APPLICATIONS OF DE MOIVER'S THEOREM:

1) Expansion of $\sin n\theta, \cos n\theta$ in powers of $\sin\theta, \cos\theta$:

By De Moivre's theorem $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta . i\sin\theta + {}^nC_2\cos^{n-2}\theta . (i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta(i\sin\theta)^3 + \ldots\ldots.$$

$$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots\ldots)$$
$$+ i({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots\ldots\ldots)$$

Comparing real imaginary part on both sides

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots\ldots\ldots\ldots\ldots$$
$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-2}\sin^3\theta + \ldots\ldots\ldots\ldots.$$

SOME SOLVED EXAMPLES:

1. Using De Moivre's Theorem, prove that, Cos $3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$

Solution: By De Moivre's theorem,

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$
$$= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$
$$= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$
$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Equating real and imaginary parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

2. Using De Moivre's Theorem express $\sin 3\theta, \cos 3\theta, \tan 3\theta$ in terms of powers of $\sin\theta, \cos\theta, \tan\theta$ respty.

Solution: continue as example (1) and obtain

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 3\sin^2\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta\,(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^2\theta$

$= 4\,cos^3\theta - 3\cos\theta$

$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$

Dividing the numerator and denominator by $\cos^3\theta$

$\tan 3\theta = \frac{(3\tan\theta - tan^3\theta)}{(1-3\,tan^2\theta)}$

3. Show that, **(i)** $\sin 5\theta = 5\sin\theta - 20\,sin^3\theta + 16\,sin^5\theta$

(ii) $\cos 5\theta = 5\cos\theta - 20cos^3\theta + 16\,cos^5\theta$

Solution: By De Moivre's Theorem, $(cos\,5\theta + i\,sin\,5\theta) = (cos\,\theta + i\,sin\,\theta)^5$

$= cos^5\theta + 5\,cos^4\theta(i\sin\theta) + 10cos^3\theta(i\sin\theta)^2 + 10cos^2\theta(i\sin\theta)^3$

$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$ … Using Binomial Theorem

$= cos^5\theta + i\,5cos^4\theta\,\sin\theta - 10cos^3\theta sin^2\theta + i.10cos^2\theta sin^3\theta +$
$5\cos\theta sin^4\theta + i\,sin^5\theta$

$= (cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta)$

$+i(5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta)$

Equating real and imaginary parts

$\cos 5\,\theta = cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta$

$\sin 5\,\theta = 5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta$

We have $\sin 5\theta = 5\,cos^4\,\theta\,\sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta$

$= 5(1-sin^2\theta)^2\sin\theta - 10(1-sin^2\theta)\,sin^3\,\theta + sin^5\theta$

$= 5(1-2\,sin^2\theta + sin^4\theta)\sin\theta - 10(1-\,sin^2\theta)sin^3\,\theta + sin^5\theta$

$= 5\sin\theta - 20sin^3\,\theta + 16\,sin^5\theta$

And $\cos 5\theta = cos^5\theta - 10\,cos^3\theta\,\,sin^2\theta + 5\cos\theta\,sin^4\theta$

$= cos^5\theta - 10cos^3\theta(1-cos^2\theta) + 5\cos\theta(1-cos^2\theta)^2$

$= cos^5\theta - 10cos^3\theta(1-cos^2\theta) + 5\cos\theta\,(1-2cos^2\theta + cos^4\theta)$

$= 5\,cos\theta - 20cos^3\theta + 16\,cos^5\theta$

4. Show that, $\frac{\sin 5\theta}{\sin\theta} = 16\, cos^4\theta - 12\, cos^2\theta + 1$

Solution: From above example (3)

$$\sin 5\theta = 5cos^4\theta\ \sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta$$

$$\therefore \frac{\sin 5\theta}{\sin\theta} = 5cos^4\theta - 10cos^2\theta sin^2\theta + sin^4\theta$$

$$= 5cos^4\theta - 10cos^2\theta(1 - cos^2\theta) + (1 - cos^2\theta)^2$$

$$= 5cos^4\theta - 10cos^2\theta + 10cos^4\theta + 1 - 2cos^2\theta + cos^4\theta$$

$$= 16\, cos^4\theta - 12\, cos^2\theta + 1$$

5. Use De Moiver's Theorem to show that $tan5\theta = \frac{5\tan\theta - 10\, tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta}$ and hence deduce that $5tan^4\frac{\pi}{10} - 10tan^2\frac{\pi}{10} + 1 = 0$

Solution: From above example (3)

$$\cos 5\,\theta = cos^5\theta - 10\ cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta$$

$$\sin 5\,\theta = 5\, cos^4\theta \sin\theta - 10cos^2\theta\, sin^3\theta + sin^5\theta$$

$$\therefore \tan 5\theta = \frac{\sin 5\,\theta}{\cos 5\,\theta} = \frac{5\, cos^4\theta\sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta}{cos^5\theta - 10\, cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta}$$

Dividing the numerator and denominator by $cos^5\theta$

$$\tan 5\theta = \frac{\tan\theta - 10tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Now, Put $\theta = \frac{\pi}{10}$.

Then $\tan 5\theta = \tan\frac{\pi}{2} = \infty$ and hence the denominator in (1) must be zero.

$$\therefore 5\, tan^4\frac{\pi}{10} - 10tan^2\frac{\pi}{10} + 1 = 0\ .$$

6. If $\sin 6\theta = a\, cos^5\theta\sin\theta + b\, cos^3\theta\, sin^3\theta + c\cos\theta\ sin^5\theta$,

find the values of a, b, c.

Solution: By De Moivre's Theorem $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

$$= (\cos\theta)^6 + 6(\cos\theta)^5(i\sin\theta) + 15(\cos\theta)^4(i\sin\theta)^2 + 20(\cos\theta)^3(i\sin\theta)^3$$

$$+15(\cos\theta)^2(i\sin\theta)^4 + 6(\cos\theta)^1(i\sin\theta)^5 + (i\sin\theta)^6$$

$$= \cos^6\theta + i6\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - i20\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta$$
$$+i6\cos\theta\sin^5\theta - \sin^6\theta$$

$$= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta)$$
$$+i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta)$$

Equating imaginary parts, $\sin 6\theta = 6\cos^6\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

Comparing with $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$

we get, $a = 6, b = -20, c = 6$

7. Prove that,

$$\cos 8\theta = cos^8\theta - 28\, cos^6\theta\, sin^2\theta + 70\, cos^4\theta\, sin^4\theta - 28\, cos^2\theta\, sin^6\theta + sin^8\theta$$
$$\sin 8\theta = 8\, cos^7\theta\sin\theta - 56\, cos^5\theta\, sin^3\theta + 56 cos^3\theta sin^5\theta - 8\cos\theta\, sin^7\theta.$$

Solution: By De Moivre's Theorem $\cos 8\theta + i\sin 8\theta = (\cos\theta + i\sin\theta)^8$

$$= (\cos\theta)^8 + 8(\cos\theta)^7(i\sin\theta) + 28(\cos\theta)^6(i\sin\theta)^2 + 56(\cos\theta)^5(i\sin\theta)^3$$
$$+70(\cos\theta)^4(i\sin\theta)^4 + 56(\cos\theta)^3(i\sin\theta)^5 + 28(\cos\theta)^2(i\sin\theta)^6$$
$$+8(\cos\theta)(i\sin\theta)^7 + (i\sin\theta)^8$$

$$= \cos^8\theta + i\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta - i56\cos^5\theta\sin^3\theta + 28\cos^4\theta\sin^4\theta$$
$$+i56\cos^3\theta\sin^5\theta - 28\cos^2\theta\sin^6\theta - i8\cos\theta\sin^7\theta + \sin^8\theta$$

$$= (\cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta)$$
$$+i(8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta + 56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta)$$

Equating real and imaginary parts

$$\cos 8\theta = cos^8\theta - 28\, cos^6\theta\, sin^2\theta + 70\, cos^4\theta\, sin^4\theta - 28\, cos^2\theta\, sin^6\theta + sin^8\theta$$
$$\sin 8\theta = 8\, cos^7\theta\sin\theta - 56\, cos^5\theta\, sin^3\theta + 56 cos^3\theta sin^5\theta - 8\cos\theta\, sin^7\theta.$$

8. Using De Moivre's theorem prove that,

$$2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2\ where\ x = 2\cos\theta.$$

Solution: $2(1 + \cos 8\theta) = 2(2cos^2 4\,\theta) = (2\cos 4\theta)^2$ ……………………(1)

To find $\cos 4\theta$ in powers of $\cos\theta$,

Consider $(\cos 4\theta + i\sin 4\theta) = (\cos\theta + i\sin\theta)^4$

$$= cos^4\theta + 4cos^3\theta i \sin\theta + 6cos^2\,\theta i^2 sin^2\theta + 4\,cos\theta i^3 sin^3\theta + i^4 sin^4\theta$$

$$= (cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta) + i(4cos^3\theta\sin\theta - 4\,cos\theta sin^3\theta)$$

Equating real Parts, $\cos 4\theta\ = cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta$

$$= cos^4\theta - 6cos^2\theta(1 - cos^2\theta) + (1 - cos^2\theta)^2$$

$$= cos^4\theta - 6cos^2\theta + 6cos^4\theta + 1 - 2cos^2\theta + cos^4\theta$$

$$= 8cos^4\theta - 8cos^2\theta + 1$$

$\therefore 2\cos 4\theta = 16cos^4\theta - 16cos^2\theta + 2$ Putting this value in (1)

$$2(1 + \cos 8\theta) = (16cos^4\theta - 16cos^2\theta + 2)^2$$

$$= [(2\cos\theta)^4 - 4(2\cos\theta)^2 + 2]^2$$

$$= (x^4 - 4x^2 + 2)^2 \quad \text{where } x\ =\ 2\,cos\,\theta$$

9. Prove that $\frac{1+\cos 9A}{1+\cos A} = [16cos^4 A - 8\,cos^3 A - 12\,cos^2 A + 4\cos A + 1]^2$

Solution: $\frac{1+\cos 9A}{1+\cos A} = \frac{2\cos^2\left(\frac{9A}{2}\right)}{2\cos^2\left(\frac{A}{2}\right)} = \left[\frac{\cos\left(\frac{9A}{2}\right)}{\cos\left(\frac{A}{2}\right)}\right]^2$

$$= \left(\frac{2\cos\left(\frac{9A}{2}\right)\sin\left(\frac{A}{2}\right)}{2\cos\left(\frac{A}{2}\right)\sin\left(\frac{A}{2}\right)}\right)^2 = \left[\frac{\sin\left(\frac{9A}{2}+\frac{A}{2}\right)-\sin\left(\frac{9A}{2}-\frac{A}{2}\right)}{\sin A}\right]^2$$

$$= \left(\frac{\sin(5A)-\sin(4A)}{\sin A}\right)^2 \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

By De Moivre's Theorem, $(cos\ 5\theta + i\ sin\ 5\theta) = (cos\ \theta + i\ sin\ \theta)^5$

$$= cos^5\theta + 5\,cos^4\theta(i\sin\theta) + 10cos^3\theta(i\sin\theta)^2 + 10cos^2\theta(i\sin\theta)^3$$

$$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 \qquad \ldots\ldots\ldots \text{Using Binomial Theorem}$$

$$cos^5\theta + i\,5cos^4\theta\ \sin\theta - 10cos^3\theta sin^2\theta - i.10cos^2\theta sin^3\theta + 5\cos\theta sin^4\theta + i\,sin^5\theta$$

$$= (cos^5\theta - 10\ cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta) + i(5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta)$$

Equating imaginary parts

$$\sin 5\,\theta = 5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Consider $(\cos 4\theta + i\sin 4\theta) = (\cos\theta + i\sin\theta)^4$

$$= cos^4\theta + 4cos^3\theta i\sin\theta + 6cos^2\,\theta i^2 sin^2\theta + 4\,cos\theta i^3 sin^3\theta + i^4 sin^4\theta$$

$$= (cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta) + i(4cos^3\theta\sin\theta - 4\,cos\theta sin^3\theta)$$

Equating imaginary parts

$\sin 4\theta = 4\,cos^3\theta \sin\theta - 4cos\theta\, sin^3\theta$ .... (3)

Put (2) & (3) in (1) we get

$$\frac{1+\cos 9A}{1+\cos A} = \left[\frac{(5\,cos^4 A \sin A - 10cos^2 A sin^3 A + sin^5 A) - (4\,cos^3 A \sin A - 4cosA sin^3 A)}{\sin A}\right]^2$$

$$= (5\cos^2 A - 10\cos^2 A\sin^2 A + \sin^4 A - 4\cos^2 A + 4\cos A\sin^2 A)^2$$

$$= [5\cos^2 A - 10\cos^2 A\,(1-\cos^2 A) + (1-\cos^2 A)^2 - 4\cos^3 A + 4\cos A\,(1-\cos^2 A)]^2$$

$$= [5\cos^2 A - 10\cos^2 A + 10\cos^4 A + 1 - 2\cos^2 A + \cos^4 A - 4\cos^3 A + 4\cos A \;-\; 4\cos^3 A]^2$$

$$= (16\cos^4 A - 8\cos^3 A - 12\cos^2 A + 4\cos A + 1)^2$$

10. Prove that $\frac{1-\cos 9A}{1-\cos A} = [16\,cos^4 A + 8\,cos^3 A - 12\,cos^2 A - 4\cos A + 1]^2$

Solution:

$$\frac{1-\cos 9A}{1-\cos A} = \frac{2\sin^2\left(\frac{9A}{2}\right)}{2\sin^2\left(\frac{A}{2}\right)} = \left(\frac{\sin\left(\frac{9A}{2}\right)}{\sin\left(\frac{A}{2}\right)}\right)^2 = \left(\frac{2\sin\left(\frac{9A}{2}\right)\cos\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}\right)^2 = \left[\frac{\sin\left(\frac{9A}{2}+\frac{A}{2}\right)+\sin\left(\frac{9A}{2}--\frac{A}{2}\right)}{\sin A}\right]^2$$

$$= \left(\frac{\sin(5A)+\sin(4A)}{\sin A}\right)^2$$ Continue as above example

SOME PRACTICE PROBLEMS

1. Using De Moivre's Theorem prove that,

$\cos 4\,\theta = cos^4\theta - 6\,cos^2\theta sin^2\theta + sin^4\theta$ and

$\sin 4\theta = 4cos^3\theta sin\theta - 4\cos\theta\, sin^3\theta$

2. Prove that, $\frac{\sin 6\theta}{\sin 2\theta} = 16\,cos^4\theta - 16\,cos^2\theta + 3$

3. If $\cos 6\theta = a\,cos^6\theta + b\,cos^4\theta\,sin^2\theta + c cos^2\theta\,sin^4\theta + d\,sin^6\theta$, find a, b, c, d.

4. Express $\sin 7\theta$ *and* $\cos 7\theta$ in terms of powers of $\sin\theta$ *and* $\cos\theta$.

5. Prove that, $\frac{\sin 7\theta}{\sin\theta} = 7 - 56\,sin^2\theta + 112\,sin^4\theta - 64\,sin^6\theta$

6. Show that $\tan 7\theta = \frac{7\tan\theta - 35tan^3\theta + 21tan^5\theta - tan^7\theta}{1 - 21tan^2\theta + 35tan^4\theta - 7tan^6\theta}$.

7. Express $\tan 7\theta$ in terms of powers of $\tan\theta$

Hence deduce $7\,tan^6\frac{\pi}{14} - 35\,tan^4\frac{\pi}{14} + 21\,tan^2\frac{\pi}{14} - 1 = 0$

8. Prove that $\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2 \; where \; x = 2\cos\theta$

9. Prove that $\frac{1-\cos 7\theta}{1-\cos\theta} = (x^3 + x^2 - 2x - 1)^2 \; where \; x = 2\cos\theta$

Expansion of $\boldsymbol{sin^n\theta, cos^n\theta}$ in term of sin $\boldsymbol{n\,\theta, cos\;n\theta}$ (n is a + ve integer):

Let $x = cos\,\theta + i\,sin\,\theta = e^{i\theta}$ $\quad \therefore \frac{1}{X} = cos\,\theta - i\,sin\,\theta = e^{-i\theta}$

Hence $x + \frac{1}{x} = 2cos\,\theta$ and $x - \frac{1}{x} = 2i\,sin\,\theta$

Again, $x^n = (cos\,\theta + \; i\,sin\,\theta)^n = cos\,n\,\theta + i\,sin\,n\theta = e^{in\theta}$

$$\frac{1}{x^n} = (cos\,\theta - i\,sin\,\theta)^n = cos\,n\theta - i\,sin\,n\theta = e^{-in\theta}$$

$$x^n + \frac{1}{x^n} = 2\cos n\,\theta \quad \text{and} \quad x^n - \frac{1}{x^n} = 2i\,sin\,n\theta$$

To expand $cos^n\theta$, write $cos^n\theta = \frac{1}{2^n}\left(x + \frac{1}{x}\right)^n$

To expand $sin^n\theta$, write $sin^n\theta = \frac{1}{(2i)^n}\left(x - \frac{1}{x}\right)^n$ and expand R.H.S. using binomial expansion $(x+a)^n = x^n \; + {}^nC_1x^{n-1}a \; + \; {}^nC_2x^{n-2}a^2 \; +$……….+ a^n

SOME SOLVED EXAMPLES:

1. Show that $sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Solution: Let $x = \cos\theta + i\sin\theta, then\; \frac{1}{x} = cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta \quad \text{…………(1)}$$

$$x^n = \cos n\theta + i\sin n\theta \;\; and \;\; \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta \quad \text{…………(2)}$$

$$\therefore (2\,i\sin\theta)^5 = \left(x - \frac{1}{x}\right)^5 \quad \text{from (1)}$$

$$= x^5 - 5x^4.\frac{1}{x} + 10x^3.\frac{1}{x^2} - 10x^2.\frac{1}{x^3} + 5x.\frac{1}{x^4} - \frac{1}{x^5}$$

$$= x^5 - 5x^3 + 10x - 10\frac{1}{x} + 5\frac{1}{x^3} - \frac{1}{x^5}$$

$$32\, i^5 sin^5\theta = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$\therefore 32\, i\, sin^5\theta = 2\, i \sin 5\,\theta - 5(2i\sin 3\theta) + 10\,(2i\, sin\theta) \quad \text{from (2)}$$

$$\therefore sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

2. Using De Moivre's Theorem prove that, $cos^6\theta + sin^6\theta = \frac{1}{8}(3\cos 4\,\theta + 5)$

Solution: Let $x = \cos\theta + i\sin\theta, then\ \frac{1}{x} = \cos\theta - i\, sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta \quad \text{............(1)}$$

$$x^n = \cos n\theta + i\sin n\theta \quad and \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta \quad \text{............(2)}$$

$$(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6 \quad \text{from (1)}$$

$$= x^6 + 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} + 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} + 6x.\frac{1}{x^5} + \frac{1}{x^6}$$

$$2^6 cos^6\theta = x^6 + 6x^4 + 15x^2 + 20 + 15.\frac{1}{x^2} + 6.\frac{1}{x^4} + \frac{1}{x^6} \quad \text{................(3)}$$

$$(2\, i\sin\theta)^6 = \left(x - \frac{1}{x}\right)^6 \quad \text{from (1)}$$

$$= x^6 - 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} - 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} - 6x.\frac{1}{x^5} + \frac{1}{x^6}$$

$$-2^6 sin^6\theta = x^6 - 6x^4 + 15x^2 - 20 + 15.\frac{1}{x^2} - 6.\frac{1}{x^4} + \frac{1}{x^6}$$

$$\therefore 2^6 sin^6\theta = -x^6 + 6x^4 - 15x^2 + 20 - 15.\frac{1}{x^2} + 6.\frac{1}{x^4} - \frac{1}{x^6} \quad \text{............(4)}$$

Adding (3) and (4) $\quad 2^6(cos^6\theta + sin^6\theta) = 12x^4 + 40 + 12.\frac{1}{x^4}$

$$2^6(cos^6\theta + sin^6\theta) = 4\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$$

$$\therefore cos^6\theta + sin^6\theta = \frac{1}{16}\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$$

$$\therefore cos^6\theta + sin^6\theta = \frac{1}{16}[6\cos 4\theta + 10] \quad \text{from (2)}$$

$$= \frac{1}{8}[3\ \cos 4\theta + 5]$$

3. Expand $sin^7\theta$ in a series of sines of multiples of θ

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta$ …………(1)

$x^n = \cos n\theta + i\sin n\theta \quad and \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta$ …………(2)

$(2\,i\sin\theta)^7 = \left(x - \frac{1}{x}\right)^7$ from (1)

$= x^7 - 7x^6.\frac{1}{x} + 21x^5.\frac{1}{x^2} - 35x^4.\frac{1}{x^3} + 35x^3.\frac{1}{x^4} - 21x^2.\frac{1}{x^5} + 7x.\frac{1}{x^6} - \frac{1}{x^7}$

$= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$

$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$

$-2^7 i\, sin^7\theta = 2i\sin 7\theta - 7.(2i\sin 5\theta) + 21.(2i\sin 3\theta) - 35.(2i\sin\theta)$ from (2)

$\therefore -2^6 sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$

$\therefore sin^7\theta = -\frac{1}{2^6}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$

4. Expand $cos^7\theta$ in a series of cosines of multiples of θ

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta$ …………(1)

$x^n = \cos n\theta + i\sin n\theta \quad and \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta$ …………(2)

$(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7$ …………. from (1)

$= x^7 + 7x^6\frac{1}{x} + 21x^5\frac{1}{x^2} + 35x^4\frac{1}{x^3} + 35x^3\frac{1}{x^4} + 21x^2\frac{1}{x^5} + 7x\frac{1}{x^6} + \frac{1}{x^7}$

$= x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$

$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$

$\therefore 2^7\cos^7\theta = 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(\cos\theta)$ From (2)

$cos^7\theta = \frac{1}{2^6}[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta]$

5. Show that $2^5 sin^4\theta cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\text{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta \;\; and \;\; \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\text{i}\sin n\theta$ ....(2)

$(2i\sin\theta)^4(2\cos\theta)^4 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^2$ From (1)

$$\therefore 2^6\sin^4\theta\cos^2\theta = \left(x - \frac{1}{x}\right)^2\left(x - \frac{1}{x}\right)^2\left(x + \frac{1}{x}\right)^2 = \left(x - \frac{1}{x}\right)^2\left(x^2 - \frac{1}{x^2}\right)^2$$

$$= \left(x^2 - 2 + \frac{1}{x^2}\right)\left(x^4 - 2 + \frac{1}{x^4}\right)$$

$$= x^6 - 2x^2 + \frac{1}{x^2} - 2x^4 + 4 - \frac{2}{x^4} + x^2 - \frac{2}{x^2} + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 2\left(x^4 + \frac{1}{x^4}\right) - \left(x^2 + \frac{1}{x^2}\right) + 4$$

$= 2\cos 6\theta - 2(2\cos 4\theta) - (2\cos 2\theta) + 4$ From (2)

$$\therefore 2^5\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$

6. Prove that $cos^5\theta sin^3\theta = -\frac{1}{2^7}\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\text{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta \;\; and \;\; \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\text{i}\sin n\theta$ ....(2)

$$(2\cos\theta)^5(2i\sin\theta)^3 = \left(x + \frac{1}{x}\right)^5\left(x - \frac{1}{x}\right)^3$$

$$2^8 i^3\cos^5\theta\sin^3\theta = \left(x + \frac{1}{x}\right)^2\left(x + \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)^3$$

$$-2^8 i\cos^5\theta\sin^3\theta = \left(x + \frac{1}{x}\right)^2\left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \left(x^2 - 2 + \frac{1}{x^2}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)$$

$$= x^8 - 3x^4 + 3 - \frac{1}{x^4} + 2x^6 - 6x^2 + \frac{6}{x^2} - \frac{2}{x^6} + x^4 - 3 + \frac{3}{x^4} - \frac{1}{x^8}$$

$$= \left(x^8 - \frac{1}{x^8}\right) + 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) - 6\left(x^2 - \frac{1}{x^2}\right)$$

$$= (2i\sin 8\theta) + 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(2i\sin 2\theta) \quad \text{From (2)}$$

$$\therefore -2^7\cos^5\theta\sin^3\theta = \sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta$$

$$\therefore \cos^5\theta\sin^3\theta = -\frac{1}{2^7}\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$$

7. If $sin^4\theta\ cos^3\theta = a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + a_7\cos 7\theta$,

Prove that $a_1 + 9a_3 + 25a_5 + 49a_7 = 0$.

Solution: Let $x = \cos\theta + i\sin\theta$ $\quad\therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta \quad \text{............(1)}$$

$$x^n = \cos n\theta + i\sin n\theta \quad and \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta \quad \text{............(2)}$$

Consider $(2\,i\sin\theta)^4(2\,\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^3$

$$= \left(x - \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)^3$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3\left(x - \frac{1}{x}\right)$$

$$= \left(x^6 - 3x^2 + 3.\frac{1}{x^2} - \frac{1}{x^6}\right)\left(x - \frac{1}{x}\right)$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$\therefore (2\,i\sin\theta)^4(2\cos\theta)^3 = 2\cos 7\theta - 2\cos 5\theta - 6cos3\theta + 6\cos\theta \quad \text{from (2)}$$

$$\therefore sin^4\theta cos^3\theta = \frac{\cos 7\theta}{2^6} - \frac{\cos 5\theta}{2^6} - \frac{3\cos 3\theta}{2^6} + \frac{3\cos\theta}{2^6}$$

Comparing this with the given equality, $a_1 = \frac{3}{2^6}, a_3 = -\frac{3}{2^6}, a_5 = -\frac{1}{2^6}, a_7 = \frac{1}{2^6}$

$$\therefore\ a_1 + 9a_3 + 25a_5 + 49a_7 = \frac{3}{2^6} - \frac{27}{2^6} - \frac{25}{2^6} + \frac{49}{2^6} = \frac{52-52}{2^6} = 0$$

SOME PRACTICE PROBLEMS:

1. Show that $cos^6\theta = \frac{1}{32}[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$

2. Prove that $cos^6\theta - sin^6\theta = \frac{1}{16}[\cos 6\theta + 15\cos 2\theta]$

3. Express $sin^8\theta$ in a series of cosines of multiples of θ.

4. Prove that, $cos^8\theta = \frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$

5. Prove that $cos^8\theta + sin^8\theta = \frac{1}{64}[\cos 8\theta + 28\cos 4\theta + 35]$.

6. Show that $2^6 sin^4\theta cos^3\theta = \cos 7\,\theta - \cos 5\,\theta - 3\cos 3\theta + 3\cos\theta$.

7. Prove that $sin^7\theta cos^3\theta = -\frac{1}{512}[\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta]$