Random Variables and Stochastic Process (AI5030)

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Question 107 (Dec 2017)

For $n \geq 1$, let X_n be a Poisson random variable with mean n^2. Which of the following is equal to-

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$$

1. $\lim_{n\to\infty} \mathrm{P}\{X_n > (n+1)^2\}$
2. $\lim_{n\to\infty} \mathrm{P}\{X_n \leq (n+1)^2\}$
3. $\lim_{n\to\infty} \mathrm{P}\{X_n < (n-1)^2\}$
4. $\lim_{n\to\infty} \mathrm{P}\{X_n < (n-2)^2\}$

Solution

We need to first find what $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$ equals to. To calculate the value of the integral, we first try to solve the following integral-

$$I' = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$$

Since, the function inside the integral is an even function, we can write-

$$I' = 2\int_{0}^{\infty} e^{\frac{-x^2}{2}} dx$$

Let y = $\frac{x^2}{2}$ which implies x = $\sqrt{2y}$ and dy = xdx. Therefore, dx = $\frac{dy}{x}$ which is $\frac{y^{\frac{-1}{2}}}{\sqrt{2}}$. Therefore,

$$I' = 2\int_{-\infty}^{\infty} e^{-y}.\frac{y^{\frac{-1}{2}}}{\sqrt{2}} dx$$

$$\Rightarrow \sqrt{2}\int_{-\infty}^{\infty} e^{-y}.y^{\frac{-1}{2}} dx$$

$$\Rightarrow \sqrt{2}.\Gamma(\frac{1}{2}) = \sqrt{2\pi}$$

As we have defined I' above, I = $\dfrac{I}{\sqrt{2\pi}}$, therefore, I = $\dfrac{\sqrt{2\pi}}{\sqrt{2\pi}} = 1$. Therefore, we have to match the options which equals to 1.

PDF of Poisson distribution is given by $f_x(x) = \dfrac{\lambda^x.e^{-x}}{x!}$.
CDF of Poisson distribution is given be P(x ≤ n) = $F_x(n) = e^{-\lambda}.\sum_{x=0}^{n}\left(\dfrac{\lambda^x}{x!}\right)$
Now,

$$\lim_{n\to\infty}\{P(x \le n)\} = \lim_{n\to\infty}\sum_{x=0}^{n}\left(\frac{\lambda^x}{x!}\right)$$

Checking for correctness of option (2), we have $\lambda = n^2$, therefore, for $P(x_n \le (n+1)^2)$ we have-

$$P(x_n \le (n+1)^2) = e^{-n^2}.\sum_{x=0}^{(n+1)^2}\frac{(n^2)^x}{x!}$$

$$\lim_{n\to\infty} P(x_n \le (n+1)^2) = \lim_{n\to\infty} e^{-n^2}.\lim_{n\to\infty}\sum_{x=0}^{(n+1)^2}\frac{(n^2)^x}{x!}$$

$$\Rightarrow \lim_{n\to\infty}\left(e^{-n^2}.\sum_{x=0}^{(n+1)^2}\frac{(n^2)^x}{x!}\right)$$

Since, $n \to \infty$, then without loss of generality $(n+1)^2 \to \infty$, therefore,

$$\lim_{n\to\infty}\left(e^{-n^2}.\sum_{x=0}^{(n+1)^2}\frac{(n^2)^x}{x!}\right) = \lim_{n\to\infty}\left(e^{-n^2}.\sum_{x=0}^{\infty}\frac{(n^2)^x}{x!}\right)$$

$$\Rightarrow e^{-n^2}.e^{n^2} = 1$$

Checking for correctness of option (1), we have $P(x_n > (n+1)^2) = 1 - P(x_n \le (n+1)^2)$ therefore-

$$1 - P(x_n \le (n+1)^2) = 1 - 1 = 0$$

Since, we already calculated the value of $P(x_n \le (n+1)^2) = 1$ while evaluating option 2 previously.
Similarly, checking for correctness of option (3), we have $P(x_n > (n-1)^2) = 1 - P(x_n \le (n-1)^2)$ therefore-

$$1 - P(x_n \le (n-1)^2) = 1 - 1 = 0$$

Similarly, checking for correctness of option (4), we have $P(x_n > (n-2)^2) = 1 - P(x_n \le (n-2)^2)$ therefore-

$$1 - P(x_n \le (n-2)^2) = 1 - 1 = 0$$

The only option that matches the integral value is option (2). Therefore, option (2) is the correct option.