Non Exact D. Eqn.

$M\,dx + N\,dy = 0$ is non exact if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

① Find an IF using rules.

② · Multiply D. Eqn by an I.F. so that the diff eqn becomes exact D. Eqn.

③ Solve exact D. Eqn using solns.

Rules .

(1) If D.E is homogeneous i.e M & N both are homogeneous of same degree then

$$IF = \frac{1}{Mx+Ny} \quad (Mx + Ny \neq 0)$$

(2) If D. Eqn is of the form
$f(xy)\, y\, dx + f(xy)\, x\, dy = 0$ then

$$IF = \frac{1}{Mx-Ny} \quad (Mx - Ny \neq 0)$$

(3) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (/ const) then $IF = e^{\int f(x)\,dx}$

(4) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ (/ const.) then $IF = e^{\int f(y)\,dy}$

$(x^2 + y^2 + 1)dx - 2xy\,dy = 0$

$M = x^2+y^2+1 \qquad N = -2xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ D. Eqn is non exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-(-2y)}{-2xy} = \frac{4\cancel{y}}{-2x\cancel{y}} = -\frac{2}{x} = f(x)$$

$$IF = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x}$$
$$= e^{\log x^{-2}} = x^{-2}$$
$$= \frac{1}{x^2}$$

Multiplying DE by I.F we get.

$$\left(\frac{x^2+y^2+1}{x^2}\right) dx - \frac{2xy}{x^2} dy = 0$$

$$\left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right] dx - \frac{2y}{x} dy = 0.$$

Clearly D. Eqn is exact now.

$M' = 1 + \frac{y^2}{x^2} + \frac{1}{x^2} \qquad N' = -\frac{2y}{x}$

$\int x^{-2}$

$\int x^n = \frac{x^{n+1}}{n+1}$

Soln is

$\int (\text{terms in } M' \text{ free from } y)\, dx + \int N'\, dy = C$

$\Rightarrow \int \left(1 + \frac{1}{x^2}\right) dx + \int -\frac{2y}{x} dy = C$

$$x + \frac{x^{-1}}{(-1)} - \frac{\cancel{2}}{x}\frac{y^2}{\cancel{2}} = C \Rightarrow x - \frac{1}{x} - \frac{y^2}{x} = C$$

$y(xy + e^x)dx - e^x dy = 0$

$M = xy^2 + ye^x$ $\qquad$ $N = -e^x$

$\frac{\partial M}{\partial y} = 2xy + e^x$ $\qquad$ $\frac{\partial N}{\partial x} = -e^x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Non-exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-e^x - (2xy + e^x)}{xy^2 + ye^x} = \frac{-2xy - 2e^x}{y(xy + e^x)}$$

$$= \frac{-2\cancel{(xy+e^x)}}{y\cancel{(xy+e^x)}}$$

$$= \frac{-2}{y} = f(y)$$

$$I.F = e^{\int f(y)dy} = e^{-\int \frac{2}{y} dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2}$$

$$= \frac{1}{y^2}$$

Multiply D-Eqn by I.F

$$\frac{y(xy + e^x)}{y^2} dx - \frac{e^x}{y^2} dy = 0$$

$$\underbrace{\left(\frac{x\cancel{y}}{\cancel{y}} + \frac{e^x}{y}\right)}_{M} dx - \underbrace{\frac{e^x}{y^2}}_{N} dy = 0 \quad = \text{this is exact.}$$

Soln is

$$\int (\text{terms in M free from } y)\, dx + \int N dy = C$$

$$\int x \, dx + \int -\frac{e^x}{y^2} dy = C \qquad\qquad \frac{1}{y^2} = y^{-2}$$

$$\frac{x^2}{2} - e^x \frac{y^{-1}}{-1} = C$$

$$\frac{x^2}{2} + \frac{e^x}{y} = C$$

$$\frac{dy}{dx} = -\frac{x^2y^3+2y}{2x-2x^3y^2} \qquad (2x-2x^3y^2)\,dy = -(x^2y^3+2y)\,dx$$

$$\underbrace{(x^2y^3+2y)}_{M}dx + \underbrace{(2x-2x^3y^2)}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = x^2 3y^2 + 2 \qquad \frac{\partial N}{\partial x} = 2 - 2y^2 3x^2$$

$$= 2 - 6x^2y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} = \text{Non exact}$$

$$y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$$

which is of the form $y f(xy)\,dx + x f(xy)\,dy = 0$

$$\therefore \quad IF = \frac{1}{Mx-Ny} = \frac{1}{x^3y^3+\cancel{2xy}-(\cancel{2xy}-2x^3y^3)}$$

$$= \frac{1}{3x^3y^3}$$

Multiplying by an I.F. we get

$$\left(\frac{x^2y^3+2y}{3x^3y^3}\right)dx + \left(\frac{2x-2x^3y^2}{3x^3y^3}\right)dy = 0$$

$$\left(\frac{x^2y^3}{3x^3y^3} + \frac{2y}{3x^3y^3}\right)dx + \left(\frac{2x}{3x^3y^3} - \frac{2x^3y^2}{3x^3y^3}\right)dy = 0$$

$$\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0.$$

clearly this is exact

$\therefore$ Solⁿ is

$$\int M\,dx + \int(\text{terms in N free from } x)\,dy = C$$

$$\Rightarrow \int\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int -\frac{2}{3y}\,dy = C$$

$$\Rightarrow \frac{1}{3}\log x + \frac{\cancel{2}}{3y^2}\frac{x^{-2}}{(-\cancel{2})} + -\frac{2}{3}\log y = C \qquad x^{-3}$$

$$\Rightarrow \frac{1}{3}\left[\log x - \log y^2\right] - \frac{1}{3x^2y^2} = C \qquad x^{-1} \neq \cancel{\frac{x^0}{0}}$$

$$\Rightarrow \frac{1}{3}\log\left(\frac{x}{y^2}\right) - \frac{1}{3x^2y^2} = C$$

Solve $(x^2 - xy + y^2)dx - xydy = 0$ $\quad M\,dx + N\,dy = 0$

Exact

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\quad M = x^2 - xy + y^2$ $\quad N = -xy$

Soln

$\int M$ $\quad \int M dx + \int (\text{terms in } N \text{ free from}) dy = c$ $\quad \frac{\partial M}{\partial y} = -x + 2y$ $\quad \frac{\partial N}{\partial x} = -y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ D. Eqn is non-exact.

Clearly D. Eqn is homogeneous as M & N both are homog. of same deg.

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{x^3 - x^2y + \cancel{xy^2} - \cancel{xy^2}} = \frac{1}{x^2(x-y)}$$

Multiplying D. Eqn by I.F we get

$$\left[\frac{x^2 - xy + y^2}{x^2(x-y)}\right] dx - \frac{xy}{x^2(x-y)} dy = 0$$

Clearly above D. Eqn is exact.

$$\left[\frac{x(x-y) + y^2}{x^2(x-y)}\right] dx - \left[\frac{y - x + x}{x(x-y)}\right] dy = 0$$

$$\left[\frac{\cancel{x(x-y)}}{x^{\cancel{2}}\cancel{(x-y)}} + \frac{y^2 - x^2 + x^2}{x^2(x-y)}\right] dx - \left[\frac{-(x-y) + x}{x(x-y)}\right] dy = 0$$

$$\left[\frac{1}{x} + \frac{-(x^2-y^2) + x^2}{x^2(x-y)}\right] dx + \left[\frac{\cancel{(x-y)}}{x\cancel{(x-y)}} - \frac{\cancel{x}}{\cancel{x}(x-y)}\right] dy = 0$$

$$\left[\frac{1}{x} - \frac{\cancel{(x-y)}(x+y)}{x^2\cancel{(x-y)}} + \frac{\cancel{x^2}}{\cancel{x^2}(x-y)}\right] dx + \int\left[\frac{1}{x} - \frac{1}{x-y}\right] dy = 0$$

$$\left[\frac{1}{x} - \left(\frac{x}{x^2} + \frac{y}{x^2}\right) + \frac{1}{x-y}\right] dx + \left[\frac{1}{x} - \frac{1}{x-y}\right] dy = 0$$

$$\left[-\frac{y}{x^2} + \frac{1}{x-y}\right] dx + \left[\frac{1}{x} - \frac{1}{x-y}\right] dy = 0$$

Soln is

$$\int (\text{terms in } M \text{ free from } y)\, dx + \int N\, dy = C$$

$$\Rightarrow \int 0\, dx + \int\left(\frac{1}{x} - \frac{1}{x-y}\right) dy = C$$

$$\frac{y}{x} - \frac{\log(-y+x)}{(-1)} = C$$

$$\frac{y}{x} + \log(x-y) = C$$

$\frac{1}{-y+x}$

$\int \frac{1}{(bx-a)} dx$

$= \frac{\log(bx-a)}{b}$

8. If $f(x)$ a function of x only is an integrating factor of $(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0$. find $f(x)$ and then solve the equation.

Given I.F $= f(x)$.

$\therefore$ multiplying D. Eqn by I.F.

$f(x)(x^4e^x - 2mxy^2)\,dx + f(x)\,2mx^2y\,dy = 0$

Clearly above D. Eqn is exact.

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ ——— (1)

$M = f(x)(x^4e^x - 2mxy^2)$ $\quad$ $N = f(x)\,2mx^2y$

$\dfrac{\partial M}{\partial y} = f(x)(0 - 2mx\,2y)$ $\quad$ $\dfrac{\partial N}{\partial x} = 2my\left(f(x)\,2x + x^2f'(x)\right)$

$= -4mxy\,f(x)$

from (1) $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$-\cancel{4}^{2}\cancel{mxy}\,f(x) = \cancel{2my}\left(f(x)\,2x + x^2f'(x)\right)$

$\Rightarrow -2x\,f(x) = 2x\,f(x) + x^2f'(x).$

$\Rightarrow -4x\,f(x) = x^2f'(x)$

$\Rightarrow -\dfrac{4\cancel{x}}{x^{\cancel{2}}} = \dfrac{f'(x)}{f(x)}$

$\Rightarrow \int -\dfrac{4}{x}\,dx = \int \dfrac{f'(x)}{f(x)}\,dx$

$-4\log x = \log(f(x))$

$\log x^{-4} = \log(f(x))$

$f(x) = x^{-4} = \dfrac{1}{x^4}$

$\therefore$ Exact D. Eqn is

$\dfrac{1}{x^4}(x^4e^x - 2mxy^2)\,dx + \dfrac{1}{x^4}\,2mx^2y\,dy = 0$

$\Rightarrow \left(e^x - 2m\dfrac{xy^2}{x^4}\right)dx + \dfrac{2my}{x^2}\,dy = 0$

Sln is

$\int(\text{terms in } M \text{ free from } y)\,dx + \int N\,dy = C$

$\int e^x\,dx + \int \dfrac{2my}{x^2}\,dy = C$

$e^x + \dfrac{\cancel{2}m}{x^2}\dfrac{y^2}{\cancel{2}} = C$

9. If $(x+y)^k$ is an integrating factor of $(4x^2 + 2xy + 6y)\,dx + (2x^2 + 9y + 3x)\,dy = 0$. find k and solve the equation.

$I.F = (x+y)^k$

Multiplying D.Eqn by I.F.

$$\underbrace{(x+y)^k (4x^2+2xy+6y)}_{M}\,dx + \underbrace{(x+y)^k (2x^2+9y+3x)}_{N}\,dy = 0$$

Clearly D.Eqn is exact. $\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\frac{\partial M}{\partial y} = (x+y)^k(2x+6) + (4x^2+2xy+6y)\,k\,(x+y)^{k-1}$$

$$= (x+y)^{k-1}\left[(x+y)(2x+6) + k(4x^2+2xy+6y)\right]$$

$$\frac{\partial N}{\partial x} = (x+y)^k(4x+3) + (2x^2+9y+3x)\,k\,(x+y)^{k-1}$$

$$= (x+y)^{k-1}\left[(x+y)(4x+3) + k(2x^2+9y+3x)\right]$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\cancel{(x+y)^{k-1}}\left[2x^2+6x+2xy+6y + k(4x^2+2xy+6y)\right]$$

$$= \cancel{(x+y)^{k-1}}\left[4x^2+3x+4xy+3y + k(2x^2+9y+3x)\right]$$

$$k\left[4x^2+2xy+6y-2x^2-9y-3x\right]$$

$$= 4x^2+3x+4xy+3y-2x^2-6x-2xy-6y$$

$$\Rightarrow k\left[2x^2+2xy-3y-3x\right] = \left(2x^2-3x+2xy-3y\right)$$

$k = 1$ $\quad$ $I.F = (x+y)$

$$\int \sinh\left(\frac{x}{y}\right) dy.$$

$$\int \sinh\left(\frac{x}{y}\right) dx$$

$$\frac{\cosh\left(\frac{x}{y}\right)}{1/y}$$

$$\left[2x \sin h\left(\frac{y}{x}\right) + 3y\cos h\left(\frac{y}{x}\right)\right] dx - 3x.\cos h\left(\frac{y}{x}\right).dy = 0$$

$$(xy \sin xy + \cos xy)\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0$$

$$(x^2y - 2xy^2)\, dx - (x^3 - 3x^2y)\, dy = 0.$$