DE MOIVRE’S THEOREM

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Statement : For any rational number n the value or one of the values of

$$(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\ \sin n\,\theta$$

1. If $z = cos\ \theta + i\sin\theta$ then

$\frac{1}{z} = z^{-1} = (\cos\theta +\ i\ sin\ \theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta -\ i\sin\theta$

i.e. $\frac{1}{z} = \cos\theta -\ i\sin\theta$

2. $(\cos\theta - i\sin\theta)^n = \cos\ n\,\theta - i\ \sin n\,\theta$

For, $(\cos\theta - i\sin\theta)^n = \{\cos\ (-\theta) + i\ \sin\ (-\theta)\}^n$

$= \cos(-n\theta) + isin(-n\theta).$

$= \cos n\,\theta - i\sin\ n\ \theta$

Note : Note carefully that ,

(1) $(\sin\theta + i\cos\theta)^n \neq \sin n\,\theta + i\cos n\,\theta$

But $(\sin\theta + i\cos\theta)^n = [cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)]^n$

$= \cos\ n\left(\frac{\pi}{2} - \theta\right) + i\sin n\ \left(\frac{\pi}{2} - \theta\right)$

(2) $(\cos\ \theta + \mathrm{i}\sin\Phi\ )^n \neq \cos\ n\,\theta +\ i\sin n\ \Phi.$

SOME SOLVED EXAMPLES:

1. Simplify $\frac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^5}{(\cos 3\theta + i\sin 3\theta)^{12}(\cos 5\theta - i\sin 5\theta)^7}$

Solution: $cos2\theta - i\sin 2\theta = (cos\theta + i\sin\theta)^{-2}$

$cos3\theta + i\sin 3\theta = (cos\theta + i\sin\theta)^3$

$cos5\theta - i\sin 5\theta = (cos\theta + i\sin\theta)^{-5}$

$\therefore$ Expression $= \frac{(cos\theta + i\sin\theta)^{-14}(cos\theta + i\sin\theta)^{15}}{(cos\theta + i\sin\theta)^{36}(cos\theta + i\sin\theta)^{-35}} = \frac{(cos\theta + i\sin\theta)^1}{(cos\theta + i\sin\theta)^1} = 1$

2. Prove that $\frac{(1+i)^8\left(\sqrt{3}-i\right)^4}{(1-i)^4\left(\sqrt{3}+i\right)^8}=-\frac{1}{4}$

Solution: $\frac{(1+i)^8\left(\sqrt{3}-i\right)^4}{(1-i)^4\left(\sqrt{3}+i\right)^8}$

$$(1+i)^8=\left[\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right]^8=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^8=\left\{\sqrt{2}e^{i\pi/4}\right\}^8=2^4.e^{i\,2\pi}$$

$$(1-i)^4=\left[\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\right]^4=\left[\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right]^4=\left\{\sqrt{2}e^{-i\pi/4}\right\}^4=2^2.e^{-i\,\pi}$$

$$\left(\sqrt{3}-i\right)^4=\left[2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\right]^4=\left[2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\right]^4=\left\{2e^{-i\pi/6}\right\}^4=2^4.e^{-i\,2\pi/3}$$

$$\left(\sqrt{3}+i\right)^8=\left[2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right]^8=\left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^8=\left\{2e^{i\pi/6}\right\}^8=2^8.e^{i\,4\pi/3}$$

$$\text{Expression}=\frac{\left(2^4.e^{i\,2\pi}\right).\left(2^4.e^{-i\,2\pi/3}\right)}{\left(2^2.e^{-i\,\pi}\right).\left(2^8.e^{i\,4\pi/3}\right)}=\frac{1}{2^2}.\frac{e^{i\,3\pi}}{e^{i\,2\pi}}=\frac{1}{4}e^{i\,\pi}=\frac{1}{4}(\cos\pi+i\sin\pi)=\frac{-1}{4}$$

3. Find the modulus and the principal value of the argument of $\frac{\left(1+i\sqrt{3}\right)^{16}}{\left(\sqrt{3}-i\right)^{17}}$

Solution: We have $1+i\sqrt{3}=2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=2\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)$

$$\sqrt{3}-i=2\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=2\left(cos\frac{\pi}{6}-isin\frac{\pi}{6}\right)$$

$$\therefore\frac{\left(1+i\sqrt{3}\right)^{16}}{\left(\sqrt{3}-i\right)^{17}}=\frac{2^{16}[cos(\pi/3)+i\sin(\pi/3)]^{16}}{2^{17}[cos(\pi/6)-i\sin(\pi/6)]^{17}}$$

$$=\frac{1}{2}\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^{16}\left(cos\frac{\pi}{6}-isin\frac{\pi}{6}\right)^{-17}$$

$$\therefore\frac{\left(1+i\sqrt{3}\right)^{16}}{\left(\sqrt{3}-i\right)^{17}}=\frac{1}{2}\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^{16}\left[cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]^{-17}$$

$$=\frac{1}{2}\left(cos\frac{16\pi}{3}+i\sin\frac{16\pi}{3}\right)\left[cos\left(\frac{17\pi}{6}\right)+i\sin\left(\frac{17\pi}{6}\right)\right]$$

$$=\frac{1}{2}\left[cos\left(\frac{16}{3}+\frac{17}{6}\right)\pi+i\sin\left(\frac{16}{3}+\frac{17}{6}\right)\pi\right]$$

$$=\frac{1}{2}\left[cos\left(\frac{49}{6}\right)\pi+i\,sin\left(\frac{49}{6}\right)\pi\right]$$

$$=\frac{1}{2}\left[cos\left(8\pi+\frac{\pi}{6}\right)+i\sin\left(8\pi+\frac{\pi}{6}\right)\right]$$

$$=\frac{1}{2}\left[cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]$$

Hence, the modulus is $\frac{1}{2}$ and principal value of the argument is $\frac{\pi}{6}$

4. Simplify $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n$

Solution: We have $1=sin^2\alpha+cos^2\alpha=sin^2\alpha-i^2cos^2\alpha$

$$=(\sin\alpha+i\cos\alpha)(\sin\alpha-i\cos\alpha)$$

$$\therefore 1+\sin\alpha+i\cos\alpha=(\sin\alpha+i\cos\alpha)(\sin\alpha-i\cos\alpha)+(\sin\alpha+i\cos\alpha)$$

$$=(\sin\alpha+i\cos\alpha)(\sin\alpha-i\cos\alpha+1)$$

$$\therefore \frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}=\sin\alpha+i\cos\alpha=cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)$$

$$\therefore \left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n=\left\{cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)\right\}^n$$

$$=\cos n\left(\frac{\pi}{2}-\alpha\right)+\ i\sin n\ \left(\frac{\pi}{2}-\alpha\right)$$

5. If $z=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\ and\ \overline{z}$ is the conjugate of z prove that $(z)^{10}+(\overline{z})^{10}=0$.

Solution: $z=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}+\ i\sin\frac{\pi}{4}$ $\therefore \bar{z}=\cos\frac{\pi}{4}-isin\ \frac{\pi}{4}$

$$\therefore (z)^{10}+(\bar{z})^{10}=\left(\cos\frac{\pi}{4}+\ i\sin\frac{\pi}{4}\right)^{10}+\left(\cos\frac{\pi}{4}-\ i\sin\frac{\pi}{4}\right)^{10}$$

$$=\left(\cos\frac{10\pi}{4}+\ i\sin\frac{10\pi}{4}\right)+\left(\cos\frac{10\pi}{4}-\ i\sin\frac{10\pi}{4}\right)$$

$$=2\cos\frac{10\pi}{4}=2\cos\left(\frac{5\pi}{2}\right)=0$$

(ii) $\left(1+i\sqrt{3}\right)^n+\left(1-i\sqrt{3}\right)^n=2^{n+1}cos(n\pi/3)$.

Solution: $1+i\sqrt{3}=2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

$$1-i\sqrt{3}=2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

$$\therefore \left(1+i\sqrt{3}\right)^n+\left(1-i\sqrt{3}\right)^n$$

$$=2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n+2^n\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n$$

$$= 2^n \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + 2^n \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$

$$= 2^n \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$

$$= 2^n \left(2\cos\frac{n\pi}{3}\right)$$

$$= 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$$

6. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2.2^{n/2}\cos n\pi/4$, Hence, deduce that $\alpha^8 + \beta^8 = 32$

Solution: The given equation is $x^2 - 2x + 2 = 0$

$$\therefore x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

$$\therefore \alpha = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\beta = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$\therefore \alpha^n + \beta^n = \left[\sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$$

$$= 2^{n/2}\left(cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + 2^{n/2}\left(cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

$$= 2^{n/2}\left(cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

$$= \left(\sqrt{2}\right)^n\left(2cos\frac{n\pi}{4}\right)$$

$$= 2 \cdot 2^{n/2} cos\frac{n\pi}{4}$$

Putting $n = 8$ $\quad \alpha^8 + \beta^8 = 2 \cdot 2^4 \cos 2\pi = 2^5 = 32$

7. If α, β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, Prove that $\alpha^3 + \beta^3 = 0$ and $\alpha^3 - \beta^3 = 16\,i$

Solution: The given equation is $x^2 - 2\sqrt{3}x + 4 = 0$

$\therefore x = \frac{2\sqrt{3} \pm \sqrt{12-16}}{2} = \sqrt{3} \pm i = 2\left(\frac{\sqrt{3}}{2} \pm i.\frac{1}{2}\right) = 2\left(cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}\right)$ are the roots

Let $\alpha = 2\left(cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right),\ \beta = 2\left(cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$$\therefore \alpha^3 + \beta^3 = 2^3\left(cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 + 2^3\left(cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^3$$

$$= 2^3\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - i\ sin\frac{\pi}{2}\right] = 2^3.2\ cos\frac{\pi}{2} = 0$$

Similarly, $\alpha^3 - \beta^3 = 2^3\left(cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 + 2^3\left(cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^3$

$$= 2^3\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{\pi}{2} + i\ sin\frac{\pi}{2}\right] = 2^3.2\ i\ sin\ \frac{\pi}{2} = 16\ i$$

8. If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$, $c = \cos 2\gamma + i\sin 2\gamma$, prove that $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$

Solution: $\frac{ab}{c} = \frac{(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + \sin 2\beta)}{(\cos 2\gamma + i\sin 2\gamma)}$

$$= \cos\ (2\alpha + 2\beta - 2\gamma) + i\sin(2\alpha + 2\beta - 2\gamma)$$

$$= \cos 2(\alpha + \beta - \gamma) + i\sin 2(\alpha + \beta - \gamma)$$

$$\sqrt{\frac{ab}{c}} = [\cos 2(\alpha + \beta - \gamma) + i\sin 2(\alpha + \beta - \gamma)]^{1/2}$$

$$= \cos(\alpha + \beta - \gamma) + i\sin(\alpha + \beta - \gamma)$$

Similarly, $\sqrt{\frac{c}{ab}} = \cos(\alpha + \beta - \gamma) - i\sin(\alpha + \beta - \gamma)$

By addition we get $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$

9. If $x - \frac{1}{x} = 2i\sin\theta$, $y - \frac{1}{y} = 2\,i\sin\Phi$, $z - \frac{1}{z} = 2i\sin\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$

(ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\,cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Solution: Since $x - \frac{1}{x} = 2i\sin\theta \quad \therefore x^2 - 2ix\sin\theta - 1 = 0$

Solving the quadratic for x, we get,

$$x = \frac{2i\sin\theta \pm \sqrt{4i^2\sin^2\theta - 4(1)(-1)}}{2(1)} = i\sin\theta \pm \sqrt{1 - \sin^2\theta} = i\sin\theta \pm \cos\theta$$

consider $x = \cos\theta + i\sin\theta$

Similarly, $= \cos\Phi + i\sin\Phi$, $z = \cos\psi + i\sin\psi$

(i) $xyz = (\cos\theta + i\sin\theta)(\cos\Phi + i\sin\Phi)(\cos\psi + i\sin\psi)$

$$= \cos(\theta + \Phi + \psi) + i\sin(\theta + \Phi + \psi)$$

$$\therefore \frac{1}{xyz} = \cos(\theta + \Phi + \psi) - i\sin(\theta + \Phi + \psi)$$

Adding we get $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$

(ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} = \frac{(\cos\theta + i\sin\theta)^{1/m}}{(\cos\Phi + i\sin\Phi)^{1/n}} = \frac{\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right)}{\left(\cos\frac{\Phi}{n} + i\sin\frac{\Phi}{n}\right)} = \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right) + i\sin\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Similarly, $\frac{\sqrt[n]{y}}{\sqrt[m]{x}} = \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right) - i\sin\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Adding we get $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

10. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, Prove that $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$.

Solution: We have $cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

$\therefore (cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$

$\therefore (\cos\alpha + i\sin\alpha) + 2(\cos\beta + i\sin\beta) + 3(\cos\gamma + i\sin\gamma) = 0$

Let $x = cos\ \alpha + i\sin\alpha, y = 2(\cos\beta + i\sin\beta), z = 3(\cos\gamma + i\sin\gamma)$

$\therefore x + y + z = 0$

$\therefore (x + y + z)^3 = 0$

$\therefore x^3 + y^3 + z^3 + 3(x + y + z)(xy + yz + zx) - 3xyz = 0$

$\therefore x^3 + y^3 + z^3 = 3\,xyz$

$\therefore (\cos\alpha + i\sin\alpha)^3 + 2^3(\cos\beta + i\sin\beta)^3 + 3^3(\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha).2.(\cos\beta + i\sin\beta).3.(\cos\gamma + i\sin\gamma)$

$\therefore$ By De Moivre's Theorem,

$(\cos 3\alpha + i\sin 3\alpha) + 8.(\cos 3\beta + i\sin 3\beta) + 27(\cos 3\gamma + i\sin 3\gamma)$

$= 18[cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

$(\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma) + i(\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma)$

$= 18[cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

Equating imaginary parts, we get the required result.

11. If $x_r = cos\frac{\pi}{3^r} + isin\frac{\pi}{3^r}$, prove that **(i)** $x_1x_2x_3 \dots ad.inf. = i$

(ii) $x_0x_1x_2 \dots ad.inf. = -i$

Solution: We have $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$

Putting $r = 0, 1, 2, 3 \dots \dots \dots$ we get $x_0 = \cos\frac{\pi}{3^0} + i\sin\frac{\pi}{3^0} = \cos\pi + i\sin\pi = -1$

$x_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, $x_2 = \cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2} \dots \dots \dots \dots$ and so on

$x_1\ x_2\ x_3 \dots \dots \dots \dots$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right) \dots \dots \dots$$

$$= \cos\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ \dots\dots\right)\pi + i\sin\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ \dots\dots\right)\pi$$

But $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ \dots\dots..\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

$x_1\ x_2\ x_3 \dots \dots \dots \dots = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

Also $x_0\ x_1\ x_2\ x_3 \dots \dots \dots \dots = x_0(i) = (-1)(i) = -i$

12. If $(cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta) \dots \dots [cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$ then show that the general value of θ is $\frac{2\,r\pi}{n^2}$

Solution:

L.H.S $= (cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta) \dots \dots [cos(2n-1)\theta + i\sin(2n-1)\theta]$

$$= cos[1 + 3 + \cdots + (2n-1)]\theta + i\sin[1 + 3 + \cdots + (2n-1)]\theta$$

But $1 + 3 + \cdots + (2n-1)$ is an A.P. with first term 1, the number of terms n and common difference 2.

$\therefore$ The Sum, $S_n = \frac{n}{2}[2a + (n-1).d] = \frac{n}{2}[2 + (n-1).2] = n^2$

$\therefore$ L.H.S = $cos(n^2\theta) + \ i\sin(n^2\theta)$

R.H.S $= 1 = \cos 2\,r\,\pi + i\sin 2\,r\,\pi$ where $r = 0,1,2 \dots \dots$

Equating the two sides, we get $n^2\theta = 2\,r\,\pi$ $\therefore \theta = \frac{2\,r\,\pi}{n^2}$

13. By using De Moivre's Theorem show that

$$\sin\alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha \sin(5\alpha/2)}{sin\alpha/2}$$

Solution: $\frac{1-z^6}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5$ ....(i)

Let $z = \cos\alpha + i\sin\alpha$, then by De Moivre's theorem, $z^n = \cos n\alpha + i\sin n\,\alpha$

$$\therefore 1 + z + z^2 + z^3 + z^4 + z^5 = 1 + (cos\alpha + i\sin\alpha) + (\cos 2\,\alpha + i\sin 2\alpha)$$
$$+(cos3\alpha + i\sin 3\alpha) + (cos4\alpha + i\sin 4\alpha) + (cos5\alpha + i\sin 5\alpha)$$
$$= (1 + cos\alpha + \cos 2\,\alpha + cos3\alpha + cos4\alpha + cos5\alpha)$$
$$+i\,(\sin\alpha + \sin 2\alpha + \sin 3\alpha + \sin\alpha\, 4 + \sin 5\alpha)$$ ....(ii)

Now, $\frac{1-z^6}{1-z} = \frac{1-(cos\alpha+i\sin\alpha)^6}{1-(cos\alpha+i\sin\alpha)} = \frac{1-\cos 6\alpha - i\sin 6\alpha}{1-cos\alpha - i\sin\alpha} = \frac{2sin^2 3\alpha - 2i\, sin3\alpha\cos 3\alpha}{2sin^2(\alpha/2) - 2isin(\alpha/2)\cos(\alpha/2)}$

$$= \frac{\sin 3\alpha(\sin 3\alpha - i\cos 3\alpha)\,[sin(\alpha/2) + i\cos(\alpha/2)]}{sin(\alpha/2)[sin(\alpha/2) - icos(\alpha/2)][\sin(\alpha/2) + i\cos(\alpha/2)]}$$
$$= \frac{\sin 3\alpha(\sin 3\alpha - i\cos 3\alpha)[\sin(\alpha/2) - i\cos(\alpha/2)]}{sin(\alpha/2)[\sin^2(\alpha/2) + \cos^2(\alpha/2)]}$$
$$= \frac{\sin 3\alpha}{sin(\alpha/2)}(\sin 3\alpha - i\cos 3\alpha)[\sin(\alpha/2) - i\cos(\alpha/2)]$$
$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(\frac{\pi}{2} - 3\alpha\right) - \ isin\left(\frac{\pi}{2} - 3\alpha\right)\right] \times \left[cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \ i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$
$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(-\frac{\pi}{2} + 3\alpha\right) + \ i\sin\left(-\frac{\pi}{2} + 3\alpha\right)\right] \times \left[cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$
$$\therefore \frac{1-z^6}{1-z} = \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(3\alpha - \frac{\alpha}{2}\right) + \ i\sin\left(3\alpha - \frac{\alpha}{2}\right)\right]$$
$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(\frac{5\alpha}{2}\right) + i\ sin\left(\frac{5\alpha}{2}\right)\right]$$ ....(iii)

Using (i) equating real parts, from (ii) and (iii), we get

$$1 + cos\alpha + \cos 2\,\alpha + \cdots . + cos5\alpha = \frac{\sin 3\alpha . cos(5\alpha/2)}{sin(\alpha/2)}$$

And equating imaginary parts, we get

$$\sin\alpha + \sin 2\alpha + \cdots . + \sin 5\alpha = \frac{\sin 3\alpha . sin(5\alpha/2)}{sin(\alpha/2)}$$

PRACTICE PROBLEMS:

1. Simplify

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7(\cos\theta - i\sin\theta)^8}$ **(ii)** $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^2}{(\cos 4\theta + i\sin 4\theta)^5(\cos 5\theta - i\sin 5\theta)^4}$

2. Prove that

(i) $\frac{(1+i)^8(1-i\sqrt{3})^3}{(1-i)^6(1+i\sqrt{3})^9} = \frac{i}{32}$ **(ii)** $\frac{(1+i\sqrt{3})^9(1-i)^4}{(\sqrt{3}+i)^{12}(1+i)^4} = -\frac{1}{8}$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$

4. Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.

5. If $x_n + iy_n = (1+i\sqrt{3})^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

6. Simplify $(1 + cos\,\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n$

7. Prove that $\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$ Hence deduct that

$\left(1 + sin\frac{\pi}{5} + i\,cos\frac{\pi}{5}\right)^5 + i\left(1 + sin\frac{\pi}{5} - i\,cos\frac{\pi}{5}\right)^5 = 0.$

8. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $and\ \overline{z}$ is the conjugate of z find the value of $(z)^{15} + (\overline{z})^{15}$.

9. Prove that, if n is a positive integer, then

(i) $(a+ib)^{m/n} + (a-ib)^{m/n} = 2\left(\sqrt{a^2+b^2}\right)^{m/n} cos\left(\frac{m}{n}tan^{-1}\frac{b}{a}\right)$

(ii) $(\sqrt{3}+i)^{120} + (\sqrt{3}-i)^{120} = 2^{121}$

10. If n is a positive integer, prove that $(1+i)^n + (1-i)^n = 2\ 2^{n/2}\cos n\ \pi/4$

Hence, deduce that $(1+i)^{10} + (1-i)^{10} = 0$

11. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$

and 2 if $n = 3k$ where k is an integer.

12. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$,

prove that $\alpha^n + \beta^n = 2^{n+1}cos(n\pi/3)$.

(i) Deduce that $\alpha^{15}+\beta^{15}=-2^{16}$ **(ii)** Deduce that $\alpha^{6}+\beta^{6}=128$

13. If α,β are the roots of the equation $z^2 sin^2\theta - z.\sin 2\theta + 1 = 0$, prove that $\alpha^n+\beta^n = 2\cos n\,\theta\, cosec^n\theta$

14. If $a=\cos 3\alpha + i\sin 3\alpha, b=\cos 3\beta + i\sin 3\beta, c = \cos 3\gamma + i\sin 3\gamma$,

prove that $\sqrt[3]{\frac{ab}{c}}+\sqrt[3]{\frac{c}{ab}} = 2cos(\alpha+\beta-\gamma)$

15. If $x+\frac{1}{x}=2\cos\theta$, $y+\frac{1}{y}=2\cos\emptyset$, $z+\frac{1}{z}=2\cos\psi$, prove that

(i) $xyz+\frac{1}{xyz}=2\cos(\theta+\Phi+\psi)$ **(ii)** $\sqrt{xyz}+\frac{1}{\sqrt{xyz}}=2\,cos\left(\frac{\theta+\Phi+\psi}{2}\right)$

(iii) $\frac{x^m}{y^n}+\frac{y^n}{x^m}=2\cos(m\theta-n\Phi)$ **(iv)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}}+\frac{\sqrt[n]{y}}{\sqrt[m]{x}}=2\,cos\left(\frac{\theta}{m}-\frac{\emptyset}{n}\right)$

16. If $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$, $c=\cos\gamma+i\sin\gamma$, prove that $\frac{(b+c)(c+a)(a+b)}{abc}=8\,cos\frac{(\alpha-\beta)}{2}cos\frac{(\beta-\gamma)}{2}cos\frac{(\gamma-\alpha)}{2}$.

17. If a, b, c are three complex numbers such that $a+b+c=0$, prove that

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ and **(ii)** $a^2+b^2+c^2=0$

18. If $\cos\alpha+\cos\beta+\cos\gamma=0\; and \sin\alpha+\sin\beta+\sin\gamma=0$, Prove that

(i) $\cos 2\,\alpha+\cos 2\beta+\cos 2\gamma=0$, $\sin 2\,\alpha+\sin 2\beta+\sin 2\gamma=0$.

(ii) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$

(iii) $cos(\alpha+\beta)+cos(\beta+\gamma)+cos(\gamma+\alpha)=0$.

(iv) $\sin(\alpha+\beta)+sin(\beta+\gamma)+sin(\gamma+\alpha)=0$.

(v) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\,cos(\alpha+\beta+\gamma)$

(vi) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

19. If $a\cos\alpha+b\cos\beta+c\cos\gamma=a\sin\alpha+b\sin\beta+c\sin\gamma=0$, Prove that

$a^3\cos 3\alpha+b^3\cos 3\beta+c^3\cos 3\gamma=3abc\cos(\alpha+\beta+\gamma)$ and

$a^3\sin 3\alpha+b^3\sin 3\beta+c^3\sin 3\gamma=3\,abc\,sin(\alpha+\beta+\gamma)$

20. If $x_r=cos\left(\frac{2}{3}\right)^r\pi+i\,sin\left(\frac{2}{3}\right)^r\pi$, prove that

(i) $x_1x_2x_3\ldots\infty=1$, **(ii)** $x_0x_1x_2\ldots\infty=-1$