On the Numerical Instability of an LCMV Beamformer for a Uniform Linear Array

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1. Introduction

Motivation

- Two well known conditions for the existence of the inverse term in LCMV solution:
  1) Noise PSD matrix should be full rank.
  2) Columns of the constraint matrix should be linearly independent.
- The conditions are generic for mathematical operations involving matrix inversions.
- No previous study provides any insight into the influence of the geometric setup of the sound scene on the stability of the LCMV solution.

Aim

- Study the solution of an LCMV beamformer for a ULA, with directional constraints given by steering vectors.
- Analyse the columns of the constraint matrix and present an analytic expression to determine the frequency where the LCMV fails to satisfy the constraints.
- Provide insight into the behaviour and applicability of the LCMV beamformer beyond the spatial aliasing frequency.

2. Problem Formulation

- Consider a uniform linear array (ULA) of M microphones located at $\mathbf{d}_{1\ldots M}$. The microphone signals are given by

$$\mathbf{y}(f) = \underbrace{\sum_{l=1}^{L} \mathbf{x}_l(f)}_{\mathbf{x}(f)} + \mathbf{x}_\text{v}(f)$$

- Assuming signal components are mutually uncorrelated

$$\begin{aligned}\mathbf{\Phi}_\mathbf{y}(f) &= \text{E}\{\mathbf{y}(f)\mathbf{y}^\text{H}(f)\} \\ &= \mathbf{\Phi}_\mathbf{x}(f) + \mathbf{\Phi}_\text{v}(f)\end{aligned}$$

- Desired signal

$$Z(f) = \sum_{l=1}^{L} Q^*(\theta_l, f) X_l(f, \mathbf{d}_1)$$

- Estimate of the desired signal

$$\hat{Z}(f) = \mathbf{w}^\text{H}(f)\mathbf{y}(f)$$

Definitions

- Microphone signal of l-th plane wave
  $\mathbf{x}_l(f) = [X_l(f, \mathbf{d}_1) \ldots X_l(f, \mathbf{d}_M)]^T$
- Sound pressure of l-th plane wave
  $\mathbf{x}_l(f) = \mathbf{a}(\theta_l, f) X_l(f, \mathbf{d}_1)$
- m-th element of steering vector
  $a_m(\theta_l, f) = \exp\{-j\kappa\,(m-1)d\cos\theta_l\}$
- Noise PSD matrix
  $\mathbf{\Phi}_\text{v} = \sigma_v^2(f)\mathbf{I}$
- Direction dependent gain
  $Q(\theta_l, f)$
- Filter weights
  $\mathbf{w}(f)$
- Constraint matrix
  $\mathbf{A}(f) = [\mathbf{a}(\theta_1, f), \ldots, \mathbf{a}(\theta_L, f)]$

LCMV filter

- Formulation: $\mathbf{w}_\text{LCMV}(f) = \arg\min_{\mathbf{w}} \mathbf{w}^\text{H}\mathbf{\Phi}_\text{v}(f)\mathbf{w}$ s.t. $\mathbf{w}^\text{H}(f)\mathbf{A}(f) = \mathbf{q}^\text{H}(f)$
- Solution: $\mathbf{w}_\text{LCMV}(f) = \mathbf{\Phi}_\text{v}^{-1}\mathbf{A}(f)\left[\mathbf{A}^\text{H}(f)\mathbf{\Phi}_\text{v}^{-1}\mathbf{A}(f)\right]^{-1}\mathbf{q}(f)$
- Analyze the columns of the constraint matrix, $\mathbf{A}(f)$, to derive an analytic expression for determining the frequency where the columns are linearly dependent, leading to instability in the LCMV solution.

3. Non-existence of solution

- Given a pair of DOAs, θ_s and θ_t, the lowest frequency where the columns of the constraint matrix are linearly dependent is given by

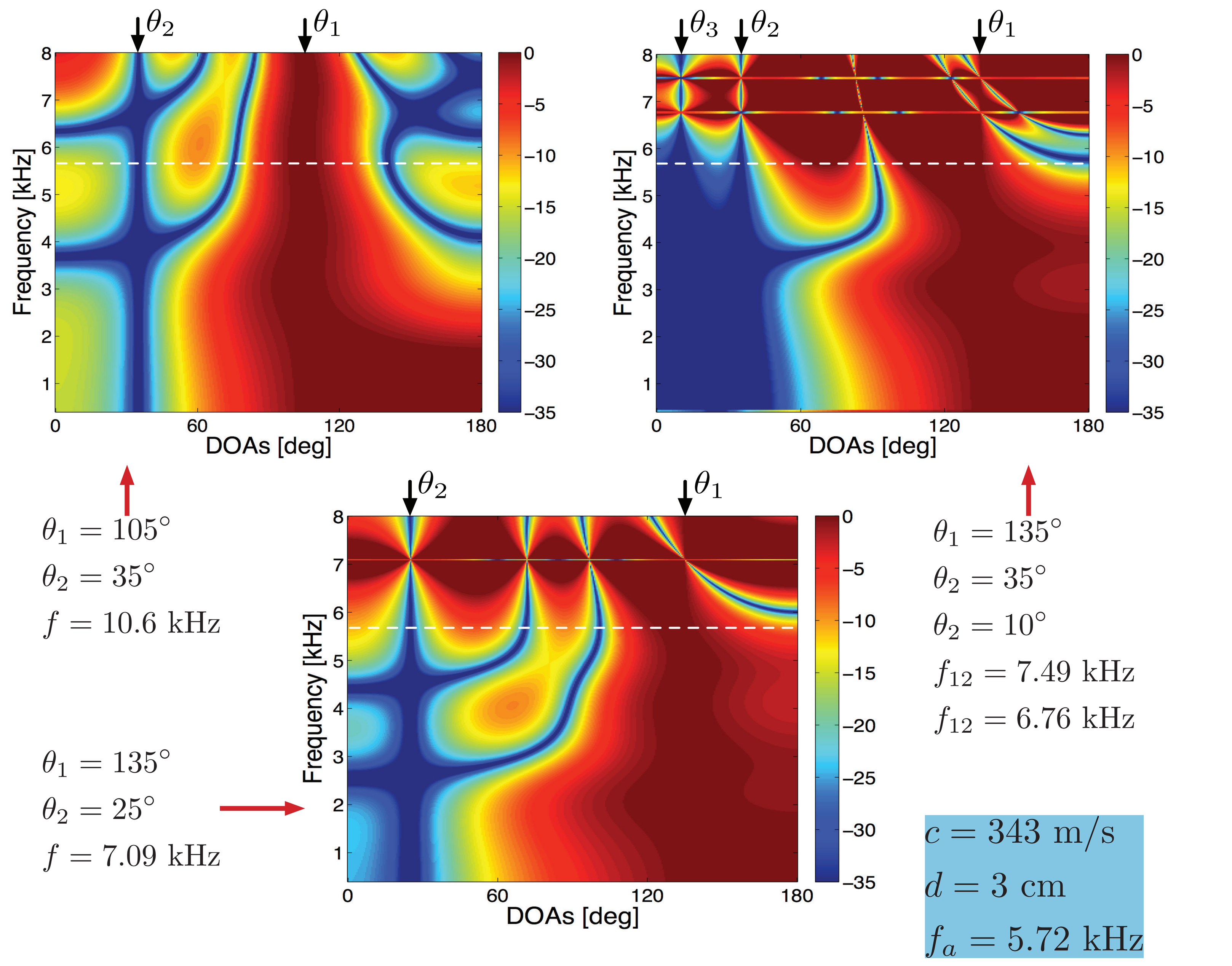
$$f = \frac{c}{d\,|\cos\theta_t - \cos\theta_s|}$$

- In terms of the spatial aliasing frequency

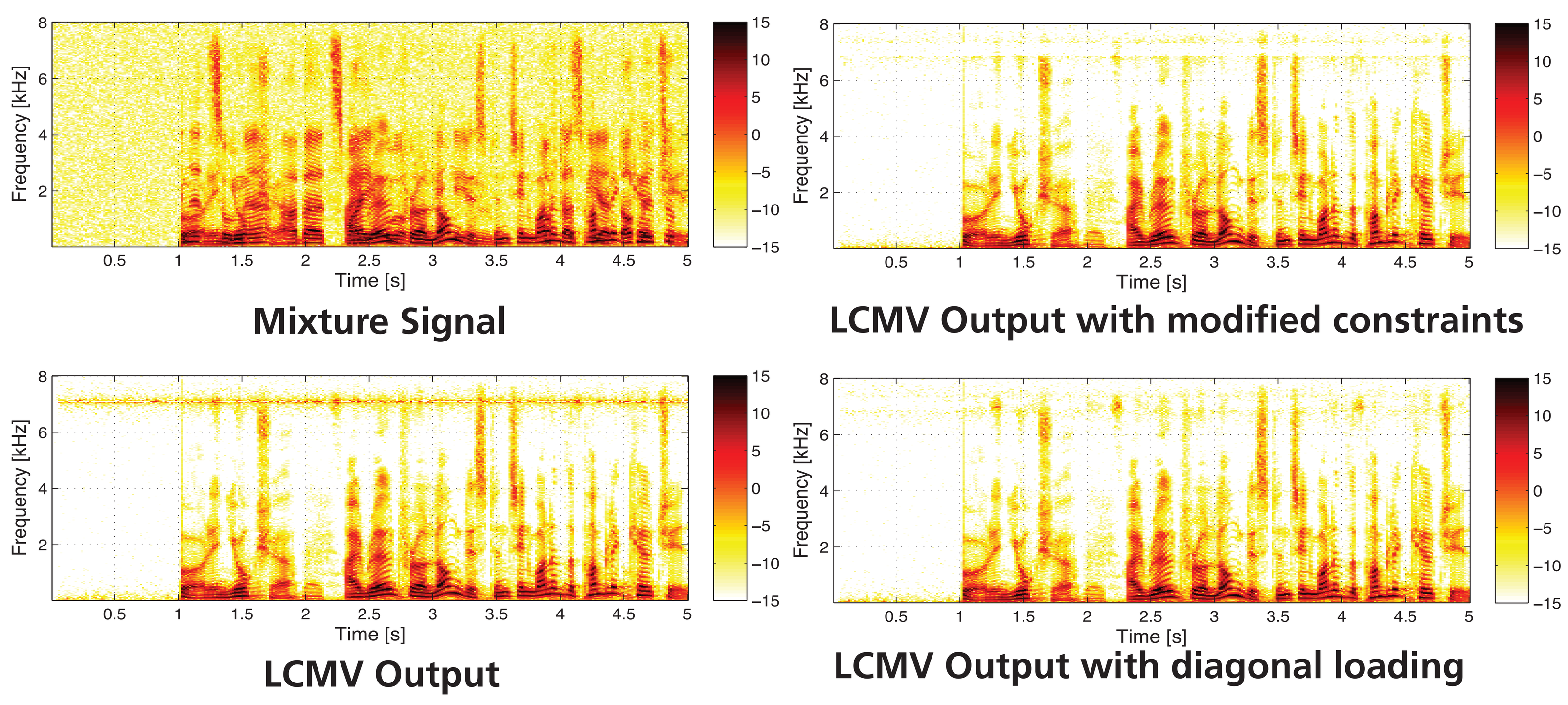
$$f = \frac{f_a}{\left|\sin\left(\frac{\theta_s+\theta_t}{2}\right)\sin\left(\frac{\theta_s-\theta_t}{2}\right)\right|} \quad \text{where } f_a = c/(2d)$$

- Two important observations:
  1) The frequency of instability lies above the spatial aliasing frequency.
  2) The instability also occurs at integer multiples of the lowest frequency of instability.

4. Power Patterns



5. Simulation Experiment



Mixture Signal — LCMV Output with modified constraints — LCMV Output — LCMV Output with diagonal loading

$M = 4$, $d = 3$ cm, $f_a = 5.72$ kHz

$\theta_1 = 135^\circ$, $\theta_2 = 25^\circ$, $f = 7.09$ kHz

Fallback solutions

- Modify the directional constraints: For our experiment, the modified main beam direction was set to be $\theta_1 = 120^\circ$ while keeping the null in the original direction.

  Problem: All directional signal components for the specific frequency are suppressed.

- Diagonal loading

$$\mathbf{w}_\text{LCMV}(f) = \mathbf{\Phi}_\text{v}^{-1}\mathbf{A}(f)\left[\mathbf{A}^\text{H}(f)\mathbf{\Phi}_\text{v}^{-1}\mathbf{A}(f) + \alpha\mathbf{I}\right]^{-1}\mathbf{q}(f)$$

where

$$\alpha = \lambda\frac{1}{L}\text{tr}\left\{\mathbf{A}^\text{H}(f)\mathbf{\Phi}_\text{v}^{-1}\mathbf{A}(f)\right\}$$

6. Conclusions

- The derived analytic expression gives the exact frequency(s) where the LCMV fails to provide a stable solution.
- The analytic expression also establishes a relation between the sound source locations and the numerical stability of the LCMV beamformer.
- The provided solutions mitigate the instability problem, however they fail to satisfy the original directional constraints.