Random Variables and Stochastic Process (AI5030)

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Q. Let X_1, X_2, X_3, X_4, X_5 be i.i.d random variables having a continuous distribution function. Then $P(X_1 > X_2 > X_3 > X_4 > X_5 \mid X_1 = \max(X_1, X_2, X_3, X_4, X_5))$ equals

1. $\frac{1}{4}$ **2.** $\frac{1}{5}$

3. $\frac{1}{4!}$ **4.** $\frac{1}{5!}$

Sol. The above problem has to be solved using the concept of conditional probability. Conditional probability is defined as the likelihood of an event or outcome occurring, based on the occurrence of a previous event or outcome. It is denoted by P(A/B) and is formulated as:

$$P(A/B) = P(A \cap B)/P(B)$$

For the given problem,

$P(X_1 > X_2 > X_3 > X_4 > X_5 \mid X_1 = \max(X_1, X_2, X_3, X_4, X_5))$ can be expressed as

$[P(X_1 > X_2 > X_3 > X_4 > X_5) \cap (X_1 = \max(X_1, X_2, X_3, X_4, X_5))]/P(X_1 = \max(X_1, X_2, X_3, X_4, X_5))$

Now, from the definition of probability we know that,

P(X) = Number of events in favour of the event X/ Total number of all possible events.

Therefore, for calculating $P(X_1 > X_2 > X_3 > X_4 > X_5)$, we have to use some logic from permutations and combinations.

Now, __ > __ > __ > __ > __ has to be filled by X_1, X_2, X_3, X_4, X_5 which can be done in 5! ways. Now, out of 5! ways, there is only one way of arranging $X_1 > X_2 > X_3 > X_4 > X_5$, therefore

$$P(X_1 > X_2 > X_3 > X_4 > X_5) = \frac{1}{5!}$$

For calculating $P(X_1 = \max(X_1, X_2, X_3, X_4, X_5))$,

The total number of ways the gaps $\_\_ > \_\_ > \_\_ > \_\_ > \_\_$ can be filled by X_1, X_2, X_3, X_4, X_5 is still 5!, for X_1 to be the largest among X_1, X_2, X_3, X_4, X_5, the gaps $\_\_ > \_\_ > \_\_ > \_\_ > \_\_$ have to be filled as $X_1 > \_\_ > \_\_ > \_\_ > \_\_$. Therefore, the remaining four places can be filled with X_2, X_3, X_4, X_5 in 4! Ways. Therefore,

$$P(X_1 = \max(X_1, X_2, X_3, X_4, X_5)) = \frac{4!}{5!}$$

Now, for calculating $P[(X_1 > X_2 > X_3 > X_4 > X_5) \cap (X_1 = \max(X_1, X_2, X_3, X_4, X_5))]$,

We have to find out the probability of a common event between the two. Now the event $X_1 = \max(X_1, X_2, X_3, X_4, X_5)$ is actually permuting X_2, X_3, X_4, X_5 in $X_1 > \_\_ > \_\_ > \_\_ > \_\_$ which occurs in 4! ways as calculated above and out of this 4! ways $X_1 > X_2 > X_3 > X_4 > X_5$ is a subset which occurs in 1 way. Therefore, the common event is actually $X_1 > X_2 > X_3 > X_4 > X_5$ and probability of $X_1 > X_2 > X_3 > X_4 > X_5$ is already calculated above as:

$$P(X_1 > X_2 > X_3 > X_4 > X_5) = \frac{1}{5!}$$

Therefore from conditional probability,

$[P(X_1 > X_2 > X_3 > X_4 > X_5) \cap (X_1 = \max(X_1, X_2, X_3, X_4, X_5))]/P(X_1 = \max(X_1, X_2, X_3, X_4, X_5))$ can be solved as $\frac{1}{5!}/\frac{4!}{5!}$ which can be simplified further to $\frac{1}{4!}$.

Therefore, option (c) $\frac{1}{4!}$ is the correct answer.