$= \left[i(0-0) - j(3z^2 - 3z^2) + k(2y\cos x - 2y\cos x) \right]$

necessary

$= \overline{0}$

$\Rightarrow \overline{F}$ is conservative.

$\Rightarrow \exists\, \phi$ s.t $F = \nabla\phi$.

$\phi \equiv$ scalar potential

$$\overline{F} = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = y^2\cos x + z^3.$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 + 2$$

But, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$\Rightarrow d\phi = (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2 + 2)dz = \overline{F}\cdot d\overline{r}$$

$$= (y^2\cos x\, dx + 2y\sin x\, dy) + (z^3 dx + 3xz^2 dz) - (4dy + 2dz)$$

$$= d(y^2\sin x) + d(xz^3) - d(4y) + d(2z)$$

$$= \underbrace{d(y^2\sin x + xz^3 - 4y + 2z)}_{= \phi}$$

$\therefore\ \phi = y^2\sin x + xz^3 - 4y + 2z + c$

x^2

xz^3

$y^2(\sin x)$

$y^2(\sin x)$

(2)

$$\text{Work done} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_C d\phi$$

$$= [\phi]_{(0,1,-1)}^{(\pi/2,-1,2)}$$

$$= \left[y^2 \sin x + xz^3 - 4y + 2z + c\right]_{(0,1,-1)}^{(\pi/2,-1,2)}$$

$$= \left[(1 + 4\pi + \cancel{4}\,4 + 4 + \cancel{c}) - (0 + 0 - 4 - 2 + \cancel{c})\right]$$

$$= 4\pi + 15$$

Q. Find the total work done by the force $\bar{F} = 3xy\,i - y\,j + 2xz\,k$ in moving a particle around a circle $x^2 + y^2 = 4$.

Solⁿ. $\text{curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & -y & 2xz \end{vmatrix}$

$$= \hat{i}(0-0) - \hat{j}(2z - 0) + \hat{k}(0 - 3x)$$

$$= (-2z)\hat{j} - (3x)\hat{k}$$

$\Rightarrow \bar{F}$ is NOT conservative

$$\text{w.d.} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_C 3xy\,dx - y\,dy + \underset{z=0}{2xz\,dz}$$

$$= \int 3xy\,dx - y\,dy$$

Put $x = 2\cos\theta \Rightarrow dx = -2\sin\theta\,d\theta$

$y = 2\sin\theta \Rightarrow dy = 2\cos\theta\,d\theta$

$\theta = 0$ to 2π

- **Green's Thm in the Plane:**

Statement: If R is a closed region of the xy-plane bounded by a simple closed curve C and if P and Q are continuous functions of x and y having continuous partial derivatives in R, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dx\,dy$$

Q. Evaluate by Green's theorem:

$\int_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy)$ where C is the rectangle whose vertices are $(0,0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(0, \pi/2)$.

Sol^n -

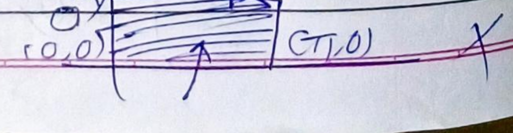
$$Q = e^{-x}\cos y$$

$$\frac{\partial P}{\partial y} = e^{-x}\cos y$$

By Green's theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\int e^{-x}\sin y\,dx + e^{-x}\cos y\,dy = \iint_R -2e^{-x}\cos y\,dx\,dy$$



In R taking a vertical strip

$y \rightarrow x$ axis to BC

$y = 0$ to $\pi/2$

$x \rightarrow y$ axis to AB

$x = 0$ to $x = \pi$

$$= \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} -2e^{-x} \cos y \, dx \, dy$$

$$= \left(\int_{x=0}^{\pi} -2e^{-x} \, dx \right) \left(\sin y \right)_0^{\pi/2}$$

$$= \left(\frac{-2e^{-x}}{-1} \right)_0^{\pi} \left(\left(\sin \pi/2 \right) - \left(\sin 0 \right) \right)$$

$$= 2\left(e^{-\pi} - e^{0}\right)\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$\cancel{= \int_0^{\pi} 2e^{-\sin y} \, dx}$$

$$= 2\left(e^{-\pi} - 1\right)(1 - 0)$$

$$= 2\left(e^{-\pi} - 1\right)$$

$$= \int_0^{\pi} -2e^{-x} \, dx$$

$$= \left(\frac{-2e^{-x}}{-1} \right)_0^{\pi}$$

$$= \underline{2\left(e^{-\pi} - 1\right)}$$

2. Evaluate Green's theorem. $\int_C \bar{F}\cdot d\bar{r}$

where $F = -xy(xi - yj)$ and C is $r = a(1+\cos\theta)$

$\underline{Sol^n}$ $\int_C \bar{F}\cdot d\bar{r} = \int_C x^2y\,dx + xy^2\,dy$

$= \int_C Pdx + Qdy$

Now, $P = -x^2y \qquad Q = xy^2$

$\therefore \frac{\partial P}{\partial y} = -x^2 \qquad \frac{\partial Q}{\partial x} = y^2$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 - (-x^2)$

$= \underline{x^2+y^2}$

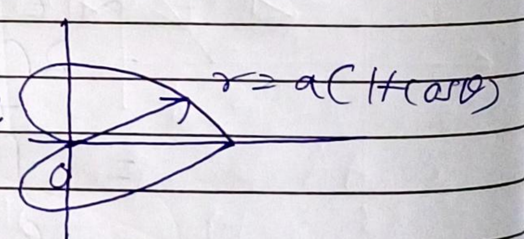
By Green's theorem $\int Pdx + Qdy$

$= \int\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

To evaluate the double integral, put $x = r\cos\theta$, $y = r\sin\theta$.

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2$



$\int -x^2y\,dx + xy^2\,dy = \iint r^2 r\,dr\,d\theta$

$r \to 0$ to $a(1+\cos\theta)$

$\underline{\theta \to 0 \text{ to } \pi}$ (twice integral)

$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\,dr\,d\theta$

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$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\, dr\, d\theta$$

$$= 2\int_0^{\pi}\left(\frac{r^4}{4}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^4 (1+\cos\theta)^4\, \cancel{\frac{}{4}}\, d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi}\left(\frac{2\cos^2\theta}{2}\right)^4 d\theta$$

$$= \frac{2^4 a^4}{2}\int_0^{\pi}\cos^8\frac{\theta}{2}\, d\theta$$

put $\frac{\theta}{2} = t$ $\quad d\theta = 2dt$

$t \to 0$ to $\pi/2$

$\cos^8 t\,(2dt)$

$$= 16a^4\int^{(\pi/2)}\cos^8 t\, dt$$

$$= 16a^4 \cdot \frac{1}{2}\beta\left(\frac{8+1}{2}, \frac{0+1}{2}\right)$$

$$= 8a^4\, \Gamma\frac{9}{2}\,\Gamma\frac{1}{2}$$

$$= \frac{8 \cdot \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{4!}$$

$$= \frac{35\pi a^4}{16}$$

3. Verify Green's theorem for $F = x^2 i - xy j$ and C the triangle with vertices A(0,2), B(2,0) & C(4,2).

Soln. $\int_C \bar F \cdot d\bar r = \int x^2 dx - xy\,dy = \int_C P\,dx + Q\,dy$

$P = x^2 \qquad Q = -xy$

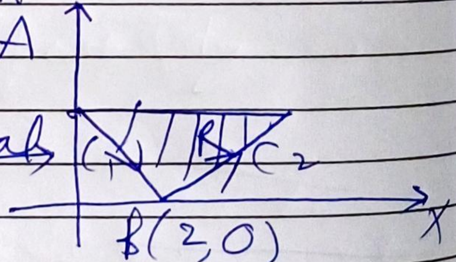
$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = -y \qquad \therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y$

By Green's thrm $\int_C P\,dx + Q\,dy$

$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$x^2 dx - xy\,dy = \iint_R -y\,dx\,dy$

(a) RHS $= \iint_R -y\,dx\,dy$



taking a horizontal strip

$x \to$ AB to BC

$x = 2-y$ to $2+y$ eq. of AB $x+y=2$

$y \to$ x axis to AC

$y = 0$ to $y = 2$ eq. of BC $x-y=2$

RHS $= \int_0^2 \int_{2-y}^{2+y} -y\,dx\,dy$

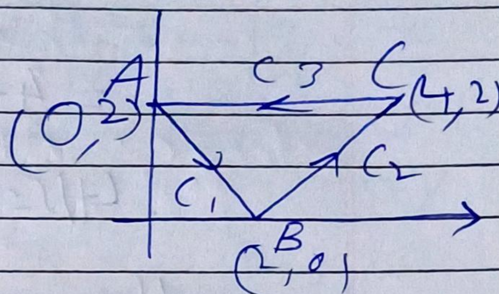
$= \int_0^2 -y\,(x)_{2-y}^{2+y}$

$= \int_0^2 -y[2y]\,dy = -2\int_0^2 y^2\,dy$

$= -2\left(\frac{y^3}{3}\right)_0^2$

RHS $= \frac{-16}{3}$

(b) LHS $= x^2 dx - xy\,dy$



$C = C_1 + C_2 + C_3$

along C_1: $x + y = 2$

$y = 2 - x$

$dy = -dx$

$x \to 0$ to 2

AB : $x + y = 2$

BC : $x - y = 2$

$$\int x^2 dx - xy\,dy = \int_0^2 x^2 dx - x(2-x)(-dx)$$

$$= \int_0^2 2x\,dx$$

$$= (x^2)_0^2 = \underline{\underline{4}}$$

Along C_2: BC $\longrightarrow x - y = 2$

$y = x - 2$

$dy = dx$

$x \to 2$ to 4

$$\int_{C_2} x^2 dx - xy\,dy = \int_2^4 x^2 dx - x(x-2)\,dx$$

$$= \int_2^4 2x\,dx = (x^2)_2^4 = \underline{\underline{12}}$$

along C_3: CA $\to y = 2$

$dy = 0$ $\quad x \to 4$ to 0

$$\int_{C_3} x^2 dx - xy\,dy = \int_4^0 x^2 dx = 0 - \frac{4^3}{3}$$

$$= \underline{\underline{-\frac{64}{3}}}$$

LHS $\int_C x^2 dx - xy\,dy$

$$= \int_{C_1} x^2 dx - xy\,dy + \int_{C_2} x^2 dx - xy\,dy + \int_{C_3} x^2 dx - xy\,dy$$

$$= 4 + 12 - \frac{64}{3} = \frac{-16}{3}$$

$\therefore$ LHS = RHS = $\frac{-16}{3}$

Hence Green's theorem is verified.

4. Verify Green's theorem for $\int_C \frac{1}{y} dx + \frac{1}{x} dy$

where C is the boundary of the region defined by $x=1$, $x=4$, $y=1$ and $y=\sqrt{x}$.

Solⁿ: $\int_C \frac{1}{y} dx + \frac{1}{x} dy = \int P\,dx + Q\,dy$

$$P = \frac{1}{y} \qquad Q = \frac{1}{x}$$

$$\frac{\partial P}{\partial y} = \frac{-1}{y^2} \qquad \frac{\partial Q}{\partial x} = \frac{-1}{x^2}$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-1}{x^2} + \frac{1}{y^2}$$

By Green's theorem

$$\int P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_C \frac{1}{y} dx + \frac{1}{x} dy = \iint_R \left(\frac{-1}{x^2} + \frac{1}{y^2}\right) dx\,dy$$

(a) RHS = $\iint_R \left(\frac{-1}{x^2}+\frac{1}{y^2}\right) dx\,dy$

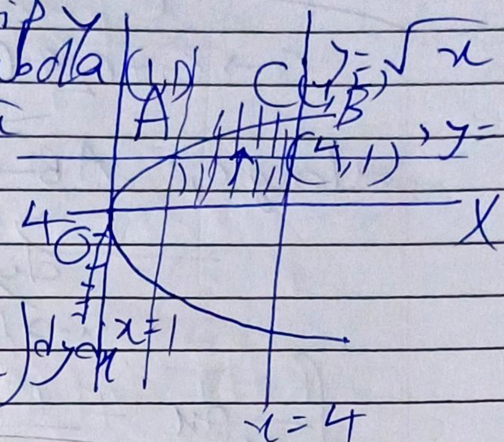
Taking a vertical strip

$y \to$ AB to parabola

$y=1$ to $y=\sqrt{x}$

$x \to$ A to BC

$x=1$ to $x=4$



$$RHS = \int_1^4 \int_1^{\sqrt{x}} \left(\frac{-1}{x^2}+\frac{1}{y^2}\right) dy\,dx$$

$$= \int_1^4 \left(\frac{-y}{x^2}-\frac{1}{y}\right)_1^{\sqrt{x}} dx$$

$$= \int_1^4 \left(\frac{-\sqrt{x}}{x^2}-\frac{1}{\sqrt{x}}\right)-\left(\frac{-1}{x^2}-1\right) dx$$

$$= \int_1^4 \left(\frac{-1}{x^{3/2}}-\frac{1}{\sqrt{x}}+\frac{1}{x^2}+1\right) dx$$

$$= \int_1^4 \left(-x^{-3/2}-x^{-1/2}+x^{-2}+1\right) dx$$

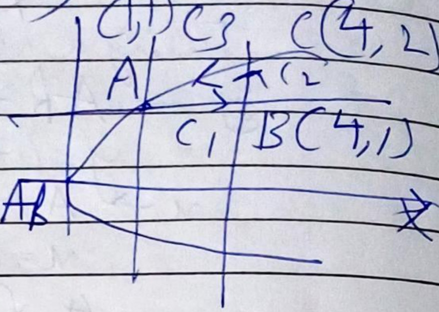
$$= \left(\frac{-x^{-1/2}}{-1/2}-\frac{x^{1/2}}{1/2}+\frac{x^{-1}}{-1}+x\right)_1^4$$

$$= \left(2(4)^{-1/2}-2(4)^{1/2}-(4)^{-1}+4\right) - \left(2(1)^{-1/2}-2(1)^{1/2}-(1)+1\right)$$

$$= \left(\frac{2}{2}-(2\times 2)-\frac{1}{4}+4\right)-\left(2-2-1+1\right)$$

$$= \frac{3}{4}$$

(b) $LHS = \oint_C \left(\frac{1}{y} dx + \frac{1}{x} dy \right)$



$C \to C_1 + C_2 + C_3$

along C_1: $AB \to$ eqⁿ of AB
$y = 1$
$dy = 0$
$x \to 1$ to 4

$$\int_{C_1} \frac{1}{y} dx + \frac{1}{x} dy = \int_1^4 \frac{1}{y} dx = \int_1^4 (1) dx$$
$$= \underline{3}$$

along C_2: $BC \to$ eqⁿ of $BC \to x = 4$
$dx = 0$
$y \to 1$ to 2

$$\int_{C_2} \frac{1}{y} dx + \frac{1}{x} dy = \int \frac{1}{x} dy = \int_1^2 \frac{1}{4} dy$$
$$= \frac{1}{4}$$

along C_3: curve CA: $y = \sqrt{x}$
$x = y^2 \to dx = 2y\,dy$
$y \to 2$ to 1

$$\frac{1}{y} dx + \frac{1}{x} dy = \int_2^1 \left(\frac{2y\,dy}{y} + \frac{1}{y^2} dy \right)$$
$$= \int_2^1 \left(2 + \frac{1}{y^2} \right) dy$$
$$= (2-1) - \left(4 - \frac{1}{2}\right) = \frac{-5}{2}$$

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$LHS = \int \frac{1}{y} dx + \frac{1}{x} dy$

$= \int_{C_1} \frac{1}{y} dx + \frac{1}{x} dy + \int_{C_2} \frac{1}{y} dx + \frac{1}{x} dy$

$+ \int_{C_3} \frac{1}{y} dx + \frac{1}{x} dy$

$= \frac{3+1}{4} - \frac{5}{2} = \boxed{-\frac{3}{4}}$

Hence $LHS = RHS = \frac{3}{4}$

$\therefore$ Green's thrm is verified

5. Verify Green's thr^n in the plane for $\int_C (xy+y^2) dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ & $y = x^2$

Sol^n. $\int (xy+y^2) dx + x^2 dy = \int_C P dx + Q dy$

$\therefore P = xy + y^2 \qquad Q = x^2$

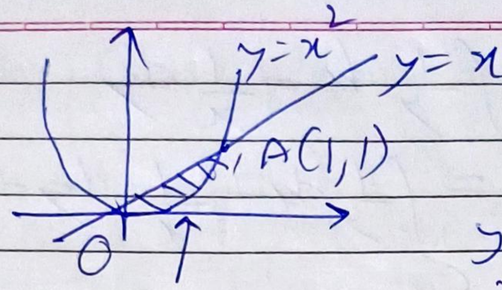
$\frac{\partial P}{\partial y} = x + 2y \qquad \frac{\partial Q}{\partial x} = 2x$

$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - (x+2y)$
$= x - 2y$

By Green's th^m

$\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$

$\int (xy+y^2) dx + x^2 dy = \iint_R (x - 2y) dx\, dy$



Pt. of intersection

$y=x$ & $y=x^2$

$x=x^2$

$x(1-x)=0$

$x=0$ or $x=1$

$y=0$ or $y=1$

$y \to$ parabola to line, $y=x^2$ to $y=x$

$x \to 0$ to 1

$$RHS = \int_0^1\int_{x^2}^{x}(x-2y)\,dy\,dx$$

$$= \int_0^1 (xy-y^2)_{x^2}^{x}\,dx$$

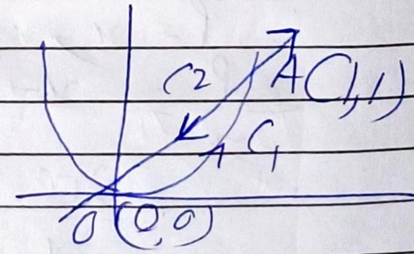
$$= \int_0^1 (x^2-x^2)-(x^3-x^4)\,dx$$

$$= \int_0^1 (x^4-x^3)\,dx = \left(\frac{x^5}{5}-\frac{x^4}{4}\right)_0^1$$

$$= \boxed{\frac{-1}{20}}$$

(b) $LHS = \int_C (xy+y^2)\,dx + x^2\,dy$

$C \to C_1 + C_2$



along C_1: parabola

$y=x^2$

$dy = 2x\,dx$

$x \to 0$ to 1

$$\int_{C_1}(xy+y^2)\,dx + x^2\,dy = \int_0^1 (x^3+x^4)\,dx + x^2(2x\,dx)$$

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$$= \int_0^1 (3x^3 + x^4)\,dx = \left(\frac{3x^4}{4} + \frac{x^5}{5}\right)_0^1$$

$$= \boxed{\frac{19}{20}}$$

along C_2: line AO: $y = x$, $dy = dx$, $x \to 1$ to 0

$$(xy + \cancel{x}y^2)\,dx + x^2 dy = \int_1^0 (x^2 + x^2)\,dx + x^2 dx$$

$$= (x^3)_1^0 = 0 - 1 = -1$$

$$LHS = \int_C (xy + y^2)\,dx + x^2 dy$$

$$= \int_{C_1} (xy + y^2)\,dx + x^2 dy$$

$$= \frac{19}{20} - 1 = \boxed{\frac{-1}{20}}$$

$\therefore$ $LHS = RHS = \dfrac{-1}{20}$

Hence verified Green's thm

6. Using Green's th^m evaluate $\oint (e^{x^2} - xy)\,dx - (y^2 - ax)\,dy$ where C is the circle $x^2 + y^2 = a^2$.

Sol^n. $\int (e^{x^2} - xy)\,dx - (y^2 - ax)\,dy$

$$= \int P\,dx + Q\,dy$$

$$P = e^{x^2} - xy \qquad Q = -y^2 + ax$$

$$\frac{\partial P}{\partial y} = -x \qquad \frac{\partial Q}{\partial x} = a$$

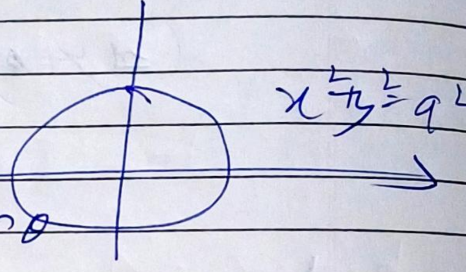
$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = a - (-x) = a + x$$

By Green's th^m

$$\int_C (e^{x^2} - xy)\,dx - (y^2 - ax)\,dy$$

$$= \iint_R (a+x)\,dx\,dy$$

$$\iint_R (a+x)\,dx\,dy$$



Using polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r \to 0$ to a

$\theta \to 0$ to 2π

$$\iint (a+x)\,dx\,dy = \int_0^{2\pi}\int_0^a (a + r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^a (ar + r^2\cos\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{a \cdot a^2}{2} + \frac{a^3}{3}\cos\theta\,d\theta$$

$$= a^3 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{3}\cos\theta\right) d\theta$$

$$= a^3 \left[\frac{\theta}{2} + \frac{1}{3}\sin\theta\right]_0^{2\pi}$$

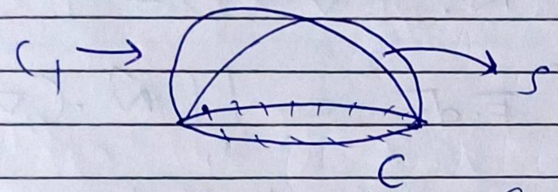
$$= a^3 \left(\frac{2\pi}{2} + \frac{1}{3}\sin 2\pi - 0 - \frac{1}{3}\sin 0\right)$$

$= \underline{a^3\pi}$

Stoke's Theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \hat{N}\cdot(\nabla\times\bar{F})\, ds$$

$\hat{N}$ = unit outward normal to an element ds of S.



Consider surface S_1, whose boundary is same as the boundary of the given surface S. Then,

$$\iint_S \hat{N}\cdot(\nabla\times\bar{F})\, ds = \iint_{S_1} \hat{N}\cdot(\nabla\times\bar{F})\, ds_1$$

$$= \oint_C \bar{F}\cdot d\bar{r}$$

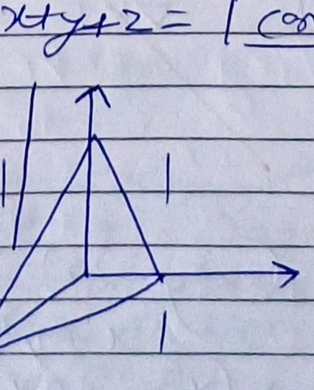
→ Evaluate $\int \bar{F}\cdot d\bar{r}$, $\bar{F} = y\,i + z\,j + x\,k$ over $S: x^2+y^2 = 1-z,\ z \geq 0$.

Solⁿ: - paraboloid : $x^2+y^2 = az$
- tetrahedron : $x+y+z=1$ cone: $x^2+y^2=z^2$

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

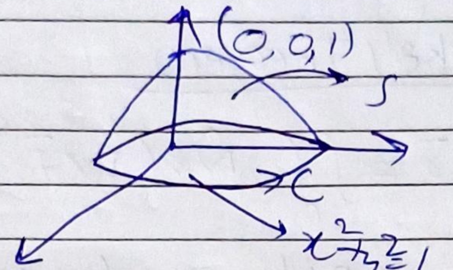
$= i(0-1) - j(1-0) + k(0-1)$

$= -i-j-k$



$S: x^2+y^2 = 1-z = -(z-1)$

$$x^2+y^2=-z$$



Consider $S_1: x^2+y^2 \le 1$

$\hat{N} = \hat{k}$

$\therefore \hat{N}\cdot(\nabla\times F) = -1$

$$\therefore \int_C F\cdot d\bar{r} = \iint_{S_1} \hat{N}\cdot(\nabla\times F)\, ds_1$$

$$= \iint_{S_1} -1\, dx\, dy$$

$$= -\iint_{S_1} dx\,dy = -\pi$$

Q. $\int_C F\cdot d\bar{r}$

$\bar{F} = (y^2+z^2-x^2)i + (z^2+x^2-y^2)j + (x^2+y^2-z^2)k$

$S: x^2+y^2-2ax+az=0$ above the plane $z=0$.

Sol^n: $\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}$

$= i(2y-2z) - j(2x-2z) + k(2x-2y)$

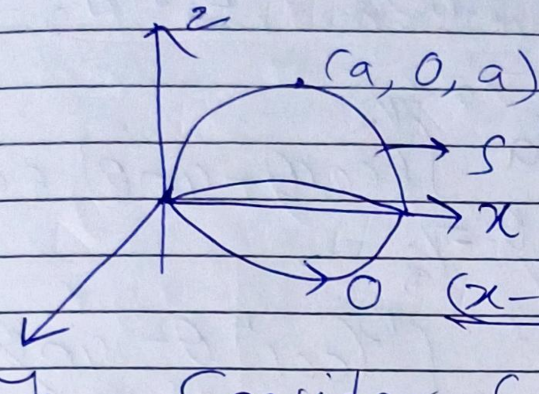
$x^2+y^2-2ax = -az$

$\Rightarrow (x^2-2ax+a^2)+y^2 = -az+a^2$

$\Rightarrow (x-a)^2+y^2 = -a(z-a)$

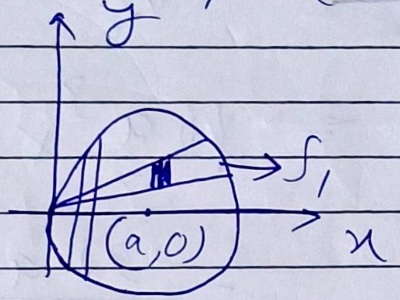
→ $x^2+y^2 = \cancel{a} - a^2$

where vertex $\equiv (a, 0, a)$



$(x-a)^2+y^2 = a^2$

Consider $S_1 : (x-a)^2+y^2 \le a^2$



$\hat{N} = \hat{k}$

$\therefore (\nabla \times F) \cdot \hat{N} = 2x - 2y$

$\therefore \int F \cdot d\bar{r} = \iint_{S_1} \hat{N} \cdot (\nabla \times F)\, ds$

$= \iint_{S} (2x-2y)\, dx\, dy$

Put $x = r\cos\theta$
$y = r\sin\theta$ } must do simple trick

$\underline{dx\,dy = r\,dr\,d\theta}$

$(x-a)^2+y^2 = a^2$

$x^2 - 2ax + a^2 + y^2 = a^2$

$\cancel{x^2}\ r^2 = 2ar\cos\theta$

~~Put $x = r\cos\theta$~~

~~$y = r\sin\theta$~~

~~$dx\,dy =$~~

$\iint_{S_1} 2x - 2y\ dx\,dy$

$= 2\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2a\cos\theta} r(\cos\theta - \sin\theta)\, r\,dr\,d\theta$

$$= 2\int_{\theta=-\frac{\pi}{2}}^{\pi/2} (\cos\theta - \sin\theta)\left[\frac{r^3}{3}\right]_{r=0}^{a\cos\theta} d\theta$$

$$= \frac{2}{3} \times 8a^3 \int_{\theta=-\pi/2}^{\pi/2} (\cos\theta - \sin\theta)\cos^3\theta \, d\theta$$

$$= \frac{16a^3}{3} \int_{-\pi/2}^{\pi/2} (\cos^4\theta - \cancel{\sin\theta\cos^3\theta}^{\,0})\, d\theta$$

$$= \frac{16a^3}{3} \times 2\int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{32a^3}{3} \times \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

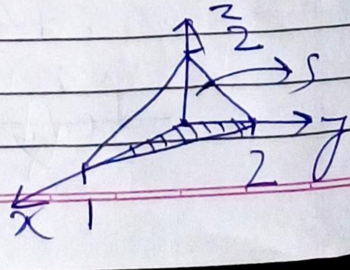
$$= 2\pi a^3$$

Q. Using Stoke's theorem, find integral $\oint_C \bar{F} \cdot d\bar{r}$ along C,

$\bar{F} = (x+y)\, i + (y+z)\, j - x\, k$

$S: 2x + y + z = 2$ in the first octant.

Sol^n $\nabla \times \bar{F} = -i + j - k$



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consider $\phi = 2x + y + z - 2$

Normal to $\phi = \nabla\phi$

$= 2i + j + k$

$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i + j + k}{\sqrt{6}}$

$\hat{N} \cdot (\nabla \times F) = \frac{(2i+j+k)\cdot(-i+j-k)}{\sqrt{6}}$

$= -\frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}$

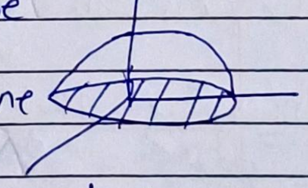
$= -\frac{2}{\sqrt{6}}$

$ds = \ldots$

(Note: Actual formula: 1 Projection of xy-plane

2 projection on yz-plane

3 projection on xz-plane



1. $\frac{dx\,dy}{|\hat{N}\cdot\hat{k}|}$ $\therefore ds = 1 \cdot \frac{dx\,dy}{\frac{1}{\sqrt{6}}} = \sqrt{6}\,dx\,dy$

2. $\frac{dy\,dz}{|\hat{N}\cdot\hat{i}|}$ $\therefore ds = \iint -\frac{2}{\sqrt{6}}\sqrt{6}\,dx\,dy$

$= -2 \iint_{\Delta OAB} dx\,dy$

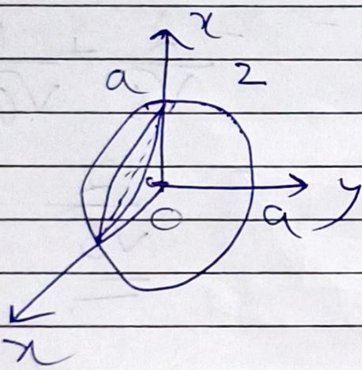
3. $\frac{dx\,dz}{|\hat{N}\cdot\hat{j}|}$ $= -2 \times \frac{1}{2} \times \frac{1}{2} \times 2$

$= -2$

Q. Apply stoke's theorem to evaluate:
$\int y\,dx + z\,dy + x\,dz$, where c is a curve of intersection of
c: $x^2+y^2+z^2 = a^2$ and $x+z=a$.

$\int d\hat{\ }$ $\qquad \bar{F} = yi + zj + xk$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$



$$= i(-1) - j(1-0) + k(0-1)$$

$$= \underline{-i-j-k}$$

Consider $\phi = x+z-a$

Normal to $\phi = \nabla\phi$

$$= \underline{i+k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{i+k}{\sqrt{2}}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{1/\sqrt{2}} = \sqrt{2}\,dx\,dy$$

$$\hat{n}\cdot(\nabla\times F) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

$$\therefore \oint_c F\cdot d\bar{r} = \iint \hat{n}\cdot(\nabla\times\bar{F})\,ds$$

$$= \iint -\sqrt{2}\,\sqrt{2}\,dx\,dy$$

$$= -2\iint dx\,dy$$

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Solving $x^2+y^2+z^2 = a^2$ & $x+z=a$.

$\Rightarrow x^2+y^2+(a-x)^2 = a^2$

$\Rightarrow 2x^2+y^2-2ax = \cancel{a^2}-\cancel{a^2} = 0$

$\cancel{(\sqrt{2}x)^2+y^2+}$

$\Rightarrow 2(x^2-ax)+y^2 = 0$

$\Rightarrow 2\left(x^2-ax+\frac{a^2}{4}\right)+y^2 = \frac{a^2}{2}$

$\Rightarrow 2\left(x-\frac{a}{2}\right)^2+y^2 = \frac{a^2}{2}$

$\Rightarrow \left(x-\frac{a}{2}\right)^2+\frac{y^2}{2} = \frac{a^2}{4}$

$\Rightarrow \frac{\left(x-\frac{a}{2}\right)^2}{(a/2)^2}+\frac{y^2}{2(a^2/4)} = 1$

$\Rightarrow \frac{(x-a/2)^2}{(a/2)^2}+\frac{y^2}{\left(\frac{a}{\sqrt{2}}\right)^2} = 1$

$= \frac{X^2}{A^2}+\frac{Y^2}{B^2} = 1$

$\int F\cdot dr = -2(\text{Area of ellipse})$

$= -2(\pi ab)$

$= -\cancel{2}\pi\left(\frac{a}{\cancel{2}}\right)\left(\frac{a}{\sqrt{2}}\right) = -\frac{\pi a^2}{\sqrt{2}}$

Note: online lec prob. on z-transform on Sunday.

- Divergence theorem - Use it to find $\iint_S \hat{N}\cdot\bar{F}\,ds$,

$\bar{F} = xi + yj + z^2 k$, S closed surface

Bounded by cone $x^2+y^2 = z^2$ & $z=1$.

Sol^n: $\iint_S \hat{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot F\,dv$

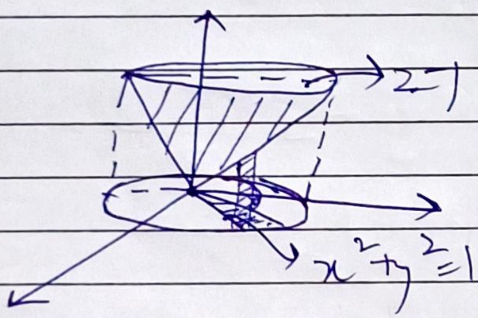
$$\nabla\cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1+1+2z$$

$$= \underline{\underline{2+2z}}$$

$$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$ds = \frac{dx\,dy}{|\hat{N}\cdot k|}$$



$$\iint_S \hat{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot F\,dv$$

$$= \iiint_V \cancel{2z}\,(2+2z)\,dx\,dy\,dz$$

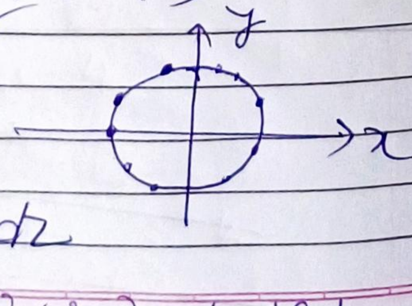
Put $x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$= \int\int_{r=0}\int_{z=r}^{1} (\cancel{z}+2z)\,r\,dr\,d\theta\,dz$$



Put $x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1}(2+2z)\,r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^1 (2z+z^2)\Big|_r^1 \, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (3-2r-r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{3r^2}{2}-\frac{2r^3}{3}-\frac{r^4}{4}\right)_0^1 d\theta$$

$$= \int_0^{2\pi}\left(\frac{3}{2}-\frac{2}{3}-\frac{1}{4}\right) d\theta$$

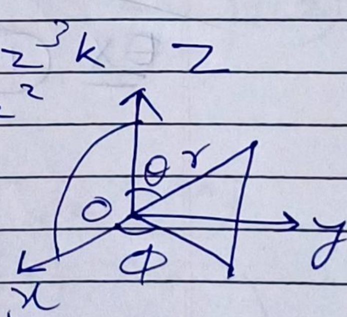
$$= \frac{7}{12}\, 2\pi$$

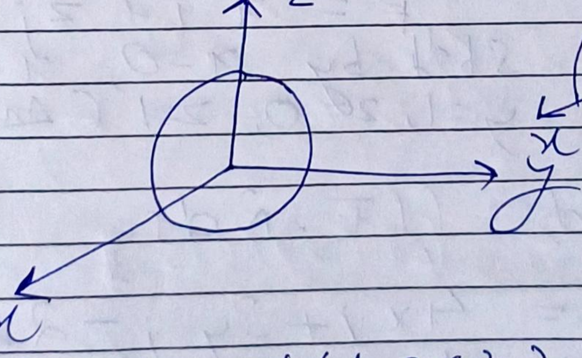
$$= \frac{7\pi}{6}$$

Q. $\iint_S \hat{N}\cdot\bar{F}\, ds$, $x^2+y^2+z^2=a^2$.

$$\bar{F} = x^3 i + y^3 j + z^3 k$$

Soln. $\nabla\cdot\bar{F} = 3x^2+3y^2+3z^2$





$$\iint_S \hat{N}\cdot\bar{F}\, ds = \iiint_V 3(x^2+y^2+z^2)\, dx\,dy\,dz$$

Put $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

entire sphere : $0 \le r \le a$ 1^{st} oct: $0 \le r \le a$

$0 \le \theta \le \pi$ $0 \le \theta \le \pi/2$

$0 \le \phi \le 2\pi$ $0 \le \phi \le \pi/2$

Hemisphere : $0 \le r \le a$

$0 \le \theta \le \pi/2$

$0 \le \phi \le 2\pi$

$$= 3 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 3 \times 2\pi \int_{\theta=0}^{\pi} \frac{a^5}{5} \sin\theta \, d\theta$$

$$= \frac{6\pi a^5}{5} [-\cos\theta]_0^{\pi}$$

$$= \frac{6\pi a^5}{5}(1+1) = \frac{12\pi a^5}{5} \text{ Ans.}$$

Ex ① $\iint \hat{n} \cdot F \, ds$

$\bar{F} = x^2 i + zj + yzk$

sbdd by $x=0$, $x=1$, $y=0$

$y=1$, $z=0$, $z=1$ (Ans. 3/2)

② Find $\iint \bar{F} \cdot \hat{n} \, ds$

$\bar{F} = 4x i + 3y j - 2z k$

sbdd by $x=0, y=0, z=0$

& $2x+2y+2z=4$

(Ans. 40/3)

③ $\bar{F} = 4x i - 2y^2 j + z^2 k$

sbdd by $x^2+y^2=4$, $z=0$, $z=3$

(Ans. 84π)

Sol^n ② $\nabla \cdot \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x & 3y & -2z \end{vmatrix}$

$4+3-2$

$= \underline{5}$

~~$= i(0-0) - j(0-0) + k(0)$~~

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$=$

Q. If $Z(f(k)) = F(z)$

then $Z(k^n f(k)) = \left(-z\frac{d}{dz}\right)^n F(z)$

Ex. Find $Z(k^2 e^{ak})$.

Sol^n. $Z(k^2 e^{ak}) = \left(-z\frac{d}{dz}\right)^2 \cdot Z(e^{ak})$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-e^a}\right)$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\left(\frac{z-e^a-z}{(z-e^a)^2}\right)\right)$$

$$= \left(-z\frac{d}{dz}\right)\left(\frac{ze^a}{(z-e^a)^2}\right)$$

$$= -ze^a\left(\frac{(z-e^a)^2 - z(2(z-e^a))}{(z-e^a)^4}\right)$$

$$= \frac{-ze^a(z-e^a)((z-e^a)-2z)}{(z-e^a)^4}$$

$$= \frac{ze^a \cancel{(z-e^a)}(z+e^a)}{(z-e^a)^3}$$

Convolution th^m

If $z^{-1}[F(z)] = f(n)$

& $z^{-1}[G(z)] = g(n)$.

Then $z^{-1}[F(z)G(z)] = \sum_{m=0}^{n} f(m)\,g(n-m)$

Ex 1: $z^{-1}\left(\frac{z^2}{(z-a)(z-b)}\right)$

Sol^n: $\sum_{n=0}^{\infty} \frac{z}{z-a}$

$F(z) = \frac{z}{z-a}$

$G(z) = \frac{z}{z-b}$

$\therefore z^{-1}(F(z)) = a^n = f(n)$

and $z^{-1}(G(z)) = b = g(n)$

$$z^{-1}[F(z)(G(z)] = \sum_{m=0}^{n} a^m b^{n-m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$a < b$

$$= b^n\left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \dots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n \frac{1(1-(a/b)^n)}{1-(a/b)}$$

$$= b^n \frac{\left(1-\left(\frac{a}{b}\right)^n\right)}{\frac{b-a}{b}} = \frac{b^n\left(\left(\frac{a}{b}\right)^{n+1}-1\right)}{\left(\frac{a}{b}-1\right)b^{n+1}}$$

$$= \frac{\frac{b^n}{b^{n+1}}(a^{n+1}-b^{n+1})}{\frac{a-b}{b}} = \frac{(a^{n+1}-b^{n+1})b/b}{a-b}$$

$$= \frac{a^{n+1}-b^{n+1}}{a-b}$$

Ans

Ex. 2: $z^{-1}\left(\frac{z^3}{(z-a)^3}\right)$

Sol^n: First we find $z^{-1}\left(\frac{z^2}{(z-a)^2}\right)$

Let $F(z) = \frac{z}{z-a} = G(z)$

$z^{-1}[F(z)] = a^n = z^{-1}[G(z)]$

$\therefore z^{-1}(F(z)(G(z))$

$$= \sum_{m=0}^{n} a^m a^{n-m}$$

$$= \sum_{m=0}^{n} a^n = \cancel{0 + (a \cdot n)\frac{(n+1)}{2}a^n}$$

$$= a^n \sum_{m=0}^{n} 1 = a^n(n+1)$$

Now we find $z^{-1}\left(\frac{z^3}{(z-a)^3}\right)$

$= \sum_{m=0}^{n} a^{m}(n+1) \cdot a^{n-m}$

$= \sum_{m=0}^{n} \frac{a^{2n}(n+1)}{a^{m}}$

$= \sum_{m=0}^{n} a^{m} a^{n-m} (m+1)$

$~~= a^{2n}(n+1) \sum_{m=0}^{n} a^{-m}~~$

$= a^{n} \sum_{m=0}^{n} (m+1)$

$~~= a^{2n}(n+1)~~$

$= a^{n} \frac{(n+1)(n+2)}{2}$

Ex.1 $z^{-1}\left(\frac{8z^2}{(2z-1)(4z-1)}\right)$

$= z^{-1}\left[\frac{8z^2}{8\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]$

$= z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$; $a = \frac{1}{2}, b = \frac{1}{4}$

Extra

Change of scale

$z\{f(k)\} = F(z)$ then

$z\{a^{k} f(k)\} = F\left(\frac{z}{a}\right)$

Shifting property:

$z\{f(k)\} = F(z) \Rightarrow z\{f(k \pm n)\}$

$= z^{\pm n} F(z)$

Multiply by k:

If $F(z) = z\{f(k)\}$ then

~~$z\{f(k+n)\} = z^{\pm n} F(z)$~~

then $z\{k^{n} f(k)\} = \left(-z\frac{d}{dz}\right) F(z)$