{w}]}$$

$$[\bar{d} \times (\bar{a} \times \bar{b})] \cdot (\bar{a} \times \bar{c}) = [\bar{a}\ \bar{b}\ \bar{c}](\bar{a} \cdot \bar{d})$$

$$LHS = [(\bar{d} \cdot \bar{b})\bar{a} - (\bar{d} \cdot \bar{a})\bar{b}] \cdot (\bar{a} \times \bar{c})$$

$$= (\bar{d} \cdot \bar{b})\cancel{(\bar{a} \cdot (\bar{a} \times \bar{c}))}^{=0}$$

$$- (\bar{d} \cdot \bar{a})(\bar{b} \cdot (\bar{a} \times \bar{c}))$$

$$= -(\bar{d} \cdot \bar{a})[\bar{b}\ \bar{a}\ \bar{c}]$$

$$= (\bar{d} \cdot \bar{a})[\bar{a}\ \bar{b}\ \bar{c}]$$

DIY Q. $(\bar{a} \times \bar{b}) \cdot (\bar{b} \times \bar{c}) \times (\bar{c} \times \bar{a}) = [\bar{a} \cdot (\bar{b} \times \bar{c})]^2$

Q5.

Solⁿ: $(\bar{a} \times \bar{b}) \cdot ((\bar{b} \times \bar{c}) \times (\bar{c} \times \bar{a}))$

$$= \cancel{[(\bar{a} \times \bar{b})}[(\bar{b} \times \bar{c})\ (\bar{c} \times \bar{a})\cancel{]}(\bar{a} \times \bar{b})]$$

Let $\bar{c} \times \bar{a} = \bar{n}$

$$LHS = [(\bar{b} \times \bar{c}) \times \bar{n}] \cdot (\bar{a} \times \bar{b})$$

$$= [(\bar{b} \cdot \bar{n})\bar{c} - (\bar{c} \cdot \bar{n})\bar{b}] \cdot (\bar{a} \times \bar{b})$$

$$= [\cancel{(\bar{b} \cdot \bar{n})}(\bar{b} \cdot (\bar{c} \times \bar{a}))\bar{c} - \cancel{(\bar{c} \cdot (\bar{c} \times \bar{a}))}^{0}\bar{b}] \cdot (\bar{a} \times \bar{b})$$

$$= ([\bar{b}\ \bar{c}\ \bar{a}]\ \bar{c}) \cdot (\bar{a} \times \bar{b})$$

$$= ([\bar{a}\ \bar{b}\ \bar{c}]\ \bar{c}) \cdot (\bar{a} \times \bar{b})$$

$$= [\bar{a}\ \bar{b}\ \bar{c}]^2$$

Scalar product of 4 vectors $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$

(3) $\boxed{(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c} \\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d} \end{vmatrix}}$

P.T $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) = 0$

Solⁿ: $LHS = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c} \\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d} \end{vmatrix} + \begin{vmatrix} \bar{b}\cdot\bar{a} & \bar{c}\cdot\bar{a} \\ \bar{b}\cdot\bar{d} & \bar{c}\cdot\bar{d} \end{vmatrix} + \begin{vmatrix} \bar{c}\cdot\bar{b} & \bar{a}\cdot\bar{b} \\ \bar{c}\cdot\bar{d} & \bar{a}\cdot\bar{d} \end{vmatrix}$

$= [(ac)(bd) - (bc)(ad)] \longrightarrow$ not allowed

$= [(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d}) - (\bar{b}\cdot\bar{c})(\bar{a}\cdot\bar{d})] + [(\bar{b}\cdot\bar{a})(\bar{c}\cdot\bar{d}) - (\bar{c}\cdot\bar{a})(\bar{b}\cdot\bar{d})] + [(\bar{c}\cdot\bar{b})(\bar{a}\cdot\bar{d}) - (\bar{a}\cdot\bar{b})(\bar{c}\cdot\bar{d})]$

$= \underline{\underline{0}}$

- Vector product of 4 vectors is given by

(4) $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\,\bar{c}\,\bar{d}]\,\bar{b} - [\bar{b}\,\bar{c}\,\bar{d}]\,\bar{a}$

or $= [\bar{a}\,\bar{b}\,\bar{d}]\,\bar{c} - [\bar{a}\,\bar{b}\,\bar{c}]\,\bar{d}$

P.T. $[(\bar{a}\times\bar{b})\times(\bar{a}\times\bar{c})]\cdot\bar{d} = (\bar{a}\cdot\bar{d})[\bar{a}\,\bar{b}\,\bar{c}]$

$LHS = \{\underbrace{[\bar{a}\,\bar{a}\,\bar{c}]}_{0}\,\bar{b} - [\bar{b}\,\bar{a}\,\bar{c}]\,\bar{a}\}\cdot\bar{d}$

$= [\bar{a}\,\bar{b}\,\bar{c}](\bar{a}\cdot\bar{d}) \quad (\because -[\bar{b}\,\bar{a}\,\bar{c}] = [\bar{a}\,\bar{b}\,\bar{c}])$

$= \underline{RHS}$

Q. Prove that
$$\bar{d}\cdot\{\bar{a}\times[\bar{b}\times(\bar{c}\times\bar{d})]\} = (\bar{b}\cdot\bar{d})[\bar{a}\,\bar{c}\,\bar{d}]$$

Solⁿ: $\bar{d}\cdot\{\bar{a}\times((\bar{b}\cdot\bar{d})\bar{c}-(\bar{b}\cdot\bar{c})\bar{d}]\}$

$\cancel{= \bar{d}\cdot\{\bar{a}[(\bar{b}\cdot\bar{d})\bar{c}-(\bar{b}\cdot\bar{c})\bar{d}\ \ d]}$

$= \bar{d}\cdot\{(\bar{a}\times\bar{c})(b\cdot d) - (\bar{a}\times\bar{d})(\bar{b}\cdot\bar{c})\}$

$= \bar{d}\cdot\{(\bar{a}\times\bar{c})(\bar{b}\cdot\bar{d}) - \underbrace{(\bar{a}\times\bar{d})(b\cdot\bar{c})\cdot\bar{d}}_{0}$

$= [\bar{a}\,\bar{c}\,\bar{d}](\bar{b}\cdot\bar{d})$

$= (b\cdot d)[\bar{a}\,\bar{c}\,\bar{d}]$ H.P.

Vector differentiation

$y=f(x)$

$\underbrace{\dfrac{dy}{dx}}_{\text{basic}}$

Gradient:

The Gradient of a vector $f(x,y)$ at the point (x_0,y_0) is given by

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j$$

Directional derivative

Directional derivative of the ~~derivative~~ function(u) in the direction of unit vector $\hat{u}$ is given by

$$D_{\hat{u}} f(a,b) = \nabla f(a,b) \cdot \hat{u} \quad (a,b)$$

Gradient vector

$$\boxed{\nabla(\Phi) = \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j + \frac{\partial \Phi}{\partial z} k} \quad (1)$$

Directional derivative (same as above but for 2 var.) (not)

$$\boxed{D_{\hat{u}} f(a) = \nabla f(a) \cdot \hat{u}} \quad (2)$$

$$= |\nabla f(a)| \underbrace{|\hat{u}|}_{=1} \cos\theta$$

$$= \underbrace{|\nabla f(a)|}_{\text{constant}} \cos\theta$$

$D_{\hat{u}} f$ is max if $\cos\theta = 1$ i.e. $\theta = 0$

~~but i.e.~~ $D_{\hat{u}} f$ is in the dir^n of $\hat{u}$.

If $\theta = 0$, then dir^n of $\hat{u}$ & ∇f is same

$\therefore$ If $\theta = 0$, $D_{\hat{u}} f$ is max in the dir^n of ∇f, and it's max mag^n = $|\nabla f|$

$$\therefore d\phi = \left(\frac{\partial \Phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right) \cdot \left(dx\, i + dy\, j + dz\, k\right)$$

$$= \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \cdot (dx, dy, dz)$$

$$\therefore \boxed{d\phi = \nabla\phi \cdot d\bar{r}} \quad (3)$$

If $\bar{r} = xi + yj + zk$
$d\bar{r} = dx\,i + dy\,j + dz\,k$

Ex Find $\Phi(r)$.
s.t. $\nabla\Phi = \frac{-\bar{r}}{r^5}$ and $\phi(1) = 0$

Solⁿ: $\bar{r} = xi + yj + zk$
$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$
Now
$$\nabla\Phi = \frac{-\bar{r}}{r^5} = \frac{-(xi+yj+zk)}{(x^2+y^2+z^2)^{5/2}}$$
$$= \frac{\partial\Phi}{\partial x} i + \frac{\partial\Phi}{\partial y} j + \frac{\partial\Phi}{\partial z} k$$
$$\therefore \frac{\partial\Phi}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{5/2}}$$
$$\frac{\partial\Phi}{\partial y} = \frac{-y}{(x^2+y^2+z^2)^{5/2}}$$
$$\frac{\partial\Phi}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{5/2}}$$

Now $d\phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$
$= \frac{-[x\,dx + y\,dy + z\,dz]}{(x^2+y^2+z^2)^{5/2}}$ (Total diff formula)
$\int d\phi = -\int \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{5/2}}$

Put $x^2+y^2+z^2 = t$
$\Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = dt$
$\therefore \Phi = \int \frac{dt}{2t^{5/2}} = \frac{-1(2)}{2\times 3} t^{-3/2} = \frac{1}{3t^{3/2}} + c$

$$= \frac{1}{3}\frac{1}{(x^2+y^2+z^2)^{3/2}} + C$$

$$\therefore \phi(r) = \frac{1}{3}\frac{1}{r^3} + C$$

but $\phi(1) = 0$

i.e. if $r=1, \phi = 0$

$$\Rightarrow 0 = \frac{1}{3} + C$$

$$C = -\frac{1}{3}$$

$$\therefore \boxed{\phi(r) = \frac{1}{3r^3} - \frac{1}{3}}$$

P.T. $\boxed{\nabla f(r) = f'(r)\frac{\bar{r}}{r}}$ —(**)

Hence find f ib. $\nabla f = 2r^4 \frac{\bar{r}}{r}$.

Solⁿ. $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} + \frac{\partial \phi}{\partial z} k$

$f \to r \to x, y, z$

$$\therefore \frac{\partial f}{\partial x} = \frac{df}{dr} \times \frac{\partial r}{\partial x}$$

$$= f'(r)\frac{\partial r}{\partial x}$$

$|||^{ly}$ $\frac{\partial f}{\partial y} = f'(r)\frac{\partial r}{\partial y}$

$$\frac{\partial f}{\partial z} = f'(r)\frac{\partial r}{\partial z}$$

$$\therefore \nabla f = f'(r)\left[\frac{\partial r}{\partial x} i + \frac{\partial r}{\partial y} j + \frac{\partial r}{\partial z} k\right]$$

$$\cancel{d}\, \cancel{4}\; r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$|||^{y} = \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \bar{r} = xi+yj+zk \quad \therefore r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \nabla f = f'(r)\left[\frac{x}{r}i+\frac{y}{r}j+\frac{z}{r}k\right]$$

$$= \frac{f'(r)}{r}[xi+yj+zk]$$

$$\boxed{\nabla f(r) = \frac{f'(r)}{r}\bar{r}}$$

Now

$$\nabla f = 2r^4\bar{r} = f'(r)\frac{\bar{r}}{r}$$

$$\Rightarrow 2r^4 = \frac{f'(r)}{r}$$

$$\Rightarrow f'(r) = 2r^5$$

$$\therefore f(r) = 2\int r^5 dr$$

$$= \frac{r^6}{3}+c$$

$$f(r) = \frac{r^6}{3}+c$$

grad $\phi = \nabla \phi$

$$= \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$\nabla[f(r)] = f'(r) \frac{\bar{r}}{r}$

Q. $\nabla\left[\frac{\bar{a}\cdot\bar{r}}{r^n}\right] = \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$

Let $\bar{a} = a_1 i + a_2 j + a_3 k$
$\bar{r} = xi + yj + zk$

Let $\phi = \frac{\bar{a}\cdot\bar{r}}{r^n} = \frac{a_1 x + a_2 y + a_3 z}{r^n}$

$$\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \quad \text{--- (1)}$$

$$\frac{\partial \phi}{\partial x} = \frac{r^n a_1 - (\bar{a}\cdot\bar{r}) n r^{n-1}\left(\frac{\partial r}{\partial x}\right)}{r^{2n}}$$

But $r^2 = x^2 + y^2 + z^2$

$\therefore 2r \frac{\partial r}{\partial x} = 2x$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \frac{\partial \phi}{\partial x} = \frac{r^n a_1 - (\bar{a}\cdot\bar{r}) n x r^{n-2}}{r^{2n}}$$

Similarly $\frac{\partial \phi}{\partial y} = \frac{r^n a_2 - (\bar{a}\cdot\bar{r}) n r^{n-2} y}{r^{2n}}$

$$\frac{\partial \phi}{\partial z} = r^n a_3 - (\bar{a}\cdot\bar{r}) n r^{n-2} z \quad (2)$$

From (1) & (2),

$$\nabla\phi = \frac{r^n (a_1 i + a_2 j + a_3 k) - (xi + yj + zk)(\bar{a}\cdot\bar{r}) n r^{n-2}}{r^{2n}}$$

$$= \frac{r^n \bar{a} - (\bar{a}\cdot\bar{r}) n r^{n-2} \bar{r}}{r^{2n}}$$

$$= \frac{\bar{a}}{r^n} - \frac{(\bar{a}\cdot\bar{r}) n \bar{r}}{r^{n+2}}$$

Note: If ϕ is a constant then $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} = 0$

$\therefore$ grad $\phi = 0$.

Results: (1) $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$

(2) $\nabla(\phi\psi) = \phi(\nabla\psi) + (\nabla\phi)\psi$

(3) $\nabla f(u) = i\frac{\partial f(u)}{\partial x} + j\frac{\partial f(u)}{\partial y} + k\frac{\partial f(u)}{\partial z}$

$= f'(u)\nabla u$

$\phi = \boxed{xyz = 3}$

$\nabla\phi \rightarrow$ normal to surface

If $\phi(x,y,z) = c_1$ and $\psi(x,y,z) = c_2$ are two surfaces $\angle^{le}$ b/w 2 surface = $\angle^{le}$ b/w the normal i.e. $\angle^{le}$ b/w $\nabla\phi$ & $\nabla\psi$. If θ is the $\angle^{le}$ b/w them then $\theta = \cos^{-1}\left|\frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi||\nabla\psi|}\right|$

If the surfaces are orthogonal then
$\nabla\phi \cdot \nabla\psi = 0$

- Directional derivative of ϕ in the direction of $a = \frac{\nabla\phi \cdot a}{|a|}$

Q1. Find $\angle$le b/w normals to the surface $xy = z^2$ at the points $(1, 4, 2)$ and $(-3, -3, 3)$.

Soln. Let $\phi = xy - z^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i(y) + j(x) + k(-2z)$$

let $\bar{a} = \nabla\phi\Big|_{(1,4,2)} = 4i + j - 4k$

let $\bar{b} = \nabla\phi\Big|_{(-3,-3,3)} = -3i - 3j - 6k$

angle betn normals $= \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} = \frac{4(-3) + (1)(-3) + (-4)(-6)}{\left(\sqrt{4^2+1^2+(-4)^2}\right) \times \sqrt{(-3)^2+(-3)^2+(-6)^2}}$

$$= \frac{-12 - 3 + 24}{\sqrt{33}\sqrt{54}} = \frac{9}{3\sqrt{198}} = \frac{3}{\sqrt{198}} = \frac{1}{\sqrt{22}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$

Q2. Find the constants a & b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1,-1,2)$.

Sol^n. Let $\phi = ax^2 - byz - (a+2)x$

$\psi = 4x^2y + z^3 - 4$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2ax - (a+2)) + \bar{j}(-bz) + \bar{k}(-by)$$

Let $\bar{p} = \nabla\phi\big|_{(1,-1,2)} = (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$

Now $$\nabla\psi = i\frac{\partial\psi}{\partial x} + j\frac{\partial\psi}{\partial y} + k\frac{\partial\psi}{\partial z}$$

$$= i(8xy) + j(4x^2) + k(3z^2)$$

Let $\bar{q} = \nabla\psi\big|_{(1,-1,2)} = -8\bar{i} + 4\bar{j} + 12\bar{k}$

The surfaces are orthogonal

$\therefore \bar{p}\cdot\bar{q} = 0 \Rightarrow -8(a-2) + 4(-2b) + 12b = 0$

$\Rightarrow -8a + 16 - 8b + 12b = 0$

$\Rightarrow -8a + 4b + 16 = 0$ — (1)

$\because (1,-1,2)$ lies on $\rightarrow ax^2 - byz = (a+2)x$

$\therefore a + 2b = (a+2)$

$\Rightarrow 2b = 2$ $\boxed{b = 1}$ Ans.

Put this in (1) $\Rightarrow -8a + 4 + 16 = 0$

$\boxed{a = 5/2}$ Ans

~~∇^2~~ If $\bar{f} = f_1 i + f_2 j + f_3 k$
then $\operatorname{div} \bar{f} = \nabla \cdot f = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

If $\nabla \cdot \bar{f} = 0 \Rightarrow \bar{f}$ is called solenoidal.

$$\operatorname{curl} \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

If $\nabla \times \bar{f} = \bar{0} \Rightarrow f$ is called irrotational.

$\nabla \cdot (\phi \bar{f}) = \phi (\nabla \cdot \bar{f}) + f (\nabla \phi)$

Ex. $|\bar{a}| = a$
P.T. $\nabla \cdot \{ (\bar{a} \cdot \bar{r}) \cdot \bar{a} \} = a^2$

proof: $\phi = \bar{a} \cdot \bar{r}$
$\bar{f} = \bar{a}$

$\therefore \nabla \cdot (\phi \bar{f}) = (\bar{a} \cdot \bar{r})(\bar{\nabla} \cdot \bar{a}) + \bar{a} \cdot \nabla (\bar{a} \cdot \bar{r})$

but. $\nabla \cdot \bar{a} = \dfrac{\partial a_1}{\partial x} + \dfrac{\partial a_2}{\partial y} + \dfrac{\partial a_3}{\partial z}$

$= 0 + 0 + 0 = \underline{\underline{0}}$

$\therefore \nabla \cdot (\phi \cdot \bar{f}) = \bar{a} \cdot \nabla (\bar{a} \cdot \bar{r})$
$= \bar{a} \cdot \nabla (a_1 x + a_2 y + a_3 z)$
$= \bar{a} \cdot (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$
$= \bar{a} \cdot \bar{a} = |\bar{a}|^2 = \underline{a^2}$

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$\nabla\cdot(\phi\bar{f}) = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot(\nabla\phi)$
(To prove)

Proof: We know that,

Ex. :

$$\nabla\left\{\nabla\cdot\frac{\bar{r}}{r}\right\} = -\frac{2\bar{r}}{r^3}$$

$\therefore$ Solⁿ: $\nabla\cdot\frac{\bar{r}}{r} = \frac{1}{r}\nabla\cdot\bar{r} + \bar{r}\cdot\nabla\left(\frac{1}{r}\right)$

no need to write description in exam

but $\nabla\cdot\bar{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

$$\therefore \nabla\cdot\frac{\bar{r}}{r} = \frac{3}{r} + \bar{r}\cdot\left(\frac{-1}{r^2}\right)\frac{\bar{r}}{r}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$

$$\therefore \nabla\left\{\nabla\cdot\frac{\bar{r}}{r}\right\} = \nabla\left(\frac{2}{r}\right) = \frac{-2}{r^2}\frac{\bar{r}}{r}$$

$$= -\frac{2\bar{r}}{r^3}$$

Prove that $\nabla\cdot\left(r\nabla\frac{1}{r^n}\right) = \frac{n(n-2)}{r^{n+1}}$

Solⁿ: $\nabla\frac{1}{r^n} = \frac{-n}{r^{n+1}}\frac{\bar{r}}{r}$

$$\therefore r\nabla\frac{1}{r^n} = \frac{-n\bar{r}}{r^{n+1}}$$

$$\therefore \nabla\cdot\left[r\nabla\frac{1}{r^n}\right] = -n\nabla\cdot\left[\frac{\bar{r}}{r^{n+1}}\right]$$

$$= -n\left[\frac{1}{r^{n+1}}\nabla\cdot\bar{r} + \bar{r}\cdot\nabla\frac{1}{r^{n+1}}\right]$$

$$= -n\left[\frac{3}{r^{n+1}} + \bar{r}\cdot\left(\frac{-(n+1)}{r^{n+2}}\right)\frac{\bar{r}}{r}\right]$$

$$= -n\left[\frac{3}{r^{n+1}} - \frac{(n+1)}{r^{n+1}}\right]$$

$$= -n\left[\frac{2-n}{r^{n+1}}\right]$$

$$= \frac{n(n-2)}{r^{n+1}}$$ Hence proved

P.T. $\nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\} = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$

Hence P.T. $\nabla\cdot(r^n\bar{r}) = (n+3)r^n$

Solⁿ. $\nabla\cdot(\phi\bar{f}) = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot\nabla\phi$

Let $\phi = \frac{f(r)}{r}$, $\bar{f} = \bar{r}$

$$\therefore \nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\} = \frac{f(r)}{r}(\bar{\nabla}\cdot\bar{r}) + \bar{r}\cdot\nabla\left(\frac{f(r)}{r}\right)$$

$$= \frac{3f(r)}{r} + \bar{r}\cdot\left[\frac{rf'(r) - f(r)}{r^2}\right]\frac{\bar{r}}{r}$$

$$= \frac{3f(r)}{r} + f'(r) - \frac{f(r)}{r}$$

$$= \frac{2f(r)}{r} + f'(r) \quad \text{--- (1)}$$

$= \text{LHS}$

$\text{RHS} = \frac{1}{r^2}\frac{d}{dr}(r^2 f(r))$

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$$= \frac{1}{r^2}\left[r^2 f'(r) + 2r f(r)\right]$$

$$= f'(r) + \frac{2}{r} f(r) \quad \text{—— (2)}$$

Now let

$$\frac{f(r)}{r} = r^n$$

$$\therefore f(r) = r^{n+1}$$

$$\therefore \nabla \cdot \left[r^n \bar{r}\right] = \frac{1}{r^2}\frac{d}{dr}\left[r^2 r^{n+1}\right]$$

$$= \frac{1}{r^2}\frac{d}{dr}\left[r^{n+3}\right]$$

$$= \frac{1}{r^2}(n+3) r^{n+2}$$

$$= \underline{\underline{r^n (n+3)}}$$

Q. Find $f(r)$ so that $f(r)\bar{r}$ is solenoidal.

Solⁿ: $\because f(r)\bar{r}$ is solenoidal.

$\Rightarrow$ Its divergence $= 0$.

Given $\nabla \cdot \left[f(r)\bar{r}\right] = 0$

$$\Rightarrow f(r)\,\nabla \cdot \bar{r} + \bar{r} \cdot \nabla f(r) = 0$$

$$\Rightarrow 3f(r) + \bar{r} \cdot f'(r)\frac{\bar{r}}{r} = 0$$

$$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$$

$$\Rightarrow \int \frac{f'(r)}{f(r)} dr = \int \frac{-3}{r} dr$$

$\therefore \; f(r) = cr^{-3}$ Ans.

- **Vector integration**

Let $\bar{f}$ be a continuous function in the region $\bar{R}$. Let c be any curve in the region $\bar{R}$.
Then, line integral of $\bar{F}$, is given by
$$\int_c \bar{F}\cdot dr = \int_c f_1\,dx + f_2\,dy + f_3\,dz \quad \text{(along c)}$$

Definition: $\bar{F}$ is called conservative if there exists, i.e. $\bar{F}$ conservative if $\exists\,\phi$ s.t. $\bar{F} = \nabla\phi$.

N.B.: Conservative iff irrotational.
$$\text{curl}\,\bar{F} = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \bar{0}.$$

Thr^m: Let $\bar{F}$ be a continuous function, then following statements are equivalent.

(1) $\rightarrow$ $\int_c \bar{F}\cdot d\bar{r}$ is independent of the path joining the end points of the curve c. But, it depends only on the end points of the curve c.

(2) $\bar{F} = \nabla\phi$, that is there exists ϕ such that $\bar{F} = \nabla\phi$.

(3) For any closed curve c,
$\oint_c \bar{F}\cdot d\bar{r} = 0$. Work done $= 0$

Q. Evaluate

$$\int_c \bar{F} \cdot d\bar{r}, \quad \bar{F} = \cos y \, \bar{i} - x \sin y \, \bar{j}$$

and, c is the curve

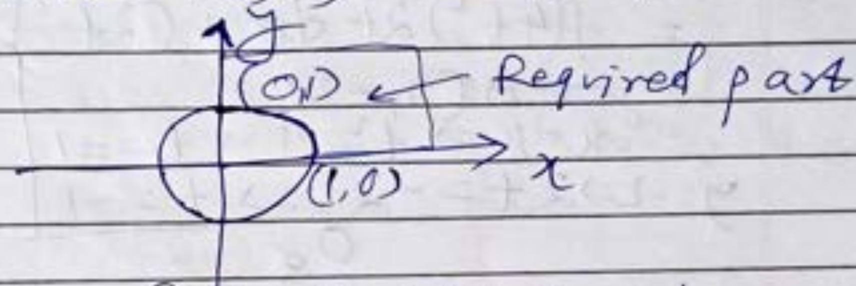
c: $y = \sqrt{1-x^2}$ (circle)

From $(1,0)$ to $(0,1)$.

Solⁿ: $y = \sqrt{1-x^2}$ squaring,

$x^2 + y^2 = 1$

centre: $(0,0)$. radius $= 1$



$$\int_c \bar{F} \cdot d\bar{r} = \int_c \cos y \, dx - x \sin y \, dy$$

$$= \int_c d\underbrace{(x \cos y)}_{= f}$$

$\because df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$\therefore d(x\cos y) = \cos y \, dx - x \sin y \, dy$

$$= [x \cos y]_{(1,0)}^{(0,1)} = (0-1) = \underline{\underline{-1}} \text{ Ans}$$

$\bar{F} \cdot d\bar{r} = df = \nabla f \cdot d\bar{r}$

$\Rightarrow \bar{F} = \nabla f \quad \nabla \phi$

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Q Evaluate $\int_A^B y^2 dx + xy\,dy$ along
$x = t^2,\ y = 2t$ from $A(1,-2)$ to $B(0,0)$.

Slf. Note: in prev. day's prblm, used formula $\int \{f(x) + x f'(x)\} dx = x f(x) + c$ except that two var. x, y.

Soln. $\int_C y^2 dx + xy\,dy$

$= \int (4t^2) 2t\,dt + (2t^3) 2\,dt$

at $A = (1,-2)$
$x = 1 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$
$y = -2 \Rightarrow 2t = -2 \Rightarrow t = -1$

at $B = (0,0)$
$x = 0 \Rightarrow t^2 = 0$
$\Rightarrow t = 0$

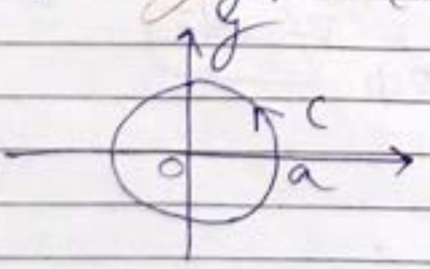
$\therefore I = \int_{t=-1}^{0} 8t^3 dt + 4t^3\,dt$

$= \left[\frac{8t^4}{4} + \frac{4t^4}{4}\right]_{-1}^{0}$

$= 2(-1) + 1(-1)$
$= -3$

Q.- Once around a circle $x^2 + y^2 = a^2,\ z = 0$.
$\bar{F} = \sin y\, i + (x + x\cos y)\, j$.

Soln.



Work done $= \int_C \bar{F} \cdot d\bar{r}$

$= \int_C (\sin y\, dx + (x + x\cos y)\, dy)$

$= \int (\sin y\, dx + x\cos y) + x$

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$$= \int_C d(x\sin y) + \int_C x\,dy$$

$$= 0 + \int_C x\,dy$$

$\because x^2 + y^2 = a^2$

$\Rightarrow x = \sqrt{a^2 - y^2}$

<u>OR</u> Put $x = a\cos t$

$y = a\sin t$

$\Rightarrow dy = a\cos t\,dt$

$$= 0 + \int_C a\cos t\,(a\cos t)\,dt$$

$$\Rightarrow \int_C a^2\cos^2 t = \int_{t=0}^{2\pi} a^2\cos^2 t\,dt$$

$$= a^2 \int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2} \times 2\pi = \underline{a^2\pi}$$

Imp. Q. $F = (y^2\cos x + z^3)i + (2y\sin x - 4)j + (3xz^2 + 2)k$

Prove that above function is conservative, and Find (1) Scalar potential for F.

(2) Work done $(0, 1, -1)$ to $(\frac{\pi}{2}, -1, 2)$.

<u>Sol^n:</u> $\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2\cos x + z^3) & (2y\sin x - 4) & (3xz^2 + 2) \end{vmatrix}$