EXERCISE – I

1. If A is a skew – symmetric matrix and X is a column matrix, then prove that $X^T AX$ is a null matrix.
2. (i) If A and B are Hermitian matrices then prove that $(A+B)$ is also Hermitian matrix.
   (ii) If A and B are skew – Hermitian matrices then prove that $(A+B)$ is skew – Hermitian matrix.
3. If A is any square matrix, then show that $A + A^\theta$ is Hermitian and $A - A^\theta$ is skew – Hermitian
4. If A is any matrix, then show that AA^θ and $A^\theta A$ are Hermitian matrices.
5. Show that the matrix $B^\theta AB$ is Hermitian or skew – Hermitian accordingly when A is Hermitian or skew – Hermitian matrix.
6. If A and B are skew – symmetric matrices of order n, then show that AB is symmetric if and only if A and B commute.
7. If $A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & -2 & 0 \end{bmatrix}$ show that $A + A^T$ is symmetric and $A - A^T$ is skew symmetric
8. Express the following matrices as the sum of symmetric and skew – symmetric matrices
   (i) $\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 8 & 6 \\ 0 & 4 & 4 \\ 2 & 10 & 12 \end{bmatrix}$ (iii) $\begin{bmatrix} 3a & 2b & 2c \\ b & c & a \\ 3c & 3a & 3b \end{bmatrix}$
9. Express the following matrices as the sum of Hermitian and skew – Hermitian matrices
   (i) $\begin{bmatrix} 2+i & -i & 3+i \\ 1+i & 3 & 6-2i \\ 3-2i & 6i & 4-3i \end{bmatrix}$. (ii) $\begin{bmatrix} 1+i & 2-3i & 2 \\ 3-4i & 4+5i & 1 \\ 5 & 3 & 3-i \end{bmatrix}$
10. Express the following Hermitian matrices as $B + iC$ where B is real symmetric and C is real skew symmetric.
   $\begin{bmatrix} 4 & 3-2i & -1+i \\ 3+2i & 2 & 5+4i \\ -1-i & 5-4i & 7 \end{bmatrix}$
11. Express the following skew – Hermitian matrices as $P + iQ$ where P is real skew – symmetric and Q is real symmetric. $\begin{bmatrix} 2i & 3+i & 2-i \\ -3+i & 0 & 6i \\ -2-i & 6i & -2i \end{bmatrix}$
12. Prove that $\overline{A}$ is Hermitian or skew – Hermitian accordingly when A is Hermitian or skew – Hermitian

ANSWERS

7. $A + A^T = \begin{bmatrix} 2 & -2 & 8 & 7 \\ -2 & 2 & 8 & -3 \\ 8 & 8 & 14 & -1 \\ 7 & -3 & -1 & 0 \end{bmatrix}$ and $A - A^T = \begin{bmatrix} 0 & 2 & 2 & -1 \\ -2 & 0 & 4 & 5 \\ -2 & -4 & 0 & 3 \\ 1 & -5 & -3 & 0 \end{bmatrix}$
8. (i) $A = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 4 & 7 \\ 4 & 7 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$
   (iii) $A = \begin{bmatrix} 3a & \frac{3b}{2} & \frac{5c}{2} \\ \frac{3b}{2} & c & 2a \\ \frac{5c}{2} & 2a & 3b \end{bmatrix} + \begin{bmatrix} 0 & \frac{b}{2} & \frac{-c}{2} \\ \frac{-b}{2} & 0 & -a \\ \frac{c}{2} & a & 0 \end{bmatrix}$

9. (i) $\begin{bmatrix} 2 & \frac{1}{2}-i & 3+\frac{3}{2}i \\ \frac{1}{2}+i & 3 & 3-4i \\ 3-\frac{3}{2}i & 3+4i & 4 \end{bmatrix} + \begin{bmatrix} i & \frac{-1}{2} & \frac{-i}{2} \\ \frac{1}{2} & 0 & 3+2i \\ \frac{-i}{2} & -3+2i & -3i \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & \frac{5}{2}+\frac{i}{2} & \frac{7}{2} \\ \frac{5}{2}-\frac{i}{2} & 4 & 2 \\ \frac{7}{2} & 2 & 3 \end{bmatrix} + \begin{bmatrix} i & \frac{-1}{2}-\frac{7}{2}i & \frac{-3}{2} \\ \frac{1}{2}-\frac{7}{2}i & 5i & -1 \\ \frac{3}{2} & 1 & -i \end{bmatrix}$

10. $B = \begin{bmatrix} 4 & 3 & -1 \\ 3 & 2 & 5 \\ -1 & 5 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix}$

11. $P = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 6 \\ -1 & 6 & -2 \end{bmatrix}$

EXERCISE – II

1. Verify that the matrix A is orthogonal, where $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ and find A^{-1}

2. Show that following matrices are orthogonal.

(i) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

(ii) $\begin{bmatrix} \cos\emptyset & 0 & \sin\emptyset \\ \sin\theta.\sin\emptyset & \cos\theta & -\sin\theta.\cos\emptyset \\ -\cos\theta.\sin\emptyset & \sin\theta & \cos\theta.\cos\emptyset \end{bmatrix}$

3. Determine the values of ∝, β, γ when the matrix given by $A = \begin{bmatrix} \alpha & \beta & -\gamma \\ \alpha & -2\beta & 0 \\ \alpha & \beta & \gamma \end{bmatrix}$ is orthogonal

4. Determine the values of a, b, c when the matrix $\frac{1}{9}\begin{bmatrix} a & 1 & b \\ c & b & 7 \\ 1 & a & c \end{bmatrix}$ is orthogonal

5. If $A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$ is orthogonal then find a, b, c. Also find A^{-1}. State the rank of A^2

6. Is the following matrix orthogonal? If not, can it be converted into an orthogonal matrix? If yes how ?

$A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

7. If $(l_r, m_r, n_r), r = 1,2,3$ are the direction cosines of three mutually perpendicular lines referred to an orthogonal Cartesian coordinate system, then prove that the matrix $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is an orthogonal matrix.

8. If A = $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal find the relations between $(l_r, m_r, n_r), r = 1,2,3$

9. Prove that the following matrices are unitary and hence find A^{-1}.

(i) $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$

(ii) $\begin{bmatrix} \frac{1+i}{2} & \frac{i}{\sqrt{3}} & \frac{3+i}{2\sqrt{15}} \\ \frac{-1}{2} & \frac{1}{\sqrt{3}} & \frac{4+3i}{2\sqrt{15}} \\ \frac{1}{2} & \frac{-i}{\sqrt{3}} & \frac{5i}{2\sqrt{15}} \end{bmatrix}$

10. If N = $\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, then show that $(I-N)(I+N)^{-1}$ is unitary.

11. Show that if A is Hermitian and P is unitary, then $P^{-1}AP$ is Hermitian.

ANSWERS

1. $\frac{1}{3}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}$

3. $\alpha = \pm\frac{1}{\sqrt{3}}, \beta = \pm\frac{1}{\sqrt{6}}$ *and* $\gamma = \pm\frac{1}{\sqrt{2}}$

4. $a = \pm 8,\ b = \pm 4,\ c = \pm 4$

5. $a = \pm\frac{2}{3},\ b = \pm\frac{2}{3},\ c = \pm\frac{1}{3}$ $A^{-1} = \frac{1}{3}\begin{bmatrix}1 & 2 & \pm 2\\ 2 & 1 & \pm 2\\ 2 & -2 & \pm 1\end{bmatrix}$, rank of $A^2 = 3$

6. $\begin{bmatrix}2/3 & 2/3 & 1/3\\ -2/3 & 1/3 & 2/3\\ 1/3 & -2/3 & 2/3\end{bmatrix}$

8. $l_1^2 + m_1^2 + n_1^2 = 1, l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ etc

EXERCISE – III

1. Find the ranks of the following matrices

(i) $\begin{bmatrix}1 & 2 & -2 & 3\\ -1 & -3 & 2 & -2\\ 0 & -1 & 0 & 1\\ -1 & -4 & 2 & -1\end{bmatrix}$ (ii) $\begin{bmatrix}3 & 4 & 5 & 6 & 7\\ 4 & 5 & 6 & 7 & 8\\ 5 & 6 & 7 & 8 & 9\\ 10 & 11 & 12 & 13 & 14\\ 15 & 16 & 17 & 18 & 19\end{bmatrix}$ (iii) $\begin{bmatrix}2 & -3 & 5\\ 4 & -6 & 10\\ -8 & 12 & -20\\ 6 & -9 & 15\end{bmatrix}$

(iv) $\begin{bmatrix}1 & 2 & 3 & 1\\ 2 & 4 & 3 & 2\\ 3 & 2 & 1 & 3\\ 6 & 8 & 7 & 6\end{bmatrix}$ (v) $\begin{bmatrix}-1 & 2 & 3 & -2\\ 2 & -5 & 1 & 2\\ 3 & -8 & 5 & 2\\ -5 & -12 & -1 & 6\end{bmatrix}$ (vi) $\begin{bmatrix}2 & 3 & 1 & 4\\ 5 & 2 & 3 & 0\\ 9 & 8 & 0 & 8\end{bmatrix}$

(vii) $\begin{bmatrix}25 & 31 & 17 & 43\\ 75 & 94 & 53 & 132\\ 75 & 94 & 54 & 134\\ 25 & 32 & 20 & 48\end{bmatrix}$

2. Reduce the following matrices to their normal form and hence obtain their ranks.

(i) $\begin{bmatrix}1 & 2 & -1 & 3\\ 3 & 4 & 0 & -1\\ -1 & 0 & -2 & 7\end{bmatrix}$ (ii) $\begin{bmatrix}3 & -3 & 0 & -1 & -7\\ 0 & 2 & 2 & 1 & -5\\ 1 & -2 & -3 & -2 & 1\\ 0 & 1 & 2 & 1 & -6\end{bmatrix}$ (iii) $\begin{bmatrix}8 & 1 & 3 & 6\\ 0 & 3 & 2 & 2\\ -8 & 1 & -3 & 4\end{bmatrix}$

(iv) $\begin{bmatrix}-2 & -1 & -3 & -1\\ 1 & 2 & 3 & -1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\end{bmatrix}$ (v) $\begin{bmatrix}0 & 1 & 2 & 3 & 4\\ 1 & 1 & 2 & 3 & 3\\ 2 & 2 & 0 & 2 & 2\\ 3 & 3 & 2 & 1 & 1\\ 4 & 3 & 2 & 1 & 0\end{bmatrix}$ (vi) $\begin{bmatrix}6 & 1 & 3 & 8\\ 4 & 2 & 6 & -1\\ 10 & 3 & 9 & 7\\ 16 & 4 & 12 & 15\end{bmatrix}$

(vii) $\begin{bmatrix}2 & -4 & 3 & 1 & 0\\ 1 & -2 & 1 & -4 & 2\\ 0 & 1 & -1 & 3 & 1\\ 4 & -7 & 4 & -4 & 5\end{bmatrix}$ (viii) $\begin{bmatrix}3 & 2 & 5 & 7 & 12\\ 1 & 1 & 2 & 3 & 5\\ 3 & 3 & 6 & 9 & 15\end{bmatrix}$ (ix) $\begin{bmatrix}1 & 2 & -2 & 3 & 1\\ 1 & 3 & -2 & 3 & 0\\ 2 & 4 & -3 & 6 & 4\\ 1 & 1 & -1 & 4 & 6\end{bmatrix}$

(x) $\begin{bmatrix}1 & 3 & 5 & 7\\ 4 & 6 & 8 & 10\\ 15 & 27 & 39 & 51\\ 6 & 12 & 18 & 24\end{bmatrix}$ (xi) $\begin{bmatrix}1 & 1 & 0\\ 2 & 2 & 0\\ 0 & 1 & 1\end{bmatrix}$ (xii) $\begin{bmatrix}2 & 1 & -3 & -6\\ 3 & -3 & 1 & 2\\ 1 & 1 & 1 & 2\end{bmatrix}$

(xiii) $\begin{bmatrix}2 & 15 & 14 & 15\\ 6 & 24 & 18 & 30\\ 1 & 4 & 2 & 5\end{bmatrix}$ (xiv) $\begin{bmatrix}1 & 0 & 2 & -2\\ 2 & -1 & 0 & -1\\ 1 & 0 & 2 & -1\\ 4 & -1 & 3 & -1\end{bmatrix}$

3. Find the rank of A by reducing it to the normal form, where $A = \begin{bmatrix}2 & 1 & 3 & 4\\ 3 & -1 & 2 & 2\\ 4 & 1 & 0 & -1\\ 9 & 1 & 5 & 6\end{bmatrix}$

Hence find the rank of A^2

4. Reduce the following matrices to Echelon Forms and hence find the ranks.

(i) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

5. Find the values of P for which the matrix $A = \begin{bmatrix} P & 2 & 2 \\ 2 & P & 2 \\ 2 & 2 & P \end{bmatrix}$ will have **(i)** rank 1, **(ii)** rank 2, **(iii)** rank 3,

6. The rank of the matrix $\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{bmatrix}$ is 2. Find the value of λ, where λ is real.

7. Find the rank of A = $\begin{bmatrix} x-1 & x+1 & x \\ -1 & x & 0 \\ 0 & 1 & 1 \end{bmatrix}$ where x is real.

8. If x is a rational number, find the rank of A – xI where I is the identity matrix of order 3 and A = $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

9. Find the normal form of following Matrices, Hence obtain rank of A where A is

(i) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 7 & -11 & -6 & 1 \\ 7 & 2 & 12 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 8 & 5 & 8 \end{bmatrix}$ (v) $\begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$ (viii) $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 2 & 3 & 6 & 4 \\ 6 & 5 & 15 & 10 \end{bmatrix}$ (ix) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(x) $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 0 & 2, & 2 \\ 4 & 1 & 9 & 7 \end{bmatrix}$ (xi) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 3 \\ 5 & 6 & 10 & 2 \end{bmatrix}$ (xii) $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$

11. If A = $[a_{ij}]$ is a square matrix of order 3 where $a_{ij} = i + j$, find the rank of A

12. Find the rank of A = $[a_{ij}]_{3\times3}$ where $a_{ij} = \frac{i}{j}$

ANSWERS

1. (i) 2 (ii) 2 (iii) 1 (iv) 3 (v) 2 (vi) 3 (vii) 3
2. (i) 2 (ii) 4 (iii) 3 (iv) 2 (v) 4 (vi) 2 (vii) 3
   (viii) 2 (ix) 4 (x) 2 (xi) **2** (xii) **3** (xiii) **3** (xiv) 2
3. $\rho(A) = 4$ $\rho(A^2) = 4$ 4. (i) 2 (ii) 2
5. rank of A = 1 for P = 2, rank of A = 2 for P = -4,
   rank of A = 3 for any value of P other than 2 and -4.
6. $\lambda = 1$ 7. 3 8. $r = 3 \; if \; x \neq 1$
9. (i) 3 (ii) 3 (iii) 3 (iv) 2 (v) 3 (vi) 3 (vii) 2
   (viii) 3 (ix) **2** (x) 3 (xi) 3 (xii) 2
   11. 2 12. 1

EXERCISE

1. Show that the system $\begin{aligned} 2x_1 - 3x_2 + 7x_3 &= 5 \\ 3x_1 + x_2 - 3x_3 &= 13 \\ 2x_1 + 19x_2 - 47x_3 &= 32 \end{aligned}$ is inconsistent.

2. Test for consistency the following set of equations and obtain the solution if consistence.

(i) $\begin{aligned} 3x + 3y + 2z &= 1 \\ x + 2y &= 4 \\ 10y + 3z &= -2 \\ 2x - 3y - z &= 5 \end{aligned}$ (ii) $\begin{aligned} 2x - y - z &= 2 \\ x + 2y + z &= 2 \\ 4x - 7y - 5z &= 2. \end{aligned}$

(iii) $\begin{aligned} 2x_1 + 2x_2 &= -11 \\ 6x_1 + 20x_2 - 6x_3 &= -3 \\ 6x_2 - 18x_3 &= -1 \end{aligned}$ (iv) $\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 5x_2 - 2x_3 &= 3 \\ x_1 + 7x_2 - 7x_3 &= 5. \end{aligned}$

(v) $\begin{aligned} 5x_1 - 3x_2 - 7x_3 + x_4 &= 10 \\ -x_1 + 2x_2 + 6x_3 - 3x_4 &= -3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 0. \end{aligned}$ (vi) $\begin{aligned} 2x_1 - x_2 + x_3 &= 4 \\ 3x_1 - x_2 + x_3 &= 6 \\ 4x_1 - x_2 + 2x_3 &= 7 \\ -x_1 + x_2 - x_3 &= 9 \end{aligned}$

(vii) $\begin{aligned} 2x - y + 3z &= 9 \\ x + y + z &= 6 \\ x - y + z &= 2 \end{aligned}$ (viii) $\begin{aligned} x + y + 4z &= 6 \\ 3x + 2y - 2z &= 9 \\ 5x + y + 2z &= 13 \end{aligned}$

3. Investigate for what values of a and b the simultaneous equations $\begin{aligned} 2x - y + 3z &= 2 \\ x + y + 2z &= 2 \\ 5x - y + az &= b \end{aligned}$ have **(i)** no solution: **(ii)** a unique solution: **(iii)** an infinite number of solutions.

4. Investigate for what values of λ and μ the simultaneous equations $\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$ have **(i)** no solution: **(ii)** a unique solution: **(iii)** an infinite number of solutions.

5. Find the values of λ for which the system of equations $\begin{aligned} x + y + 4z &= 1 \\ x + 2y - 2z &= 1 \\ \lambda x + y + z &= 1 \end{aligned}$ will have **(i)** a unique solutions **(ii)** no solution

6. Find what values of λ of the set of equations $\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ x_1 + x_2 + x_3 &= \lambda \\ 3x_1 + x_2 + 3x_3 &= \lambda^2 \end{aligned}$ are consistent and solve them.

7. For what value of λ the equations $x + y + z = 1, x + 2y + 4z = \lambda, x + 4y + 10z = \lambda^2$ have a solution and solve them completely in each case.

8. Show that the system of equation $\begin{aligned} -2x + y + z &= a \\ x - 2y + z &= b \\ x + y - 2z &= c \end{aligned}$ have no solution unless $a + b + c = 0$, in which case they have infinitely many solutions. Find these solutions when $a = 1, b = 1, c = -2$.

9. Solve the system of equations.

(i) $\begin{aligned} x + 2y &= 1 \\ -3x + 2y &= -2 \\ -x + 6y &= 0 \end{aligned}$ (ii) $\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 5x_2 - 2x_3 &= 3 \end{aligned}$

10. Test the consistency the following equation and solve them if consistent
$x - 2y + 3t = 0, 2x + y + z + t = -4, 4x - 3y + z + 7t = 8$

ANSWERS

2. (i) Consistent and unique solution: $x = 2, y = 1, z = -4$

(ii) Consistent and infinite solution: $x = \frac{6+k}{5}, y = \frac{2-3k}{5}, z = k$

(iii) Inconsistent (iv) Inconsistent

(v) Consistent and infinite solutions: $x_1 = 5 - 7k, x_2 = 19 - 44k, x_3 = -6 + 14k, x_4 = k,$

(vi) Inconsistent (vii) Consistent and unique solution: $x = 1, y = 2, z = 3.$

(viii) Consistent and unique solution: $x = 2, y = 2, z = \frac{1}{2}$

3. (i) $a = 8, b \neq 6$ the system has no solution;

(ii) $a \neq 8$ the system has unique solution.

(iii) $a = 8, b = 6$ the system has infinite solution

4. (i) $\lambda = 3, \mu \neq 10$ the system has no solution

(ii) $\lambda \neq 3$ the system has unique solution.

(iii) $\lambda = 3, \mu = 10$ the system has infinite solution

5. (i) For $\lambda \neq 7/10$ the system has unique solution;

(ii) For $\lambda = 7/10$ no solution,

6. For $\lambda = 2$ the infinite number of solutions are: $x_1 = 1 - k_1, x_2 = 1, x_3 = k_1$

For $\lambda = 3$ the infinite number of solutions are: $x_1 = 3 - k, x_2 = 0, x_3 = 0$

7. For $\lambda = 2$ the infinite number of solutions are: $x = 2t, y = 1 - 3t, z = t$

For $\lambda = 1$ the infinite number of solutions are: $x = 1 + 2t, y = -3t, z = t$

8. $x = k - 1, y = k - 1, z = k$

9. (i) $x = \frac{3}{4} and\ y = \frac{1}{8}$ (iii) $\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{matrix} \frac{17}{3} - \frac{7}{3}a \\ -\frac{5}{3} + \frac{4}{3}a \\ a \end{matrix}$

10. $x = 2 - \frac{2}{5}t_2 - t_1,\ y = -\frac{1}{5}t_2 + t_1,\ z = t_2,\ t = t_1$

EXERCISE

1. Find the solution of the system given by
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 - x_3 &= 0 \\ 2x_1 - 4x_2 - 5x_3 &= 0 \end{aligned}$$
Also find the relation between column vectors of coefficient matrix.

2. Solve the following system of linear equation
$$\begin{aligned} x_1 - 2x_2 - x_3 &= 0 \\ -2x_1 + 4x_2 + 2x_3 &= 0 \\ -3x_1 - x_2 + 7x_3 &= 0 \\ 4x_1 + 3x_2 + 6x_3 &= 0 \end{aligned}$$

3. If the following system has non – trivial, prove that $a + b + c = 0$ or $a = b = c$, Where $ax + by + cz = 0, bx + cy + az = 0, cx + ay + bz = 0$. Find the non – trivial solution when the condition is satisfied.

4. Find the values of λ for which the following equations have non – zero solution. Obtain the general solution in each case.
$$\begin{aligned} 2x - 2y + z &= \lambda x \\ 2x - 3y + 2z &= \lambda y \\ -x + 2y &= \lambda z \end{aligned}$$

5. Find (trivial or non trivial) solutions of the following linear equations.

(i) $\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$

(ii) $\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 - x_4 &= 0 \\ 3x_1 - x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$

(ii) $\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 - x_3 &= 0 \\ 2x_1 - 4x_2 - 5x_3 &= 0 \end{aligned}$

(iv) $\begin{aligned} 2x_1 + 3x_2 - x_3 + x_4 &= 0 \\ 3x_1 + 2x_2 - 2x_3 + 2x_4 &= 0 \\ 5x_1 - 4x_3 + 4x_4 &= 0 \end{aligned}$

6. Find k if the system $\begin{aligned} 2x - 3y + 4z &= 0 \\ 3x + 4y + 6z &= 0 \\ 4x + 5y + kz &= 0 \end{aligned}$ has non trivial solution

ANSWERS

1. $x_1 = 2k,\ x_2 = k\ ,\ x_3 = 0$ Relationship $X_2 = -2X_1 + 0X_3$
2. $x_1 = x_2 = x_3 = 0$ is the trivial solution.
3. If $a + b + c = 0$ then $x = (ab - c^2)t, y = (bc - a^2)t,\ z = (ac - b^2)t$

   If $a = b = c$ then $x = -(t_1 + t_2),\ \ y = t_1,\ z = t_2$
4. For $\lambda = 1, \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 2k_1 - k_2 \\ k_1 \\ k_2 \end{vmatrix}$ and for $\lambda = -3, \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} -k \\ -2k \\ k \end{vmatrix}$
5. (i) Trivial Solution

   (ii) Infinite solutions: $x_1 = -\frac{1}{3}a, x_2 = \frac{2}{3}a, x_3 = -\frac{2}{3}a, x_4 = a.$

   (iii) Infinite solutions: $x_1 = 2b,\ x_2 = b,\ x_3 = 0.$

   (iv) Infinite solution: $x_1 = \frac{4}{5}k_1 - \frac{4}{5}k_2\ ,\ x_2 = -\frac{1}{5}k_1 + \frac{1}{5}k_2, x_3 = k_1,\ x_4 = k_2.$
6. $k = 8$

EXERCISE

1. Are the following vectors linearly dependent? If so find the relation between them.

   (i) $X_1 = [1\ 1\ 1\ 3], X_2 = [1\ 2\ 3\ 4], X_3 = [2\ 3\ 4\ 9]$

   (ii) $X_1 = [2\ 3\ 4 - 2], X_2 = [-1 - 2 - 2\ 1], X_3 = [1\ 1\ 2 - 1]$

   (iii) $X_1 = [1\ 2\ 1\ ], X_2 = [2\ 1\ 4], X_3 = [4\ 5\ 6], X_4 = [1\ 8 - 3]$

   (iv) $X_1 = [1 - 1\ 1\ ], X_2 = [2\ 1\ 1], X_3 = [3\ 0\ 2]$

   (v) $X_1 = [1\ 2\ 3\ ], X_2 = [2 - 2\ 6]$

   (vi) $X_1 = [3\ 1 - 4\ ], X_2 = [2\ \ 2 - 3], X_3 = [0 - 4\ \ 1]$

   (vii) $X_1 = [1\ 1\ 1\ 3], X_2 = [1\ 2\ 3\ 4], X_3 = [2\ 3\ 4\ 7]$

   (viii) $X_1 = [1\ 1 - 1\ 1], X_2 = [1 - 1\ 2 - 1], X_3 = [3\ 1\ 0\ 1]$

   (ix) $X_1 = [1 - 1\ 2\ 0], X_2 = [2\ 1\ 1\ 1], X_3 = [3 - 1\ 2 - 1], X_4 = [3\ 0\ 3\ 1]$
2. Show that the following set of vectors are mutually orthogonal vectors

   (i) $X_1 = [1\ 2\ 2],\ X_2 = [2\ 1 - 2],\ X_3 = [2 - 2\ 1]$

   (ii) $X_1 = [-2\ \ 1\ \ - 1],\ X_2 = [0\ \ 1\ \ 1],\ X_3 = [1\ \ 1\ \ - 1]$
3. Show that the rows of the matrix $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 1 \\ 4 & 6 & 2 & -4 \\ -6 & 0 & -3 & -4 \end{bmatrix}$ are linearly dependent and find the relationship between them.

ANSWERS

1. (i) Independent.
(ii) Dependent. $x_1 + x_2 - x_3 = 0$
(iii) Dependent. $x_3 = 2x_1 + x_2,\ \ x_4 = 5x_1 - 2x_2$
(iv) Dependent. $x_1 + x_2 - x_3 = 0$
(v) Independent.
(vi) Dependent. $2x_1 - 3x_2 - x_3 = 0$
(vii) Dependent. $x_1 + x_2 - x_3 = 0$
(viii) Dependent. $2x_1 + x_2 - x_3 = 0$
(ix) Dependent. $x_1 + x_2 - x_4 = 0$

3. $X_3 = -2X_1 + 6X_2 + 2X_4$

EXERCISE

1. Solve the following equations by Jacobi's method
   (i) $15x + y - z = 14\,, x + 20y + z = 23, 2x - 3y + 18z = 37$
   (ii) $12x + 2y + z = 27\,, 2x + 15y - 3z = 16,\ 2x - 3y + 25z = 23$
   (iii) $14x - y + 3z = 18\,, 2x - 14y + 3z = 19\,, x - 3y + 16z = 20$
2. Solve the following equations by Gauss – Seidel method
   (i) $27x + 6y - z = 85\ \,, 6x + 15y + 2z = 72\,, x + y + 54z = 110$
   (ii) $10x_1 + x_2 + x_3 = 12\,, 2x_1 + 10x_2 + x_3 = 13, 2x_1 + 2x_2 + 10x_3 = 14$
   (iii) $5x + y - z = 10\,, 2x + 4y + z = 14\,, x + y + 8z = 20$

ANSWERS

1. (i) $x = 1, y = 1, z = 2$
   (ii) $x = 2, y = 1, z = 1$
   (iii) $x = 1, y = -1, z = 1$
2. (i) $x = 2.43,\ \ y = \ 3.57,\ \ z = 1.93$
   (ii) $x_1 = 1, x_2 = 1, x_3 = 1$
   (iii) $x = 2, y = 2, z = 2.$