Module :1
Complex Numbers

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MODULE INFORMATION

- **Contribution in Course Outcome:**
  - CO1: Solve problems involving different forms and properties of complex numbers, hyperbolic functions and logarithm of complex numbers.

Module No.	Unit No.	Details	Hrs.	CO
1	**Complex Numbers, Hyperbolic Functions and Logarithm of Complex Number**		12	CO1
	1.1	Statement of De Moivre's theorem and related examples		
	1.2	Powers and roots of complex numbers		
	1.3	Circular functions of complex number and hyperbolic functions		
	1.4	Inverse circular and inverse hyperbolic functions		
	1.5	Logarithmic functions		
	1.6	Separation of real and imaginary parts		
		#Self-learning topics: Expansion of $sin^n\theta$, $cos^n\theta$ in terms of sine and cosine of multiples of angle θ and expansion of $sinn\theta, cosn\theta$ in powers of $sin\theta, cos\theta$		

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ACTIVITY: Recall And Share
Whatever you know about Complex Number?

- What is complex Number?
  - A complex number z is an ordered pair (x, y) of real numbers x and y. It is written as $z = x + i\,y$ or $z = x + j\,y$.
- What is i?
  - i or $j = \sqrt{-1}$ is known as **imaginary unit**.
- What is re(z)?
  - x is called the Real part of z and is written as "**Re (z)**"
- What is Im(z)?
  - y is called the Imaginary part of z and is written as "**Im(z)**".

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POWERS OF j (or i)

- We know that $j = \sqrt{-1}$,
- $j^2 = j \times j = -1,$
- $j^3 = j^2 \times j = -j$
- $j^4 = (j^2)^2 = (-1)^2 = 1,$
- $j^5 = j \times j^4 = j$ etc.
- This shows that even power of i is either 1 or -1 and odd power of j is either j or $-j$.

EQUALITY OF COMPLEX NUMBERS

- If $z_1 = z_2$ then, $x_1 + j\, y_1 = x_2 + j\, y_2$ Comparing real and imaginary parts $x_1 = x_2$ and $y_1 = y_2$
- This shows that two complex numbers are equal if and only if their corresponding real and imaginary parts are equal.

INEQUALITY OF COMPLEX NUMBERS

- Which of the following is correct?

a. 2+3i=3+2i b. 2+3i<3+2i c. 2+3i>3+2i

CONJUGATE OF COMPLEX NUMBER

- If $z = x + j\,y$ is a complex number then its conjugate or complex conjugate is defined as $\bar{z} = x - j\,y$.
- Also $z\,\bar{z} = (x + j\,y)\,(x - j\,y) = x^2 + y^2$

  Note: To write the conjugate of a complex number, replace j by – j in the complex number.

ALGEBRA OF COMPLEX NUMBER

Let $z_1 = x_1 + j\,y_1$ and $z_2 = x_2 + j\,y_2$ be two complex numbers. Then

- **(a) Addition:** $z_1 + z_2 = (x_1 + j\,y_1) + (x_2 + j\,y_2)$

  $= (x_1 + x_2) + j\,(y_1 + y_2)$

- **(b) Subtraction:** $z_1 - z_2 = (x_1 + j\,y_1) - (x_2 + j\,y_2)$

  $= (x_1 - x_2) + j\,(y_1 - y_2)$

- **(c) Multiplication:** $z_1 \cdot z_2 = (x_1 + j\,y_1) \cdot (x_2 + j\,y_2)$

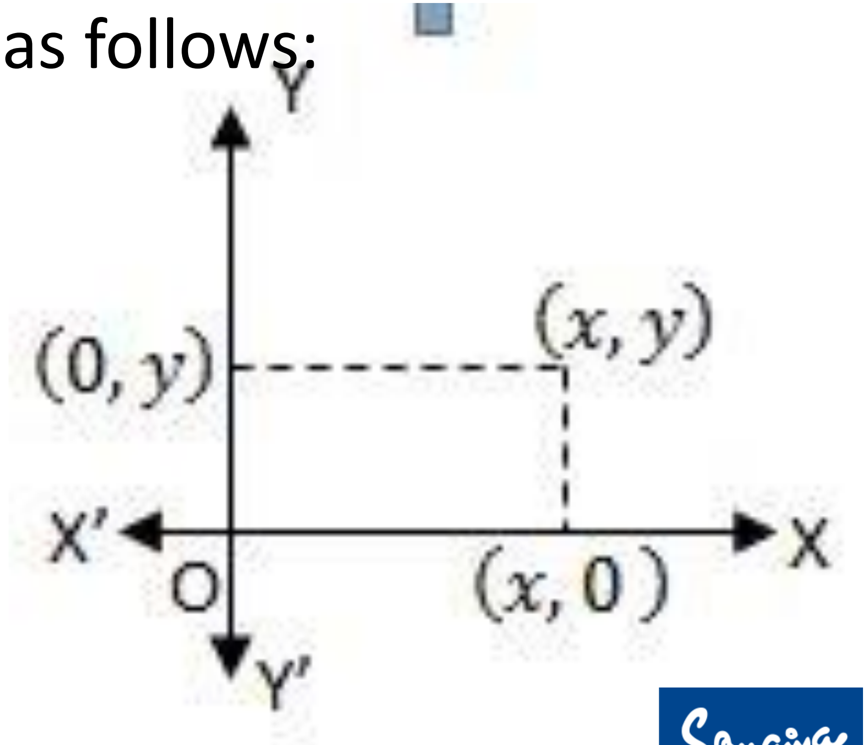
  $= (x_1 x_2 - y_1\,y_2) + j\,(x_2\,y_1 + y_2\,x_1)\ [j^2 = -1]$

- **(d) Division:** $\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2}$

$$= \frac{(x_1 + jy_1)}{(x_2 + jy_2)} \cdot \frac{(x_2 - jy_2)}{(x_2 - jy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{(y_1x_2 - x_1y_2)}{(x_2^2 + y_2^2)}$$

GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

- **Argand's Diagram**: We know that the real numbers can be represented by point on a line in such a way that corresponding to every real number, there is one and only one point on the line and corresponding to every point on the line, there is one and only one real number.
- Similarly, we can represent a complex number as follows:
- Consider a complex number $z = x + iy$,
- where $x, y \in R$ and $i = \sqrt{-1}$.
- Then the point (x, y) represents the complex number $x + iy$, i.e., $x + iy = (x, y)$
- Such a representation of complex numbers by points in a plane is called **Argand's diagram**.



Geometrical Meaning of Modulus and Argument

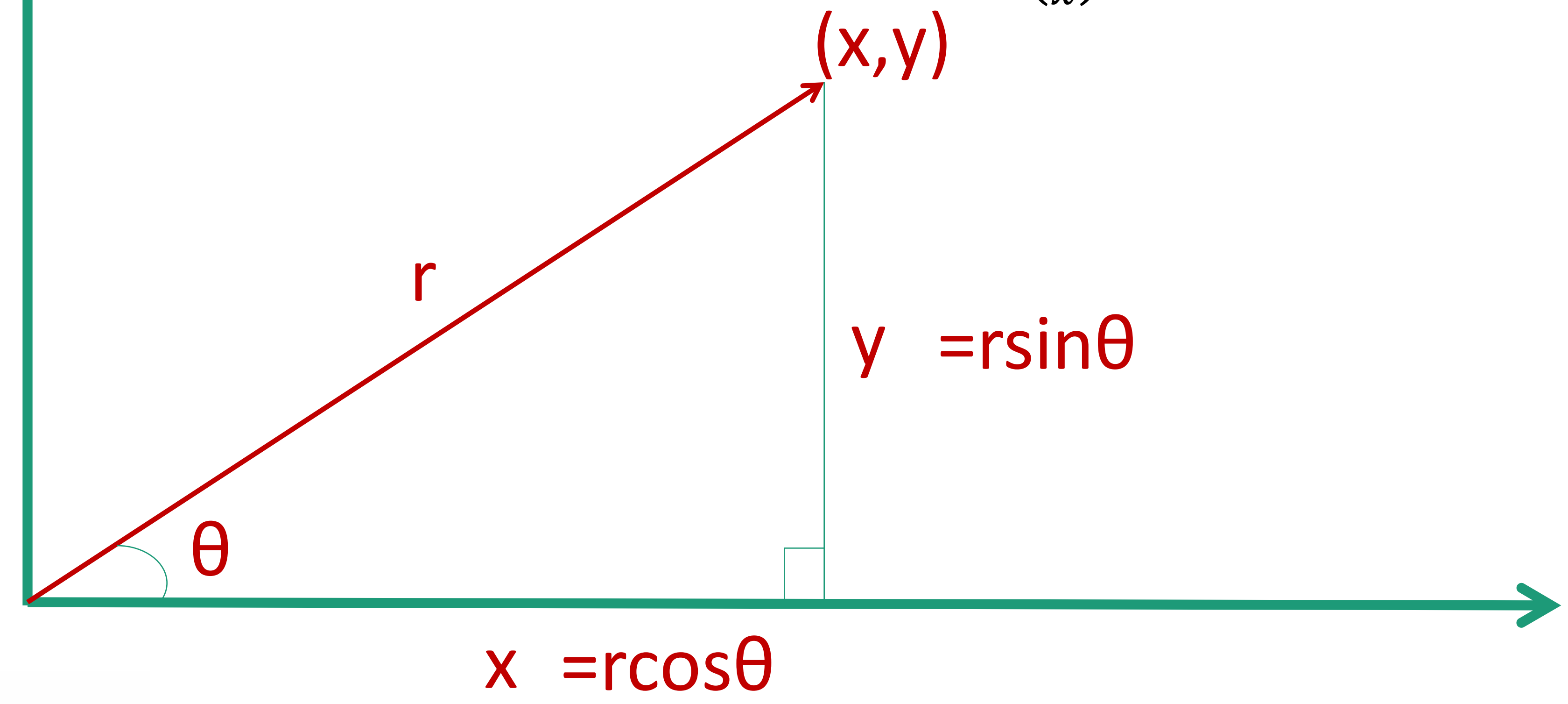
- If $z = x + jy$ is a complex number, then the modulus of z is $|z| = \sqrt{x^2 + y^2}$ and is denoted by $|z|$ or mod (z) .
- Line segment from origin to point makes an angle θ with the positive direction of X-axis.
- θ is called the amplitude or argument of the complex number $z = x + jy$ and is denoted by arg (z) or amp (z).

$\therefore \sin\theta = \frac{y}{|z|}$ and $\cos\theta = \frac{x}{|z|}, |z| \neq 0$

$\therefore \tan\theta = \frac{y}{x}$, if $x \neq 0$

Hence, $|z| = r = \sqrt{x^2 + y^2}$ &

$\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$



- **Note:** The value of θ which satisfies both the equation x = r cos θ and y = r sin θ, gives the argument of z.
- Argument θ has infinite number of values. The value of θ lying between $-\pi$ and π is called the **principal value** of Argument.

POLAR FORM & EXPONENTIAL FORM OF A COMPLEX NUMBER

- $z = x + jy$ can be written as $z = r\cos\theta + j(r\sin\theta) = r(\cos\theta + j\sin\theta)$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$

- We know $e^{i\theta} = \cos\theta + j\sin\theta$.
- $z = r\,(cos\,\theta + j\,sin\,\theta) = re^{j\theta}$: Exponential form /Euler's form of a complex number z.
- $z \quad = x + jy$ **(Cartesian form)**

$= r\,(cos\,\theta + j\,sin\,\theta)$ **(Polar form)**

$= re^{j\theta}$ **(Exponential form)**

- **Note:** $e^{j\theta} = \cos\theta + j\ \sin\theta,\ e^{-j\theta} = \cos\theta - j\,sin\,\theta$
- Hence, $cos\,\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$ and $sin\,\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$

$$Z = x + iy$$
$$(x, y)$$

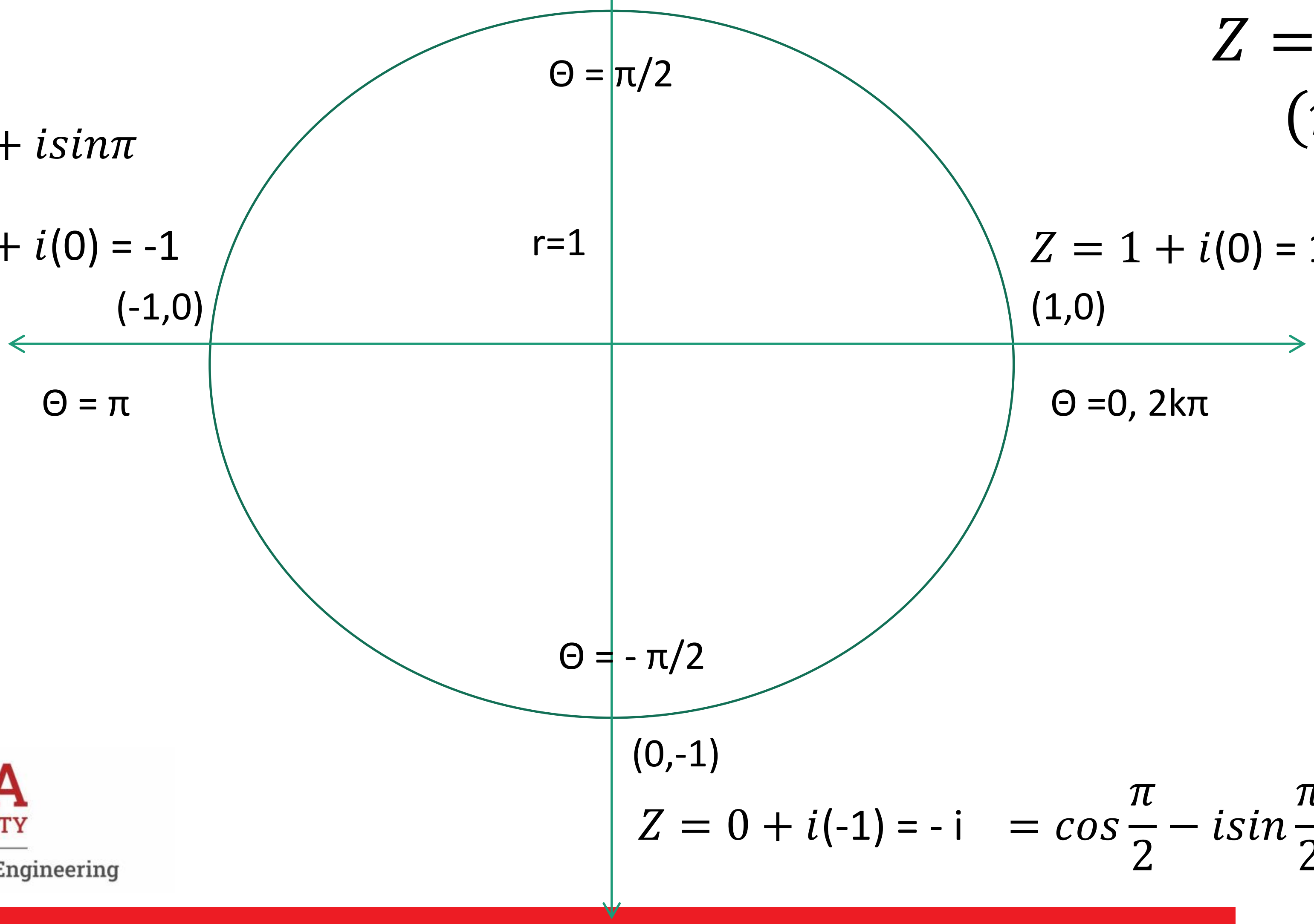
$$Z = r\cos\theta + ir\sin\theta$$
$$(r\cos\theta, r\sin\theta)$$

$Z = 0 + i(1) =$ i
$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= \cos\pi + i\sin\pi$

$Z = -1 + i(0)$ = -1

$Z = 1 + i(0)$ = 1 $\quad = \cos 0 + i\sin 0$



$Z = 0 + i$(-1) = - i $\quad = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

Remember

- $1 = cos0 + isin0$
- $-1 = cos\pi + isin\pi$
- $i = \cos\frac{\pi}{2} + isin\frac{\pi}{2}$
- $-i = \cos\frac{\pi}{2} - isin\frac{\pi}{2}$

Quick Answer

(i) $z = 3$, then amp $(z) =$
0

(ii) $z = -300$, then amp $(z) =$
π

(iii) $z = 525\, i$, then amp $(z) =$
$\pi/2$

(iv) $z = -5000i$, then amp $(z) =$
$\frac{3\pi}{2}$ or - $\pi/2$

Quick Answer

(v) $polar\ form\ of\ 100i$

$$polar\ form\ of\ 100(i) = (100)\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

(vi) $polar\ form\ of\ -4i$

$$polar\ form\ of\ 4(-i) = (4)\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right]$$

(vii) $polar\ form\ of\ -900$

$$polar\ form\ of\ 900(-1) = (900)[\cos\pi + i\sin\pi]$$

Think

Polar form of $1 + \boldsymbol{i} =$

Consider, $Z = x + iy = rcos\theta + irsin\theta$

$Where\ r{=}|z| = \sqrt{x^2 + y^2}\ \ and\ \theta = \tan^{-1}(y/x).$

For $\mathbf{1} + \boldsymbol{i}$ **;** $r{=}|z| = \sqrt{2}$ $\theta = \tan^{-1} 1 = \frac{\pi}{4}$

Polar form of $1 + \boldsymbol{i} = \sqrt{2}\,(cos\frac{\pi}{4} + irsin\frac{\pi}{4})$

What is the Polar form of $-1 - \boldsymbol{i}$?

Is it Same as 1+i?

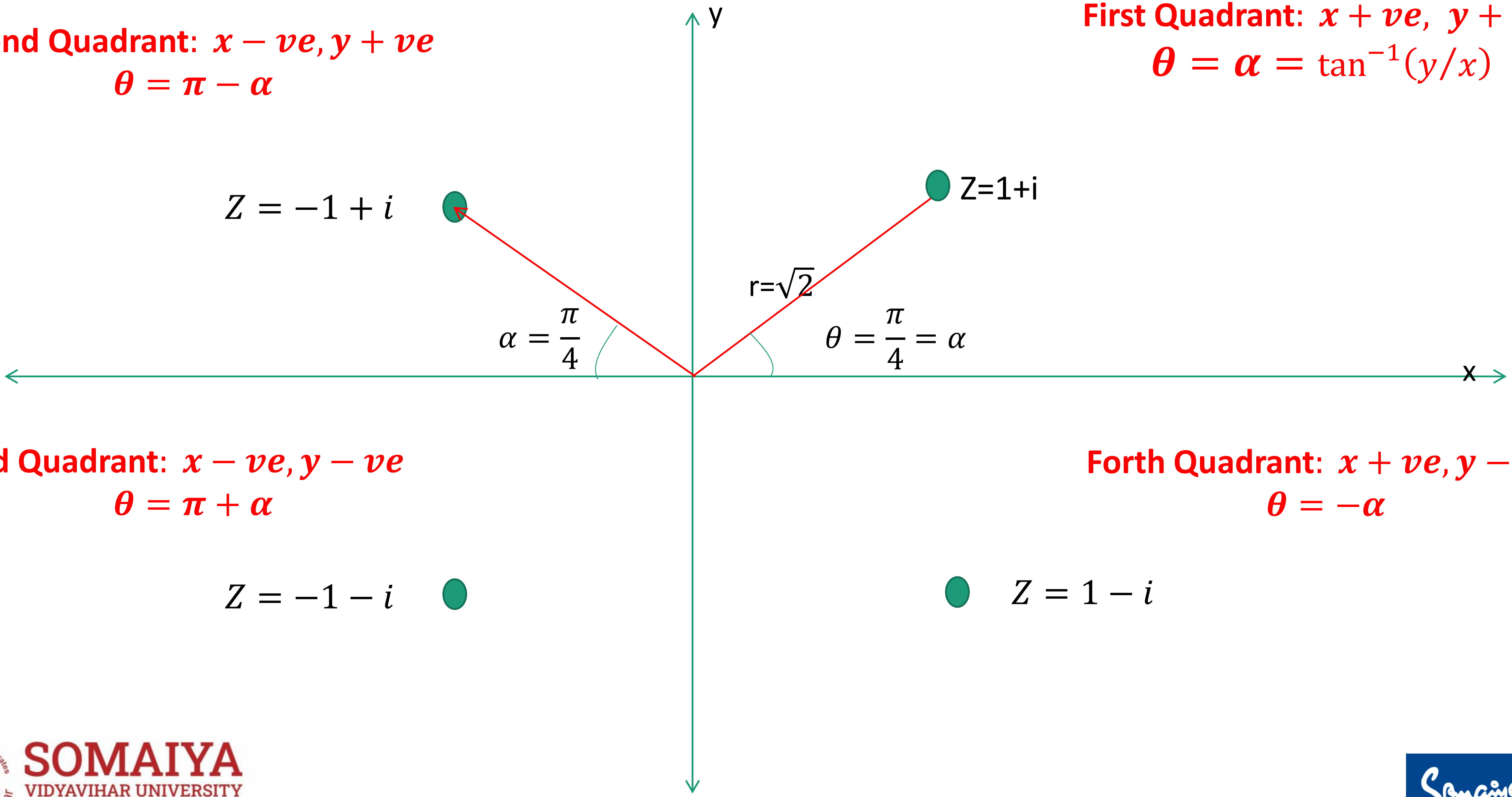
What about polar form of -1+i and 1-i?

Second Quadrant: $\boldsymbol{x-ve, y+ve}$

$$\boldsymbol{\theta = \pi - \alpha}$$

First Quadrant: $\boldsymbol{x+ve,\ y+ve}$

$$\boldsymbol{\theta = \alpha = \tan^{-1}(y/x)}$$



Third Quadrant: $\boldsymbol{x-ve, y-ve}$

$$\boldsymbol{\theta = \pi + \alpha}$$

Forth Quadrant: $\boldsymbol{x+ve, y-ve}$

$$\boldsymbol{\theta = -\alpha}$$

PROPERTIES OF COMPLEX NUMBER

- Let $z = x + iy$ and $\bar{z} = x - iy$
- **(a)** $Re(z) = x = \frac{1}{2}(z + \bar{z})$
- **(b)** $Im(z) = y = \frac{1}{2i}(z - \bar{z})$
- **(c)** $\overline{(z_1 + z_2)} = \bar{z_1} + \bar{z_2}$
- **(d)** $\overline{(z_1 \,.\, z_2)} = \bar{z_1} \,.\bar{z_2}$
- **(e)** $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
- **(f)** $z\bar{z} = |z|^2 = |\bar{z}|^2$
- since $|z| = |\bar{z}| = \sqrt{x^2 + y^2}$

PROPERTIES OF COMPLEX NUMBER

- **(g)** $|z_1 z_2| = |z_1||z_2|$ & arg $(z_1 z_2)$ = arg $(z_1) + \arg(z_2)$
- Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$
- $z_1 z_2 = r_1 e^{i\theta_1} . r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)}$
- Comparing with exponential form
- $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ And
- arg.$(z_1 z_2) = \theta_1 + \theta_2$ = arg $(z_1) + \arg(z_2)$

- **(h)** $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ & arg $\left(\frac{z_1}{z_2}\right)$ = arg $(z_1) - \arg(z_2)$
- Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$
- $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_1}\right) e^{i(\theta_1 - \theta_2)}$
- Comparing with exponential form
- $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_1} = \frac{|z_1|}{|z_2|}$ And arg $\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$ =arg $(z_1) - \arg(z_2)$

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EXAMPLE-1

- Express $\alpha = \frac{(1+i)^3}{(2+i)(1+2i)}$ in the form $a + ib$. Also find α^2.
- **Solution:** $\alpha = \frac{(1+i)^3}{(2+i)(1+2i)}$
- $= \frac{1+3i+3i^2+i^3}{2+4i+i+2i^2}$
- $= \frac{1+3i-3-i}{2+4i+i-2}$
- $= \frac{-2+2i}{5i}$
- $= \frac{-2+2i}{5i} \cdot \frac{i}{i}$
- $= \frac{-2i+2i^2}{5\, i^2}$
- $= \frac{-2-2i}{-5}$
- $= \frac{2}{5} + \frac{2}{5} i$
- $\alpha^2 = \left(\frac{2}{5}(1+i)\right)^2$
- $= \frac{4}{25}(1 + 2i + i^2)$
- $= \frac{8i}{25}$

EXAMPLE-2

- Find the value of $z^4 - 4z^3 + 6z^2 - 4z - 12$ when $z = 1 + 2i$
- **Solution:** Since $z = 1 + 2i$ i.e $z - 1 = 2i$
- $\therefore (z-1)^2 = 4i^2$
- $\therefore z^2 - 2z + 1 = -4$
- $\therefore z^2 - 2z + 5 = 0$

We express the give expressions in terms of $z^2 - 2z + 5$.

For this we divide the given expressions by $z^2 - 2z + 5$

- **Expressions** $= (z^2 - 2z + 5)(z^2 - 2z - 3) + 3$
- $= 0(z^2 - 2z - 3) + 3$
- $= 0 + 3 = 3$

EXAMPLE-3

- Find the modulus and the principal argument of
- $\dfrac{\left(1+i\sqrt{3}\right)^3(1+i)^{-2}\left(\sqrt{3}+i\right)^{-1}}{2}$
- **Solution:** $z = \dfrac{\left(1+i\sqrt{3}\right)^3(1+i)^{-2}\left(\sqrt{3}+i\right)^{-1}}{2}$
- $= \dfrac{\left(1+i\sqrt{3}\right)^3}{2(1+i)^2\left(\sqrt{3}+i\right)}$
- $= \dfrac{1+i3\sqrt{3}-3(3)-i3\sqrt{3}}{2(1+2i-1)(\sqrt{3}+i)}$
- $\therefore z = -\dfrac{8}{2(2i)\left(\sqrt{3}+i\right)} = -\dfrac{2}{i\left(\sqrt{3}+i\right)}$
- $= -\dfrac{2}{i\sqrt{3}-1} = \dfrac{2}{1-i\sqrt{3}}$
- $= \dfrac{2}{1-i\sqrt{3}} \cdot \dfrac{1+i\sqrt{3}}{1+i\sqrt{3}}$
- $= \dfrac{2\left(1+i\sqrt{3}\right)}{4}$
- $= \dfrac{\left(1+i\sqrt{3}\right)}{2} \quad = \dfrac{1}{2} + i\,\dfrac{\sqrt{3}}{2}$
- $\therefore x = \dfrac{1}{2}, y = \dfrac{\sqrt{3}}{2},$
- $r = \sqrt{x^2+y^2} = \sqrt{\dfrac{1}{4}+\dfrac{3}{4}} = 1$
- $\cos\theta = \dfrac{1}{2}, \sin\theta = \dfrac{\sqrt{3}}{2} \quad \therefore \ \theta = \dfrac{\pi}{3}$
- $\therefore$ Modulus $z = 1$, Amplitude $z = \dfrac{\pi}{3}$

EXAMPLE-4

- Find the square root of $21 - 20i$
- **Solution:** Let $\sqrt{21 - 20i} = x + iy$
- $\therefore (x + iy)^2 = 21 - 20i$
- $\therefore x^2 - y^2 + 2ixy = 21 - 20i$
- Equating real and imaginary parts
  $x^2 - y^2 = 21$ and $xy = -10$
- Putting $y = \frac{-10}{x}$ in $x^2 - y^2 = 21$
- We get, $x^2 - \left(\frac{-10}{x}\right)^2 = 21$
- $\therefore x^2 - \frac{100}{x^2} = 21$
- $\therefore x^4 - 100 = 21x^2$
- $\therefore x^4 - 21x^2 - 100 = 0$
- $\therefore (x^2 - 25)(x^2 + 4) = 0$
- $\therefore x^2 = 25$ or $x^2 = -4$
- Since x is real $x^2 = 25$
- $\therefore x = \pm 5$
- When $x = 5$, $y = \frac{-10}{x} = \frac{-10}{5} = -2$
- When $x = -5$, $y = \frac{-10}{x} = \frac{-10}{-5} = 2$
- $\therefore \sqrt{21 - 20i}$ is $5 - 2i$ or $-5 + 2i$

EXAMPLE-5

- If $x+iy=\sqrt[3]{a+ib}$, prove that $\frac{a}{x}+\frac{b}{y}=4(x^2-y^2)$
- **Solution:** $x+iy=\sqrt[3]{a+ib}$
- $\therefore (x+iy)^3=a+ib$
- $\therefore x^3-3ix^2y-3xy^2-iy^3=a+ib$
- $(x^3-3xy^2)+i(3x^2y-y^3)=a+ib$
- Comparing real and imaginary parts
- $a=x^3-3xy^2, \quad b=3x^2y-y^3$
- $\frac{a}{x}=x^2-3y^2, \quad \frac{b}{y}=3x^2-y^2$
- $\therefore \frac{a}{x}+\frac{b}{y}=(x^2-3y^2)+(3x^2-y^2)$
- $=4x^2-4y^2=4(x^2-y^2)$

EXAMPLE-6

- Find the complex number z if $\arg(z+1)=\frac{\pi}{6}$ *and* $\arg(z-1)=\frac{2\pi}{3}$
- **Solution:** Let $z=x+iy$

$\therefore z+1=(x+1)+iy$ and

$z-1=(x-1)+iy$

We are given that, $\arg(z+1)=\frac{\pi}{6}$

$$\therefore tan^{-1}\left(\frac{y}{x+1}\right)=\frac{\pi}{6}$$

$$\therefore \frac{y}{x+1}=\tan 30°=\frac{1}{\sqrt{3}}$$

$\therefore \sqrt{3}\cdot y=x+1$.....(1)

$\arg(z-1)=\frac{2\pi}{3}\quad \therefore tan^{-1}\left(\frac{y}{x-1}\right)$

$=\frac{2\pi}{3}$

$$\therefore \frac{y}{x-1}=\tan 120°=-\sqrt{3}$$

$\therefore y=-\sqrt{3}x+\sqrt{3}$......(2)

Adding both equations, we get,

$0=4x-2\qquad \therefore x=1/2$

Now $\sqrt{3}.y=x+1$ gives $\sqrt{3}.y=\frac{3}{2}$

$\therefore y=\frac{\sqrt{3}}{2}\qquad\qquad \therefore z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

EXAMPLE-7

- Find two complex numbers such that their difference is $10i$ and their product is 29.

Solution: Let z_1 and z_2 be two complex numbers such that, $z_1 - z_2 = 10i$ and $z_1 z_2 = 29$

Now, $(z_1 + z_2)^2 = (z_1 - z_2)^2 - 4z_1 z_2 = (10i)^2 + 4(29)$

$= -100 + 116 = 16$

$\therefore\ |z_1 + z_2| = 4$

$\therefore$ z_1 and z_2 are roots of quadratic equation

$x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$

i.e. $x^2 - (z_1 + z_2)x + z_1 z_2 = 0$ i.e., $x^2 - 4x + 29 = 0$

Solving $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(29)}}{2(1)} = \frac{4 \pm \sqrt{-100}}{2} = \frac{4 \pm 10i}{2} = 2 \pm 5i$

$\therefore\ z_1 = 2 + 5i$ and $z_2 = 2 - 5i$

EXAMPLE-8

If $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$

show that $\frac{1}{2i}\left(\frac{z_1}{z_2} - \frac{z_2}{z_1}\right) = sin(\alpha - \beta).$

Solution: We have $\frac{z_1}{z_2} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} = \frac{e^{i\alpha}}{e^{i\beta}}$

$$= e^{i(\alpha-\beta)} = cos(\alpha - \beta) + isin(\alpha - \beta)$$

Similarly, $\frac{z_2}{z_1} = \frac{\cos\beta + i\sin\beta}{\cos\alpha + i\sin\alpha} = \frac{e^{i\beta}}{e^{i\alpha}} = e^{i(\beta-\alpha)} = e^{-i(\alpha-\beta)}$

$$= cos(\alpha - \beta) - i\sin(\alpha - \beta)$$

$\therefore \frac{z_1}{z_2} - \frac{z_2}{z_1} = 2\, i\sin(\alpha - \beta).$ Hence, the result

EXAMPLE-9

- If $z = \cos\theta + i\sin\theta$, prove that **(i)** $\frac{2}{1+z} = 1 - i\tan(\theta/2)$. **(ii)** $\frac{1+z}{1-z} = i\cot\left(\frac{\theta}{2}\right)$.
- **Solution:**
- **(i)** $\frac{2}{1+z} = \frac{2}{1+\cos\theta+i\sin\theta} = \frac{2}{2cos^2(\theta/2)+2\,i\sin(\theta/2)\cos(\theta/2)}$
- $= \frac{1}{cos(\theta/2)\cdot(cos(\theta/2)+i\sin(\theta/2))}$
- $= \frac{1}{cos(\theta/2)\cdot e^{i\left(\frac{\theta}{2}\right)}}$
- $= \frac{e^{-i\left(\frac{\theta}{2}\right)}}{cos(\theta/2)}$
- $= \frac{cos(\theta/2)-i\sin(\theta/2)}{cos(\theta/2)} = 1 - i\tan(\theta/2)$

EXAMPLE-9

- **(ii)** $\frac{1+z}{1-z} = \frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}$
- $= \frac{2cos^2(\theta/2)+2\,i\sin(\theta/2)\cos(\theta/2)}{2sin^2(\theta/2)-2\,i\sin(\theta/2)\cos(\theta/2)}$
- $= \frac{cos(\theta/2)}{sin(\theta/2)} \cdot \frac{cos(\theta/2)+i\sin(\theta/2)}{sin(\theta/2)-\,i\cos(\theta/2)}$
- $= \cot\left(\frac{\theta}{2}\right) \cdot \frac{cos(\theta/2)+i\sin(\theta/2)}{sin(\theta/2)-\,i\cos(\theta/2)}$
- $= \cot\left(\frac{\theta}{2}\right) \cdot \frac{cos(\theta/2)+i\sin(\theta/2)}{-i^2 sin(\theta/2)-\,i\cos(\theta/2)}$
- $= \cot\left(\frac{\theta}{2}\right) \cdot \frac{1}{-i}\left[\frac{cos(\theta/2)+i\sin(\theta/2)}{cos(\theta/2)+i\sin(\theta/2)}\right]$
- $= \frac{1}{-i} \cdot \cot\left(\frac{\theta}{2}\right) = \frac{1}{-i} \cdot \frac{i}{i} cot\left(\frac{\theta}{2}\right) = i\ \cot\left(\frac{\theta}{2}\right)$

EXAMPLE-10

- If $(1+\cos\theta+i\sin\theta)(1+\cos 2\,\theta+i\sin 2\theta)=u+iv$, prove that (i) $u^2+v^2=16\,cos^2\left(\frac{\theta}{2}\right)cos^2\theta$ **(ii)** $\frac{v}{u}=tan\left(\frac{3\theta}{2}\right)$
- **Solution:** We have to find u and v.
- Now from data $(1+\cos\theta+i\sin\theta)(1+\cos 2\,\theta+i\sin 2\theta)=u+iv,$
- $\therefore \left[2cos^2\left(\frac{\theta}{2}\right)+2\,i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right][2\,cos^2\theta+2\,i\sin\theta\cos\theta]=u+iv$
- $\therefore 2\cos\left(\frac{\theta}{2}\right)\left[cos\left(\frac{\theta}{2}\right)+\ i\sin\left(\frac{\theta}{2}\right)\right].2\cos\theta[\cos\theta+i\sin\theta]=u+iv$
- $\therefore 4\cos\left(\frac{\theta}{2}\right)\cos\theta\cdot e^{i\left(\frac{\theta}{2}\right)}\cdot e^{i\theta}=u+iv$
- $\therefore 4\cos\left(\frac{\theta}{2}\right)\cos\theta e^{i\left(\frac{3\theta}{2}\right)}=u+iv$
- $\therefore 4\cos\left(\frac{\theta}{2}\right)\cos\theta\left[\cos\left(\frac{3\theta}{2}\right)+\ i\sin\left(\frac{3\theta}{2}\right)\right]=u+iv$
- Equating real and imaginary parts $u=4\cos\left(\frac{\theta}{2}\right)cos\theta cos\left(\frac{3\theta}{2}\right)$ and
- $v=4\cos\left(\frac{\theta}{2}\right)cos\theta\ sin\left(\frac{3\theta}{2}\right)$

$\therefore u^2+v^2=16\,cos^2\left(\frac{\theta}{2}\right)cos^2\theta\ \ and\ \frac{v}{u}=tan\left(\frac{3\theta}{2}\right)$

EXAMPLE-11

- If $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$ $\;where\; 0 < \alpha, \beta < \pi/2$ find the polar form of $\frac{1+{z_1}^2}{1-i\,z_1 z_2}$
- **Solution**: Expression $= \frac{1+{z_1}^2}{1-i\,z_1 z_2} = \frac{(1/z_1)+z_1}{(1/z_1)-iz_2}$
- Putting $z_1 = \cos\alpha + i\sin\alpha \;\; and \;\; \frac{1}{z_1} = cos\alpha - i\; sin\alpha$
- Expression $= \frac{(\cos\alpha - i\sin\alpha)+(\cos\alpha + i\sin\alpha)}{(\cos\alpha - i\sin\alpha) - i(\cos\beta - i\sin\beta)}$
- $= \frac{2\cos\alpha}{(\cos\alpha+\sin\beta)-i(\sin\alpha+\cos\beta)} = \frac{2\cos\alpha}{\left[\cos\alpha+\cos\left(\frac{\pi}{2}-\beta\right)\right]-i\left[\sin\alpha+\sin\left(\frac{\pi}{2}-\beta\right)\right]}$

$$= \frac{2\cos\alpha}{2\cos\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)\cos\left(-\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right) - i2\, sin\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)\cos\left(-\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right)}$$

EXAMPLE-11

- $= \dfrac{2\cos\alpha}{2\cos\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)\cos\left(-\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right)-i2\,sin\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)\cos\left(-\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right)}$
- But $\cos\left(-\dfrac{\pi}{4}+\dfrac{\alpha+\beta}{2}\right)=\cos\left(\dfrac{\pi}{4}-\dfrac{\alpha+\beta}{2}\right)$
- $\therefore$Expression $= \dfrac{\cos\alpha}{\cos\left(\frac{\pi}{4}-\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)-isin\left(\frac{\pi}{4}+\frac{\alpha-\beta}{2}\right)\right]}$
- $=\left[cos\alpha.\sec\left(\dfrac{\pi}{4}-\dfrac{\alpha+\beta}{2}\right)\right]\left[cos\left(\dfrac{\pi}{4}+\dfrac{\alpha-\beta}{2}\right)+\ i\sin\left(\dfrac{\pi}{4}+\dfrac{\alpha-\beta}{2}\right)\right]$
- $= r[\cos\theta+i\sin\theta]$
- Where $r=\cos\alpha\ \ sec\left(\dfrac{\pi}{4}-\dfrac{\alpha+\beta}{2}\right)$ and $\theta=\dfrac{\pi}{4}+\dfrac{\alpha-\beta}{2}$

EXAMPLE-12

- If z_1 *and* z_2 are any two complex numbers, prove that
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$
- **Solution:** Let $z_1 = x_1 + iy_1, \;\; z_2 = x_2 + iy_2$
- $\therefore |z_1 + z_2|^2 = |(x_1 + iy_1) + (x_2 + iy_2)|^2 = |(x_1 + x_2) + i(y_1 + y_2)|^2$
- $= (x_1 + x_2)^2 + (y_1 + y_2)^2$
- Similarly, $|z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
- and $|z_1|^2 = x_1^2 + y_1^2 \qquad |z_2|^2 = x_2^2 + y_2^2$
- l.h.s. $= |z_1 + z_2|^2 + |z_1 - z_2|^2$
- $= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2$
  $= (x_1^2 + x_2^2 + 2x_1x_2) + (y_1^2 + y_2^2 + 2y_1y_2) + (x_1^2 + x_2^2 - 2x_1x_2) + (y_1^2 + y_2^2 - 2y_1y_2)$

$= 2[x_1^2 + x_2^2 + y_1^2 + y_2^2]$

- Now, r.h.s$= 2[|z_1|^2 + |z_2|^2] = 2[x_1^2 + x_2^2 + y_1^2 + y_2^2]$
- $\therefore \; l.h.s = r.h.s$

EXAMPLE-13

- If $|z-1| < |z+1|$, prove that Re $z > 0$.
- **Solution:** We have $|z-1| < |z+1|$
- $\therefore |x+iy-1| < |x+iy+1|$
- $\therefore |(x-1)+iy| < |(x+1)+iy|$
- $\therefore \sqrt{(x-1)^2+y^2} < \sqrt{(x+1)^2+y^2}$
- $\therefore (x-1)^2+y^2 < (x+1)^2+y^2$
- $\therefore x^2-2x+1+y^2 < x^2+2x+1+y^2$
- $\therefore -2x < 2x$
- $\therefore -4x < 0 \quad \therefore 4x > 0 \quad \therefore x > 0$

$\therefore$ The real Part of $z > 0$

EXAMPLE-14

- If $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$ Prove that $\frac{a+ib}{1+c} = \frac{1+iz}{1-iz}$.
- **Solution:** By data, $z = \frac{(b+ic)}{(1+a)}$ $\therefore iz = \frac{ib-c}{1+a}$
- $\therefore$ By componendo and dividendo,
- $\frac{1+iz}{1-iz} = \frac{1+a+ib-c}{1+a-ib+c} = \frac{(1+a-c)+ib}{(1+a+c)-ib} \cdot \frac{(1+a+c)+ib}{(1+a+c)+ib}$
- $= \frac{[(1+a+ib)-c].[(1+a+ib)+c]}{[(1+a+c)-ib][(1+a+c)+ib]} = \frac{1+a^2-b^2+2a+2ib+2aib-c^2}{1+a^2+c^2+2a+2c+2ac+b^2}$
- Since by data $a^2 + b^2 + c^2 = 1$, in the numerator, we put $1 - b^2 - c^2 = a^2$
- and in the denominator, we put $a^2 + b^2 + c^2 = 1$
- $\frac{1+iz}{1-iz} = \frac{2a^2+2a+2ib+2aib}{2+2a+2c+2ac}$
- $= \frac{a(a+1)+ib(1+a)}{1(a+1)+c(1+a)} = \frac{(1+a)(a+ib)}{(1+a)(1+c)} = \frac{a+ib}{1+c} = l.h.s$

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- Complex numbers, quadrants, conjugate etc.
- https://www.collegemathgames.com/games/complex-quadrants/

- Modulus & arguments
- https://quizizz.com/join/game/U2FsdGVkX19Z7CpHkv8AssNvYNlp6MzhJh5YtyhMi9k%252BEdWE6Vtw4VRSy1Mxn%252FZY?gameType=solo