Random Variables and Stochastic Process (AI5030)

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April 1, 2022

Question 118 Dec (2017)

Twenty items are put in a life testing experiment starting at time 0. The failure times of the items are recorded in a sequential manner. The experiment stops if all the twenty items fails or a pre-fixed time T$\geqslant$ 0 is reached, whichever is earlier. If the lifetimes of the items are independent and identically distributed exponential random variables with mean θ, where $0 < \theta < 10$, then which of the following statements are correct ?

1. The MLE of θ always exists.
2. The MLE of θ does not exist.
3. The MLE of θ is an unbiased estimator of θ if it exists.
4. The MLE of θ is bounded with probability 1, if it exists.

Solution

An exponentially distributed Random Variable is given by the equation:

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{1}$$

The graph of exponential Random Variable plot is:

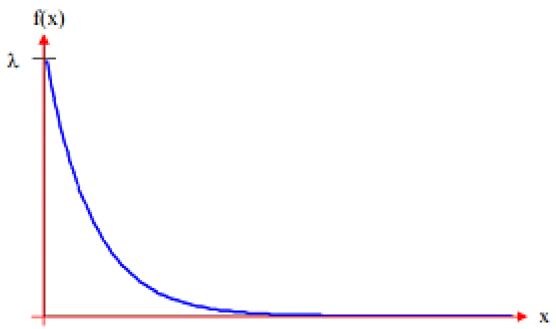

Figure 1: Exponential Random Variable

The CDF of exponential random variable is given by:

$$F(x, \lambda) = (1 - e^{-\lambda x}) \tag{2}$$

Here, the independent variable is time and mean θ, therefore the CDF with respect to time t and mean θ is F(t, θ) = (1 - $e^{-\theta t}$)

Out of the 20 items let the first item survive till time t_1 before failing.

Therefore, the probability that the item survives till time t_1 before T where t is exponentially distributed is given by P(t < T, θ) which is equal to the CDF of F(t_1, θ).

Therefore, the probability the item fails is given by 1 - F(t_1, θ) which is 1 - (1 - $e^{-\theta t_1}$) = $e^{-\theta t_1}$

Similarly, let the second item survive till time t_2 and the probability it survives till time t_2 is P(t_2 < T, θ) is given by the CDF F(t_2, θ).

Therefore, the probability the item fails is given by 1 - F(t_2, θ) which is 1 - (1 - $e^{-\theta t_2}$) = $e^{-\theta t_2}$

Therefore, the probability of all 20 items failing sequentially will be:

$$P = e^{-\theta t_1}.e^{-\theta t_2}.e^{-\theta t_3}.e^{-\theta t_4}.e^{-\theta t_5}.e^{-\theta t_6}.....e^{-\theta t_{20}}$$

Which will be equal to P= $e^{-\theta(t_1+t_2+t_3+t_4+......t_{20})}$, writing $t_1+t_2+t_3+t_4+......t_{20}$ as some constant τ, we can rewrite the value of P= $e^{-\theta\tau}$.

From the definition of maximum likelihood estimation, we have to find that value of θ in the range $0 < \theta < 10$ that maximizes the probability P.

To find the maximum value of the likelihood with respect to θ we can calculate $\dfrac{dP}{d\theta} = 0$ to find the value of θ that maximizes P.

However, P = $e^{-}\theta\tau$ is an exponential function, its derivative is also an exponential function which will never be exactly equal to 0.

Therefore, since $\dfrac{dP}{d\theta}$ will not be zero for any θ, we cannot find the optimal θ that maximizes P. Therefore, the MLE for this problem does not exist.

Therefore, option (2) is the correct option.