K. J. Somaiya College of Engineering, Mumbai-77
(A Constituent college of Somaiya Vidyavihar University)

F.Y. B. Tech SEM-I (2020-21)
Applied Mathematics-I
Eigenvalues and Eigenvectors

Some Practice Problems

Q: For the following matrices:

i. **Find Characteristic equation.**
ii. **Find Eigenvalues and Eigenvectors.**
iii. **Prove that eigenvectors are linearly independent.**
iv. **Verify Cayley-Hamilton Theorem. Hence, find A^{-1} and A^4 if exists.**
v. **Check weather the matrix is diagonalisable. If yes, find the transforming matrix M and the diagonal matrix D./ Check whether the given matrix is similar to diagonal matrix. If similar to diagonal matrix, express in form of $D = M^{-1}AM$.**
vi. **Find the minimal polynomial and check weather the matrix is derogatory.**

1. $\begin{bmatrix} 3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

13. $\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix}$

15. $\begin{bmatrix} 3 & -2 & 3 \\ 10 & -3 & 5 \\ 5 & -4 & 7 \end{bmatrix}$

Function of a square matrix

1. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then find f(A) = $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ and eigenvalues of f(A).
2. If $A = \begin{bmatrix} sin\theta & cos\theta \\ -cos\theta & sin\theta \end{bmatrix}$ then find f(A) = A^{-1} using Cayley-Hamilton Theorem. Also find eigenvalues of f(A).
3. If $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$ then find f(A) = $2A^5 - 3A^4 + A^2 - 4I$ and eigenvalues of f(A).
4. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then find f(A) = $e^{A\pi/2}$ and eigenvalues of f(A).
5. Show that cos $O_{3\times3} = I_{3\times3}$ where cos $O_{3\times3}$ and $I_{3\times3}$ are respectively the zero matrix and the identity matrix of are order 3.
6. If $A = \begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix}$ then for any positive integer n, prove that $A^{2n+1} = A$
7. If $A = \begin{bmatrix} \theta & \theta \\ \theta & \theta \end{bmatrix}$ then prove that $e^A = e^\theta \begin{bmatrix} cosh\theta & sinh\theta \\ sinh\theta & cosh\theta \end{bmatrix}$
8. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 3, A^n = A^{n-2} + A^2 - I$. Hence find A^{50}.
9. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, that find f(A) = $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$ and eigenvalues of f(A).
10. If $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, that find f(A) = A^{100} and eigenvalues of f(A).

Similarity of Matrices

1. Determine the diagonal matrix unitarily similar to the Hermitian matrix. Also find the transformation matrix.

   i. $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$. ii. $A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$

2. Diagonalise the matrix $A = \begin{bmatrix} -3 & 2+2i \\ 2-2i & 4 \end{bmatrix}$

3. Find symmetric matrix $A_{3\times3}$ having the eigenvalues 3,6 & 9 with corresponding eigenvectors [1 2 2]' , [2 2 -1]' and X3.
4. Is the matrix diagonalisable? Justify your answer. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.
5. Transform the matrix in to diagonal form. $A = \begin{bmatrix} 8 & 15 & 5 \\ -12 & -25 & 11 \\ 5 & -42 & 19 \end{bmatrix}$.
6. If A is similar to B and B is similar to C, then A is similar to C.
7. If A is similar to B then A^n is similar to B^n.

Properties

1. Prove that the eigenvalues of a unitary matrix are of unit modulus.
2. Show that the matrices A and A' have the same eigenvalues.
3. Prove that eigenvalues of a Hermitian matrix are real.
4. Prove that if λ is an eigenvalue of matrix A then $\lambda \pm K$ is an eigenvalue for $A \pm KI$.
5. Show that the eigenvalues of the triangular/ diagonal matrix are its diagonal elements.
6. Show that eigenvalues of the orthogonal matrix are of unit modulus.
7. Show that eigenvalues of a skew Hermitian matrix are either purely imaginary or zero.
8. Eigenvectors corresponding to distinct eigenvalues of the matrix are linearly independent.
9. Eigenvectors corresponding to distinct eigenvalues of the real symmetric matrix are orthogonal.
10. Eigenvectors corresponding to distinct eigenvalues of a unitary matrix are mutually orthogonal.