Assignment 2.

1. $|b| \leq a$ iff $-a \leq b \leq a$.

Suppose that $|b| \leq a$.

If $b \geq 0$ then $|b| = b \leq a$. — (a)

If $b < 0$ then $|b| = -b \leq a$

$\Rightarrow -a \leq b$ — (a^*)

Hence (a) and (a^*) together imply

$-a \leq b \leq a$ whenever $|b| \leq a$.

Conversely, suppose $-a \leq b \leq a$.

If $b \geq 0$ then $|b| = b \leq a$ — $(*)$

If $b < 0$ then $|b| = -b$ and

since $-a \leq b \Rightarrow -b \leq a$

$\therefore$ in this case $|b| \leq a$ — $(*^2)$

$\therefore$ $(*)$ and $(*^2)$ imply that $|b| \leq a$

whenever $-a \leq b \leq a$.

(ii) By Triangle inequality we have

$|x+y| \leq |x| + |y| \quad \forall\, x, y \in \mathbb{R}$.

substituting $x + y = a$, $y = b$

we get $x = a - b$,

$\therefore \quad |a| \leq |a-b| + |b|$

$\Rightarrow \quad |a| - |b| \leq |a-b|$ — $(**)$

Again, substituting $x+y=b$, $y=a$ we get

$$|b| \le |b-a| + |a|$$

$$\Rightarrow -(|a|-|b|) \le |b-a| = |a-b|$$

$$\therefore \quad -|a-b| \le |a|-|b| \quad \text{---} (**^2)$$

i.e by $(**)$ and $(**^2)$,

$$-|a-b| \le |a|-|b| \le |a-b|$$

Hence by part (i) of the problem it follows that

$$\big||a|-|b|\big| \le |a-b|.$$

2. Let $a, b \in \mathbb{R}$. If $a \le b_1$ for every $b_1 > b$, then $a \le b$.

On the contrary assume that $a > b$.

Then $a-b > 0$ and hence $\frac{a-b}{2} > 0$.

$\Rightarrow \quad b + \frac{a-b}{2} > b$, hence by given condition,

$$a \le b + \frac{a-b}{2}$$

i.e $a \le \frac{a+b}{2} < \frac{a+a}{2}$ ($\because$ by assumption $b<a$.)

This implies $a < a$ which is weird.

Hence our assumption that $a > b$ is wrong.

3.(1) Supremum and infimum of a set are uniquely defined.

Let S be a subset of $\mathbb{R}$ that is bounded above. Then by completeness axiom, supremum S exists.

Let $s_1, s_2 \in \mathbb{R}$ be such that

$$S_1 = \sup S, \quad S_2 = \sup S.$$

If $S_1 \neq S_2$, then by order axiom, either $S_1 < S_2$ or $S_1 > S_2$.

Without loss of generality assume that $S_2 > S_1$. i.e $S_2 - S_1 > 0$.

Since s_2 is a supremum and $\epsilon = s_2 - s_1 > 0$, $\exists$ $s \in S$ such that

$$s_2 - \epsilon < s \leq s_2$$

$$\Rightarrow s_2 - (s_2 - s_1) < s \leq s_2$$

$$\Rightarrow s_1 < s \leq s_2 \quad — \quad (*).$$

By $(*)$ $\exists$ $s \in S$ such that $s > s_1$ which contradicts that s_1 is an upper bound of S.

Hence s_1 and s_2 have to be equal.

Proof for uniqueness of infimum is

similar. (complete it yourself! In case you get stuck discuss with me or tutors.)

3(ii). Let S be a finite subset of $\mathbb{R}$.
By order axiom the elements of S can be written in strictly increasing order

$$s_1 < s_2 < \dots < s_N \quad \text{where } N = \# S.$$

Since $s \geq s_1 \ \forall s \in S$ and for $\epsilon > 0$ $\quad s_1 < s_1 + \epsilon \ , \ s_1 \in S$

$\therefore \quad s_1 = \inf S.$

Likewise, $\because \ s_N \geq s \ \ \forall s \in S$ and for $\epsilon > 0$, $\quad s_N - \epsilon < s_N \ , \ s_N \in S$,

$\therefore \quad s_N = \sup S.$

This show $\sup S, \inf S \in S$ for S finite.

iii. If $S \neq \emptyset$ and $b \in \mathbb{R}$ st $b < s$ for all $s \in S$, then S is bounded below. Hence using completeness axiom we know $\inf S$ exists.

Let $\quad m = \inf S.$

Then $\quad m \leq s \quad \forall s \in S \quad — *$

and for $\epsilon > 0$, $\exists \ s \in S$ such that

$$m \leq s < m + \epsilon. \quad — *^2$$

From $*$, it follows that

$$-s \leq -m \quad \forall\, s \in S. \quad — (a)$$

$\therefore -m$ is an upper bound for the set $-S = \{-s : s \in S\}$.

From $*^2$, it follows that for $\epsilon > 0$, $\exists$ $s \in S$ s.t,

$$-m - \epsilon < -s \leq -m. \quad — (a^2)$$

From (a) and (a^2) it follows that

$$-m = \sup(-S)$$

i.e $\quad \inf S = m = -\sup(-S)$.

4. (i) Let $S = \left\{ \frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N} \right\}$

$$= \bigcup_{m \in \mathbb{N}} \left\{ \frac{1}{n} - \frac{1}{m} : n \in \mathbb{N} \right\}.$$

$$= \bigcup_{n \in \mathbb{N}} \left\{ \frac{1}{n} - \frac{1}{m} : m \in \mathbb{N} \right\}.$$

Let $S_m = \left\{ \frac{1}{n} - \frac{1}{m} : n \in \mathbb{N} \right\}$

and $S^n = \left\{ \frac{1}{n} - \frac{1}{m} : m \in \mathbb{N} \right\}$.

For a fixed $m \in \mathbb{N}$, observe that the set S_m is bounded below by $-\frac{1}{m}$.

$\because \frac{1}{n} > 0 \quad \forall\, n \in \mathbb{N}$

and $\quad -\frac{1}{m} < \frac{1}{n} - \frac{1}{m}$

claim: $\quad -\frac{1}{m} = \inf S_m$.

Given $\epsilon > 0$, by Archimedean property,

$\exists \quad K \in \mathbb{N}$ st $\quad K\epsilon > 1$ i.e $\epsilon > \frac{1}{K}$.

$$\therefore \quad \frac{1}{K} - \frac{1}{m} < -\frac{1}{m} + \epsilon$$

which shows that $-1/m = \inf S_m$.

i.e $\quad -\frac{1}{m} < \frac{1}{n} - \frac{1}{m} \quad \forall \; n \in \mathbb{N}$.

Now notice that

Since $\quad -1 < -\frac{1}{2} < -\frac{1}{3} \ldots\ldots$

we see that

$$\inf S_1 < \inf S_2 < \ldots\ldots$$

$\therefore$ $\inf S_1$ is a lower bound for $\bigcup_{m \in \mathbb{N}} S_m$

$\Rightarrow \quad -1 \leq s \quad \forall \; s \in \bigcup_{m \in \mathbb{N}} S_m = S$ —(b)

claim: $\quad -1 = \inf S$.

Given $\epsilon > 0$, by Archimedean property

$\exists \quad K \in \mathbb{N}$ st $\quad K\epsilon > 1 \Rightarrow \epsilon > \frac{1}{K}$.

$\Rightarrow \quad \exists \; K \in \mathbb{N}$ st $\quad (K+1)\epsilon > 1 \Rightarrow \epsilon > \frac{1}{K+1}$

$$\therefore \quad -1 + \frac{1}{K+1} < -1 + \epsilon$$

$$\because \quad -1 + \frac{1}{K+1} \in S_1 \subseteq S$$

we see that for every $\epsilon > 0$, $\exists$ $s \in S$ such that $-1 < s < -1+\epsilon$

Hence $-1 = \inf S$.

Now for fixed n, observe that the set $S^n = \{ \frac{1}{n} - \frac{1}{m} : m \in \mathbb{N} \}$ is bdd above by $1/n$. $\therefore$ by completeness axiom $\sup S^n$ exists.

claim: $\sup S^n = \frac{1}{n}$.

given $\epsilon > 0$, $\exists$ K st $K\epsilon > 1 \Rightarrow \epsilon > \frac{1}{K}$.

$$\therefore \quad \frac{1}{n} - \epsilon < \frac{1}{n} - \frac{1}{K} < \frac{1}{n}$$

$\because$ $\frac{1}{n} - \frac{1}{K} \in S^n$, it follows that

$$\sup S^n = \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

$$\because \quad \frac{1}{n} < \frac{1}{n-1} < \cdots < 1$$

$$\therefore \quad \sup S^1 > \sup S^n \quad \forall n \in \mathbb{N}.$$

implying $\sup S^1 > s$ $\forall s \in \bigcup_{n \in \mathbb{N}} S^n$.

$\because$ $1 = \sup S^1$, for $\epsilon > 0$ $\exists$ $s \in S^1 \subseteq S$ st

$$1 - \epsilon < s < 1$$

Hence $1 = \sup S$.

(ii). $S = \left\{\cos \frac{n\pi}{3} : n \in \mathbb{N}\right\}$

$= \left\{\cos \frac{\pi}{3}, \cos \frac{2\pi}{3}, \cos \pi, \cos \frac{4\pi}{3} \ldots\right\}$

$= \left\{\frac{1}{2}, -\frac{1}{2}, -1\right\}$

$\therefore$ $\sup S = \frac{1}{2}$ and $\inf S = -1$.

(iii). $S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

$= S_e \cup S_o$

where $S_e = \left\{1 - \frac{1}{2n} : n \in \mathbb{N}\right\}$

$S_o = \left\{1 + \frac{1}{2n+1} : n \in \mathbb{N}\right\}$

$\because$ $2n \geq 2 \quad \forall \quad n \in \mathbb{N}$.

$\therefore$ $\frac{1}{2} \geq \frac{1}{2n}$

$\Rightarrow 1 - \frac{1}{2n} \geq 1 - \frac{1}{2} = \frac{1}{2} \quad \forall \quad n \in \mathbb{N}$

$\therefore$ $\frac{1}{2}$ is a lower bound for S_e.

$\because$ $\frac{1}{2} \in S_e$ and for $\epsilon > 0$ $\frac{1}{2} < \frac{1}{2} + \epsilon$

$\therefore$ $\frac{1}{2} = \inf S_e$

Further every elt of S_o is of the form $1 + \frac{1}{2n+1} > 1 > \frac{1}{2}$

$\therefore \frac{1}{2} \leq s \quad \forall \ s \in S_e \cup S_o = S$

implying $\frac{1}{2} = \inf S$.

On the other hand notice that

$$2n+1 \geq 3 \qquad \forall \ n \in \mathbb{N}.$$

$$\therefore \frac{1}{3} \geq \frac{1}{2n+1} \qquad \forall \ n \in \mathbb{N}$$

$$\Rightarrow 1 + \frac{1}{3} \geq 1 + \frac{1}{2n+1} \qquad \forall \ n \in \mathbb{N}.$$

$$\Rightarrow \frac{4}{3} \geq 1 + \frac{1}{2n+1} \qquad \forall \ n \in \mathbb{N}.$$

$\Rightarrow \frac{4}{3}$ is an upper bound for S_o.

for all $\epsilon > 0$, $\quad \frac{4}{3} < \frac{4}{3} + \epsilon$ and

$\frac{4}{3} \in S_o \quad \therefore \frac{4}{3} = \sup S_o$.

$\because$ every elt of S_e is of the form

$1 - \frac{1}{2n}$, $n \in \mathbb{N}$ and $1 - \frac{1}{2n} < 1 < \frac{4}{3}$

$\therefore \sup S = \sup S_o = \frac{4}{3}$.

5. S is a non-empty bounded subset of $\mathbb{R}$ i.e $\exists$ $M, m \in \mathbb{R}$ such that

$$m \leq s \leq M \quad \forall\ s \in S. \quad — ①.$$

Hence $\sup S$ and $\inf S$ exists and

$$\inf S \leq s \leq \sup S \quad \forall\ s \in S$$

For $b < 0$, $-b > 0$, therefore it follows from ① that

$$\inf S\,(-b) \leq (-b)s \leq (-b) \sup S \quad — (*)$$

$$\Rightarrow \quad b \sup S \leq bs \leq b \inf S \ \forall\ s \in S.$$

i.e the set $bS = \{ sb : s \in S \}$ is bounded with $b \inf S$ as an upper bd and $b \sup S$ as a lower bound.

$\therefore$ by completeness axiom,

$$\sup bS \leq b \inf S, \quad \inf bS \geq b \sup S.$$

$\because$ for $\epsilon > 0$, $b < 0$, $\dfrac{\epsilon}{-b} > 0$,

$\exists$ $s, s' \in S$ such that

$$\sup S - \frac{\epsilon}{(-b)} < s \leq \sup S \quad \text{and}$$

$$\inf S \leq s' < \inf S + \frac{\epsilon}{(-b)}.$$

$$\Rightarrow \quad (-b) \sup S - \epsilon < (-b)s \leq (-b) \sup S$$

and $(-b)\inf S \leq (-b)s' < (-b)\inf S + \epsilon$

$\Rightarrow$ for $\epsilon > 0$, $\exists$ $s, s' \in S$ st

$$\left.\begin{array}{l} b\sup S \leq bs < b\sup S + \epsilon \\ \text{and } b\inf S - \epsilon < bs' \leq b\inf S \end{array}\right\} *^2$$

From $*$ and $*^2$ it thus follow that

$$b\sup S = \inf bS$$

and $$b\inf S = \sup bS.$$

6. Let A and B be two sets of positive real numbers which are bounded above. Then by completeness axiom sup A and sup B exists.

Let $a = \sup A$, $b = \sup B$.

Let $C = \{xy : x \in A, y \in B\}$.

$\because$ $x > 0$, $y > 0$ $\forall$ $x \in A$, $y \in B$,

$x \leq a$, $y \leq b$ $\forall$ $x \in A$, $y \in B$

$\therefore 0 < xy < xb < ab$ $\forall$ $xy \in C$.

$\Rightarrow$ C is bdd above, $c < ab$ $\forall$ $c \in C$

and by completeness axiom sup C exists.

$\because$ ab is an upper bd for set C,

$\sup C \leq ab$.

Claim: $\sup C = ab$

$\because$ A is a set of positive numbers $\sup A > 0$.
B " " " " " $\sup B > 0$.

Given $\epsilon > 0$, by Archimedean property $\exists$ $n_1, n_2, n_3 \in \mathbb{N}$ such that

$$n_1 a > \epsilon, \quad n_2 b > \epsilon, \quad n_3 > a+b$$

Let $n_0 = \max\{n_1, n_2, n_3\}$.

Then $a - \dfrac{\epsilon}{n_0} > 0$, $b - \dfrac{\epsilon}{n_0} > 0$, $1 > \dfrac{a+b}{n_0}$.

$\because$ $a = \sup A$ and $b = \sup B$,

for $\epsilon' = \epsilon/n_0 > 0$ $\exists$ $x \in A$, $y \in B$ such that

$$a - \epsilon' < x \le a \quad \text{—} \quad (*)$$

$$b - \epsilon' < y \le b. \quad \text{——} \quad (**)$$

$\because$ by choice of n_0, $b - \epsilon' > 0$,

multiplying inequality $(*)$ by $b - \epsilon'$ we get,

$$(a-\epsilon')(b-\epsilon') < x(b-\epsilon') \le a(b-\epsilon')$$

but $b - \epsilon' < y \le b$ and $x > 0 \; \forall x \in A$

$\therefore$ $(b-\epsilon')x < yx \le bx \le ba$.

So we have

$$(a-\epsilon')(b-\epsilon') < xy \le ab$$

$\Rightarrow$ $ab - \epsilon'(a+b) + \epsilon'^2 < xy \le ab.$ —$(*^3)$

$\because$ $\epsilon' > 0$, $\epsilon'^2 > 0$, hence

$$ab - \epsilon'(a+b) < ab - \epsilon'(a+b) + \epsilon'^2 \quad -(*^4)$$

Further since $\quad n_0 > n_3$

$$\frac{1}{n_3} > \frac{1}{n_0}$$

$$\therefore 1 > \frac{a+b}{n_3} > \frac{a+b}{n_0}$$

and $$\epsilon > \frac{\epsilon(a+b)}{n_3} > \epsilon\left(\frac{a+b}{n_0}\right)$$

$$\Rightarrow \quad -\epsilon'(a+b) > -\epsilon$$

$$\Rightarrow \quad ab - \epsilon'(a+b) > ab - \epsilon \quad -(*^5)$$

$\therefore$ from $(*^3)$, $(*^4)$ and $(*^5)$ we get

$$ab - \epsilon < ab - \epsilon'(a+b) < xy \leq ab$$

i.e $\quad ab - \epsilon < xy \leq ab.$

Hence it follows that $ab = \sup C$.