PRACTICE PROBLEMS ON PARTIAL DIFFERENTIATION

TYPE-I

1. If $u = sin\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$.

2. If $u = e^{xyz}$, prove that, $\frac{\partial^3 u}{\partial x\partial y\partial z}=(1+3xyz+x^2y^2z^2)e^{xyz}$

3. If $z = x^2tan^{-1}\left(\frac{y}{x}\right)-y^2tan^{-1}\left(\frac{x}{y}\right)$,prove that $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{x^2-y^2}{x^2+y^2}$

4. If $u=f\left(\frac{x^2}{y}\right)$, prove that $x\frac{\partial u}{\partial x}+2y\frac{\partial u}{\partial y}=0$ and $x^2\frac{\partial^2 u}{\partial x^2}+3xy\frac{\partial^2 u}{\partial x\partial y}+2y^2\frac{\partial^2 u}{\partial y^2}=0$.

5. If $u=\log(x^3+y^3-x^2y-xy^2)$, prove that **(i)** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$. **(ii)** $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=-\frac{4}{(x+y)^2}$

6. If $u=\log r, r=(x^3+y^3-x^2y-xy^2)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$ & $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2u=-\frac{4}{(x+y)^2}$

7. If $u=(1-2xy+y^2)^{-1/2}$, prove that $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=y^2u^3$.

8. If $u=(1-2xy+y^2)^{-1/2}$, prove that $\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right]=0$.

9. If $u=\log(\tan x+\tan y+\tan z)$, prove that $\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2$.

10. If $z=ct^{-1/2}e^{-x^2/4a^2t}$, prove that $\frac{\partial z}{\partial t}=a^2\frac{\partial^2 z}{\partial x^2}$

11. If $u=(x^2+y^2+z^2)^{-1/2}$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

or If $\frac{1}{u^2}=x^2+y^2+z^2$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

or If $u=\frac{1}{r}, r=\sqrt{x^2+y^2+z^2}$ prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

12. If $x=r\cos\theta, y=r\sin\theta$, prove that

(i) $\frac{\partial x}{\partial r}=\frac{\partial r}{\partial x}$ **(ii)** $\frac{\partial x}{\partial \theta}=r^2\frac{\partial \theta}{\partial x}$ **(iii)** $\left[x\left(\frac{\partial x}{\partial r}\right)+y\left(\frac{\partial y}{\partial r}\right)\right]^2=x^2+y^2$

13. If $x=r\cos\theta, y=r\sin\theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$, **(ii)** $\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0$.

14 If $u=f(r)$ and $r=\sqrt{x^2+y^2}$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$. or

If $u=f\left(\sqrt{x^2+y^2}\right)$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{\sqrt{x^2+y^2}}f'\left(\sqrt{x^2+y^2}\right)+f''\left(\sqrt{x^2+y^2}\right)$

15. If $u=f(r^2)$ where $r^2=x^2+y^2+z^2$prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=4r^2f''(r^2)+6f'(r^2)$

16. If $u=e^{xyz}f\left(\frac{xy}{z}\right)$, prove that $x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=2xyzu$; $y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2xyzu$.

Hence, show that $x\frac{\partial^2 u}{\partial z\,\partial x}=y\frac{\partial^2 u}{\partial z\,\partial y}$

17. If $z = xf(x+y) + yg(x+y)$, prove that $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$.

18. If $z = x\log(x+r) - r$ where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x+r}$, $\frac{\partial^3 z}{\partial x^3} = -\frac{x}{r^3}$.

19. Find that value of n so that $V = r^n(3\,cos^2\theta - 1)$ satisfies the equation,

$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0.$ **Ans :** $n = 2\ \ or\ -3.$

20. If $u = Ae^{-gx} sin(nt - gx)$ where A, g, n are constants, satisfies the equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

Show that $ag = \sqrt{\frac{n}{2}}$

TYPE – II

1. If $z = x^2 + y^2, x = at^2, y = 2at$, verify that $\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$.

2. If $z = xy^2 + x^2y, x = at^2, y = 2at, find\ \frac{dz}{dt}$. **Ans :** $a^3(16t^3 + 10t^4)$

3. If $z = tan^{-1}\left(\frac{x}{y}\right), x = 2t, y = 1 - t^2$, prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$.

4. If $z = sin^{-1}(x - y), x = 3t, y = 4t^3$, prove that $\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}$.

5. If $u = x^2 + y^2 + z^2, x = e^t, y = e^t \sin t\,, z = e^t \cos t,\ find\ \frac{du}{dt}$. **Ans :** $4e^{2t}$

TYPE – III

1. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$.

2. If $z = f(x,y), x = e^u + e^{-v}, y = e^{-u} - e^v$, prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

3. If $z = f\ (x,y), x = ucoshv, y = usinhv$, prove that $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2$

4. If $z = f(x,y), x = e^u\ cosv, y = e^u \sin v$, prove that ,

(i) $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$. **(ii)** $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

5. If $z = f(x,y) and\ x = e^u \sec v, y = e^u \tan v$, prove that $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - cos^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$

6. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

7. If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

8. If $x = u + v + w,\ \ y = uv + vw + wu,\ \ z = uvw$, prove that

$x\frac{\partial \Phi}{\partial x} + 2y\frac{\partial \Phi}{\partial y} + 3z\frac{\partial \Phi}{\partial z} = u\frac{\partial \Phi}{\partial u} + v\frac{\partial \Phi}{\partial v} + w\frac{\partial \Phi}{\partial w}$ where Φ is a function of x, y, z.