K. J. Somaiya College of Engineering, Mumbai-77
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F.Y. Btech SEM-I

APPLIED MATHEMATICS-I

QUESTION BANK -1

TOPIC – COMPLEX NUMBERS

Type -1: Review

1. Express the following in the form $x + iy$

(i) $\frac{(2+i)(1+2i)}{3+4i}$ **(ii)** $\frac{(2+3i)^2}{1+i}$

2. Find the complex conjugate of **(i)** $\frac{3+5i}{1+2i}$ **(ii)** $\frac{1+i}{1-i}$

3. Find the value of $x^4 - 4x^3 + 4x^2 + 8x + 46 \; when \; x = 3 + 2i$

4. Find the modulus and the principal argument of

(i) $-1 + \sqrt{3}.i$ **(ii)** $\frac{(2-3i)(5+3i)}{3-2i}$

5. Find the real part, imaginary part, modulus and argument of $(4 + 2i)(-3 + \sqrt{2}i)$

6. Express the following in polar form and find their arguments

(i) $\sqrt{3} + i$ **(ii)** $\sin\theta + \mathrm{i}\cos\theta$

7. Find the square root of $-5 + 12i$

8. If $z_1 = 1 + i, \; z_2 = 2 - i, \; z_3 = 3 + 2i$, find

(i) $\left|\frac{z_1 - z_2 - i}{z_1 + z_2 + i}\right|$ **(ii)** $|\overline{z}_2 - z_1|^2 + |\overline{z}_3 - z_1|^2$ **(iii)** $\frac{z_3}{z_1} + \frac{z_2}{z_3}$

(iv) $\frac{z_1}{\overline{z}_1} - \frac{\overline{z}_1}{z_1}$ **(v)** $\frac{1}{(z_2+z_3)(z_2-z_3)}$ **(vi)** $(z_2 - \overline{z}_2)^5$

9. If $z = x + iy$, prove that $\left(\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right) = 2\left(\frac{x^2-y^2}{x^2+y^2}\right)$

10. If $z = a\cos\theta + ia\sin\theta$, prove that $\left(\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right) = 2\cos 2\,\theta$

11. Prove that $\left|\frac{z-1}{\overline{z}-1}\right| = 1$

12. If $\alpha - i\beta = \frac{1}{a-ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1.$

13. If p is real and the complex number $\frac{1+i}{2+pi} + \frac{2+3i}{3+i}$ represents a point on the line $y = x$, prove that $p = -5 \pm \sqrt{21}$

14. If $x + iy = \sqrt{a + ib}$, prove that $(x^2 + y^2)^2 = a^2 + b^2$

15. If $arg.(z + 2i) = \pi/4 \; and \; arg.(z - 2i) = 3\pi/4, find \; z$

16. If $|z+i| = |z|$ and $arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$, find z

17. Find two complex numbers such that their sum is 6 and their product is 13.

18. If $sin\psi = i\tan\theta$, prove that $\cos\theta + i\sin\theta = \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$

19. Prove that $\frac{1+\cos\alpha+i\sin\alpha}{1-\cos\alpha+i\sin\alpha} = cot\left(\frac{\alpha}{2}\right).e^{i(\alpha-\pi/2)}$

20. If $p = \cos\theta + i\sin\theta, q = \cos\Phi + i\sin\Phi$, Show that $\frac{(p+q)(pq-1)}{(p-q)(pq+1)} = \frac{\sin\theta+\sin\Phi}{\sin\theta-\sin\Phi}$.

21. If $(a_1 + i\,b_1)(a_2 + i\,b_2)\ldots\ldots\ldots\ldots(a_n + i\,b_n) = A + i\,B$, prove that
$(a_1{}^2 + b_1{}^2)(a_2{}^2 + b_2{}^2)\ldots\ldots\ldots\ldots(a_n{}^2 + b_n{}^2) = A^2 + B^2$ and
$tan^{-1}\frac{b_1}{a_1} + tan^{-1}\frac{b_2}{a_2} + \cdots\ldots\ldots\ldots + tan^{-1}\frac{b_n}{a_n} = tan^{-1}\frac{B}{A}$.

22. If z_1 and z_2 are two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$,
prove that $arg.z_1 - arg.z_2 = \frac{\pi}{2}$

23. Prove that, if $|z-i| > |z+i|$ $then$ $Im(z) < 0$.

24. If $|z-1| = |z+1|$ then prove that Re $z = 0$

25. If $x^2 + y^2 = 1$, prove that $\frac{1+x+iy}{1+x-iy} = x + iy$

26. If $x + iy = \frac{3}{2+\cos\theta+i\sin\theta}$, prove that $(x-1)(x-3) + y^2 = 0$.

27. If z_1, z_2 are non – zero complex numbers of equal modulus and $z_1 \neq z_2$
then prove that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

28. If $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = k$ show that $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$.

29. If $z = x + iy$, prove that
(i) If $\frac{z+i}{z+2}$ is real, then locus of (x, y) is a straight line.
(ii) If it is pure imaginary, then the locus of a point (x, y) is a circle. Also find radius and centre.

Type – 2: De-Moivre's Theorem

1. Simplify

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7(\cos\theta - i\sin\theta)^8}$ **(ii)** $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^2}{(\cos 4\theta + i\sin 4\theta)^5(\cos 5\theta - i\sin 5\theta)^4}$

2. Prove that

(i) $\frac{(1+i)^8(1-i\sqrt{3})^3}{(1-i)^6(1+i\sqrt{3})^9} = \frac{i}{32}$ **(ii)** $\frac{(1+i\sqrt{3})^9(1-i)^4}{(\sqrt{3}+i)^{12}(1+i)^4} = -\frac{1}{8}$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$

4. Express in the form $a + ib$, $\frac{(1+i)^{10}}{(1+i\sqrt{3})^5}$

5. Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.

6. If $x_n + iy_n = \left(1 + i\sqrt{3}\right)^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

7. Simplify

(i) $(1 + cos\,\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n$ **(ii)** $\left(\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n$

8. Prove that $\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \sin\theta + i\cos\theta$ Hence deduct that

$\left(1 + sin\frac{\pi}{5} + i\,cos\frac{\pi}{5}\right)^5 + i\left(1 + sin\frac{\pi}{5} - i\,cos\frac{\pi}{5}\right)^5 = 0.$

9. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $and\ \bar{z}$ is the conjugate of z find the value of $(z)^{15} + (\bar{z})^{15}$.

10. Prove that, if n is a positive integer, then

(i) $(a + ib)^{m/n} + (a - ib)^{m/n} = 2\left(\sqrt{a^2 + b^2}\right)^{m/n} cos\left(\frac{m}{n}tan^{-1}\frac{b}{a}\right)$

(ii) $\left(\sqrt{3} + i\right)^{120} + \left(\sqrt{3} - i\right)^{120} = 2^{121}$

11. If n is a positive integer, prove that $(1 + i)^n + (1 - i)^n = 2\ \ 2^{n/2}\cos n\ \pi/4$

Hence, deduce that $(1 + i)^{10} + (1 - i)^{10} = 0$

12. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$ and 2 if $n = 3k$ where k is an integer.

13. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}cos(n\pi/3)$.

(i) Deduce that $\alpha^{15} + \beta^{15} = -2^{16}$ **(ii)** Deduce that $\alpha^6 + \beta^6 = 128$

14. If α, β are the roots of the equation $z^2 sin^2\theta - z.\sin 2\theta + 1 = 0$, prove that

$\alpha^n + \beta^n = 2\cos n\,\theta\, cosec^n\theta$

15. If $a = \cos 3\alpha + i\sin 3\alpha, b = \cos 3\beta + i\sin 3\beta, c = \cos 3\gamma + \ i\sin 3\gamma$, prove that

$\sqrt[3]{\frac{ab}{c}} + \sqrt[3]{\frac{c}{ab}} = 2cos(\alpha + \beta - \gamma)$

16. If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\emptyset$, $z + \frac{1}{z} = 2\cos\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$ **(ii)** $\sqrt{xyz} + \frac{1}{\sqrt{xyz}} = 2\ cos\left(\frac{\theta+\Phi+\psi}{2}\right)$

(iii) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\Phi)$ **(iv)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\ cos\left(\frac{\theta}{m} - \frac{\emptyset}{n}\right)$

17. If $x + \frac{1}{x} = 2\cos\theta$ then prove that $\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\cos n\theta}{\cos(n-1)\theta}$ and $\frac{x^{2n}-1}{x^{2n-1}-x} = \frac{\sin n\theta}{\sin(n-1)\theta}$

18. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$, prove that

$\frac{(b+c)(c+a)(a+b)}{abc} = 8\,cos\frac{(\alpha-\beta)}{2}cos\frac{(\beta-\gamma)}{2}cos\frac{(\gamma-\alpha)}{2}.$

19. If a, b, c are three complex numbers such that $a+b+c=0$, prove that

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ and **(ii)** $a^2+b^2+c^2=0$

20. If $\cos\alpha+\cos\beta+\cos\gamma=0 \ and \sin\alpha+\sin\beta+\sin\gamma=0$, Prove that

(i) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$, $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$.

(ii) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$

(iii) $cos(\alpha+\beta)+cos(\beta+\gamma)+cos(\gamma+\alpha)=0.$

(iv) $\sin(\alpha+\beta)+sin(\beta+\gamma)+sin(\gamma+\alpha)=0.$

(v) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\ cos(\alpha+\beta+\gamma)$

(vi) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

21. If $a\cos\alpha+b\cos\beta+c\cos\gamma=a\sin\alpha+b\sin\beta+c\sin\gamma=0$, Prove that

$a^3\cos 3\alpha+b^3\cos 3\beta+c^3\cos 3\gamma=3abc\cos(\alpha+\beta+\gamma)$ and

$a^3\sin 3\alpha+b^3\sin 3\beta+c^3\sin 3\gamma=3\ abc\ sin(\alpha+\beta+\gamma)$

22. If $x_r=cos\left(\frac{2}{3}\right)^r\pi+i\ sin\left(\frac{2}{3}\right)^r\pi$, prove that

(i) $x_1x_2x_3\ldots\infty=1$, **(ii)** $x_0x_1x_2\ldots\infty=-1$

23. If $(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)\ldots\ldots\ldots\ldots(\cos n\,\theta+i\sin n\,\theta)=i$, then show that the general value of $\theta=\left[2r+\frac{1}{n(n+1)}\right]\pi$

Type -3: Roots of Complex numbers

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1+\omega+\omega^2=0$ **(ii)** $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$

2. Prove that the n nth roots of unity are in geometric progression.

3. Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1-i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9+8x^6+x^3+8=0$ **(ii)** $x^4-x^3+x^2-x+1=0$

(iii) $(x+1)^8+x^8=0$

9. If $(x+1)^6=x^6, show\ that\ x=-\frac{1}{2}-i\ cot\frac{\theta}{2}\ where\ \theta=\frac{2k\pi}{6}, k=0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7=(x-1)^7$are given by $\pm i\ cot\frac{r\pi}{7}, r=1,2,3.$

11. If $\alpha, \alpha^2, \alpha^3, \dots \alpha^6$ are the roots of $x^7 - 1 = 0$, find them and prove that $(1-\alpha)(1-\alpha^2) \dots \dots \dots \dots (1-\alpha^6) = 7$.

12. Prove that $x^5 - 1 = (x-1)\left(x^2 + 2x\, cos\frac{\pi}{5} + 1\right)\left(x^2 + 2x\, cos\frac{3\pi}{5} + 1\right) = 0$.

13. Solve the equation $z^n = (z+1)^n$ and show that the real part of all the roots is $-1/2$.

14. If $a = e^{i\,2\pi/7}$ and $b = a + a^2 + a^4,\ c = a^3 + a^5 + a^6$**.** then prove that b & c are roots of quadratic equation $x^2 + x + 2 = 0$.

15. Prove that **(i)** $\sqrt{1 - cosce(\theta/2)} = \left(1 - e^{i\theta}\right)^{-1/2} - \left(1 - e^{-i\theta}\right)^{-1/2}$

(iv) $\sqrt{1 - sce(\theta/2)} = \left(1 + e^{i\theta}\right)^{-1/2} - \left(1 + e^{-i\theta}\right)^{-1/2}$

16. If $1 + 2i$ is a root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find all the other roots.

17. Find the roots common to $x^{12} - 1 = 0$ and $x^4 - x^2 + 1 = 0$