Higher Order Linear D.Eqⁿ with const. coefficients.

A.D.Eqⁿ is of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X \quad \text{—— (1)}$$

Here, $P_0, P_1, \ldots P_n$ are const.
X is funⁿ of x.

$\frac{d}{dx} = D \qquad \left(\frac{d}{dx} y = Dy \right.$

$P_0 D^n y + P_1 D^{n-1} y + \cdots + P_n y = X$

$(P_0 D^n + P_1 D^{n-1} + \text{——} + P_n)\, y = X$

$f(D)\, y = X \quad \text{—— (2)}$

general solⁿ of (2) is given by

y = complementary funⁿ + Particular Integral
(CF) (PI)

$y = y_c + y_p$

* C.F. y_c

Consider $X = 0$ in (2)

$f(D) y = 0$

Auxiliary eqⁿ is. $f(D) = 0$

Then find roots of A.eqⁿ.
& C.F is depends on nature of the roots.

(1) Roots are real & distinct.

Consider $m_1, m_2, \ldots m_n$ are all real & distinct roots
then $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \text{——} + c_n e^{m_n x}$

eg. suppose $2, -2, 3$ are roots
$y_c = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{3x}$

(2) Roots are real & repeated.

Consider m_1 is repeated thrice & remaining roots are m_2, m_3, m_4

$y_c = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + c_4 e^{m_2 x} + c_5 e^{m_3 x} + c_6 e^{m_4 x}$

eg. consider roots are $2, 2, 2, 2, 3$
$y_c = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4 x^3 e^{2x} + c_5 e^{3x}$

(3) Complex & distinct roots say $a \pm ib$, $(m_1 = a+ib,\ m_2 = a-ib)$

$y_c = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

e^{mx}
$e^{(a+ib)x}$
$e^{ax} \cdot e^{ibx}$
$e^{ax}(c_1 \cos bx + c_2 \sin bx)$
$+ e^{ax}(c_3 \cos bx + c_4 \sin bx)$
$e^{ax}((c_1 + c_3)\cos bx + (c_2 + c_4)\sin bx)$

(4) Roots are complex & repeated.

Consider $m_1 = m_2 = a + ib$
$m_3 = m_4 = a - ib$

$y_c = e^{ax}(c_1 \cos bx + c_2 \sin bx) + x e^{ax}(c_3 \cos bx + c_4 \sin bx)$

eg. suppose roots are $2 \pm 3i,\ 2 \pm 3i$
$a = 2,\ b = 3$
$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + e^{2x} x\,(c_3 \cos 3x + c_4 \sin 3x).$

① $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$ $D = \frac{d}{dx}$

$D^3y - 6D^2y + 11Dy - 6y = 0.$

$(D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary eqn : $D^3 - 6D^2 + 11D - 6 = 0$

$D = 1, 3, 2$

$y_c = c_1e^{x} + c_2e^{3x} + c_3e^{2x}$

② $(D^4 - 18D^2 + 81)y = 0$

A. eqn $=$ $D^4 - 18D^2 + 81 = 0$

$D^2 = m$

$m^2 - 18m + 81 = 0$

$m = 9, 9$

$D^2 = 9, 9$

$D = \pm 3, \pm 3$

$D = 3, 3, -3, -3$

$D^2 = 9$

$D = \pm\sqrt{9}$

$y_c = c_1e^{3x} + c_2xe^{3x} + c_3e^{-3x} + c_4xe^{-3x}$

(3) $$\frac{d^3y}{dx^3} + y = 0.$$

$$D^3y + y = 0$$

$$(D^3+1)y = 0$$

A. eq^n $\quad D^3 + 1 = 0$

$$D = -1, \ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\underset{a}{\frac{1}{2}} + \underset{b}{\frac{\sqrt{3}}{2}}i$$

$$y_c = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

(4) $$(D^4+4)y = 0.$$

A eq^n. $\quad D^4 + 4 = 0.$

$$(D^2)^2 + 2^2 = 0$$

$$\underline{(D^2)^2 + 2D^2\cdot 2 + 2^2} - 2D^2\cdot 2 = 0$$

$$(D^2+2)^2 - (2D)^2 = 0$$

$$(D^2+2-2D)(D^2+2+2D) = 0$$

$$\underline{(D^2-2D+2)} \quad (D^2+2D+2) = 0$$

$a=1 \quad b=1$

$$D = 1 \pm i, \ -1 \pm i$$

$1 + i1$

$-1 + i$

$$y_c = e^x(c_1\cos x + c_2 \sin x) + e^{-x}(c_3\cos x + c_4 \sin x)$$

$$(D^2+1)^3(D^2+D+1)^2 y = 0.$$

A. eqn $(D^2+1)^3 \quad (D^2+D+1)^2 = 0.$

$D^2+1=0$ or $D^2+D+1=0$

$D^2=-1$ $\qquad$ $D = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$D = \pm i$

$D = \pm i, \pm i, \pm i, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\overset{0}{i}$

$a=-1/2 \quad b=\frac{\sqrt{3}}{2}$

$a=0$

$b=1$

$$y_c = e^{0x}(c_1\cos x + c_2\sin x)$$

$-\frac{a}{2} + i\frac{\sqrt{3}}{2}$

$$+ x(c_3\cos x + c_4\sin x)$$

$$+ x^2(c_5\cos x + c_6\sin x)$$

$$+ e^{-1/2 x}\left(c_7\cos\frac{\sqrt{3}}{2}x + c_8\sin\frac{\sqrt{3}}{2}x\right)$$

$$+ x e^{-1/2 x}\left(c_9\cos\frac{\sqrt{3}}{2}x + c_{10}\sin\frac{\sqrt{3}}{2}x\right)$$

$$(D-1)^4 (D^2+2D+2)^2 y = 0$$

$$(D-1)^4 \cdot (D^2+2D+2)^2 = 0$$

$$(D-1)^4 = 0 \quad \text{or} \quad (D^2+2D+2)^2 = 0$$

$$D-1 = 0 \qquad\qquad D^2+2D+2 = 0$$

$$D = 1 \qquad\qquad D = -1 \pm i$$

$$\therefore D = 1, 1, 1, 1, -1 \pm i, -1 \pm i$$

$-1+i$

$a=-1,\ b=1$

$$y_c = c_1 e^{x} + c_2 x e^{x} + c_3 x^2 e^{x} + c_4 x^3 e^{x}$$

$$+ e^{-x}(c_5 \cos x + c_6 \sin x)$$

$$+ e^{-x} x (c_7 \cos x + c_8 \sin x)$$

★ particular Integral (y_p) | P.I.

$$D = \frac{d}{dx}$$

$$Dy \neq \cancel{yD} \qquad \cancel{D\,\frac{1}{D}}$$

$$f(D)y = X$$

$$y_p = \frac{1}{f(D)} X$$

Note $D = \frac{d}{dx}$ $\qquad Dy = \frac{d}{dx} y$

$$\frac{1}{D} X = \int X \, dx$$

① If $X = 0$ $\qquad y_p = \frac{1}{f(D)} X = 0$

② If $X = e^{ax}$

$$y_p = \frac{1}{f(D)} X = \frac{1}{f(D)} e^{ax}$$

$$= \begin{cases} \frac{1}{f(a)} e^{ax} & \text{— } f(a) \neq 0 \\ x \frac{1}{f'(a)} e^{ax} & \text{— } f(a) = 0, \; f'(a) \neq 0. \end{cases}$$

eg. $f(D) = D^2 - 4$

$$y_p = \frac{1}{D^2-4} e^{3x} = \frac{1}{9-4} e^{3x} = \frac{1}{5} e^{3x}$$

$$y_p = \frac{1}{D^2-4} e^{2x} = x \frac{1}{2D} e^{2x} = \frac{x}{4} e^{2x}$$

1. $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-3x/2} + 2^x + 3.$

$\frac{d}{dx} = D$

$6D^2y + 17Dy + 12y = e^{-3x/2} + e^{\log 2^x} + 3e^{0x}$

$(6D^2 + 17D + 12)y = e^{-3x/2} + e^{x(\log 2)} + 3e^{0x}$

$\Rightarrow f(D)y = X$

Consider, $(6D^2+17D+12)y = 0$

A eqn $6D^2 + 17D + 12 = 0$

$D = -4/3, -3/2$

$y_c = c_1 e^{-4/3x} + c_2 e^{-3/2x}$

$y_p = \frac{1}{f(D)} X = \frac{1}{6D^2+17D+12}\left(e^{-3x/2} + e^{(\log 2)x} + 3e^{0x}\right)$

$y_p = \frac{1}{6D^2+17D+12} e^{-3x/2} + \frac{1}{6D^2+17D+12} e^{(\log 2)x} + \frac{1}{6D^2+17D+12} 3e^{0x}$

as $f(-3/2) = 0$

$= x\frac{1}{12D+17} e^{-3x/2} + \frac{1}{6(\log 2)^2 + 17(\log 2) + 12} e^{(\log 2)x} + 3\frac{1}{12} e^{0x}$

$= x\frac{1}{\cancel{12}(-3/2)+17} e^{-3x/2} + \frac{e^{\log 2^x}}{6(\log 2)^2 + 17\log 2 + 12} + \frac{1}{4}$

$= \frac{x}{(-1)} e^{-3x/2} + \frac{2^x}{6(\log 2)^2 + 17\log 2 + 12} + \frac{1}{4}$

$y = y_c + y_p$

$= c_1 e^{-4/3x} + c_2 e^{-3/2x} - x e^{-3x/2} + \frac{2^x}{6(\log 2)^2 + 17\log 2 + 12} + \frac{1}{4}$

(D)

$$\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cosh^2 2x$$

$$D^3y - 4Dy = 2\left(\frac{e^{2x}+e^{-2x}}{2}\right)^2$$

$$(D^3-4D)y = 2\left(\frac{e^{4x}+2e^{2x}.e^{-2x}+e^{-4x}}{4}\right)$$

$$(D^3-4D)y = \frac{1}{2}\left[e^{4x}+2+e^{-4x}\right]$$

$$f(D)y = X$$

Consider $(D^3-4D)y=0$

A. eqn $D^3-4D=0$

$$D(D^2-4)=0$$

$$D=0 \quad D=\pm 2$$

$$y_c = c_1e^{0x} + c_2e^{2x} + c_3e^{-2x}$$

$$y_p = \frac{1}{f(D)}X$$

$$= \frac{1}{D^3-4D}\frac{1}{2}(e^{4x}+2+e^{-4x})$$

$$= \frac{1}{2}\left[\frac{1}{D^3-4D}e^{4x} + \frac{1}{D^3-4D}2e^{0x} + \frac{1}{D^3-4D}e^{-4x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4^3-4^2}e^{4x} + 2.x\frac{1}{3D^2-4}e^{0x} + \frac{1}{(-4)^3+4^2}e^{-4x}\right]$$

$$= \frac{1}{2}\left[\frac{e^{4x}}{48} + 2x\frac{1}{(-4)}e^{0x} + \frac{1}{(-48)}e^{-4x}\right]$$

$$= \frac{1}{2}\left[\frac{e^{4x}}{48} - \frac{x}{2} - \frac{e^{-4x}}{48}\right]$$

$$y = y_c + y_p$$

$$= c_1 + c_2e^{2x} + c_3e^{-2x} + \frac{1}{2}\left[\frac{e^{4x}}{48} - \frac{x}{2} - \frac{e^{-4x}}{48}\right]$$