LOGARITHMS OF COMPLEX NUMBERS

Let $z = x + iy$ and also let $x = r\cos\theta\,, y = r\sin\theta$ so that $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}(y/x)$.

Hence, $\log z = \log(r(\cos\theta + i\sin\theta)) = \log\big(r.e^{i\theta}\big)$

$= \log r + \log e^{i\theta} = \log r + i\theta$

$\therefore \quad \log(x+iy) = \log r + i\theta$

$\therefore \quad \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$ .... (1)

This is called **principal value** of log (x + iy)

The general value of log (x + iy) is denoted by Log (x + iy) and is given by

$\therefore \text{Log}\ (\mathrm{x} + \mathrm{iy}) = 2\mathrm{n}\pi\mathrm{i} + \log(\mathrm{x} + \mathrm{iy})$

$\therefore \text{Log}\ (\mathrm{x} + \mathrm{iy}) = 2\mathrm{n}\pi\mathrm{i} + \frac{1}{2}\log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$

$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i(2n\pi + tan^{-1}\frac{y}{x})$ .... (2)

Caution: $\theta = tan^{-1}y/x$ only when x and y are both positive.

In any other case θ is to be determined from $x = r\cos\theta\,, y = r\sin\theta\,, -\pi \le \theta \le \pi$.

SOLVED EXAMPLES:

$\log(-1) = ?$

$z = -1 = cos\pi + isin\pi$

$\therefore \log(-1) = \log(1) + i\pi = i\pi$

$\log(i) = ?$

$$\log(i) = \log(1) + \frac{i\pi}{2} = \frac{i\pi}{2}$$

$Log\ (\ -100) = ?$

$\text{Log}(-100) = \log(100) + i(\pi + 2n\pi)$

$Log(-40i) = ?$

$$\text{Log}(-40i) = \log(40) + i(-\frac{\pi}{2} + 2n\pi)$$

1. Considering the principal value only prove that $\log_2(-3) = \frac{\log 3 + i\,\pi}{\log 2}$

Solution: Since $log(x+iy) = \frac{1}{2}log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$

Putting $x = -3, y = 0$

we have $\log(-3) = \frac{1}{2}(9) + i\,tan^{-1}\left(\frac{0}{-3}\right) = \frac{1}{2}\log 3^2 + i\pi = \log 3 + i\pi$

$log_2(-3) = \frac{log_e(-3)}{log_e 2} = \frac{\log 3 + i\pi}{\log 2}$

2. Find the general value of $\mathrm{Log}(1+\mathrm{i})+\mathrm{Log}\,(1-\mathrm{i})$

Solution: $\log(1+i)=\frac{1}{2}\log 2+i\,\frac{\pi}{4}=log\sqrt{2}+i\frac{\pi}{4}$

$\therefore \mathrm{Log}(1+i)=log\sqrt{2}+i\left(2n\pi+\frac{\pi}{4}\right)$ (General value)

Changing the sign of i,

$\mathrm{Log}(1-i)=log\sqrt{2}-i\left(2n\pi+\frac{\pi}{4}\right)$

By addition, we get $\mathrm{Log}(1+i)+\mathrm{Log}(1-i)=2\,log\sqrt{2}=2.\frac{1}{2}\log 2=\log 2$

3. Prove that $\log\left(1+e^{2\,i\,\theta}\right)=\log(2\cos\theta)+i\,\theta$

Solution: $\log\left(1+e^{2\,i\,\theta}\right)=\log(1+\cos 2\theta+i\sin 2\theta)$

$$=\log(2\cos^2\theta+i2\sin\theta\cos\theta)$$

$$=\log\left(2\cos\theta\,(\cos\theta+i\sin\theta)\right)$$

$$=\log\left(2\cos\theta\cdot e^{i\theta}\right)$$

$$=\log(2\cos\theta)+\log\left(e^{i\theta}\right)$$

$$=\log(2\cos\theta)+i\theta$$

4. Prove that $\log\frac{1}{1-e^{\,i\,\theta}}=\log(\frac{1}{2}\mathrm{cosec}\frac{\theta}{2})+i(\frac{\pi}{2}-\frac{\theta}{2})$

Solution: $\log\left(\frac{1}{1-e^{\,i\,\theta}}\right)=\log\left(\frac{1}{1-(\cos\theta+i\sin\theta)}\right)$

$$=\log\left(\frac{1}{(1-\cos\theta)-i\sin\theta)}\right)$$

$$=\log\left(\frac{1}{2\sin^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\log\left(\frac{1}{2\sin\frac{\theta}{2}(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2})}\right)$$

$$=\log\left(\frac{1}{2}\mathrm{cosec}\frac{\theta}{2}\cdot e^{i(\frac{\pi}{2}-\frac{\theta}{2})}\right)$$

$$=\log(\frac{1}{2}\mathrm{cosec}\frac{\theta}{2})+i(\frac{\pi}{2}-\frac{\theta}{2})$$

Exercise:

- Prove that $\log\frac{1}{1+e^{\,i\,\theta}}=\log(\frac{1}{2}\sec\frac{\theta}{2})-i\frac{\theta}{2}$
- Prove that $\log\left(1+e^{\,i\,\theta}\right)=\log(\cos\frac{\theta}{2})+i\frac{\theta}{2}$
- Prove that $\log(1+cos\theta+isin\theta)=\log(\cos\frac{\theta}{2})+i\frac{\theta}{2}$

5. Find the value of $\log\,[\sin(x+i\,y)]$

Solution: We have, $\sin(x+iy)=\sin x\cosh y+icos\,x\,sin\,hy$

$\therefore \log\sin(x+iy)=\frac{1}{2}\log(sin^2x\cos h^2y+\,cos^2x\sin h^2y)+i\,tan^{-1}\left(\frac{\cos x\sin hy}{\sin x\cos hy}\right)$

Now, $sin^2x\cos h^2y+\,cos^2x\,sinh^2y=(1-cos^2x)\cos h^2y+cos^2x\,(\cos h^2y-1)$

$$=cosh^2y-cos^2x$$

$$= \left(\frac{1+\cos h\,2\,y}{2}\right) - \left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}(\cos h\,2\,y - \cos 2x)$$

$$\therefore \log\sin(x+iy) = \frac{1}{2}log\left(\frac{\cos h\,2\,y - \cos 2x}{2}\right) + i\,tan^{-1}(\cot x \tan hy)$$

6. Prove that $\log\frac{\sin(x+iy)}{\sin(x-iy)} = 2i\,tan^{-1}(\cot x \tan hy)$

Solution: We have, $\log\frac{\sin(x+iy)}{\sin(x-iy)} = \log\sin(x+iy) - \log\sin(x-iy)$.......1

$$\sin(x+iy) = \sin x\cos hy + icos\,x\,sin\,hy$$

$\therefore \log\sin(x+iy) = \frac{1}{2}\log(sin^2 x\cos h^2 y + cos^2 x\sin h^2 y) + i\,tan^{-1}\left(\frac{\cos x\sin hy}{\sin x\cos hy}\right)$...2

$\log\sin(x-iy) = \frac{1}{2}\log(sin^2 x\cos h^2 y + cos^2 x\sin h^2 y) - i\,tan^{-1}\left(\frac{\cos x\sin hy}{\sin x\cos hy}\right)$ ...3

Using (2) & (3) in (1)

We get, $\log\frac{\sin(x+iy)}{\sin(x-iy)} = 2i\tan^{-1}(\cot x\tan hy)$

Exercise:

- prove that $\log\frac{\cos(x-iy)}{\cos(x+iy)} = 2i\,tan^{-1}(\tan x\tan hy)$
- prove that $\log\frac{(x+iy)}{(x-iy)} = 2i\,tan^{-1}(y/x)$
- prove that $i\ \log\frac{(x-i)}{(x+i)} = \pi - 2\,tan^{-1}(x)$
- Separate in to real and imaginary parts $tanh^{-1}(x+iy)$

Hint: $\tan h^{-1}(x+iy) = \frac{1}{2}\log\frac{1+x+iy}{1-x-iy}$

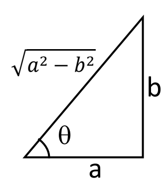
7. Show that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2\,ab}{a^2-b^2}$

Solution: We have $\log(a-bi) = \frac{1}{2}log(a^2+b^2) - itan^{-1}\frac{b}{a}$

And $\log(a+bi) = \frac{1}{2}log(a^2+b^2) + itan^{-1}\frac{b}{a}$

$$\therefore log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi) = -2itan^{-1}\,\frac{b}{a}$$

$$\therefore i\,log\left(\frac{a-bi}{a+bi}\right) = -2i^2tan^{-1}\frac{b}{a} = 2tan^{-1}\,\frac{b}{a}$$



$$\therefore tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\} = \tan\left(2tan^{-1}\frac{b}{a}\right)$$

$$\therefore tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\} = \tan 2\theta \quad where\ tan^{-1}\frac{b}{a} = \theta$$

$$= \frac{2\tan\theta}{1-tan^2\theta} = \frac{2(b/a)}{1-(b^2/a^2)} = \frac{2ab}{a^2-b^2}$$

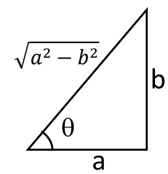
8. Prove that $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$

Solution: We have $\log(a-bi) = \frac{1}{2}log(a^2+b^2) - itan^{-1}\frac{b}{a}$

And $\log(a+bi) = \frac{1}{2} log(a^2+b^2) + itan^{-1}\frac{b}{a}$

$\therefore\ log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi) = -2itan^{-1}\ \frac{b}{a}$

$\therefore\ i\ log\left(\frac{a-bi}{a+bi}\right) = -2i^2tan^{-1}\frac{b}{a} = 2tan^{-1}\ \frac{b}{a}$



$cos\left[i\ log\left(\frac{a-ib}{a+ib}\right)\right] = cos\left(2tan^{-1}\frac{b}{a}\right)$

$cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = cos2\theta \quad where\ tan^{-1}\frac{b}{a} = \theta$

$= cos^2\theta - sin^2\theta = \frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$

Exercise

- Prove that $\sin\left[\mathrm{i}\log\left(\frac{\mathrm{a-ib}}{\mathrm{a+ib}}\right)\right] = \frac{2ab}{\mathrm{a^2+b^2}}$

9. Separate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$

Solution: We have $\sqrt{i} = i^{1/2} = \left(cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}$

Also $\sqrt{i} = \left(cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \left(e^{i\,\pi/2}\right)^{1/2} = e^{i\,\pi/4}$

$\therefore\ \left(\sqrt{i}\right)^{\sqrt{i}} = \left\{e^{i\,\pi/4}\right\}^{\left(\frac{1}{\sqrt{2}}+i.\frac{1}{\sqrt{2}}\right)} = e^{i\,\pi/4\sqrt{2}-\pi/4\sqrt{2}} = e^{-\,\pi/4\sqrt{2}}.e^{i\,\pi/4\sqrt{2}}$

$= e^{-\,\pi/4\sqrt{2}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$

$\therefore$ Real Part $= e^{-\,\pi/4\sqrt{2}}\cos\left(\frac{\pi}{4\sqrt{2}}\right)$ & Imaginary Part $= e^{-\,\pi/4\sqrt{2}}\sin\left(\frac{\pi}{4\sqrt{2}}\right)$

10. Find the principal value of $(1+i)^{1-i}$

Solution: $z = (1+i)^{1-i}$

$\therefore \log z = (1-i)\log(1+i)$

$\therefore \log z = (1-i)\left[\log\sqrt{1+1} + \mathrm{i}\tan^{-1}1\right]$

$= (1-i)\left[\frac{1}{2}\log 2 + i.\frac{\pi}{4}\right]$

$= \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}log2\right) = x + iy\ say$

$\therefore\ z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$

$= e^{\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)}\left[cos\left(\frac{\pi}{4} - \frac{1}{2}log2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}log2\right)\right]$

$= \sqrt{2}e^{\pi/4}\left[cos\left(\frac{\pi}{4} - \frac{1}{2}log2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}log2\right)\right] \qquad \because e^{\frac{1}{2}\log 2} = e^{log\sqrt{2}} = \sqrt{2}$

Exercise:

- Separate into real and imaginary parts i^i
- Separate into real and imaginary parts $(1+i)^i$
- Separate into real and imaginary parts $(i)^{(1-i)}$

11. Prove that the general value of $(1+\mathrm{i}\tan\alpha)^{-\mathrm{i}}$ is

$e^{2\,m\,\pi+\alpha}[\cos(\log\cos\alpha)+\mathrm{i}\sin(\log\cos\alpha)]$

Solution: Let $1+i\tan\alpha = r\,e^{\,i\theta}$

$\therefore r^2 = 1+tan^2\alpha = sec^2\alpha \qquad \therefore r = \sec\alpha$

And $\theta = \tan^{-1}\left(\frac{\tan\alpha}{1}\right) = \tan^{-1}(\tan\alpha) = \alpha$

Now, $Log\,(1+i\,\tan\alpha) = log\left(r\,e^{\,i\theta}\right) = \log r + (2m\pi+\theta)i$

$= \log\sec\alpha + (2m\pi+\alpha)i$

$\therefore 1+i\,\tan\alpha = e^{[\log\sec\alpha+(2m\pi+\alpha)i]}$

$\therefore (1+i\,\tan\alpha)^{-i} = e^{-i[\log\sec\alpha+(2m\pi+\alpha)i]}$

$= e^{2m\pi+\alpha}.e^{-ilog\sec\alpha}$

$= e^{2m\pi+\alpha}.e^{i\,(log\cos\alpha)}$

$= e^{2m\pi+\alpha}.[\cos\,(\log\cos\alpha)+i\sin(\log\cos\alpha)]$

12. Considering only principal value, if $(1+i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{sec^2\beta}$

Solution: Let $z = (1+i\,tan\alpha)^{1+i\tan\beta}$

Taking logarithms of both sides,

$\text{Log}\,z = (1+i\tan\beta)log(1+i\,tan\alpha)$

$= (1+i\tan\beta)\left[\frac{1}{2}log(1+tan^2\alpha)+itan^{-1}tan\alpha\right]$

$= (1+i\tan\beta)[\log\sec\alpha+i\alpha]$

$\therefore \log z = (\log\sec\alpha-\alpha\tan\beta)+i(\alpha+tan\beta\log\sec\alpha) = x+iy$ say

Where $x = \log\sec\alpha-\alpha\tan\beta$ $and\ y = \alpha+tan\beta\log secc\alpha$ ....(i)

Now, $z = e^{x+iy} = e^x.e^{iy} = e^x(cosy+i\sin y)$

Since by data z is real

$\therefore e^x\sin y = 0 \qquad \therefore y = 0 \qquad \therefore \cos y = 1$

$\therefore z = e^x\cos y = e^x = e^{\log sec\alpha-\alpha\tan\beta}$ from (i)

$\therefore z = e^{\log sec\alpha}.e^{-\alpha\tan\beta} = sec\alpha.e^{-\alpha\tan\beta}$ ....(ii)

But since $y = 0$, from (i) $\alpha+tan\beta\log\sec\alpha = 0$

$\therefore -\alpha = tan\beta\log sec\alpha$

$\therefore -\alpha\ tan\beta = tan^2\beta.\log sec\ \alpha = log(sec\ \alpha)^{tan^2\beta}$

$\therefore e^{-\alpha\tan\beta} = (sec\ \alpha)^{tan^2\beta}$

$\therefore$from (ii) $z = sec\alpha.(sec\ \alpha)^{tan^2\beta} = (sec\ \alpha)^{(1+tan^2\beta)} = (sec\ \alpha)^{sec^2\beta}$

13. If $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha+i\,\beta$, find α and β

Solution: $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha+i\,\beta,$

Taking logarithms of both sides, $log\left(\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}\right) = log(\alpha + i\,\beta)$

$\log(\alpha + i\beta) = (x+iy)\log(a+ib) - (x-iy)\log(a-ib)$

$log(\alpha + i\beta) = (x+iy)\left[\frac{1}{2}log(a^2+b^2) + i\,tan^{-1}\left(\frac{b}{a}\right)\right] - (x-iy)\left[\frac{1}{2}log(a^2+b^2) - i\,tan^{-1}\left(\frac{b}{a}\right)\right]$

$log(\alpha + i\beta) = 2i\left[xtan^{-1}\frac{b}{a} + \frac{y}{2}log(a^2+b^2)\right]$

$= 2ik\ say \qquad where\ k = \left[xtan^{-1}\frac{b}{a} + \frac{y}{2}log(a^2+b^2)\right]$

$\therefore (\alpha + i\beta) = e^{2ik} = cos2k + isin\,2k$

$\therefore \alpha = cos2k,\ \beta = sin\,2k \qquad where\ k = \left[xtan^{-1}\frac{b}{a} + \frac{y}{2}log(a^2+b^2)\right]$

14. If $\mathrm{i}^{\alpha+\mathrm{i}\,\beta} = \alpha + \mathrm{i}\,\beta$ $\left(\text{or } \mathrm{i}^{\mathrm{i}^{\mathrm{i}\ldots\ldots\infty}} = \alpha + \mathrm{i}\,\beta\right)$, prove that $\alpha^2 + \beta^2 = \mathrm{e}^{-(4\mathrm{n}+1)\pi\beta}$ Where n is any positive integer

Solution: Since $i = \cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$

we have $\mathrm{i}^{\alpha+\mathrm{i}\,\beta} = \alpha + \mathrm{i}\,\beta$

$\left[\cos\left(2n\pi + \frac{\pi}{2}\right) + \ i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{\alpha+i\beta} = \alpha + i\beta$

$\therefore e^{i\left(2n\pi+\frac{\pi}{2}\right)(\alpha+i\beta)} = \alpha + i\beta$

$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta + i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha + i\beta$

$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}.e^{i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha + i\beta$

$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\left[\cos\left(2n\pi + \frac{\pi}{2}\right)\alpha + i\sin\left(2n\pi + \frac{\pi}{2}\right)\alpha\right] = \alpha + i\beta$

Equating real and imaginary parts

$e^{-(4n+1)\frac{\pi}{2}\beta}\cos\left(2n\pi + \frac{\pi}{2}\right)\alpha = \alpha$ and $e^{-(4n+1)\frac{\pi}{2}\beta}\sin\left(2n\pi + \frac{\pi}{2}\right)\alpha = \beta$

Squaring and adding, we get $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$

15. Prove that $\log tan\left(\frac{\pi}{4} + i\frac{x}{2}\right) = i\,tan^{-1}(\sinh x)$.

Solution: $\log\tan\left(\frac{\pi}{4} + \frac{ix}{2}\right) = log\left\{\frac{1+tan(ix/2)}{1-tan(ix/2)}\right\}$

$= log\left\{\frac{1+\mathrm{i}\tan h\,(x/2)}{1-\mathrm{i}\tan h(x/2)}\right\}$

$= log[1 + i\tan h\,(x/2)] - log[1 - i\tan h(x/2)\,]$

$= \left[\frac{1}{2}log\left(1 + tanh^2\left(\frac{x}{2}\right)\right) + itan^{-1}tanh\left(\frac{x}{2}\right)\right]$

$- \left[\frac{1}{2}log\left(1 + \tan h^2\left(\frac{x}{2}\right)\right) - i\,tan^{-1}\tan h\left(\frac{x}{2}\right)\right]$

$= 2i\,tan^{-1}\tan h\left(\frac{x}{2}\right) = i.tan^{-1}\left\{\frac{2\tan h(x/2)}{1-tanh^2(x/2)}\right\} = itan^{-1}(\sin hx)$

$\because 2tan^{-1}\alpha = tan^{-1}\left\{\frac{2\alpha}{1-\alpha^2}\right\}$