HYPERBOLIC FUNCTIONS

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

If $z = x + iy$ is complex number, then $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These are called circular function of complex numbers.

HYPERBOLIC FUNCTIONS:

If x is real or complex, then sine hyperbolic of x is denoted by $sinh\ x$ and is given as, $\sinh x = \frac{e^{x}-e^{-x}}{2}$ and

Cosine hyperbolic of x is denoted by $cosh\ x$ and is given as, $\cosh x = \frac{e^{x}+e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as $\tan \mathrm{hx} = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$\mathrm{cosechx} = \frac{1}{\sinh x} = \frac{2}{e^{x}-e^{-x}}$, $\mathrm{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^{x}+e^{-x}}$, and $\coth x = \frac{1}{\tanh x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinh x, cosh x, tanh x, we can obtain the following values of hyperbolic function.

x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

Note: since $\tanh(-\infty) = -1$, $\tanh(0) = 0$, $\tanh(\infty) = 1$ $\quad \therefore |\tanh x| \le 1$

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i\sin ix$	$\sinh ix = i\sin x$ & $\sin x = -i\sinh ix$
(ii)	$\cos ix = \cosh x$	$\cosh ix = \cos x$
(iii)	$\tan ix = i\tanh x$ & $\tanh x = -i\tan ix$	$\tanh ix = i\tan x$ & $\tan x = -i\tanh ix$

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	HYPERBOLIC FUNCTIONS
1	$\sin(-x) = -(\sin x)$	$\sinh(-x) = -\sinh x,$
2	$\cos(-x) = (cosx)$	$\cosh(-x) = \cosh x$

3	$e^{ix} = \cos x + i\sin x$	$e^{x} = \cosh x + \sinh x$
4	$e^{-ix} = \cos x - i\sin x$	$e^{-x} = \cosh x - \sinh x$
5	$sin^2x + cos^2x = 1$	$\cosh^2 x - \sinh^2 x = 1$
6	$1 + \tan^2 x = \sec^2 x$	$\text{sech}^2 x + \tanh^2 x = 1$
7	$1 + \cot^2 x = \text{cosec}^2 x$	$\coth^2 x - \text{cosech}^2 x = 1$
8	$\sin 2x = 2\sin x\cos x$ $= \frac{2\tan x}{1+\tan^2 x}$	$\sinh 2x = 2\sinh x\cosh x$ $= \frac{2\tanh x}{1-\tanh^2 x}$
9	$\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \frac{1-\tan^2 x}{1+\tan^2 x}$	$\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \frac{1+\tanh^2 x}{1-\tanh^2 x}$
10	$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$	$\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$
11	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
12	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
13	$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$	$\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$
14	$\sin(x \pm y) = \sin x\cos y \pm \cos x\, sin\, y$	$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x\sinh y$
15	$\cos(x \pm y) = \cos x\cos y \mp \sin x\, sin\, y$	$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y$
16	$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x\tanh y}$
17	$\cot(x \pm y) = \dfrac{\cot x\cot y \mp 1}{\cot y \pm \cot x}$	$\coth(x \pm y) = \dfrac{-\coth x\coth y \mp 1}{\coth y \pm \coth x}$
18	$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$
19	$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$
20	$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$
21	$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$
22	$2\sin x\cos y = \sin(x+y) + \sin(x-y)$	$2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y)$
23	$2\cos x\sin y = \sin(x+y) - \sin(x-y)$	$2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y)$

24	$2\cos x \cos y = \cos(x+y) + \cos(x-y)$	$2\cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$
25	$2\sin x \sin y = \cos(x-y) - \cos(x+y)$	$2\sinh x \sinh y = \cos h(x+y) - \cos h(x-y)$

PERIOD OF HYPERBOLIC FUNTIONS:

$$\begin{aligned}\sinh(2\pi i + x) &= \sinh(2\pi i)\cosh x + \cosh(2\pi i)\sinh x \\ &= i\sin 2\pi \cosh x + \cos 2\pi \sinh x \\ &= 0 + \sinh x \\ &= \sinh x\end{aligned}$$

Hence sinh x is a periodic function of period 2πi

Similarly we can prove that cosh x and tanh x are periodic functions of period 2πi and πi .

DIFFERENTIATION AND INTRGRATION :

(i) If $y = \sinh x$, $y = \frac{e^x - e^{-x}}{2}$ $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$

If $y = \sinh x$, $\frac{dy}{dx} = \cosh x$

(ii) If $y = cosh\ x$, $y = \frac{e^x + e^{-x}}{2}$, $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$

If $y = cosh\ x$, $\frac{dy}{dx} = \sinh x$

(iii) If $y = \tanh x$, $y = \frac{\sinh x}{\cosh x}$ $\quad \therefore \frac{dy}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$

If $y = \tanh x$, $\frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$\int \cosh x\, dx = \sinh x$, $\quad \int \sinh x\, dx = \cosh x$, $\quad \int sech^2 x dx = \tanh x$

SOME SOLVED EXAMPLES:

1. If tanh x $= \frac{1}{2}, find \sinh 2x\ and \cosh 2x$ -

Solution: $\tan hx = \frac{e^x - e^{-x}}{e^x + e^x} = \frac{1}{2}$ $\quad \therefore \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1}{2}$ $\quad \therefore 2\, e^{2x} - 2 = e^{2x} + 1$ $\quad \therefore\ e^{2x} = 3$

Now, $sin\ h2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3 - (1/3)}{2} = \frac{4}{3}$

Now, $\cos h2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3 + (1/3)}{2} = \frac{5}{3}$

2. Solve the equation $7cosh\ x + 8sinh\ x = 1$ for real values of x.

Solution: $7cosh\ x + 8sinh\ x = 1$

Putting the values of $coshx$ and $sin\ hx$, we get

$\therefore 7\left(\frac{e^x+e^{-x}}{2}\right)+8\left(\frac{e^x-e^{-x}}{2}\right)=1$

$\therefore 7e^x+7e^{-x}+8e^x-8e^{-x}=2$

$\therefore 15e^x-e^{-x}=2$

$\therefore 15e^{2x}-2e^x-1=0$ Solving it as a quadratic equation in e^x,

$e^x=\frac{2\pm\sqrt{4-4(15)(-1)}}{2(15)}=\frac{2\pm 8}{30}=\frac{1}{3}\ \ or-\frac{1}{5}$

$\therefore x=\log\left(\frac{1}{3}\right)\ \ or\ x=\log\left(-\frac{1}{5}\right)$

Since x is real, $x=log\left(\frac{1}{3}\right)=-\log 3$