Prof. Sushma Kadge
16/09/24, 11:30am
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Fourier series (FS)

Introduction :-

- Journey into frequency begins with Fourier series → only periodic s/g
- Periodic s/g representation unfolds → frequency content / spectrum

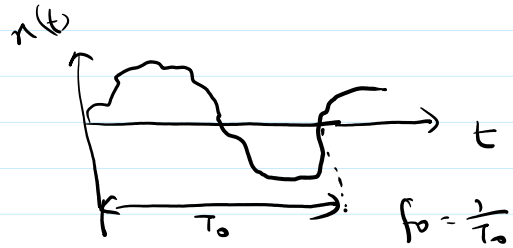
→ Synthesis → Combine 2 or more s/g → another periodic s/g
Analysis → Seperation of periodic s/g into its periodic components.

FS :- describes a periodic s/g $x_p(t)$ as sum of harmonics at fundamental freq (f_0) of $x_p(t)$ & its multiples nf_0

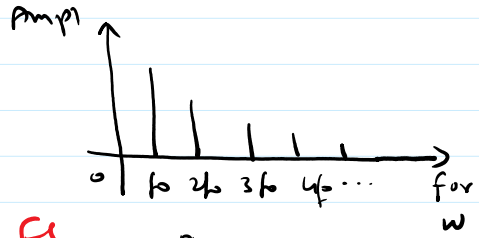
[note :- Analysis of a s/g is more convenient in freq. domain]

3 important class — FS → Fourier series → Periodic s/g
FT " Transform → Both (Nonperiodic)
LT Laplace " → Both

In frequency domain representation → x axis → freq (ω or f)
↓
Line spectrum graph — Amplitude Spectrum (mag.)
Phase Spectrum.

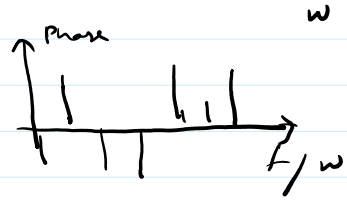


freq. ? compo. ?



Information
→ Type of s/g (Per / Aperi)
→ T_0
→ f_0
→ Shape of s/g

Present FS → FT
↓
line spectrum all info.



Line & phase Spectrum :-

$$x(t) = A\cos(\omega_0 t + \phi)$$

$\omega_0 = 2\pi f_0$; $f_0 \rightarrow$ fundamental freq.

$T_0 = \frac{1}{f_0}$

$$f_0 = \frac{2\pi}{\omega_0}$$

One sided spectrum
Double sided spectrum. [mag. $\rightarrow$ even symmetry
phase $\rightarrow$ odd symmetry.

eg: Plot the line spectrum of sig

$$x(t) = 7 - 10\cos(40\pi t - 60^\circ) + 4\sin(120\pi t)$$

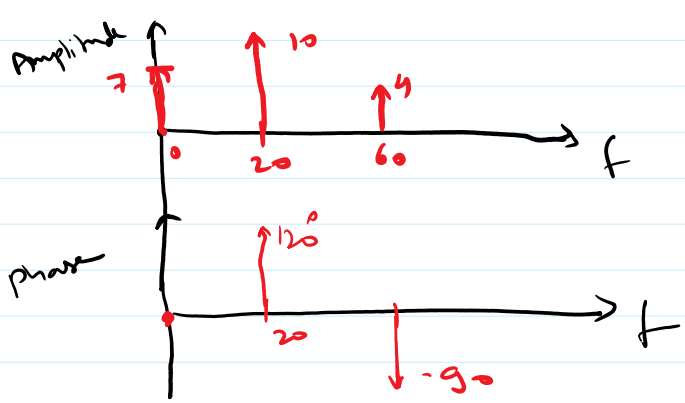
$\omega t \rightarrow 2\pi f t$

$2\times\pi\times 20\times t$

$40\pi t$

$$\Rightarrow x(t) = 7\cos(2\pi 0 t) + 10\cos(2\times 20\pi t + 120^\circ) + 4\cos(2\times 60\pi t - 90^\circ)$$

$f_0 =$ zero freq	$f_1 = 20$	$f_2 = 60$ Hz
Ampl $\rightarrow 7$	$\phi_1 = -60^\circ, -10$ or $120^\circ, +10$	$\phi_2 = -90^\circ$
$\phi_0 \rightarrow 0^\circ$		
dC component	1st harmonic	2nd harmonic



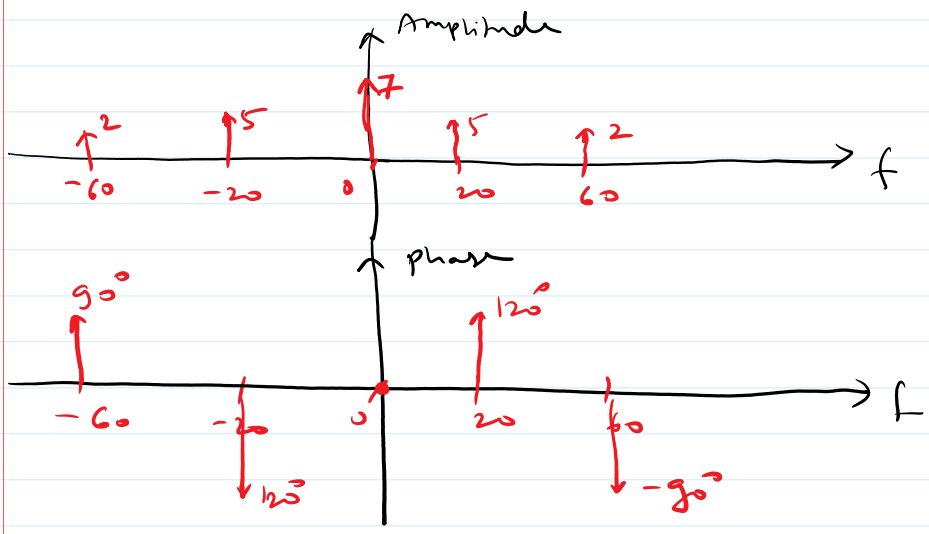
one sided mag & phase spectrum

Draw Double sided spectrum for the above sig

$\uparrow$ Amplitude

Ampl. $\rightarrow$ half.

Draw double sided spectrum for the above sig



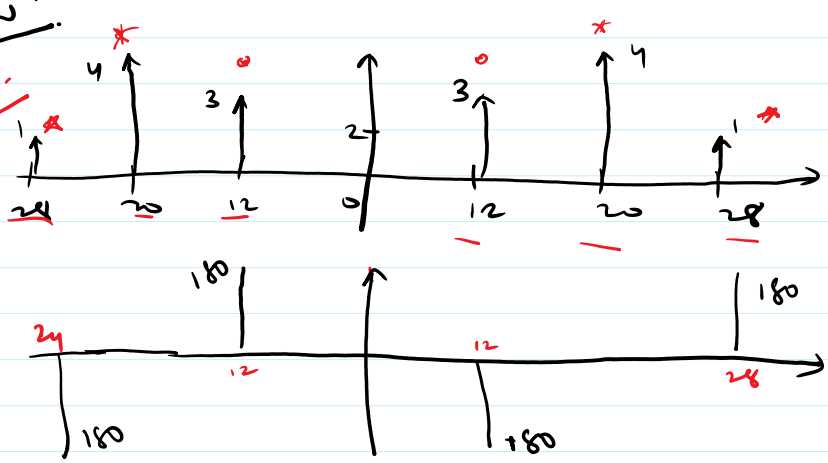
Ampl. → half.

Even symm.

Odd Symmetry

H.W.

Eg 2:



$x(t) = ?$

fundamental freq = ?

→ fundamental freq → GCD(12, 20, 28)

= 4 Hz

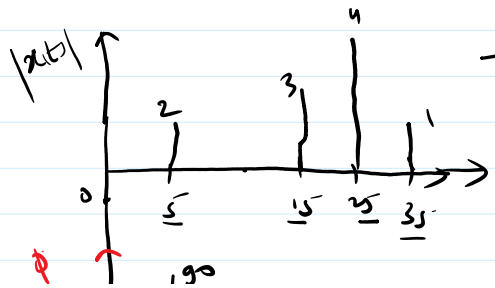
$\omega = 2\pi f$

$x(t) = 2\cos(\omega t) + 6\cos(\omega t + \phi) + 8\cos(\omega t) + 2\cos(\omega t)$

$f = 0 \qquad f = 12 \qquad f = 20 \qquad f = 28$

$$\boxed{x(t) = 2 + 6\cos(24\pi t - 180^\circ) + 8\cos(40\pi t) + 2\cos(56\pi t + 180^\circ)}$$

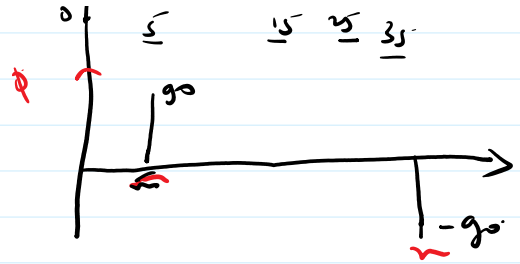
Eg:



→ single sided spectrum.

$f_0 = \text{GCD}(\overset{\omega_0}{5}, \overset{\omega_1}{15}, \overset{\omega_2}{25}, \overset{\omega_3}{35})$

$f_0 = 5\,Hz$



$f_0 = 5\,Hz$

$x(t) = A\cos(\omega t)$ $\quad \omega = 2\pi f$ $\quad f = 5, 15, 25, 35$ (2, 3, 4, 1)

$$x(t) = 2\cos(10\pi t + 90^\circ) + 3\cos(30\pi t + 0^\circ) + 4\cos(50\pi t) + 1\cos(70\pi t - 90^\circ)$$

Orthogonality :-

Two functions $f(t)$ & $g(t)$

Two functions are said to be orthogonal if area of their product $f(t) \cdot g^*(t)$ over the interval $\alpha \le t \le \beta$ equal to zero

$$\int_\alpha^\beta f(t) \cdot g^*(t)\, dt = 0$$

* Conjugation is req^d only complex signals

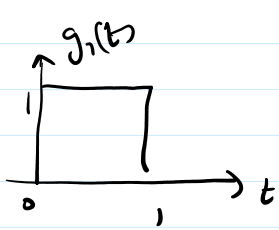
$$\int_\alpha^\beta f(t) \cdot g(t)\, dt = 0$$

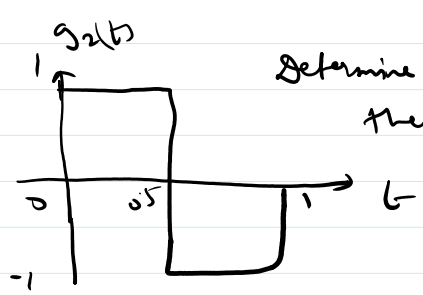
If the functions are same

$$\int_\alpha^\beta x_1(t)\, x_2(t) \neq 0 \quad \text{const ?}$$

Kn.

Eg:-





Determine whether they are orthogonal

$g_1(t) = 1 \qquad 0 \le t < 1$

$g_2(t) = 1 \qquad 0 \le t < 0.5$

$\quad\;\; = -1 \qquad 0.5 \le t < 1$

$$g(t) = \int_{\alpha}^{\beta} g_1(t) \cdot g_2(t)$$

$$= \int_0^{0.5} 1 \cdot 1 \, dt + \int_{0.5}^{1} 1 \cdot (-1) \, dt$$

$$g(t) = 0.5 - 0.5 = \underline{\underline{0}}$$

Eg:- S.T following s/g are orthogonal over an interval $[0, 1]$

$f(t) = 1 \qquad g(t) = \sqrt{3}(1-2t)$

$$\Rightarrow \int_{\alpha}^{\beta} f(t)\, g(t) = 0$$

$$\int_0^1 1 \cdot \sqrt{3}(1-2t)\, dt$$

$$\int_0^1 \sqrt{3}\, dt - \sqrt{3}\int_0^1 2t \, dt$$

$$\sqrt{3}\left[t - \frac{2t^2}{2}\right]_0^1$$

$$\underline{\underline{0}}$$

Eg:- S.T set of s/g $\{\sin n\omega_0 t\}$ forms an orthogonal set over the interval $\left\{0, \frac{2\pi}{\omega_0}\right\}$, Also find k_n.

$$\Rightarrow \int_0^{\frac{2\pi}{\omega_0}} \sin n\omega_0 t \cdot \sin m\omega_0 t \, dt = 0$$

$$\frac{1}{2}\int_0^{\frac{2\pi}{\omega_0}} 2 \sin n\omega_0 t \cdot \sin m\omega_0 t \, dt$$

$2\sin A \cdot \sin B$

$\cos(A-B) - \cos(A+B)$

$$\frac{1}{2}\int \cos(m-n)\omega_0 t - \cos(m+n)\omega_0 t \; dt$$

$$\frac{1}{2}\left[\frac{\sin m-n\,\omega_0 t}{(m-n)\omega_0} - \frac{\sin(m+n)\omega_0 t}{m+n\,\omega_0}\right]_0^{\frac{2\pi}{\omega_0}}$$

$\sin p(2\pi) = 0$

$= 0$ Hence verified.

$$K_n = \int_0^{2\pi/\omega_0} \sin^2 n\omega_0 t \; dt$$

$$= \int_0^{2\pi/\omega_0} \frac{1-\cos 2n\omega_0 t}{2}$$

$$= \int_0^{2\pi/\omega_0} \frac{1}{2}\,dt - \frac{1}{2}\int_0^{2\pi/\omega_0} \cos 2n\omega_0 t \; dt$$

$\sin. = 0$

$$= \frac{1}{2}\,\frac{2\pi}{\omega_0}.$$

$$\boxed{K_n = \frac{\pi}{\omega_0}}$$

H.W. (1) $\int_0^{\pi} \cos mt \cdot \cos nt \; dt \Rightarrow$ S.T. they are orthogonal.

(2) $\int_0^{\pi} \sin nt \cdot \sin mt \; dt$

(3) P.T $\{\sin nt\}$ $\{0, 2\pi\}$ they are orthogonal

Also find $|c_n| = \pi$

(4) P.T $\{\cos nt\}$ $\{0, 2\pi\}$ orthogonality

$K_n = \pi$

Fourier series :- Sum of sinusoids at fundamental frequency f_0 & its multiples of nf_0.

Any periodic signal $f(t)$ can be represented by

$$f(t) = a_0 \overbrace{\cos(0\omega t)}^{1} + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \cdots$$

zero

$$\cdots + a_n \cos(n\omega_0 t) + b_0 \sin(0\omega_0 t) + b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \cdots + b_n \sin(n\omega_0 t)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

a_0, a_n & b_n = coefficient of trigonometric f.s

a_0 : dc component or offset.

1. Trigonometric f.s
2. Polar f.s
3. Exponential f.s

1. Trigonometric f.s.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$\omega = 2\pi f_0 = \frac{2\pi}{T_0} \quad (1)$$

$$a_0 = \frac{1}{T_0} \int^{T_0} x(t)\, dt \quad \rightarrow (2)$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} x(t) \cos(n\omega t)\, dt \rightarrow (3)$$

$$b_n = \frac{2}{T_0} \int_0^{T_0} x(t) \sin(n\omega t)\, dt \rightarrow (4)$$

(2) <u>Polar series</u> :- (D_0 & D_n)

$$D_0 = a_0$$

Spectrum - mag / phase

$$D_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = -\tan^{-1} \frac{b_n}{a_n}$$

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega t + \phi_n)$$

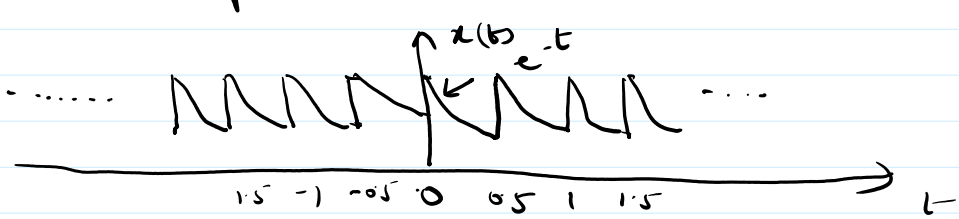
(3) Exponential F.S

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

$$c_n = \frac{1}{T_0} \int x(t)\, e^{-jn\omega t}\, dt.$$

eg:-

Find trigonometric F-S for the periodic signal



$\Rightarrow$ $T_0 = 0.5$

$f_0 = 2$

$a_0, \; a_n, \; b_n = ?$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt$$

$$= \frac{1}{0.5}\int_0^{0.5} e^{-t}\, dt$$

$$= \frac{1}{0.5}\left[-e^{-t}\right]_0^{0.5}$$

$$= \frac{1}{0.5}\left[1 - e^{-0.5}\right] = 0.786g$$

$$\boxed{a_0 = 0.786g}$$

$a_n = ?$

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(n\omega t)\, dt$$

$\omega = 2\pi f_0$
$= 2\pi \times 2$
$= 4\pi$

$$= \frac{2}{0.5}\int_0^{0.5} e^{-t}\cos(n4\pi t)\, dt$$

$$\boxed{\int e^{ax}\cdot\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)}$$

$a = -1 \qquad b = 4\pi n$

$$a_n = 4\left\{\frac{e^{-t}}{1+(4\pi n)^2}\left(-1\cos 4\pi n t + 4\pi n \sin 4\pi n\right)t\right]_0^{0.5}$$

$$a_n = 4\left\{\frac{e^{-0.5}}{1+(4\pi n)^2}\left[-\cos 4\pi n(0.5) + 4\pi n \sin 4\pi n(0.5)\right]\right.$$

$$\left. - e^{0}\left(-1\cos 0 + 4\pi n \sin 0\right)\right\}$$

$$- \frac{e^{0}}{1+(4\pi n)^2}\left(-1\cos 0 + 4\pi n \sin 0\right)\Big\}$$

$$= \frac{4}{1+(4\pi n)^2}\Big\{0.606\left[-\cos(2\pi n) + 4\pi n \sin(2\pi n)\right]$$

$$- \left[-\cos 0 + 4\pi n \sin 0\right]\Big\}$$

$$= \frac{4}{1+(4\pi n)^2}\left\{-0.606 + 0 + 1 + 0\right\}$$

$$\boxed{a_n = \frac{1.576}{1+(4\pi n)^2}}$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(n\omega t)\,dt$$

$$= \frac{2}{0.5}\int_0^{0.5} e^{-t}\sin(n4\pi t)\,dt$$

$$\boxed{\int e^{-ax}\sin bx\,dx = \frac{e^{-ax}}{a^2+b^2}\left[-a\sin bx + b\cos bx\right]}$$

$$b_n = \frac{e^{-t}}{1+(4\pi n)^2}\left[1\sin 4\pi n t + 4\pi n \sin 4\pi n t\right]_0^{0.5}$$

$$\vdots$$

$$\boxed{b_n = \frac{6.32\pi n}{1+(4\pi n)^2}}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega t) + \sum_{n=1}^{\infty} b_n\sin(n\omega t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$\boxed{x(t) = 0.7869 + \sum_{n=1}^{\infty} \frac{1.576}{1+(4\pi n)^2} \cos(4\pi n t) + \sum_{n=1}^{\infty} \frac{6.32\pi n}{1+(4\pi n)^2} \sin 4\pi n}$$

Eg 2:

Find Polar series of the above eq 1. & plot its magnitude & phase spectrum.

$\Rightarrow$

D_0 & D_n = ?

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega t + \phi_n)$$

$$\boxed{D_0 = a_0 = 0.7869}$$

$$D_n = \sqrt{a_n^2 + b_n^2}$$

$$= \sqrt{\frac{(1.576)^2}{[1+(4\pi n)^2]^2} + \frac{(6.32\pi n)^2}{[1+(4\pi n)^2]^2}}$$

$$= \sqrt{\frac{(1.576)^2 \cancel{[1+(4\pi n)^2]}}{1+(4\pi n)^2}}$$

$$\boxed{D_n = \frac{1.576}{\sqrt{1+(4\pi n)^2}}}$$

$$\phi_n = -\tan^{-1} \frac{b_n}{a_n}$$

$$= -\tan^{-1} \frac{6.32\pi n \,/\, \cancel{1+(4\pi n)^2}}{1.576 \,/\, \cancel{1+(4\pi n)^2}}$$

$1.576 / 1 + (4\pi n)^2$

$$\boxed{\phi_n = -\tan^{-1}(4\pi n)}$$

$$\boxed{x(t) = 0.786 + \sum_{n=1}^{\infty} \frac{1.576}{\sqrt{1+(4\pi n)^2}} \cos\left[4\pi n - \tan^{-1}(4\pi n)\right]}$$

Magnitude & Phase spectrum :-

we know that $T_0 = 0.5$ $f_0 = 2$.

$$\omega_0 = 2\pi f_0 = 4\pi$$

Magnitude spectrum is given by

$$|D_n| = \frac{1.576}{\sqrt{1+(4\pi n)^2}}$$

Plot $|D_n|$ vs ω.

$\omega = n\omega_0$

$n = 1, 2, 3 \ldots\ldots$

✓ $\omega = 4\pi n$.

$= 0, 4\pi, 8\pi, 12\pi \ldots\ldots\ldots$

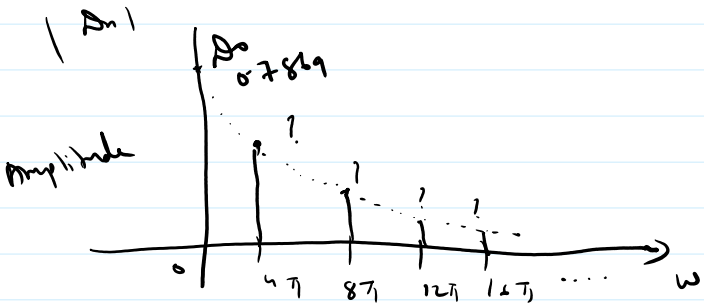
$$|D_n| = \frac{1.576}{\sqrt{1+\omega^2}}$$

$$\phi_n = -\tan^{-1}(4\pi n)$$

$$= -\tan(\omega)$$

$\omega = n\omega_0 = 4\pi n$	$\lvert D_n\rvert = \frac{1.576}{\sqrt{1+\omega^2}}$	$\phi_n = -\tan^{-1}\omega$
$n=1$ 4π	$\frac{1.576}{\sqrt{1+(4\pi)^2}} =$	$\phi_n = -\tan^{-1}(4\pi) =$

n=1	4π	$\frac{1.576}{\sqrt{1+(4\pi)^2}} =$	$\phi_n = -\tan^{-1}(4\pi) =$
n=2	8π	$\frac{1.576}{\sqrt{1+(8\pi)^2}} =$	$\phi_n = -\tan^{-1}(8\pi)$
n=3	12π	$\frac{1.576}{\sqrt{1+(12\pi)^2}} =$	$\phi_n = -\tan^{-1}(12\pi)$
n=4	16π	$\frac{1.576}{\sqrt{1+(16\pi)^2}} =$	$\phi_n = -\tan^{-1}(16\pi)$



85 to 90°

(3) Find the exponential f.s for eq (1)

$\Rightarrow$ $$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}$$

$\omega = 2\pi f$

$= \frac{2\pi}{T_0}$

$$c_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j\omega_n t}\, dt$$

$\omega = 4\pi$

$T_0 = 0.5 \qquad f_0 = 2$

$\int_0^{0.5} \ldots e^{-j4\pi n t} \ldots dt$

$t_0 = 0.5 \qquad \omega_0 = 2$

$$C_n = \frac{1}{0.5}\int_0^{0.5} e^{-t}\, e^{-j4\pi n t}\, dt$$

$$= \frac{1}{0.5}\int_0^{0.5} e^{-(1+j4\pi n)t}\, dt$$

$$= 2\left[\frac{e^{-(1+j4\pi n)t}}{-(1+j4\pi n)}\right]_0^{0.5}$$

$$= \frac{2}{1+j4\pi n}\left[-e^{-(1+j4\pi n)0.5} + e^{0}\right]$$

$$= \frac{2}{1+j4\pi n}\left[1 - e^{-0.5}\cdot e^{-j2\pi n}\right]$$

$$= \frac{2}{1+j4\pi n}\left[1 - 0.606\, e^{-j2\pi n}\right]$$

$e^{-j2\pi n} = 1 \rightarrow \cos 2\pi n - j\sin 2\pi n$, $\sin 2\pi n = 0$

$$\boxed{C_n = \frac{0.789}{1+j4\pi n}}$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{j\omega_n t}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{0.789}{1+j4\pi n}\, e^{j\omega_n t}}$$

Magnitude & phase spectrum of C_n.

$$C_n = \frac{0.789}{1+j4\pi n} \times \frac{1-j4\pi n}{1-j4\pi n}$$

$$= \frac{0.789 - j\,0.789 \times 4\pi n}{1-(j4\pi n)^2}$$

$$\frac{\quad}{1-(j4\pi n)^2}$$

$$\boxed{j^2=-1}$$

$$= \underbrace{\frac{0.789}{1+(4\pi n)^2}}_{\text{Real}} - j\underbrace{\frac{0.789\times 4\pi n}{1+(4\pi n)^2}}_{\text{Imag.}}$$

Magnitude of c_n

$$|c_n| = \sqrt{\left(\frac{0.789}{1+(4\pi n)^2}\right)^2 + \left(\frac{0.789\times 4\pi n}{1+(4\pi n)^2}\right)^2}$$

$$= \sqrt{\frac{(0.789)^2 + 0.789^2(4\pi n)^2}{\left[1+(4\pi n)^2\right]^2}}$$

$$= 0.789\sqrt{\frac{1+(4\pi n)^2}{\left[(1+4\pi n)^2\right]^2}}$$

$$|c_n| = \frac{0.789}{\sqrt{1+(4\pi n)^2}}$$

<u>Plot</u> c_n vs ω

$$\omega_0 = 2\pi f_0 = 4\pi$$

$$\omega = n\omega_0 = 4\pi n \qquad n = \pm 1, \pm 2, \pm 3 \ldots$$

$$|c_n| = \frac{0.789}{\sqrt{1+\omega^2}} \qquad \omega = 0, \pm 4\pi, \pm 8\pi, \pm 12\pi \ldots$$

$$\phi_n = -\tan^{-1}\frac{\text{Imag}}{\text{Real}}$$

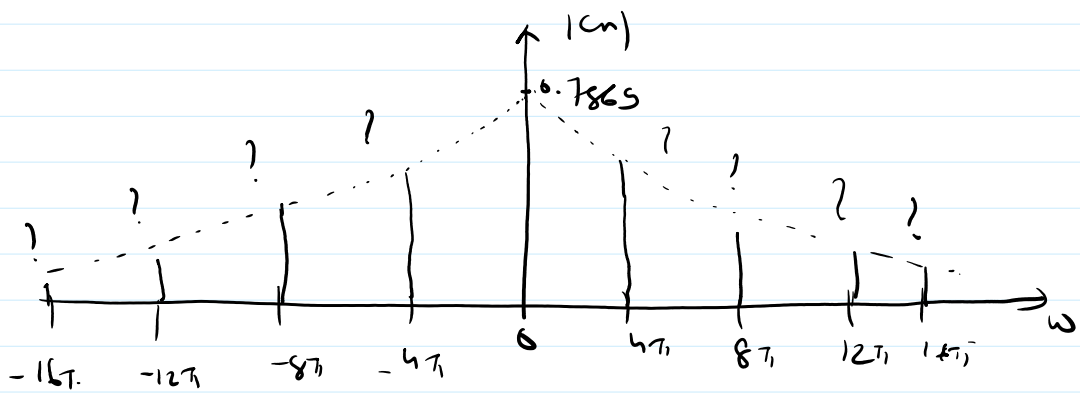
$$= -\tan^{-1}\frac{0.789 \times 4\pi n}{\left[1+(4\pi n)^2\right]}$$

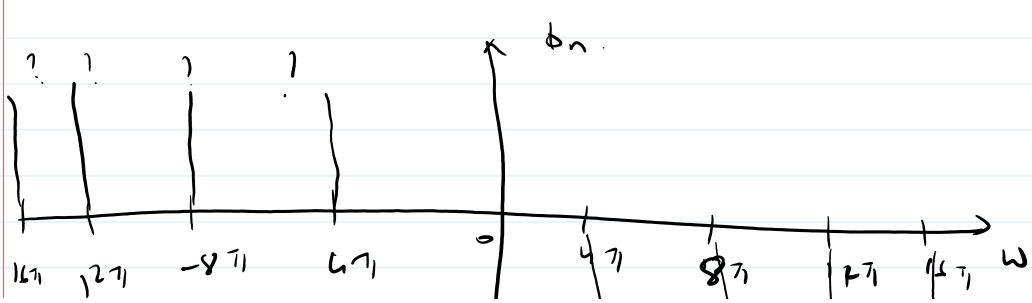
$$= -\tan^{-1} \frac{\cancel{0.7869} \times 4\pi n \Big/ \left[1+(4\pi n)^2\right]}{\cancel{0.7869}\left[1+(4\pi n)^2\right]}$$

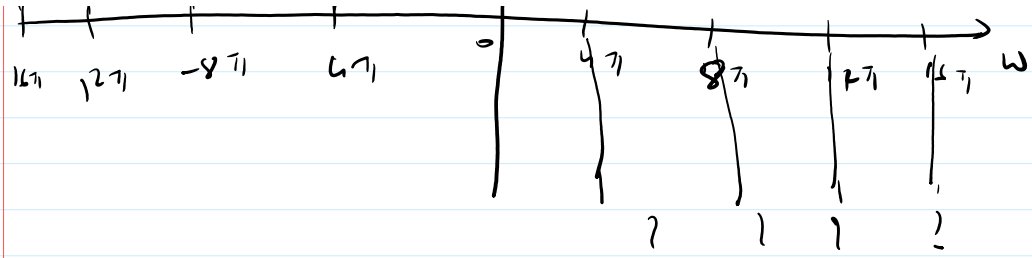
$$= -\tan^{-1} 4\pi n$$

$$\phi_n = -\tan^{-1}(w)$$

	$w = n\omega_0 = 4\pi n$	$\lvert c_n \rvert = \frac{0.7869}{\sqrt{1+w^2}}$	$\phi_n = -\tan^{-1}(w)$
0	0	$c_0 = 0.7869 =$	$-\tan^{-1}(0) =$
±1	$\pm 4\pi$	$= \frac{0.7869}{\sqrt{1+(4\pi)^2}} = ?$	$-\tan^{-1}(4\pi) =$ $-\tan^{-1}(-4\pi) =$
±2	$\pm 8\pi$	$\frac{0.7869}{\sqrt{1+(8\pi)^2}} = ?$	$-\tan^{-1}(8\pi)$ $-\tan^{-1}(-8\pi)$
±3	$\pm 12\pi$	$\frac{0.7869}{\sqrt{1+(12\pi)^2}} = ?$	$-\tan^{-1}(12\pi)$ $-\tan^{-1}(-12\pi)$







on comparison of both (polar & exps) spectrum

① $D_0 = C_0 = 0.7869$

② $D_1 = 2C_1 =$

$D_2 = 2C_2$

$D_3 = 2C_3$

$$\boxed{\text{ie } |D_n| = 2|C_n|}$$

③ magnitude spectrum has even symmetry ✓

④ ϕ_n is same

⑤ Phase spectrum has odd symmetry. ✓

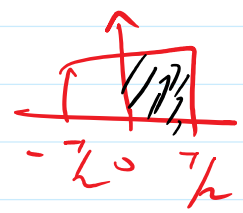
Symmetry of the signals

Even function * Even Function = Even Function

Odd function * odd Function = Even Function

Even function * Odd Function = Odd Function

Odd function * Even Function = Odd Function



$$\int_{-T/2}^{T/2} x_e(t)\,dt = \frac{2}{}\int_{0}^{T/2} x_e(t)\,dt$$

$$\int_{-T/2}^{T/2} x_o(t)\,dt = 0$$

* Even periodic function

$$a_0 = \frac{1\cdot 2}{T_0}\int_{0}^{T_0/2} x(t)\,dt$$

$$a_n = \frac{2\cdot 2}{T_0}\int_{0}^{T_0/2} \underbrace{x(t)}_{\text{even}}\,\underbrace{\cos(n\omega_0 t)}_{\text{even}}\,dt.$$

$$b_n = \frac{2}{T_0}\int \underbrace{x(t)}_{\text{even}}\,\underbrace{\sin(n\omega_0 t)}_{\text{odd}}\,dt = 0$$

only find a_0 & a_n

* **odd periodic function**

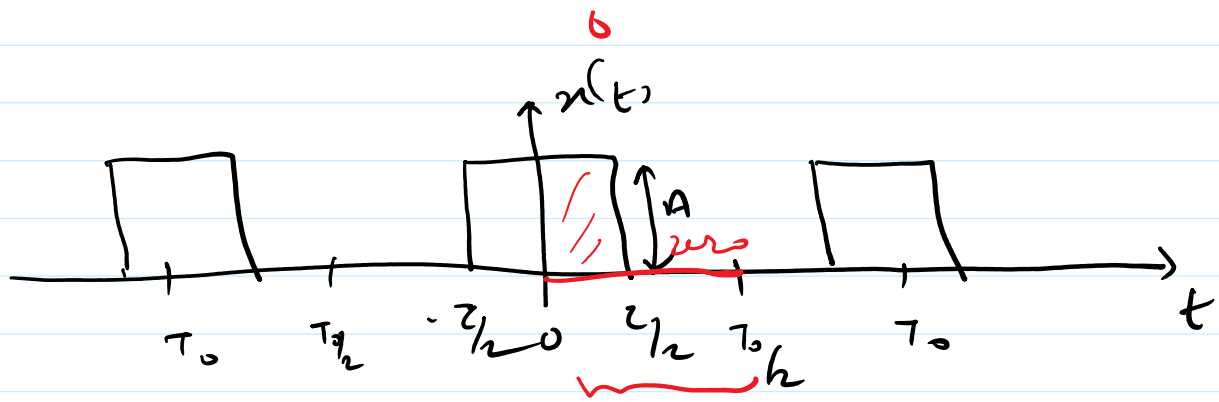
$$a_n = \frac{2}{T_0}\int x(t)\cdot \cos n\omega_0 t \, dt = 0$$

odd · even

$$\boxed{a_n = 0}$$

$$b_n = \frac{2\cdot 2}{T_0}\int_0^{T_0/2} x(t)\cdot \sin n\omega_0 t \, dt$$

*



⇒ we can use symmetry → even

$x_e(t) = x_e(-t)$

$a_0, a_n = ?$ $\qquad b_n = 0$

$$a_0 = \frac{2}{T_0}\int_0^{T_0/2} x(t)\, dt$$

$$= \frac{2}{T_0}\int_0^{\tau/2} x(t)\, dt + \int_{\tau/2}^{T_0} x(t)\, dt$$

zero

$$= \frac{2}{\quad}\int_0^{\tau/2} A\, dt = 2A\cdot t\Big|^{\tau/2}$$

$$= \frac{2}{T_0} \int_0^{\tau/2} A\,dt \qquad = \frac{2A}{T_0} \cdot t \Big|_0^{\tau/2}$$

$$\boxed{a_0 = \frac{A\tau}{T_0}}$$

$$a_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \cos n\omega_0 t \, dt$$

$$= \frac{4}{T_0} \int_0^{\tau/2} A \cdot \cos n\omega_0 t \, dt + \int_{\tau/2}^{T_0/2} 0 \cdot \cos n\omega_0 t \, dt$$

$$= \frac{4A}{T_0} \left[\frac{\sin n\omega_0 t}{n\omega_0} \right]_0^{\tau/2} + 0$$

$$= \frac{4A}{nT_0\omega_0} \left[\sin n\omega_0 \frac{\tau}{2} - \sin 0 \right]$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$= \frac{{}^{2}\cancel{4}A}{n\cancel{T_0}\,\frac{\cancel{2}\pi}{\cancel{T_0}}} \left[\sin n \frac{\cancel{2}\pi}{T_0} \cdot \frac{\tau}{\cancel{2}} \right]$$

$$\boxed{a_n = \frac{2A}{\quad} \left[\sin n\pi\tau \right]}$$

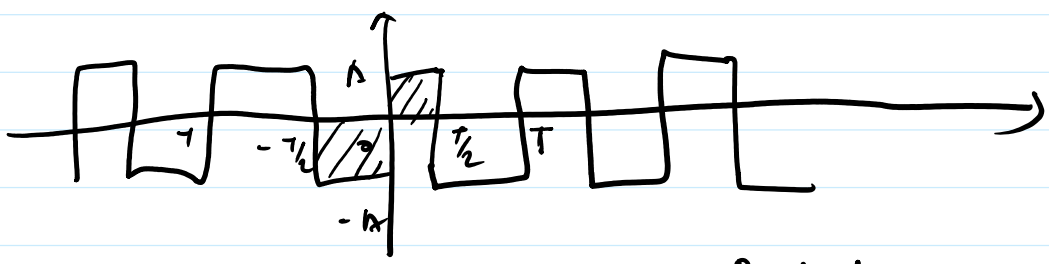
$$\boxed{a_n = \frac{2A}{n\pi}\left[\sin\frac{n\pi\tau}{T_0}\right]}$$

$b_n = 0$ [even symmetry]

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$\boxed{x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi}\sin\left(\frac{\pi n\tau}{T_0}\right)\cdot\cos(n\omega_0 t)}$$

*



Period T

→ Identify whether Symmetry exists?

odd or even

$$x(t) = -x(-t) \checkmark$$

$a_n = 0$ odd symmetry.

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cdot\sin(n\omega_0 t)\,dt$$

$$\frac{4}{\ } \quad T/2$$

$$b_n = \frac{4}{T}\int_0^{T/2} A \cdot \sin(n\omega t)\,dt$$

$$= \frac{4A}{T}\left[-\frac{\cos n\omega t}{n\omega}\right]_0^{T/2}$$

$$\omega = 2\pi/T$$

$$= \frac{\cancel{4}^{2}A}{\cancel{T}\,n\,\frac{\cancel{2}\pi}{\cancel{T}}}\left[-\cos n\frac{\cancel{2}\pi}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{2}} + \cos 0\right]$$

$$= \frac{2A}{n\pi}\left[-\cos n\pi + 1\right]$$

$$\boxed{b_n = \frac{2A}{n\pi}\left[1 - \cos n\pi\right]}$$

Dirichlet Conditions:

1. The function $x(t)$ should be absolutely integrable i.e. $\int x(t)\,dt < \infty$
2. The function should have finite number of maxima & minima in the interval T_0
3. The function $x(t)$ should have at the most finite number of discontinuities in the interval T_0

(4) The function $x(t)$ should be single valued within the interval T_0.

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Parseval power theorem:-

It states that the total average power of the periodic s/g $x(t)$, is equal to sum of the average power of its individual components.

F.S $\rightarrow$ expands $x(t)$ $\rightarrow$ Sum of phasor $C_n e^{j2\pi n f_0 t}$

The avg. power of each phasor component

$$P = \left| C_0^2 + C_1^2 + \ldots\ldots \right|$$

$$\left| C_n e^{j2\pi n f_0} \right|^2 = |C_n|^2$$

$$\boxed{\text{Total avg. power } P = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt$$

$$|x(t)|^2 = x(t) \cdot x^*(t)$$

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot x^*(t) \, dt$$

Expo. series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi n f_0 t}$$

$$x^*(t) = \left[\sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi n f_0 t} \right]^*$$

$$x^*(t) = \sum_{n=-\infty}^{\infty} c_n^* \, e^{j2\pi n f_0 t}$$

$$P = \frac{1}{T_0} \int x(t) \underline{\left[\sum_{n=-\infty}^{\infty} c_n^* \, e^{-j2\pi n f_0 t} \right]} dt.$$

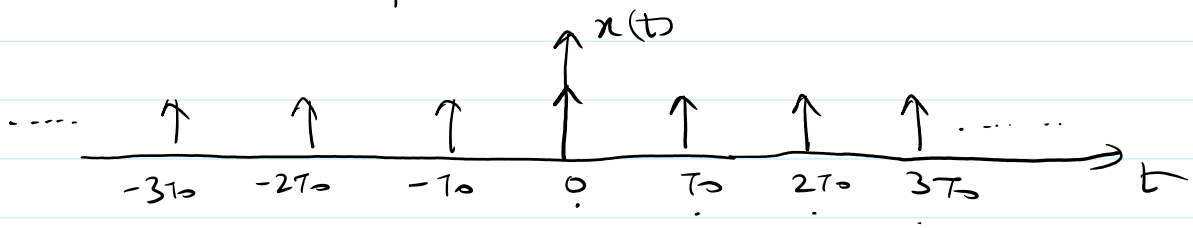
$$P = \sum_{n=-\infty}^{\infty} \underbrace{\left[\frac{1}{T_0} \int x(t) \, e^{-j2\pi n f_0 t} \, dt \right]}_{c_n} c_n^*$$

$$P = \sum_{n=-\infty}^{\infty} c_n \cdot c_n^*$$

$$\boxed{P = \sum_{n=-\infty}^{\infty} |c_n|^2}$$

Parseval's power theorem

Eg:- Find the total normalized avg. power of the unit impulse train



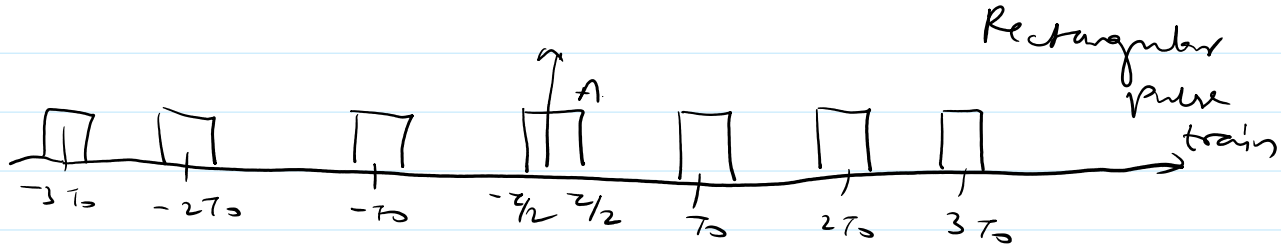
$$\Rightarrow \quad P = \sum_{n=-\infty}^{\infty} |c_n|^2 \qquad \text{step 1}$$

$$c_n = ?$$

$c_n \rightarrow$ Expo. F.S coefficient

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

$$c_n = \frac{1}{T_0} \int x(t)\, e^{-j\frac{2\pi n}{T_0} \cdot t}\, dt$$



width $\rightarrow \tau$

Ampl $\rightarrow A$

period $\rightarrow T_0$

$$c_n = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A \cdot e^{-j\frac{2\pi n t}{T_0}}\, dt$$

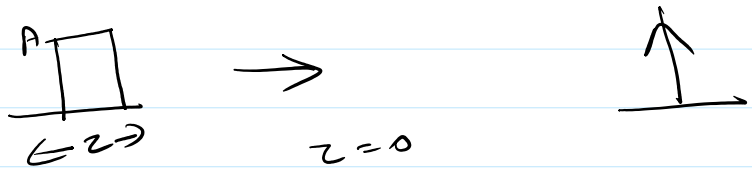
$$= \frac{A}{\cancel{T_0}} \left[\frac{e^{-j2\pi n t/T_0}}{\frac{-j2\pi n}{\cancel{T_0}}} \right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{\pi n} \left[\frac{-e^{-j\frac{2\pi n}{T_0}\tau/2} + e^{+j\frac{2\pi n}{T_0}\cdot\frac{\tau}{2}}}{2j} \right]$$

$$= \frac{A}{\pi n} \left[\sin\left(\frac{\pi n \tau}{T_0}\right) \right]$$

$$= \frac{A}{\cancel{\pi n}} \cdot \frac{\cancel{\pi n}\tau}{T_0} \frac{\sin\left(\frac{\pi n \tau}{T_0}\right)}{\frac{\pi n \tau}{T_0}}$$

$$C_n = \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{\pi n \tau}{T_0}\right)$$



$\tau = 0$

$A\tau = 1$

$$C_n = \lim_{\substack{A\tau \to 1 \\ \tau \to 0}} \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{n\pi\tau}{T_0}\right)$$

$$\boxed{\operatorname{sinc} 0 = 1}$$

$$\boxed{C_n = \frac{1}{T_0}}$$

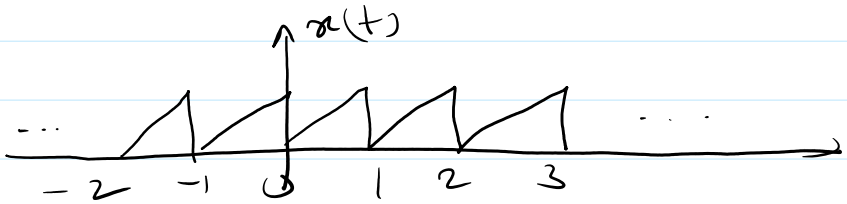
$\boxed{\sin c\ 0 = 1}$

By Parseval's theorem, total avg. power

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{T_0^2}$$

$$\boxed{P = \frac{1}{T_0^2}}$$

Eg H.W



Ans $|C_n| = \frac{1}{2T_1 n}$

$$\boxed{P = \sum_{n=-\infty}^{\infty} \frac{1}{4T_1^2 n^2}}$$

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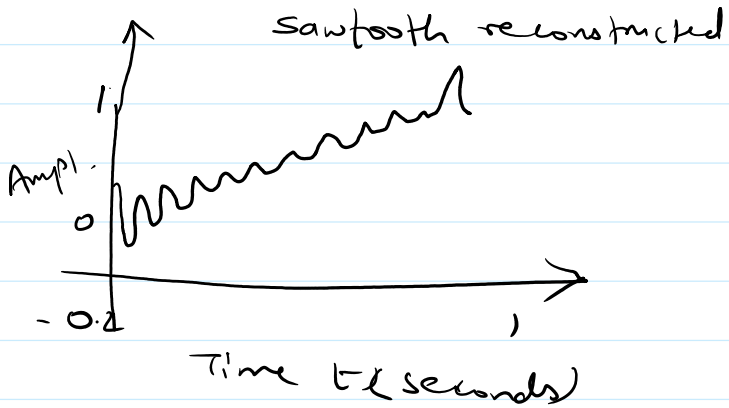
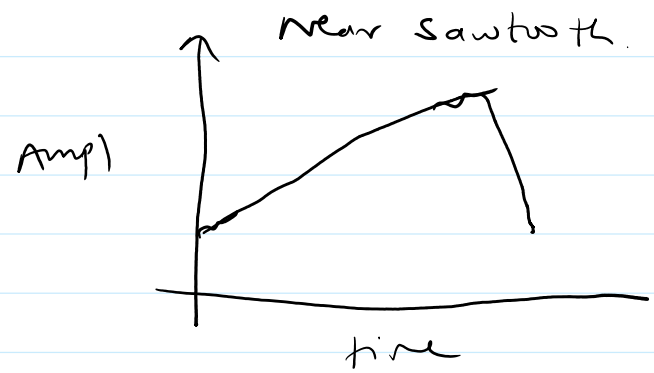
Signal reconstruction and the Gibbs effect

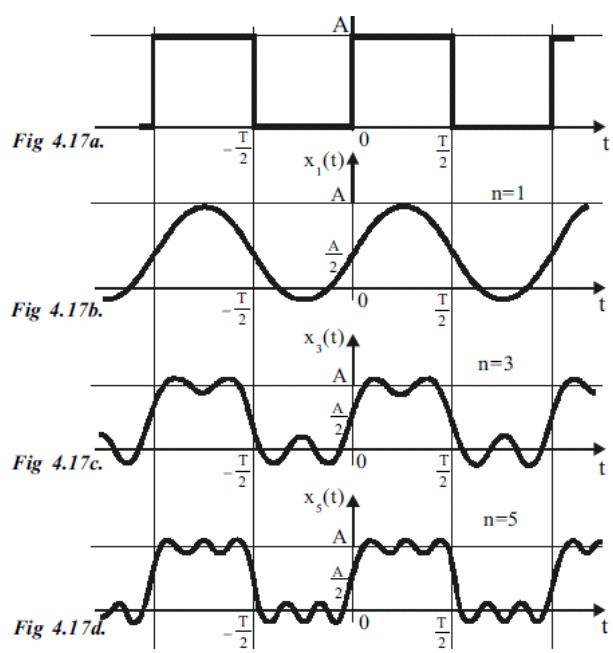
* Peaks in the harmonics used in s/g reconstruction occur at different times & lead to oscillations ie maxima & minima or ripples in the reconstruction.

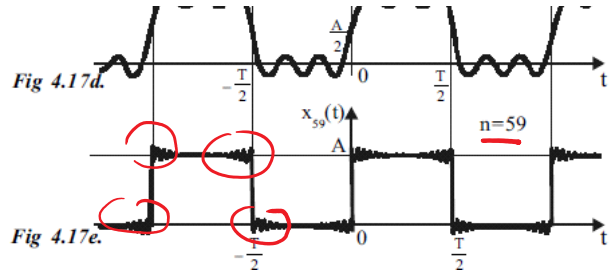
* No. of peaks in the ripples → indication of non-zero harmonics N → reconstruction.

* Can you count 20 peaks in the reconstruction









Gibbs phenomenon:-

The persistence of overshoot & imperfect reconstruction for signals with jumps → describes the Gibbs phenomenon / Gibbs effect.

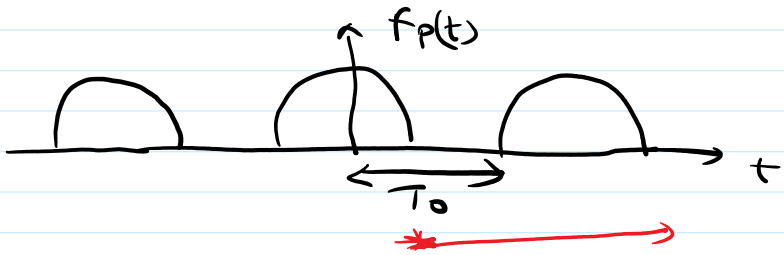
$N \to \infty$ 9% upward or downward.

Prof. Sushma Kadge

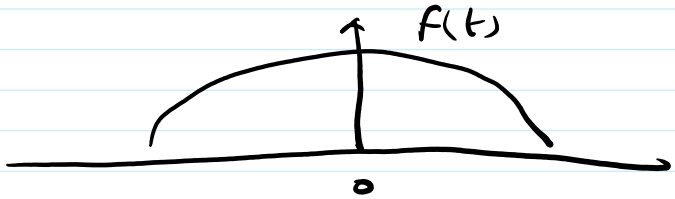
Tuesday, October 12, 2021 10:30 AM

* Fourier Transform * (FT)

* Scope & objective
  - freq. domain. description of time domain signals
  - Extension of FS
  - Aperiodic signal.
  - transforming a periodic s/g to an aperiodic s/g by stretching the period without limit.



periodic s/g



Non periodic s/g

FT → transformation technique, which transforms the signals from CT domain to corresponding freq domain & v$ce versa, → applied to both periodic & aperiodic (study) signals.

CTFT → (FT) ✓

✓ DTFT } higher sem.
✓ DFT

→ FT → extremely useful mathematical tool (LTI)

Application:-

- Signal Analysis (ECG, image, video)
- Signal Processing "
- Cryptography
- astronomy
- Biomedical.
- RAdar
- Communication.

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

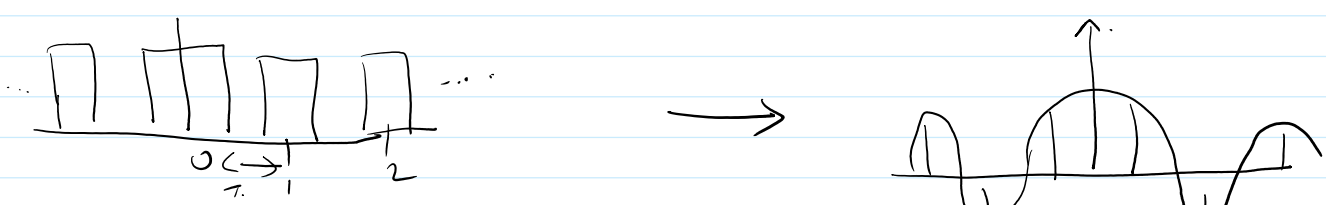
$$c_n = \frac{1}{T} \int x(t)\, e^{-jn\omega_0 t}\, dt$$

F.S → F.T?

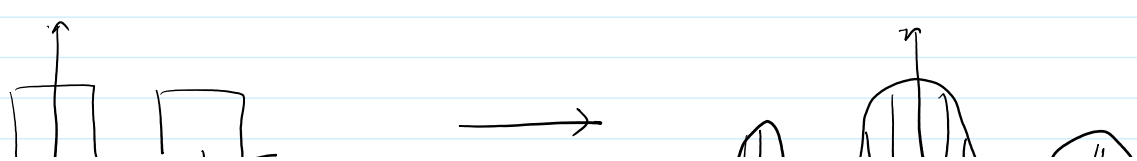
$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int x(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}$$

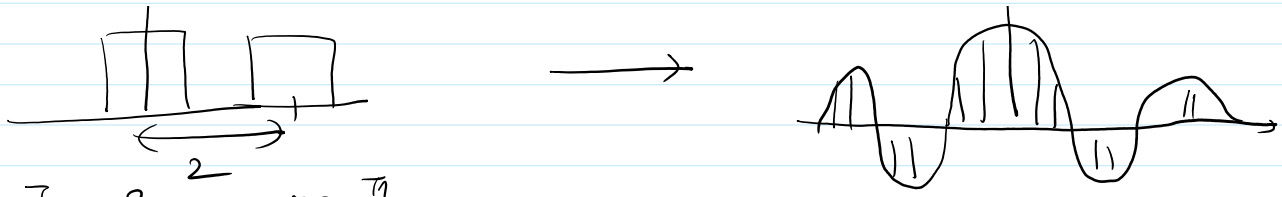
$$\omega_0 = 2\pi f = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega_0}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int x(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}\, \omega_0$$

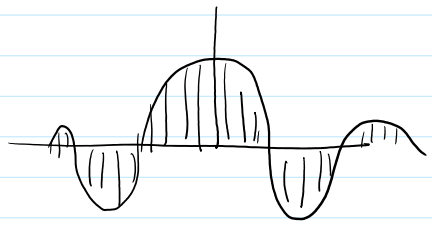


$T_0 = 1 \quad f_0 = 1 \quad \omega_0 = 2\pi$

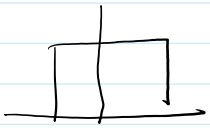




$T_0 = 2 \qquad \omega_0 = \pi$



$T_0 = 4 \qquad \omega_0 = \pi/2$



$T_0 \to \infty \qquad \omega_0 \to 0$ (infinitesimal small)

$$\omega_0 = \Delta\omega_0 \to d\omega \qquad \Sigma \to \int$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]}_{F.T.} e^{j\omega t}\, d\omega$$

F.T.

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt} \qquad (1)$$

IFT

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega} \qquad (2)$$

$$F\{x(t)\} \xrightarrow[F.T]{} X(\omega)$$

$$F^{-1}\{X(\omega)\} \xrightarrow[IFT]{} x(t)$$

$$\boxed{x(t) \longleftrightarrow X(\omega)}$$

Condition for existence F.T

① The fn $x(t)$ should be absolutely integrable fn

That means $\int_{-\infty}^{\infty} x(t)\,dt < \infty$

eg. $v^2 \rightarrow$ Resistor energy dissipated.

② The function $x(t)$ should be single value in any finite time interval T

③ finite number of discontinuities in any finite interval T

④ " " " maxima & minima " " "

* Amplitude & phase spectrums.

* " " " are continuous in nature

$$X(\omega) = X_R(\omega) + j X_I(\omega)$$

$$|X(\omega)| = \sqrt{X_R(\omega^2) + X_I(\omega)^2}$$

$$\angle X(\omega) = -\tan^{-1} \frac{X_I(\omega)}{X_R(\omega)}$$

Plot $\rightarrow |X(\omega)|$ vs ω $\rightarrow$ Amplitude spectrum } Freq spectrum

$\angle X(\omega)$ vs ω $\rightarrow$ Phase " }

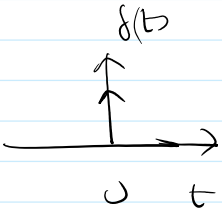
Fourier Transform of Standard signals:-

→ Impulse function $\delta(t)$ $\qquad x(t) = \delta(t)$ → given

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$X(\omega) = ?$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$



$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} \, dt$$

$$= 1 \cdot e^{-j\omega t} \Big|_{t=0}$$

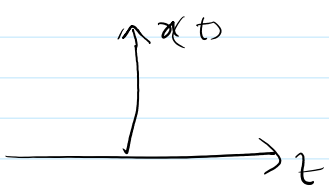
$$\boxed{X(\omega) = 1}$$

$$\boxed{\delta(t) \xleftrightarrow{F.T} 1}$$

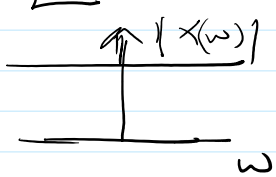
Spectrum:-

$$|X(\omega)| = 1 \quad \text{for all } \omega$$

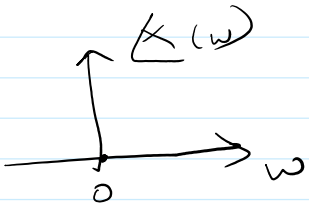
$$\angle X(\omega) = 0 \quad \text{for all } \omega$$



Impulse fn



mag. spectrum



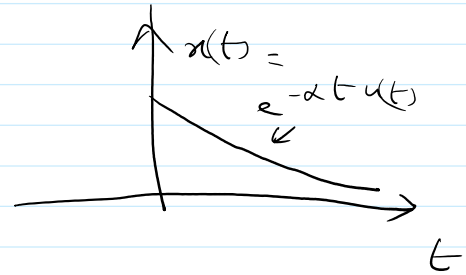
phase spectrum.

$$\boxed{F\,\delta(t - t_0) \longrightarrow e^{-j\omega t_0}}$$

(2) Single sided exponential function $e^{-\alpha t}\, u(t)$

$\Rightarrow$

$$x(t) = e^{-\alpha t} \qquad t \geq 0$$
$$= 0 \qquad t < 0$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\alpha t}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + j\omega)t}\, dt$$

$$= \left. \frac{e^{-(\alpha + j\omega)t}}{-(\alpha + j\omega)} \right|_{0}^{\infty}$$

$$= \frac{-1}{\alpha + j\omega}\left[e^{-(\alpha + j\omega)\infty} - e^{0} \right]$$

$$\boxed{X(\omega) = \frac{1}{\alpha + j\omega}}$$

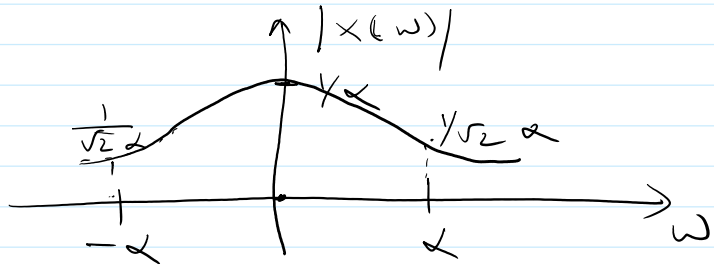
$$\boxed{e^{-\alpha t}\, u(t) \longleftrightarrow \frac{1}{\alpha + j\omega}}$$

Spectrum:-

$$\boxed{|X(\omega)| = \frac{1}{\sqrt{\alpha^2 + \omega^2}}}$$ mag.

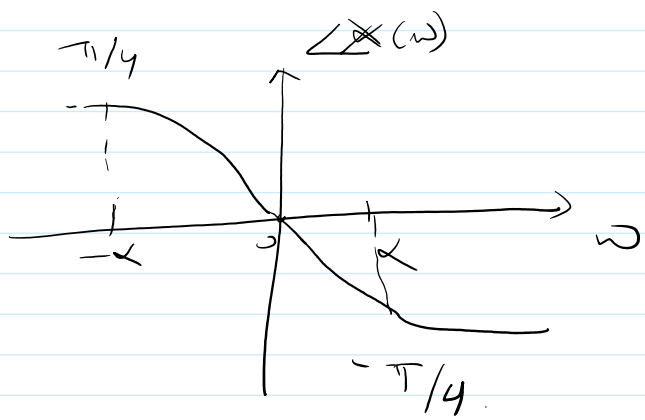
$$\boxed{\angle X(\omega) = -\tan^{-1}\frac{\omega}{\alpha}}$$ phase

ω	$\lvert X(\omega)\rvert$
0	$1/\alpha$
$\pm\alpha$	$\frac{1}{\sqrt{2}\,\alpha}$



Phase Spectrum

ω	$\angle X(\omega) = -\tan^{-1}\frac{\omega}{\alpha}$
0	0
$+\alpha$	$-\pi/4$
$-\alpha$	$\pi/4$
∞	$-\pi/2$



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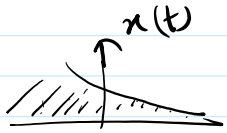
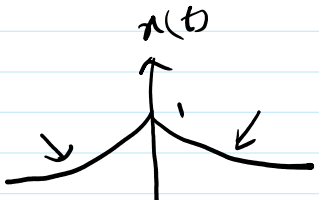
Note :  $e^{-\alpha t}$ $\longrightarrow$ area under curve is not finite $\rightarrow$ hence application of FT formula give absurd result.

Integral doesnot converge. $\therefore$ does not FT

(3) Double sided exponential signal

F.T of $e^{-\alpha|t|}$



$\Rightarrow$

$\Rightarrow$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{\alpha t}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-\alpha t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(\alpha - j\omega)t}\, dt + \int_{0}^{\infty} e^{-(\alpha + j\omega)t}\, dt$$

$$= \left.\frac{e^{(\alpha - j\omega)t}}{\alpha - j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(\alpha + j\omega)t}}{-(\alpha + j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega}$$

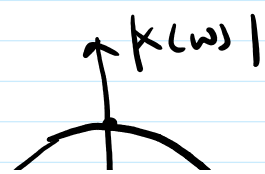
$$= \frac{\alpha + \cancel{j\omega} + \alpha - \cancel{j\omega}}{\alpha^2 + \omega^2} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$\boxed{X(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}}$$

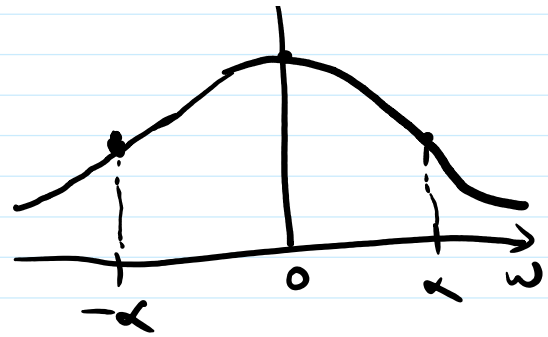
Spectrum:-

ω	$\lvert X(\omega) \rvert$

Amplitude Spectrum.



0	$\rightarrow 2/\alpha$
$\mp\alpha$	$\mp \frac{1}{\alpha}$



Phase Spectrum :-

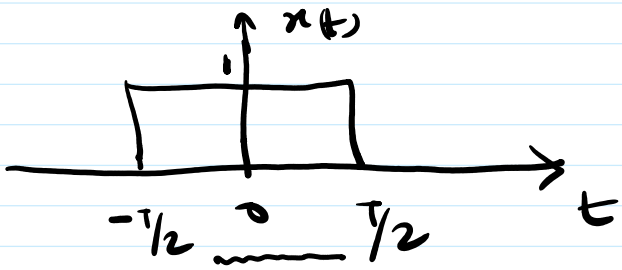
phase is zero at all values of ω

$$\angle X(\omega) = -\tan^{-1}\frac{\text{Imag}}{\text{Real}} \rightarrow$$

(4) F.T of gate function / rectangular pulse

$\Rightarrow$ Function is expressed as

$$G_T(t) = \begin{cases} 1 & T/2 \leq t < T/2 \\ 0 & \text{otherwise} \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int^{T/2} 1 \cdot e^{-j\omega t}\, dt$$

$$= \int_{-T/2} 1 \cdot e^{-j\omega t}\, dt$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2}$$

$$= \frac{1}{j\omega}\left[-e^{-j\omega T/2} - \underbrace{\left(-e^{+j\omega T/2}\right)}_{+} \right]$$

$$X(\omega) = \frac{1}{j\omega}\left[e^{+j\omega T/2} - e^{-j\omega T/2} \right] \times \frac{2}{2}$$

$$= \frac{2}{\omega}\left[\frac{e^{+j\omega T/2} - e^{-j\omega T/2}}{2j} \right]$$

$$\boxed{X(\omega) = \frac{2}{\omega} \sin \frac{\omega T}{2}} \quad *$$

This fn is written using sinc(x) or Sa(x) $\longrightarrow$ Sampling fn. & it is

$$\text{Sinc } x = \frac{\sin x}{x} \quad *$$

$$X(\omega) = \cancel{2} \cdot \frac{T}{\cancel{2}} \frac{\sin \omega T/2}{\omega\, T/2} \xrightarrow{\text{form.}} \sin \frac{x}{x}$$

$$X(\omega) = T \cdot \frac{\sin \omega T/2}{\omega T/2}$$

$\omega T/2$

$$\boxed{X(\omega) = T \cdot \operatorname{sinc}(\omega T/2)}$$

$$\operatorname{sinc}(x) = \frac{\sin x}{x} \qquad x = \frac{\omega T}{2}$$

$$\operatorname{sinc}(0) = 1$$

$$\operatorname{sinc}(\pi) = 0$$

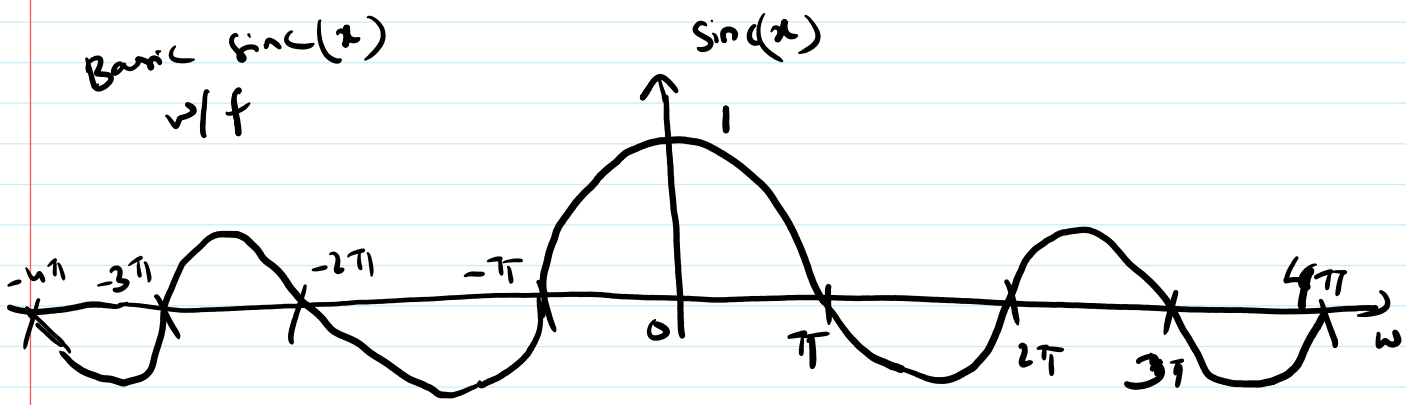
$x = \pm p\pi$ where 'p' is any integer

$$\frac{\omega T}{2} = \pm p\pi$$

$$\omega = \pm \frac{2p\pi}{T}$$

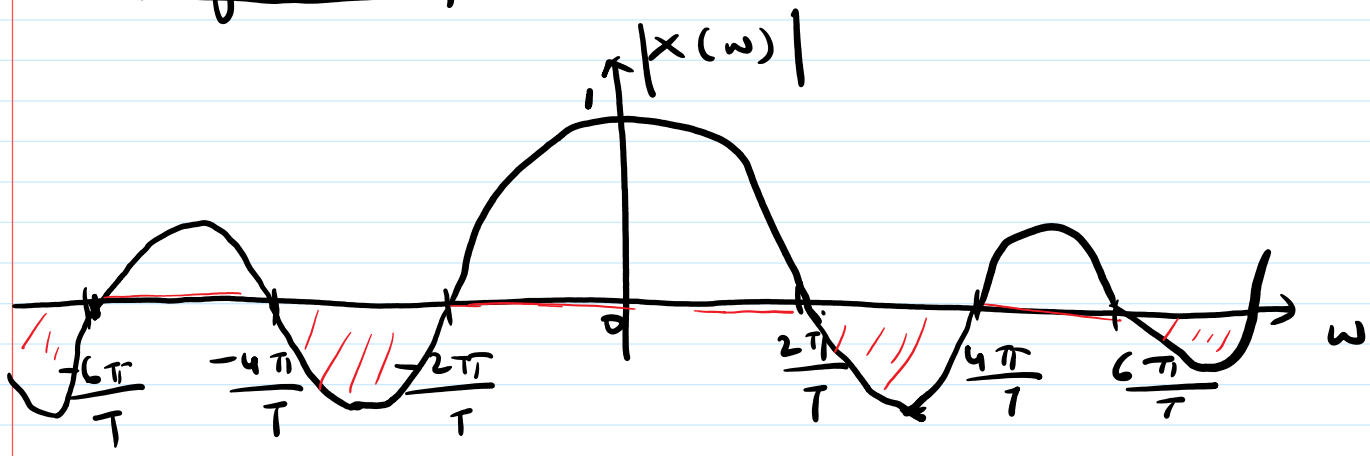
$$= \pm \frac{2\pi}{T}, \frac{4\pi}{T}, \frac{6\pi}{T} \ldots\ldots$$

Basic sinc(x) w/f



Rectangular pulse/gate fn.

magnitude spectrum :-

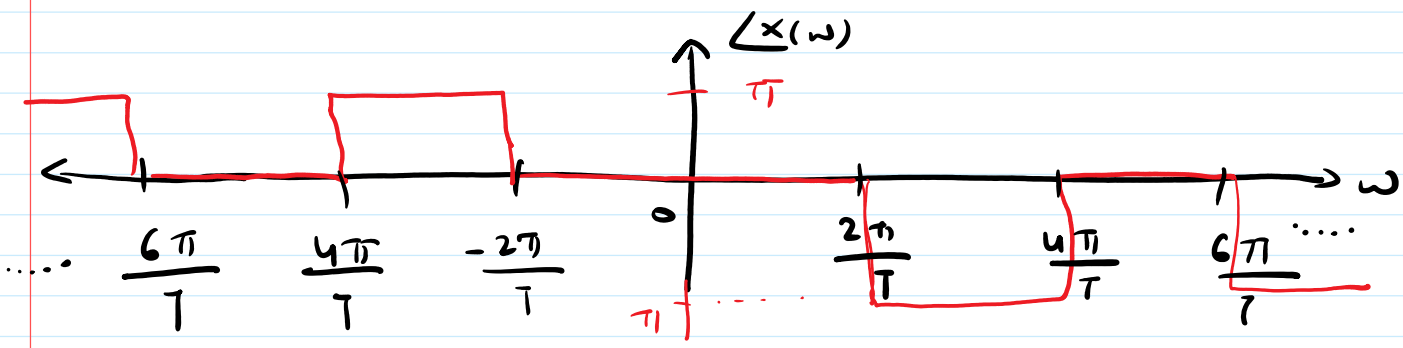


phase spectrum :-

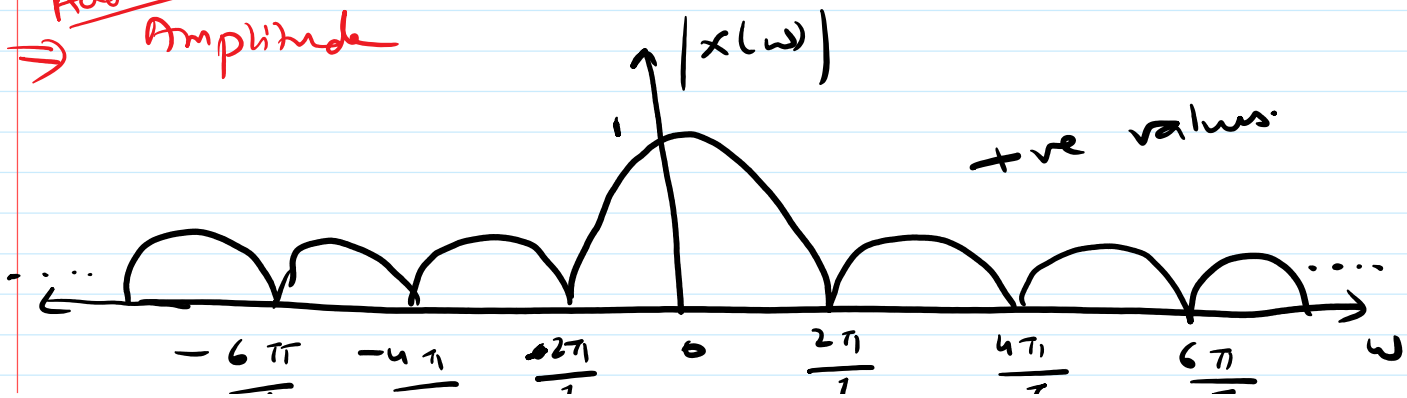
$\angle X(\omega)$ → Considering the angle $\pm\pi$

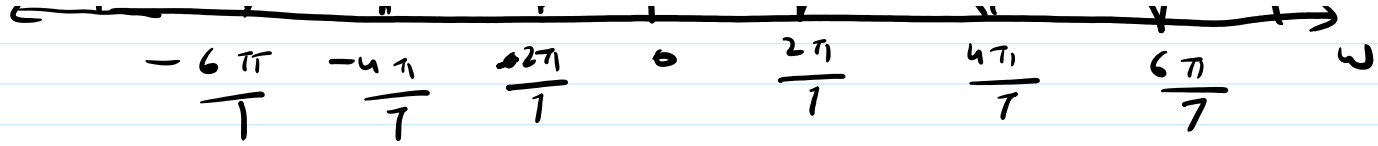
Angle is zero → $|X(\omega)|$ = +ve

Agle is $\pm\pi$ $|X(\omega)|$ = −ve

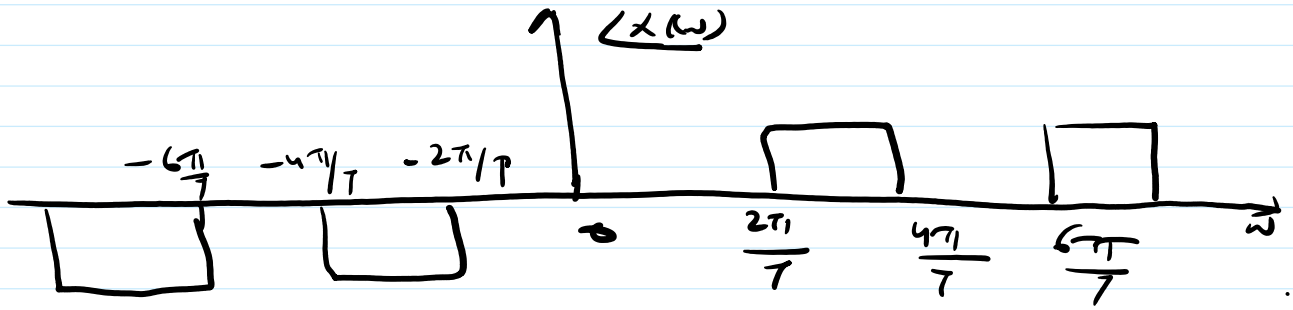


⇒ Additional Amplitude





−ve values of amplitude → +ve by phase shift of ± 180 in the phase spectrum

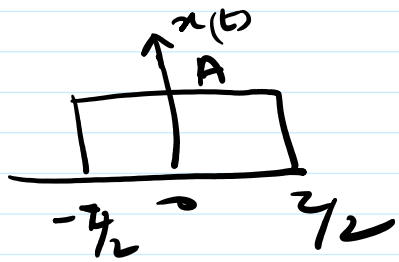


(5) Shifted Rectangular pulse → determine F.T

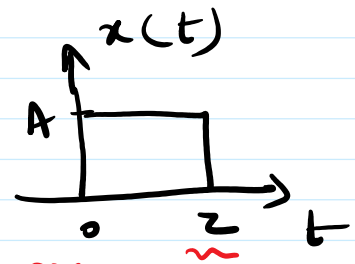
↓

delay by $\tau/2$ or $T/2$

⇒



Shifted ⇒ pulse



⇓

F.T = ?

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\tau} A \cdot e^{-j\omega t}\, dt$$

$$= A \cdot \left. \frac{e^{-j\omega t}}{-j\omega} \right]_{0}^{\tau}$$

Shifted pulse

↓

$$= \left[\quad e^{-j\omega \tau} \right]$$

$$= \frac{A}{j\omega}\left[1 - e^{-j\omega\tau}\right]$$

F.T. sinc

$$\frac{\sin x}{x}$$

$$= \frac{A}{j\omega}\, e^{-j\omega\tau/2}\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]$$

multiply & divide by 2

$$X(\omega) = \frac{2A}{\omega}\, e^{-j\omega\tau/2} \cdot \sin \omega\tau/2$$

shift

$$X(\omega) = 2A \cdot \tau/2\; e^{-j\omega\tau/2}\; \frac{\sin \omega\tau/2}{\omega\tau/2}$$

$$\boxed{X(\omega) = A\tau\, e^{-j\omega\tau/2} \sin(\omega\tau/2)}$$

shift by $\tau/2$

F.W spectrum.

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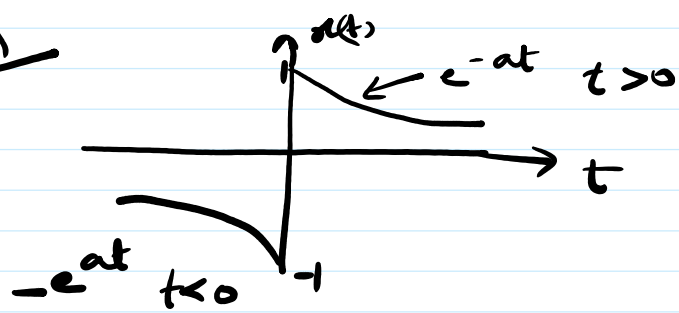


$$\Rightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

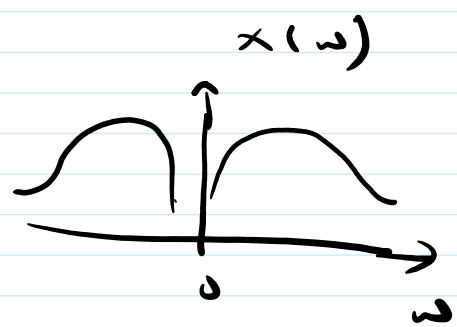
$$= \int_{-\infty}^{0} -1 \cdot e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} -e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

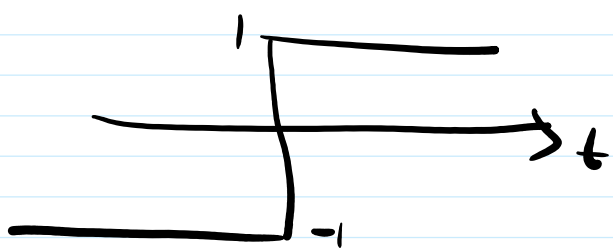
$$= -\frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega}$$

$$X(\omega) = \frac{\cancel{-\alpha} - j\omega + \cancel{\alpha} - j\omega}{\alpha^2 + \omega^2} = \frac{-2j\omega}{\alpha^2 + \omega^2}$$

$$\boxed{X(\omega) = \frac{-2j\omega}{\alpha^2 + \omega^2}}$$



*.



$\Rightarrow$ signum function $\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$

Function is not absolutely integrable.

so cannot directly find its F.T

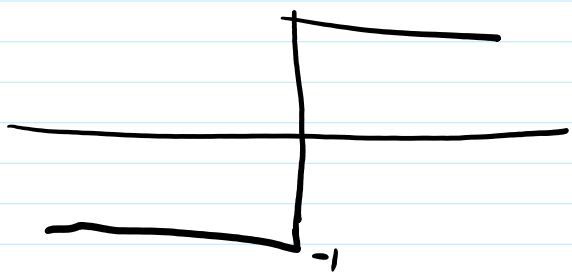
how =?

multiplying fn by expo.

<u>step</u>

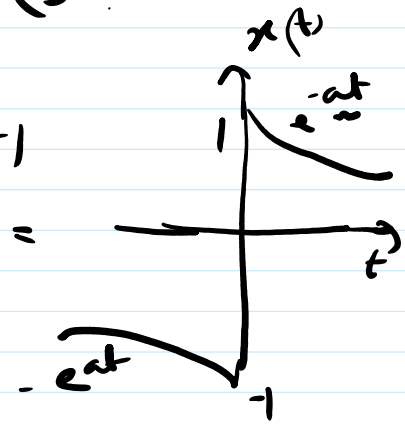
① $x(t) = e^{-a|t|}\,\mathrm{sgn}(t)$.

② $a \to 0 \rightarrow$ obtain $\mathrm{sgn}(t)$



$\times\ e^{-a|t|}$

$=$



$$X(\omega) = \lim_{a \to 0} \left[\mathrm{sgn}(t)\, e^{-a|t|} \right]$$ from previous problem.

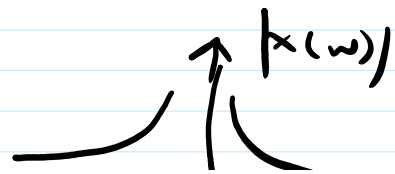
$$= \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2}$$

$$= \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

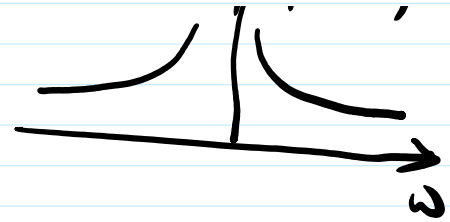
$$\boxed{X(\omega) = \frac{2}{j\omega}} \quad \text{or} \quad \boxed{\mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}}$$

<u>magnitude spectrum</u>

$$|X(\omega)| = \frac{2}{\omega}$$



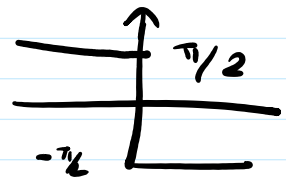
$$|X(\omega)| = \frac{2}{\omega}$$



<u>phase spectrum</u>

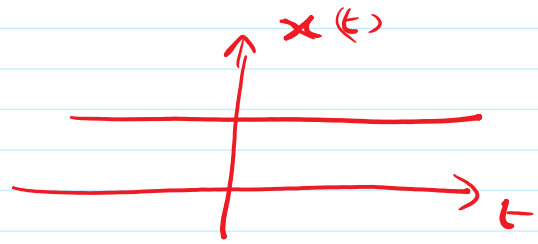
$$\angle X(\omega) = \frac{-\pi}{2} \quad \text{for } \omega > 0$$

$$= \frac{+\pi}{2} \quad \text{for } \omega < 0$$



* <u>F.T of a constant</u>

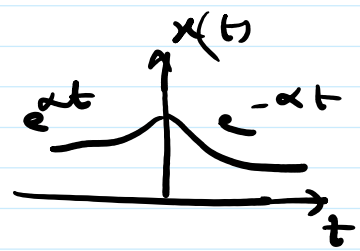
⇒ Let $f(t) = 1$



- Not absolutely integrable
- area under $|x(t)|$ infinite → attemp to evaluate F.T integral fails.

<u>Step</u>

① So → consider for $e^{-\alpha|t|}$



② $\alpha \to 0$

$$x(t) = 1 = \lim_{\alpha \to 0} F\left[e^{-\alpha|t|}\right]$$

$$F\left(e^{-\alpha|t|}\right) \longleftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$X(\omega) = \lim_{\alpha \to 0} \frac{2\alpha}{\alpha^2 + \omega^2}$$

$= 0$ except at $\omega = 0$

[Case 1:- $\alpha \to 0$ $\omega =$ Some value $(\neq 0)$

$$\lim_{\alpha \to 0} \frac{2}{\alpha + \frac{\omega^2}{\alpha}} = \underline{\underline{0}}$$

Case 2 $\omega = 0$

$$\lim_{\alpha \to 0} \frac{2}{\alpha + \frac{\omega^2}{\alpha}} = \text{Some const value. depending } \alpha.$$

$$F\{x(t) = 1\} = A\,\delta(\omega)$$

$$\int_{-\infty}^{\infty} A\,\delta(t)\,dt = A$$

$$A = \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + \omega^2}\,d\omega = 2 \cdot \tan^{-1}\frac{\omega}{\alpha}\Big|_{-\infty}^{\infty}$$

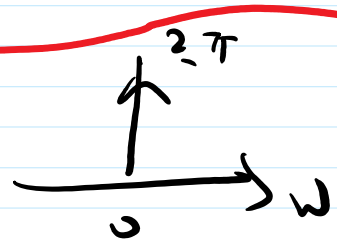
$$= 2\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right]$$

$$= 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

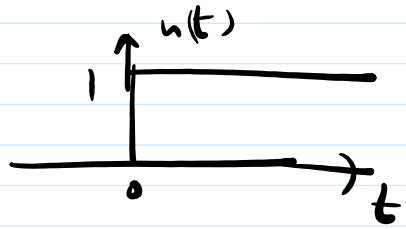
$$= 2\pi$$

$$\therefore \boxed{F\{x(t) = 1\} = 2\pi\,\delta(\omega)}$$

$$\therefore \boxed{F\left[x(t)=1\right] = 2\pi\delta(\omega)}$$

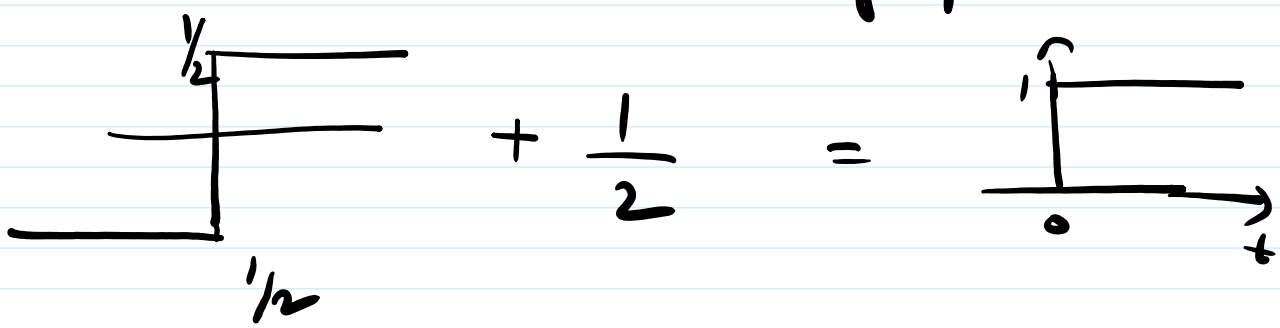


★ F.T of unit step $u(t)$



$\Rightarrow$ - Not absolutely integrable

- express $u(t)$ in terms of $sgn(t)$



$$\frac{1}{2}\,sgn(t) + \frac{1}{2} = u(t)$$

$$X(\omega) = F\left[\frac{1}{2}\left(1 + sgn(t)\right)\right]$$

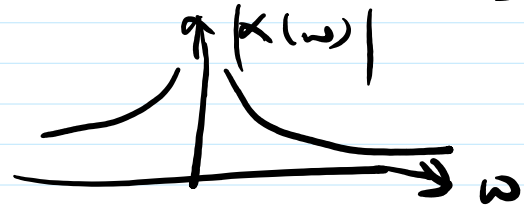
$$= \frac{1}{2}\left[2\pi\delta(\omega) + \frac{2}{j\omega}\right]$$

$$\boxed{X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}}$$

$$\boxed{u(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}}$$

$|X(\omega)| = \infty$ at $\omega = 0$ & zero at $\omega = \pm\infty$

Spectrum.



Let, $\mathcal{F}\{x(t)\} = X(j\Omega)$; $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

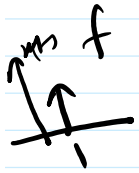
$j\Omega$ (ω)

Property	Time domain signal	Frequency domain signal
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(j\Omega) + a_2 X_2(j\Omega)$
Time shifting	$x(t - t_0)$	$e^{-j\Omega t_0} X(j\Omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{j\Omega}{a}\right)$
Time reversal	$x(-t)$	$X(-j\Omega)$
Conjugation	$x^*(t)$	$X^*(-j\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(j(\Omega - \Omega_0))$
Time differentiation	$\frac{d}{dt} x(t)$	$j\Omega\, X(j\Omega)$ $\rightarrow (j\omega\, X(\omega))$
Time integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(j\Omega)}{j\Omega} = \pi X(0)\, \delta(\Omega)$
Frequency differentiation	$t\, x(t)$	$j \frac{d}{d\Omega} X(j\Omega)$
Time convolution	$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$	$X_1(j\Omega) \cdot X_2(j\Omega)$
Frequency convolution (or Multiplication)	$x_1(t)\, x_2(t)$	$\frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega-\lambda))\, d\lambda$

$(j\omega)^2 \times (\omega)$

$(j\omega)^3 \times (\omega)$

imp



Symmetry of real signals	$x(t)$ is real	$X(j\Omega) = X^*(j\Omega)$ $\lvert X(j\Omega) \rvert = \lvert X(-j\Omega) \rvert$; $\angle X(j\Omega) = -\angle X(-j\Omega)$ $\mathrm{Re}\{X(j\Omega)\} = \mathrm{Re}\{X(-j\Omega)\}$ $\mathrm{Im}\{X(j\Omega)\} = -\mathrm{Im}\{X(-j\Omega)\}$
Real and even	$x(t)$ is real and even	$X(j\Omega)$ are real and even
Real and odd	$x(t)$ is real and odd	$X(j\Omega)$ are imaginary and odd
Duality	If $x_2(t) \equiv X_1(j\Omega)$ [i.e., $x_2(t)$ and $X_1(j\Omega)$ are similar functions] then $X_2(j\Omega) \equiv 2\pi x_1(-j\Omega)$ [i.e., $X_2(j\Omega)$ and $2\pi x_1(-j\Omega)$ are similar functions]	
Area under a frequency domain signal	$\int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi\, x(0)$	
Area under a time domain signal	$\int_{-\infty}^{+\infty} x(t)\, dt = X(0)$	
Parseval's relation	Energy in time domain is, $E = \int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt$	Energy in frequency domain is, $E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\Omega) \rvert^2 d\Omega$
	$\int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\Omega) \rvert^2\, d\Omega$	

Examples

(1) If $m(t - t_0) + m(t + t_0) = p(t)$.

F.T of $P(t)$ using time shifting property.

→ Time shifting.

$$x(t \mp t_0) \longleftrightarrow e^{\mp j\omega t_0} X(\omega)$$

$$m(t - t_0) \overset{f}{\longleftrightarrow} e^{-j\omega t_0} M(\omega) \quad \rightarrow (1)$$

$$m(t + t_0) \overset{f}{\longleftrightarrow} e^{+j\omega t_0} M(\omega) \quad \rightarrow (2)$$

$$p(t) \longleftrightarrow \underline{\underline{P(\omega)}} = ?$$

$$P(\omega) = e^{-j\omega t_0} M(\omega) + e^{+j\omega t_0} M(\omega)$$

$$= 2 \underbrace{\left[\frac{e^{+j\omega t_0} + e^{-j\omega t_0}}{2} \right]}_{} M(\omega)$$

$$\boxed{P(\omega) = 2 \cos(\omega t_0) M(\omega)}$$

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11:30am

(2)

LTI system, initially at rest is described by differential eqn.

$$\underbrace{\frac{d^2 y(t)}{dt^2}}_{2^{nd}} + 3 \underbrace{\frac{d y(t)}{dt}}_{1^{st}} + 2 y(t) = \frac{d x(t)}{dt} + 3 x(t)$$

Determine transfer function & impulse response of the system.

→ $H(\omega) = \frac{Y(\omega)}{X(\omega)}$ → $h(t)$

⇒ Same problem by F.T / L.T

$x(t)$ i/p → $\boxed{\text{system } h(t)}$ → o/p $y(t) \Rightarrow$ conv.

$X(\omega) \cdot \text{———} \cdot H(\omega)$ $\qquad$ $Y(\omega)$ = multiplication

$X(s)$ $\qquad$ $H(s)$ $\qquad$ $Y(s) \rightarrow$

$H(s) = \dfrac{Y(s)}{X(s)}$?

$Y(s) \quad X(s) \cdot H(s)$

$y(t) \rightarrow$ o/p of system

$$\frac{d^2}{dt^2} = (j\omega)^2$$

$$\frac{d}{dt} = j\omega$$

$$y(t) \overset{F.T}{\longleftrightarrow} Y(\omega)$$

$$x(t) \longleftrightarrow X(\omega)$$

By time differentiation property.

$$(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega) = j\omega X(\omega) + 3X(\omega)$$

$$\boxed{H(\omega) = ?}$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$$

$$\boxed{H(\omega) = \frac{j\omega + 3}{[(j\omega)^2 + 3j\omega + 2]}}$$

(1) $h(t) = ?$ $\qquad$ root of denominator

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= -1, -2$$

$$H(\omega) = \frac{3 + j\omega}{(j\omega+1)(j\omega+2)}$$

$$H(\omega) \xleftrightarrow[\text{IFT}]{} h(t)$$

<u>PFE</u> $\quad H(\omega) = \dfrac{A}{j\omega+1} + \dfrac{B}{j\omega+2}$

2 Coeff: $\quad A = (j\omega+1) \cdot H(\omega) \Big|_{j\omega=-1} = 2$

$$B = (j\omega+2)\, H(\omega) \Big|_{j\omega=-2} = -1$$

$$H(\omega) = \frac{2}{j\omega+1} - \frac{1}{j\omega+2}$$

3 $\quad$ IFT $\quad e^{-at} u(t) \longleftrightarrow \dfrac{1}{a+j\omega}$

$$\boxed{h(t) = 2 \cdot e^{-t} u(t) - 1 \cdot e^{-2t} u(t)}$$

This is impulse response of system.

Eg

$$\frac{d^2 y(t)}{dt^2} + \frac{3\, dy(t)}{dt} + 2y(t) = x(t)$$

Calculate o/p $\rightarrow y(t)$ if input $x(t) = e^{-3t} u(t)$

$\Rightarrow$ (1) Take F.T on D.S

$$(j\omega)^2 Y(\omega) + 3(j\omega)Y(\omega) + 2Y(\omega) = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3j\omega + 2}$$

(2) $x(t) = e^{-3t}u(t)$

$$X(\omega) = \frac{1}{3+j\omega}$$

$$Y(\omega) = \frac{1}{(j\omega)^2 + 3j\omega + 2} \cdot \frac{1}{3+j\omega}$$

(3) $y(t) = ?$

$$Y(\omega) \xrightarrow{IFT} y(t)$$

(a) roots, PFE

$$Y(\omega) = \frac{1}{(1+j\omega)(2+j\omega)} \cdot \frac{1}{3+j\omega}$$

$$Y(\omega) = \frac{A}{1+j\omega} + \frac{B}{2+j\omega} + \frac{C}{3+j\omega}$$

Calculate A, B, C. = *

$$A = 1/2 \quad B = -1, \quad C = 1/2$$

$$Y(\omega) = \frac{1}{2}\frac{1}{1+j\omega} - 1 \cdot \frac{1}{2+j\omega} + 1/2 \cdot \frac{1}{3+j\omega}$$

IFT

$$y(t) = \frac{1}{2}e^{-t}u(t) - e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

IFT

$$\boxed{y(t) = \frac{1}{2} e^{-t} u(t) - e^{-2t} u(t) + \frac{1}{2} e^{-3t} u(t)}$$

This is the o/p of system

Determine convolution of $x_1(t) = e^{-2t} u(t)$

$x_2(t) = e^{-6t} u(t)$.

$\Rightarrow$ $X_1(\omega) = \frac{1}{2+j\omega}$ $\quad X_2(\omega) = \frac{1}{6+j\omega}$.

convolution property. $\rightarrow$ conv $\rightarrow$ multiplicn freq.

$$X(\omega) = X_1(\omega) \cdot X_2(\omega)$$

$$X(\omega) = \frac{1}{2+j\omega} \cdot \frac{1}{6+j\omega}$$

PFE $\quad X(\omega) = \frac{A}{2+j\omega} + \frac{B}{6+j\omega}$

$$X(\omega) = \frac{0.25}{2+j\omega} - \frac{0.25}{6+j\omega}.$$

$$\boxed{x(t) = 0.25 e^{-2t} u(t) - 0.25 e^{-6t} u(t)}$$

Eg: use Time shifting property find F.T of

$$x(t) = e^{-a|t-t_0|}$$

$\Rightarrow$ $X(\omega) = ?$

$$x(t-t_0) = e^{-j\omega t_0} X(\omega)$$

$$x(t - t_0) = e^{-j\omega t_0} X(\omega)$$

$$e^{-a|t|} = \frac{2a}{a^2 + \omega^2}$$

$$\boxed{e^{-a|t - t_0|} \xleftrightarrow[\text{property}]{\text{Time}} e^{-j\omega t_0} \cdot \frac{2a}{a^2 + \omega^2}}$$