Practice problems on Homogeneous functions

Type-1

1. If $u = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^n$,show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

2. If $u = sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

3. If $u = (x/y)^{y/x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

4. Verify Euler's Theorem for **(i)** $u = 3x^2yz + 5xy^2z + 4xyz^2$

(ii) $u = \frac{x(x^3-y^3)}{(x^3+y^3)}$. **(iii)** $u = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$.

5. If $u = \log(x^2 + y^2) + \frac{x^2+y^2}{x+y} - 2\log(x + y)$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. **Ans :** $\frac{x^2+y^2}{x+y}$

6. If $z = f(x, y)$ and u,v are homogeneous function of degree n in x , y then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$.

7. If $u = \frac{x^2}{y}f\left(\frac{y}{x}\right) + \frac{y^2}{x}g\left(\frac{x}{y}\right)$, prove that $x^2\left(y\frac{\partial u}{\partial x} - x.f\left(\frac{y}{x}\right)\right) + y^2\left(x\frac{\partial u}{\partial y} - y.g\left(\frac{x}{y}\right)\right) = 0.$

8. If $u = sin^{-1}\left(\frac{x}{y}\right) + cos^{-1}\left(\frac{y}{z}\right) + tan^{-1}\left(\frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

9. If $u = xy\, f\left(\frac{y}{x}\right) + yz\, \Phi\left(\frac{y}{z}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u.$

10. If $u = x^2.f\left(\frac{y}{x}\right) + y^2.g\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$

11. If $u = x^3 sin^{-1}\left[\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}}\right]$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3\, sin^{-1}\left[\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}}\right].$

12. If $u = \frac{x-y-z}{x^2+y^2+z^2}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + u = 0.$

13. If $u = e^{x+y+z} + \sin\left(\frac{x^2+y^2+z^2}{xy+yz+zx}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = (x + y + z)e^{x+y+z}$

Type-2

1. If $u = x^3e^{-x/y}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 6u$

2. If $y = x\cos u$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

3. If $u = tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{x+y}\right]$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

4. If $u = \frac{x^2y+xy^2}{x^2+y^2}$, $prove\ that\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

5. If $u = x^2 sin^{-1}\frac{y}{x} - y^2 cos^{-1}\frac{x}{y}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$

6. If $u = xsin^{-1}\frac{y}{x} + tan^{-1}\frac{y}{x}$, find that value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

7. If $u = \frac{(x^2+y^2)^m}{2m(2m-1)} + xf\left(\frac{y}{x}\right) + \Phi\left(\frac{x}{y}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

8. If $u = x^3\left(tan^{-1}\left(\frac{y}{x}\right) + \frac{y}{x}e^{-y/x}\right) + y^{-3}\left(sin^{-1}\frac{x}{y} + \frac{x}{y}log\frac{x}{y}\right)$,

Prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9u$

9. If $u = x^4 sin^{-1}\frac{y}{x} + x^6 tan^{-1}\frac{y}{x}$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ $at\ x = 1, y = 1.$

ANSWERS

6. 0 **7.** $(x^2+y^2)^m$ **9.** 17π

Type-3

1. If $u = sin^{-1}\left\{\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right\}^{1/2}$, prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u$

2. If $u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

3. If $u = \log\left(\frac{x^2+y^2}{x-y}\right)$, prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

4. If $u = tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that , $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$

5. If $u = cos^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right)$, prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\cot u$

6. If $u = tan^{-1}\left(\frac{x^4+y^4}{x^2+y^2}\right)$, prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

7. If $u = \cos ec^{-1}\left(\frac{x+y}{x^2+y^2}\right)$, prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

8. If $u = \log\left(\frac{x^2+y^2}{x^3+y^3}\right)$, prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 1 = 0.$

9. If $u = sin^{-1}(x^2+y^2)^{2/5}$, show that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{4}{5}\tan u$

10. If $u = cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

11. If $u = \tan\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right) + \sin\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$$

Type-4

1. If $u = tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\sin u\cos 3u = \sin 2u\left[1 - 4sin^2 u\right] = \sin 4u - \sin 2u$$

2. If $u = cosec^{-1}\sqrt{\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left[\frac{13}{12} + \frac{1}{12}tan^2 u\right]$

3. If $u = sin^{-1}(x^2+y^2)^{1/5}$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{2}{25}\tan u\,(2tan^2 u - 3)$

4. If $u = sec^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ **Ans :** $-\cot u\,[2 + cot^2 u]$

5. If $u = sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{400}\tan u\,(tan^2 u - 19)$.

6. If $u = log\left[\frac{x^3+y^3}{x^2+y^2}\right]$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -1$.

7. If $u = \log\frac{x+y}{\sqrt{x^2+y^2}} + sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin w \cos 2w}{4\,cos^3 w}$

Where $w = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$