2nd Edition

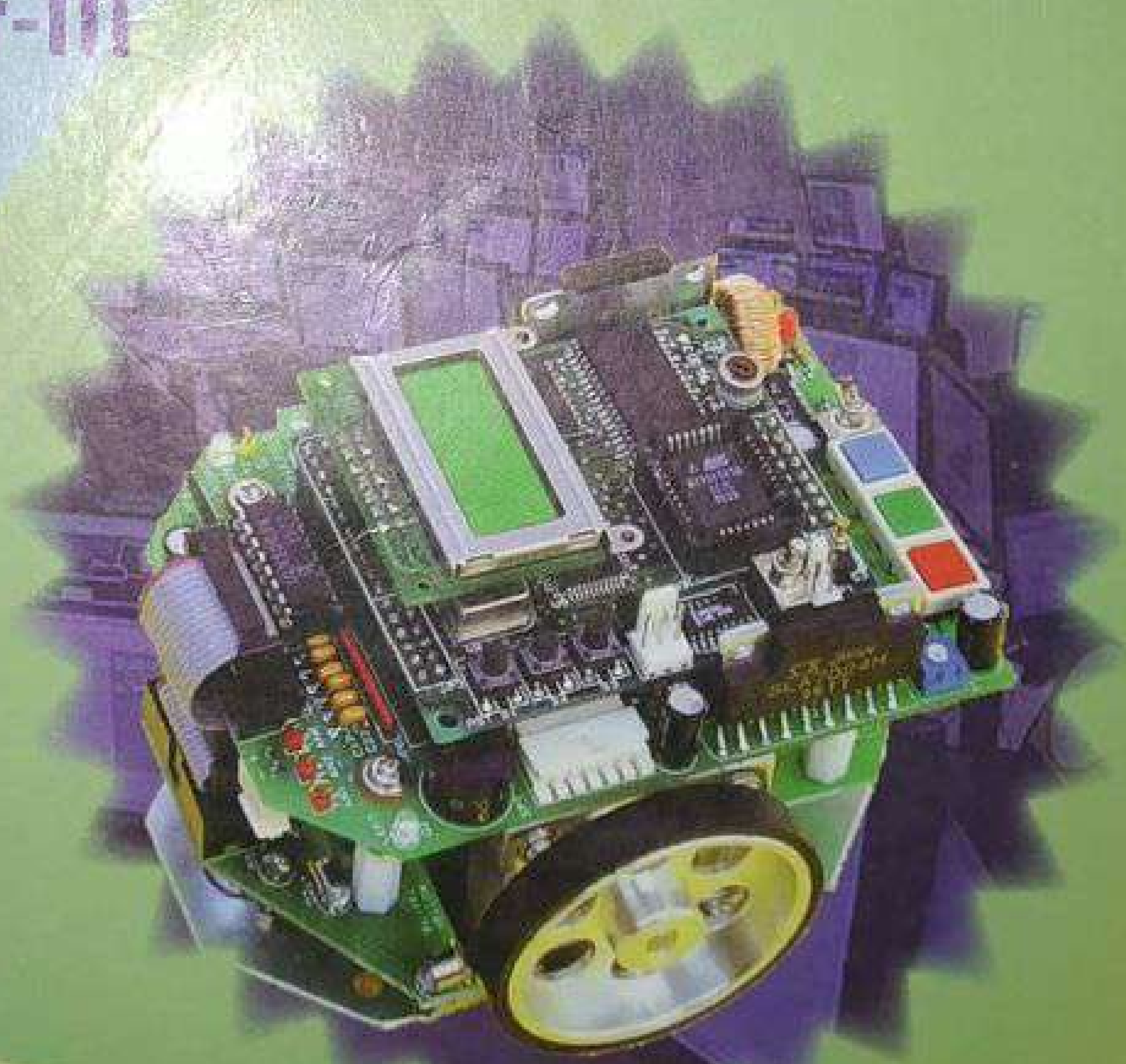
As per New Syllabus 2017-18

G. V. Kumbhojkar

Applied Mathematics-III

Computer Engineering

S. E. Semester-III



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A Text Book of "**Applied Mathematics - III**" for S. E. Semester - III (Computer Engineering) of "**University of Mumbai**" written according to the Latest and Revised Syllabus effective from June 2017

APPLIED MATHEMATICS - III

(Second Edition)

Computer Engineering

By

G. V. Kumbhojkar
M.Sc.

P. JAMNADAS LLP.

Educational Publishers
Shoppe Link (Dosti Acres), 2nd Floor, Office No. 19,
Antop Hill, Wadala (East), **MUMBAI** - 400 037
Phone : 2417 1118 / 2417 1119
E-mail : pjamnadasllp@gmail.com

Preface to the Second Edition

I am pleased to place this second edition of **Applied Mathematics - III** in the hands of Second Year (Computer Engineering) Students of University of Mumbai in particular and other Indian Universities in general. The book is written strictly according to the syllabus of ***Applied Mathematics - III (Computer Engineering) of Mumbai University*** in force from 2017-18. The book is based on my earlier books Applied Mathematics III and IV and is modified according to the new syllabus. It contains a number of additional examples–solved and unsolved–taken mostly from recent examination papers. I have taken this opportunity to rewrite some old articles in some chapters so as to make the discussion complete and easier to understand. Also I have taken care to remove the printing mistakes that were brought to my notice by my colleagues.

The book is an outcome of over thirty-five years of teaching of Mathematics to students of Engineering and Science, of writing over thirty books on Mathematics and statistics for degree and diploma courses, of careful study of the syllabus of University of Mumbai, of clear understanding of the nature of question papers of more than twenty years of Examinations of **all branches** of Engineering of University of Mumbai. I have also made extensive use of question papers of the past twenty years of Pune, Shivaji and other Indian Universities.

Mathematics, no doubt, is a tool in the hands of an engineer. But to make it effective, it is imperative for him to know clearly how the tool works. This, in turn, requires clear understanding of the concepts and methods of higher mathematics. A teacher of Engineering Mathematics has to strike a golden mean between rigorous mathematical proofs of the theorems, which often tend to be tedious, and mere applications of these theorems to engineering problems which tend to be difficult to grasp. With this difficulty in mind, an attempt has been made in this book to explain the theory through illustrations, diagrams and examples whenever it was found sufficient for understanding the concepts involved. Some concepts and theorems are discussed through examples, and rigorous mathematical proofs are given where it was found unavoidable. In short, every care has been taken to see that unnecessary material is not presented and necessary material is not left out, so that the book becomes student-friendly.

To make the matter easy for understanding, main topics have been divided into sub-topics, and illustrative examples with complete solutions are given after theoretical discussion of each topic. Whenever possible, alternative methods of solving problems are given. These examples are mostly taken from the university question papers and some are newly constructed according to the need. I have also given sufficient number of examples for practice after each sub-topic. This will help students to know the nature of problems they are expected to face and the methods of solving such problems. In addition to this, I have given a large number of miscellaneous examples with complete solutions at the end of each chapter, and an equally large number of examples in the exercise for practice with sufficient hints. These examples are properly graded and carefully classified so that a student can have sufficient drill-work requiring a particular technique. In short, the book is so designed that it becomes complete in all respects and meets the demands of the students of both average and above-average calibre. It also fulfils the expectations of the university syllabus in letter and spirit.

I am extremely and sincerely thankful to Prof. A. N. Nakra; M.Sc. P. G. (BARC); M.Sc. (Nuclear Engg.) Canada, (Formerly Senior Scientist, BARC), Prof. Nelson Periera, M.Sc. (Pure Mathematics) and Prof. A. S. Desai for their continuous assistance, valuable guidance and constant encouragement in writing the whole series of books for Mumbai University. I also take this opportunity to express my sincere thanks to Prof. (Dr.) A. V. Dubewar, Prof. A. B. Pawar, Prof. Mrs. S. Hegade, Prof. M. A. Rafik, Prof. Mrs. Seema Latkar, Prof. A. V. Deshmukh, Prof. Uday Kashid, Prof. Rachana Desai and Prof. N. R. Dasre for their help and appreciation of my books. My thanks are also due to Shri Y. B. Thorat for his prompt help.

I take this opportunity to express my sincere thanks and sense of gratitude to Late Parimal J. Shah (M.A.) and Shri Rushabh P. Shah (B. Com.) who wholeheartedly accepted the responsibility of publishing the entire series. I also thank the printer for printing nicely and in time the book and the whole series. I am thankful to Shri. Rajan Bhate and other members of the staff of P. Jamnadas LLP. for sparing no efforts to see that the books reach the students and teachers in time and for their cooperation and assistance in production and distribution of the series. My thanks are also due to Shri Anil M. Vhatkar of Mahalaxmi D.T.P. Center for neat, careful and swift type-setting and to Shri. Kaushal Kulkarni of Creative Concepts for beautiful cover design.

I hope that the students and teachers of Applied Mathematics of Mumbai and other Universities will appreciate my efforts and will receive this edition enthusiastically. Any suggestions for enhancing the utility of the book and for removing errors that might have gone unnoticed will be gratefully acknowledged.

1st June, 2018

Gajanan Vishnu Kumbhojkar

'Sukrut'
Residency Colony, Sagarmal,
Kolhapur - 416 008
Phone : (0231) 2690988, 2692150
Mobile : 9326052278
Email : gv_kumbhojkar@yahoo.co.in

Syllabus
Applied Mathematics - III
S. E. Semester - III
(University of Mumbai - Effective From June 2017)

Computer Engineering

Module 1 : Laplace Transform **(09)**

1.1 Laplace Transform of Standard Functions

Introduction, Definition of Laplace transform, Laplace transform of 1, e^{at}, sin at, cos at, sin h at, cos h at, t^n erf t, Heaviside unit step, Dirac-delta function, Laplace transform of periodic function.

1.2 Properties of Laplace Transform

Linearity, First shifting theorem, Second shifting property, Multiplication by t^n, division by t, Laplace transform of derivatives and integrals, Change of scale property (without proof).

Module 2 : Inverse Laplace Transform **(08)**

2.1 Inverse Laplace transform by Partial fraction method, Convolution theorem.

2.2 Application to solve initial and boundary value problem involving ordinary differential equations with one dependent variable and constant coefficients.

Module 3 : Fourier Series **(10)**

3.1 Dirichlet's conditions, Fourier series of periodic functions with period 2π and $2L$, Fourier series for even and odd functions.

3.2 Half range sine and cosine Fourier series, Parseval's identities (without proof).

3.3 Complex form of Fourier series, Orthogonal and Orthonormal set of functions.

Module 4 : Complex Variable and Mapping **(09)**

4.1 Functions of a complex variable, Analytic functions, Cauchy-Riemann equations in Cartesian co-ordinates, Polar co-ordinates.

4.2 Harmonic functions, Analytic method and Milne-Thomson method to find $f(z)$, Orthogonal trajectories.

4.3 **Mapping :** Conformal mapping, Bilinear transformations, Cross ratio, fixed points, bilinear transformation of straight lines and circles.

Module 5 : Z-transform **(06)**

5.1 Z-transform of standard functions such as $Z(a^n)$, $Z(n^p)$.

5.2 **Properties of Z-transform :** Linearity, Change of scale, Shifting property, Multiplication of K, Initial and final value, Convolution theorem (without proof).

5.3 **Inverse Z-transform :** Binomial expansion and Method of partial fraction.

Module 6 : Correlation and Regression, Curve Fitting (10)

6.1 Scattered diagrams, Karl Pearson's coefficient of correlation, Covariance, Spearman's Rank correlation (non-repeated and repeated ranks)

6.2 Regression coefficients and Lines of regression.

6.3 Fitting of curves : Least square method, Fitting of the straight line $y = a + bx$, Parabolic curve $y = a + bx + cx^2$ and Exponential curve $y = ab^x$.

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✦ Salient Features of this Book ✦

1. Covers entire syllabus of Applied Mathematics - III of Mumbai University. (Computer Engineering)
2. Large Number of solved examples.
3. Sufficient examples for self-study.
4. Solved examples as well as examples in exercises are classified according to their types.
5. Printing mistakes and also errors of commission and omission are carefully removed. Hopefully there is no mistake in the book.
6. **Contains solutions of the examples from Mumbai University Examinations (Computer Engineering) upto May 2017.**
7. **Contains solutions of the examples from Mumbai University Examinations of November 2017 and May 2018 at the end of the book.**

CONTENTS

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CHAPTER

1 Laplace Transforms - I

1. Introduction

By using a particular type of definite integral as an operator a new function can be defined. One such operator is called **Laplace Transform**. Laplace transform changes a function of one variable denoted by t into a function of another variable denoted by s.

Pierre-Simon Laplace (1749 - 1827)

Laplace was a well-known French mathematician and astronomer whose work contributed greatly to the development of mathematical astronomy and statistics. He summarised the work of his predecessors in five volumes titled 'Me'canique Ce'leste' (Celestial Mechanics). He formulated Laplace's equation, pioneered and developed Laplace transforms, used in many branches of science. The Laplacian differential operator is named after him. He was one of the first scientists who postulated the existence of black holes. He became a count of the First French Empire in 1806. He is remembered as one of the greatest scientist of all time and is called **Newton Of France.**

2. Definition

If $f(t)$ is a function of t satisfying certain conditions, then the definite integral

$$\boxed{\Phi(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt} \quad \text{......................} (1)$$

when it exists, is called the **Laplace Transform** of $f(t)$ and is written as $L[f(t)]$. Thus,

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt} \quad \text{......................} (2)$$

Laplace transform exists if the integral on the right hand side is convergent.

There is one to one correspondence between $f(t)$ which **is a function of t** and $\Phi(s)$ **which is a function of s.**

Note ....

Some authors use $F(s)$ in place of $\Phi(s)$ and p in place of s.

Sufficient Conditions For Existence of Laplace Transforms

We first learn two concepts *viz.* (i) piecewise continuity and (ii) exponential order.

(a) Piecewise continuity : A function $f(t)$ is said to be piecewise continuous in an interval if the interval can be divided into a finite number of subintervals such that in each of these subintervals (i) $f(t)$ is continuous and (ii) the limits of $f(t)$ as t tends to the end points are finite.

In the adjoining figure we have shown a piecewise continuous function.

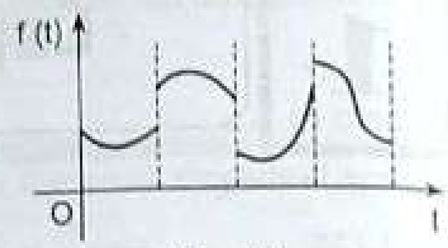

Fig. 1.1

(b) Exponential order : A function $f(t)$ is said to be of exponential order for $t > T$ if we can find constant M and a such that $|f(t)| \le M e^{at}$ for $t > T$.

For example, $f(t) = t^2$ is a function of exponential order because $|f(t)| = |t^2| < e^{3t}$ for all positive t. Similarly t, e^{-2t}, $\cos t$ are all of exponential order because $|t| < e^t$, $|e^{-2t}| < e^t$, $|\cos t| < e^t$.

Theorem : If $f(t)$ is piecewise continuous in every finite interval $[0, T]$ and is of exponential order for $t > T$ then $L[f(t)]$ exists for $s > a$.

It should be noted carefully that the above conditions are only sufficient (and not necessary). If the conditions are not satisfied $L[f(t)]$ may still exist. For example, $L(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$ [See Example, on page 1-6] exists eventhough $t^{-1/2}$ is not continuous in $0 \le t \le T$. [$\because\ t^{-1/2} = \frac{1}{\sqrt{t}} \to \infty$ as $t \to 0$]

Example 1 : Find the Laplace transform of $f(t)$, where

$f(t) = t$, for $0 < t < 4$ and $f(t) = 5$, for $t > 4$. **(M.U. 2006)**

Sol. :
$$L[f(t)] = \int_0^\infty e^{-st}\cdot f(t)\,dt = \int_0^4 e^{-st}\cdot f(t)\,dt + \int_4^\infty e^{-st}\cdot f(t)\,dt$$
$$= \int_0^4 e^{-st}\cdot t\,dt + \int_4^\infty e^{-st}\cdot 5\,dt$$
$$= \left[(t)\cdot\left(\frac{e^{-st}}{-s}\right) - (1)\cdot\left(\frac{e^{-st}}{s^2}\right)\right]_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$
$$= -\frac{4}{s}\cdot e^{-4s} - \frac{1}{s^2}\cdot e^{-4s} + \frac{1}{s^2} + \frac{5}{s}\cdot e^{-4s}$$
$$= \frac{1}{s^2} + \left(\frac{1}{s} - \frac{1}{s^2}\right)\cdot e^{-4s}.$$

Example 2 : Find the Laplace transform of $f(t)$, where

(1) $f(t) = a$, $0 < t < b$ and $f(t) = 0$, $t > b$.

(2) $f(t) = \cos(t - \alpha)$, $t > \alpha$ and $f(t) = 0$, $t < \alpha$.

Sol. : (1)
$$L[f(t)] = \int_0^\infty e^{-st}\cdot f(t)\,dt = \int_0^b e^{-st}\cdot f(t)\,dt + \int_b^\infty e^{-st}\cdot f(t)\,dt$$
$$= \int_0^b e^{-st}\cdot a\,dt + \int_b^\infty e^{-st}\cdot 0\,dt = a\int_0^b e^{-st}dt = a\left[\frac{-e^{-st}}{s}\right]_0^b = \frac{a}{s}(1 - e^{-bt})$$

(2)
$$L[f(t)] = \int_0^\infty e^{-st}\cdot f(t)\,dt = \int_0^\alpha 0\cdot dt + \int_\alpha^\infty e^{-st}\cdot\cos(t-\alpha)\,dt$$

But
$$\int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}\,e^{ax}(a\cos bx + b\sin bx) \quad \text{.............. (A)}$$
$$\therefore\ L[f(t)] = \left[\frac{1}{s^2+1}\cdot e^{-st}\{-s\cos(t-\alpha) + \sin(t-\alpha)\}\right]_\alpha^\infty$$
$$= \frac{se^{-\alpha s}}{s^2+1} \quad \text{[See also Ex. (i), page 1-27]}$$

Example 3 : Find the Laplace transform of $f(t) = \cos t$, for $0 < t < \pi$ and $f(t) = \sin t$, for $t > \pi$. **(M.U. 1993, 2002, 05, 09)**

Sol. :
$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st}\cos t\,dt + \int_\pi^\infty e^{-st}\sin t\,dt.$$

But
$$\int e^{ax}\cos bx\,dx = \frac{1}{(a^2+b^2)}\cdot e^{ax}(a\cos bx + b\sin bx) \text{ and}$$
$$\int e^{ax}\sin bx\,dx = \frac{1}{(a^2+b^2)}\,e^{ax}(a\sin bx - b\cos bx) \quad \text{.............. (A)}$$
$$\therefore\ L f(t) = \frac{1}{s^2+1}\cdot\left[e^{-st}(-s\cos t + \sin t)\right]_0^\pi + \frac{1}{s^2+1}\cdot\left[e^{-st}(-s\sin t - \cos t)\right]_\pi^\infty$$
$$= \frac{1}{s^2+1}\cdot\left[e^{-s\pi}(s) - (-s)\right] + \frac{1}{s^2+1}\cdot\left[-e^{-s\pi}\right]$$
$$= \frac{1}{s^2+1}\cdot\left[s + (s-1)\,e^{-s\pi}\right]$$

Example 4 : If $L[f(t)] = \Phi(s)$ and if

$$g(t) = \begin{cases} 0, & 0 < t < a \\ f(t-a), & t > a \end{cases}, \text{ then } L[g(t)] = e^{-as}\,\Phi(s). \quad \textbf{(M.U. 2007)}$$

Sol. :
$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt = \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$
$$= \int_0^a 0\cdot dt + \int_a^\infty e^{-st} f(t-a)\,dt = \int_a^\infty e^{-st} f(t-a)\,dt$$

Now, put $t - a = u$ *i.e.* $t = a + u$, $dt = du$
$$\therefore\ L[g(t)] = \int_0^\infty e^{-s(a+u)} f(u)\,du = e^{-as}\int_0^\infty e^{-su} f(u)\,du$$
$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt = e^{-as}\,L[f(t)] = e^{-as}\,\Phi(s)$$

[$\because$ The definite integral does not depend upon the letter used.]

EXERCISE - I

Find the Laplace transform of $f(t)$, where,

1. $f(t) = 3$, $0 < t < 5$; $\quad f(t) = 0$, $t > 5$.
2. $f(t) = (t-1)^2$, $t > 1$; $\quad f(t) = 0$, $0 < t < 1$.

3. $f(t) = (t-1)^3,\ t > 1$; $f(t) = 0,\ 0 < t < 1.$

4. $f(t) = t,\ 0 < t < 1/2$; $f(t) = t - 1,\ (1/2) < t < 1$; $f(t) = 0,\ t > 1.$ (M.U. 2003)

5. $f(t) = \sin 2t,\ 0 < t < \pi$; $f(t) = 0,\ t > \pi.$ (M.U. 2007)

6. $f(t) = 0,\ 0 \le t \le 1$; $f(t) = t,\ 1 < t < 2,\quad f(t) = 0,\ t > 2.$

7. $f(t) = t,\ 0 < t < 3$; $f(t) = 6,\ t > 3.$ (M.U. 2006)

8. $f(t) = \cos t,\ 0 < t < 2\pi$; $f(t) = 0,\ t > 2\pi.$ (M.U. 2002, 04)

9. $f(t) = t^2,\ 0 < t < 1$; $f(t) = 1,\ t > 1.$ (M.U. 2003)

10. $f(t) = (t-a)^3,\ t > a$; $f(t) = 0,\ t < a.$

11. $f(t) = t,\ 0 < t < a$; $f(t) = b,\ t > a.$

12. $f(t) = 0,\ 0 < t < \pi$; $f(t) = \sin^2(t-\pi),\ t > \pi.$ (M.U. 2003)

[**Ans. :** (1) $\frac{3}{s}(1-e^{-5s})$, (2) $\frac{2e^{-s}}{s^3}$, (3) $\frac{6}{s^4}e^{-s}$, (4) $\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s/2}}{s}$,

(5) $\frac{2(1-e^{-\pi s})}{s^2+4}$, (6) $\left(\frac{1}{s^2}+\frac{1}{s}\right)\cdot e^{-s} - \left(\frac{1}{s^2}+\frac{2}{s}\right)\cdot e^{-2s}$,

(7) $\frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-3s}$, (8) $(1-e^{-2\pi s})\frac{s}{s^2+1}$,

(9) $\frac{1}{s}(1-e^{-s}) - 2\frac{e^{-s}}{s^2} + \frac{2}{s^3}(1-e^{-s})$, (10) $\frac{6}{s^4}e^{-as}$,

(11) $\frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right]e^{-as}$, (12) $\frac{e^{-\pi s}}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$]

(**Hint :** For integration, use generalised rule of integration by parts.)

3. Linearity Property

If k_1 and k_2 are constants then,

$$\boxed{L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)]} \qquad \text{.......... (3)}$$

The result can be easily proved by using the above definition.

4. Laplace Transforms of Standard Functions

By using the definition, we find below Laplace Transforms of some standard functions.

$$\boxed{L(k) = \frac{k}{s}} \qquad \text{.......... (3A)}$$

Proof : We have $L(k) = \int_0^\infty e^{-st}\, k\, dt = k\int_0^\infty e^{-st}\, dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{-s}[0-1] = \frac{k}{s}$$

(1) $$\boxed{L(e^{at}) = \frac{1}{s-a}\ (s > a)} \qquad \text{.......... (4)}$$

Proof : We have $L(e^{at}) = \int_0^\infty e^{-st}\cdot e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\cdot dt$

$$= \frac{1}{-(s-a)}\left[e^{-(s-a)t}\right]_0^\infty = \frac{1}{s-a} \qquad \text{.......... (A)}$$

Cor. 1 : If $a = 0$, $(s > 0)$ $\boxed{L(1) = \frac{1}{s}}$ .... (5)

Cor. 2 : Changing the sign of a, $\boxed{L(e^{-at}) = \frac{1}{s+a}}$ .... (6)

Cor. 3 : Since $c^{at} = e^{at\log c}$, $(c > 0,\ s > a\log c)$

$$\boxed{L(c^{at}) = \frac{1}{s - a\log c}}$$

$$\boxed{L(2^{3t}) = \frac{1}{s - 3\log 2}} \qquad \text{(M.U. 2008)}$$

Note ....

Note that the integral in (A) is defined only if $(s-a) > 0$ i.e. $s > a$.

(2) $\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$ and $\boxed{L(\cos at) = \frac{s}{s^2+a^2}\ (s > 0)}$ .... (7)

Proof : Consider, $L(\cos at + i\sin at) = L(e^{iat}) = \frac{1}{s-ai}$ by (i) $= \frac{s+ai}{s^2+a^2}$

Equating real and imaginary parts we get (4).

Note ....

The above transforms (4) can also be obtained directly by using the definition.

(3) $\boxed{L(\sin h\, at) = \frac{a}{s^2-a^2}}$ and $\boxed{L(\cos h\, at) = \frac{s}{s^2-a^2}\ (s > |a|)}$ .... (8)

Proof : We have, $L(\sin h\, t) = L\left(\frac{e^{at}-e^{-at}}{2}\right)$

$$\therefore\quad L(\sin h\, t) = \frac{1}{2}\left\{L(e^{at}) - L(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2-a^2} \qquad \text{[By (4)]}$$

Similarly, we can prove $L(\cos h\, t) = \frac{s}{s^2-a^2}$

(4) $\boxed{L(t^n) = \frac{\overline{|(n+1)}}{s^{n+1}}}$ [$(n+1) > 0$ and $s > 0$] .... (9)

Proof : We have, $L(t^n) = \int_0^\infty e^{-st}\cdot t^n\, dt$. Put $z = st$, $dz = s\, dt$.

$$= \int_0^\infty e^{-z}\cdot\frac{z^n}{s^n}\cdot\frac{dz}{s} = \frac{\overline{|n+1}}{s^{n+1}}$$

Example : Find $L\left[\frac{1}{\sqrt{\pi t}}\right]$.

Sol. : We have $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{\pi}} L \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{\pi}} L(t^{-1/2})$

$$\therefore \quad L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma((-1/2)+1)}{s^{(-1/2)+1}} = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(1/2)}{s^{1/2}}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{1}{\sqrt{s}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Cor. 4 : If n is a positive integer, $\Gamma(n+1) = n!$

$$\boxed{L(t^n) = \frac{n!}{s^{n+1}} \text{ if } n \text{ is a +ve integer.}} \qquad \text{.................... (10)}$$

e.g. $L(1) = \frac{1}{s}$, $L(t) = \frac{1}{s^2}$, $L(t^2) = \frac{2}{s^3}$, $L(t^3) = \frac{6}{s^4}$

(5) $\boxed{L(\text{erf}\,\sqrt{t}) = \frac{1}{s\sqrt{s+1}}}$ **(M.U. 1996, 97, 2003, 05, 09)** .... (11)

Proof : We know that $\text{erf}\, t = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx$

$$\therefore \quad \text{erf}\,\sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\, dx = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right] dx$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \ldots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5(2!)} - \frac{t^{7/2}}{7(3!)} + \ldots\right]$$

Taking Laplace transform of both sides and using $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

$$L(\text{erf}\,\sqrt{t}) = \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3} \frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5(2!)} \frac{\Gamma(7/2)}{s^{7/2}} + \ldots\right]$$

But $\Gamma(3/2) = (1/2)\Gamma(1/2) = (1/2)\sqrt{\pi}$

$\Gamma(5/2) = (3/2)(\Gamma(3/2)) = (3/2)(1/2)\sqrt{\pi}$ etc.

$$\therefore \quad L(\text{erf}\,\sqrt{t}) = \frac{1}{s^{3/2}} \left[1 - \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{s^2} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{s^3} + \ldots\right]$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s^{3/2}} \frac{\sqrt{s}}{\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}$$

(We shall obtain this result again in Ex. 3 on page 1-56.)

(6) $\boxed{L[\text{erf}_c\,\sqrt{t}\,] = \frac{1}{\sqrt{s+1}\,[\sqrt{s+1}+1]}}$ .... (12)

Proof : We know that $\quad \text{erf}\, x + \text{erf}_c\, x = 1$

$\therefore \quad \text{erf}_c\, x = 1 - \text{erf}\, x \qquad \therefore \quad \text{erf}_c\,\sqrt{t} = 1 - \text{erf}\,\sqrt{t}$

Taking Laplace transforms of both sides,

$$L[\text{erf}_c\,\sqrt{t}\,] = L(1) - L[\text{erf}\,\sqrt{t}\,] = \frac{1}{s} - \frac{1}{s\sqrt{s+1}} = \frac{\sqrt{s+1}-1}{s\sqrt{s+1}}$$

$$= \frac{\sqrt{s+1}-1}{s\sqrt{s+1}} \cdot \frac{\sqrt{s+1}+1}{\sqrt{s+1}+1} \qquad \text{[Rationalising the numerator]}$$

$$= \frac{s+1-1}{s\sqrt{s+1}\,[\sqrt{s+1}+1]} = \frac{1}{\sqrt{s+1}\,[\sqrt{s+1}+1]}$$

Note ....

For existence of the above Laplace transforms the conditions given in the brackets are necessary.

List of Transforms

(1) $L(e^{at}) = \frac{1}{s-a}$	(2) $L(e^{-at}) = \frac{1}{s+a}$	(3) $L(\sin at) = \frac{a}{s^2+a^2}$
(4) $L(\cos at) = \frac{s}{s^2+a^2}$	(5) $L(\sin h\, at) = \frac{a}{s^2-a^2}$	(6) $L(\cos h\, at) = \frac{s}{s^2-a^2}$
(7) $L(t^n) = \frac{\Gamma(n+1)}{s^n}$	(8) $L(t^n) = \frac{n!}{s^n}$ if n is a +ve integer	
(9) $L(1) = \frac{1}{s}$	(10) $L(\text{erf}\,\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$	(11) $L[\text{erf}_c\,\sqrt{t}\,] = \frac{1}{\sqrt{s+1}\,[\sqrt{s+1}+1]}$

EXERCISE - II

Write down the Laplace transforms of the following.

1. t^2, 2. $t^{3/2}$, 3. t^4, 4. $t^{1/2}$, 5. 1, 6. e^{2t}, 7. e^{-4t}, 8. $\sin 2t$, 9. $\cos 3t$, 10. $\sin 5t$, 11. $\cos t$, 12. 10^{2t}, 13. $\sin h\, 3t$, 14. $\cos h\, 2t$, 15. 5^{3t}, 16. $1/\sqrt{\pi t}$.

[**Ans. :** Answers not given for obvious reason.]

Example 1 : Find the Laplace transforms of

(i) $4t^2 + \sin 3t + e^{2t}$ (ii) $(\sin 2t - \cos 2t)^2$ (iii) $\cos h^2 4t$ **(M.U. 2004)**

Sol. : (i) $L[4t^2 + \sin 3t + e^{2t}] = 4L(t^2) + L(\sin 3t) + L(e^{2t})$

$$= 4 \cdot \frac{2}{s^3} + \frac{3}{s^2+3^2} + \frac{1}{s-2}$$

(ii) $L[(\sin 2t - \cos 2t)^2] = L(1 - 2\sin 2t \cos 2t)$

$$= L(1 - \sin 4t) = L(1) - L(\sin 4t) = \frac{1}{s} - \frac{4}{s^2 + 4^2}$$

(iii) $L[\cosh^2 4t] = L[\frac{1}{2}(1 + \cosh 8t)] = \frac{1}{2}[L(1) + L(\cosh 8t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 - 8^2}\right]$

Example 2 : Find the Laplace transforms of

(i) $\sin(\omega t + \alpha)$. (ii) $t^2 - e^{-2t} + \cosh^2 3t$.

Sol. : (i) $L[\sin(\omega t + \alpha)] = L[\sin \omega t \cos \alpha + \cos \omega t \sin \alpha]$

$$= \cos\alpha \, L(\sin \omega t) + \sin \alpha \, L(\cos \omega t)$$

$$= \cos\alpha \cdot \frac{\omega}{s^2 + \omega^2} + \sin\alpha \cdot \frac{s}{s^2 + \omega^2}.$$

(ii) $L[t^2 - e^{-2t} + \cosh^2 3t] = L[t^2] - L[e^{-2t}] + L\frac{1}{2}[1 + \cosh 6t]$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 - 6^2}\right]$$

Example 3 : Find the Laplace transform of the following.

(i) $\sin^5 t$ **(M.U. 2003, 05, 07, 10, 12)** (ii) $\sinh^5 t$

(iii) $\cos t \cos 2t \cos 3t$ **(M.U. 1995, 2003)** (iv) $(\cosh t - \sinh t)^n$

Sol. : (i) First we find the expression for $\sin^5 t$. (Refer to Applied Mathematics I by the same author.)

Let $x = \cos t + i \sin t \quad \therefore \frac{1}{x} = \cos t - i \sin t$

$\therefore (2i \sin t)^5 = \left(x - \frac{1}{x}\right)^5$ [See Ex. 3, § 4, page 2-25 of App. Maths - I]

$$= x^5 - 5x^4 \cdot \frac{1}{x} + 10x^3 \cdot \frac{1}{x^2} - 10x^2 \cdot \frac{1}{x^4} + 5x\frac{1}{x^4} - \frac{1}{x^5}$$

$$\therefore 32 i^5 \sin^5 t = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$= 2i \sin 5t - 5(2i \sin 3t) + 10(2i \sin t)$$

$$\therefore \sin^5 t = \frac{1}{16}(\sin 5t - 5\sin 3t + 10 \sin t)$$

$$\therefore L(\sin^5 t) = \frac{1}{16}[L(\sin 5t) - 5L\sin(3t) + 10L\sin t]$$

$$= \frac{1}{16}\left[\frac{5}{s^2+25} - 5\cdot\frac{3}{s^2+9} + 10\cdot\frac{1}{s^2+1}\right]$$

$$= \frac{5}{16}\left[\frac{1}{s^2+25} - \frac{3}{s^2+9} + \frac{2}{s^2+1}\right] = \frac{5}{16}\left[\frac{s^2+9-3s^2-75}{(s^2+25)(s^2+9)} + \frac{2}{s^2+1}\right]$$

$$= \frac{5}{16}\left[\frac{-2s^2-66}{(s^2+25)(s^2+9)} + \frac{2}{s^2+1}\right] = \frac{5}{8}\left[\frac{-s^2-33}{(s^2+25)(s^2+9)} + \frac{1}{s^2+1}\right] \quad (1)$$

$$= \frac{5}{8}\left[\frac{-s^4 - s^2 - 33s^2 - 33 + s^4 + 34s^2 + 225}{(s^2+1)(s^2+9)(s^2+25)}\right]$$

$$= \frac{5}{8}\cdot\frac{192}{(s^2+1)(s^2+9)(s^2+25)} = \frac{120}{(s^2+1)(s^2+9)(s^2+25)}$$

$$\therefore L(\sin^5 t) = \frac{5!}{(s^2+1)(s^2+9)(s^2+25)}$$

Aliter : $\sin^5 t = \left(\frac{e^{it} - e^{-it}}{2i}\right)^5$

$$= \frac{1}{32i}\left[e^{5it} - 5^{4it}\cdot e^{-it} + 10\cdot e^{3it}\cdot e^{-2it} - 10e^{2it}\cdot e^{-3it} + 5e^{it}\cdot e^{-4it} - e^{-5it}\right]$$

$$= \frac{1}{32i}\left[(e^{5it} - e^{-5it}) - 5(e^{3it} - e^{-3it}) + 10(e^{it} - e^{-it})\right]$$

$$= \frac{1}{16}\left[\left(\frac{e^{5it} - e^{-5it}}{2i}\right) - 5\left(\frac{e^{3it} - e^{-3it}}{2i}\right) + 10\left(\frac{e^{it} - e^{-it}}{2i}\right)\right]$$

$$\therefore \sin^5 t = \frac{1}{16}[\sin 5t - 5\sin 3t + 10 \sin t]$$

$$\therefore L[\sin^5 t] = \frac{1}{16}[L\sin 5t - 5L\sin 3t + 10L\sin t]$$

$$= \frac{1}{16}\left[\frac{5}{s^2+5^2} - 5\cdot\frac{3}{s^2+3^2} + 10\cdot\frac{1}{s^2+1}\right]$$

Same as (1).

(ii) We have, $\sinh^5 t = \left(\frac{e^t - e^{-t}}{2}\right)^5$

$$= \frac{1}{32}\left[e^{5t} - 5e^{4t}\cdot e^{-t} + 10e^{3t}\cdot e^{-2t} - 10e^{2t}\cdot e^{-3t} + 5e^t\cdot e^{-4t} - e^{-5t}\right]$$

$$\therefore \sinh^5 t = \frac{2}{32}\left[\left(\frac{e^{5t} - e^{-5t}}{2}\right) - 5\left(\frac{e^{3t} - e^{-3t}}{2}\right) + 10\left(\frac{e^t - e^{-t}}{2}\right)\right]$$

$$= \frac{1}{16}[\sinh 5t - 5\sinh 3t + 10 \sinh t]$$

$$\therefore L[\sinh^5 t] = \frac{1}{16}[L\sinh 5t - 5\sinh 3t + 10\sinh t]$$

$$= \frac{1}{16}\left[\frac{5}{s^2-25} - \frac{15}{s^2-9} + \frac{10}{s^2-1}\right] = \frac{5}{16}\left[\frac{1}{s^2-25} - \frac{3}{s^2-9} + \frac{2}{s^2-1}\right]$$

$$= \frac{5}{16}\left[\frac{s^2-9-3s^2+75}{(s^2-25)(s^2-9)} + \frac{2}{s^2-1}\right] = \frac{5}{16}\left[\frac{-2s^2+66}{(s^2-25)(s^2-9)} + \frac{2}{s^2-1}\right]$$

$$= \frac{5}{16}\left[\frac{-2s^4 + 66s^2 + 2s^2 - 66 + 2s^4 - 68s^2 + 450}{(s^2-1)(s^2-9)(s^2-25)}\right]$$

$$= \frac{120}{(s^2-1)(s^2-9)(s^2-25)} = \frac{5!}{(s^2-1)(s^2-9)(s^2-25)}$$

(iii) $L(\cos t \cos 2t \cos 3t) = L\left[\frac{1}{2}(\cos 3t + \cos t)\cos 3t\right]$

$$= \frac{1}{2}L\left[\cos^2 3t + \cos 3t \cos t\right] = \frac{1}{2}L\left[\frac{1}{2}(1+\cos 6t) + \frac{1}{2}\{\cos 4t + \cos 2t\}\right]$$

$$= \frac{1}{4}L[1 + \cos 2t + \cos 4t + \cos 6t]$$

$$= \frac{1}{4}[L(1) + L(\cos 2t) + L(\cos 4t) + L(\cos 6t)]$$

$$= \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+2^2} + \frac{s}{s^2+4^2} + \frac{s}{s^2+6^2}\right]$$

(iv) $(\cosh t - \sinh t)^n = \left(\frac{e^t + e^{-t}}{2} - \frac{e^t - e^{-t}}{2}\right)^n = (e^{-t})^n = e^{-nt}$

$$\therefore\ L[(\cosh t - \sinh t)^n] = L(e^{-nt}) = \frac{1}{s+n}$$

EXERCISE - III

Find the Laplace transforms of

1. $(t^2 + a)^2$, 2. $2t^3 + \cos 4t + e^{-2t}$, 3. $e^{2t} + 4t^3 - \sin 2t \cos 3t$
4. $\cos(wt + \beta)$, **(M.U. 1999)** 5. $\cos^2 2bt$, 6. $e^t + \sin 2t \cdot \sin 3t$,
7. $3t^2 + e^{-t} + \sin^3 2t$, 8. $\cos^3 2t$, 9. $\sin h^3 3t$,
10. $\cos h\, at - \cos bt$, 11. $\sin h\, at - \sin bt$, 12. $\cos h^3 2t$,
13. $\cos^5 t$, 14. $\cos h^5 t$, 15. $\cos^4 t$,
16. $\sin^4 t$ **(M.U. 2002, 03)** 17. $\cos h^4 t$ **(M.U. 2004)** 18. $(\cos h + \sin ht)^n$
19. $\frac{1+2t}{\sqrt{t}}$ 20. $\sqrt{1+\sin t}$ **(M.U. 2004, 08)** 21. $(\sqrt{t}-1)^2$ **(M.U. 2004)**
22. $\left(\sqrt{t} \pm \frac{1}{\sqrt{t}}\right)^3$ **(M.U. 2007)**

Ans. : (1) $\frac{a^2s^4 + 4as^2 + 24}{s^5}$, (2) $2\cdot\frac{3!}{s^4} + \frac{s}{s^2+4^2} + \frac{1}{s+2}$,

(3) $\frac{1}{s-2} + 4\cdot\frac{3!}{s^4} - \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]$, (4) $\cos\beta\cdot\frac{5}{s^2+w^2} - \sin\beta\cdot\frac{w}{s^2+w^2}$,

(5) $\frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16b^2}\right]$, (6) $\frac{1}{s-1} + \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right]$,

(7) $3\cdot\frac{2!}{s^3} + \frac{1}{s+1} + \frac{3}{4}\cdot\frac{2}{s^2+2^2} - \frac{1}{4}\cdot\frac{6}{s^2+6^2}$, (8) $\frac{s(s^2+28)}{(s^2+36)(s^2+4)}$,

(9) $\frac{162}{(s^2-81)(s^2-9)}$, (10) $\frac{s}{s^2-a^2} - \frac{s}{s^2+b^2}$, (11) $\frac{a}{s^2-a^2} - \frac{b}{s^2+b^2}$,

(12) $\frac{s(s^2-28)}{(s^2-36)(s^2-4)}$, (13) $\frac{s}{16}\left[\frac{1}{s^2+25} + \frac{5}{s^2+9} + \frac{10}{s^2+1}\right]$,

(14) $\frac{s}{16}\left[\frac{1}{s^2-25} + \frac{5}{s^2-9} + \frac{10}{s^2-1}\right]$, (15) $\frac{1}{8}\left[\frac{s}{s^2-16} + \frac{4s}{s^2-4} + \frac{6}{s}\right]$

(16) $\frac{1}{8}\left[\frac{3}{s} - \frac{4s}{s^2+4} + \frac{s}{s^2+16}\right]$, (17) $\frac{1}{8}\left[\frac{3}{s} + \frac{4s}{s^2-4} + \frac{s}{s^2-16}\right]$

(18) $\frac{1}{s-n}$, (19) $\sqrt{\frac{\pi}{s}}\left(1+\frac{1}{s}\right)$ (20) $\frac{s}{s^2+(1/2)^2} + \frac{1/2}{s^2+(1/2)^2}$

(21) $\frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{3/2}} + \frac{1}{s}$, (22) $\frac{\overline{|5/2}}{s^{5/2}} \pm \frac{3\overline{|3/2}}{s^{3/2}} + \frac{3\overline{|1/2}}{s^{1/2}} - \frac{\overline{|-1/2}}{s^{-1/2}}$.

Miscellaneous Examples

Example 1 : Find the Laplace transform of the following.

(i) $\sin\sqrt{t}$ **(M.U. 1996, 2013)** (ii) $\frac{\cos\sqrt{t}}{\sqrt{t}}$ **(M.U. 2004, 09)**

Sol. : (i) Since, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$; $\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \ldots$

$$\therefore\ L\left[\sin\sqrt{t}\right] = L(t^{1/2}) - \frac{1}{3!}L(t^{3/2}) + \frac{1}{5!}L(t^{5/2}) + \ldots$$

But $L(t^n) = \frac{\overline{|n+1}}{s^{n+1}}$ and $\overline{|n} = n\overline{|n-1}$, $\overline{|1/2} = \sqrt{\pi}$

$$\therefore\ L\left[\sin\sqrt{t}\right] = \frac{\overline{|3/2}}{s^{3/2}} - \frac{1}{3!}\cdot\frac{\overline{|5/2}}{s^{5/2}} + \frac{1}{5!}\cdot\frac{\overline{|7/2}}{s^{7/2}} - \ldots$$

$$= \frac{(1/2)\overline{|1/2}}{s^{3/2}} - \frac{1}{3!}\cdot\frac{(3/2)(1/2)\overline{|1/2}}{s^{5/2}} + \frac{1}{5!}\cdot\frac{(5/2)(3/2)(1/2)\overline{|1/2}}{s^{7/2}}$$

$$= \frac{\overline{|1/2}}{2s^{3/2}}\left[1 - \left(\frac{1}{2^2\cdot s}\right) + \frac{1}{2!}\left(\frac{1}{2^2\cdot s}\right)^2 - \ldots\right] \quad \left[\because \overline{|n} = n\overline{|n-1}\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\cdot e^{-1/(4s)} \quad \left[\because \overline{|1/2} = \sqrt{\pi}\right]$$

(ii) We know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$$\therefore\ \cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \ldots; \quad \frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \ldots$$

$$\therefore\ L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\overline{|1/2}}{s^{1/2}} - \frac{1}{2!}\cdot\frac{\overline{|3/2}}{s^{3/2}} + \frac{1}{4!}\cdot\frac{\overline{|5/2}}{s^{5/2}} - \frac{1}{6!}\cdot\frac{\overline{|7/2}}{s^{7/2}} + \ldots$$

$$\therefore \quad L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\overline{|1/2}}{s^{1/2}} - \frac{1}{2!}\cdot\frac{(1/2)\overline{|1/2}}{s^{3/2}} + \frac{1}{4!}\cdot\frac{(3/2)(1/2)\overline{|1/2}}{s^{5/2}} - \frac{1}{6!}\cdot\frac{(5/2)(3/2)(1/2)\overline{|1/2}}{s^{7/2}} + \ldots$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \frac{1}{3!(4s)^3} + \ldots\right] = \sqrt{\frac{\pi}{s}}\cdot e^{-1/(4s)}.$$

Example 2 : If $J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$, find $L[J_0(t)]$. **(M.U. 2010, 11)**

Sol. : By data $\quad J_0(t) = 1 - \left(\frac{t}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{t}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{t}{2}\right)^6 + \ldots$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \ldots$$

$$\therefore \quad L[J_0(t)] = L(1) - \frac{1}{2^2}L(t^2) + \frac{1}{2^2\cdot 4^2}\cdot L(t^4) - \frac{1}{2^2\cdot 4^2\cdot 6^2}\cdot L(t^6) - \ldots$$

$$= \frac{1}{s} - \frac{1}{2^2}\cdot\frac{2!}{s^3} + \frac{1}{2^2\cdot 4^2}\cdot\frac{4!}{s^5} - \frac{1}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{s^2}\right)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{s^2}\right)^3 + \ldots\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{1+s^2}}.$$

Example 3 : If $J_0(t) = \frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta$, prove that $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$.

Hence, evaluate $\frac{1}{\pi}\int_0^{\infty} e^{-t}\left[\int_0^{\pi}\cos t\cos\theta\ d\theta\right]dt$. **(M.U. 2003, 05)**

Sol. : We have $\quad J_0(t) = \frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta$

Taking Laplace transforms of both sides,

$$L[J_0(t)] = \frac{2}{\pi}\int_0^{\infty} e^{-st}\left[\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta\right]dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^{\infty} e^{-st}\cos(t\cos\theta)\,d\theta\right]dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}[L\cos(t\cos\theta)]\,dt = \frac{2}{\pi}\int_0^{\pi/2}\frac{s}{s^2+\cos^2\theta}\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta}{s^2\sec^2\theta+1}\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta}{(s^2+1)+s^2\tan^2\theta}\,d\theta$$

Put $\quad s\tan\theta = t \qquad \therefore\ s\sec^2\theta\,d\theta = dt$

When $\theta = 0$, $t = 0$; when $\theta = \pi/2$, $t = \infty$.

$$\therefore \quad L[J_0(t)] = \frac{2}{\pi}\int_0^{\infty}\frac{dt}{t^2+(s^2+1)} = \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\cdot\left[\tan^{-1}\left(\frac{t}{\sqrt{s^2+1}}\right)\right]_0^{\infty}$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\left[\frac{\pi}{2} - 0\right] = \frac{1}{\sqrt{s^2+1}}$$

By definition of Laplace transform this means

$$\int_0^{\infty} e^{-st}\cdot J_0(t)\,dt = \frac{1}{\sqrt{s^2+1}}$$

i.e. $\quad \int_0^{\infty} e^{-st}\cdot\left[\frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta\right]dt = \frac{1}{\sqrt{s^2+1}}$

Putting $s = 1$, we get

$$\frac{1}{\pi}\int_0^{\infty} e^{-t}\left[\int_0^{\pi}\cos(t\cos\theta)\,d\theta\right]dt = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

Example 4 : If $J_0(t) = \frac{1}{\pi}\int_0^{\pi}\cos(t\sin\theta)\,d\theta$, prove that $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$.

Sol. : Prove it yourself on the above lines.

5. Evaluation of the Integral $\int_0^{\infty} e^{-at} f(t)\,dt$

We have earlier remarked that Laplace transforms are highly useful to find certain integrals which cannot be evaluated by usual methods. Consider the following integral.

Find $\int_0^{\infty} e^{-at} f(t)\,dt$

To find $\int_0^{\infty} e^{-at} f(t)\,dt$, we first find $L\,f(t)\ dt$. Say it is $\Phi(s)$.

This means, $\quad \int_0^{\infty} e^{-st} f(t)\,dt = \Phi(s) \quad$ .... (1)

Comparing with the given integral we see that we have to replace s by a.

Replacing s by a in (1), we get $\int_0^{\infty} e^{-at} f(t)\,dt = \Phi(a)$

For example, by definition,

$$\int_0^{\infty} e^{-st}\sin t\,dt = \frac{1}{s^2+1^2} \qquad \therefore\ \int_0^{\infty} e^{-3t}\sin t\,dt = \frac{1}{10} \text{ for } s = 3.$$

Also $\quad \int_0^{\infty} e^{-st}\cos 3t\,dt = \frac{s}{s^2+9} \qquad \therefore\ \int_0^{\infty} e^{-2t}\cos 3t\,dt = \frac{2}{13}$ for $s = 2$.

Thus, Laplace transforms can be used to evaluate some integrals of the above type and of some more types very easily. (See also pages 1-24, 1-35, 1-43, 1-65)

Example 1 : Evaluate $\int_0^\infty e^{-2t}\sin^2 2t\,dt$.

Sol. : We shall first obtain Laplace transform of $\sin^2 2t$.

Now, $L(\sin^2 2t) = L\left[\frac{1}{2}(1-\cos 4t)\right]$

$$= \frac{1}{2}[L(1) - L(\cos 4t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

This means $\int_0^\infty e^{-st}\sin^2 2t\,dt = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$

Now, putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\sin^2 2t\,dt = \frac{1}{2}\left[\frac{1}{2} - \frac{2}{2^2+4^2}\right] = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{10}\right] = \frac{1}{5}.$$

Example 2 : Evaluate $\int_0^\infty e^{-2t}\cdot\sin^3 t\,dt$. **(M.U. 1985, 2002, 09)**

Sol. : We have $L(\sin^3 t) = L\left[\frac{3}{4}\cdot(\sin t) - \frac{1}{4}\cdot(\sin 3t)\right]$

$$= \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\cdot\frac{3}{s^2+9}$$

This means $\int_0^\infty e^{-st}\sin^3 t\,dt = \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\cdot\frac{3}{s^2+9}$

Putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\sin^3 t\,dt = \frac{3}{4}\cdot\frac{1}{4+1} - \frac{1}{4}\cdot\frac{3}{4+9} = \frac{6}{65}.$$

Example 3 : Evaluate $\int_0^\infty e^{-2t}t^5\cosh t\,dt$. **(M.U. 2016)**

Sol. : Consider $L(t^5\cosh t)$

$$L[t^5\cosh t] = L\left[t^5\left(\frac{e^t+e^{-t}}{2}\right)\right] = \frac{1}{2}[L(e^t t^5) + L(e^{-t}t^5)]$$

$$= \frac{1}{2}\left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6}\right] = \frac{5!}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

This means,

$$\int_0^\infty e^{-st}t^5\cosh t\,dt = \frac{5!}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

Now, put $s = 2$,

$$\therefore \int_0^\infty e^{-st}t^5\cosh t\,dt = \frac{5!}{2}\left[\frac{1}{(2-1)^6} + \frac{1}{(2+1)^6}\right] = 60\left[1+\frac{1}{3^6}\right]$$

Alternatively :

$$\int_0^\infty e^{-st}t^5\cosh t\,dt = \int_0^\infty e^{-2t}t^5\left(\frac{e^t+e^{-t}}{2}\right)dt$$

$$\therefore \int_0^\infty e^{-st}t^5\cosh t\,dt = \frac{1}{2}\int_0^\infty (e^{-t}t^5 + e^{-3t}t^5)\,dt$$

Consider $L(t^5) = \frac{5!}{s^6}$.

This means, $\int_0^\infty e^{-st}t^5\,dt = \frac{5!}{s^6}$ .... (A)

Hence, we get

$$\int_0^\infty e^{-st}t^5\cosh t\,dt = \frac{1}{2}\int_0^\infty e^{-t}t^5\,dt + \frac{1}{2}\int_0^\infty e^{-3t}t^5\,dt$$

$$= \frac{1}{2}\cdot\frac{5!}{16} + \frac{1}{2}\cdot\frac{5!}{3^6}$$

$$= \frac{5!}{2}\left[1+\frac{1}{3^6}\right] \quad \text{[Putting } s = 1,\ s = 3 \text{]}$$

$$= 60\left[1+\frac{1}{3^6}\right]$$

Example 4 : Evaluate $\int_0^\infty e^{-2t}\cosh^5 t\,dt$.

Sol. : We have $\cosh^5 x = \left(\frac{e^x+e^{-x}}{2}\right)^5$

$$= \frac{1}{32}\left[e^{5x} + 5e^{4x}\cdot e^{-x} + 10e^{3x}\cdot e^{-2x} + 10e^{2x}\cdot e^{-3x} + 5e^x\cdot e^{-4x} + e^{-5x}\right]$$

$$= \frac{2}{32}\left[\left(\frac{e^{5x}+e^{-5x}}{2}\right) + 5\left(\frac{e^{3x}+e^{-3x}}{2}\right) + 10\left(\frac{e^x+e^{-x}}{2}\right)\right]$$

$$= \frac{1}{16}[\cosh 5x + 5\cosh 3x + 10\cosh x]$$

$$L(\cosh^5 t) = \frac{1}{16}[L(\cosh 5t) + 5L(\cosh 3t) + 10L(\cosh t)]$$

$$= \frac{1}{16}\left[\frac{s}{s^2-25} + 5\cdot\frac{s}{s^2-9} + 10\cdot\frac{s}{s^2-1}\right]$$

This means $\int_0^\infty e^{-st}\cosh^5 t\,dt = \frac{1}{16}\left[\frac{s}{s^2-25} + 5\cdot\frac{s}{s^2-9} + 10\cdot\frac{s}{s^2-1}\right]$

Now putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\cosh^5 t\,dt = \frac{1}{16}\left[\frac{2}{4-25} + \frac{10}{4-9} + \frac{20}{4-1}\right]$$

$$\therefore \int_0^\infty e^{-2t}\cosh^5 t\,dt = \frac{1}{16}\left[-\frac{2}{21} - \frac{10}{5} + \frac{20}{3}\right] = \frac{1}{16}\left[\frac{32}{7}\right] = \frac{2}{7}.$$

Example 5 : Find $L(\mathrm{erf}\sqrt{t})$ and hence, obtain $\int_0^\infty \mathrm{erf}\sqrt{t}\cdot e^{-t}\,dt$. **(M.U. 2000)**

Sol. : We have proved in (11), page 1-6 that $L(\mathrm{erf}\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$.

This means $\int_0^{\infty} e^{-st}\, erf\sqrt{t}\, dt = \dfrac{1}{s\sqrt{s+1}}$

Now, putting $s = 1$, we get $\int_0^{\infty} e^{-t} erf\sqrt{t}\, dt = \dfrac{1}{1\sqrt{1+1}} = \dfrac{1}{\sqrt{2}}$.

Example 6 : Find $\int_0^{\infty} e^{-t}\, erf_c\sqrt{t}\, dt$. **(M.U. 1996)**

Sol. : By (12), page 1-7, we have

$$L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}.$$

By definition of Laplace transform this means

$$\int_0^{\infty} e^{-st}\, erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$$

Now putting $s = 1$, we get

$$\int_0^{\infty} e^{-t}\, erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{2}(\sqrt{2}+1)} = \frac{1}{\sqrt{6}}.$$

Example 7 : Find $\int_0^{\infty} erf_c\sqrt{t}\, dt$.

Sol. : As seen above $L[erf_c\sqrt{t}] = \dfrac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$.

This means $\int_0^{\infty} e^{-st} erf_c\sqrt{t}\, dt = \dfrac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$

Now, putting $s = 0$, we get $\int_0^{\infty} erf_c\sqrt{t}\, dt = \dfrac{1}{2}$.

Example 8 : If $\int_0^{\infty} e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\, dt = \dfrac{1}{4}$, then find α.

Sol. : Now, $L[\sin(t+\alpha)\cos(t-\alpha)] = \dfrac{1}{2}[L(\sin 2t) + \sin 2\alpha \cdot L(1)]$

$$= \frac{1}{2}\cdot\left[\frac{2}{s^2+4} + \sin 2\alpha \cdot \frac{1}{s}\right]$$

This means $\int_0^{\infty} e^{-st} \sin(t+\alpha)\cos(t-\alpha)\, dt = \dfrac{1}{2}\cdot\left[\dfrac{2}{s^2+4} + \sin 2\alpha\cdot\dfrac{1}{s}\right]$

Now, putting $s = 2$, we get

$$\int_0^{\infty} e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\, dt = \frac{1}{2}\left[\frac{2}{4+4} + \frac{1}{2}\sin 2\alpha\right] = \frac{1}{8} + \frac{1}{4}\sin 2\alpha$$

But this is equal to $\dfrac{1}{4}$.

$$\therefore \frac{1}{8} + \frac{1}{4}\sin 2\alpha = \frac{1}{4} \quad \therefore \frac{1}{4}\sin 2\alpha = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

$$\therefore \sin 2\alpha = \frac{1}{2} \quad \therefore 2\alpha = \frac{\pi}{6} \quad \therefore \alpha = \frac{\pi}{12}.$$

EXERCISE - IV

(A) Evaluate by Laplace transform,

1. $\int_0^{\infty} e^{-3t} \cdot \cos^2 t\, dt$, 2. $\int_0^{\infty} e^{-4t} \cdot \cosh^3 t\, dt$, 3. $\int_0^{\infty} e^{-5t} \cdot \sinh^3 dt$,

4. $\int_0^{\infty} e^{-3t}\, t^5\, dt$, 5. $\int_0^{\infty} e^{-4t} \cdot \sin^3 t\, dt$, 6. $\int_0^{\infty} e^{-3t} \cdot \cos^3 t\, dt$. **(M.U. 2003)**

(B) If $\int_0^{\infty} e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\, dt = \dfrac{3}{8}$, then find α. **(M.U. 2004, 06, 07, 09, 14)**

(C) Evaluate the following :

1. $\int_0^{\infty} e^{-8t}\, erf\sqrt{t}\, dt$ 2. $\int_0^{\infty} e^{-3t}\, erf_c\sqrt{t}\, dt$

[**Ans :** (A) (1) $\dfrac{11}{39}$, (2) $\dfrac{12}{35}$, (3) $\dfrac{1}{64}$, (4) $\dfrac{40}{243}$, (5) $\dfrac{6}{425}$, (6) $\dfrac{4}{15}$. (B) $\alpha = \dfrac{\pi}{4}$

(C) (1) $\dfrac{1}{24}$, (2) $\dfrac{1}{6}$.]

6. Change of Scale Property

If $L[f(t)] = \Phi(s)$, then $\boxed{L[f(at)] = \dfrac{1}{a}\Phi\left(\dfrac{s}{a}\right)}$ **(M.U. 2002, 06, 09)** .... (13)

Proof : We have $L[f(at)] = \int_0^{\infty} e^{-st} f(at)\, dt$. Now, put $u = at$.

$$= \int_0^{\infty} e^{-s(u/a)} f(u)\cdot\frac{du}{a} = \frac{1}{a}\int_0^{\infty} e^{-(s/a)u} f(u)\, du = \frac{1}{a}\Phi\left(\frac{s}{a}\right).$$

e.g. (i) If $L f(t) = \dfrac{2s}{s^2+4}$, then $L[f(2t)] = \dfrac{1}{2}\left[\dfrac{2(s/2)}{(s/2)^2+4}\right] = \dfrac{2s}{s^2+16}$.

(ii) If $L f(t) = \dfrac{s^2-s}{s^2+3s+5}$, then $L[f(3t)] = \dfrac{1}{3}\left[\dfrac{(s/3)^2-(s/3)}{(s/3)^2+3(s/3)+5}\right] = \dfrac{1}{3}\left[\dfrac{s^2-3s}{s^2+9s+45}\right]$

Example 1 : (i) If $L f(t) = \dfrac{1}{s}e^{-1/s}$, find $L f(3t)$. (ii) If $L\left(\dfrac{1}{\sqrt{\pi t}}\right) = \dfrac{1}{\sqrt{s}}$, find $L\left(\sqrt{\dfrac{\pi}{t}}\right)$.

(iii) If $L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$, find $L(erf\, 3\sqrt{t})$. (iv) If $L J_0(t) = \dfrac{1}{\sqrt{s^2+1}}$, find $L J_0(at)$.

(M.U. 2003)

Sol. : We have if $L f(t) = \Phi(s)$, then $L f(at) = \dfrac{1}{a}\Phi\left(\dfrac{s}{a}\right)$.

(i) Since, $L f(t) = \dfrac{1}{s}e^{-1/s}$, $L f(3t) = \dfrac{1}{3}\cdot\dfrac{1}{(s/3)}\cdot e^{-1/(s/3)}$

$$\therefore L f(3t) = \frac{1}{s}\cdot e^{-3/s}.$$

(ii) Since, $L\left(\frac{1}{\sqrt{\pi t}}\right)=\frac{1}{\sqrt{s}}$, $L\left(\sqrt{\frac{\pi}{t}}\right)=L\left(\pi\sqrt{\frac{1}{\pi t}}\right)$

$$\therefore\ L\left(\sqrt{\frac{\pi}{t}}\right)=\frac{1}{\pi\sqrt{s/\pi}}=\sqrt{\frac{\pi}{s}}$$

(iii) Since, $L(erf\sqrt{t})=\frac{1}{s\sqrt{s+1}}$ $\therefore\ L(erf\ 3\sqrt{t})=L(erf\sqrt{9t})$

$$\therefore\ L(erf\,3\sqrt{t})=\frac{1}{3}\cdot\frac{1}{(s/3)\sqrt{(s/9)+1}}=\frac{3}{s\sqrt{s+9}}$$

(iv) Since, $LJ_0(t)=\frac{1}{\sqrt{s^2+1}}$, $\therefore\ LJ_0(at)=\frac{1}{a}\cdot\frac{1}{\sqrt{(s^2/a^2)+1}}=\frac{1}{\sqrt{s^2+a^2}}$.

Example 2 : Find $L(erf\sqrt{t})$ and then evaluate $\int_0^\infty e^{-5t}\,erf(2\sqrt{t})\,dt$. **(M.U. 2008)**

Sol. : We have proved (11), page 1-6 that $L(erf\sqrt{t})=\frac{1}{s\sqrt{s+1}}$.

But $L\{f(at)\}=\frac{1}{a}\Phi\left(\frac{s}{a}\right)$

$$\therefore\ L\{erf(2\sqrt{t})\}=L\{erf.(\sqrt{4t})\}=\frac{1}{4}\cdot\frac{1}{(s/4)\sqrt{(s/4)+1}}=\frac{2}{s\sqrt{s+4}}$$

$$\therefore\ \int_0^\infty e^{-st}\cdot erf(2\sqrt{t})\,dt=\frac{2}{s\sqrt{s+4}}$$

Putting $s=5$, $\int_0^\infty e^{-5t}erf(2\sqrt{t})\,dt=\frac{2}{5\sqrt{5+4}}=\frac{2}{15}$.

EXERCISE - V

1. If $Lf(t)=\frac{2}{s^3}e^{-s}$, find $Lf(2t)$. **[Ans. :** $\frac{8}{s^3}e^{-s/2}$ **]**
2. If $L\,erf_c(\sqrt{t})=\frac{1}{\sqrt{s+1}+1}$, find $L\,erf_c(2\sqrt{t})$. **[Ans. :** $\frac{1}{2\sqrt{s+4}+4}$ **]**
3. If $Lf(t)=\log\left(\frac{s+3}{s+1}\right)$, find $Lf(2t)$. **[Ans. :** $\log\sqrt{\frac{s+6}{s+2}}$ **]**
4. If $L\left[\sin\sqrt{t}\right]=\frac{\sqrt{\pi}}{2s\sqrt{s}}\cdot e^{-1/(4s)}$, find $L\left[\sin 2\sqrt{t}\right]$. **(M.U. 2004, 06, 09) [Ans. :** $\frac{\sqrt{\pi}}{s\sqrt{s}}\cdot e^{-1/s}$ **]**
5. If $L[f(t)]=\frac{8(s-3)}{(s^2-6s+25)^2}$, find $L[f(2t)]$. **(M.U. 2005) [Ans. :** $\frac{s-6}{(s^2-12s+100)^2}$ **]**
6. If $L[f(t)]=\frac{20-4s}{s^2-4s+20}$, find $L[f(3t)]$. **(M.U. 2009) [Ans. :** $\frac{60-4s}{s^2-12s+180}$ **]**

7. First Shifting Theorem

If $L[f(t)]=\Phi(s)$, then $\boxed{L[e^{-at}f(t)]=\Phi(s+a)}$ .... (14)

(M.U. 1997, 2003, 06, 09, 11)

Proof : We have

$$L[e^{-at}f(t)]=\int_0^\infty e^{-st}\left\{e^{-at}f(t)\right\}dt=\int_0^\infty e^{-(s+a)t}f(t)\,dt=\Phi(s+a)$$

e.g. $L[\sin at]=\frac{a}{s^2+a^2}$ $\therefore\ L\left[e^{-bt}\sin at\right]=\frac{a}{(s+b)^2+a^2}$.

Cor. : Changing sign of a, we get,

If $L[f(t)]=\Phi(s)$, then $\boxed{L\left[e^{at}f(t)\right]=\Phi(s-a)}$

1. $\because$ $L(\sin at)=\frac{a}{s^2+a^2}$, $L(e^{bt}\sin at)=\frac{a}{(s-b)^2+a^2}$
2. $\because$ $L(\cos at)=\frac{s}{s^2+a^2}$, $L(e^{bt}\cos at)=\frac{s-b}{(s-b)^2+a^2}$
3. $\because$ $L(\sin h\,at)=\frac{a}{s^2-a^2}$, $L(e^{bt}\sin h\,at)=\frac{a}{(s-b)^2-a^2}$
4. $\because$ $L(\cos h\,at)=\frac{s}{s^2-a^2}$, $L(e^{bt}\cos h\,at)=\frac{s-b}{(s-b)^2-a^2}$
5. $\because$ $L(t^n)=\frac{n!}{s^{n+1}}$, $L(e^{bt}\cdot t^n)=\frac{n!}{(s-b)^{n+1}}$ (n is an integer)
6. $\because$ $L(t^n)=\frac{\overline{\lvert n+1}}{s^{n+1}}$, $L(e^{bt}\cdot t^n)=\frac{\overline{\lvert n+1}}{(s-b)^{n+1}}$

EXERCISE - VI

Write down the Laplace transforms of the following.

1. $e^{3t}\sin 2t$, 2. $e^{-2t}\cos 3t$, 3. $e^{-t}\sin 4t$, 4. $e^{t}\cos t$,
5. $e^{2t}\sin ht$, 6. $e^{-t}\cos h2t$, 7. $e^{-2t}\sin h4t$, 8. $e^{-3t}\cos h3t$,
9. $e^{-3t}t^4$, 10. $e^{2t}t^{3/2}$, 11. $e^{-t}\sqrt{t}$, 12. $e^{t}\cos 3t$ **(M.U. 2003)**

[Ans. : Answers not given for obvious reason. **]**

Example 1 : If $L[f(t)]=\frac{s}{s^2+s+4}$, find $L[e^{-3t}f(2t)]$. **(M.U. 2014)**

Sol. : By change of scale property,

$$L[f(2t)]=\frac{1}{2}\cdot\frac{(s/2)}{(s/2)^2+(s/2)+4}=\frac{s}{s^2+2s+16}$$

By first shifting theorem,

$$L[e^{-3t}f(2t)]=\frac{(s+3)}{(s+3)^2+2(s+3)+16}=\frac{s+3}{s^2+8s+31}.$$

Example 2 : Find the Laplace transform of (i) $e^{4t}\sin^3 t$, (ii) $\cos h\,2t\cos 2t$. **(M.U. 2013)**

Sol. : (i) $L[\sin^3 t] = L\left[\frac{1}{4}(3\sin t - \sin 3t)\right] = \frac{3}{4}L(\sin t) - \frac{1}{4}L(\sin 3t)$

$$\therefore\ L[\sin^3 t] = \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\cdot\frac{3}{s^2+9}.$$

By first shifting theorem,

$$L\left[e^{4t}\sin^3 t\right] = \frac{3}{4}\cdot\frac{1}{(s-4)^2+1} - \frac{1}{4}\cdot\frac{3}{(s-4)^2+9}$$

$$= \frac{3}{4}\left[\frac{1}{s^2-8s+17} - \frac{1}{s^2-8s+25}\right]$$

$$= \frac{6}{(s^2-8s+17)(s^2-8s+25)}.$$

(ii) $L[\cos h\,2t\cdot\cos 2t] = L\left[\frac{1}{2}\left(e^{2t}+e^{-2t}\right)\cos 2t\right] = \frac{1}{2}\left[L(e^{2t}\cos 2t) + L(e^{-2t}\cos 2t)\right]$

But $L\cos 2t = \frac{s}{s^2+4}$. $\therefore$ By shifting theorem,

$$L[\cos h\,2t\cdot\cos 2t] = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} + \frac{s+2}{(s+2)^2+4}\right] = \frac{s^3}{s^4+64}.$$

Example 3 : Find $L[(t^2\sin h\,t)^2]$.

Sol. : We have $(t^2\sin ht)^2 = t^4\left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{t^4}{4}[e^{2t}-2+e^{-2t}]$

$$\therefore\ L[(t^2\sin h\,t)^2] = L\left[\frac{t^4}{4}(e^{2t}-2+e^{-2t})\right] = \frac{1}{4}\left[L(e^{2t}t^4) - 2L(t^4) + L(e^{-2t}t^4)\right]$$

But $L[t^4] = \frac{4!}{s^5}$.

$$\therefore\ L[(t^2\sin h\,t)^2] = \frac{1}{4}\left[\frac{4!}{(s-2)^5} - 2\cdot\frac{4!}{s^5} + \frac{4!}{(s+2)^5}\right] = 6\left[\frac{1}{(s-2)^5} - \frac{2}{s^5} + \frac{1}{(s+2)^2}\right]$$

Example 4 : Find $L\left[e^t\left(1+\sqrt{t}\right)^4\right]$.

Sol. : We have

$$L\left(1+\sqrt{t}\right)^4 = L\left[1+4\sqrt{t}+6\left(\sqrt{t}\right)^2+4\left(\sqrt{t}\right)^3+\left(\sqrt{t}\right)^4\right] \quad \text{[By Binomial Theorem]}$$

$$= L\left[1+4\sqrt{t}+6t+4t^{3/2}+t^2\right]$$

$$= \frac{1}{s} + \frac{4\overline{|3/2}}{s^{3/2}} + \frac{6\overline{|2}}{s^2} + \frac{4\overline{|5/2}}{s^{5/2}} + \frac{\overline{|3}}{s^3}$$

$$= \frac{1}{s} + \frac{4(1/2)\overline{|1/2}}{s^{3/2}} + 6\cdot\frac{1}{s^2} + \frac{4(3/2)(1/2)\overline{|1/2}}{s^{5/2}} + \frac{2}{s^2}$$

(For Gamma functions refer to Applied Mathematics - II)

$$\therefore\ L\left[\left(1+\sqrt{t}\right)^4\right] = \frac{1}{s} + \frac{2\sqrt{\pi}}{s^{3/2}} + \frac{6}{s^2} + 3\cdot\frac{\sqrt{\pi}}{s^{5/2}} + \frac{2}{s^2}$$

Now, by first shifting theorem,

$$L\left[e^t\left(1+\sqrt{t}\right)^4\right] = \frac{1}{s-1} + \frac{2\sqrt{\pi}}{(s-1)^{3/2}} + \frac{6}{(s-1)^2} + \frac{3\sqrt{\pi}}{(s-1)^{5/2}} + \frac{2}{(s-1)^2}$$

Example 5 : Find Laplace transform of sin *h at* sin *at*. **(M.U. 2003, 05, 09)**

Sol. : We have

$$\sin h\,at\sin at = \left(\frac{e^{at}-e^{-at}}{2}\right)\sin at = \frac{1}{2}\left[e^{at}\sin at - e^{-at}\sin at\right]$$

Now, $L(\sin at) = \frac{a}{s^2+a^2}$.

By first Shifting Theorem,

$$L(e^{at}\sin at) = \frac{a}{(s-a)^2+a^2} = \frac{a}{s^2-2as+2a^2}$$

$$L(e^{-at}\sin at) = \frac{a}{(s+a)^2+a^2} = \frac{a}{s^2+2as+2a^2}$$

$$\therefore\ L(\sin h\,at\sin at) = \frac{1}{2}\left[L(e^{at}\sin at) - L(e^{-at}\sin at)\right]$$

$$\therefore\ L(\sin h\,at\sin at) = \frac{1}{2}\cdot\left[\frac{a}{s^2-2as+2a^2} - \frac{a}{s^2+2as+2a^2}\right]$$

$$= \frac{a}{2}\cdot\left[\frac{4as}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{2a^2s}{(s^2+2a^2-2as)(s^2+2a^2+2as)} = \frac{2a^2s}{s^4+4a^4}$$

Example 6 : Show that $L[\sin h(t/2)\sin(t/2)] = \frac{2s}{4s^4+1}$.

Sol. : Putting $a = 1/2$ in the above example, we get

$$L[\sin h(t/2)\sin(t/2)] = \frac{2(1/4)s}{s^4+4\cdot(1/2)^4} = \frac{2s}{4s^4+1}$$

Or solve it independently.

Example 7 : Show that $L\left[\sin h\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right] = \frac{\sqrt{3}}{2}\cdot\frac{s}{(s^4+s^2+1)}$.

(M.U. 1993, 2002, 03, 09)

Sol. : We have, $\sin h(t/2)\cdot\sin(\sqrt{3}t/2) = \left(\frac{e^{t/2}-e^{-t/2}}{2}\right)\cdot\sin\frac{\sqrt{3}}{2}t$

Now $L\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{s^2+(3/4)}$.

By first shifting theorem,

$$\therefore \quad L\,e^{t/2}\cdot\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{[s-(1/2)]^2+3/4} = \frac{\sqrt{3}/2}{s^2+1-s}$$

and $$L\,e^{-t/2}\cdot\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{[s+(1/2)]^2+3/4} = \frac{\sqrt{3}/2}{s^2+1+s}$$

$$\therefore \quad L\left[\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right] = \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s^2+1)-s} - \frac{\sqrt{3}/2}{(s^2+1)+s}\right]$$

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\left[\frac{(s^2+1+s)-(s^2+1-s)}{(s^2+1)^2-s^2}\right]$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{s}{s^4+s^2+1}.$$

Example 8 : Find the Laplace transform of the following.

(i) $\dfrac{\cos 2t\sin t}{e^t}$, **(M.U. 1993, 2003, 04, 10)** (ii) $e^{-4t}\sinh t\sin t$, **(M.U. 1995)**

(iii) $e^t\sin 2t\sin 3t$, **(M.U. 1997)** (iv) $e^{-3t}\cosh 5t\sin 4t$. **(M.U. 1998)**

Sol. : (i) $$\cos 2t\sin t = \frac{1}{2}\cdot 2\sin t\cos 2t = \frac{1}{2}[\sin(1+2)t+\sin(1-2)t]$$

$$= \frac{1}{2}[\sin 3t - \sin t]$$

$$\therefore \quad L(\cos 2t\sin t) = \frac{1}{2}[L(\sin 3t) - L(\sin t)] = \frac{1}{2}\cdot\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$$

Now by first shifting theorem,

$$L\left[e^{-t}(\cos 2t\sin t)\right] = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+2s+10} - \frac{1}{s^2+2s+2}\right]$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

(ii) $$e^{-4t}\sinh t\sin t = e^{-4t}\cdot\left(\frac{e^t-e^{-t}}{2}\right)\sin t = \frac{1}{2}e^{-3t}\sin t - \frac{1}{2}e^{-5t}\sin t$$

$$\therefore \quad L\left(e^{-4t}\sinh t\sin t\right) = \frac{1}{2}L\left(e^{-3t}\sin t\right) - \frac{1}{2}L\left(e^{-5t}\sin t\right)$$

$$= \frac{1}{2}\cdot\frac{1}{(s+3)^2+1^2} - \frac{1}{2}\cdot\frac{1}{(s+5)^2+1^2}$$

$$= \frac{1}{2}\left[\frac{s^2+10s+26-s^2-6s-10}{(s^2+6s+10)(s^2+10s+26)}\right]$$

$$= \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}.$$

In some of the following examples, we need the following trigonometric results.

1. $\sin A\cos B = \dfrac{1}{2}[\sin(A+B)+\sin(A-B)]$
2. $\cos A\sin B = \dfrac{1}{2}[\sin(A+B)-\sin(A-B)]$
3. $\cos A\cos B = \dfrac{1}{2}[\cos(A+B)+\cos(A-B)]$
4. $\sin A\sin B = -\dfrac{1}{2}[\cos(A+B)-\cos(A-B)]$

(iii) $$\sin 2t\sin 3t = \frac{1}{2}\cdot 2\sin 3t\sin 2t = -\frac{1}{2}[\cos 5t - \cos t]$$

$$\therefore \quad L(\sin 2t\sin 3t) = -\frac{1}{2}[L(\cos 5t) - L(\cos t)] = -\frac{1}{2}\left[\frac{s}{s^2+5^2} - \frac{s}{s^2+1^2}\right]$$

Now by first shifting theorem,

$$L\left[e^t\sin 2t\sin 3t\right] = -\frac{1}{2}\left[\frac{s-1}{(s-1)^2+5^2} - \frac{s-1}{(s-1)^2+1^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-1}{s^2-2s+26} - \frac{s-1}{s^2-2s+2}\right]$$

$$= \frac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}$$

(iv) We have

$$e^{-3t}\cosh 5t\sin 4t = e^{-3t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\sin 4t = \frac{1}{2}\left(e^{2t}+e^{-8t}\right)\sin 4t$$

Now $$L(\sin 4t) = \frac{4}{s^2+16}$$

$\therefore$ By first shifting theorem,

$$L\left[e^{-3t}\cosh 5t\sin 4t\right] = \frac{1}{2}L\left[\left(e^{2t}+e^{-8t}\right)\sin 4t\right] = \frac{1}{2}\left[\frac{4}{(s-2)^2+16} + \frac{4}{(s+8)^2+16}\right]$$

$$= \frac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+80)}$$

Example 9 : Find Laplace transform of $\sin 2t\cos t\cosh 2t$. **(M.U. 1994, 2004, 11, 14)**

Sol. : $$\sin 2t\cos t = \frac{1}{2}\cdot 2\sin 2t\cos t = \frac{1}{2}[\sin 3t + \sin t]$$

$$\cosh 2t = \frac{e^{2t}+e^{-2t}}{2}$$

$$\therefore \quad \sin 2t\cos t\cosh 2t = \frac{1}{2}\left(e^{2t}+e^{-2t}\right)(\sin 3t + \sin t) \quad \text{.......................... (1)}$$

$$\therefore \quad L\sin 3t = \frac{3}{s^2+9}$$

By first shifting theorem,

$$\therefore \quad L\left[e^{2t}\sin 3t\right] = \frac{3}{(s-2)^2+9}, \quad L\left[e^{-2t}\sin 3t\right] = \frac{3}{(s+2)^2+9}$$

$$\therefore \quad L\left(e^{2t}\sin 3t\right) + L\left(e^{-2t}\sin 3t\right) = 3\left[\frac{1}{(s-2)^2+9} + \frac{1}{(s+2)^2+9}\right]$$

$$= \frac{3\cdot 2(s^2+13)}{s^4+10s^2+13^2} \quad \ldots\ldots(2)$$

Now $\sin t = \dfrac{1}{s^2+1}$

By first shifting theorem,

$$\therefore \quad L(e^{2t}\sin t) = \frac{1}{(s-2)^2+1}, \quad L(e^{-2t}\sin t) = \frac{1}{(s+2)^2+1}$$

$$\therefore \quad L(e^{2t}\sin t) + L(e^{-2t}\sin t) = \frac{2(s^2+5)}{s^4-6s^2+5^2} \quad \ldots\ldots(3)$$

From (1), (2) and (3), we get

$$L[\sin 2t\cos t\cos h\,2t] = \frac{3(s^2+13)}{s^4+10s^2+13^2} + \frac{s^2+5}{s^4-6s^2+5^2}$$

8. Evaluation of the Integral $\int_0^\infty e^{-at}\cdot e^{bt} f(t)\,dt$

To find $\int_0^\infty e^{-at}\cdot e^{bt} f(t)\,dt$ we first find $L[e^{bt} f(t)]$, say $\Phi(s)$.

This means $\quad \int_0^\infty e^{-st}\cdot e^{bt} f(t)\,dt = \Phi(s)$

Now, we put $s = a$ in both sides.

Example 1 : Evaluate $\int_0^\infty e^{-t}\sin h\,2t\sin 3t\,dt$.

Sol. : Now, $L(\sin 3t) = \dfrac{3}{s^2+9}$. But $\quad \sin h\,2t = \dfrac{e^{2t}-e^{-2t}}{2}$

$$\therefore \quad L\sin h\,2t\cdot\sin 3t = \frac{1}{2}L\left[(e^{2t}-e^{-2t})\cdot\sin 3t\right] = \frac{1}{2}\left\{L(e^{2t}\sin 3t) - L(e^{-2t}\sin 3t)\right\}$$

By first shifting theorem,

$$L\sin h\,2t\cdot\sin 3t = \frac{1}{2}\left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right]$$

This means, $\quad \displaystyle\int_0^\infty e^{-st}\sin h\,2t\sin 3t\,dt = \frac{1}{2}\left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right]$

Putting $s = 1$, $\quad \displaystyle\int_0^\infty e^{-t}\sin h\,2t\sin 3t\,dt = \frac{1}{2}\left[\frac{3}{(1-2)^2+9} - \frac{3}{(1+2)^2+9}\right]$

$$= \frac{1}{2}\left[\frac{3}{1+9} - \frac{3}{9+9}\right] = \frac{1}{2}\left[\frac{3}{10} - \frac{3}{18}\right] = \frac{1}{15}.$$

Alternatively : We have to find

$$\int_0^\infty e^{-t}\left(\frac{e^{2t}-e^{-2t}}{2}\right)\sin 3t\,dt = \frac{1}{2}\left[\int_0^\infty e^{t}\sin 3t\,dt - \int_0^\infty e^{-3t}\sin 3t\,dt\right]$$

But $\quad L\sin 3t = \dfrac{3}{s^2+9}$. $\quad$ This means, $\quad \displaystyle\int_0^\infty e^{-st}\sin 3t\,dt = \frac{3}{s^2+9}$

Putting $s = -1$, we get $\quad \displaystyle\int_0^\infty e^{t}\sin 3t\,dt = \frac{3}{1+9} = \frac{3}{10}$

Putting $s = 3$, we get $\quad \displaystyle\int_0^\infty e^{-3t}\sin 3t\,dt = \frac{3}{9+9} = \frac{3}{18} = \frac{1}{6}$

$\therefore$ Required integral, $I = \dfrac{1}{2}\left(\dfrac{3}{10} - \dfrac{1}{6}\right) = \dfrac{1}{15}$.

Example 2 : Evaluate $\int_0^\infty e^{-t}\sin\frac{t}{2}\sin h\frac{\sqrt{3}}{2}\,dt$. **(M.U. 2012)**

Sol. : Now $L\left(\sin\dfrac{t}{2}\right) = \dfrac{1/2}{s^2+(1/4)}$. $\quad \sin h\dfrac{\sqrt{3}}{2}t = \dfrac{e^{(\sqrt{3}/2)t} - e^{-(\sqrt{3}/2)t}}{2}$

$$L\sin h\left(\frac{\sqrt{3}}{2}t\right)\sin\frac{t}{2} = \frac{1}{2}\left[L\left\{e^{(\sqrt{3}/2)t}\sin\frac{t}{2} - e^{-(\sqrt{3}/2)t}\sin\frac{t}{2}\right\}\right]$$

By first shifting theorem,

$$L\sin h\left(\frac{\sqrt{3}}{2}t\right)\sin\frac{t}{2} = \frac{1}{2}\left[\frac{1/2}{[s-(\sqrt{3}/2)]^2+(1/4)} - \frac{1/2}{[s+(\sqrt{3}/2)]^2+(1/4)}\right]$$

$$\therefore \quad \int_0^\infty e^{-st}\sin\frac{t}{2}\sin h\frac{\sqrt{3}}{2}t\,dt = \frac{1}{2}\left[\frac{1/2}{[s-(\sqrt{3}/2)]^2+(1/4)} - \frac{1/2}{[s+(\sqrt{3}/2)]^2+(1/4)}\right]$$

Putting $s = 1$,

$$\therefore \quad \int_0^\infty e^{-t}\sin\frac{t}{2}\sin h\frac{\sqrt{3}}{2}t\,dt = \frac{1}{2}\left[\frac{1/2}{[1-(\sqrt{3}/2)]^2+(1/4)} - \frac{1/2}{[1+(\sqrt{3}/2)]^2+(1/4)}\right]$$

$$= \frac{1}{4}\left[\frac{1}{1-\sqrt{3}+1} - \frac{1}{1+\sqrt{3}+1}\right] = \frac{1}{4}\left[\frac{1}{2-\sqrt{3}} - \frac{1}{2+\sqrt{3}}\right]$$

$$= \frac{1}{4}\left[\frac{2+\sqrt{3}-2+\sqrt{3}}{4-3}\right] = \frac{\sqrt{3}}{2}.$$

Alternatively : We have to find

$$\int_0^\infty e^{-t}\left(\frac{e^{(\sqrt{3}/2)t} - e^{-(\sqrt{3}/2)t}}{2}\right)\sin\frac{t}{2}\,dt = \frac{1}{2}\int_0^\infty\left[e^{-(1-\sqrt{3}/2)t} - e^{-(1+\sqrt{3}/2)t}\right]\cdot\sin\frac{t}{2}\,dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(1-\sqrt{3}/2)t}\cdot\sin\frac{t}{2}\,dt - \int_0^\infty e^{-(1+\sqrt{3}/2)t}\cdot\sin\frac{t}{2}\,dt\right]$$

But $\quad L\left(\sin\dfrac{t}{2}\right) = \dfrac{1/2}{s^2+(1/4)} = \dfrac{2}{4s^2+1}$

That means, $\quad \displaystyle\int_0^\infty e^{-st}\cdot\sin\frac{t}{2}\,dt = \frac{2}{4s^2+1}$

Putting $s = 1 - \frac{\sqrt{3}}{2}$, we get

$$\int_0^\infty e^{-(1-\sqrt{3}/2)t} \cdot \sin\frac{t}{2}\, dt = \frac{2}{4(1-\sqrt{3}/2)^2+1} = \frac{2}{4[1-\sqrt{3}+(3/4)]+1}$$

$$= \frac{2}{4-4\sqrt{3}+3+1} = \frac{2}{8-4\sqrt{3}} = \frac{1}{4-2\sqrt{3}}$$

Similarly, putting $s = 1 + \frac{\sqrt{3}}{2}$, we get

$$\int_0^\infty e^{-(1+\sqrt{3}/2)t} \cdot \sin\frac{t}{2}\, dt = \frac{2}{4[1+\sqrt{3}+(3/4)]+1} = \frac{2}{8+4\sqrt{3}} = \frac{1}{4+2\sqrt{3}}$$

$\therefore$ Required integral,

$$I = \frac{1}{2}\left[\frac{1}{4-2\sqrt{3}} - \frac{1}{4+2\sqrt{3}}\right] = \frac{1}{2}\left[\frac{4\sqrt{3}}{16-12}\right] = \frac{\sqrt{3}}{2}.$$

EXERCISE - VII

(A) Find the Laplace transform of

1. $\cos at \cdot \sin h\, at$
2. $e^t \sin 4t \cdot \cos 2t$
3. $e^{-2t}(2\cos 3t - 3\sin 3t)$,
4. $\cos h\, at \sin at$
5. $\cos 3t \cdot \cos h\, 4t$
6. $t^n e^{-at} + \sin^3 t$
7. $\cos h\, at \cos at$
8. $t^2 - 3t + 5 + e^{2t} t^2$
9. $\cos h\, at \cos bt$
10. $e^{4t} t^{3/2}$
11. $e^{-4t} \cos ht \sin t$ **(M.U. 1995)**
12. $e^{-3t} \cos h\, 4t \sin 3t$ **(M.U. 1999)**
13. $e^{2t} \cos 2t \cos t$ **(M.U. 2002, 03)**
14. $e^{2t}(1+t)^2$ **(M.U. 2002, 03)**
15. $e^{2t} \sin^4 t$ **(M.U. 2007)**
16. $e^{-2t} \cos h\, t \sin t$ **(M.U. 2003, 06)**
17. $e^{-t} \sin^2 t$ **(M.U. 2009)**

[**Ans. :** (1) $\frac{a(s^2-2a^2)}{s^4+4a^4}$, (2) $\frac{3}{(s-1)^2+6^2} + \frac{1}{(s-1)^2+2^2}$, (3) $\frac{2s-5}{s^2+4s+13}$,

(4) $\frac{a(s^2+2a^2)}{s^4+2a^4}$, (5) $\frac{s(s^2-7)}{s^4+25^2-14s^2}$, (6) $\frac{\overline{|n+1}}{(s+a)^{n+1}} \quad \frac{6}{(s^2+1)(s^2+9)}$,

(7) $\frac{s^3}{s^4+4a^4}$, (8) $\frac{5s^2-3s+2}{s^3} + \frac{2}{(s-2)^3}$, (9) $\frac{s[s^2+(b^2-a^2)]}{s^4+2(b^2-a^2)s^2+(a^2+b^2)^2}$,

(10) $\frac{\overline{|5/2}}{(s-4)^{5/2}}$, (11) $\frac{s^2+8s+18}{(s^2+6s+10)(s^2+10s+26)}$,

(12) $\frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$, (13) $\frac{(s-2)(s^2-4s+9)}{(s^2-4s+13)(s^2-4s+5)}$

(14) $\frac{1}{s-2} + \frac{2}{(s-2)^2} + \frac{2}{(s-2)^3}$, (15) $\frac{1}{8}\left[\frac{3}{s-2} - \frac{4(s-2)}{s^2-4s+8} + \frac{s-2}{s^2-8s+20}\right]$

(16) $\frac{1}{2}\left[\frac{1}{s^2+2s+2} + \frac{1}{s^2+6s+10}\right]$ (17) $\frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{s^2+2s+5}\right]$

(B) 1. If $L[f(t)] = \frac{s}{s^2+s+4}$, find $L[e^{-3t} f(2t)]$. **(M.U. 2003)** [**Ans. :** $\frac{s+3}{s^2+8s+10}$]

2. If $L[f(t)] = \frac{1}{s(s^2+1)}$, find $L[e^{-t} \cdot f(2t)]$. **(M.U. 2007)** [**Ans. :** $\frac{4}{(s+1)(s^2+2s+5)}$]

(C) Evaluate the following integrals.

1. $\int_0^\infty e^{-t}(t^2 - 3t + 5 + e^{2t} t^2)\, dt$
2. $\int_0^\infty e^{-2t} \cos h \sin t\, dt$
3. $\int_0^\infty e^{-t} \cos h\, t \cos 2t\, dt$

[**Ans. :** (1) 2, (2) $\frac{3}{10}$, (3) $\frac{1}{8}$]

9. Second Shifting Theorem

If $L[f(t)] = \Phi(s)$ and $g(t) = f(t-a)$ when $t > a$ and $g(t) = 0$ when $t < a$, then prove that

$$\boxed{L[g(t)] = e^{-as}\,\Phi(s)} \quad \text{(M.U. 1995, 2002, 11)} \quad \ldots\ldots (15)$$

Proof : By definition of Laplace transform,

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)\, dt = \int_0^a e^{-st} g(t)\, dt + \int_a^\infty e^{-st} g(t)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

Now put $t - a = p \quad \therefore \ dt = dp$

$$\therefore \ L\{g(t)\} = \int_0^\infty e^{-s(a+p)} f(p)\, dp = e^{-as} \int_0^\infty e^{-sp} f(p)\, dp$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\, dt = e^{-as} L[f(t)] = e^{-as}\,\Phi(s)$$

Example : Using second shifting theorem find,

(i) $L[f(t)]$ where $f(t) = \cos(t-\alpha)$, $t > \alpha$ and $f(t) = 0$, $t < \alpha$.

(ii) $L[f(t)]$ where $f(t) = e^{t-k}$, $t > k$ and $f(t) = 0$, $t < k$.

(iii) $L[f(t)]$ where $f(t) = \sin(t - \pi/3)$, $t > \pi/3$ and $f(t) = 0$, $t < \pi/3$.

Sol. : (i) We have $L\cos t = \frac{s}{s^2+1}$. Hence, by second shifting theorem,

$$L\cos(t-\alpha) = e^{-\alpha s} \cdot \frac{s}{s^2+1}.$$

Note ....

We have obtained this result in Ex. 2(2) on page 1-2.

(ii) We have $L(e^t) = \frac{1}{s-1}$. Hence, by second shifting theorem,

$$L(e^{t-k}) = e^{-ks}\frac{1}{s-1}.$$

(iii) We have $L \sin t = \frac{1}{s^2+1}$. Hence, by second shifting theorem

$$L \sin(t - \pi/3) = e^{-\pi s/3} \cdot \frac{1}{s^2+1}.$$

Remark ....

Solve the above examples (ii) and (iii) by using the definition of $L[f(t)]$.

EXERCISE - VIII

Using second shifting theorem find $L f(t)$ where,

1. $f(t) = \cos[t - (2\pi/3)]$, $t > 2\pi/3$ and $f(t) = 0$, $t < 2\pi/3$.
2. $f(t) = \sin[t - (2\pi/3)]$, $t > 2\pi/3$ and $f(t) = 0$, $t < 2\pi/3$.
3. $f(t) = (t-1)^3$, $t > 1$ and $f(t) = 0$, $t < 1$.
4. $f(t) = (t-2)^2$, $t > 2$ and $f(t) = 0$, $t < 2$.
5. $f(t) = (t-1)^4$, $t > 1$ and $f(t) = 0$, $t < 1$.

[**Ans. :** (1) $e^{-2\pi s/3} \cdot \frac{s}{s^2+1}$ (2) $e^{-2\pi s/3} \cdot \frac{1}{s^2+1}$ (3) $e^{-s} \cdot \frac{3!}{s^4}$

(4) $e^{-2s} \cdot \frac{2}{s^3}$ (5) $e^{-s} \cdot \frac{24}{s^5}$.]

10. Effect of Multiplication by t

If $L[f(t)] = \Phi(s)$, then $\boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\Phi(s)}$ **(M.U. 2010, 11, 12)** .... (1)

Proof : We shall prove this property by the method of induction.

Step 1 : Let $\Phi(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

Differentiating both sides w.r.t. s and applying the rule of differentiation under integral sign

$$\Phi'(s) = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\, dt\right] = -\int_0^\infty e^{-st}\, t\, f(t)\, dt = -L[t\, f(t)]$$

$$\therefore \quad L[t\, f(t)] = (-1)\frac{d}{ds}\Phi(s)$$

Thus, the rule is true for $n = 1$.

Step 2 : Now we assume the rule is true for $n = m$ and prove that it is true for $n = m+1$. i.e. assume that

$$L[t^m f(t)] = (-1)^m \frac{d^m}{ds^m}\Phi(s)$$

$$\therefore \quad (-1)^m \frac{d^m}{ds^m}\Phi(s) = L[t^m f(t)] = \int_0^\infty e^{-st}\, t^m f(t)\, dt$$

Differentiating both sides w.r.t. s and applying the rule of differentiation under the integral sign

$$(-1)^m \frac{d^{m+1}}{ds^{m+1}}\Phi(s) = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} \cdot t^m f(t)\, dt\right]$$

$$\therefore \quad (-1)^m \frac{d^{m+1}}{ds^{m+1}}\Phi(s) = -\int_0^\infty e^{-st} \cdot t^{m+1} f(t)\, dt = -L[t^{m+1} f(t)]$$

$$\therefore \quad L\left[t^{m+1} f(t)\right] = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\Phi(s)$$

Thus, if the property is true form $n = m$ then it is true for $n = m+1$.

Step 3 : Since, the property is true for $n = 1$, it is by step (3) true for $n = 1 + 1 = 2$. Since, it is true for $n = 2$, by step (3) again it is true for $n = 2 + 1 = 3$. Hence, it is true for any value of n.

Note ....

In particular, if $L[f(t)] = \Phi(s)$, then $L[t\, f(t)] = -\Phi'(s)$, $L[t^2 f(t)] = \Phi''(s)$.

Example 1 : Find the Laplace transforms of

(i) $t\, e^{at}$ (ii) $t\, e^{-t} \cos h\, 2t$ **(M.U. 2003, 15)** (iii) $t \sin at$

(iv) $(1 + t\, e^{-t})^3$ **(M.U. 1997, 2009)** (v) $t^2 \sin at$

(vi) $t\, e^{-4t} \cdot \sin 3t$ (vii) $t^2 e^{-t} \sin 4t$. **(M.U. 2008)**

Sol. : (i) $L(e^{at}) = \frac{1}{s-a}$ $\quad\therefore\quad L(t\, e^{at}) = -\frac{d}{ds}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}$

(ii) $e^{-t} \cos h\, 2t = e^{-t} \cdot \frac{e^{2t} + e^{-2t}}{2} = \frac{e^{t} + e^{-3t}}{2}$

$$\therefore \quad L(e^{-t} \cos h\, 2t) = \frac{1}{2} \cdot L(e^{t} + e^{-3t}) = \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+3}\right]$$

$$\therefore \quad L(t\, e^{t} \cos h\, 2t) = -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+3}\right)\right] = \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+3)^2}\right].$$

(iii) $L(t \sin at) = (-1) \cdot \frac{d}{ds}[L(\sin at)] = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = +\frac{2as}{(s^2+a^2)^2}$

(iv)
$$L\left[(1 + te^{-t})^3\right] = L\left[1 + 3\, t\, e^{-t} + 3t^2 e^{-2t} + t^3 e^{-3t}\right]$$
$$= L(1) + 3L(t\, e^{-t}) + 3L(t^2 e^{-2t}) + L(t^3 e^{-3t})$$
$$= L(1) + 3(-1)\frac{d}{ds}\left[L(e^{-t})\right] + 3(-1)^2\frac{d^2}{ds^2}\left[L(e^{-2t})\right] + (-1)^3\frac{d^3}{ds^3}\left[L(e^{-3t})\right]$$
$$= \frac{1}{s} - 3\frac{d}{ds}\left(\frac{1}{s+1}\right) + 3\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) - \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)$$
$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

Alternatively we can obtain the Laplace transform of $(1 + t\, e^{-t})^3$ by using first shifting theorem.

$$L[(1 + t\, e^{-t})^3] = L[1 + 3\, t\, e^{-t} + 3\, t^2 e^{-2t} + t^3 e^{-3t}]$$
$$= L[1] + 3L[e^{-t}\, t] + 3L[e^{-2t} \cdot t^2] + L[e^{-3t}\, t^3]$$

But $L[t^n] = \frac{n!}{s^{n+1}}$ and $L[e^{bt}\, t^n] = \frac{n!}{(s-b)^{n+1}}$

$$\therefore \quad L[(1 + t\, e^{-t})^3] = \frac{1}{s} + 3 \cdot \frac{1}{(s+1)^2} + 3 \cdot \frac{2!}{(s+2)^3} + \frac{3!}{(s+3)^4}.$$

(v) $$L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2} L(\sin at) = \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$$

$$= -\frac{(s^2+a^2)^2 \cdot 2a - 2as \cdot 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

(vi) $$L\left[t \cdot e^{-4t} \sin 3t\right] = (-1)\frac{d}{ds} L\left[e^{-4t} \sin 3t\right]$$

$$= -\frac{d}{ds}\left[\frac{3}{(s+4)^2+9}\right] \qquad \text{[By shifting theorem]}$$

$$= -\frac{d}{ds}\left[\frac{3}{s^2+8s+25}\right] = \frac{6(s+4)}{(s^2+8s+25)^2}$$

(vii) $$L\left[t^2 \cdot e^{-t} \sin 4t\right] = (-1)^2 \cdot \frac{d^2}{ds^2} L\left[e^{-t} \sin 4t\right]$$

$$= \frac{d^2}{ds^2}\left[\frac{4}{(s+1)^2+16}\right] \qquad \text{[By shifting theorem]}$$

$$= \frac{d^2}{ds^2}\left[\frac{4}{s^2+2s+17}\right] = \frac{8(3s^2+6s-13)}{(s^2+2s+17)^3}.$$

Example 2 : Find the Laplace transforms of the following.

(i) $t \sin^3 t$ **(M.U. 1994, 2015)** (ii) $t \sin 2t \cos ht$ **(M.U. 1995)**

(iii) $t \cos^2 t$ **(M.U. 1993, 2012)** (iv) $t^5 \cos ht$ **(M.U. 1996)**

(v) $t\, e^{3t} \sin 4t$ **(M.U. 1997, 2016)** (vi) $t^n e^{at}$.

Sol. : (i) $\sin 3t = 3 \sin t - 4 \sin^3 t$

$$L\left[\sin^3 t\right] = \frac{1}{4}\left[L(3\sin t) - L(\sin 3t)\right] = \frac{1}{4}\cdot\left[\frac{3}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$\therefore \quad L\left[t \sin^3 t\right] = -\frac{3}{4}\frac{d}{ds}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] = -\frac{3}{4}\left[-\frac{2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\cdot\frac{16(s+5)}{(s^2+1)^2(s^2+9)^2} = 24\cdot\frac{s(s+5)}{(s^2+1)^2(s^2+9)^2}$$

(ii) Since $\cos ht = \dfrac{e^t + e^{-t}}{2}$, $\quad \sin 2t \cos ht = \dfrac{e^t + e^{-t}}{2} \cdot \sin 2t$

Now $$t \sin 2t \cos ht = (-1)\frac{d}{ds} L(\sin 2t \cos ht) = (-1)\frac{d}{ds}\left[\frac{1}{2}\left(e^t \sin 2t + e^{-t} \sin 2t\right)\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{2}{(s-1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\right]$$

$$= \frac{2s-2}{(s^2-2s+5)^2} + \frac{2s+2}{(s^2+2s+5)^2}$$

$$\therefore \quad t \sin 2t \cos ht = 2\left[\frac{s-1}{(s^2-2s+5)^2} + \frac{s+1}{(s^2+2s+5)^2}\right]$$

(iii) $$L(t\cos^2 t) = L\,t\left(\frac{1+\cos 2t}{2}\right) = \frac{1}{2}L(t) + \frac{1}{2}L(t\cos 2t)$$

$$= \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\cdot\frac{d}{ds}L(\cos 2t) = \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\frac{d}{ds}\cdot\frac{s}{s^2+2^2}$$

$$= \frac{1}{2s^2} - \frac{1}{2}\left[\frac{s^2+2^2-s\cdot 2s}{(s^2+2^2)^2}\right] = \frac{1}{2s^2} + \frac{1}{2}\cdot\frac{s^2-2^2}{(s^2+2^2)^2}$$

Similarly, $$L(t\sin^2 t) = \frac{1}{2s^2} - \frac{1}{2}\left[\frac{s^2-2^2}{(s^2+2^2)^2}\right]$$

(iv) $$L(t^5 \cos ht) = L\,t^5\left(\frac{e^t+e^{-t}}{2}\right) = \frac{1}{2}L\,t^5(e^t) + \frac{1}{2}L\,t^5(e^{-t})$$

$$= \frac{1}{2}\cdot(-1)^5\cdot\frac{d^5}{ds^5}\left(\frac{1}{s-1}\right) + \frac{1}{2}\cdot(-1)^5\cdot\frac{d^5}{ds^5}\left(\frac{1}{s+1}\right)$$

But $$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \frac{d^2}{dx^2}\left(\frac{1}{x}\right) = \frac{2}{x^3}, \quad \frac{d^3}{dx^3}\left(\frac{1}{x}\right) = -\frac{2\cdot 3}{x^4},$$

$$\frac{d^4}{dx^4}\left(\frac{1}{x}\right) = \frac{2\cdot 3\cdot 4}{x^5} = \frac{4!}{x^5}, \quad \frac{d^5}{dx^5}\left(\frac{1}{x}\right) = -\frac{5!}{x^6}.$$

$$\therefore \quad L(t^5 \cos ht) = -\frac{1}{2}\cdot\frac{(-5!)}{(s-1)^6} - \frac{1}{2}\cdot\frac{(-5!)}{(s+1)^6} = 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

(v) $$L(\sin 4t) = \frac{4}{s^2+4^2} \qquad \therefore \text{ By first shifting theorem}$$

$$L\left[e^{3t}\sin 4t\right] = \frac{4}{(s-3)^2+4^2} = \frac{4}{s^2-6s+25}$$

$$\therefore \quad L\left[t\,e^{3t}\sin 4t\right] = -\frac{d}{ds}\left(\frac{4}{s^2-6s+25}\right) = \frac{4(2s-6)}{(s^2-6s+25)^2} = \frac{8(s-3)}{(s^2-6s+25)^2}.$$

(vi) $$L\,e^{at} = \frac{1}{s-a}$$

$$L\left[t^n e^{at}\right] = (-1)^n\cdot\frac{d^n}{ds^n}\cdot\frac{1}{(s-a)} = (-1)^n\cdot\frac{n!}{(s-a)^{n+1}}$$

Example 3 : Find the Laplace transforms of the following.

(i) $t\,e^{3t}\sin t$ **(M.U. 1998, 2002, 13)** (ii) $t\sqrt{1+\sin t}$ **(M.U. 1999, 2000, 02, 07, 13, 15)**

(iii) $t^3 \cos t$ **(M.U. 2014)** (iv) $t\,e^{3t}\mathrm{erf}\,\sqrt{t}$. **(M.U. 2000)** (v) $t\left(\dfrac{\sin t}{e^t}\right)^2$ **(M.U. 2004)**

Sol. : (i) We have $$L\sin t = \frac{1}{s^2+1}$$

$$\therefore \quad L(e^{3t}\sin t) = \frac{1}{(s-3)^2+1} = \frac{1}{s^2-6s+10} \qquad \text{[By first shifting theorem]}$$

$$\therefore \quad L(t e^{3t} \sin t) = -\frac{d}{ds}\left(\frac{1}{s^2-6s+10}\right) = \frac{2s-6}{(s^2-6s+10)^2}.$$

(ii) We have

$$\sqrt{1+\sin t} = \sqrt{\left[\sin^2(t/2)+\cos^2(t/2)+2\sin(t/2)\cos(t/2)\right]}$$

$$= \sqrt{\left[\sin(t/2)+\cos(t/2)\right]^2} = \sin(t/2)+\cos(t/2)$$

$$\therefore \quad L\sqrt{1+\sin t} = L\left[\sin(t/2)+\cos(t/2)\right] = \frac{1/2}{s^2+(1/2)^2}+\frac{s}{s^2+(1/2)^2}$$

$$= \frac{1}{2}\cdot\frac{4}{(4s^2+1)}+\frac{4s}{(4s^2+1)} = \frac{4s+2}{(4s^2+1)} = \frac{2(2s+1)}{(4s^2+1)}$$

$$\therefore \quad L\left[t\sqrt{1+\sin t}\right] = -\frac{d}{ds}\left[\frac{2(2s+1)}{(4s^2+1)}\right] = -2\left[\frac{(4s^2+1)2-(2s+1)8s}{(4s^2+1)^2}\right]$$

$$= -2\frac{[-8s^2-8s+2]}{(4s^2+1)^2} = 4\frac{(4s^2+4s-1)}{(4s^2+1)^2}$$

(iii) $L\cos t = \frac{s}{s^2+1}$

$$\therefore \quad \frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1)\cdot 1 - s\cdot 2s}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$$

$$\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1)^2(-2s)-(1-s^2)\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}$$

$$\therefore \quad \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3} = \frac{2s^3-6s}{(s^2+1)^3}$$

$$\frac{d^3}{ds^3}\left(\frac{d}{s^2+1}\right) = \frac{(s^2+1)^3(6s^2-6)-(2s^3-6s)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}$$

$$= \frac{6\left[(s^2+1)(s^2-1)-(2s^3-6s)s\right]}{(s^2+1)^4} = -\frac{6(s^4-6s^2+1)}{(s^2+1)^4}$$

$$\therefore \quad L\left[t^3\cos t\right] = \frac{6(s^4-6s^2+1)}{(s^2+1)^4}.$$

(iv) We have $L(\mathrm{erf}\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$ $\quad\therefore\quad L(e^{3t}\mathrm{erf}\sqrt{t}) = \frac{1}{(s-3)\sqrt{s-2}}$

$$L(t e^{2t}\mathrm{erf}\sqrt{t}) = -\frac{d}{ds}\left[\frac{1}{(s-3)\sqrt{s-2}}\right] = -\left[-\frac{1}{(s-3)^2(s-2)}\frac{d}{ds}\left\{(s-3)\sqrt{s-2}\right\}\right]$$

$$= \frac{1}{(s-3)^2(s-2)}\left[(s-3)\frac{1}{2\sqrt{s-2}}+\sqrt{s-2}\right]$$

$$= \frac{1}{(s-3)^2(s-2)}\cdot\frac{(s-3+2s-4)}{2\sqrt{s-2}} = \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}$$

(v) We have

$$t\left(\frac{\sin t}{e^t}\right)^2 = t\cdot e^{-2t}\sin^2 t = t\cdot e^{-2t}\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\cdot t\cdot e^{-2t}\left[1-\cos 2t\right]$$

Now $L(1-\cos 2t) = L(1)-L(\cos 2t) = \frac{1}{s}-\frac{s}{s^2+2^2}$

By first shifting theorem,

$$L\{e^{-2t}(1-\cos 2t)\} = \frac{1}{s+2}-\frac{s+2}{(s+2)^2+2^2}$$

$$\therefore \quad L\{e^{-2t}(1-\cos 2t)\} = \frac{1}{s+2}-\frac{s+2}{s^2+4s+8}$$

$$\therefore \quad L\left\{\frac{1}{2}\cdot t\cdot e^{-2t}(1-\cos 2t)\right\} = -\frac{1}{2}\frac{d}{ds}\left\{\frac{1}{s+2}-\frac{s+2}{s^2+4s+8}\right\}$$

$$= -\frac{1}{2}\left[-\frac{1}{(s+2)^2}-\frac{(s^2+4s+8)\cdot 1-(s+2)(2s+4)}{(s^2+4s+8)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s+2)^2}+\frac{s^2+4s}{(s^2+4s+8)^2}\right]$$

Example 4 : If $L[\mathrm{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$, find

(i) $L[t\,\mathrm{erf}\,2\sqrt{t}]$ **(M.U. 2003, 06)** (ii) $L[t e^{3t}\mathrm{erf}\sqrt{t}]$

Sol. : (i) By change of scale property if $L[f(t)] = \Phi(s)$, then $L[f(at)] = \frac{1}{a}\Phi\left(\frac{s}{a}\right)$.

Since, $L\left[\mathrm{erf}\sqrt{t}\right] = \frac{1}{s\sqrt{s+1}}$.

$$L\left[\mathrm{erf}\,2\sqrt{t}\right] = L\left[\mathrm{erf}\sqrt{4t}\right] = \frac{1}{4}\frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}} = \Phi(s)$$

(We have found $L\left[\mathrm{erf}\sqrt{t}\right]$ in (11), page 1-6.)

By the effect of multiplication of t

$$L\left[t f(t)\right] = -\Phi'(s) = -\frac{d}{ds}\left[2(s^3+4s^2)^{-1/2}\right]$$

$$\therefore \quad L\left[t f(t)\right] = -2\cdot\left(-\frac{1}{2}\right)(s^3+4s^2)^{-3/2}(3s^2+8s)$$

$$\therefore \quad L\left[t\,\mathrm{erf}\,2\sqrt{t}\right] = \frac{s(3s+8)}{s^3(s+4)^{3/2}} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

(ii) Since, $L\left(\mathrm{erf}\sqrt{t}\right) = \frac{1}{s\sqrt{s+1}}$

$$L\left[e^{3t}\cdot\mathrm{erf}\sqrt{t}\right] = \frac{1}{(s-3)\sqrt{(s-3)+1}} = \frac{1}{(s-3)\sqrt{s-2}}$$

$$= -\frac{d}{dt}\left[\frac{1}{(s-3)\sqrt{s-2}}\right] = \frac{1}{(s-3)^2(s-2)}\frac{d}{dt}\left[(s-3)\sqrt{s-2}\right]$$

$$\therefore\ L\left[t e^{3t}\cdot \operatorname{erf}\sqrt{t}\right]=\frac{1}{(s-3)^2(s-2)}\frac{d}{dt}\left[(s-3)\sqrt{s-2}\right]$$

$$=\frac{1}{(s-3)^2(s-2)}\left[(s-3)\cdot\frac{1}{2\sqrt{s-2}}+\sqrt{s-2}\cdot 1\right]$$

$$=\frac{1}{2(s-3)^2(s-2)^{3/2}}[(s-3)+2(s-2)]$$

$$\therefore\ L\left[t e^{3t}\cdot \operatorname{erf}\sqrt{t}\right]=\frac{3s-7}{2(s-3)^2(s-2)^{3/2}}.$$

Example 5 : Find $L[(t+e^{-t}+\sin t)^2]$. (M.U. 20..)

Sol. : We have

$$(t+e^{-t}+\sin t)^2=t^2+e^{-2t}+\sin^2 t+2te^{-t}+2t\sin t+2e^{-t}\sin t$$

$$\therefore\ L[(t+e^{-t}+\sin t)^2]=L(t^2)+L(e^{-2t})+\frac{1}{2}[L(1)-L(\cos 2t)]$$
$$+2[L(e^{-t}t)+L(t\sin t)+L(e^{-t}\sin t)]$$

$$\because\ L(t^2)=\frac{2}{s^3},\quad L(e^{-2t})=\frac{1}{s+2},\quad L(1)=\frac{1}{s},$$

$$L[\cos 2t]=\frac{s}{s^2+2^2},\quad L(t)=\frac{1}{s^2},\quad L(e^{-t}t)=\frac{1}{(s+1)^2},$$

$$L(t\sin t)=-\frac{d}{ds}\left[\frac{1}{s^2+1}\right]=\frac{2s}{(s^2+1)^2},\quad L(e^{-t}\sin t)=\frac{1}{(s+1)^2+1}$$

$$\therefore\ L[(t+e^{-t}+\sin t)^2]=\frac{2}{s^3}+\frac{1}{s+2}+\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+2^2}\right]$$
$$+2\left[\frac{1}{(s+1)^2}+\frac{2s}{(s^2+1)^2}+\frac{1}{s^2+2s+2}\right].$$

EXERCISE - IX

(A) 1. If $L[f(t)]=\dfrac{s+3}{s^2+s+1}$, find $L[t f(2t)]$. (M.U. 2004) [Ans. : $\dfrac{s^2+12s+\ldots}{(s^2+2s+\ldots}$

2. If $L\left[\operatorname{erf}\sqrt{t}\right]=\dfrac{1}{s\sqrt{s+1}}$, find $L\left[t\ \operatorname{erf}\ 3\sqrt{t}\right]$. (M.U. 2002, 11) [Ans. : $\dfrac{9}{2}\cdot\dfrac{s+6}{s^2(s+9\ldots}$

(B) Find the Laplace transforms of the following.

1. $t^2\sin 3t$, 2. $t\cos^3 t$, 3. $t e^{-2t}\sin t$, 4. $t e^{3t}\sin 2t$, (M.U. 2005)

5. $t e^{3t}\sin 3t$, (M.U. 2004) 6. $t e^{-3t}\cos 2t$ 7. $t e^{3t}\sin^2 t$, 8. $t(2\sin 3t+e^{2t})$,

9. $t(2\sin 3t-3\cos 3t)$, 10. $t e^{2t}(\cos t-\sin t)$,

11. $\frac{1}{4}e^{-2t}(2t\cos 2t+\sin 2t)$, 12. $t e^{t}\sin 2t\cos t$, (M.U. 2003)

13. $t\sqrt{1-\sin t}$, (M.U. 2000) 14. $t e^{-2t}\sinh 4t$. (M.U. 2000, 06)

15. $t e^{3t}\sinh 2t$ (M.U. 2003) 16. $t\ \operatorname{erf}\ 2\sqrt{t}$

17. $t\sqrt{1+\sin 2t}$ (M.U. 2003, 05, 07, 13)

18. $t\cos(wt-\alpha)$ (M.U. 2003) 19. $(t+\sin 2t)^2$ (M.U. 2004)

20. $(t\sinh 2t)^2$ (M.U. 2003) 21. $t e^{3t}\sinh 2t$ (M.U. 2003)

22. $t e^{-3t}\cos 2t\cos 3t$ (M.U. 2014) 23. $t e^{-t}\cosh 2t$ (M.U. 2015)

[**Ans. :** (1) $18\dfrac{(s^2-3)}{(s^2+9)^3}$, (2) $\dfrac{1}{4}\left[\dfrac{s^2-9}{(s^2+9)^2}+\dfrac{3(s^2+1)}{(s^2+1)^2}\right]$, (3) $\dfrac{2(s+2)}{(s^2+4s+5)^2}$,

(4) $4\cdot\dfrac{s-3}{(s^2-6s+13)^2}$, (5) $6\cdot\dfrac{(s-3)}{(s^2-6s+18)^2}$, (6) $\dfrac{s^2+6s+5}{(s^2+6s+13)^2}$,

(7) $\dfrac{1}{2}\left[\dfrac{1}{(s-3)^2}-\dfrac{(s^2-6s+5)}{(s^2-6s+13)^2}\right]$, (8) $\dfrac{12s}{(s^2+9)^2}+\dfrac{1}{(s-2)^2}$, (9) $\dfrac{3(9+4s-s^2)}{(s^2+9)^2}$,

(10) $\dfrac{s^2-6s+7}{(s^2-4s+5)^2}$, (11) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$, (12) $\dfrac{3(s-1)}{(s^2-2s+10)^2}+\dfrac{s-1}{(s^2-2s+2)}$,

(13) $4\cdot\dfrac{(1+4s-4s^2)}{(4s^2+1)^2}$, (14) $\dfrac{8(s+2)}{(s^2+4s-12)^2}$, (15) $\dfrac{4(s-3)}{(s^2-6s+5)^2}$

(16) $\dfrac{3s+8}{s^2(s+4)^{3/2}}$, (17) $\dfrac{s^2+2s-1}{(s^2+1)^2}$, (18) $\dfrac{(s^2-w^2)\cos\alpha+2ws\sin\alpha}{(s^2+w^2)^2}$,

(19) $\dfrac{2}{s^3}+\dfrac{8s}{(s^2+2^2)^2}+\dfrac{1}{2s}-\dfrac{s}{2(s^2+4^2)}$, (20) $\dfrac{1}{2}\left[\dfrac{1}{(s-4)^3}-\dfrac{2}{s^3}+\dfrac{1}{(s+4)^3}\right]$,

(21) $\dfrac{1}{2}\left[\dfrac{1}{(s-5)^2}-\dfrac{1}{(s-1)^2}\right]$ (22) $\dfrac{1}{2}\left[\dfrac{s^2+6s-16}{(s^2+6s+34)^2}+\dfrac{s^2+6s+8}{(s^2+6s+10)^2}\right]$ (23) $\dfrac{s^2+2s+5}{(s^2+2s-3)^2}$

11. Evaluation of the Integral $\int_0^\infty e^{-at}\, t f(t)\, dt$

To find the above integral, we first find $L[t f(t)]$, say $\Phi(s)$.

This means, $\int_0^\infty e^{-st}\cdot t f(t)\,dt=\Phi(s)$.

Now, we put $s=a$.

Example 1 : Evaluate $\int_0^\infty e^{-3t}t\sin t\cdot dt$. (M.U. 2002, 05)

Sol. : Consider $L[t\sin t]=(-1)\dfrac{d}{ds}[L(\sin t)]=-\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right)=\dfrac{2s}{(s^2+1)^2}$

That means, $\int_0^\infty e^{-st}t\sin t\,dt=\dfrac{2s}{(s^2+1)^2}$.

Putting $s=3$, $\int_0^\infty e^{-3t}\cdot t\sin t\cdot dt=\dfrac{6}{(9+1)^2}=\dfrac{3}{50}$

Example 2 : Evaluate $\int_0^\infty e^{-t}\left(t\sqrt{1+\sin t}\right)\cdot dt$. (M.U. 2016)

Sol. : We have proved in Ex. 3 (ii), page 1-31 that

$$L(t\sqrt{1+\sin t}) = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

By definition of Laplace transform this means,

$$\int_0^{\infty} e^{-st}[t\sqrt{1+\sin t}]\,dt = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

Putting $s = 1$ on both sides, we get

$$\int_0^{\infty} e^{-t}(t\sqrt{1+\sin t})\cdot dt = \frac{4(4+4-1)}{(4+1)^2} = \frac{28}{25}.$$

Example 3 : Evaluate $\int_0^{\infty} e^{-3t}\, t\cos t \cdot dt$. **(M.U. 2004, 06, 15)**

Sol. : Consider $L[t\cos t] = -\frac{d}{ds}L(\cos t) = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$

$$= -\left[\frac{(s^2+1)-s\cdot 2s}{(s^2+1)^2}\right] = \frac{s^2-1}{(s^2+1)^2}$$

That means, $\int_0^{\infty} e^{-st}\, t\cos t\, dt = \frac{s^2-1}{(s^2+1)^2}$

Putting $s = 3$, $\int_0^{\infty} e^{-3t}\, t\cos t \cdot dt = \frac{8}{100} = \frac{2}{25}$.

Example 4 : Evaluate $\int_0^{\infty} e^{-t}\, t^3 \sin t\, dt$. **(M.U. 1996)**

Sol. : Consider $L[t^3 \sin t] = (-1)^3 \frac{d^3}{ds^3} L(\sin t) = (-1)\frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$

Now $\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = -\frac{2s}{(s^2+1)^2}$

$$\frac{d}{ds}\left[-\frac{2s}{(s^2+1)^2}\right] = -2\left[\frac{(s^2+1)^2\cdot 1 - s\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{(s^2+1)-4s^2}{(s^2+1)^3}\right] = -2\frac{(-3s^2+1)}{(s^2+1)^3} = 2\frac{(3s^2-1)}{(s^2+1)^3}$$

$$\frac{d}{ds}\left[2\frac{(3s^2-1)}{(s^2+1)^3}\right] = 2\left[\frac{(s^2+1)^3\cdot 6s - (3s^2-1)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}\right]$$

$$= 2\left[\frac{6s(s^2+1)-6s(3s^2-1)}{(s^2+1)^4}\right] = -24s\cdot\frac{(s^2-1)}{(s^2+1)^4}$$

This means, $\int_0^{\infty} e^{-st}\, t^3 \sin t\, dt = 24s\cdot\frac{(s^2-1)}{(s^2+1)^4}$

To find the value of the given integral we put $s = 1$.

$$\therefore \int_0^{\infty} e^{-t}\, t^3 \sin t\, dt = 0.$$

Example 5 : If $L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$, prove that

(i) $\int_0^{\infty} J_0(t)\, dt = 1$. (ii) $L[t\, J_0(at)] = \frac{s}{(s^2+a^2)^{3/2}}$.

(iii) $L\left[e^{-bt} J_0(at)\right] = \frac{1}{\sqrt{(s+b)^2+a^2}}$. **(M.U. 1998)**

(iv) $\int_0^{\infty} t\, e^{-3t} J_0(4t)\, dt = \frac{3}{125}$. **(M.U. 1996, 2003)**

Sol. : (i) By definition of Laplace transform since $L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$,

$$\int_0^{\infty} e^{-st} J_0(t)\, dt = \frac{1}{\sqrt{1+s^2}}.$$

Putting $s = 0$, we get $\int_0^{\infty} J_0(t)\, dt = \frac{1}{\sqrt{1+0}} = \frac{1}{1} = 1$.

(ii) Since $J_0(t) = \frac{1}{\sqrt{1+s^2}}$, by change of scale property (§ 6 page 1-17).

$$J_0(at) = \frac{1}{a}\cdot\frac{1}{\sqrt{1+(s/a)^2}} = \frac{1}{\sqrt{s^2+a^2}}$$

$\therefore$ $L[t\, J_0(at)] = (-1)\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}}\right)$ [By § 10 (16) page 1-28]

$$= (-1)\left(-\frac{1}{2}\right)(s^2+a^2)^{-3/2}\cdot 2s = \frac{s}{(s^2+a^2)^{3/2}}$$

(iii) Since $J_0(at) = \frac{1}{\sqrt{s^2+a^2}}$, by first shifting theorem (§ 7 page 1-19).

$$L\left[e^{-bt} J_0(at)\right] = \frac{1}{\sqrt{(s+b)^2+a^2}}.$$

(iv) Now putting $a = 4$ in (ii) above $L[t\, J_0(4t)] = \frac{s}{(s^2+16)^{3/2}}$.

By definition of Laplace transform this means

$$\int_0^{\infty} e^{-st}\, t\, J_0(4t)\, dt = \frac{s}{(s^2+16)^{3/2}}$$

Putting $s = 3$, we get $\int_0^{\infty} e^{-3t}\, t\, J_0(4t)\, dt = \frac{3}{(9+16)^{3/2}} = \frac{3}{125}$.

Example 6 : Evaluate $\int_0^{\infty} \frac{t^2 \sin 3t}{e^{2t}}\, dt$. **(M.U. 2002, 07, 09)**

Sol. : Consider $\int_0^{\infty} e^{-st}\, t^2 \sin 3t\, dt = L(t^2 \sin 3t)$

$$\therefore \int_0^\infty e^{-st} t^2 \sin 3t\, dt = (-1)^2 \frac{d^2}{ds^2}[L(\sin 3t)] = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{d}{ds}\left(-\frac{3 \cdot 2s}{(s^2+9)^2}\right) = -6\left[\frac{(s^2+9)^2 \cdot 1 - s \cdot 2(s^2+9) \cdot 2s}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right] = \frac{-6 \cdot (-3s^2+9)}{(s^2+9)^3} = \frac{18(s^2-3)}{(s^2+9)^3}$$

This means, $\int_0^\infty e^{-st} t^2 \sin 3t\, dt = \frac{18(s^2-3)}{(s^2+9)^3}$

Now putting $s = 2$, we get

$$\int_0^\infty e^{-2t} t^2 \sin 3t\ dt = \frac{18(4-3)}{(4+9)^3} = \frac{18}{2197}.$$

Example 7 : Given $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$, evaluate $\int_0^\infty t \cdot e^{-t} erf(\sqrt{t})\, dt$. **(M.U. 2009)**

Sol. : Since $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$,

$$L(t\, erf\sqrt{t}) = -\frac{d}{ds}\left(\frac{1}{s\sqrt{s+1}}\right) = -\frac{d}{ds}[(s^2+s)^{-1/2}]$$

$$\therefore\ L(t\, erf\sqrt{t}) = -\left[-\frac{1}{2}(s^2+s)^{-3/2}(2s+1)\right] = \frac{1}{2}(s^2+s)^{-3/2}(2s+1)$$

$$\therefore\ \int_0^\infty e^{-st} \cdot t \cdot erf(\sqrt{t})\, dt = \frac{1}{2}(s^2+s)^{-3/2}(2s+1)$$

Now, put $s = 1$,

$$\therefore\ \int_0^\infty e^{-t} \cdot t \cdot erf(\sqrt{t})\, dt = \frac{1}{2}(1+1)^{-3/2}(2+1) = \frac{1}{2 \cdot 2\sqrt{2}} \cdot (3) = \frac{3}{4\sqrt{2}}.?$$

EXERCISE - X

(A) Evaluate the following using Laplace transform.

1. $\int_0^\infty e^{-2t} t^3 \sin t\, dt.$ **(M.U. 2004)**
2. $\int_0^\infty e^{-2t} t \cos t\, dt.$
3. $\int_0^\infty e^{-2t} t \sin^2 t\, dt.$
4. $\int_0^\infty e^{-3t} t^2 \sin h\, 2t\, dt.$

[**Ans. :** (1) $\frac{144}{625}$, (2) $\frac{3}{25}$, (3) $\frac{1}{8}$, (4) $\frac{124}{125}$

(B) If $L[J_o(t)] = \frac{1}{\sqrt{s^2+1}}$, then evaluate the following

1. $\int_0^\infty e^{-t} J_o(t)\, dt.$
2. $\int_0^\infty e^{-4t} t\, J_o(3t)\, dt.$

[**Ans.** (1) $\frac{1}{\sqrt{2}}$, (2) $\frac{4}{125}$

(C) If $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$, evaluate $\int_0^\infty e^{-2t} erf(2\sqrt{t})\, dt.$ **(M.U. 2003)** [**Ans. :** $1/\sqrt{6}$]

12. Effect of Division by t

If $L[f(t)] = \Phi(s)$, then $\boxed{L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \Phi(s)\, ds}$ **(M.U. 1994, 2005, 08)** .... **(17)**

Proof : We have $\Phi(s) = \int_0^\infty e^{-st} f(t)\, dt$

Integrating both sides w.r.t. s between the limits s, ∞ and changing the order of integration on r.h.s.

$$\int_s^\infty \Phi(s)\, ds = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds\right] dt = \int_0^\infty \left[\frac{e^{-st}}{-t} f(t)\right]_s^\infty dt$$

$$= \int_s^\infty e^{-st} \frac{f(t)}{t}\, dt = L\left[\frac{1}{t} f(t)\right]$$

Example 1 : Find the Laplace transform of the following.

(i) $\frac{1}{t}(1 - \cos t)$ **(M.U. 1995, 2004)** (ii) $\frac{1}{t} e^{-t} \sin t$ **(M.U. 2003)**

(iii) $\frac{1}{t}(e^{-at} - e^{-bt})$ **(M.U. 1997)**

Sol. : (i) $L[1 - \cos t] = L(1) - L(\cos t) = \frac{1}{s} - \frac{s}{s^2+1}$

By (17), we get

$$L\left[\frac{1}{t}(1 - \cos t)\right] = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds = \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log(s^2+1) - \log s^2\right]_s^\infty = -\frac{1}{2}\left[\log\frac{s^2+1}{s^2}\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log\left(1 + \frac{1}{s^2}\right)\right]_s^\infty \qquad \textbf{[Note this carefully]}$$

$$= -\frac{1}{2}\left[\log 1 - \log\left(1 + \frac{1}{s^2}\right)\right] = \frac{1}{2}\log\left(1 + \frac{1}{s^2}\right) = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

(ii) $L(e^{-t} \sin t) = \frac{1}{(s+1)^2+1}$

By (17), we get

$$L\left[\frac{1}{t}(e^{-t} \sin t)\right] = \int_s^\infty \frac{1}{(s+1)^2+1}\, ds = \left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

(iii) $L\left(e^{-at} - e^{-bt}\right) = \frac{1}{s+a} - \frac{1}{s+b}$

By (17), we get

$$L\left[\frac{1}{t}\left(e^{-at}-e^{-bt}\right)\right] = \int_s^\infty \left[\frac{1}{s+a}-\frac{1}{s+b}\right]ds = \left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$= \left[\log\left\{\frac{s+a}{s+b}\right\}\right]_s^\infty = -\log\left(\frac{s+a}{s+b}\right) = \log\left(\frac{s+b}{s+a}\right)$$

Example 2 : Find the Laplace transform of the following.

(i) $\dfrac{\sin^2 2t}{t}$ **(M.U. 1993, 2016)** (ii) $\dfrac{1-\cos t}{t^2}$ **(S.U. 2011)**

(iii) $\dfrac{e^{-2t}\sin 2t \cos ht}{t}$ **(M.U. 1996, 2003)**

Sol. : (i) $\sin^2 2t = \dfrac{1-\cos 4t}{2}$.

$$\therefore \quad L\sin^2 2t = L\frac{1}{2}(1-\cos 4t) = \frac{1}{2}\left[L(1)-L(\cos 4t)\right] = \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4^2}\right]$$

By (17), we get

$$L\left[\frac{\sin^2 2t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{1}{s}-\frac{s}{s^2+4^2}\right]ds = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4^2)\right]_s^\infty$$

$$= -\frac{1}{4}\left[\log(s^2+4^2)-\log s^2\right]_s^\infty = -\frac{1}{4}\left[\log\left(\frac{s^2+4^2}{s^2}\right)\right]_s^\infty = \frac{1}{4}\log\left(\frac{s^2+4^2}{s^2}\right)$$

(ii) $L[1-\cos t] = L(1) - L(\cos t) = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

By (17), we get

$$L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty \left[\frac{1}{s}-\frac{s}{s^2+1}\right]ds = \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty = -\frac{1}{2}\log\frac{s^2}{s^2+1} = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

By (17) again, we get

$$L\left(\frac{1-\cos t}{t^2}\right) = \int_s^\infty \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)ds$$

Integrating by parts

$$L\left(\frac{1-\cos t}{t^2}\right) = \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2}\right)\cdot s - \int s\cdot\left(\frac{s^2}{s^2+1}\right)\left(\frac{s^2\cdot 2s-(s^2+1)2s}{s^4}\right)ds\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right) + 2\int\frac{ds}{s^2+1}\right]_s^\infty = \frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right) + 2\tan^{-1}s\right]_s^\infty$$

$$\therefore \quad L\left(\frac{1-\cos t}{t^2}\right) = \frac{1}{2}\left[0+2\cdot\frac{\pi}{2} - s\log\left(\frac{s^2+1}{s^2}\right) - 2\tan^{-1}s\right]$$

$$= \frac{\pi}{2} - \frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right) - \tan^{-1}s.$$

(iii) We have, $e^{-2t}\sin 2t\cos ht = e^{-2t}\sin 2t\,\dfrac{(e^t+e^{-t})}{2} = \dfrac{1}{2}\left(e^{-t}\sin 2t + e^{-3t}\sin 2t\right)$

$$\therefore \quad L\left[e^{-2t}\sin 2t\cos ht\right] = \frac{1}{2}L\left(e^{-t}\sin 2t + e^{-3t}\sin 2t\right)$$

But $\quad L\sin 2t = \dfrac{2}{s^2+2^2}$

$$\therefore \quad L\left(e^{-t}\sin 2t\right) = \frac{2}{(s+1)^2+2^2} = \frac{2}{s^2+2s+5}$$

$$L\left(e^{-3t}\sin 2t\right) = \frac{2}{(s+3)^2+2^2} = \frac{2}{s^2+6s+13}$$

$$\therefore \quad L\left[e^{-2t}\sin 2t\cos ht\right] = \frac{1}{s^2+2s+5} + \frac{1}{s^2+6s+13}$$

$$\therefore \quad L\left[\frac{e^{-2t}\sin 2t\cos ht}{t}\right] = \int_s^\infty\left(\frac{1}{s^2+2s+5}+\frac{1}{s^2+6s+13}\right)ds$$

$$= \int_s^\infty\left[\frac{1}{(s+1)^2+2^2}+\frac{1}{(s+3)^2+2^2}\right]ds$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{s+1}{2}+\frac{1}{2}\tan^{-1}\frac{s+3}{2}\right]_s^\infty$$

$$= \frac{1}{2}\cdot\left(\frac{\pi}{2}-\tan^{-1}\frac{s+1}{2}\right)+\frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}\frac{s+3}{2}\right)$$

$$= \frac{\pi}{2}-\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right)-\frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right).$$

Example 3 : Find the Laplace transform of $\dfrac{\sin at}{t}$. Does Laplace transform of $\dfrac{\cos at}{t}$ exist? **(M.U. 2015)**

Sol. : Consider $f(t) = \sin at \quad \therefore \quad L[f(t)] = \dfrac{a}{s^2+a^2} = \Phi(s)$

$$\therefore \quad L\left[\frac{\sin at}{t}\right] = \int_s^\infty \Phi(s)\,ds \qquad \text{[By (17), page 1-39]}$$

$$= \int_s^\infty \frac{a}{s^2+a^2}ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2}-\tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}.$$

Now consider, $f(t) = \cos at$ $\quad \therefore \quad L[f(t)] = \dfrac{s}{s^2+a^2} = \Phi(s)$

$$\therefore \quad L\left[\frac{\cos at}{t}\right] = \int_s^\infty \Phi(s)\, ds = \int_s^\infty \frac{s}{s^2+a^2}\, ds = \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty$$

Since $\log(s^2+a^2)$ is infinite when $s \to \infty$, $L\left[\dfrac{\cos at}{t}\right]$ does not exist.

Example 4 : Find $L\left[\dfrac{\sin^2 t}{t^2}\right]$. (M.U. 2009)

Sol. : $$L(\sin^2 t) = L\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\left[L(1) - L\cos 2t\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\therefore \quad L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty = -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right) = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

$$\therefore \quad L\left[\frac{\sin^2 t}{t^2}\right] = \int_s^\infty \frac{1}{4}\cdot\log\left(\frac{s^2+4}{s^2}\right) ds$$

Integrating by parts,

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left[\log\left(\frac{s^2+4}{s^2}\right)\cdot s - \int s \cdot \frac{s^2}{s^2+4}\left(\frac{s^2\cdot 2s - (s^2+4)\cdot 2s}{s^4}\right) ds\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 8\int\frac{ds}{s^2+4}\right]_s^\infty = \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 2\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[0 + 2\cdot\frac{\pi}{2} - s\log\left(\frac{s^2+4}{s^2}\right) - 2\tan^{-1}\frac{s}{2}\right]$$

$$\therefore \quad L\left[\frac{\sin^2 t}{t^2}\right] = \frac{\pi}{4} - \frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right) - \frac{1}{2}\cdot\tan^{-1}\left(\frac{s}{2}\right).$$

EXERCISE - XI

Find the Laplace transform of the following.

1. $\dfrac{\sin t}{t}$ (M.U. 1994, 2015)
2. $\dfrac{1}{t}[1-\cos at]$
3. $\dfrac{1}{t}\left[e^{-t}\sin at\right]$ (M.U. 2013)
4. $\dfrac{1}{t}\left[1-e^{2t}\right]$
5. $\dfrac{1}{t}\left[e^{-3t}\sin 2t\right]$
6. $\dfrac{1}{t}\left[\sin^2 t\right]$ (M.U. 2003, 04)
7. $\dfrac{1}{t}\left[e^{2t}\sin^3 t\right]$ (M.U. 2006)
8. $\dfrac{1}{t}[\cos at - \cos bt]$ (M.U. 2010, 13)
9. $\dfrac{1}{t}[1-\cos 3t]$
10. $\dfrac{1}{t}\left[e^{-2t}\sin 3t\right]$
11. $\dfrac{1}{t}(\sin^3 t)$
12. $\dfrac{1}{t}\left[e^{-t}\sin^3 t\right]$

13. $\dfrac{e^{2t}\sin t}{t}$ (M.U. 1999)
14. $\dfrac{\cosh 2t \sin 2t}{t}$ (M.U. 1999, 2009, 10)
15. $\dfrac{2\sin t\sin 2t}{t}$ (M.U. 2004, 11)
16. $\dfrac{\sin t\sin 5t}{t}$ (M.U. 2003, 07, 08)
17. $\dfrac{(1-\cos 2t)}{t}$ (M.U. 2003)
18. $\dfrac{\sinh at}{t}$ (M.U. 2004, 05)

[**Ans. :** (1) $\cot^{-1} s$, (2) $\dfrac{1}{2}\log\dfrac{(s^2+a^2)}{s^2}$, (3) $\cot^{-1}\left(\dfrac{s+1}{a}\right)$, (4) $\log\left(\dfrac{s-2}{s}\right)$,

(5) $\cot^{-1}\left(\dfrac{s+3}{2}\right)$, (6) $\dfrac{1}{4}\log\left(\dfrac{s^2+4}{s^2}\right)$, (7) $\dfrac{3}{4}\cot^{-1}(s-2) - \dfrac{1}{4}\cot^{-1}\left(\dfrac{s-2}{3}\right)$,

(8) $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$, (9) $\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2}\right)$, (10) $\cot^{-1}\left(\dfrac{s+2}{3}\right)$,

(11) $\dfrac{1}{4}\left[3\cot^{-1} s - \cot^{-1}\dfrac{s}{3}\right]$, (12) $\dfrac{1}{4}\left[3\cot^{-1}(s+1) - \cot^{-1}\left(\dfrac{s+1}{3}\right)\right]$,

(13) $\cot^{-1}(s-2)$, (14) $\dfrac{\pi}{2} - \dfrac{1}{2}\tan^{-1}\left(\dfrac{s-2}{2}\right) - \dfrac{1}{2}\tan^{-1}\left(\dfrac{s+2}{2}\right)$,

(15) $\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+1}\right)$, (16) $\dfrac{1}{4}\log\left(\dfrac{s^2+36}{s^2+16}\right)$,

(17) $\dfrac{1}{2}\log\left(\dfrac{s^2+4}{s^2}\right)$, (18) $\dfrac{1}{2}\log\left(\dfrac{s+a}{s-a}\right)$.]

13. Evaluation of the Integral $\int_0^\infty e^{-at}\dfrac{f(t)}{t}\,dt$

To evaluate the integral $\int_0^\infty e^{-at}\dfrac{f(t)}{t}\,dt$, we first find $L\left[\dfrac{f(t)}{t}\right]$, say $\Phi(s)$.

This means, $\int_0^\infty e^{-st}\dfrac{f(t)}{t}\,dt = \Phi(s)$

Now, we put $s = a$ on both sides.

Example 1 : Evaluate $\int_0^\infty \dfrac{e^{-t}\sin t}{t}\,dt$. (M.U. 2003, 08)

Sol. : We have proved in Ex. 3, page 1-41 that $L\left[\dfrac{\sin at}{t}\right] = \cot^{-1}\left(\dfrac{s}{a}\right)$.

This means $\int_0^\infty e^{-st}\cdot\dfrac{\sin at}{t}\,dt = \cot^{-1}\dfrac{s}{a}$

Now, put $s = 1$, $a = 1$, then $\int_0^\infty \dfrac{e^{-t}\sin t}{t}\,dt = \cot^{-1}(1) = \dfrac{\pi}{4}$.

Example 2 : Show that $\int_0^\infty \dfrac{\sin at}{t}\,dt = \dfrac{\pi}{2}$. (M.U. 1997, 2010)

Sol. : We have proved in Ex. 3, page 1-41 that $L\left[\dfrac{\sin at}{t}\right] = \cot^{-1}\left(\dfrac{s}{a}\right)$.

This means $\int_0^{\infty} e^{-st}\left(\frac{\sin at}{t}\right) dt = \cot^{-1}\frac{s}{a}$

Now put $s = 0$, $\therefore \int_0^{\infty} \frac{\sin at}{t}\, dt = \cot^{-1}(0) = \frac{\pi}{2}$.

Example 3 : Evaluate $\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t}\, dt$. (M.U. 20...)

Sol. : Consider $f(t) = e^{-at} - e^{-bt}$ $\therefore L[f(t)] = \frac{1}{s+a} - \frac{1}{s+b}$

$$\therefore L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty}\left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds = [\log(s+a) - \log(s+b)]_s^{\infty}$$

$$\therefore L\left[\frac{1}{t} f(t)\right] = -\left[\log\left(\frac{s+b}{s+a}\right)\right]_s^{\infty} = -\left[\log\frac{1+(b/s)}{1+(a/s)}\right]_s^{\infty}$$

$$= -\left[\log 1 - \log\frac{1+(b/s)}{1+(a/s)}\right] = \log\left(\frac{s+b}{s+a}\right)$$

This means $\int_0^{\infty} e^{-st}\left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \log\left(\frac{s+b}{s+a}\right)$

Putting $s = 0$, $\int_0^{\infty}\left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \log\frac{b}{a}$.

Example 4 : Evaluate $\int_0^{\infty} \frac{\cos at - \cos bt}{t}\, dt$. **(M.U. 2004, 07, 201...)**

Sol. : Consider $f(t) = \cos at - \cos bt$ $\therefore L[f(t)] = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$

$$\therefore L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty}\left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2 + a^2) - \log(s^2 + b^2)\right]_s^{\infty}$$

$$= -\frac{1}{2}\left[\log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)\right]_s^{\infty} = \frac{1}{2}\log\left(\frac{s^2 + b^2}{s^2 + a^2}\right) \quad \text{.............(1)}$$

By definition of Laplace transform the equation (1) means

$$\int_0^{\infty} e^{-st}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}\log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)$$

Putting $s = 0$, $\int_0^{\infty} \frac{\cos at - \cos bt}{t}\, dt = \frac{1}{2}\log\frac{b^2}{a^2} = \log\frac{b}{a}$.

Cor. : Putting $s = 1$, we get

$$\int_0^{\infty} e^{-t}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}\log\left(\frac{b^2 + 1}{a^2 + 1}\right)$$ **(M.U. 2005, 09...)**

Example 5 : Evaluate $\int_0^{\infty} e^{-t}\left(\frac{\cos 3t - \cos 2t}{t}\right) dt$. **(M.U. 2005, 07, 15)**

Sol. : Consider $f(t) = \cos 3t - \cos 2t$ $\therefore L[f(t)] = \frac{s}{s^2 + 9} - \frac{s}{s^2 + 4}$

$$\therefore L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty}\left(\frac{s}{s^2 + 9} - \frac{s}{s^2 + 4}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2 + 9) - \log(s^2 + 4)\right]_s^{\infty} = \frac{1}{2}\left[\log\left(\frac{s^2 + 9}{s^2 + 4}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{1 + (9/s^2)}{1 + (4/s^2)}\right)\right]_s^{\infty} = \frac{1}{2}\left[\log 1 - \log\left(\frac{1 + (9/s^2)}{1 + (4/s^2)}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2 + 9}{s^2 + 4}\right) = \frac{1}{2}\log\left(\frac{s^2 + 4}{s^2 + 9}\right)$$

This means,

$$\int_0^{\infty} e^{-st}\left(\frac{\cos 3t - \cos 2t}{t}\right) dt = \frac{1}{2}\log\left(\frac{s^2 + 4}{s^2 + 9}\right)$$

Putting $s = 1$, we get

$$\int_0^{\infty} e^{-t}\left(\frac{\cos 3t - \cos 2t}{t}\right) dt = \frac{1}{2}\log\left(\frac{5}{10}\right) = \frac{1}{2}\log\left(\frac{1}{2}\right)$$

Example 6 : Evaluate $\int_0^{\infty} e^{-st} \cdot \frac{\sin^2(at/2)}{t} \cdot dt$.

Sol. : We have $\sin^2\left(\frac{at}{2}\right) = \frac{1 - \cos at}{2}$

$$\therefore L\left(\sin^2\frac{at}{2}\right) = \frac{1}{2} L(1) - \frac{1}{2} L(\cos at) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s}{(s^2 + a^2)}$$

$$\therefore L\left(\frac{\sin^2(at/2)}{t}\right) = \int_s^{\infty}\frac{1}{2}\cdot\frac{1}{s}\, ds - \int_s^{\infty}\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2s}{s^2 + a^2}\, ds$$

$$= \left[\frac{1}{2}\log s - \frac{1}{4}\log(s^2 + a^2)\right]_s^{\infty} = \frac{1}{4}\left[\log\frac{s^2}{s^2 + a^2}\right]_s^{\infty}$$

$$= -\frac{1}{4}\log\left(\frac{s^2}{s^2 + a^2}\right) = \frac{1}{4}\log\left(\frac{s^2 + a^2}{s^2}\right)$$

This means $\int_0^{\infty} e^{-st} \cdot \frac{\sin^2(at/2)}{t}\, dt = \frac{1}{4}\log\left(\frac{s^2 + a^2}{s^2}\right)$.

Example 7 : Prove that $\int_0^{\infty} e^{-t} \cdot \frac{\sin^2 t}{t}\, dt = \frac{1}{4}\log 5$. **(M.U. 1999, 2000, 03, 04, 05, 2009)**

Sol. : In the above example put $s = 1$, $a = 2$.

$$\therefore \int_0^\infty e^{-t}\cdot\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log\left(\frac{1+4}{1}\right) = \frac{1}{4}\log 5$$

Or independently, consider $\sin^2 t = \dfrac{1-\cos 2t}{2}$.

$$\therefore L\sin^2 t = \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2+4}$$

$$\therefore L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty \frac{1}{2}\cdot\frac{1}{s}\,ds - \int_s^\infty \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2s}{s^2+4}\,ds$$

$$= \left[\frac{1}{2}\log s - \frac{1}{4}\log(s^2+4)\right]_s^\infty = \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= -\frac{1}{4}\log\frac{s^2}{s^2+4} = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right).$$

This means $\displaystyle\int_0^\infty e^{-st}\cdot\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

Now put $s = 1$, $\quad \therefore \displaystyle\int_0^\infty e^{-t}\cdot\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log 5.$

Example 8 : Prove that $\displaystyle\int_0^\infty e^{-st}\cdot\frac{\sin t\sin h\,at}{t} = \frac{1}{2}\tan^{-1}\left[\frac{2a}{1+(s^2-a^2)}\right].$

Sol. : We have, $\sin t \sin h\,at = \left(\dfrac{e^{at}-e^{-at}}{2}\right)\sin t$. Now $L\sin t = \dfrac{1}{s^2+1}$

$$\therefore L\left(e^{at}\sin t\right) = \frac{1}{(s-a)^2+1},\quad L\left(e^{-at}\sin t\right) = \frac{1}{(s+a)^2+1}$$

$$\therefore L\sin t\sin h\,at = \frac{1}{2}\left[\frac{1}{(s-a)^2+1} - \frac{1}{(s+a)^2+1}\right]$$

$$\therefore L\left(\frac{\sin t\sin h\,at}{t}\right) = \frac{1}{2}\int_s^\infty\left[\frac{1}{(s-a)^2+1} - \frac{1}{(s+a)^2+1}\right]ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s-a) - \tan^{-1}(s+a)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left\{\frac{\pi}{2} - \tan^{-1}(s-a)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(s+a)\right\}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+a) - \tan^{-1}(s-a)\right]$$

Now let $\tan^{-1}(s+a) = \alpha$, $\tan^{-1}(s-a) = \beta$

$\therefore \tan\alpha = s+a$, $\tan\beta = s-a$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{(s+a)-(s-a)}{1+(s^2-a^2)} = \frac{2a}{1+(s^2-a^2)}.$$

$$\therefore \alpha-\beta = \tan^{-1}\frac{2a}{1+(s^2-a^2)}$$

$$\therefore \tan^{-1}(s+a) - \tan^{-1}(s-a) = \tan^{-1}\frac{2a}{1+(s^2-a^2)}$$

$$\therefore L\left(\frac{\sin t\sin h\,at}{t}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1+(s^2-a^2)}\right)$$

This means $\displaystyle\int_0^\infty e^{-st}\cdot\frac{\sin t\sin h\,at}{t}\,dt = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1+(s^2-a^2)}\right).$

Example 9 : Evaluate $\displaystyle\int_0^\infty e^{-2t}\sin h\,t\,\frac{\sin t}{t}\,dt.$ **(M.U. 2002, 16)**

Sol. : We have $\sin h\,t\sin t = \left(\dfrac{e^t-e^{-t}}{2}\right)\sin t$

Now, $L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore L(e^t\sin t) = \frac{1}{(s-1)^2+1} \quad\text{and}\quad L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1}$$

$$\therefore L(\sin h\,t\sin t) = \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore L\left[\frac{\sin h\,t\cdot\sin t}{t}\right] = \frac{1}{2}\left[\int_0^\infty\frac{ds}{(s-1)^2+1} - \frac{ds}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s-1) - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left\{\frac{\pi}{2} - \tan^{-1}(s-1)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(s+1)\right\}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right]$$

But $\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right] = \tan^{-1}\left(\dfrac{2}{s^2}\right)$

[Let $\tan^{-1}(s+1) = \alpha$ and $\tan^{-1}(s-1) = \beta$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{(s+1)-(s-1)}{1+(s^2-1)} = \frac{2}{s^2}$$

$$\therefore \alpha-\beta = \tan^{-1}\left(\frac{2}{s^2}\right).]$$

$$\therefore L\left[\frac{\sin h\,t\sin t}{t}\right] = \frac{1}{2}\tan^{-1}\left(\frac{2}{s^2}\right)$$

This means,

$$\int_0^\infty e^{-st}\cdot\frac{\sin h\,t\sin t}{t}\,dt = \frac{1}{2}\tan^{-1}\left(\frac{2}{s^2}\right)$$

Putting $s = 2$, we get

$$\int_0^{\infty} e^{-2t} \cdot \frac{\sin h\, t \sin t}{t}\, dt = \frac{1}{2} \tan^{-1}\left(\frac{1}{2}\right).$$

Example 10 : Prove that $\int_0^{\infty} e^{-\sqrt{2}\, t} \frac{\sin t \sin h\, t}{t} = \frac{\pi}{8}$. **(M.U. 2002, 05, 07, 11, ...)**

Sol. : In the above example put $s = \sqrt{2}$, $a = 1$.

$$\therefore \int_0^{\infty} e^{-\sqrt{2}\, t} \frac{\sin t \sin h\, t}{t}\, dt = \frac{1}{2} \tan^{-1}\left(\frac{2}{1 + (2-1)}\right) = \frac{1}{2} \tan^{-1}\left(\frac{2}{2}\right)$$

$$= \frac{1}{2} \tan^{-1} 1 = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$$

Or independently consider again,

$$\sin t \sin h\, t = \left(\frac{e^t - e^{-t}}{2}\right) \sin t.$$

Now, $L \sin t = \dfrac{1}{t^2 + 1}$

$$\therefore L\left(e^t \sin t\right) = \frac{1}{(s-1)^2 + 1}, \quad L\left(e^{-t} \sin t\right) = \frac{1}{(s+1)^2 + 1}$$

$$\therefore L \sin t \sin h\, t = \frac{1}{2}\left[\frac{1}{(s-1)^2 + 1} - \frac{1}{(s+1)^2 + 1}\right]$$

$$\therefore L\left(\frac{\sin t \sin h\, t}{t}\right) = \frac{1}{2} \int_s^{\infty} \left[\frac{1}{(s-1)^2 + 1} - \frac{1}{(s+1)^2 + 1}\right] ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s-1) - \tan^{-1}(s+1)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\left\{\frac{\pi}{2} - \tan^{-1}(s-1)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(s+1)\right\}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right].$$

Now let $\tan^{-1}(s+1) = \alpha$, $\quad \tan^{-1}(s-1) = \beta$.

$$\therefore \tan \alpha = s + 1, \quad \tan \beta = s - 1$$

$$\therefore \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \alpha} = \frac{(s+1) - (s-1)}{1 + (s^2 - 1)} = \frac{2}{1 + (s^2 - 1)} = \frac{2}{s^2}$$

$$\therefore \alpha - \beta = \tan^{-1}\left(\frac{2}{s^2}\right) \qquad \therefore \tan^{-1}(s+1) - \tan^{-1}(s-1) = \tan^{-1} \frac{2}{s^2}$$

$$\therefore L\left(\frac{\sin t \sin ht}{t}\right) = \frac{1}{2} \tan^{-1} \frac{2}{s^2}$$

This means $\displaystyle\int_0^{\infty} e^{-st} \cdot \frac{\sin t \sin ht}{t} = \frac{1}{2} \tan^{-1} \frac{2}{s^2}$

Now put $s = \sqrt{2}$

$$\therefore \int_0^{\infty} e^{-\sqrt{2}\, t} \left(\frac{\sin t \sin ht}{t}\right) = \frac{1}{2} \tan^{-1} \frac{2}{2} = \frac{1}{2} \tan^{-1} 1 = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}.$$

Example 11 : Prove that $\displaystyle\int_0^{\infty} e^{-st} \left(\frac{\sin at + \sin bt}{t}\right) dt = \pi - \tan^{-1}\left[\frac{s(a+b)}{ab - s^2}\right]$.

Sol. : We have $L(\sin at) = \dfrac{a}{s^2 + a^2}$, $L(\sin bt) = \dfrac{b}{s^2 + b^2}$

$$\therefore L\left(\frac{\sin at + \sin bt}{t}\right) = \int_s^{\infty} \frac{a}{s^2 + a^2}\, ds + \int_s^{\infty} \frac{b}{s^2 + b^2}\, ds = \left[\tan^{-1} \frac{s}{a}\right]_s^{\infty} + \left[\tan^{-1} \frac{s}{b}\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1} \frac{s}{a} + \frac{\pi}{2} - \tan^{-1} \frac{s}{b} = \pi - \left(\tan^{-1} \frac{s}{a} + \tan^{-1} \frac{s}{b}\right)$$

Now let $\tan^{-1} \dfrac{s}{a} = \alpha$, $\tan^{-1} \dfrac{s}{b} = \beta$ $\quad \therefore \tan \alpha = \dfrac{s}{a}$, $\tan \beta = \dfrac{s}{b}$

$$\therefore \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{(s/a) + (s/b)}{1 - (s/a)(s/b)} = \frac{s(a+b)}{ab - s^2}$$

$$\alpha + \beta = \tan^{-1}\left[\frac{s(a+b)}{ab - s^2}\right]$$

$$\therefore L\left(\frac{\sin at + \sin bt}{t}\right) = \pi - \tan^{-1}\left[\frac{s(a+b)}{ab - s^2}\right]$$

This means $\displaystyle\int_0^{\infty} e^{-st} \cdot \left(\frac{\sin at + \sin bt}{t}\right) dt = \pi - \tan^{-1}\left[\frac{s(a+b)}{ab - s^2}\right]$.

Example 12 : Evaluate $\displaystyle\int_0^{\infty} e^{-2t} \cdot \frac{\cos 2t \sin 3t}{t}\, dt$. **(M.U. 2015)**

Sol. : We have

$$L(\sin 3t \cos 2t) = L \frac{1}{2}[\sin 5t + \sin t] = \frac{1}{2}\left[\frac{5}{s^2 + 5^2} + \frac{1}{s^2 + 1}\right]$$

$$\therefore L\left[\frac{\sin 3t \cos 2t}{t}\right] = \frac{1}{2} \int_s^{\infty} \left[\left(\frac{5}{s^2 + 5^2}\right) + \left(\frac{1}{s^2 + 1}\right)\right] ds$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{5}\right) + \tan^{-1}\left(\frac{s}{1}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{5}\right) + \frac{\pi}{2} - \tan^{-1} s\right]$$

$$= \frac{\pi}{2} - \frac{1}{2} \tan^{-1}\left(\frac{s}{5}\right) - \frac{1}{2} \tan^{-1} s$$

This means,

$$\int_0^{\infty} e^{-st} \cdot \frac{\cos 2t \sin 3t}{t}\, dt = \frac{\pi}{2} - \frac{1}{2} \tan^{-1}\left(\frac{s}{5}\right) - \frac{1}{2} \tan^{-1} s$$

Putting $s = 2$, we get

$$\int_0^\infty e^{-2t} \cdot \frac{\cos 2t \sin 3t}{t}\, dt = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{2}{5}\right) - \frac{1}{2}\tan^{-1} 2$$

$$= \frac{\pi}{2} - \frac{1}{2}\left[\tan^{-1}\left(\frac{2}{5}\right) + \tan^{-1} 2\right]$$

Now, let $\tan^{-1}\frac{2}{5} = \alpha$, $\tan^{-1} 2 = \beta$.

$$\therefore \quad \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{(2/5) + 2}{1 - (2/5) \cdot 2} = 12$$

$$\therefore \quad \alpha + \beta = \tan^{-1} 12.$$

$$\therefore \quad I = \frac{\pi}{2} - \frac{1}{2}\tan^{-1} 12.$$

[In Ex. 11 above, put $a = 5$ and $b = 1$ and $s = 2$ after dividing by 2.]

Example 13 : Find the Laplace transform of $\dfrac{e^{-at} - \cos at}{t}$.

Hence, evaluate $\displaystyle\int_0^\infty \frac{e^{-t} - \cos t}{t\, e^{4t}}\, dt$. **(M.U. 20**

Sol. : We have $L(e^{-at}) = \dfrac{1}{s+a}$ and $L(\cos at) = \dfrac{s}{s^2 + a^2}$

$$\therefore \quad L[e^{-at} - \cos at] = \frac{1}{s+a} - \frac{s}{s^2 + a^2}$$

$$\therefore \quad L\left[\frac{e^{-at} - \cos at}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{s}{s^2 + a^2}\right) ds = \left[\log(s+a) - \frac{1}{2}\log(s^2 + a^2)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{\sqrt{s^2 + a^2}}\right)\right]_s^\infty = \left[\log\left(\frac{1 + (a/s)}{\sqrt{1 + (a^2/s^2)}}\right)\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{1 + (a/s)}{\sqrt{1 + (a^2/s^2)}}\right) = \log\left(\frac{\sqrt{s^2 + a^2}}{s+a}\right)$$

This means, $\displaystyle\int_0^\infty e^{-st}\left(\frac{e^{-at} - \cos at}{t}\right) dt = \log\left(\frac{\sqrt{s^2 + a^2}}{s+a}\right)$ .... (1)

Now, $\displaystyle\int_0^\infty \frac{e^{-t} - \cos t}{t\, e^{4t}}\, dt = \int_0^\infty e^{-4t}\left(\frac{e^{-t} - \cos t}{t}\right) dt$

Putting $s = 4$ and $a = 1$ in (1), we get

$$\int_0^\infty \frac{e^{-t} - \cos t}{t\, e^{4t}}\, dt = \log\left(\frac{\sqrt{16+1}}{4+1}\right) = \log\frac{\sqrt{17}}{5}.$$

Example 14 : Prove that $\displaystyle\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{t\, e^t}\right) dt = \frac{3\pi}{4}$. **(M.U. 1995, 2003, 06, 12)**

Sol. : In Ex. 11 above put $s = 1$, $a = 2$, $b = 3$.

$$\therefore \quad \int_0^\infty e^{-t} \cdot \left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \tan^{-1}\left[\frac{(1)(5)}{6-1}\right] = \pi - \tan^{-1}\left(\frac{5}{5}\right)$$

$$= \pi - \tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Or independently, we have,

$$L(\sin 2t) = \frac{2}{s^2 + 4}, \quad L(\sin 3t) = \frac{3}{s^2 + 9}$$

$$\therefore \quad L\left(\frac{\sin 2t + \sin 3t}{t}\right) = \int_s^\infty \left[\frac{2}{(s^2+4)} + \frac{3}{(s^2+9)}\right] ds = \left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty + \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right) + \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right) = \pi - \left(\tan^{-1}\frac{s}{2} + \tan^{-1}\frac{s}{3}\right)$$

Now let $\tan^{-1}\frac{s}{2} = \alpha$, $\tan^{-1}\frac{s}{3} = \beta$ $\quad \therefore \quad \frac{s}{2} = \tan\alpha$, $\frac{s}{3} = \tan\beta$

$$\therefore \quad \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{(s/2) + (s/3)}{1 - (s/2)(s/3)} = \frac{5s}{6 - s^2}$$

$$\therefore \quad \alpha + \beta = \tan^{-1}\left(\frac{5s}{6 - s^2}\right)$$

$$\therefore \quad L\left(\frac{\sin 2t + \sin 3t}{t}\right) = \pi - \tan^{-1}\left(\frac{5s}{6 - s^2}\right)$$

This means $\displaystyle\int_0^\infty e^{-st} \cdot \left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \tan^{-1}\left(\frac{5s}{6 - s^2}\right)$

Now put $s = 1$,

$$\int_0^\infty e^{-t} \cdot \left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \tan^{-1}\left(\frac{5}{6-1}\right) = \pi - \tan^{-1}\left(\frac{5}{5}\right)$$

$$= \pi - \tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Example 15 : Evaluate $\displaystyle\int_0^\infty t\, e^{-t^2} \operatorname{erf}(t)\, dt$. **(M.U. 2009)**

Sol. : Putting $t^2 = y$, $2t\, dt = dy$.

$$\int_0^\infty t\, e^{-t^2} \operatorname{erf}(t)\, dt = \int_0^\infty e^{-y} \operatorname{erf}(\sqrt{y})\, \frac{dy}{2} \qquad \text{.......... (1)}$$

But $L\left[\operatorname{erf}(\sqrt{t})\right] = \dfrac{1}{s\sqrt{s+1}}$. This means, $\displaystyle\int_0^\infty e^{-st} \operatorname{erf}(\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$.

Putting $s = 1$, $\displaystyle\int_0^\infty e^{-t} \operatorname{erf}(\sqrt{t}) = \frac{1}{\sqrt{2}}$

Hence, from (1) $\displaystyle\int_0^\infty t\, e^{-t^2} \operatorname{erf}(t)\, dt = \frac{1}{2\sqrt{2}}$.

EXERCISE - XII

Evaluate the following integrals by using Laplace transform.

1. $\int_0^\infty e^{-t} \cdot \frac{\sin 3t}{t}\, dt$ (M.U. 2014)
2. $\int_0^\infty \frac{e^{-2t} - e^{-3t}}{t}\, dt$
3. $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt$ (M.U. 1997, 2002, 04)
4. $\int_0^\infty \frac{\cos 4t - \cos 3t}{t}\, dt$
5. $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt$ (M.U. 2005, 07, 08, 13, 14)
6. $\int_0^\infty \frac{\sin 2t}{t}\, dt$ (M.U. 2010)
7. $\int_0^\infty \frac{e^{-t} \sin t}{t}\, dt$ (M.U. 2004, 05)
8. $\int_0^\infty \frac{\sin 3t + \sin 2t}{t e^t}\, dt$ (M.U. 2012)
9. $\int_0^\infty e^{-t} \cdot \frac{(1 - \cos 2t)}{2t}\, dt$
10. $\int_0^\infty \frac{e^{-t} \cdot \sin \sqrt{3} \cdot t}{t}\, dt$ (M.U. 2003)
11. $\int_0^\infty e^{-2t} \cdot \frac{\sinh t}{t}\, dt$ (M.U. 2008)

[**Ans.** : (1) $\cot^{-1}\left(\frac{1}{3}\right)$, (2) $\log\frac{3}{2}$, (3) $\log 3$, (4) $\log\frac{3}{4}$, (5) $\log\frac{2}{3}$, (6) $\frac{\pi}{2}$, (7) $\frac{\pi}{4}$, (8) $\frac{3\pi}{4}$, (9) $\frac{1}{4}\log 5$, (10) $\frac{\pi}{3}$, (11) $\frac{1}{2}\log 3$.

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14. Laplace Transforms of Derivatives

$$\boxed{Lf'(t) = -f(0) + sLf(t)} \quad \text{(M.U. 2004, 13)} \quad \text{.......... (1)}$$

Proof : By definition of Laplace transform,

$Lf'(t) = \int_0^\infty e^{-st} f'(t)\, dt.$ Integrating by parts

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\, dt$$

$$\therefore \; Lf'(t) = -f(0) + s\int_0^\infty e^{-st} f(t)\, dt = -f(0) + sLf(t)$$

Applying (1) again,

$$Lf''(t) = -f'(0) + s[Lf'(t)]$$
$$= -f'(0) + s[-f(0) + sLf(t)]$$
$$\therefore \; Lf''(t) = -f'(0) - sf(0) + s^2 Lf(t)$$

Further, $Lf'''(t) = -f''(0) - sf'(0) - s^2 f(0) + s^3 Lf(t)$

In general

$$Lf^n(t) = -f^{n-1}(0) - s \cdot f^{n-2}(0) - s^2 \cdot f^{n-3}(0) + \ldots\ldots + s^n Lf(t)$$

Cor. 1 : If $f(0) = f'(0) = f''(0) = \ldots\ldots = 0$, we get

$$\boxed{\begin{array}{l} Lf'(t) = s \cdot Lf(t), \quad Lf''(t) = s^2 Lf(t), \\ L^3 f'''(t) = s^3 Lf(t) \ldots\ldots Lf^n(t) = s^n Lf(t) \end{array}} \quad \text{.......... (4)}$$

Note ....

These results are going to be highly useful to solve differential equations.

Example 1 : Given $f(t) = t + 1,\; 0 \le t \le 2$ and $f(t) = 3,\; t > 2$, find $L[f(t)]$, $L[f'(t)]$ and $L[f''(t)]$ **(M.U. 2003, 04, 05, 06)**

Sol. : By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt.$

$$\therefore \; L[f(t)] = \int_0^2 e^{-st}(t+1)\, dt + \int_2^\infty e^{-st} \cdot 3\, dt$$

$$= \left[(t+1) \cdot \left(\frac{e^{-st}}{-s}\right) - (1) \cdot \frac{e^{-st}}{s^2}\right]_0^2 + 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-2s}\right)$$

Now, as shown above, $L[f'(t)] = -f(0) + sL[f(t)]$

But by data $f(0) = 1$,

$$\therefore \; L[f'(t)] = -1 + s\left[\frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-2s}\right)\right] = -1 + 1 + \frac{1}{s}\left(1 - e^{-2s}\right) = \frac{1}{s}\left(1 - e^{-2s}\right)$$

Further, $L[f''(t)] = s^2 L[f(t)] - s[f(0)] - f'(0)$

$$= s^2\left[\frac{1}{s} + \frac{1}{s^2}(1 - e^{-2s})\right] - s - 1$$

$$= s + 1 - e^{-2s} - s - 1 = -e^{-2s}.$$

$[\because f(t) = t + 1,\; f'(t) = 1 \quad \therefore \; f(0) = 1,\; f'(0) = 1]$

Example 2 : Given $f(t) = \frac{\sin t}{t}$, find $L[f'(t)]$. **(M.U. 2002, 13)**

Sol. : Let $L \sin t = \frac{1}{s^2 + 1}$

$$\therefore \; L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2 + 1} = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

But $L[f'(t)] = sLf(t) - f(0) = s\cot^{-1} s - 1.$

Example 3 : Given $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-(1/4s)}$, prove that $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}} \cdot e^{-(1/4s)}$ **(M.U. 2009)**

Sol. : Let $f(t) = \sin\sqrt{t}$ $\quad \therefore \; L[f'(t)] = L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right]$

Also $f(0) = \sin 0 = 0$

$\therefore$ By (4) above, $L[f'(t)] = s \cdot [L f(t)]$

$$\therefore L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = s \cdot \frac{\sqrt{\pi}}{2 \cdot s^{3/2}} e^{-(1/4s)} \quad \therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}} \cdot e^{-(1/4s)}.$$

Example 4 : Find Laplace transform of $\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)$. (M.U. 2...)

Sol. : We have $L(1-\cos 2t) = L(1) - L(\cos 2t) = \frac{1}{s} - \frac{s}{s^2+2^2}$

$$\therefore L\left(\frac{1-\cos 2t}{t}\right) = \int_s^\infty \Phi(s)\, ds \qquad \text{[By (17), page 1-39]}$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+2^2}\right] ds = \left[\log s - \frac{1}{2}\log(s^2+2^2)\right]_s^\infty$$

$$\therefore L\left(\frac{1-\cos 2t}{t}\right) = \left[\log\frac{s}{\sqrt{s^2+2^2}}\right] = \left[0 - \log\frac{s}{\sqrt{s^2+2^2}}\right] = \log\left(\frac{\sqrt{s^2+2^2}}{s}\right)$$

But $L[f'(t)] = s\,L[f(t)] - f(0)$ where $f(t) = \frac{1-\cos 2t}{t} = \frac{2\sin^2 t}{t}$

$$\therefore f(0) = 2\sin t\left(\frac{\sin t}{t}\right) = 0$$

$$\therefore L\left[\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)\right] = s\log\left(\frac{\sqrt{s^2+2^2}}{s}\right)$$

Alternatively : We may differentiate $\left(\frac{1-\cos 2t}{t}\right)$ w.r.t. t and then obtain the Laplace transf... of the derivative.

Example 5 : Find $L\left[\frac{d}{dt}\left(\frac{\sin 3t}{t}\right)\right]$. (M.U. 2...)

Sol. : $L(\sin 3t) = \frac{3}{s^2+9}$

$$\therefore L\left[\frac{\sin 3t}{t}\right] = 3\int_s^\infty \frac{ds}{s^2+9} = 3 \cdot \frac{1}{3}\left[\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right) = \cot^{-1}\left(\frac{s}{3}\right).$$

$\therefore L[f'(t)] = -f(0) - s\,L[f(t)]$ where $f(t) = \frac{\sin 3t}{t}$.

Now, $\lim_{t\to 0}\left(\frac{\sin 3t}{t}\right) = 3 \cdot \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right) = 3$

$$\therefore L[f'(t)] = -3 - s\cot^{-1}\left(\frac{s}{3}\right)$$

Example 6 : If $L(t\sin\omega t) = \frac{2\omega s}{(s^2+\omega^2)}$, find $L[\omega t\cos\omega t + \sin\omega t]$. **(M.U. 2011)**

Sol. : Let $f(t) = t\sin\omega t \quad \therefore f'(t) = \sin\omega t + \omega t\cos\omega t$.

Further since $f(0) = 0$ and $f'(0) = 0$.

By the above result (4), page 1-53, $\quad L[f'(t)] = s\,L[f(t)]$

$$\therefore L[\omega t\cos\omega t + \sin\omega t] = s \cdot \frac{2\omega s}{(s^2+\omega^2)^2} = \frac{2\omega s^2}{(s^2+\omega^2)^2}.$$

EXERCISE - XIII

1. Given $f(t) = 3,\ 0 \le t < 5,\ f(t) = 0,\ t > 5$. Find $L[f(t)]$ and also $L[f'(t)]$.
2. Given $f(t) = t,\ 0 \le t < 3,\ f(t) = 6,\ t > 3$. Find $L[f(t)]$ and also $L[f'(t)]$. **(M.U. 2004)**
3. Find $L\left[\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right]$. **(M.U. 2002)**

[**Ans. :** **(1)** $\frac{3}{s}\left[1-e^{-5s}\right]$, $\frac{3}{s}\left(1-s-e^{-5s}\right)$.

(2) $\frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$, $\frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$. **(3)** $s\cot^{-1}s - 1$]

15. Laplace Transforms of Integrals

If $L[f(t)] = \Phi(s)$, then $\boxed{L\int_0^t f(u)\, du = \frac{1}{s}\Phi(s)}$ **(M.U. 1996, 2013)** .... (19)

Proof : By definition,

$$L\left[\int_0^t f(u)\, du\right] = \int_0^\infty e^{-st}\left[\int_0^t f(u)\, du\right] dt \qquad \text{.................... (1)}$$

$$= \left[\int_0^t f(u)\, du \cdot \left(\frac{-e^{-st}}{s}\right)\right]_0^\infty - \int_0^\infty \left[\left(\frac{-e^{-st}}{s}\right)\frac{d}{dt}\int_0^t f(u)\, du\right]$$

[By integration by parts]

But $\frac{d}{dt}\int_0^t f(u)\, du = f(t)$

$$\therefore L\left[\int_0^t f(u)\, du\right] = \int_0^\infty \frac{1}{s} \cdot e^{-st} f(t)\, dt = \frac{1}{s} \cdot L[f(t)] = \frac{1}{s}\Phi(s)$$

Since $\Phi(s) = L[f(t)]$, we have

$$L\int_0^t f(u)\, du = \frac{1}{s} L\{f(t)\}$$

Corollary : The above result can be generalised as follows.

$$\boxed{L\left[\int_0^t \int_0^t \cdots \int_0^t f(u)(du)^n\right] = \frac{1}{s^n} L[f(t)]}$$

Example 1 : Find the Laplace transform of

(i) $\int_0^t \sin 2u\, du$ (ii) $\int_0^t u \cos h u \cdot du$. (M.U. 1998, 99)

Sol. : (i) Since $L \sin 2t = \dfrac{2}{s^2+4} = \Phi(s)$, say

$$\therefore \quad L \int_0^t \sin 2u\, du = \frac{1}{s}\Phi(s) = \frac{2}{s(s^2+4)}$$

(ii) Now, $L \cos ht = \dfrac{s}{s^2-a^2}$ $\quad \therefore \quad L t \cos ht = -\dfrac{d}{ds}\left[\dfrac{s}{s^2-a^2}\right]$ [By (16) § 10, page 1-2]

$$\therefore \quad L t \cos ht = \frac{s^2+a^2}{(s^2-a^2)^2} = \Phi(s), \text{ say}$$

$$\therefore \quad L \int_0^t u \cos hu\, du = \frac{1}{s}\Phi(s) = \frac{s^2+a^2}{s(s^2-a^2)^2}.$$

Example 2 : Find the Laplace transform of $\int_0^t \sin u \cos 2u\, du$. (M.U. 2...)

Sol. : We have $\sin u \cos 2u = \dfrac{1}{2}\cdot 2 \sin u \cos 2u = \dfrac{1}{2}[\sin 3u - \sin u]$

Now, $L\left[\dfrac{1}{2}(\sin 3u)\right] = \dfrac{1}{2}\cdot\dfrac{3}{s^2+9}$ and $L\left[\dfrac{1}{2}(\sin u)\right] = \dfrac{1}{2}\cdot\dfrac{1}{s^2+1}$

$$\therefore \quad L\left[\int_0^t \sin u \cos 2u\, du\right] = \frac{1}{s}\cdot\Phi(s) = \frac{1}{2s}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right].$$

Example 3 : Find the Laplace transform of the following.

(i) $erf \sqrt{t}$ **(M.U. 1996, 97, 2003, 06)**

(ii) $erf \sqrt{2t}$ and evaluate $\int_0^\infty erf(2\sqrt{t}) \cdot e^{-5t}\, dt$. **(M.U. 2005)**

(iii) $e^{3t} erf \sqrt{t}$ **(M.U. 1998)**

(iv) $e^{-3t} erf \sqrt{t}$ **(M.U. 1997, 99)**

Sol. : (i) We have $erf\, x = \dfrac{2}{\sqrt{\pi}}\int_0^x e^{-u^2} du$. Hence, $erf \sqrt{t} = \dfrac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2} du$

Now put $u^2 = v$ $\quad \therefore \quad u = \sqrt{v}$ $\quad \therefore \quad du = \dfrac{1}{2\sqrt{v}} dv$

When $u = 0$, $v = 0$; When $u = \sqrt{t}$, $v = t$

$$\therefore \quad erf \sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^t e^{-v}\cdot\frac{1}{2\sqrt{v}} dv = \frac{1}{\sqrt{\pi}}\int_0^t e^{-v} v^{-1/2} dv$$

By (9), page 1-5.

$$L(v^{-1/2}) = \frac{\lceil 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} \quad \therefore \quad L(e^{-v} v^{-1/2}) = \frac{\sqrt{\pi}}{\sqrt{s+1}}$$ [By first shifting theorem]

$$\therefore \quad L\int_0^t e^{-v} v^{-1/2} dv = \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$\therefore \quad L\, erf \sqrt{t} = \frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{s\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}.$$ [See also (11), page 1-6]

(ii) Now $erf\, 2\sqrt{t} = erf \sqrt{4t}$. But $erf \sqrt{t} = \dfrac{1}{s\sqrt{s+1}}$

Hence, by (13), § 6 page 1-17.

$$erf \sqrt{4t} = \frac{1}{4}\cdot\frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}} \qquad \therefore \quad erf\, 2\sqrt{t} = \frac{2}{s\sqrt{s+4}}$$

By definition of Laplace transform

$$\int_0^\infty e^{-st}\, erf\, 2\sqrt{t}\, dt = \frac{2}{s\sqrt{s+4}}$$

Put $s = 5$, $\displaystyle\int_0^\infty e^{-5t}\, erf\, 2\sqrt{t}\, dt = \frac{2}{5\sqrt{5+4}} = \frac{2}{15}$.

(iii) Since, $L(erf \sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$

$$L\left(e^{3t} erf \sqrt{t}\right) = \frac{1}{(s-3)\sqrt{s-3+1}} = \frac{1}{(s-3)\sqrt{s-2}}$$ [By (14) § 7, page 1-19]

(iv) Similarly, $L\left(e^{-3t} erf \sqrt{t}\right) = \dfrac{1}{(s+3)\sqrt{s+3+1}} = \dfrac{1}{(s+3)\sqrt{s+4}}$.

Example 4 : Find $\int_0^\infty e^{-t} erf \sqrt{t}\, dt$. **(M.U. 1997)**

Sol. : As proved above $L[erf \sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}$

This means $\displaystyle\int_0^\infty e^{-st}\cdot erf \sqrt{t}\, dt = \frac{1}{s\sqrt{s+1}}$

Now put $s = 1$, $\quad \therefore \quad \displaystyle\int_0^\infty e^{-t}\cdot erf \sqrt{t}\, dt = \frac{1}{\sqrt{2}}$

Example 5 : Find $L\, e^{3t}\cdot t\, erf \sqrt{t}$. **(M.U. 2000)**

Sol. :

$$L\, t\, erf \sqrt{t} = -\frac{d}{ds}\left(\frac{1}{s\sqrt{s+1}}\right) = -\left[-\frac{1}{s^2(s+1)}\left\{\frac{s}{2\sqrt{s+1}} + \sqrt{s+1}\right\}\right]$$

$$\therefore \quad L\, t\, erf \sqrt{t} = \frac{1}{s^2(s+1)}\cdot\frac{s+2(s+1)}{2\sqrt{s+1}} = \frac{3s+2}{2s^2(s+1)^{3/2}}$$

$$\therefore \quad L\, e^{3t}\cdot t \cdot erf \sqrt{t} = \frac{3(s-3)+2}{2(s-3)^2(s-3+1)^{3/2}} = \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}.$$

Example 6 : Find $\int_0^\infty e^{-t} erf_c \sqrt{t}\, dt$. **(M.U. 1996)**

Sol. : By (12), page 1-7, we have

$$L[erf_c \sqrt{t}] = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}.$$

By definition of Laplace transform this means

$$\int_0^\infty e^{-st}\, erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$$

Put $s = 1$, we get $\int_0^\infty e^{-t}\, erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{2}(\sqrt{2}+1)} = \frac{1}{\sqrt{6}}$.

Example 7 : Find $\int_0^\infty erf_c\sqrt{t}\, dt$.

Sol. : As proved above $L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$.

This means $\int_0^\infty e^{-st} erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$

Putting $s = 0$, we get $\int_0^\infty erf_c\sqrt{t}\, dt = \frac{1}{2}$.

Example 8 : Find $L\left[\int_0^t \int_0^t \int_0^t t \sin t\, (dt)^3\right]$. (M.U. 20

Sol. : By the corollary

$$L\left[\int_0^t \int_0^t \int_0^t t \sin t\, (dt)^3\right] = \frac{1}{s^3} L[t \sin t]$$

But by Ex. (1) (iii), page 1-29,

$$L[t \sin t] = \frac{2s}{(s^2+1)^2}$$

$$\therefore\ L\left[\int_0^t \int_0^t \int_0^t t \sin t\, (dt)^3\right] = \frac{1}{s^3} \cdot \frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}.$$

Example 9 : Find the Laplace transform of the following.

(i) $\int_0^t u \cos^2 u\, du$. **(M.U. 1995, 2010)** (ii) $\int_0^t u\, e^{-3u} \cos^2 2u\, du$. **(M.U. 1993)**

(iii) $\int_0^t u^{-1} e^{-u} \sin u\, du$. **(M.U. 1996)** (iv) $\int_0^t \frac{1-e^{au}}{u}\, du$. **(M.U. 1997, 2003)**

(v) $\int_0^t \frac{\sin u}{u}\, du$. **(M.U. 1997, 2002)** (vi) $\int_t^\infty \frac{\cos u}{u}\, du$. **(M.U. 1997)**

(vii) $\int_0^t u^2 \sin u\, du$. **(M.U. 2003)** (viii) $\int_0^t e^{-2u} \cos^2 u\, du$ **(M.U. 2008)**

Sol. : (i) $\cos^2 u = \frac{1+\cos 2u}{2}$

$$L\cos^2 u = L\frac{1}{2}(1+\cos 2u) = \frac{1}{2}L(1) + \frac{1}{2}L\cos 2u = \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+2^2}$$

$$L(u\cos^2 u) = -\frac{d}{ds}\cdot\left(\frac{1}{2}\cdot\frac{1}{s}\right) - \frac{d}{ds}\cdot\frac{1}{2}\cdot\left(\frac{s}{s^2+2^2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\frac{(s^2+2^2)\cdot 1 - s\cdot 2s}{(s^2+2^2)^2} \quad \text{[By (16) § 10, page 1-28]}$$

$$\therefore\ L(u\cos^2 u) = \frac{1}{2s^2} + \frac{1}{2}\cdot\frac{s^2-2^2}{(s^2+2^2)^2} = \Phi(s),\ \text{say}$$

$$\therefore\ L\int_0^t u\cos^2 u\, du = \frac{1}{s}\cdot\Phi(s) = \frac{1}{2s^3} + \frac{1}{2}\cdot\frac{s^2-2^2}{s(s^2+2^2)^2} \quad \text{[By (19) § 15, page 1-55]}$$

(ii) $\cos^2 2u = \frac{1+\cos 4u}{2}$

$$L\cos^2 2u = L\frac{1}{2}(1+\cos 4u) = \frac{1}{2}L(1) + \frac{1}{2}L(\cos 4u)$$

$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+4^2}$$

$$L\, e^{-3u}\cos^2 2u = \frac{1}{2}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{s+3}{(s+3)^2+4^2}$$

$$= \frac{1}{2}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{s+3}{s^2+6s+25}$$

$$L\, u\, e^{-3u}\cos^2 2u = -\frac{d}{ds}\left[\frac{1}{2}\cdot\frac{1}{s+3}\right] - \frac{d}{ds}\left[\frac{1}{2}\cdot\frac{s+3}{s^2+6s+25}\right] \quad \text{[By (16) § 10, page 1-28]}$$

$$= \frac{1}{2}\cdot\frac{1}{(s+3)^2} - \frac{1}{2}\left[\frac{(s^2+6s+25)-(s+3)(2s+6)}{(s^2+6s+25)^2}\right]$$

$$= \frac{1}{2(s+3)^2} + \frac{1}{2}\cdot\frac{s^2+6s-7}{(s^2+6s+25)^2} = \Phi(s),\ \text{say}$$

$$\therefore\ L\int_0^t u\, e^{-3u}\cos^2 2u\, du = \frac{1}{s}\Phi(s) \quad \text{[By (19) § 15, page 1-55]}$$

$$= \frac{1}{2s(s+3)^2} + \frac{1}{2}\cdot\frac{s^2+6s-7}{s(s^2+6s+25)^2}$$

(iii) $L\sin u = \frac{1}{s^2+1}$; $L\, e^{-u}\sin u = \frac{1}{(s+1)^2+1}$ [By shifting theorem]

$$\therefore\ L\left(\frac{1}{u}e^{-u}\sin u\right) = \int_s^\infty \frac{ds}{(s+1)^2+1} \quad \text{[By (17) § 12, page 1-39]}$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1) = \Phi(s),\ \text{say}$$

$$\therefore\ L\int_0^t \frac{e^{-u}\sin u}{u}\, du = \frac{1}{s}\Phi(s) = \frac{1}{s}\cot^{-1}(s+1).$$

(iv) $L(1-e^{au}) = L(1) - L(e^{au}) = \frac{1}{s} - \frac{1}{s-a}$

$$\therefore\ L\left(\frac{1-e^{au}}{u}\right) = \int_s^\infty\left[\frac{1}{s} - \frac{1}{s-a}\right]ds = \left[\log\frac{s}{s-a}\right]_s^\infty$$

$$= \left[0 - \log\frac{s}{s-a}\right] = \log\left(\frac{s-a}{s}\right)$$

$$\therefore\ L\int_0^t \left(\frac{1-e^u}{u}\right) du = \frac{1}{s}\Phi(s) = \frac{1}{s}\log\left(\frac{s-a}{s}\right).$$

(v) $L\sin u = \dfrac{1}{s^2+1}$

$$L\frac{\sin u}{u} = \int_s^\infty \Phi(s)\,ds = \int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore\ L\int_0^t \frac{\sin u}{u}\,du = \frac{1}{s}\Phi(s) = \frac{1}{s}\cot^{-1}s.$$

(vi) Let $f(t) = \displaystyle\int_t^\infty \frac{\cos u}{u}\,du$

Now put $u = vt$, $du = t\,dv$

When $u = t$, $v = 1$; when $u \to \infty$, $v \to \infty$

$$\therefore\ f(t) = \int_1^\infty \frac{\cos vt}{vt}\cdot t\,dv = \int_1^\infty \frac{\cos vt}{v}\,dv$$

$$\therefore\ L[f(t)] = \int_0^\infty e^{-st}\left[\int_1^\infty \frac{\cos vt}{v}\,dv\right]dt = \int_1^\infty \frac{dv}{v}\int_0^\infty e^{-st}\cos vt\,dt$$

$$= \int_1^\infty \frac{dv}{v} L[\cos vt]\,dt = \int_1^\infty \frac{s\,dv}{v(s^2+v^2)} = \frac{1}{s}\int_1^\infty\left[\frac{1}{v} - \frac{v}{s^2+v^2}\right]dv$$

$$= \frac{1}{s}\left[\log v - \frac{1}{2}\log(s^2+v^2)\right]_1^\infty = \frac{1}{2s}\left[\log\frac{v^2}{s^2+v^2}\right]_1^\infty$$

$$= \frac{1}{2s}\left[0 - \log\frac{1}{s^2+1}\right] = \frac{1}{2s}\log(s^2+1)$$

(vii) $L(\sin u) = \dfrac{1}{s^2+1}$

$$L(u^2\sin u) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\left(-\frac{2s}{(s^2+1)^2}\right)$$

$$= -2\frac{d}{ds}\cdot\left[\frac{s}{(s^2+1)^2}\right] = -2\left[\frac{(s^2+1)^2\cdot 1 - s\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{s^2+1-4s^2}{(s^2+1)^3}\right] = -2\cdot\frac{(1-3s^2)}{(s^2+1)^3}$$

$$\therefore\ L\left[\int_0^t u^2\sin u\,du\right] = -\frac{2}{s}\cdot\frac{(1-3s^2)}{(s^2+1)^3}.$$

(viii) $\cos^2 u = \dfrac{1+\cos 2u}{2}$

$$L\cos^2 u = L\left[\frac{1}{2}(1+\cos 2u)\right] = \frac{1}{2}L(1) + \frac{1}{2}L\cos 2u$$

$$\therefore\ L\cos^2 u = \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+2^2}$$

$$L(e^{-2u}\cos^2 u) = \frac{1}{2}\cdot\frac{1}{(s+2)} + \frac{1}{2}\cdot\frac{s+2}{(s+2)^2+2^2} = \frac{1}{2}\cdot\frac{1}{(s+2)} + \frac{1}{2}\cdot\frac{s+2}{s^2+4s+8}$$

$$L\int_0^t e^{-2u}\cos^2 u\,du = \frac{1}{s}\Phi(s) = \frac{1}{2s(s+2)} + \frac{1}{2}\cdot\frac{s+2}{s(s^2+4s+8)}.$$

Example 10 : Find the Laplace transforms of the following.

(i) $t\displaystyle\int_0^t e^{-4u}\sin 3u\,du$ (ii) $e^{-t}\displaystyle\int_0^t \frac{\sin u}{u}\,du$ **(M.U. 2015)**

(iii) $t^{-1}\displaystyle\int_0^t e^{-u}\sin u\,du$ (iv) $e^{-4t}\displaystyle\int_0^t u\sin 3u\,du$ **(M.U. 2003, 04, 15, 16)**

(v) $e^{-t}\displaystyle\int_0^t e^u\cos h u\,du$ **(M.U. 2012)** (vi) $\cos ht\displaystyle\int_0^t e^u\cos hu\,du$ **(M.U. 1996, 2008, 09)**

(vii) $t\displaystyle\int_0^t e^u\sin u\,du$ **(M.U. 2008)**

Sol. : (i) We have $L(\sin 3u) = \dfrac{3}{s^2+9}$ [By (7), page 1-5]

$\therefore\ L(e^{-4u}\sin 3u) = \dfrac{3}{(s+4)^2+9}$ [By (14), page 1-19]

$\therefore\ L\left[\displaystyle\int_0^t e^{-4u}\sin 3u\,du\right] = \dfrac{1}{s}\cdot\dfrac{3}{(s+4)^2+9}$ [By (19), page 1-55]

$\therefore\ L\left[t\displaystyle\int_0^t e^{-4u}\sin 3u\,du\right] = (-1)\dfrac{d}{ds}\left[\dfrac{3}{s^3+8s^2+25s}\right]$ [By (16), page 1-28]

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

(ii) As proved in (v) of the above Ex. 8,

$$L\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}\cdot\cot^{-1}s$$

$\therefore\ L\left[e^{-t}\displaystyle\int_0^t \frac{\sin u}{u}\,du\right] = \dfrac{1}{(s+1)}\cot(s+1)$ [By (14), page 1-19]

(iii) We have $L(\sin u) = \dfrac{1}{s^2+1}$ [By (7), page 1-5]

$L(e^{-u}\sin u) = \dfrac{1}{(s+1)^2+1}$ [By (14), page 1-19]

$\therefore\ L\left[\displaystyle\int_0^t e^{-u}\sin u\,du\right] = \dfrac{1}{s}\cdot\dfrac{1}{s^2+2s+2}$ [By (19), page 1-55]

$\therefore\ L\left[\dfrac{1}{t}\displaystyle\int_0^t e^{-u}\sin u\,du\right] = \displaystyle\int_s^\infty \dfrac{ds}{s(s^2+2s+2)}$ [By (17), page 1-39]

$= \dfrac{1}{2}\displaystyle\int_s^\infty\left[\dfrac{1}{s} - \dfrac{s+2}{s^2+2s+2}\right]ds$ [By partial fractions]

$$\therefore\ L\left[\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right]=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{1}{2}\cdot\frac{2(s+2)}{(s^2+2s+2)}\right]ds$$

$$=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{1}{2}\cdot\frac{2s+2}{(s^2+2s+2)}-\frac{2}{2(s^2+2s+2)}\right]ds$$

$$=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{1}{2}\cdot\frac{2s+2}{s^2+2s+2}-\frac{1}{(s+1)^2+1}\right]ds$$

$$=\frac{1}{2}\left[\log s-\frac{1}{2}\log(s^2+2s+2)-\tan^{-1}(s+1)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+2s+2}}-\tan^{-1}(s+1)\right]_s^\infty$$

$$=\frac{1}{2}\left[\left(\log 1-\frac{\pi}{2}\right)-\left(\log\frac{s}{\sqrt{s^2+2s+2}}-\tan^{-1}(s+1)\right)\right]$$

$$=\frac{1}{2}\left[\log\frac{\sqrt{s^2+2s+2}}{s}-\left(\frac{\pi}{2}-\tan^{-1}(s+1)\right)\right]$$

$$\therefore\ L\left[\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right]=\left[\frac{1}{4}\log\left(\frac{s^2+2s+2}{s^2}\right)-\frac{1}{2}\cot^{-1}(s+1)\right]$$

(iv) We have $L(\sin 3u)=\dfrac{3}{s^2+9}$ [By (7), page 1-5]

$$\therefore\ L(u\sin 3u)=-\frac{d}{ds}\left(\frac{3}{s^2+9}\right)=\frac{3\cdot 2s}{(s^2+9)^2}\qquad\text{[By (16), page 1-28]}$$

$$\therefore\ L\left[\int_0^t u\sin 3u\,du\right]=\frac{1}{s}\cdot\frac{6s}{(s^2+9)^2}=\frac{6}{(s^2+9)^2}\qquad\text{[By (19), page 1-55]}$$

$$\therefore\ L\left[e^{-4t}\int_0^t u\sin 3u\,du\right]=\frac{6}{[(s+4)^2+9]^2}=\frac{6}{(s^2+8s+25)^2}\qquad\text{[By (14), page 1-19]}$$

(v) We have $L(\cos hu)=\dfrac{s}{s^2-1}$ [By (8), page 1-5]

$$L(e^u\cos hu)=\frac{s-1}{(s-1)^2-1}=\frac{s-1}{s^2-2s+1-1}=\frac{s-1}{s(s-2)}\qquad\text{[By (14), page 1-19]}$$

$$\therefore\ L\left[\int_0^t e^u\cos hu\,du\right]=\frac{1}{s}\cdot\frac{s-1}{s(s-2)}=\frac{s-1}{s^2(s-2)}\qquad\text{[By (19), page 1-55]}$$

$$\therefore\ L\left[e^{-t}\int_0^t e^u\cos hu\,du\right]=\frac{(s+1)-1}{(s+1)^2[(s+1)-2]}\qquad\text{[By (14), page 1-19]}$$

$$=\frac{s}{(s+1)^2(s-1)}$$

(vi) We have $L(\cos hu)=\dfrac{s}{s^2-1}$ [By (8), page 1-5]

$$\therefore\ L(e^u\cos hu)=\frac{s-1}{(s-1)^2-1}\qquad\text{[By (14), page 1-19]}$$

$$\therefore\ L(e^u\cos hu)=\frac{s-1}{s^2-2s+1-1}=\frac{s-1}{s(s-2)}$$

$$\therefore\ L\left[\int_0^t e^u\cos hu\,du\right]=\frac{1}{s}\cdot\frac{s-1}{s(s-2)}=\frac{s-1}{s^2(s-2)}\qquad\text{[By (19), page 1-55]}$$

$$\therefore\ L\left[\cos ht\cdot\int_0^t e^u\cos hu\,du\right]=L\left[\frac{e^t+e^{-t}}{2}\int_0^t e^u\cos hu\,du\right]$$

$$=\frac{1}{2}\left[L\left\{e^t\int_0^t e^u\cos hu\,du\right\}+L\left\{e^{-t}\int_0^t e^u\cos hu\,du\right\}\right]$$

$$=\frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)}+\frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]\qquad\text{[By (14), page 1-19]}$$

$$=\frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)}+\frac{s}{(s+1)^2(s-1)}\right]$$

(vii) We have $L(\sin u)=\dfrac{1}{s^2+1}$

$$L[e^u\sin u]=\frac{1}{(s-1)^2+1}=\frac{1}{s^2-2s+2}\qquad\text{[By (14), page 1-19]}$$

$$\therefore\ L\left[\int_0^t e^u\sin u\,du\right]=\frac{1}{s}\cdot\frac{1}{s^2-2s+2}\qquad\text{[By (19), page 1-55]}$$

$$\therefore\ L\left[t\int_0^t e^u\sin u\,du\right]=-\frac{d}{ds}\left[\frac{1}{s^3-2s^2+2s}\right]=\frac{3s^2-4s+2}{(s^3-2s^2+2s)^2}\qquad\text{[By (16), page 1-28]}$$

Example 11 : Find the Laplace transform of $\cos h t\int_0^t e^u\sin hu\,du$. **(M.U. 2015)**

Sol. : We have $L[\sin hu]=\dfrac{1}{s^2-1}$

$$\therefore\ L[e^u\sin hu]=\frac{1}{(s-1)^2-1}=\frac{1}{s^2-2s+1-1}=\frac{1}{s^2-2s}=\frac{1}{s(s-2)}$$

$$\therefore\ L\left[\int_0^t e^u\sin hu\,du\right]=\frac{1}{s}\cdot\frac{1}{s(s-2)}=\frac{1}{s^2(s-2)}$$

$$\therefore\ L\left[\cos h t\int_0^t e^u\sin hu\,du\right]=L\left[\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^u\sin hu\,du\right]$$

$$=\frac{1}{2}\left[L\left\{e^t\int_0^t e^u\sin u\,du\right\}+L\left\{e^{-t}\int_0^t\sin u\,du\right\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{(s-1)^2(s-1-2)}+\frac{1}{(s+1)^2(s+1-2)}\right]\qquad\text{[By (14), page 1-19]}$$

$$=\frac{1}{2}\left[\frac{1}{(s-1)^2(s-3)}+\frac{1}{(s+1)^2(s-1)}\right]$$

Example 12 : Find the Laplace transform of $t\int_0^t e^{-2u}\cos^2 u\,du$. (M.U. 20...

Sol. : We have obtained in Ex. 9 (viii), page 1-58

$$L\left[\int_0^t e^{-2u}\cos^2 u\,du\right]=\frac{1}{2s(s+2)}+\frac{1}{2}\cdot\frac{s+2}{s(s^2+4s+8)}$$

$\therefore$ By (16), page 1-28

$$L\left[t\int_0^t e^{-2u}\cos^2 u\,du\right]=(-1)\cdot\frac{d}{ds}[\Phi(s)]$$

$$=-\frac{1}{2}\left[-\frac{1}{s^2(s+2)^2}[s\cdot 1+(s+2)\cdot 1]\right]$$

$$-\frac{1}{2}\left[\frac{s(s^2+4s+8)\cdot 1-(s+2)(3s^2+8s+8)}{s^2(s^2+4s+8)^2}\right]$$

$$=\frac{s+1}{s^2(s+2)^2}+\frac{s^3+5s^2+8s+8}{s^2(s^2+4s+8)^2}$$

Note ....

Note that in all these examples we have used more than one rules to find the Laplace transform. We start with the Laplace transform of the innermost function and arrive at the Laplace transform of the given function step by step.

EXERCISE - XIV

(A) Find the Laplace transform of

1. $\int_0^t e^{-3u}\sin 4u\,du$
2. $\int_0^t \frac{1}{u}\sin 3u\,du$
3. $\int_0^t ue^{-3u}\sin 4u\,du$ **(M.U. 2002, 12)**
4. $\int_0^t \frac{1-e^{-au}}{u}\,du$ **(M.U. 2003)**

[**Ans. :** (1) $\frac{1}{s}\cdot\frac{4}{s^2+6s+25}$, (2) $\frac{1}{s}\cot^{-1}s$, (3) $\frac{8(s+3)}{s(s^2+6s+25)^2}$, (4) $\frac{1}{s}\log\left(\frac{s+a}{s}\right)$]

(B) Find the Laplace transform of

1. $\int_0^t e^{-u}u^4\,du$
2. $\int_0^t e^{-u}\cos u\,du$
3. $\int_0^t \frac{1+e^{-u}}{u}\,du$
4. $\int_0^t \frac{e^{-u}\sin u}{u}\,du$ **(M.U. 1997, 2003, 09)**
5. $e^{-3t}\int_0^t u\sin 3u\,du$ **(M.U. 2006, 07, 08, 10)**
6. $\int_0^t e^u\cdot\frac{\sin 4u}{u}\,du$ **(M.U. 2000)**
7. $\int_0^t u\cdot e^{-2u}\sin 3u\,du$ **(M.U. 2002)**
8. $\int_0^t u\cdot e^{-3u}\sin^2 u\,du$ **(M.U. 2004)**
9. $\int_0^t u\cosh au\,du$

[**Ans. :** (1) $\frac{4!}{s(s+1)^5}$, (2) $\frac{1}{s}\cdot\frac{s+1}{(s^2+2s+2)}$, (3) $\frac{1}{s}\log\left[s\left(\frac{s+1}{s}\right)\right]$, (4) $\frac{1}{s}\cot^{-1}(s+1)$,

(5) $\frac{6}{(s^2+6s+18)^2}$, (6) $\frac{1}{s}\cdot\cot^{-1}\left(\frac{s+1}{4}\right)$, (7) $\frac{1}{s}\cdot\frac{6(s+2)}{(s^2+4s+13)^2}$,

(8) $\frac{1}{2s}\left[\frac{1}{(s+3)^2}+\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$ (9) $\frac{s^2+a^2}{s(s^2-a^2)^2}$.]

(C) Find the following.

1. $\int_0^\infty e^{-8t}\,erf\sqrt{t}\,dt$
2. $\int_0^\infty e^{-3t}\,erf_c\sqrt{t}\,dt$

[**Ans. :** (1) $\frac{1}{24}$, (2) $\frac{1}{6}$]

(D) Find 1. $L\left[\int_0^t\int_0^t t\cdot\sin t\,(dt)^2\right]$ 2. $L\left[\int_0^t\int_0^t\int_0^t \cos 2t\,(dt)^3\right]$

[**Ans. :** (1) $\frac{2}{s(s^2+1)^2}$, (2) $\frac{1}{s^2(s^2+4)}$]

16. Evaluation of the Integral $\int_0^\infty e^{-at}\left(\int_0^t e^{-u}f(u)\,du\right)dt$

As in the earlier cases, § 5, page 1-13; § 8, page 1-24; § 11, page 1-35, § 13, page 1-43, we can find the value of certain definite integrals, using the Laplace transform. This is illustrated in the following examples.

To find the above integral we first find $L\left[\int_0^t e^{-u}f(u)\,du\right]$, say $\Phi(s)$.

By definition of Laplace transform this means $\int_0^\infty e^{-st}\left[\int_0^t e^{-u}f(u)\,du\right]=\Phi(s)$.

Now, we put $s=a$, on both sides.

Example 1 : Use Laplace transform to evaluate $\int_0^\infty e^{-t}\int_0^t\frac{\sin u}{u}\,du\,dt$. **(M.U. 2004, 13)**

Sol. : We have $L(\sin u)=\frac{1}{s^2+1}$

$$L\left(\frac{\sin u}{u}\right)=\int_s^\infty\frac{ds}{s^2+1}=\left[\tan^{-1}s\right]_s^\infty=\frac{\pi}{2}-\tan^{-1}s=\cot^{-1}s$$

$$\therefore\ L\left[\int_0^t\frac{\sin u}{u}\,du\right]=\frac{1}{s}\cot^{-1}s$$

But this means $\int_0^\infty e^{-st}\left[\int_0^t\frac{\sin u}{u}\,du\right]dt=\frac{1}{s}\cot^{-1}s$

Put $s=1$, $\therefore$ $\int_0^\infty e^{-t}\int_0^t\frac{\sin u}{u}\,du\,dt=\cot^{-1}1=\frac{\pi}{4}$.

Example 2 : Evaluate the following integral by using Laplace transforms.

(i) $\int_0^\infty e^{-2t}\left(\int_0^t\frac{e^{-u}\sin u}{u}\,du\right)dt$ (ii) $\int_0^\infty e^{-t}\left(\int_0^t u^2\sin hu\cos hu\,du\right)dt$ **(M.U. 2003, 11)**

Sol. : (i) As seen in Ex. 9(iii) above, page 1-58.

$$L\left[\int_0^t \frac{e^{-u}\sin u}{u}\,du\right] = \frac{1}{s}\cot^{-1}(s+1)$$

By definition of Laplace transform this means

$$\int_0^\infty e^{-st}\left[\int_0^t \frac{e^{-u}\sin u}{u}\,du\right]dt = \frac{1}{s}\cdot\cot^{-1}(s+1)$$

Putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\left[\int_0^t \frac{e^{-u}\sin u}{u}\,du\right]dt = \frac{1}{2}\cot^{-1}(3).$$

(ii) We have $L(\sin hu\cos hu) = L\left(\frac{1}{2}\sin h\,2u\right) = \frac{1}{2}\cdot\frac{2}{s^2+2^2} = \frac{1}{s^2+4}$

$$\therefore\ L(u^2\sin hu\cos hu) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s^2+4}\right) = \frac{d}{ds}\left(-\frac{2s}{(s^2+4)^2}\right)$$

$$= -2\left[\frac{(s^2+4)^2 - s\cdot 2(s^2+4)\cdot 2s}{(s^2+4)^4}\right] = -2\left[\frac{s^2+4-4s^2}{(s^2+4)^3}\right]$$

$$\therefore\ L(u^2\sin hu\cos hu) = 2\cdot\frac{(3s^2-4)}{(s^2+4)^3} = \Phi(s)\ \text{say}$$

$$\therefore\ L\left[\int_0^t u^2\sin hu\cos hu\,du\right] = \frac{1}{s}\cdot\Phi(s) = \frac{2}{s}\cdot\frac{(3s^2-4)}{(s^2+4)^3}$$ [By (19), page 1-6]

By definition of Laplace transform, this means,

$$\int_0^\infty e^{-st}\left[\int_0^t u^2\sin hu\cos hu\,du\right]dt = \frac{2}{s}\frac{(3s^2-4)}{(s^2+4)^3}$$

Putting $s = 1$, we get

$$\int_0^\infty e^{-t}\left[\int_0^t u^2\sin hu\cos hu\,du\right]dt = \frac{2}{1}\cdot\frac{(3\cdot 1-4)}{(1+4)^3} = -\frac{2}{125}$$

EXERCISE - XV

Evaluate the following integrals by using Laplace transforms.

1. $\int_s^\infty e^{-t}\left(\int_0^t u\cos^2 u\,du\right)dt$
2. $\int_0^\infty e^{-2t}\left[\int_0^t\left(\frac{1-e^{-u}}{u}\right)du\right]dt$
3. $\int_0^\infty e^{-4t}\left(\cos ht\int_0^t e^u\cos hu\,du\right)dt$
4. $\int_0^\infty e^{-t}\left(t\int_0^t e^{-4u}\cos u\,du\right)dt$
5. $\int_0^\infty e^{-t}\left(\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right)dt$

[**Ans. :** **(1)** 11 / 25, **(2)** (1 / 2) log (3 / 2), **(3)** 31 / 225, **(4)** 77 / 338, **(5)** (1 / 4) log 5 – (1 / 2) $\cot^{-1}$ (2)]

EXERCISE - XVI

Theory

1. Define Laplace transform of $f(t)$, $t > 0$. **(M.U. 1995, 2002)**
   Also state the conditions for its existence. **(M.U. 2005)**
2. Define Laplace transform of a function of t and state the rule of change of scale with one example. **(M.U. 2002, 04)**
3. Find **(i)** $L(\sin t)$ **(M.U. 2003)** **(ii)** $L(\cos t)$ **(iii)** $L(e^{at})$ **(iv)** $L(t^n)$
4. State and prove first shifting theorem. Hence, find $L(e^{2t}\cos t\cos 2t)$. **(M.U. 2003, 09)**
   [**Ans. :** $\frac{(s-2)(s^2-4s+9)}{(s^2-4s+13)(s^2-4s+5)}$]
5. State and prove change of scale property of Laplace Transform. **(M.U. 2004, 09)**
6. If $L[f(t)] = \Phi(s)$, prove that $L\left[e^{-at}f(t)\right] = \Phi(s+a)$. **(M.U. 1995)**
7. If $g(t) = f(t-a)$ $t > a$
   $= 0$ $t < a$
   and $L[f(t)] = \Phi(s)$, prove that $L[g(t)] = e^{-as}\Phi(s)$. **(M.U. 1995)**
8. If $L[f(t)] = \Phi(s)$, prove that $L\left[t^n f(t)\right] = (-1)^n\frac{d^n}{ds^n}\Phi(s)$. **(M.U. 1997, 2005)**
9. State and prove First Shifting Theorem. **(M.U. 1997, 99, 2003)**
10. If $L[f(t)] = \Phi(s)$, prove that $L[f(at)] = \frac{1}{a}\Phi\left(\frac{s}{a}\right)$.
11. If $L[f(t)] = \Phi(s)$, prove that $L[t\,f(t)] = -\Phi'(s)$. **(M.U. 1993)**
12. If $L[f(t)] = \Phi(s)$, prove that $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \Phi(s)\,ds$. **(M.U. 1994, 98, 2000)**
13. Prove that $L[f'(t)] = -f(0) + s\,L\,f(t)$.
14. If $L[f(t)] = \Phi(s)$, prove that $L\int_0^t f(u)\,du = \frac{1}{s}\Phi(s)$. **(M.U. 1996)**
15. State and prove second shifting theorem. **(M.U. 2002)**
16. If $J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \ldots$, prove that $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$.
17. If $L[f(t)]$ is known and $f(0) = f'(0) = \ldots = f^n(0) = 0$, write the formula for $L[f^n(t)]$. **(M.U. 2011)**

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CHAPTER

2

Laplace Transforms - II

1. Introduction

In this chapter we shall study the methods of obtaining inverse Laplace transform. We further study Laplace transforms of some special functions. And then we shall apply the La transforms to solve differential equations and boundary value problems.

2. Inverse Laplace Transforms

Definition : If $L[f(t)] = \Phi(s) = \int_0^\infty e^{-st} f(t)\, dt$ then $f(t)$ is called the inverse Lap transform of $\Phi(s)$ and can be denoted as $L^{-1}\Phi(s) = f(t)$.

We get the following inverse transforms of some standard functions from this definition

Table of Inverse Transforms

(i) $L(1) = \dfrac{1}{s}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{1}{s}\right) = 1}$ ....

(ii) $L(e^{-at}) = \dfrac{1}{s+a}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{1}{s+a}\right) = e^{-at}}$ ....

(iii) $L(e^{at}) = \dfrac{1}{s-a}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}}$ ....

(iv) $L(t^{n-1}) = \dfrac{\overline{|n}}{s^n}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{1}{s^n}\right) = \dfrac{t^{n-1}}{\overline{|n}}}$ ....

(v) $L(t^{n-1}) = \dfrac{(n-1)!}{s^n}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{1}{s^n}\right) = \dfrac{t^{n-1}}{(n-1)!}}$ ....

if n is a positive integer.

e.g. $L^{-1}\left(\dfrac{1}{s}\right) = 1,\ L^{-1}\left(\dfrac{1}{s^2}\right) = t,\ L^{-1}\left(\dfrac{1}{s^3}\right) = \dfrac{t^2}{2}$.

(vi) $L(\sin at) = \dfrac{a}{s^2+a^2}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{1}{a}\sin at}$ ....

(vii) $L(\cos at) = \dfrac{s}{s^2+a^2}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at}$ ....

(viii) $L(\sin h\, at) = \dfrac{a}{s^2-a^2}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{1}{s^2-a^2}\right) = \dfrac{1}{a}\sin h\, at}$ .... (8)

(ix) $L(\cos h\, at) = \dfrac{s}{s^2-a^2}$ $\quad \therefore \quad \boxed{L^{-1}\left(\dfrac{s}{s^2-a^2}\right) = \cos h\, at}$ .... (9)

EXERCISE - I

Write down the inverse Laplace transforms of the following.

1. $\dfrac{1}{s^5}$, 2. $\dfrac{1}{s^{3/2}}$, 3. $\dfrac{1}{s^{1/2}}$, 4. $\dfrac{1}{s^6}$, 5. $\dfrac{1}{s^2+9}$, 6. $\dfrac{1}{s^2+25}$, 7. $\dfrac{1}{s-4}$, 8. $\dfrac{1}{s+4}$,

9. $\dfrac{2s}{s^2+4}$, (M.U. 2004) 10. $\dfrac{s}{s^2+16}$, 11. $\dfrac{1}{s^2-1}$, 12. $\dfrac{1}{s^2-4}$, 13. $\dfrac{s}{s^2-4}$,

14. $\dfrac{s}{s^2-36}$, 15. $\dfrac{1}{s}$. [Ans. : Not given for obvious reason.]

Example 1 : Find $L^{-1}\left(\dfrac{1-\sqrt{s}}{s^2}\right)^2$. **(M.U. 2011)**

Sol. : We have

$$L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2 = L^{-1}\left(\frac{1-2\sqrt{s}+s}{s^4}\right) = L^{-1}\left(\frac{1}{s^4}\right) - 2L^{-1}\left(\frac{1}{s^{7/2}}\right) + L^{-1}\left(\frac{1}{s^3}\right)$$

But $L^{-1}\left(\dfrac{1}{s^n}\right) = \dfrac{t^{n-1}}{\overline{|n}}$ or $\dfrac{t^{n-1}}{(n-1)!}$

$$\therefore \quad L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2 = \frac{t^3}{3!} - 2\cdot\frac{t^{5/2}}{\overline{|7/2}} + \frac{t^2}{2!} = \frac{t^3}{6} - \frac{16}{15\sqrt{\pi}}\cdot t^{5/2} + \frac{t^2}{2}.$$

Example 2 : If $L[f(t)] = \dfrac{s+2}{s^2+2}$, find $L[f'(t)]$. **(M.U. 2003)**

Sol. : We have obtained $L[f'(t)]$ and $L[f''(t)]$ on page 1-52, when $f(t)$ was given as a function of t. Here, we are given $L[f(t)]$ in terms of s.

$$\because \quad L[f(t)] = \frac{s+2}{s^2+2} \qquad \therefore\ f(t) = L^{-1}\left[\frac{s+2}{s^2+2}\right]$$

$$\therefore \quad f(t) = L^{-1}\left[\frac{s}{s^2+2}\right] + L^{-1}\left[\frac{2}{s^2+2}\right] = \cos\sqrt{2}\,t + 2\cdot\frac{1}{\sqrt{2}}\sin\sqrt{2}\cdot t$$

Putting $t = 0$, $f(0) = \cos 0 = 1$.

Now, $L[f'(t)] = -f(0) + s\,L[f(t)]$ [By (18), page 1-52]

$$\therefore \quad L[f'(t)] = -1 + s\cdot\frac{s+2}{s^2+2} = \frac{s^2+2s}{s^2+2} - 1 = \frac{2s-2}{s^2+2} = \frac{2(s-1)}{s^2+2}.$$

Example 3 : If $L[f(t)] = \frac{s+3}{s^2+4}$, find $L[f'(t)]$.

[Ans. : $\frac{3s \ldots}{s^2 \ldots}$

Sol. : Do it yourself.

Example 4 : Prove that $L^{-1}\left[\frac{1}{s}\cos\frac{1}{s}\right] = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots$

Sol. : We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\therefore \cos\frac{1}{s} = 1 - \frac{1}{2!}\cdot\frac{1}{s^2} + \frac{1}{4!}\cdot\frac{1}{s^4} - \frac{1}{6!}\cdot\frac{1}{s^6} + \ldots$$

$$\therefore \frac{1}{s}\cos\frac{1}{s} = \frac{1}{s} - \frac{1}{2!}\cdot\frac{1}{s^3} + \frac{1}{4!}\cdot\frac{1}{s^5} - \frac{1}{6!}\cdot\frac{1}{s^7} + \ldots$$

$$\therefore L^{-1}\left[\frac{1}{s}\cos\frac{1}{s}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{2!}\cdot\frac{1}{s^3} + \frac{1}{4!}\cdot\frac{1}{s^5} - \frac{1}{6}\cdot\frac{1}{s^7} + \ldots\right]$$

$$= 1 - \frac{1}{2!}\cdot\frac{t^2}{2!} + \frac{1}{4!}\cdot\frac{t^4}{4!} - \frac{1}{6!}\cdot\frac{t^6}{6!}$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots$$

3. Methods of Obtaining Inverse Laplace Transforms

There are various methods of obtaining inverse Laplace transform. The choice of the method depends upon the nature of the problem. Sometimes an example can be solved by more than one method. You will find below that some solved problems are repeated but note that they are solved by different methods. You are advised to find which method you find more easy.

(a) Use of Standard Results

Using the standard results (given on page 2-1), inverse Laplace transforms of some standard functions can be obtained as illustrated below :

Example : Find the inverse Laplace transform of (i) $\frac{2}{s} + \frac{1}{s^3} + \frac{1}{s+4}$, (ii) $\frac{3s+4}{s^2+16}$.

Sol. : (i) From the above table, we get,

$$L^{-1}\left[\frac{2}{s} + \frac{1}{s^3} + \frac{1}{s+4}\right] = 2L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s+4}\right)$$

$$= 2 + \frac{t^2}{2} + e^{-4t}$$

(ii) $$L^{-1}\left(\frac{3s+4}{s^2+16}\right) = 3L^{-1}\left(\frac{s}{s^2+4^2}\right) + L^{-1}\left(\frac{4}{s^2+4^2}\right)$$

$$= 3\cdot\cos 4t + \sin 4t$$

EXERCISE - II

Find the inverse Laplace transform of

1. $\frac{3+2s+s^2}{s^3}$, 2. $\frac{2s+3}{s^2+9}$, 3. $\frac{3s+5}{9s^2-25}$, 4. $\frac{1}{4s+5}$,

5. $\frac{1}{4s-5}$, 6. $\frac{4s+15}{16s^2-25}$, 7. $\frac{(s^2-1)^2}{s^5}$, 8. $\frac{s+3}{s^2+4}$

[**Ans. :** (1) $3\cdot\frac{t^2}{2} + 2\cdot t + 1$, (2) $2\cos 3t + \sin 3t$, (3) $\frac{1}{3}\cosh\left(\frac{5}{3}\right)t + \frac{1}{3}\sinh\left(\frac{5}{3}\right)t$, (4) $\frac{1}{4}e^{-(5/4)t}$, (5) $\frac{1}{4}e^{(5/4)t}$, (6) $\frac{1}{4}\cosh\left[\frac{5}{4}\cdot t\right] + \frac{3}{4}\sinh\left[\frac{5}{4}\cdot t\right]$, (7) $1 - t^2 + \frac{t^4}{24}$, (8) $\cos 2t + \frac{3}{2}\sin 2t$.]

(b) Use of Shifting Theorem

We know that,

if $L[f(t)] = \Phi(s)$, then $Le^{-at}f(t) = \Phi(s+a)$.

This means if $f(t) = L^{-1}\Phi(s)$ then $L^{-1}[\Phi(s+a)] = e^{-at}f(t)$.

i.e. $$\boxed{L^{-1}[\Phi(s+a)] = e^{-at}L^{-1}\Phi(s)}$$

e.g. $$L^{-1}\frac{1}{(s+2)^2} = e^{-2t}\cdot L^{-1}\frac{1}{s^2} = e^{-2t}\cdot t$$

$$L^{-1}\left[\frac{(s+1)}{(s+1)^2+3^2}\right] = e^{-t}L^{-1}\frac{s}{s^2+3^2} = e^{-t}\cdot\cos 3t$$

Note the following results

(i) $$L^{-1}\frac{1}{(s-b)^n} = e^{bt}L^{-1}\frac{1}{s^n} = e^{bt}\cdot\frac{t^{n-1}}{\lfloor n} \quad \ldots\ldots (10)$$

(ii) $$L^{-1}\frac{1}{(s-b)^n} = e^{bt}L^{-1}\frac{1}{s^n} = e^{bt}\cdot\frac{t^{n-1}}{(n-1)!} \quad \ldots\ldots (11)$$

(if n is an integer)

(iii) $$L^{-1}\frac{1}{(s-b)^2+a^2} = e^{bt}L^{-1}\frac{1}{s^2+a^2} = \frac{e^{bt}}{a}\sin at \quad \ldots\ldots (12)$$

(iv) $$L^{-1}\frac{(s-b)}{(s-b)^2+a^2} = e^{bt}L^{-1}\frac{s}{s^2+a^2} = e^{bt}\cos at \quad \ldots\ldots (13)$$

(v) $$L^{-1}\frac{1}{(s-b)^2-a^2} = e^{bt}L^{-1}\frac{1}{s^2-a^2} = \frac{e^{bt}}{a}\sinh at \quad \ldots\ldots (14)$$

(vi) $$L^{-1}\frac{(s-b)}{(s-b)^2-a^2} = e^{bt}L^{-1}\frac{s}{s^2-a^2} = e^{bt}\cosh at \quad \ldots\ldots (15)$$

EXERCISE - III

Write down the inverse Laplace transform of the following.

1. $\dfrac{1}{s-2}$, 2. $\dfrac{1}{s+3}$, 3. $\dfrac{1}{(s-2)^{3/2}}$, 4. $\dfrac{1}{(s+3)^2}$, 5. $\dfrac{1}{(s-2)^3}$, 6. $\dfrac{1}{(s+5)^{5/2}}$.

7. $\dfrac{1}{(s-2)^2+2^2}$, 8. $\dfrac{(s-3)}{(s-3)^2+2^2}$, 9. $\dfrac{1}{(s-3)^2+4^2}$, 10. $\dfrac{(s-4)}{(s-4)^2+5^2}$.

11. $\dfrac{1}{(s-2)^2-2^2}$, 12. $\dfrac{(s-3)}{(s-3)^2-2^2}$, 13. $\dfrac{1}{(s-3)^2-4^2}$, 14. $\dfrac{(s-4)}{(s-4)^2-5^2}$.

[**Ans. :** Answers not given for obvious rea

Example 1 : Find the inverse Laplace transform of $\dfrac{s}{(s-2)^6}$. (M.U. 2

Sol. :

$$L^{-1}\frac{s}{(s-2)^6}=L^{-1}\left[\frac{(s-2)+2}{(s-2)^6}\right]=L^{-1}\left[\frac{1}{(s-2)^5}+\frac{2}{(s-2)^6}\right]$$

$$=L^{-1}\frac{1}{(s-2)^5}+2L^{-1}\frac{1}{(s-2)^6}=e^{2t}L^{-1}\frac{1}{s^5}+2e^{2t}L^{-1}\frac{1}{s^6}$$

$$=e^{2t}\left[\frac{t^4}{4!}\right]+2e^{2t}\left[\frac{t^5}{5!}\right]=e^{2t}\left[\frac{t^4}{4!}+2\frac{t^5}{5!}\right]$$

Example 2 : Find the inverse Laplace transform of $\dfrac{1}{\sqrt{2s+1}}$.

Sol. :

$$L^{-1}\left[\frac{1}{\sqrt{2s+1}}\right]=L^{-1}\left[\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{s+(1/2)}}\right]$$

$$=\frac{1}{\sqrt{2}}e^{-(1/2)t}\cdot L^{-1}\left[\frac{1}{\sqrt{s}}\right]$$

$$=\frac{1}{\sqrt{2}}e^{-t/2}\cdot\frac{t^{-1/2}}{\lfloor 1/2}$$

$$=\frac{1}{\sqrt{2\pi}}\cdot e^{-t/2}\cdot t^{-1/2}$$

Example 3 : Find the inverse Laplace transform of

(i) $\dfrac{s+2}{s^2-4s+13}$ **(M.U. 2015, 16)** (ii) $\dfrac{4s+12}{s^2+8s+12}$ **(M.U. 2002, 03)**

Sol. : (i)

$$L^{-1}\left[\frac{s+2}{s^2-4s+13}\right]=L^{-1}\left[\frac{s+2}{(s-2)^2+3^2}\right]=L^{-1}\left[\frac{(s-2)+4}{(s-2)^2+3^2}\right]$$

$$=e^{2t}L^{-1}\left[\frac{s+4}{s^2+3^2}\right]=e^{2t}L^{-1}\left(\frac{s}{s^2+3^2}\right)+4\cdot e^{2t}L^{-1}\left(\frac{1}{s^2+3^2}\right)$$

$$=e^{2t}\cdot\cos 3t+\frac{4}{3}e^{2t}\sin 3t$$

(ii)

$$L^{-1}\left[\frac{4s+12}{s^2+8s+12}\right]=L^{-1}\left[\frac{4(s+4)-2^2}{(s+4)^2-2^2}\right]$$

$$=L^{-1}\left[\frac{4(s+4)}{(s+4)^2-2^2}\right]-L^{-1}\left[\frac{2^2}{(s+4)^2-2^2}\right]$$

$$=4e^{-4t}L^{-1}\left[\frac{s}{s^2-2^2}\right]-4e^{-4t}L^{-1}\left[\frac{1}{s^2-2^2}\right]$$

$$=4e^{-4t}\cosh 2t-4e^{-4t}\cdot\frac{1}{4}\sinh 2t$$

$$=e^{-4t}(4\cosh 2t-\sinh 2t)$$

Example 4 : Find :

(i) $L^{-1}\left[\dfrac{s+2}{s^2+4s+7}\right]$ **(M.U. 1993, 2008)** (ii) $L^{-1}\left[\dfrac{2s+3}{s^2+2s+2}\right]$, **(M.U. 1994)** (iii) $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right]$. **(M.U. 1999, 2003)**

(iv) $L^{-1}\left[\dfrac{2s-1}{s^2+4s+29}\right]$ (v) $L^{-1}\left[\dfrac{s+2}{s^2-2s+17}\right]$ **(M.U. 2016)**

Sol. : (i)

$$L^{-1}\left[\frac{s+2}{s^2+4s+7}\right]=L^{-1}\left[\frac{(s+2)}{(s+2)^2+(\sqrt{3})^2}\right]$$

$$=e^{-2t}L^{-1}\frac{s}{s^2+(\sqrt{3})^2}=e^{-2t}\cdot\cos\sqrt{3}\,t.$$

(ii)

$$L^{-1}\left[\frac{2s+3}{s^2+2s+2}\right]=L^{-1}\left[\frac{2(s+1)+1}{(s+1)^2+1}\right]$$

$$=2L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right]+L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=2e^{-t}L^{-1}\left[\frac{s}{s^2+1}\right]+e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$=2e^{-t}\cos t+e^{-t}\cdot\sin t=e^{-t}[2\cos t+\sin t].$$

(iii)

$$L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right]=L^{-1}\left[\frac{3(s-1)+10}{(s-1)^2-2^2}\right]$$

$$=3L^{-1}\left[\frac{s-1}{(s-1)^2-(2)^2}\right]+10L^{-1}\frac{1}{(s-1)^2-(2)^2}$$

$$\therefore\quad L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right]=3e^{t}L^{-1}\left[\frac{s}{s^2-(2)^2}\right]+10e^{t}L^{-1}\left[\frac{1}{s^2-(2)^2}\right]$$

$$=3e^{t}\cosh 2t+\frac{10}{2}e^{t}\sinh 2t$$

$$=e^{t}(3\cosh 2t+5\sinh 2t).$$

(iv) $$L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right]=L^{-1}\left[\frac{2s+4-5}{s^2+4s+4+25}\right]$$
$$=L^{-1}\left[\frac{2(s+2)}{(s+2)^2+5^2}-\frac{5}{(s+2)^2+5^2}\right]$$
$$=2L^{-1}\left[\frac{s+2}{(s+2)^2+5^2}\right]-5L^{-1}\left[\frac{1}{(s+2)^2+5^2}\right]$$
$$=2e^{-2t}L^{-1}\left[\frac{s}{s^2+5^2}\right]-5e^{-2t}\cdot L^{-1}\left[\frac{1}{s^2+5^2}\right]$$
$$=2e^{-2t}\cos 5t-5e^{-2t}\cdot\frac{1}{5}\sin 5t$$
$$=2e^{-2t}\cos 5t-e^{-2t}\sin 5t$$

(v) $$L^{-1}\left[\frac{s+2}{s^2-2s+17}\right]=L^{-1}\left[\frac{(s-1)+3}{s^2-2s+1+16}\right]=L^{-1}\left[\frac{(s-1)+3}{(s-1)^2+4^2}\right]$$
$$=L^{-1}\left[\frac{s-1}{(s-1)^2+4^2}\right]+L^{-1}\left[\frac{3}{(s-1)^2+4^2}\right]$$
$$=e^{t}\cdot L^{-1}\left(\frac{s}{s^2+4^2}\right)+e^{t}\cdot 3\cdot L^{-1}\left(\frac{1}{s^2+4^2}\right)$$
$$=e^{t}\cos 4t+\frac{3}{4}e^{t}\sin 4t$$

EXERCISE - IV

Find the inverse Laplace transform of

1. $\dfrac{6s-4}{s^2-4s+20}$, (M.U. 2005) 2. $\dfrac{3s-7}{s^2-6s+8}$, 3. $\dfrac{1}{(s-3)^3}$,

4. $\dfrac{1}{(s+2)^4}$, 5. $\dfrac{1}{(s+1)^2}+\dfrac{s-2}{s^2-4s+5}+\dfrac{s-2}{s^3-4s+3}$,

6. $\dfrac{s}{s^2+2s+2}$ (M.U. 2004) 7. $\dfrac{s}{(2s+1)^2}$ (M.U. 2003)

[Ans. : (1) $6\cdot e^{2t}\cos 4t+2e^{2t}\sin 4t$, (2) $3\cdot e^{3t}\cdot\cos ht+2\cdot e^{3t}\sin ht$,
(3) $\frac{1}{2}e^{3t}t^2$, (4) $\frac{1}{3!}e^{-2t}\cdot t^3$, (5) $e^{-t}\cdot t+e^{2t}\cos t+e^{2t}\cos ht$,
(6) $e^{-t}(\cos t-\sin t)$, (7) $[e^{-t/2}(2-t)]/8$.]

(c) Method of Partial Fractions

Whenever possible we express the given function $\Phi(s)$ into the sum of linear or qu partial fractions as $\Phi(s)=\dfrac{A}{(s+a)^r}+\dfrac{Bs+C}{(s^2+a^2)^r}$ and then use the standard results given ab find L^{-1}.

Example 1 : Find inverse Laplace transform of

(i) $\dfrac{s^2+16s-24}{s^4+20s^2+64}$ (M.U. 2005) (ii) $\dfrac{s+29}{(s+4)(s^2+9)}$ (M.U. 1999, 2009, 15)

(iii) $\dfrac{s+2}{(s+3)(s+1)^3}$ (M.U. 2005, 10) (iv) $\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$ (M.U. 2005)

(v) $\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$ (vi) $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)}$ (M.U. 2007)

Sol. : (i) $$\Phi(s)=\frac{s^2+16s-24}{s^4+20s^2+64}=\frac{s^2+16s-24}{(s^2+4)(s^2+16)}$$

Let $$\Phi(s)=\frac{as+b}{s^2+4}+\frac{cs+d}{s^2+16}$$
$$\therefore\ s^2+16s-24=(as+b)(s^2+16)+(cs+d)(s^2+4)$$
$$=(a+c)s^3+(b+d)s^2+(16a+4c)s+(16b+4d)$$

Equating the coefficients of like powers of s we get, $a+c=0$, $b+d=1$, $16a+4c=16$, $16b+4d=-24$.

$\therefore\ a=4/3,\ c=-4/3,\ b=-7/3,\ d=10/3$

$$\therefore\ \Phi(s)=\frac{1}{3}\cdot\frac{4s-7}{s^2+4}-\frac{1}{3}\cdot\frac{4s-10}{s^2+16}$$
$$\therefore\ L^{-1}[\Phi(s)]=\frac{4}{3}L^{-1}\left(\frac{s}{s^2+4}\right)-\frac{7}{3}L^{-1}\left(\frac{1}{s^2+4}\right)-\frac{4}{3}L^{-1}\left(\frac{s}{s^2+16}\right)+\frac{10}{3}L^{-1}\left(\frac{1}{s^2+16}\right)$$
$$=\frac{4}{3}\cos 2t-\frac{7}{3}\cdot\frac{1}{2}\sin 2t-\frac{4}{3}\cdot\cos 4t+\frac{10}{3}\cdot\frac{1}{4}\sin 4t$$

(ii) $$\Phi(s)=\frac{s+29}{(s+4)(s^2+9)}=\frac{a}{s+4}+\frac{bs+c}{s^2+9},\ \text{say}$$
$$\therefore\ s+29=a(s^2+9)+(bs+c)(s+4)$$
$$\therefore\ s+29=(a+b)s^2+(c+4b)s+(9a+4c).$$

Equating the coefficients of like powers of s, we get,

$\therefore\ a+b=0,\ c+4b=1,\ 9a+4c=29$ $\quad\therefore\ a=1,\ b=-1,\ c=5.$

$$\therefore\ \Phi(s)=\frac{1}{s+4}-\frac{s-5}{s^2+9}$$
$$\therefore\ L^{-1}[\Phi(s)]=L^{-1}\left(\frac{1}{s+4}\right)-L^{-1}\left(\frac{s}{s^2+9}\right)+5L^{-1}\left(\frac{1}{s^2+9}\right)=e^{-4t}-\cos 3t+5\cdot\frac{1}{3}\sin 3t$$

(iii) $$\Phi(s)=\frac{s+2}{(s+3)(s+1)^3}$$
$$=\frac{1}{8}\cdot\frac{1}{(s+3)}-\frac{1}{8}\cdot\frac{1}{(s+1)}+\frac{1}{4}\cdot\frac{1}{(s+1)^2}+\frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

(after expressing the function in partial fractions)

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$$\therefore L^{-1}[\Phi(s)] = \frac{1}{8}\cdot L^{-1}\left(\frac{1}{s+3}\right) - \frac{1}{8}\cdot L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{4}\cdot L^{-1}\frac{1}{(s+1)^2} + \frac{1}{2}\cdot L^{-1}\frac{1}{(s+1)^3}$$

But $L^{-1}\left[\frac{1}{(s+b)^{n+1}}\right] = \frac{1}{n!}\cdot e^{-bt}\cdot t^n$

$$\therefore L^{-1}[\Phi(s)] = \frac{1}{8}\cdot e^{-3t} - \frac{1}{8}\cdot e^{-t} + \frac{1}{4}\cdot\frac{1}{1!}e^{-t}\cdot t + \frac{1}{2}\cdot\frac{1}{2!}e^{-t}\cdot t^2$$

$$= \frac{1}{8}\left[(2t^2+2t-1)e^{-t} + e^{-3t}\right]$$

(iv) $$\Phi(s) = \frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \frac{as+b}{(s^2+2s+5)} + \frac{cs+d}{(s^2+2s+2)}$$

Simplifying and equating like powers of s on both sides, we get, $a = 0$, $b = 3$, $c = 0$, $d = $

$$\therefore \Phi(s) = \frac{3}{s^2+2s+5} - \frac{2}{s^2+2s+2} = \frac{3}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+1^2}$$

$$\therefore L^{-1}\Phi(s) = 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] - 2\cdot L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$= 3e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right] - 2e^{-t}L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= 3e^{-t}\cdot\frac{1}{2}\sin 2t - 2e^{t}\cdot 1\cdot\sin t$$

(v) Let us first consider

$$\frac{1}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1/24}{s^2+1} - \frac{1/15}{s^2+4} + \frac{1/40}{s^2+9}$$ [Note thi

Now, $$\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1}{24}\cdot\frac{s}{s^2+1} - \frac{1}{15}\cdot\frac{s}{s^2+4} + \frac{1}{40}\cdot\frac{s}{s^2+9}$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right] = \frac{1}{24}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{15}L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{40}L^{-1}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{1}{24}\cos t - \frac{1}{15}\cos 2t + \frac{1}{40}\cos 3t$$

(vi) $$\frac{s+2}{(s^2+4s+8)(s^2+4s+13)} = \frac{s+2}{[(s+2)^2+2^2][(s+2)^2+3^2]}$$

$$\therefore L^{-1}\frac{s+2}{(s^2+4s+8)(s^2+4s+13)} = L^{-1}\frac{1}{5}\left[\frac{s+2}{(s+2)^2+2^2} - \frac{s+2}{(s+2)^2+3^2}\right]$$

$$= \frac{1}{5}\left[e^{-2t}L^{-1}\frac{s}{s^2+2^2} - e^{-2t}L^{-1}\frac{s}{s^2+3^2}\right] = \frac{e^{-2t}}{5}(\cos 2t - \cos 3t)$$

Example 2 : Find the inverse Laplace transform of the following.

(i) $\frac{3s+1}{(s+1)(s^2+2)}$. **(M.U. 1995, 2010)**

(ii) $\frac{5s^2-15s-11}{(s+1)(s-2)^2}$ **(M.U. 1995, 2007, 11)**

(iii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ **(M.U. 1996, 98, 2000, 03)**

(iv) $\frac{s}{(s^2+a^2)(s^2+b^2)}$ **(M.U. 2004)**

(v) $\frac{5s^2+8s-1}{(s+3)(s^2+1)}$ **(M.U. 1996, 2002, 07)**

(vi) $\frac{2s}{s^4+4}$ **(M.U. 1997, 2002, 10, 15)**

(vii) $\frac{s+2}{s^2(s+3)}$ **(M.U. 1997, 2003)**

(viii) $\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$ **(M.U. 2003, 04, 05, 08)**

(ix) $\frac{s}{s^4+4a^4}$ **(M.U. 2004, 08)**

Sol. : (i) Let $\frac{3s+1}{(s+1)(s^2+2)} = \frac{a}{s+1} + \frac{bs+c}{s^2+2}$

$$\therefore 3s+1 = a(s^2+2)+(s+1)(bs+c) = (a+b)s^2+(b+c)s+2a+c.$$

Equating the coefficients of like powers of s.

$$\therefore a+b=0,\ b+c=3,\ 2a+c=1$$

Solving these equations, we get $a = -\frac{2}{3}$, $b = \frac{2}{3}$, $c = \frac{7}{3}$.

$$\therefore L^{-1}\left[\frac{3s+1}{(s+1)(s^2+2)}\right] = -\frac{2}{3}L^{-1}\frac{1}{s+1} + \frac{2}{3}L^{-1}\frac{s}{s^2+2} + \frac{7}{3}L^{-1}\frac{1}{s^2+2}$$

$$= -\left(\frac{2}{3}\right)e^{-t}L^{-1}\frac{1}{s} + \frac{2}{3}L^{-1}\frac{s}{s^2+(\sqrt{2})^2} + \frac{7}{3}L^{-1}\frac{1}{s^2+(\sqrt{2})^2}$$

$$= -\left(\frac{2}{3}\right)e^{-t}\cdot 1 + \frac{2}{3}\cos\sqrt{2}\,t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}\,t.$$

(ii) Let $\frac{5s^2-15s-11}{(s+1)(s-2)^2} = \frac{a}{s+1} + \frac{b}{s-2} + \frac{c}{(s-2)^2}$

$$\therefore 5s^2-15s-11 = a(s-2)^2 + b(s+1)(s-2) + c(s+1)$$

Equating the coefficient of like power of s

$$a+b=5,\ -4a-b-c=-15,\ 4a-2b+c=-11 \quad \therefore a=1,\ b=4,\ c=-7.$$

$$\therefore L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^2}\right] = L^{-1}\frac{1}{s+1} + 4L^{-1}\frac{1}{s-2} - 7L^{-1}\frac{1}{(s-2)^2}$$

$$= e^{-t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s} - 7e^{2t}L^{-1}\frac{1}{s^2}$$

$$= e^{-t} + 4e^{2t} - 7e^{2t}t.$$

(iii) Let $s^2 = x$ $\therefore \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{x}{(x+a^2)(x+b^2)}$ **[Note this]**

Let $\frac{x}{(x+a^2)(x+b^2)} = \frac{A}{x+a^2} + \frac{B}{x+b^2}$ $\therefore x = A(x+b^2) + B(x+a^2)$

Put $x = -a^2$ $\quad \therefore \; -a^2 = A(-a + B^2)$ $\quad \therefore \; A = a^2/(a^2 - b^2)$

Put $x = -b^2$ $\quad \therefore \; -b^2 = B(-b^2 + a^2)$ $\quad \therefore \; B = -b^2/(a^2 - b^2)$

$$\therefore \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2}\left[L^{-1}\left(\frac{a^2}{s^2+a^2}\right) - L^{-1}\left(\frac{b^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[a^2 \cdot L^{-1}\left(\frac{1}{s^2+a^2}\right) - b^2 \cdot L^{-1}\left(\frac{1}{s^2+b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{a^2}{a} \cdot \sin at - b^2 \cdot \frac{1}{b}\sin bt\right]$$

$$= \frac{1}{a^2-b^2}(a \sin at - b \sin bt)$$

[For another method see Ex. 3 (ii), page 2-20]

(iv) Let us first consider

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{1}{s+a^2} - \frac{1}{s^2+b^2}\right] \qquad \textbf{[Note this]}$$

$$\therefore L^{-1}\frac{s}{(s^2+a^2)(s^2+b^2)} = L^{-1}\frac{1}{b^2-a^2}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{(b^2-a^2)}\left[L^{-1}\frac{s}{s^2+a^2} - L^{-1}\frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2}(\cos at - \cos bt)$$

[For another method see Ex. 3 (iii), page 2-20]

(v) Let $\dfrac{5s^2+8s-1}{(s+3)(s^2+1)} = \dfrac{a}{s+3} + \dfrac{bs+c}{s^2+1}$

$$\therefore 5s^2 + 8s - 1 = a(s^2+1) + (bs+c)(s+3)$$

$$= (a+b)s^2 + (3b+c)s + (a+3c)$$

Equating the coefficients of like powers of s, we get,

$a + b = 5,\; 3b + c = 8,\; a + 3c = -1$

$\therefore \; a = 2,\; b = 3,\; c = -1$

$$\therefore L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = L^{-1}\left[\frac{2}{s+3} + 3 \cdot \frac{s}{s^2+1} - \frac{1}{s^2+1}\right].$$

$$\therefore L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = 2L^{-1}\frac{1}{s+3} + 3L^{-1}\frac{s}{s^2+1} - L^{-1}\frac{1}{s^2+1}$$

$$= 2e^{-3t} + 3\cos t - \sin t$$

(vi) $\dfrac{2s}{s^4+4} = \dfrac{2s}{(s^4+4s^2+4)-4s^2} = \dfrac{2s}{(s^2+2)^2-(2s)^2}$ **[Note this]**

$$= \frac{1}{2}\left[\frac{1}{(s^2-2s+2)} - \frac{1}{(s^2+2s+2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore L^{-1}\frac{2s}{s^4+4} = \frac{1}{2}\left[L^{-1}\frac{1}{(s-1)^2+1} - L^{-1}\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[e^t L^{-1}\frac{1}{s^2+1} - e^{-t}L^{-1}\frac{1}{s^2+1}\right]$$

$$\therefore L^{-1}\frac{2s}{s^4+4} = \frac{1}{2}\left[e^t \sin t - e^{-t}\sin t\right] = \sin t\left(\frac{e^t - e^{-t}}{2}\right) = \sin t \sin ht.$$

(vii) Let $\dfrac{s+2}{s^2(s+3)} = \dfrac{a}{s} + \dfrac{b}{s^2} + \dfrac{c}{s+3}$

$$\therefore s + 2 = as(s+3) + b(s+3) + cs^2$$

Putting $s = 0$, $2 = 3b$; putting $s = -3$, $-1 = 9c$.

Equating the coefficients of s^2, $a + c = 0$. $\therefore \; a = 1/9$.

$$\therefore L^{-1}\frac{s+2}{s^2(s+3)} = \frac{1}{9}L^{-1}\left(\frac{1}{s}\right) + \frac{2}{3}L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{9}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{9}(1) + \frac{2}{3}t - \frac{1}{9}e^{-3t} = \frac{1}{9}(1 + 6t - e^{-3t})$$

(viii) $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \dfrac{(s+1)^2+2}{\left[(s+1)^2+2^2\right]\left[(s+1)^2+1^2\right]}$ **[Note this]**

$$\therefore L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = e^{-t}L^{-1}\frac{s^2+2}{(s^2+4)(s^2+1)}$$

Let $s^2 = x$ and $\dfrac{x+2}{(x+4)(x+1)} = \dfrac{a}{x+4} + \dfrac{b}{x+1}$ **[Note this]**

$$\therefore x + 2 = a(x+1) + b(x+4)$$

When $x = -1$, $1 = 3b$; when $x = -4$, $-2 = -3a$.

$$\therefore \frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{2}{3} \cdot \frac{1}{s^2+4} + \frac{1}{3} \cdot \frac{1}{s^2+1}$$

$$\therefore L^{-1}\frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{2}{3}L^{-1}\frac{1}{s^2+4} + \frac{1}{3}L^{-1}\frac{1}{s^2+1} = \frac{2}{3} \cdot \frac{1}{2}\sin 2t + \frac{1}{3}\sin t$$

$$\therefore L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

(For another method see Ex. 4 (iii), page 2-23.)

(ix) We have, by completing the square,

$$\frac{s}{s^4+4a^4}=\frac{s}{(s^4+4s^2a^2+4a^4)-(4s^2a^2)}$$

$$\therefore \frac{s}{s^4+4a^4}=\frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$$

[Note this]

First consider that (by inspection)

$$\frac{s}{s^4+4a^4}=\frac{1}{4a}\left[\frac{1}{s^2+2a^2-2as}-\frac{1}{s^2+2a^2+2as}\right]$$

$$\therefore L^{-1}\frac{s}{s^4+4a^4}=\frac{1}{4a}\left\{L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]-L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right\}$$

$$=\frac{1}{4a}\left[e^{at}L^{-1}\left(\frac{1}{s^2+a^2}\right)-e^{-at}L^{-1}\left(\frac{1}{s^2+a^2}\right)\right]$$

$$\therefore L^{-1}\frac{s}{s^4+4a^4}=\frac{1}{4a}\left[e^{at}\frac{\sin at}{a}-e^{-at}\frac{\sin at}{a}\right]$$

$$=\frac{\sin at}{2a^2}\left[\frac{e^{at}-e^{-at}}{2}\right]=\frac{1}{2a^2}\sin at\sinh at.$$

EXERCISE - V

Find the inverse Laplace transform of

1. $\frac{s+1}{s^2-4}$ (M.U. 2013)
2. $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$
3. $\frac{s+4}{(s^2-1)(s+1)}$ (M.U. 2003)
4. $\frac{s^2+1}{s^3+3s^2+2s}$ (M.U.
5. $\frac{s}{(s^2+5s+6)}$
6. $\frac{3s+7}{s^2-2s-3}$ (M.U. 2002, 10)
7. $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (M.U. 2003, 05)
8. $\frac{s^2+10s+1}{(s-1)(s^2-5s}$
9. $\frac{1}{s^2(s+1)}$
10. $\frac{1}{s^2(s+2)}$
11. $\frac{s^2}{(s+4)^3}$
12. $\frac{s^2}{(s+1)^3}$ (M.U.
13. $\frac{1}{s^2(s+3)^2}$
14. $\frac{1}{(s-2)(s+2)^2}$
15. $\frac{4s+4}{(s-1)^2(s+2)}$
16. $\frac{2s+3}{(s+1)^2(s+2}$

*17. $\frac{1}{(s-2)^4(s+3)}$ (M.U. 2011)

18. $\frac{1}{s(s^2-a^2)}$
19. $\frac{2s^2-1}{(s^2+1)(s^2+4)}$ (M.U. 2011)
20. $\frac{s}{(s^2+1)(}$ (M.U.
21. $\frac{s+29}{(s+4)(s^2+9)}$ (M.U. 2007, 13)
22. $\frac{5s+3}{(s-1)(s^2+2s+5)}$ (M.U. 2002, 06)

*23. $\frac{11s^2-2s+5}{2s^3-3s^2-3s+2}$

24. $\frac{s}{(s-3)(s^2+4)}$

*25. $\frac{s^2}{(s^2+1)(s^2+4)}$

*26. $\frac{2s-1}{(s^4+s^2+1)}$ (M.U. 2007)

27. $\frac{s^2}{(s^2+9)(}$

28. $\frac{1}{(s-2)^2(s+3)}$ (M.U. 2003)

*29. $\frac{1}{s^3+1}$ (M.U. 2003)

30. $\frac{1}{s^3(s-1)}$ (M.U. 2003, 14)

31. $\frac{s}{(s+1)^2(s^2+1)}$ (M.U. 2004)

*32. $\frac{s+2}{(s+3)(s+1)^3}$ (M.U. 2005)

*33. $\frac{s(s^2+2a^2)}{s^4+4a^4}$ (M.U. 2009)

*34. $\frac{s^2}{(s+a)^3}$ (M.U. 2005)

(* Star indicates harder problems.)

[Ans. : (1) $\frac{1}{4}[e^{-2t}+3e^{2t}]$, (2) $-\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}$, (3) $\frac{5}{4}e^{t}-\frac{5}{4}e^{-t}-\frac{3}{2}te^{-t}$

(4) $\frac{1}{2}-2e^{-t}+\frac{5}{2}e^{-2t}$, (5) $3e^{-3t}-2e^{-2t}$, (6) $4e^{3t}-e^{-t}$,

(7) $\frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}$, (8) $12e^{t}-37e^{2t}+26e^{3t}$, (9) $-1+t+e^{-t}$,

(10) $-\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}$, (11) $e^{-4t}(1-8t+8t^2)$, (12) $e^{-t}\left\{1-2t+\frac{t^2}{2}\right\}$,

(13) $\frac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})$, (14) $\frac{1}{16}(e^{2t}-e^{-2t}-4te^{-2t})$,

(15) $\frac{4}{9}(e^{t}-e^{-2t}+6te^{t})$, (16) $e^{-t}-e^{-2t}+te^{-t}$,

(17) $\frac{e^{2t}}{6}\left[\frac{t^3}{5}-\frac{3}{25}t^2+\frac{6}{125}t-\frac{6}{625}\right]+\frac{1}{625}e^{-3t}$,

(18) $\frac{1}{2a^2}[e^{at}+e^{-at}-2]$, (19) $\frac{3}{2}\cdot\sin 2t-\sin t$, (20) $\frac{1}{3}(\cos t-\cos 2t)$,

(21) $e^{-4t}-\cos 3t+\frac{5}{3}\sin 3t$, (22) $e^{t}-e^{-t}\cos 2t+\frac{3}{2}e^{-t}\sin 2t$,

(23) $2e^{-t}+5e^{2t}-\frac{3}{2}e^{t/2}$, (24) $\frac{3}{13}e^{3t}-\frac{3}{13}\cos 2t+\frac{2}{13}\sin 2t$,

(25) $-\frac{1}{3}\sin t+\frac{2}{3}\sin 2t$,

(26) $\frac{1}{2}e^{t/2}\cos\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{2}e^{t/2}\sin\frac{\sqrt{3}}{2}t-\frac{1}{2}e^{-t/2}\cos\frac{\sqrt{3}}{2}t-\frac{5}{2\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$

(27) $\frac{1}{5}(3\sin 3t-2\sin 2t)$, (28) $-\frac{1}{25}e^{2t}+\frac{t}{5}e^{2t}+\frac{1}{25}e^{-3t}$,

(29) $\frac{1}{3}e^{-t}-\frac{e^{t/2}}{3}\cdot\cos\left(\frac{\sqrt{3}}{2}\cdot t\right)+\frac{e^{t/2}}{\sqrt{3}}\cdot\sin\left(\frac{\sqrt{3}}{2}\cdot t\right)$,

(30) $e^{-t}-\left[1+t+\frac{t^2}{2}\right]$, (31) $\frac{1}{2}\left[\sin t-t\cdot e^{-t}\right]$,

(32) $\frac{1}{6}e^{-3t}-\frac{1}{8}e^{-t}+\frac{1}{4}t\cdot e^{-t}+\frac{1}{4}t^2\cdot e^{-t}$, (33) $\sin at\cos h\,at$.

(34) $e^{-at}\left[1-2at+\frac{a^2t^2}{2}\right]$

(d) Use of by Convolution Theorem

Definition : If $f_1(t)$ and $f_2(t)$ are two functions then the following integral

$$\int_0^t f_1(u)\, f_2(t-u)\, du$$

is called the convolution (= twisting, coiling, winding together) of $f_1(t)$ and $f_2(t)$ and is denoted by $f_1(t) * f_2(t)$. Thus,

$$f_1(t) * f_2(t) = \int_0^t f_1(t)\, f_2(t-u)\, du.$$

Theorem : Let $L\, f_1(t) = \Phi_1(s)$ and $L\, f_2(t) = \Phi_2(s)$, then

$$\boxed{L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t f_1(u)\cdot f_2(t-u)\, du} \quad \ldots\ldots\ldots (1$$

where, $f_1(t) = L^{-1}\Phi_1(s)$ and $f_2(t) = L^{-1}\Phi_2(s)$

Proof. : Let $\Phi(t) = \int_0^t f_1(u)\cdot f_2(t-u)\, du.$

$$\therefore\ L[\Phi(t)] = \int_0^\infty e^{-st}\,\Phi(t)\, dt = \int_0^\infty e^{-st}\int_0^t f_1(u)\cdot f_2(t-u)\, du\, dt$$

$$= \int_0^\infty\int_0^t e^{-st} f_1(u)\cdot f_2(t-u)\, du\, dt$$

On the r.h.s. integration is first carried out w.r.t. u and then w.r.t. t. Now, we change the order of integration. We integrate first with respect to t and then w.r.t. u. But now t varies from u to ∞ and then u varies from 0 to ∞.

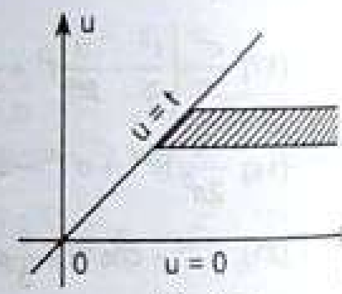


Fig. 2.1

$$\therefore\ L[\Phi(t)] = \int_0^\infty\int_u^\infty e^{-st} f_1(u)\, f_2(t-u)\, dt\, du$$

$$= \int_0^\infty\int_u^\infty e^{-su} e^{-s(t-u)} f_1(u)\, f_2(t-u)\, dt\, du$$

$$= \int_0^\infty e^{-su} f_1(u)\, du \int_u^\infty e^{-s(t-u)} f_2(t-u)\, dt$$

Putting $t-u=p,\ dt=dp$

$$\therefore\ L[\Phi(t)] = \int_0^\infty e^{-su} f_1(u)\, du \int_0^\infty e^{-sp} f_2(p)\, dp$$

Since for definite integral, the variable used does not matter, we replace u by t and p by t

$$\therefore\ L[\Phi(t)] = \int_0^\infty e^{-st} f_1(t)\, dt\cdot\int_0^\infty e^{-st} f_2(t)\, dt$$

$$\therefore\ L\Phi(t) = L f_1(t)\cdot L f_2(t) \quad \therefore\ L\Phi(t) = \Phi_1(s)\cdot\Phi_2(s)$$

$$\therefore\ L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \Phi(t) = \int_0^t f_1(u)\cdot f_2(t-u)\, du$$

Cor. : If $\Phi_1(s) = \Phi(s)$ and $\Phi_2(s) = \frac{1}{s}$ then $f_2(t) = L^{-1}\Phi_2(s) = 1.$

$$\therefore\ \boxed{L^{-1}\left[\frac{1}{s}\Phi(s)\right] = \int_0^t f_1(u)\, du = \int_0^t L^{-1}[\Phi(s)]\, du} \quad \ldots\ldots\ldots (16$$

Notes ....

1. Taking Laplace transforms of both sides of (16), we get

$$\Phi_1(s)\,\Phi_2(s) = L\left[\int_0^t f_1(u)\, f_2(t-u)\, du\right]$$

But $\Phi_1(s) = L[f_1(t)]$ and $\Phi_2(s) = L[f_2(t)]$

$$\therefore\ L\left[\int_0^t f_1(u)\cdot f_2(t-u)\, du\right] = L[f_1(t)]\cdot L[f_2(t)]$$

This means the Laplace transform of the convolution of two functions is equal to the product of the Laplace transforms of the two functions.

2. The above result given in the corollary is obtained independently in § (f) (18), page 2-35.

Procedure of Applying Convolution Theorem

To find $L^{-1}\Phi_1(s)\cdot\Phi_2(s)$:-

1. Find $L^{-1}\Phi_1(s) = f_1(u)$, say, putting u in place of t.
2. Find $L^{-1}\Phi_2(s) = f_2(u)$, say, putting u in place of t.
3. Find $L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)\, du.$

Example 1 : Find the inverse Laplace Transforms by using convolution theorem.

(i) $\frac{1}{s(s+a)}$, **(ii)** $\frac{1}{s(s+a)^2}$, **(iii)** $\frac{1}{s(s^2+a^2)}$, **(iv)** $\frac{1}{s(s^2-a^2)}$.

Sol. : In all these examples put $\Phi_2(s) = \frac{1}{s}$, then by the corollary

$$\boxed{L^{-1}\left[\frac{1}{s}\cdot\Phi(s)\right] = \int_0^t \left[L^{-1}\Phi(s)\right] du}$$

(i) $$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+a}\right] = \int_0^t\left[L^{-1}\left(\frac{1}{s+a}\right)\right] du = \int_0^t e^{-au}\, du = \left[-\frac{e^{-au}}{a}\right]_0^t$$

$$= -\left[\frac{e^{-at}-1}{a}\right] = \frac{1-e^{-at}}{a}.$$

(ii) $$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s+a)^2}\right] = \int_0^t\left[L^{-1}\frac{1}{(s+a)^2}\right] du = \int_0^t e^{-au}\left[L^{-1}\left(\frac{1}{s^2}\right)\right] du$$

$$= \int_0^t e^{-au}\cdot u\, du = \int_0^t u\cdot e^{-au}\, du = \left[u\cdot\frac{e^{-au}}{-a} - \frac{e^{-au}}{(-a)^2}\cdot 1\right]_0^t$$

$$\therefore\ L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s+a)^2}\right] = \left[-\frac{t\,e^{-at}}{a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}\right] = \frac{1}{a^2}\left[1-(1+at)\,e^{-at}\right]$$

(iii) $$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s^2+a^2)}\right] = \int_0^t\left[L^{-1}\frac{1}{s^2+a^2}\right] du = \int_0^t\left(\frac{\sin au}{a}\right) du$$

$$\therefore \; L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s^2+a^2)}\right] = -\frac{1}{a^2}\left[\cos au\right]_0^t = -\frac{1}{a^2}\left[\cos at - 1\right] = \frac{1-\cos at}{a^2}$$

(iv) $$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s^2-a^2)}\right] = \int_0^t \left[L^{-1}\frac{1}{s^2-a^2}\right] du = \int_0^t \frac{\sinh au}{a}\, du$$

$$= \frac{1}{a^2}\left[\cosh au\right]_0^t = \frac{1}{a^2}\left[\cosh at - 1\right].$$

EXERCISE - VI

Find the inverse of the following functions using convolution theorem.

(i) $\frac{1}{s(s^2+4)}$, (ii) $\frac{1}{s(s+2)}$, (iii) $\frac{1}{s(s^2-9)}$, (iv) $\frac{1}{s(s+4)^2}$.

[**Ans. :** (i) $\frac{1}{4}(1-\cos 2t)$, (ii) $\frac{e^{2t}-1}{2}$, (iii) $\frac{1}{9}[\cosh 3t - 1]$, (iv) $\frac{1}{16}[1-(1+4t)e^{-}$

Example 2 : Find the inverse Laplace Transforms by using convolution theorem

(i) $\frac{s^2}{(s^2+a^2)^2}$ **(M.U. 1995, 2000, 02, 07)**

(ii) $\frac{s^2}{(s^2-a^2)^2}$ **(M.U. 2002, 03, 09, 14)**

(iii) $\frac{1}{(s-a)(s+a)^2}$

(iv) $\frac{s}{(s^2+a^2)^2}$ **(M.U. 2000, 03, 11, 12, 13)**

(v) $\frac{s}{(s^2-a^2)^2}$ **(M.U. 2003, 15)**

(vi) $\frac{1}{(s-1)(s^2+4)}$ **(M.U. 2014)**

Sol. : (i) Let $\Phi_1(s) = \frac{s}{s^2+a^2}$, $\Phi_2(s) = \frac{s}{s^2+a^2}$.

$$L^{-1}\Phi_1(s) = L^{-1}\frac{s}{s^2+a^2} = \cos at, \quad L^{-1}\Phi_2(s) = \cos at$$

$$\therefore \; L^{-1}[\Phi(s)] = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\left(\frac{s}{s^2+a^2}\right)\right] = \int_0^t \cos au\cdot\cos a(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos a(2u-t)]\,du = \frac{1}{2}\left[u\cos at + \frac{1}{2a}\sin a(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at + \frac{1}{2a}\sin at + \frac{1}{2a}\sin at\right] = \frac{1}{2a}[\sin at + at\cos at]$$

(ii) Let $\Phi_1(s) = \frac{s}{s^2-a^2}$, $\Phi_2(s) = \frac{s}{s^2-a^2}$

$$\therefore \; L^{-1}\Phi_1(s) = L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at, \qquad L^{-1}\Phi_2(s) = \cosh at$$

$$\therefore \; L^{-1}[\Phi(s)] = \int_0^t \cosh au\cdot\cosh a(t-u)\,du = \frac{1}{2}\int_0^t [\cosh at + \cosh a(2u-t)]\,du$$

$$\therefore \; L^{-1}[\Phi(s)] = \frac{1}{2}\left[u\cosh at + \frac{1}{2a}\sinh a(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cosh at + \frac{1}{2a}\sinh at + \frac{1}{2a}\sinh at\right]$$

$$= \frac{1}{2a}[\sinh at + at\cosh at]$$

(iii) Let $\Phi_1 = \frac{1}{s-a}$, $\Phi_2(s) = \frac{1}{(s+a)^2}$

$$\therefore \; L^{-1}[\Phi_1(s)] = L^{-1}\left[\frac{1}{s-a}\right] = e^{at},$$

$$L^{-1}[\Phi_2(s)] = L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at}L^{-1}\left(\frac{1}{s^2}\right) = e^{-at}\cdot t$$

$$\therefore \; L^{-1}[\Phi(s)] = \int_0^t e^{au}\cdot e^{-a(t-u)}\cdot(t-u)\,du$$

$$= \int_0^t e^{2au-at}\cdot(t-u)\,du = e^{-at}\int_0^t e^{2au}(t-u)\,du$$

$$= e^{-at}\left[(t-u)\cdot\frac{e^{2au}}{2a} - \frac{e^{2au}}{4a^2}(-1)\right]_0^t = e^{-at}\left[(t-u)\cdot\frac{e^{2au}}{2a} + \frac{e^{2au}}{4a^2}\right]_0^t$$

$$= e^{-at}\left[0 + \frac{e^{2at}}{4a^2} - t\cdot\frac{e^0}{2a} - \frac{e^0}{4a^2}\right] = \frac{e^{-at}}{4a^2}\left[e^{2at} - 2at - 1\right]$$

$$= \frac{1}{4a^2}\left[e^{at} - 2at\cdot e^{-at} - e^{-at}\right]$$

(iv) $$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos au, \quad L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin au$$

$$\therefore \; L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \cos au\cdot\frac{1}{a}\sin a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin a(t-2u)]\,du$$

$$= \frac{1}{2a}\left[u\sin at + \frac{1}{2a}\cos a(t-2u)\right]_0^t = \frac{1}{2a}t\sin at.$$

(v) $$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh au, \quad L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh au$$

$$\therefore \; L^{-1}\left[\frac{s}{(s^2-a^2)}\cdot\frac{1}{(s^2-a^2)}\right] = \int_0^t \cosh au\cdot\frac{1}{a}\sinh a(t-u)\,du$$

$$\therefore \; L^{-1}\left[\frac{s}{(s^2-a^2)}\cdot\frac{1}{(s^2-a^2)}\right] = \frac{1}{2a}\int_0^t [\sinh at + a\sinh a(t-2u)]\,du$$

$$= \frac{1}{2a}\left[u\sinh at - \frac{1}{2a}\cosh a(t-2u)\right]_0^t = \frac{1}{2a}\cdot t\cdot\sinh at.$$

$\int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b$

(vi) Let $\Phi_1(s) = \frac{1}{s-1}$, $\Phi_2(s) = \frac{1}{s^2+4}$

$\therefore\ L^{-1}[\Phi_1(s)] = e^t,\quad L^{-1}[\Phi_2(s)] = \frac{1}{2}\sin 2t$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t e^u \cdot \frac{1}{2}\sin 2(t-u)\,du = -\frac{1}{2}\int_0^t e^u \sin 2(u-t)\,du$$

$$= -\frac{1}{2}\left[\frac{1}{5}\cdot\left\{e^u(\sin 2(u-t) - 2\cos 2(u-t))\right\}\right]_0^t$$

$$= -\frac{1}{10}\left[e^t(0-2) - e^0(-\sin 2t - 2\cos 2t)\right]$$

$$= -\frac{1}{10}\left[-2e^t + \sin 2t + 2\cos 2t\right]$$

$$= \frac{1}{5}e^t - \frac{1}{10}\sin 2t - \frac{1}{5}\cos 2t$$

EXERCISE - VII

Find the inverse Laplace Transforms of the following functions by using convolution theorem.

1. $\frac{1}{(s-2)(s+2)^2}$ (M.U. 1994, 2006)
2. $\frac{1}{(s-3)(s+3)^2}$
3. $\frac{s^2}{(s^2+2^2)^2}$ (M.U. 1998, 99, 2004, 12)
4. $\frac{s^2}{(s^2+9)^2}$
5. $\frac{s}{(s^2+1)^2}$ (M.U. 2004)
6. $\frac{s}{(s^2+4)^2}$
7. $\frac{s}{s^4+8s^2+16}$ (M.U. 2003, 13, 15)

[**Ans. :** (1) $\frac{1}{16}[e^{2t} - 4te^{-2t} - e^{-2t}]$, (2) $\frac{1}{36}[e^{3t} - 6te^{-3t} - e^{-3t}]$, (3) $\frac{1}{4}(\sin 2t + 2t\cos$ (4) $\frac{1}{6}(\sin 3t + 3t\cos 3t)$, (5) $\frac{1}{2}t\sin t$, (6) $\frac{1}{4}t\sin 2t$, (7) $\frac{1}{4}t\sin 2t$.]

Example 3 : Find inverse Laplace transforms of the following by using convolution theo

(i) $\frac{1}{s^2(s+a)^2}$ (M.U. 2014) (ii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ (M.U. 2004, 07, 11, 13) (iii) $\frac{s}{(s^2+a^2)(s^2+b^2)}$ (M.U. 2003, 09, 10)

(iv) $\frac{1}{(s-a)(s-b)}$ (v) $\frac{1}{(s^2+a^2)(s^2+b^2)}$ (vi) $\frac{1}{(s+a)}\cdot\frac{1}{(s+b)^2}$ (M.U. 2

Sol. : (i) Let $\Phi_1(s) = \frac{1}{(s+a)^2}$, $\Phi_2(s) = \frac{1}{s^2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at}\cdot L^{-1}\left(\frac{1}{s^2}\right) = e^{-at}\cdot t$$

$$L^{-1}[\Phi_2(s)] = L^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t e^{-au}\cdot u\cdot(t-u)\,du = t\int_0^t e^{-au}u\,du - \int_0^t e^{-au}u^2\,du$$

$$= t\left[u\cdot\frac{e^{-au}}{-a} - \frac{e^{-au}}{a^2}\cdot 1\right]_0^t - \left[u^2\cdot\frac{e^{-au}}{-a} - \frac{e^{-au}}{a^2}\cdot 2u + \frac{e^{-au}}{-a^3}\cdot 2\right]_0^t$$

$$= t\left[\frac{t\cdot e^{-at}}{-a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}\right] - \left[t^2\cdot\frac{e^{-at}}{-a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3} + \frac{2}{a^3}\right]$$

$$= -t^2\cdot\frac{e^{-at}}{a} - \frac{te^{-at}}{a^2} + \frac{t}{a^2} + \frac{t^2e^{-at}}{a} + \frac{2te^{-at}}{a^2} + \frac{2e^{-at}}{a^3} - \frac{2}{a^3}$$

$$= \frac{te^{-at}}{a^2} + \frac{2e^{-at}}{a^3} + \frac{t}{a^2} - \frac{2}{a^3}$$

$$= \frac{1}{a^3}\left[e^{-at}\cdot at + 2e^{-at} + at - 2\right]$$

(ii) Let $\Phi_1(s) = \frac{s}{s^2+a^2}$, $\Phi_2(s) = \frac{s}{s^2+b^2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at;\quad L^{-1}[\Phi_2(s)] = L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left[\frac{s}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}\right] = \int_0^t \cos au\cdot\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \cos[(a-b)u + bt] + \cos[(a+b)u - bt]\,du$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b+a-b)\sin at}{a^2-b^2} + \frac{(a-b-a-b)\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}. \qquad \text{.................... (A)}$$

[For another method see Ex. 2 (iii), page 2-10.]

(iii) Let $\Phi_1(s) = \frac{s}{s^2+a^2}$, $\Phi_2(s) = \frac{1}{s^2+b^2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = \cos at,\quad L^{-1}[\Phi_2(s)] = \frac{1}{b}\sin bt$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t \cos au\cdot\frac{1}{b}\sin b(t-u)\,du$$

$$= \frac{1}{2b}\int_0^t \{\sin[(a-b)u+bt] - \sin[(a+b)u-bt]\}\,du$$

$$= \frac{1}{2b}\left[-\frac{\cos[(a-b)u+bt]}{a-b} + \frac{\cos[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2b}\left[-\frac{\cos at}{a-b} + \frac{\cos at}{a+b} + \frac{\cos bt}{a-b} - \frac{\cos bt}{a+b}\right]$$

$$\therefore L^{-1}[\Phi(s)] = \frac{1}{2b}\left[\frac{(a-b-a-b)\cos at}{a^2-b^2} + \frac{(a+b-a+b)\cos bt}{a^2-b^2}\right]$$

$$= \frac{1}{2b}\left[\frac{-2b\cos bt + 2b\cos bt}{a^2-b^2}\right] = \frac{\cos bt - \cos at}{a^2-b^2}.$$

[For another method see Ex. 2 (iv), page 2-10.]

(iv) Let $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$, $L^{-1}\left(\frac{1}{s-b}\right) = e^{bt}$

$$\therefore L^{-1}\left[\frac{1}{(s-a)}\cdot\frac{1}{(s-b)}\right] = \int_0^t e^{au}\cdot e^{b(t-u)}\,du = e^{bt}\int_0^t e^{(a-b)u}\,du$$

$$= e^{bt}\cdot\left[\frac{e^{(a-b)u}}{a-b}\right]_0^t = \frac{e^{bt}}{a-b}\left[e^{(a-b)t} - 1\right] = \frac{e^{at} - e^{bt}}{a-b}.$$

[You can also use the method of partial fractions.

$$\because \frac{1}{(s-a)(s-b)} = \frac{1}{(a-b)}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] \,]$$

(v) Let $\Phi_1(s) = \frac{1}{s^2+a^2}$, $\Phi_2(s) = \frac{1}{s^2+b^2}$

$$\therefore L^{-1}[\Phi_1(s)] = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at; \quad L^{-1}[\Phi_2(s)] = L^{-1}\left(\frac{1}{s^2+b^2}\right) = \frac{1}{b}\sin bt$$

$$\therefore L^{-1}[\Phi(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+b^2}\right] = \int_0^t \frac{1}{a}\sin au\cdot\frac{1}{b}\sin b(t-u)\,du$$

$$= \frac{1}{ab}\int_0^t \sin au\cdot\sin b(t-u)\,du$$

$$= -\frac{1}{2ab}\int_0^t \{\cos[a-b]u + bt - \cos[(a+b)u - bt]\}\,du$$

$$= -\frac{1}{2ab}\left[\frac{\sin[(a-b)u + bt]}{a-b} - \frac{\sin[(a+b)u - bt]}{a+b}\right]_0^t$$

$$= -\frac{1}{2ab}\left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right]$$

$$= -\frac{1}{2ab}\left[2b\cdot\frac{\sin at}{a^2-b^2} - \frac{2a\sin bt}{a^2-b^2}\right] = \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}.$$

[You can also use the method of partial fractions.

$$\because \frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{(a^2-b^2)}\left[\frac{1}{s^2+b^2} - \frac{1}{s^2+a^2}\right] \,]$$

(vi) Let $\Phi_1(s) = \frac{1}{(s+b)^2}$ and $\Phi_2(s) = \frac{1}{s+a}$

$$\therefore L^{-1}[\Phi_1(s)] = L^{-1}\frac{1}{(s+b)^2} = e^{-bt}\cdot L^{-1}\left(\frac{1}{s^2}\right) = e^{-bt}\,t$$

$$L^{-1}[\Phi_2(s)] = L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$\therefore L^{-1}[\Phi(s)] = \int_0^t e^{-bu}\cdot u\cdot e^{-a(t-u)}\,du = e^{-at}\int_0^t u\cdot e^{(a-b)u}\,du$$

$$= e^{-at}\left[u\cdot\frac{e^{(a-b)u}}{a-b} - \int\frac{e^{(a-b)u}}{(a-b)}\cdot 1\cdot du\right]_0^t$$

$$= e^{-at}\left[\frac{u\cdot e^{(a-b)u}}{a-b} - \frac{e^{(a-b)u}}{(a-b)^2}\right]_0^t$$

$$= e^{-at}\left[\left(\frac{t\cdot e^{(a-b)t}}{(a-b)} - \frac{e^{(a-b)t}}{(a-b)^2}\right) - \left(0 - \frac{1}{(a-b)^2}\right)\right]$$

$$= e^{-at}\left[\frac{1}{(a-b)^2} + \frac{t\cdot e^{(a-b)t}}{(a-b)} - \frac{e^{(a-b)t}}{(a-b)^2}\right]$$

$$= \frac{1}{(a-b)^2}\left[e^{-at} + (a-b)e^{-bt}t - e^{-bt}\right]$$

EXERCISE - VIII

Find the inverse Laplace Transforms of the following functions by using convolution theorem.

1. $\frac{1}{s^2(s+1)^2}$ **(M.U. 1997, 99, 2014)**
2. $\frac{s^2}{(s^2+1)(s^2+4)}$ **(M.U. 2004, 06)**
3. $\frac{s}{(s^2+4)(s^2+1)}$ **(M.U. 1995, 2002, 03)**
4. $\frac{1}{(s-3)(s-1)}$
5. $\frac{1}{(s+3)(s-1)}$
6. $\frac{s^2}{(s^2+9)(s^2+4)}$
7. $\frac{1}{(s^2+9)(s^2+4)}$
8. $\frac{1}{(s^2+1)(s^2+9)}$
9. $\frac{1}{(s^2+1)(s^2+4)}$
10. $\frac{1}{s^2-s-6}$ **(M.U. 2004)**
11. $\frac{16}{(s-2)(s+2)}$ **(M.U. 2009)**
12. $\frac{1}{(s^2+1)^2}$ **(M.U. 2011)**
13. $\frac{1}{(s-3)(s+4)^2}$ **(M.U. 2014)**
14. $\frac{1}{s-2}\cdot\frac{1}{(s+2)^2}$ **(M.U. 2003)**

[**Ans. :** (1) $te^{-t} + 2e^{-t} + t - 2$, (2) $\frac{1}{3}(2\sin 2t - \sin t)$, (3) $\frac{1}{3}(\cos t - \cos 2t)$,

(4) $\frac{e^{3t} - e^{t}}{2}$, (5) $\frac{e^{t} - e^{-3t}}{4}$, (6) $\frac{1}{5}(3\sin 3t - 2\sin 2t)$,

(7) $\frac{3\sin 2t - 2\sin 3t}{30}$, (8) $\frac{3\sin t - \sin 3t}{24}$, (9) $\frac{2\sin t - \sin 2t}{6}$,

(10) $\frac{e^{3t} - e^{-2t}}{5}$, (11) $4(e^{2t} - e^{-2t})$, (12) $\frac{1}{2}(\sin t - t\cos t)$.]

(13) $\frac{1}{49}[e^{-3t} - 7te^{-4t} - e^{-4t}]$ (14) $\frac{1}{16}[e^{2t} - 4te^{-2t} - e^{-2t}]$

Example 4 : Find the inverse of the following by using convolution theorem.

(i) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ (M.U. 2004, 07) (ii) $\dfrac{1}{(s+3)(s^2+2s+2)}$ (M.U. 2004) (iii) $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ (M.U. 1996, 2004, 07, 0

Sol. : (i) $L^{-1}\left\{\dfrac{(s+2)^2}{(s^2+4s+8)^2}\right\} = L^{-1}\left\{\dfrac{(s+2)^2}{[(s+2)^2+2^2]^2}\right\} = e^{-2t}\, L^{-1}\dfrac{s^2}{(s^2+2^2)^2}$

$$\therefore\ L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$$

$$\therefore\ L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right] = \int_0^t \cos 2u \cdot \cos 2(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos 2t + \cos(4u-2t)]\, du = \frac{1}{2}\left[u\cos 2t + \frac{1}{4}\sin(4u-2t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t\right] = \frac{1}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right]$$

$$\therefore\ L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = \frac{e^{-2t}}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right] = \frac{e^{-2t}}{4}[2t\cos 2t + \sin 2t]$$

(ii) $L^{-1}\left[\dfrac{1}{(s+3)(s^2+2s+2)}\right] = L^{-1}\left[\dfrac{1}{[(s+1)+2][(s+1)^2+1]}\right]$

$$= e^{-t} L^{-1}\left[\frac{1}{(s+2)(s^2+1)}\right]$$

Let $\Phi_1(s) = \dfrac{1}{s^2+1}$ and $\Phi_2(s) = \dfrac{1}{s+2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = \sin at \text{ and } L^{-1}[\Phi_2(s)] = e^{-2t}$$

$$\therefore\ L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t \sin u \cdot e^{-2(t-u)} du$$

$$= \int_0^t \sin u \cdot e^{-2t}\cdot e^{2u}\, du = e^{-2t}\int_0^t e^{2u}\cdot \sin u\, du$$

$$= e^{-2t}\cdot\frac{1}{4+1}\left[e^{2u}(2\sin u - \cos u)\right]_0^t$$

$$= \frac{1}{5}e^{-2t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$\therefore\ L^{-1}\left[\frac{1}{(s+3)(s^2+2s+2)}\right] = e^{-t}\cdot L^{-1}[\Phi_1(s)\Phi_2(s)]$$

$$= e^{-t}\cdot\frac{1}{5}\cdot e^{-2t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$= \frac{1}{5}\cdot e^{-3t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$= \frac{1}{5}\left[e^{-t}(2\sin t - \cos t) + e^{-3t}\right]$$

(iii) $L^{-1}\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = L^{-1}\dfrac{(s+1)^2+2}{[(s+1)^2+1][(s+1)^2+4]}$

$$= e^{-t} L^{-1}\frac{s^2+2}{(s^2+1)(s^2+4)}$$

$$= e^{-t}\cdot L^{-1}\frac{s^2}{(s^2+1)(s^2+4)} + e^{-t}L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)}\right]$$

$$= e^{-t}\cdot L^{-1}\frac{s^2}{(s^2+1)(s^2+4)} + \frac{2e^{-t}}{3}\cdot L^{-1}\left[\frac{1}{(s^2+1)} - \frac{1}{(s^2+4)}\right]$$

By the result obtained above in (A), page 2-20, putting $a = 2$, $b = 1$, we get

$$= \frac{e^{-t}}{3}[2\sin 2t - \sin t] + \frac{2}{3}e^{-t}\left[\sin t - \frac{1}{2}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}[\sin 2t + \sin t]$$

[See that we have solved this example by another method in Ex. 2 (viii), page 2-12.]

Example 5 : Using convolution theorem find the inverse Laplace transform of the following.

(i) $\dfrac{1}{(s-2)^4(s+3)}$ (M.U. 1993, 98, 99, 2004, 07, 16) (ii) $\dfrac{1}{(s^2+4s+13)^2}$ (M.U. 2002, 08, 12, 13) (iii) $\dfrac{s+3}{(s^2+6s+10)^2}$

(iv) $\dfrac{s+2}{(s^2+4s+5)^2}$ (M.U. 2013) (v) $\dfrac{s^2+5}{(s^2+4s+13)^2}$ (M.U. 2014)

Sol. : (i) Let $\Phi_1(s) = \dfrac{1}{s+3}$, $\Phi_2(s) = \dfrac{1}{(s-2)^4}$

$$\therefore\ L^{-1}\Phi_1(s) = e^{-3t},\quad L^{-1}\Phi_2(s) = e^{2t}L^{-1}\frac{1}{s^4} = e^{2t}\cdot\frac{t^3}{6}$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t e^{-3u}\cdot e^{2(t-u)}\cdot\frac{(t-u)^3}{6}du$$

$$= \int_0^t e^{(2t-5u)}\cdot\frac{(t-u)^3}{6}du = e^{2t}\int_0^t e^{-5u}\frac{(t-u)^3}{6}du$$

$$= e^{2t}\left[\frac{(t-u)^3}{6}\left(\frac{-e^{-5u}}{5}\right) - \frac{(t-u)^2}{2}(-1)\left(\frac{e^{-5u}}{25}\right) + (t-u)(+1)\left(\frac{-e^{-5u}}{125}\right) - (-1)\left(\frac{e^{-5u}}{625}\right)\right]_0^t$$

$$= e^{2t}\left[\frac{e^{-5t}}{625} - \left\{-\frac{t^3}{30} + \frac{t^2}{50} - \frac{t}{125} + \frac{1}{625}\right\}\right]$$

$$= \frac{e^{-3t}}{625} - e^{2t}\left[\frac{1}{625} - \frac{t}{125} + \frac{t^2}{50} - \frac{t^3}{30}\right]$$

(ii) $L^{-1}\left[\dfrac{1}{(s^2+4s+13)^2}\right] = L^{-1}\left[\dfrac{1}{(s+2)^2+3^2}\right]^2 = e^{-2t}\cdot L^{-1}\left[\dfrac{1}{(s^2+3^2)}\right]^2$

We find $L^{-1}\left[\dfrac{1}{(s^2+3^2)^2}\right]$ by convolution theorem.

Let $\Phi_1(s) = \dfrac{1}{s^2+3^2}$ and $\Phi_2(s) = \dfrac{1}{s^2+3^2}$ and $\Phi(s) = \dfrac{1}{(s^2+3^2)^2}$

$\therefore\ L^{-1}[\Phi_1(s)] = \dfrac{1}{3}\sin 3t$ and $L^{-1}[\Phi_2(s)] = \dfrac{1}{3}\sin 3t$

$$\therefore\ L^{-1}[\Phi(s)] = \frac{1}{9}\int_0^t \sin 3u\cdot\sin 3(t-u)\,du$$

$$= -\frac{1}{18}\int_0^t [\cos 3t - \cos(6u-3t)]\,du$$

$$= -\frac{1}{18}\left[u\cos 3t - \frac{\sin(6u-3t)}{6}\right]_0^t$$

$$= -\frac{1}{18}\left[t\cos 3t - \frac{\sin 3t}{6} - \frac{\sin 3t}{6}\right] = \frac{1}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

$$\therefore\ L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos t3t\right]$$

(iii) $L^{-1}\dfrac{s+3}{(s^2+6s+10)^2} = L^{-1}\dfrac{s+3}{\left[(s+3)^2+1^2\right]^2} = e^{-3t}\cdot L^{-1}\dfrac{s}{(s^2+1^2)^2}$ [By first shifting theorem

Now, putting $a = 1$, in Ex. 2 (iv), page 2-17 or independently, we get

$$L^{-1}\left[\frac{s+3}{(s^2+6s+10)^2}\right] = e^{-3t}\cdot\frac{1}{2}\cdot t\sin t$$

(iv) $L^{-1}\left[\dfrac{s+2}{(s^2+4s+5)^2}\right] = L^{-1}\left[\dfrac{s+2}{(s^2+4s+4+1)^2}\right]$

$$= L^{-1}\left\{\frac{s+2}{[(s+2)^2+1]^2}\right\} = e^{-2t}\cdot L^{-1}\frac{s}{(s^2+1)^2}$$

Now, putting $a = 1$ in Ex. 2 (iv), page 2-17 or independently, we get

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = e^{-2t}\cdot\frac{1}{2}t\sin t.$$

(v) $L^{-1}\left[\dfrac{s^2+5}{(s^2+4s+13)^2}\right] = L^{-1}\left[\dfrac{(s^2+4s+13)-(4s+8)}{(s^2+4s+13)^2}\right]$

$$= L^{-1}\left[\frac{1}{s^2+4s+13}\right] - 4L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right] - 4L^{-1}\left[\frac{s+2}{[(s+2)^2+3^2]^2}\right]$$

$$L^{-1}\left[\frac{s^2+5}{(s^2+4s+13)^2}\right] = e^{-2t}\cdot L^{-1}\left[\frac{1}{s^2+3^2}\right] - 4e^{-2t}\cdot L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right]$$

Now, putting $a = 3$ in Ex. 2 (iv), page 2-17

$$= e^{-2t}\cdot\frac{1}{3}\sin 3t - 4e^{-2t}\,t\cdot\frac{1}{2\cdot 3}\sin 3t$$

$$= \frac{e^{-2t}}{3}(1-2t)\sin 3t.$$

Example 6 : Find the inverse Laplace transform of the following by convolution theorem.

(i) $\dfrac{1}{s\sqrt{s+4}}$ **(M.U. 2002, 05)**

(ii) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$ **(M.U. 2002)**

(iii) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ **(M.U. 2007, 09, 16)**

(iv) $\dfrac{(s+3)^2}{(s^2+6s+5)^2}$ **(M.U. 2003, 12)**

Sol. : (i) $L^{-1}\dfrac{1}{(s+4)^{1/2}} = e^{-4u}\cdot L^{-1}\dfrac{1}{s^{1/2}} = e^{-4u}\cdot\dfrac{u^{-1/2}}{\Gamma 1/2}$. And $L^{-1}\dfrac{1}{s} = 1$

$$\therefore\ L^{-1}\left[\frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}\right] = \int_0^t \frac{e^{-4u}\cdot u^{-1/2}}{\sqrt{\pi}}\cdot 1\cdot du$$

Put $4u = x^2$, $du = \dfrac{x\,dx}{2}$, $\sqrt{u} = x/2$

$$\therefore\ L^{-1}\left[\frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}\right] = \int_0^{2\sqrt{t}} \frac{e^{-x^2}}{\sqrt{\pi}}\,dx = \frac{1}{2}\,erf(2\sqrt{t}) \qquad \left[\because\ erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx\right]$$

[For another method see Ex. 1 (v), page 2-36.]

(ii) Let $\Phi_1(s) = \dfrac{s+1}{s^2+2s+2}$ and $\Phi_2(s) = \dfrac{s}{s^2+1}$

$$L^{-1}\frac{s+1}{(s+1)^2+1} = e^{-u}L^{-1}\frac{s}{s^2+1} = e^{-u}\cos u, \qquad L^{-1}\frac{s}{s^2+1} = \cos u$$

$$\therefore\ L^{-1}\left[\frac{s+1}{(s+1)^2+1}\cdot\frac{s}{s^2+1}\right] = \int_0^t e^{-u}\cos u\cdot\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}[\cos(2u-t) + \cos t]\,du$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-u}[-\cos(2u-t) + 2\sin(2u-t)] - e^{-u}\cos t\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-t}\{-\cos t + 2\sin t\} - e^{-t}\cos t - \frac{1}{5}\{-\cos t - 2\sin t\} + \cos t\right]$$

$$= \frac{1}{10}\left[e^{-t}(2\sin t - 6\cos t) + (2\sin t + 6\cos t)\right]$$

(iii) $L^{-1}\dfrac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\dfrac{(s+2)^2}{[(s+2)^2+2^2]^2} = e^{-2t}\,L^{-1}\dfrac{s^2}{(s^2+2^2)^2}$

Let $\Phi_1(s)=\dfrac{s}{s^2+2^2}$, $\Phi_2(s)=\dfrac{s}{s^2+2^2}$

$\therefore\ L^{-1}\Phi_1(s)=L^{-1}\dfrac{s}{s^2+2^2}=\cos 2t$, $L^{-1}\Phi_2(s)=\cos 2t$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right] = \int_0^t \cos 2u\cdot\cos 2(t-u)\,du$$

$$=\frac12\int_0^t[\cos 2t+\cos 2(2u-t)]\,du$$

$$=\frac12\left[u\cos 2t+\frac14\sin 2(2u-t)\right]_0^t$$

$$=\frac12\left[t\cos 2t+\frac14\sin 2t+\frac14\sin 2t\right]$$

$$=\frac14[\sin 2t+2t\cos 2t]$$

$$\therefore\ L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2}=\frac14 e^{-2t}[\sin 2t+2t\cos 2t]$$

(iv) $L^{-1}\dfrac{(s+3)^2}{(s^2+6s+5)^2}=L^{-1}\dfrac{(s+3)^2}{[(s+3)^2-2^2]^2}=e^{-3t}\cdot L^{-1}\dfrac{s^2}{(s^2-2^2)^2}$ [By first shifting theore

We find $L^{-1}\dfrac{s^2}{(s^2-2^2)^2}$ by convolution theorem.

Let $\Phi_1(s)=\dfrac{s}{s^2-2^2}$, $\Phi_2(s)=\dfrac{s}{s^2-2^2}$.

$\therefore\ L^{-1}\Phi_1(s)=\cosh 2u$, $L^{-1}\Phi_2(s)=\cosh 2u$

$$\therefore\ L^{-1}[\Phi(s)]=\int_0^t\cosh 2u\cdot\cosh 2(t-u)\,du$$

$$=\frac12\int_0^t[\cosh 2t+\cosh 2(2u-t)]\,du$$

$$=\frac12\left[u\cosh 2t+\frac{\sinh 2(2u-t)}{4}\right]_0^t$$

$$=\frac12\left[t\cosh 2t+\frac14\sinh 2t+\frac14\sinh 2t\right]$$

$$=\frac14[2t\cosh 2t+\sinh 2t]$$

$$\therefore\ L^{-1}\frac{(s+3)^2}{(s^2+6s+5)^2}=\frac14 e^{-3t}[2t\cosh 2t+\sinh 2t]$$

Example 7 : Using convolution theorem, find the inverse Laplace transform of $\dfrac{s}{s^4+8s^2+16}$. **(M.U. 2003, 15)**

Sol. : We have $\Phi=\dfrac{s}{s^4+8s^2+16}=\dfrac{s}{(s^2+4)^2}=\dfrac{s}{s^2+4}\cdot\dfrac{1}{s^2+4}$

Let $\Phi_1(s)=\dfrac{s}{s^2+4}$ and $\Phi_2(s)=\dfrac{1}{s^2+4}$.

$\therefore\ L^{-1}\Phi_1(s)=\cos 24$, $L^{-1}\Phi_2(s)=\dfrac12\sin 2u$

$$\therefore\ \Phi(s)=\int_0^t\cos 2u\cdot\frac12\sin 2(t-u)\,du=\frac14\int_0^t[\sin 2t-\sin(4u-2t)]\,du$$

$$=\frac14\left[\sin 2t\cdot u+\frac{\cos(4u-2t)}{4}\right]_0^t$$

$$=\frac14\left[t\cdot\sin 2t+\frac{\cos 2t}{4}-\frac{\cos 2t}{4}\right]=\frac14 t\sin 2t.$$

(Verify the answer from Ex. 1 (iii), page 1-29.)

EXERCISE - IX

Using convolution theorem find the inverse Laplace transform of,

1. $\dfrac{(s+3)^2}{(s^2+6s+18)^2}$
2. $\dfrac{(s+5)^2}{(s^2+10s+16)^2}$ **(M.U. 2006)**
3. $\dfrac{1}{(s-3)(s+4)^2}$
4. $\dfrac{1}{(s-2)}\cdot\dfrac{1}{(s+2)^2}$ **(M.U. 2003)**
5. $\dfrac{s+3}{(s^2+6s+13)^2}$ **(M.U. 2004, 06)**
6. $\dfrac{(s-1)^2}{(s^2-2s+5)^2}$ **(M.U. 2006, 14)**
7. $\dfrac{s+1}{(s^2+2s+2)^2}$ **(M.U. 2003)**

[**Ans. :** (1) $\dfrac16 e^{-3t}\cdot[\sin 3t+3t\cos 3t]$, (2) $\dfrac16 e^{-5t}\cdot[\sinh 3t+3t\cosh 3t]$,

(3) $\dfrac1{49}\left[e^{3t}-e^{-4t}-7te^{-4t}\right]$, (4) $\dfrac1{16}(e^{2t}-4e^{-2t}-e^{-2t})$,

(5) $\dfrac14 e^{-3t}\cdot t\sin 2t$, (6) $\dfrac{e^t}{4}[\sin 2t+2t\cos 2t]$,

(7) $e^{-t}\cdot\dfrac12\cdot t\sin t$.]

(e) Use of differentiation of $\Phi(s)$

We know that [(16) § 10, page 1-28] if $L[f(t)]=\Phi(s)$, then $-\Phi'(s)=L[t\,f(t)]$.

i.e., if $L\left[\dfrac{-1}{t}f(t)\right]=\Phi(s)$, then $\Phi'(s)=L[f(t)]$.

Hence, if $L^{-1}[\Phi'(s)] = f(t)$, then $L^{-1}[\Phi(s)] = -\frac{1}{t} f(t)$.

i.e., $\boxed{L^{-1}\Phi(s) = -\frac{1}{t} L^{-1}[\Phi'(s)] \quad i.e. \quad L^{-1}[\Phi'(s)] = -t\, L^{-1}\Phi(s)}$ .... (17)

Remarks ....

1. The result can be extended further.

$$L^{-1}\Phi(s) = \frac{1}{t^2} L^{-1}[\Phi''(s)] \quad \text{or} \quad L^{-1}[\Phi''(s)] = t^2 \cdot L^{-1}[\Phi(s)]$$

2. These results can be profitably used to find $L^{-1}[F(s)]$ if we know $L^{-1}[\Phi'(s)]$ i.e., if $\Phi'(s)$ comes out to be a standard result. [See Ex. 1].

Note ....

The above result is particularly useful when

$\Phi(s) = \log f(s)$ or $\Phi(s) = \tan^{-1} f(s)$ or $\Phi(s) = \tan h^{-1} f(s)$. See Ex. 1 and 2 below.

Example 1 : Find inverse Laplace transform of,

(i) $\log\left(\frac{s+a}{s+b}\right)$ **(M.U. 1993, 96, 13)** (ii) $\log\left(1+\frac{a^2}{s^2}\right)$ **(M.U. 1999, 2002, 05)** (iii) $\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)$ **(M.U. 1996, 2016)**

(iv) $\log\left[\frac{s^2+a^2}{\sqrt{s+b}}\right]$ **(M.U. 2003, 04, 06, 14)** (v) $\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$ **(M.U. 2007, 08, 13)** (vi) $2\tan h^{-1} s$ **(M.U. 2003, 05, 14)**

(vii) $s\log\left(\frac{s+1}{s-1}\right)$ **(M.U. 2016)** (viii) $\log\left(\frac{s^2+1}{s(s+1)}\right)$ **(M.U. 2008)**

Sol. : We have $L^{-1}[\Phi(s)] = -\frac{1}{t} L^{-1}[\Phi'(s)]$

(i) $\therefore L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\{\log(s+a) - \log(s+b)\}\right]$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] = -\frac{1}{t}(e^{-at} - e^{-bt}) \text{ [See Ex. 1 (iii), page 1-3]}$$

(ii) $L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(1+\frac{a^2}{s^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{a^2+s^2}{s^2}\right)\right]$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\{\log(a^2+s^2) - \log s^2\}\right] = -\frac{1}{t} L^{-1}\left[\frac{2s}{a^2+s^2} - \frac{2}{s}\right]$$

$$= -\frac{1}{t}[2\cos at - 2 \cdot 1] = \frac{2}{t}[1 - \cos at] \quad \text{[See Ex. 1 (i), page 1-3]}$$

(iii) $L^{-1}\log\left(1+\frac{1}{s^2}\right) = -\frac{1}{u} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+1}{s^2}\right)\right]$ [By (17) above]

$$= -\frac{1}{u} L^{-1}\left[\frac{d}{ds}\{\log(s^2+1) - \log s^2\}\right]$$

$\therefore L^{-1}\log\left(1+\frac{1}{s^2}\right) = -\frac{1}{u} L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s}\right] = -\frac{2}{u}(\cos u - 1)$ And $L^{-1}\frac{1}{s} = 1$

$\therefore L^{-1}\left[\frac{1}{s}\cdot\log\left(1+\frac{1}{s^2}\right)\right] = \int_0^t -\frac{2}{u}(\cos u - 1)\cdot 1 \cdot du$ [By Cor., page 2-15]

(**Note** that the result is expressed in terms of the integral only.)

(iv) $L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left\{\log(s^2+a^2) - \frac{1}{2}\log(s+b)\right\}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{1}{2}\cdot\frac{1}{s+b}\right]$$

$$= -\frac{1}{t}\left[2\cos at - \frac{1}{2}e^{-bt}\right] = \frac{1}{t}\left[\frac{1}{2}e^{-bt} - 2\cos at\right]$$

(v) $L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}\right]$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\{\log(s^2+a^2) - \log(s^2+b^2)\}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right]$$

$$= -\frac{1}{t}[2\cos at - 2\cos bt] = \frac{2}{t}(\cos bt - \cos at)$$

(See Ex. 8 of Exercise - XI, page 1-42.)

(vi) $L^{-1}(2\tan h^{-1}s) = L^{-1}\left[2\cdot\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)\right] = L^{-1}\left[\log\left(\frac{1+s}{1-s}\right)\right]$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{1+s}{1-s}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{1}{1+s} + \frac{1}{1-s}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = -\frac{1}{t}\left[e^{-t} - e^{t}\right]$$

$$= \frac{2}{t}\left(\frac{e^t - e^{-t}}{2}\right) = \frac{2}{t}\sin ht$$

(vii) $L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}s\log\left(\frac{s+1}{s-1}\right)\right]$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\{s\log(s+1) - s\log(s-1)\}\right]$$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{s}{s+1} + \log(s+1) - \frac{s}{s-1} - \log(s-1)\right]$$

$$\therefore\ L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{s}{s+1}-\frac{s}{s-1}\right] - \frac{1}{t}\cdot L^{-1}[\log(s+1)-\log(s-1)]$$

$$= -\frac{1}{t}\cdot L^{-1}\left(-\frac{2s}{s^2-1}\right) - \frac{1}{t}\cdot\left(-\frac{1}{t}\right)\cdot L^{-1}\left\{\frac{d}{ds}[\log(s+1)-\log(s-1)]\right\}$$

$$= \frac{2}{t}\cdot L^{-1}\left(\frac{s}{s^2-1}\right) + \frac{1}{t^2}\cdot L^{-1}\left[\frac{1}{s+1}-\frac{1}{s-1}\right]$$

Note .... $L^{-1}\left(\frac{s}{s+1}\right)$ and $L^{-1}\left(\frac{s}{s-1}\right)$ do not exist. Hence, we have combined these two terms as ab

$$= \frac{2}{t}\cosh t + \frac{1}{t^2}(e^{-t}-e^{t}) = \frac{2}{t}\cosh t - \frac{2}{t^2}\left(\frac{e^t-e^{-t}}{2}\right)$$

$$= \frac{2}{t}\cosh t - \frac{2}{t^2}\sinh t$$

(viii) $$L^{-1}\log\left[\frac{s^2+1}{s(s+1)}\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log\frac{s^2+1}{s(s+1)}\right\}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\{\log(s^2+1)-\log s-\log(s+1)\}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}\right] = -\frac{1}{t}\left[2L^{-1}\frac{s}{s^2+1} - L^{-1}\frac{1}{s} - L^{-1}\frac{1}{s+1}\right]$$

$$= -\frac{1}{t}[2\cos t - 1 - e^{-t}]$$

Example 2 : Find inverse Laplace transform of

(i) $\tan^{-1}\left(\frac{2}{s^2}\right)$ **(M.U. 1995, 2004, 05, 06, 07)** (ii) $\cot^{-1} as$

(iii) $\tan^{-1}\left(\frac{a}{s}\right)$ **(M.U. 2003, 04, 2007, 12, 13)**

(iv) $\cot^{-1}(s+1)$ **(M.U. 2006, 16)** (v) $\tan^{-1}\left(\frac{s+a}{b}\right)$ **(M.U. 2004, 05**

(vi) $\cot^{-1}\left(\frac{s+3}{2}\right)$ **(S.U. 2015)**

Sol. : We have $L^{-1}[\phi(s)] = -\frac{1}{t}L^{-1}[\phi'(s)]$

(i) $$\therefore\ L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}\frac{2}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{1}{1+(4/s^4)}\left(-\frac{4}{s^3}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[-\frac{4s}{s^4+4}\right] = \frac{4}{t}L^{-1}\left[\frac{s}{s^4+4}\right] = \frac{4}{t}\cdot L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{4}{t}\cdot\frac{1}{4}L^{-1}\left[\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right]$$

$$\therefore\ L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{1}{t}\cdot L^{-1}\left[\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{t}\left[e^t\cdot L^{-1}\left(\frac{1}{s^2+1}\right) - e^{-t}\cdot L^{-1}\left(\frac{1}{s^2+1}\right)\right] = \frac{1}{t}[e^t\sin t - e^{-t}\sin t]$$

$$= \frac{2\sin t}{t}\left(\frac{e^t-e^{-t}}{2}\right) = \frac{2\sin t\sinh t}{t}$$

(ii) $$L^{-1}[\cot^{-1}(as)] = -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\cot^{-1}(as)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{-a}{1+a^2s^2}\right] = \frac{a}{t}\cdot L^{-1}\left(\frac{1}{1+a^2s^2}\right)$$

$$= \frac{a}{t}\cdot L^{-1}\left[\frac{1}{a^2}\cdot\frac{1}{s^2+(1/a)^2}\right] = \frac{1}{t}\cdot L^{-1}\left[\frac{1/a}{s^2+(1/a)^2}\right] = \frac{1}{t}\sin\frac{t}{a}.$$

(iii) $$L^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{a}{s}\right)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{1}{1+(a/s)^2}\left(-\frac{a}{s^2}\right)\right]$$

$$= \frac{a}{t}\cdot L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{a}{t}\cdot\frac{1}{a}\sin at = \frac{1}{t}\sin at$$

(iv) $$L^{-1}[\cot^{-1}(s+1)] = -\frac{1}{t}\frac{d}{ds}[\cot^{-1}(s+1)] = -\frac{1}{t}\cdot L^{-1}\left[\frac{-1}{1+(s+1)^2}\right]$$

$$= \frac{1}{t}\cdot L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = \frac{1}{t}\cdot e^{-t}\cdot L^{-1}\left(\frac{1}{s^2+1}\right) = \frac{1}{t}e^{-t}\sin t.$$

(v) $$L^{-1}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{s+a}{b}\right)\right]$$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{1}{1+[(s+a)/b]^2}\cdot\frac{1}{b}\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$$

$$= -\frac{1}{t}\cdot e^{-at}\cdot L^{-1}\left(\frac{b}{s^2+b^2}\right) = -\frac{1}{t}\cdot e^{-at}\sin bt.$$

(vi) $$L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

$$= -\frac{1}{t}\cdot L^{-1}\left[-\frac{1}{1+[(s+3)/2]^2}\cdot\frac{1}{2}\right] = \frac{1}{t}\cdot L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]$$

$$= \frac{1}{t}\cdot e^{-3t}\cdot L^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{1}{t}\cdot e^{-3t}\sin 2t.$$

EXERCISE - X

Find the inverse Laplace transform of,

1. $\log\left[\frac{s^2-4}{(s-3)^2}\right]$ **(M.U. 2003)**
2. $\log\left[1+\frac{4}{s^2}\right]$
3. $\log\left[\frac{s-1}{s+1}\right]$ **(M.U. 2016)**
4. $\log\left[1+\frac{1}{s^2}\right]$ **(M.U. 2004)**

5. $\frac{1}{2}\log\left(\frac{s^2+1}{s^2+4}\right)$ 6. $\frac{1}{2}\log\left(1-\frac{a^2}{s^2}\right)$ (M.U. 1999) 7. $\tan^{-1}(s+1)$ (M.U. 2003) *8. $\cot^{-1}\left(\frac{2}{s^2}\right)$

9. $\log\sqrt{\frac{s^2+1}{s^2}}$ (M.U. 1998, 2003) 10. $\log\left[\frac{s^2+a^2}{(s+b)^2}\right]$ (M.U. 2003, 04) 11. $\log\left(1+\frac{a}{s}\right)$ (M.U. 2003) 12. $\tan^{-1}\frac{1}{s}$ (M.U. 20

13. $\cot^{-1} s$ (M.U. 1996, 98, 2011) 14. $\log\sqrt{\frac{s^2+a^2}{s^2}}$ (M.U. 1995, 98, 2004) 15. $\log\left(1-\frac{1}{s^2}\right)$ (M.U. 1997) 16. $\tan^{-1}\left(\frac{s}{2}\right)$ (M.U. 20

[**Ans. :** (1) $\frac{2}{t}\left[e^{3t}-\cos h2t\right]$, (2) $\frac{2}{t}\left[1-\cos 2t\right]$, (3) $-\frac{1}{t}\sin ht$, (4) $\frac{2}{t}(1-\cos t)$,

(5) $\frac{1}{t}(\cos 2t-\cos t)$, (6) $\frac{1}{t}(1-\cos h\,at)$, (7) $-\frac{1}{t}e^{-t}\sin t$, (8) $-\frac{2}{t}[\sin t \sin ht]$,

(9) $\frac{1}{t}(1-\cos t)$, (10) $\frac{2}{t}(e^{-bt}-\cos at)$, (11) $\frac{1}{t}(1-e^{-at})$, (12) $\frac{1}{t}\sin 2t$,

(13) $-\frac{1}{t}\sin t$, (14) $\frac{1}{t}(1-\cos at)$, (15) $\frac{2}{t}(1-\cos ht)$, (16) $-\frac{1}{t}\sin 2t$.]

Example 1 : Using convolution theorem, prove that

(i) $L^{-1}\left[\frac{1}{s}\log\left(a+\frac{b}{s^2}\right)\right]=\int_0^t \frac{2}{u}\left[1-\cos\left(\frac{b}{a}\right)u\right]du.$ (M.U. 19

(ii) $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right]=\int_0^t \frac{e^{-bu}-e^{-au}}{u}\,du.$ (M.U. 2004

Sol. : We first note that both the functions are of the form $\Phi_1(s)=\frac{1}{s}\log f(s)$ and as such we s use (17), page 2-29. [See note, page 2-29]

(i) By (17), page 2-29, we have

$$L^{-1}[\Phi(s)]=-\frac{1}{t}\cdot L^{-1}[\Phi'(s)] \quad \text{where } \Phi(s)=\log f(s).$$

$$\therefore\ L^{-1}\left[\log\left(a+\frac{b}{s^2}\right)\right]=-\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\log\left(a+\frac{b}{s^2}\right)\right] \qquad \text{[By (17), page 2-29]}$$

$$=-\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\log\left(\frac{as^2+b}{s^2}\right)\right]=-\frac{1}{t}\cdot L^{-1}\frac{d}{ds}[\log(as^2+b)-\log s^2]$$

$$=-\frac{1}{t}\cdot L^{-1}\left(\frac{2as}{as^2+b}-\frac{2s}{s^2}\right)=-\frac{2}{t}\cdot L^{-1}\left[\frac{s}{s^2+(b/a)}-\frac{1}{s}\right]$$

$$=-\frac{2}{t}\left[\cos\left(\frac{b}{a}\cdot t\right)-1\right]=\frac{2}{t}\left[1-\cos\left(\frac{b}{a}\right)t\right]$$

By cor. (16A) of convolution theorem, page 2-15

$$L^{-1}\left[\frac{1}{s}\log\left(a+\frac{b}{s^2}\right)\right]=\int_0^t \frac{2}{u}\left[1-\cos\left(\frac{b}{a}\right)u\right]du$$

(ii) As above, we have

$$L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right]=-\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\log\left(\frac{s+a}{s+b}\right)\right] \qquad \text{[By (17), page 2-29]}$$

$$=-\frac{1}{t}\cdot L^{-1}\frac{d}{ds}[\log(s+a)-\log(s+b)]=-\frac{1}{t}\cdot L^{-1}\left[\frac{1}{s+a}-\frac{1}{s+b}\right]$$

$$=-\frac{1}{t}(e^{-at}-e^{-bt})=\frac{1}{t}(e^{-bt}-e^{-at})$$

By cor. (16A) of convolution theorem, page 2-15

$$L^{-1}\left[\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right]=\int_0^t \frac{1}{u}(e^{-bu}-e^{-au})\,du.$$

Example 2 : Using convolution theorem, prove that

(i) $L^{-1}\left[\frac{1}{s}\tan^{-1}\frac{2}{s}\right]=\int_0^t \frac{1}{u}\cdot\sin 2u\,du$ (ii) $L^{-1}\left[\frac{1}{s}\cot^{-1}(s+1)\right]=\int_0^t \frac{1}{u}\cdot e^{-u}\sin u\,du$

(iii) $L^{-1}\frac{1}{s}\tan h^{-1}s=\int_0^t \frac{\sin ht}{t}\,dt$

Sol. : We first note that both the functions are of the form $\Phi_1(s)=\frac{1}{s}\tan^{-1}f(s)$ or $\frac{1}{s}\cos^{-1}f(s)$ and as such we shall use (17), page 2-29. [See note, page 2-29]

(i) As in Example 2 (iii), page 2-31, we have

$$L^{-1}[\Phi(s)]=-\frac{1}{t}\cdot L^{-1}[\Phi'(s)] \quad \text{where, } \Phi(s)=\tan^{-1}\left(\frac{2}{s}\right).$$

$$\therefore\ L^{-1}\left[\tan^{-1}\left(\frac{2}{s}\right)\right]=-\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\tan^{-1}\left(\frac{2}{s}\right)\right] \qquad \text{[By (17), page 2-29]}$$

$$=-\frac{1}{t}\cdot L^{-1}\left[\frac{1}{1+(4/s^2)}\cdot\left(-\frac{2}{s^2}\right)\right]$$

$$=\frac{1}{t}\cdot L^{-1}\left[\frac{2}{s^2+4}\right]=\frac{1}{t}\sin 2t$$

By cor. (16A) of convolution theorem, page 2-15.

$$L^{-1}\left[\frac{1}{s}\tan^{-1}\left(\frac{2}{s}\right)\right]=\int_0^t \frac{1}{u}\sin 2u\,du.$$

(ii) As above,

$$L^{-1}[\cot^{-1}(s+1)]=-\frac{1}{t}\cdot L^{-1}\frac{d}{ds}[\cot^{-1}(s+1)] \qquad \text{[By (17), page 2-29]}$$

$$=-\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\frac{\pi}{2}-\tan^{-1}(s+1)\right]$$

$$=-\frac{1}{t}\cdot L^{-1}\left[-\frac{1}{1+(s+1)^2}\right]=\frac{1}{t}\cdot L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$\therefore\ L^{-1}\{\cot^{-1}(s+1)\} = \frac{1}{t}\cdot e^{-t}\cdot L^{-1}\left(\frac{1}{s^2+1}\right) = \frac{1}{t}\cdot e^{-t}\sin t.$$

By cor. (16A) of convolution theorem, page 2-15.

$$L^{-1}\left[\frac{1}{s}\cot^{-1}(s+1)\right] = \int_0^t \frac{1}{u}\cdot e^{-u}\sin u\,du.$$

(iii) Now, $L^{-1}(\tan h^{-1} s) = L^{-1}\left[\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)\right]$

$$= -\frac{1}{2t}\cdot L^{-1}\left[\frac{d}{ds}\log\left(\frac{1+s}{1-s}\right)\right] = -\frac{1}{2t}\cdot L^{-1}\left[\frac{1}{1+s}+\frac{1}{1-s}\right] \quad \text{[By (17), page 2-}$$

$$= -\frac{1}{2t}\cdot L^{-1}\left[\frac{1}{s+1}-\frac{1}{s-1}\right] = -\frac{1}{2t}\left[e^{-t}-e^{t}\right]$$

$$= \frac{1}{t}\left[\frac{e^{t}-e^{-t}}{2}\right] = \frac{\sin h\,t}{t}$$

By cor. (16A) of convolution theorem, page 2-15.

$$\therefore\ L^{-1}\left(\frac{1}{s}\tan h^{-1} s\right) = \int_0^t \frac{\sin h\,t}{t}\,dt$$

EXERCISE - XI

Using convolution theorem prove that,

1. $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+3}{s+4}\right)\right] = \int_0^t \frac{e^{-4u}-e^{-3u}}{u}\,du.$
2. $L^{-1}\left[\frac{1}{s}\tan^{-1}\frac{a}{s}\right] = \int_0^t \frac{1}{u}\cdot\sin au\,du.$

(M.U. 2008)

3. $L^{-1}\left[\frac{1}{s}\cot^{-1}s\right] = \int_0^t \frac{1}{u}\cdot\sin u\,du.$
4. $L^{-1}\left[\frac{1}{s}\cdot\log\left(\frac{s+1}{s+2}\right)\right] = \int_0^t \left(\frac{e^{-2u}-e^{-u}}{u}\right)du$ **(M.U. 2**
5. $L^{-1}\left[\frac{1}{s}\cdot\log\left(\frac{s+2}{s+3}\right)\right] = \int_0^t \left(\frac{e^{-3u}-e^{-2u}}{u}\right)du$
6. $L^{-1}\left[\frac{1}{s}\log\left(1+\frac{2}{s^2}\right)\right] = \int_0^t \frac{2}{u}[1-\cos 2u]$

(M.U. 2001)

7. $L^{-1}\left[\frac{1}{s}\tan^{-1}\left(\frac{s+a}{b}\right)\right] = \int_0^t -\frac{1}{u}e^{-au}\sin bu\,du$ **(M.U. 2006)**

(f) Use of integration of Φ (s) (Division by s)

We have proved in § 15, page 1-55 that if $L[f(t)] = \Phi(s)$ then $L\int_0^t f(u)\,du = \frac{1}{s}\Phi(s)$.

Taking the inverse Laplace transform of both sides.

$$\int_0^t f(u)\,du = L^{-1}\frac{1}{s}\Phi(s).$$

But $f(u) = L^{-1}\Phi(s)$

$$\therefore\ \boxed{L^{-1}\frac{1}{s}\Phi(s) = \int_0^t L^{-1}\Phi(s)\,ds} \quad \text{.................... (18)}$$

Note ....

See that we have obtained this result as a corollary to convolution theorem in (16A) on page 2-15.

Example 1 : Find : **(i)** $L^{-1}\left[\frac{1}{s(s^2+4)}\right]$ **(ii)** $L^{-1}\frac{1}{s^2(s+1)}$

(iii) $L^{-1}\frac{1}{s^3(s^2+a^2)}$ **(iv)** $L^{-1}\frac{s^2-a^2-s^3}{s^3(s^2-a^2)}$ **(v)** $L^{-1}\frac{1}{s\sqrt{s+4}}$

(M.U. 1993) **(M.U. 2002, 09)**

Sol. : (i) $L^{-1}\left[\frac{1}{s(s^2+4)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2+4)}\right]\cdot du = \int_0^t \frac{1}{2}\cdot\sin 2u\,du$

$$= \frac{1}{2}\left[\frac{-\cos 2u}{2}\right]_0^t = \frac{1}{4}[1-\cos 2t] \quad \text{[See Ex. 1, Exercise - VI, page 2-17]}$$

(ii) $$L^{-1}\left[\frac{1}{s^2(s+1)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s(s+1)}\right] = \int_0^t L^{-1}\left[\frac{1}{s(s+1)}\right]\cdot du$$

$$= \int_0^t L^{-1}\left[\frac{1}{s}-\frac{1}{s+1}\right]du = \int_0^t L^{-1}\left[\frac{1}{s}\right]du - \int_0^t L^{-1}\left[\frac{1}{s+1}\right]du$$

$$= \int_0^t 1\cdot dt - \int_0^t e^{-u}\,du = [u]_0^t + \left[e^{-u}\right]_0^t = t + e^{-t} - 1.$$

Aliter : We can use (18), page 2-36 repeatedly.

$$\therefore\ L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s+1}\right] = \int_0^t L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+1}\right]du = \int_0^t\int_0^t L^{-1}\left(\frac{1}{s+1}\right)(du)^2$$

$$\therefore\ L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s+1}\right] = \int_0^t\int_0^t (e^{-u})(du)^2 = \int_0^t\left[-e^{-u}\right]_0^t du = \int_0^t (1-e^{-u})\,du$$

$$= \left[u+e^{-u}\right]_0^t = t + e^{-t} - 1.$$

(iii) $$L^{-1}\left[\frac{1}{s^3(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]du$$

$$= \frac{1}{a^2}\int_0^t L^{-1}\left[\frac{1}{s^2}-\frac{1}{s^2+a^2}\right]du = \frac{1}{a^2}\int_0^t\left[u-\frac{1}{a}\sin au\right]du$$

$$= \frac{1}{a^2}\left[\left\{\frac{u^2}{2}\right\}_0^t + \frac{1}{a^2}\{\cos au\}_0^t\right] = \frac{1}{a^2}\left[\frac{t^2}{2}\right] + \frac{1}{a^4}[\cos at - 1]$$

Aliter : We may also use (18), page 2-36 repeatedly.

$$\therefore\ L^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]du$$

$$\therefore L^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{(s^2+a^2)}\right]=\int_0^t L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s(s^2+a^2)}\right]du=\int_0^t\int_0^t L^{-1}\left[\frac{1}{s(s^2+a^2)}\right](du)^2$$

$$=\int_0^t\int_0^t\int_0^t L^{-1}\frac{1}{(s^2+a^2)}(du)^3=\int_0^t\int_0^t\int_0^t\frac{1}{a}\sin(au)\,(du)^3$$

$$=\frac{1}{a}\int_0^t\int_0^t\left[-\frac{\cos au}{a}\right]_0^t(du)^2=\frac{1}{a^2}\int_0^t\int_0^t(1-\cos au)\,(du)^2$$

$$=\frac{1}{a^2}\int_0^t\left[u-\frac{\sin au}{a}\right]_0^t du=\frac{1}{a^2}\int_0^t\left(u-\frac{\sin au}{a}\right)du$$

$$=\frac{1}{a^2}\left[\frac{u^2}{2}+\frac{\cos au}{a^2}\right]_0^t=\frac{1}{a^2}\left[\frac{t^2}{2}+\frac{\cos at}{a^2}-\frac{1}{a^2}\right]$$

$$=\frac{1}{a^2}\left[\frac{t^2}{2}\right]+\frac{1}{a^4}(\cos at-1)$$

(iv) $$L^{-1}\frac{s^2-a^2-s^3}{s^3(s^2-a^2)}=L^{-1}\left[\frac{1}{s}\cdot\frac{s^2-a^2-s^3}{s^2(s^2-a^2)}\right]=\int_0^t L^{-1}\left[\frac{s^2-a^2-s^3}{s^2(s^2-a^2)}\right]du$$

$$\therefore L^{-1}\frac{s^2-a^2-s^3}{s^3(s^2-a^2)}=\int_0^t L^{-1}\left[\frac{1}{s^2}-\frac{s}{s^2-a^2}\right]du=\int_0^t u\,du-\int_0^t\cos h\,au\,du$$

$$=\left[\frac{u^2}{2}\right]_0^t-\left[\frac{\sin h\,au}{a}\right]_0^t=\frac{t^2}{2}-\frac{1}{a}\sin h\,at.$$

(v) $$L^{-1}\frac{1}{s\sqrt{s+4}}=\int_0^t L^{-1}\frac{1}{\sqrt{s+4}}du=\int_0^t e^{-4u}L^{-1}\left(\frac{1}{\sqrt{s}}\right)du$$

$$=\int_0^t e^{-4u}\cdot\frac{u^{-1/2}}{\overline{|1/2}}du=\frac{1}{\sqrt{\pi}}\int_0^t e^{-4u}u^{-1/2}du$$

Now put $4u=x^2$, $2du=x\,dx$, $\sqrt{u}=\frac{x}{2}$.

$$\therefore L^{-1}\frac{1}{s\sqrt{s+4}}=\frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}}e^{-x^2}\cdot\frac{2}{x}\cdot\frac{x}{2}dx=\frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}}e^{-x^2}dx$$

$$=\frac{1}{2}\cdot\frac{2}{\sqrt{\pi}}\int_0^{2\sqrt{t}}e^{-x^2}du=\frac{1}{2}erf(2\sqrt{t})\qquad\left[\because\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx=erf(t)\right]$$

[For another method see Ex. 6 (i), page 2-26.]

EXERCISE - XII

Find : 1. $L^{-1}\left[\frac{1}{s(s^2+9)}\right]$ 2. $L^{-1}\left[\frac{1}{s^2(s+2)}\right]$ 3. $L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$ (M.U. 1

4. $L^{-1}\frac{1}{s(s^2-a^2)}$ 5. $L^{-1}\frac{54}{s^3(s-3)}$ **(M.U. 2003)**

[**Ans. : (1)** $\frac{1}{9}[1-\cos 3t]$, **(2)** $\frac{1}{4}\left[2t+e^{-2t}-1\right]$, **(3)** $\frac{t^2}{2}+\cos t-1$,

(4) $\frac{1}{a}[\cos h\,at-1]$, **(5)** $[-2-6t-9t^2+2e^{3t}]$.]

Miscellaneous Examples

Theorem : From (17), page 1-39, we get the following corollary easily.

$$\text{If } L^{-1}[\Phi(s)]=f(t),\ \text{ then } L^{-1}\left[\int_s^\infty\Phi(s)\,ds\right]=\frac{1}{t}f(t).$$

Example 1 : Find $L^{-1}\left[\int_s^\infty\left(\frac{u}{u^2+a^2}-\frac{u}{u^2+b^2}\right)du\right]$. **(M.U. 2009)**

Sol. : Let $\Phi(s)=\frac{s}{s^2+a^2}-\frac{s}{u^2+b^2}$

$$\therefore L^{-1}[\Phi(s)]=L^{-1}\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]=\cos at-\cos bt$$

Hence, by the above corollary,

$$L^{-1}\left[\int_s^\infty\left(\frac{u}{u^2+a^2}-\frac{u}{u^2+b^2}\right)du\right]=\frac{\cos at-\cos bt}{t}$$

Example 2 : Find $\int_0^\infty\cos(tx^2)\,dx$ and hence, find $\int_0^\infty\cos x^2\cdot dx$. **(M.U. 2010)**

Sol. : Let $f(t)=\int_0^\infty\cos(tx^2)\,dx$

$$\therefore L\,f(t)=\int_0^\infty e^{-st}f(t)\,dt=\int_0^\infty e^{-st}\int_0^\infty[\cos(tx^2)\,dx]\,dt$$

$$\therefore L\,f(t)=\int_0^\infty\left[\int_0^\infty e^{-st}\cos(tx^2)\,dt\right]dx=\int_0^\infty[L\cos(tx^2)]\,dx=\int_0^\infty\frac{s}{s^2+x^4}dx$$

Now put $x=\sqrt{s\tan\theta}$ $\quad\therefore\ dx=\frac{s\cdot\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$

$$\therefore L[f(t)]=\int_0^{\pi/2}\frac{s}{s^2+s^2\tan^2\theta}\cdot\frac{s\cdot\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$$

$$=\int_0^{\pi/2}\frac{1}{2\sqrt{s\tan\theta}}d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}(\sin\theta)^{-1/2}(\cos\theta)^{1/2}d\theta$$

$$=\frac{1}{2\sqrt{s}}\,\frac{\overline{|1/4}\,\overline{|3/4}}{2\cdot 1!}\left[\int_0^{\pi/2}\sin^p\theta\cos^p\theta\,d\theta=\frac{\overline{|(p+1)/2}\ \overline{|(q+1)/2}}{2\overline{|(p+q+2)/2}}\right]$$

$$=\frac{1}{2\sqrt{s}}\cdot\frac{\sqrt{2}\cdot\pi}{2}\qquad\left[\because\overline{|1/4}\ \overline{|3/4}=\sqrt{2}\cdot\pi\right]$$

$$\therefore L(f(t)=\frac{\pi}{2\sqrt{2}\sqrt{s}}$$

$$\therefore\ f(t)=\frac{\pi}{2\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right)=\frac{\pi}{2\sqrt{2}}\cdot\frac{t^{-1/2}}{\lceil 1/2}=\frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{t}}$$

Now put $t=1$, $\therefore\ \int_0^\infty \cos x^2\,dx=\frac{\sqrt{\pi}}{2\sqrt{2}}=\frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Example 3 : Find $\int_0^\infty \sin(tx^2)$ and hence, find $\int_0^\infty \sin x^2\,dx$. (M.U. 200

Sol. : Let $f(t)=\int_0^\infty \sin(tx^2)\,dx$

$$\therefore\ L\,f(t)=\int_0^\infty e^{-st}f(t)\,dt=\int_0^\infty e^{-st}\left[\int_0^\infty \sin(tx^2)\,dx\right]dt$$

$$=\int_0^\infty\left[\int_0^\infty e^{-st}\sin(tx^2)\,dt\right]dx=\int_0^\infty[L\sin(tx^2)]\,dx=\int_0^\infty\frac{x^2}{s^2+x^4}\,dx$$

Now put $x=\sqrt{s\tan\theta}$, $dx=\frac{s\cdot\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$

$$\therefore\ L\,f(t)=\int_0^{\pi/2}\frac{s\tan\theta}{s^2+s^2\tan^2\theta}\cdot\frac{s\cdot\sec^2\theta}{2\sqrt{s\tan\theta}}\cdot d\theta$$

$$=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$$=\frac{1}{2\sqrt{s}}\cdot\frac{\lceil 3/4\ \lceil 1/4}{2\cdot 1}=\frac{1}{2\sqrt{s}}\cdot\frac{\sqrt{2}\cdot\pi}{2}=\frac{\pi}{2\sqrt{2}}\cdot\frac{1}{\sqrt{s}}$$

$$\therefore\ f(t)=\frac{\pi}{2\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right)=\frac{\pi}{2\sqrt{2}}\cdot\frac{t^{-1/2}}{\lceil 1/2}=\frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{t}}$$

Now put $t=1$, $\therefore\ \int_0^\infty \sin x^2\,dx=\frac{\sqrt{\pi}}{2\sqrt{2}}=\frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Example 4 : Find $\int_0^\infty e^{-tx^2}\,dx$ and hence, find $\int_0^\infty e^{-x^2}\,dx$. (M.U. 200

Sol. : Let $f(t)=\int_0^\infty e^{-tx^2}\,dx$

$$\therefore\ L\,f(t)=\int_0^\infty e^{-st}\left[\int_0^\infty e^{-tx^2}\,dx\right]dt=\int_0^\infty\left[\int_0^\infty e^{-st}\cdot e^{-tx^2}\,dt\right]dx$$

$$=\int_0^\infty\left[L(e^{-tx^2})\right]dx=\int_0^\infty\frac{dx}{s+x^2}\qquad\left[\because\ L\,e^{-at}=\frac{1}{s+a}\right]$$

$$=\left[\frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\right]_0^\infty=\frac{\pi}{2\sqrt{s}}$$

$$\therefore\ f(t)=\frac{\pi}{2}L^{-1}\left(\frac{1}{\sqrt{s}}\right)=\frac{\pi}{2}\frac{t^{-1/2}}{\lceil 1/2}=\frac{\pi}{2}\frac{1}{\sqrt{\pi}\sqrt{t}}=\frac{1}{2}\sqrt{\frac{\pi}{t}}.$$

Now, put $t=1$, $\therefore\ \int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$.

Example 5 : If $J_0(t)=\frac{1}{\pi}\int_0^\pi \cos(t\cos\theta)\,d\theta$, prove that $L[J_0(t)]=\frac{1}{\sqrt{s^2+1}}$. **(M.U. 2003)**

Sol. : We have $J_0(t)=\frac{1}{\pi}\int_0^\pi\cos(t\cos\theta)\,d\theta=\frac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta$

Taking Laplace transforms of both sides.

$$L[J_0(t)]=\frac{2}{\pi}\int_0^\infty e^{-st}\left[\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta\right]dt$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^\infty e^{-st}\cos(t\cos\theta)\,dt\right]d\theta=\frac{2}{\pi}\int_0^{\pi/2}[L\cos(t\cos\theta)]\,d\theta$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\left[\frac{s}{s^2+\cos^2\theta}\right]d\theta=\frac{2}{\pi}\int_0^{\pi/2}\frac{s\sec^2\theta}{s^2\sec^2\theta+1}\,d\theta$$

$$\therefore\ L[J_0(t)]=\frac{2}{\pi}\int_0^{\pi/2}\frac{s\sec^2\theta}{(s^2+1)+s^2\tan^2\theta}\,d\theta$$

Put $s\tan\theta=t$ $\quad\therefore\ s\sec^2\theta\,d\theta=dt$

$$\therefore\ L[J_0(t)]=\frac{2}{\pi}\int_0^\infty\frac{dt}{t^2+(s^2+1)}=\frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\left[\tan^{-1}\left(\frac{t}{\sqrt{s^2+1}}\right)\right]_0^\infty$$

$$=\frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\cdot\frac{\pi}{2}=\frac{1}{\sqrt{s^2+1}}.$$

Example 6 : If $J_0(t)=\frac{1}{\pi}\int_0^\pi\cos(t\sin\theta)\,d\theta$, prove that $L[J_0(t)]=\frac{1}{\sqrt{s^2+1}}$. **(M.U. 2003)**

Sol. : Do it yourself.

Example 7 : Prove that $L^{-1}\left[\frac{1}{s}\cos\frac{1}{s}\right]=1-\frac{t^2}{(2!)^2}+\frac{t^4}{(4!)^2}-\frac{t^6}{(6!)^2}+\ldots..$

Sol. : We know that $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots..$

$$\therefore\ \cos\left(\frac{1}{s}\right)=1-\frac{1}{2!\,s^2}+\frac{1}{4!}\cdot\frac{1}{s^4}-\frac{1}{6!}\cdot\frac{1}{s^6}+\ldots..$$

$$\therefore\ \frac{1}{s}\left\{\cos\frac{1}{s}\right\}=\frac{1}{s}\left\{1-\frac{1}{2!\,s^2}+\frac{1}{4!}\cdot\frac{1}{s^4}-\frac{1}{6!}\cdot\frac{1}{s^6}+\ldots..\right\}$$

$$\therefore\ L^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\}=L^{-1}\frac{1}{s}\left\{1-\frac{1}{2!\,s^2}+\frac{1}{4!\,s^4}-\frac{1}{6!\,s^6}+\ldots..\right\}$$

$$=L^{-1}\left\{1-\frac{1}{2!\,s^3}+\frac{1}{4!\,s^5}-\frac{1}{6!\,s^7}+\ldots..\right\}$$

$$=1-\frac{1}{2!}\cdot\frac{t^2}{2!}+\frac{1}{4!}\cdot\frac{t^4}{4!}-\frac{1}{6!}\cdot\frac{t^6}{6!}+\ldots..=1-\frac{t^2}{(2!)^2}+\frac{t^4}{(4!)^2}-\frac{t^6}{(6!)^2}+\ldots..$$

4. Laplace Transform of Periodic Functions

If $f(t)$ is a periodic function of period a, show that

$$\boxed{L f(t) = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\, dt}$$ **(M.U. 1995, 96, 2006, 07)** ....(1

Proof. : Since $f(t)$ is periodic with period a, $f(t) = f(t+a) = f(t+2a) = ...$

$$\therefore\ L f(t) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^a e^{-st} f(t)\, dt + \int_a^{2a} e^{-st} f(t)\, dt + ...\infty$$

Now $$\int_a^{2a} e^{-st} f(t)\, dt = \int_0^a e^{-s(u+a)} f(u+a)\, du \qquad [\text{where } t = u+a]$$

$$= e^{-as}\int_0^a e^{-su} f(u+a)\, du$$

$$= e^{-as}\int_0^a e^{-st} f(t+a)\, dt \qquad (\text{Changing } u \text{ to } t)$$

$$= e^{-as}\int_0^a e^{-st} f(t)\, dt \qquad [f(t+a) = f(t)]$$

Similarly, we can show that the other integrals are also equal to $e^{-2as}\int_0^a e^{-st} f(t)\, dt$ and so

$$\therefore\ L f(t) = \left(1 + e^{-as} + e^{-2as} + ...\infty\right)\int_0^a e^{-st} f(t)\, dt.$$

$$\therefore\ \boxed{L f(t) = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\, dt} \qquad \left[\text{For a G.P. } S_\infty = \frac{a}{1-r}\right]$$ ....(2

Example 1 : Find Laplace transform of

(i) $f(t) = K\frac{t}{T}$ for $0 < t < T$ and $f(t) = f(t+T)$. **(M.U. 20**

(ii) $f(t) = 1$, for $0 \le t < a$ and $f(t) = -1$, $a < t < 2a$ and $f(t)$ is periodic with period $2a$. **(M.U. 1999, 2003, 06, 1**

(iii) $f(t) = e^t$ for $0 < t < \pi$ and $f(t) = f(t+2\pi)$

Sol. : (i) Since $f(t)$ is periodic with period T

$$L f(t) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\cdot\frac{Kt}{T}\, dt$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\int_a^T e^{-st}\cdot t\cdot dt$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[-t\cdot\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^T$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[-\frac{Te^{-sT}}{s} - \frac{e^{-st}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[\frac{1}{s^2}\left(1-e^{-sT}\right) - \frac{Te^{-sT}}{s}\right] = K\left[\frac{1}{Ts^2} - \frac{e^{-sT}}{s(1-e^{-st})}\right]$$

The function $f(t) = k\cdot\frac{t}{T}$ is known as "saw-tooth wave" function with period T and its graph is as shown below.

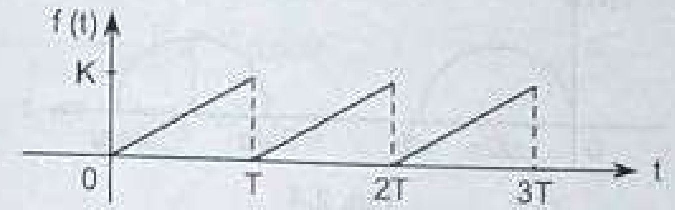


Fig. 2.2

(ii) Since $f(t)$ is periodic with period $2a$.

$$L f(t) = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(1)\, dt + \int_a^{2a} e^{-st}(-1)\, dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left\{-\frac{e^{-st}}{s}\right\}_0^a + \left\{\frac{e^{-st}}{s}\right\}_a^{2a}\right]$$

$$= \frac{1}{s}\cdot\frac{1}{1-e^{-2as}}\cdot(1-e^{-as})^2 = \frac{1}{s}\cdot\frac{1-e^{-as}}{1+e^{-as}}$$

$$= \frac{1}{s}\cdot\left[\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right] = \frac{1}{s}\tanh\left\{\frac{as}{2}\right\}.$$

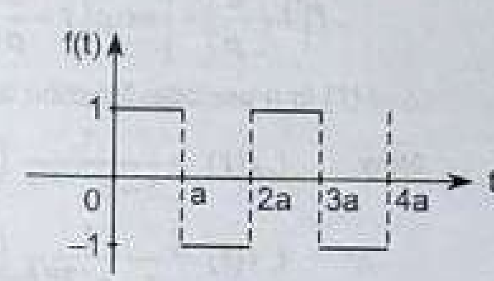


Fig. 2.3

The function is known as "square wave" function and its graph is shown in Fig. 2.3.

(iii) Since $f(t)$ is periodic with period 2π.

$$L f(t) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}\cdot f(t)\, dt = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}\cdot e^t\, dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{(1-s)t}\, dt = \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)t}}{1-s}\right]_0^{2\pi}$$

$$= \frac{1}{(1-e^{-2\pi s})(1-s)}\left[e^{(1-s)2\pi} - 1\right]$$

Example 2 : Find Laplace transform of $f(t) = a\sin pt$, $0 < t < \pi/p$, $f(t) = 0$, $\pi/p < t < 2\pi/p$ and $f(t) = f(t + 2\pi/p)$. **(M.U. 1995, 2002, 08, 13)**

Sol. : $$L f(t) = \frac{1}{1-e^{-(2\pi/p)s}}\int_0^{\pi/p} e^{-st} a\sin pt\, dt$$

$$= \frac{a}{1-e^{-(2\pi/p)s}}\left[\frac{1}{s^2+p^2}\cdot e^{-st}(-s\sin pt - p\cos pt)\right]_0^{\pi/p}$$

$$= \frac{a}{1-e^{-(2\pi/p)s}}\cdot\frac{1}{(s^2+p^2)}\cdot\left[e^{-s\pi/p}(-s\sin\pi - p\cos\pi) - e^0(-s\sin 0 - p\cos 0)\right]$$

$$= \frac{a}{1-e^{-(2\pi/p)s}}\cdot\frac{1}{s^2+p^2}\left[p\, e^{-s\pi/p} + p\right]$$

$$= \frac{a\, p}{1-e^{-(s\pi/p)}}\cdot\frac{1}{s^2+p^2}$$

The function $f(t) = a \sin pt$ is known as "half-sine wave rectifier" function with period 2π and its graph is shown below.

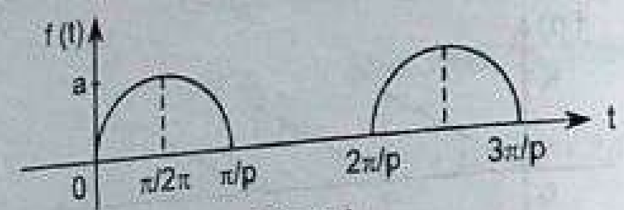


Fig. 2.4

Example 3 : Find the Laplace transform of $f(t) = |\sin pt|,\ t \ge 0$. **(M.U. 2003,**

Sol. : We first note that

$$f\left(t+\frac{\pi}{p}\right) = \left|\sin p\left(t+\frac{\pi}{p}\right)\right| = |\sin(pt+\pi)| = |\sin pt|$$

$\therefore$ $f(t)$ is a periodic function with period π / p.

Now, $$L\,f(t) = \frac{1}{1-e^{-\pi s/p}}\int_0^{\pi/p} e^{-st}\,|\sin pt|\,dt$$

$$\therefore \quad L\,f(t) = \frac{1}{1-e^{-\pi s/p}}\int_0^{\pi/p} e^{-st}\sin pt\,dt \qquad \left[\because \sin pt > 0, \text{ for } 0 \le t \le \frac{\pi}{p}\right]$$

$$= \frac{1}{1-e^{-\pi s/p}}\left[\frac{e^{-st}}{s^2+p^2}(-s\sin pt - p\cos pt)\right]_0^{\pi/p}$$

$$\therefore \quad L[f(t)] = \frac{1}{1-e^{-\pi s/p}}\cdot\frac{1}{(s^2+p^2)}\left[e^{-s\pi/p}(0+p)-(0-p)\right]$$

$$= \frac{1}{s^2+p^2}\cdot\frac{1}{1-e^{-\pi s/p}}\cdot p\cdot(1+e^{-\pi s/p})$$

$$= \frac{p}{s^2+p^2}\cdot\left(\frac{e^{\pi s/2p}+e^{-\pi s/2p}}{e^{\pi s/2p}-e^{-\pi s/2p}}\right)$$

$$= \frac{p}{s^2+p^2}\cdot\coth\left(\frac{\pi s}{2p}\right)$$

The function $f(t) = |\sin pt|$ is known as "full-sine wave rectifier" and its graph is shown below.

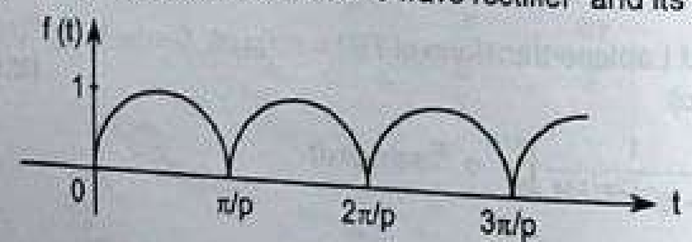


Fig. 2.5

Example 4 : Find Laplace transform of $f(t) = \sin 2t,\ 0 < t < \pi/2$, $f(t) = 0,\ \pi/2 < t < \pi$ and $f(t) = f(t+\pi)$. **(M.U. 1995, 2002, 05, 12)**

Sol. : In the above Ex. 2, put $p = 2,\ a = 1$.

$$\therefore \quad L\,f(t) = \frac{1}{1-e^{-\pi s}}\cdot\frac{1}{s^2+2^2}\cdot\left[2e^{-\pi s/2}+2\right] = \frac{2}{1-e^{-\pi s/2}}\cdot\frac{1}{s^2+2^2}.$$

Example 5 : Find $L\,f(t)$ where, $f(t) = t,\ 0 < t < 1$; $f(t) = 0,\ 1 < t < 2$ and $f(t+2) = f(t)$ for $t > 0$. **(M.U. 1996)**

Sol. : Since $f(t)$ is periodic with period $a = 2$, we have

$$L\,f(t) = \frac{1}{1-e^{-as}}\int_0^a e^{-st}\,f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^a e^{-st}\,f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st}\cdot 0\cdot dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[t\left(-\frac{e^{-st}}{s}\right) - \left(\frac{e^{-st}}{s^2}\right)(1)\right]_0^1$$

$$= \frac{1}{1-e^{-2s}}\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$\therefore \quad L\,f(t) = \frac{1}{s^2(1-e^{-2s})}(1-e^{-s}-s\,e^{-s}).$$

EXERCISE - XIII

Find the Laplace transform of

1. $f(t) = \dfrac{t}{a},\ 0 < t \le a$; $f(t) = \dfrac{1}{a}(2a-t),\ a < t < 2a$ and $f(t) = f(t+2a)$. **(M.U. 2002, 04, 11)**
2. $f(t) = t,\ 0 < t < \pi$; $f(t) = \pi - t,\ \pi < t < 2\pi$ and $f(t) = f(t+2\pi)$.
3. $f(t) = t^2,\ 0 < t < 2$, where $f(t)$ is a periodic function with period 2.
4. $f(t) = t,\ 0 < t < 1$ and $f(t)$ is of period 1.
5. $f(t) = \begin{cases} E, & 0 \le t \le (p/2) \\ -E, & (p/2) \le t \le p \end{cases},\ f(t+p) = f(t)$. **(M.U. 2002, 03, 14)**

[**Ans. :** **(1)** $\dfrac{1}{as^2}\tanh\left(\dfrac{as}{2}\right)$, **(2)** $\dfrac{1-(1+\pi s)e^{-\pi s}}{s^2(1+e^{-\pi s})}$,

(3) $\dfrac{2}{1-e^{-2s}}\left[\dfrac{1}{s^3} - \dfrac{1}{s^3}\cdot e^{-2s} - \dfrac{2}{s}\cdot e^{-2s} - \dfrac{2}{s^2}\cdot e^{-2s}\right]$,

(4) $\dfrac{1}{s^2} - \dfrac{e^{-s}}{s(1-e^{-s})}$, **(5)** $\dfrac{E}{s}\tanh\left(\dfrac{sp}{4}\right)$.]

5. Laplace Transforms of Two Special Functions

We shall now obtain Laplace transforms of some special functions which we come across in discussions of engineering problems. These are

(i) Heaviside's Unit Step Function,

(ii) Dirac-Delta Function

6. Heaviside's Unit Step Function

The function takes only two values 0 and 1. When x is negative the value of the function is zero and when x is positive its value is 1. It is denoted by $H(t)$ [or $U(t)$], H for Heaviside (U for unit). Thus, the value of $H(t)$ is one to the right of the origin and is zero to the left of the origin. Obviously, it is a discontinuous function.

We define it as $H(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1 \end{cases}$

The function takes a jump of unit magnitude and remains there thereafter. The graph of the function is shown in the Fig. 2.6.

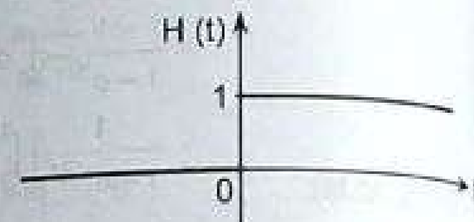


Fig. 2.6

Displaced Unit Step Function

If the origin is shifted to a point $t = a$ *i.e.* if the function is zero before $t = a$ and takes a jump of unit magnitude at $t = a$ and remains there thereafter, the function is called **displaced unit step function.** It is defined by

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

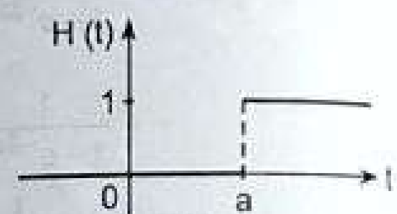


Fig. 2.7

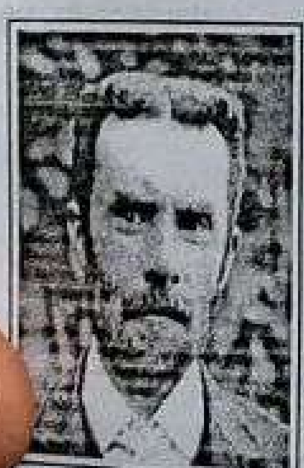
The graph of the function is shown in the Fig. 2.7.

Remark ....

As remarked earlier the Heavisides unit step function is also denoted by $u(t)$.

Thus, $u(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1 \end{cases}$ and $u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$

Oliver Heaviside (1850 - 1925)



Although he was a bright student while in school, his parents could not keep him at school after he was 16, so he continued studying by himself and had no further formal education. He adapted complex numbers to the study of electrical circuits, invented mathematical techniques for the solution of differential equations, reformulated Maxwell's field equations and independently coformulated vector analysis. He was, thus, an electrical engineer, mathematician and physicist. He invented the Heaviside step function and employed it to model the current in an electric circuit. He developed the transmission line theory and advanced the idea that the Earth's uppermost atmosphere contained an ionised layer known as ionosphere.

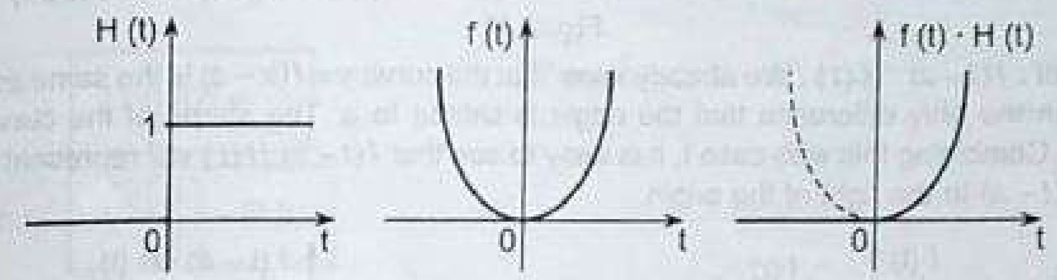
Functions Represented by Unit Step Functions

In many engineering discussions, it is convenient to represent a function with the help of unit step function. We shall denote the given function by $f(t)$ and consider the following five cases.

Case I : $f(t)\,H(t)$: Since $H(t)$ is zero for $t < 0$ and $H(t)$ is unity for $t \geq 0$, it is easy to see that the product $f(t)\,H(t)$ will be zero for $t < 0$ and $f(t) \cdot H(t)$ will remain $f(t)$ for $t \geq 0$. This means by taking the product $f(t) \cdot H(t)$ **the part of $f(t)$ to the left of the origin is cut off**. Thus

$$f(t) \cdot H(t) = \begin{cases} 0, & t < 0 \\ f(t), & t \geq 0 \end{cases}$$

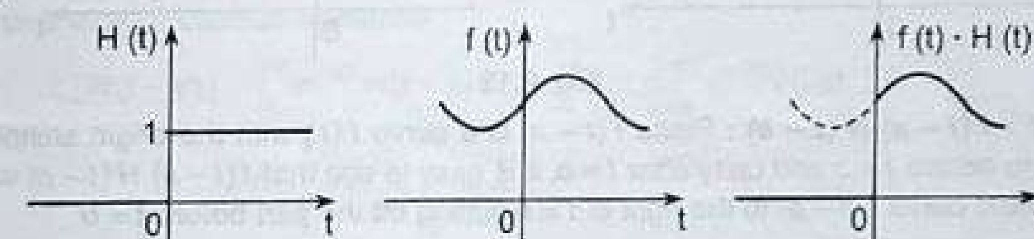
Suppose $f(t) = t^2$, a parabola then $f(t)\,H(t)$ will give only the branch of the parabola on the right. In the following diagram we shall show both the curves $f(t) = t^2$ and $f(t) \cdot H(t)$.



(The dotted part is cut-off)

Fig. 2.8

If $f(t)$ is some **general function**, we get the following diagrams.



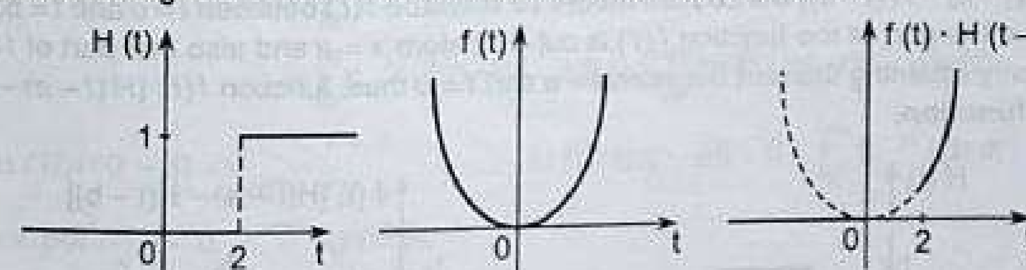
(The dotted part is cut-off)

Fig. 2.9

Case II : $f(t)\,H(t-a)$: Since $H(t-a)$ is zero for $t < a$ and $H(t-a)$ is unity for $t \geq a$, it is easy to see that the product $f(t)\,H(t-a)$ will be zero for $t < a$ and $f(t) \cdot H(t-a)$ will remain $f(t)$ for $t \geq a$. This means by taking the product $f(t) \cdot H(t-a)$, **the part of $f(t)$ to the left of $t = a$ is cut-off.** Thus,

$$f(t)\,H(t-a) = \begin{cases} 0, & t < a \\ f(t), & t \leq a \end{cases}$$

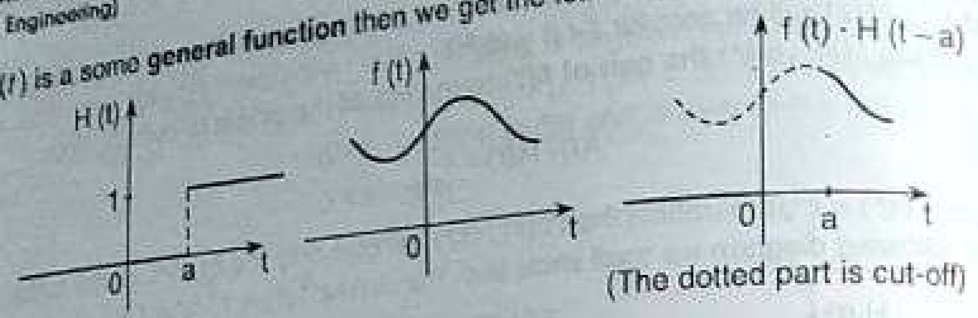
Suppose as before $f(t) = t^2$ is a parabola and $a = 2$ then $f(t)\,H(t-2)$ will give the branch of the parabola to the right of $t = 2$. This is shown in the following diagram.



(The dotted part is cut-off)

Fig. 2.10

If $f(t)$ is a some **general function** then we get the following diagrams.



(The dotted part is cut-off)

Fig. 2.11

Case III : $f(t-a) \cdot H(t)$: We already know that the curve $y = f(x-a)$ is the same as the $y = f(x)$ with the only difference that the origin is shifted to a. The shape of the curve rem unchanged. Combining this with case I, it is easy to see that $f(t-a)\,H(t)$ will represent the p the curve $f(t-a)$ to the right of the origin.

Fig. 2.12

Case IV : $f(t-a)\,H(t-b)$: Since $f(t-a)$ is a curve $f(t)$ with the origin shifted to a $H(t-b)$ is zero before $t = b$ and unity after $t = b$, it is easy to see that $f(t-a)\,H(t-b)$ will giv part of the shifted curve $f(t-a)$ to the right of $t = b$ cutting off the part before $t = b$.

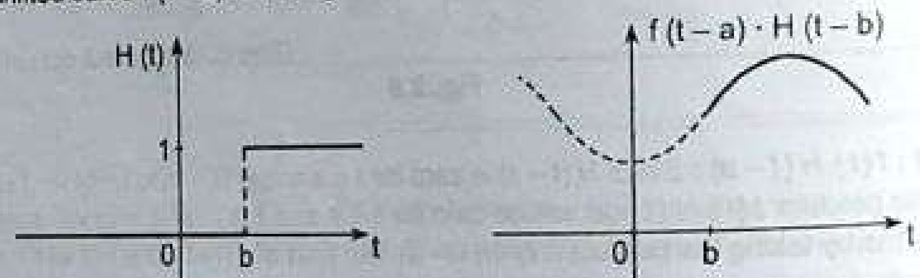


Fig. 2.13

Case V : Representation of the part of the curve $f(t)$ from $t = a$ to $t = b$

We first see that $H(t-a)$ is a unit function to the right of $t = a$ and $H(t-b)$ is a unit func the right of $t = b$. Hence $H(t-a) - H(t-b)$ is the part of the unit function between $t = a$ and Hence, $f(t)\,[H(t-a) - H(t-b)]$ will be zero before $t = a$, will be $f(t)$ between $t = a$ and $t = b$, an after $t = b$. Since the part of the function $f(t)$ is cut-off before $x = a$ and also that part of $f(t)$ is off after $t = b$, only retaining the part between $t = a$ and $t = b$ thus, function $f(t)\,[H(t-a) - H($ is called **filter function.**

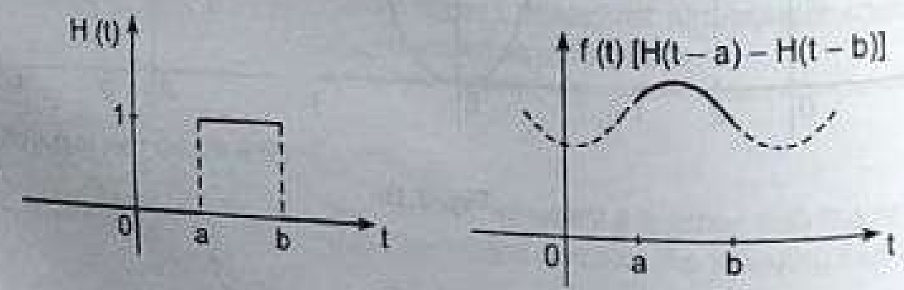


Fig. 2.14

7. Laplace Transform of Heaviside's Unit Step Function $H(t)$

Since, $H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

By definition of Laplace transform

$$L[H(t)] = \int_0^\infty e^{-st} H(t)\,dt = \int_0^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

$$\therefore \quad \boxed{L[H(t)] = \frac{1}{s}} \qquad \text{.................... (21)}$$

Cor. : $$\boxed{L^{-1}\left[\frac{1}{s}\right] = H(t)} \qquad \text{.................... (22)}$$

Laplace Transform of $H(t-a)$

Since, $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$

By definition of Laplace transform

$$L[H(t-a)] = \int_0^\infty e^{-st} H(t-a)\,dt = \int_0^a 0\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= 0 + \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{1}{s}e^{-as}$$

$$\therefore \quad \boxed{L[H(t-a)] = \frac{1}{s}\cdot e^{-as}} \qquad \text{.................... (23)}$$

Cor. : $$\boxed{L^{-1}\left[\frac{1}{s}\cdot e^{-as}\right] = H(t-a)} \qquad \text{.................... (24)}$$

Laplace Transform of $f(t)\,H(t-a)$

Since, $f(t)H(t-a) = \begin{cases} 0, & t < a \\ f(t), & t \geq a \end{cases}$

By definition of Laplace transform, we get

$$L[f(t)H(t-a)] = \int_0^\infty e^{-st}\,[f(t)H(t-a)]\,dt$$

$$= \int_0^a e^{-st}\,[f(t)H(t-a)]\,dt + \int_a^\infty e^{-st}\,[f(t)H(t-a)]\,dt$$

But $f(t)H(t-a) = \begin{cases} 0, & t < a \\ f(t), & t \geq a \end{cases}$ $\quad \therefore \; L[f(t)H(t-a)] = 0 + \int_a^\infty e^{-st} f(t)\,dt$

Now, put $t - a = u$ $\quad \therefore \; dt = du$

When $t = a$, $u = 0$; when $t = \infty$, $u = \infty$.

$$\therefore \quad L[f(t)\cdot H(t-a)] = \int_0^\infty e^{-s(a+u)} f(a+u)\,du$$

$$= e^{-as}\int_0^\infty e^{-su} f(u+a)\,du$$

$$= e^{-as}\int_0^{\infty} e^{-st} f(t+a)\,dt$$

$$\therefore \quad \boxed{L[f(t)\cdot H(t-a)] = e^{-as}\,L[f(t+a)]}$$

Putting $a = 0$, we got

$$\boxed{L[f(t)\,H(t)] = L[f(t)]}$$

Procedure to find $L[f(t)\,H(t-a)]$

(i) First find $f(t+a)$.

(ii) Then find $L[f(t+a)]$.

(iii) Multiply the result by e^{-as}.

This is the required Laplace transform.

Example 1 : Find $L[t^2 H(t-3)]$.

Sol. : (i) Here $f(t) = t^2$ and $a = 3$.

$$\therefore \quad f(t+3) = (t+3)^2 = t^2 + 6t + 9$$

(ii) $$L[f(t+3)] = L(t^2+6t+9) = L(t^2) + 6\,L(t) + 9\,L(1) = \frac{2}{s^3} + 6\cdot\frac{1}{s^2} + 9\cdot\frac{1}{s}$$

(iii) $$L[t^2 H(t-3)] = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Example 2 : Find $L[(1+2t-t^2+t^3)\,H(t-1)]$.

Sol. : (i) Here $f(t) = 1+2t-t^2+t^3$ and $a = 1$.

$$\begin{aligned}\therefore \quad f(t+1) &= 1 + 2(t+1) - (t+1)^2 + (t+1)^3\\ &= 1 + 2t + 2 - t^2 - 2t - 1 + t^3 + 3t^2 + 3t + 1\\ &= t^3 + 2t^2 + 3t + 3\end{aligned}$$

(ii) $$\begin{aligned}L[f(t+1)] &= L(t^3+2t^2+3t+3)\\ &= L(t^3) + 2L(t^2) + 3L(t) + 3L(1)\\ &= \frac{3!}{s^4} + 2\cdot\frac{2!}{s^3} + 3\cdot\frac{1}{s^2} + 3\cdot\frac{1}{s}\end{aligned}$$

(iii) $$L[(1+2t-t^2+t^3)\,H(t-1)] = e^{-s}\left[\frac{6}{s^4} + \frac{4}{s^3} + \frac{3}{s^2} + \frac{3}{s}\right]$$

Example 3 : Find Laplace transform of $(1+2t-3t^2+4t^3)\,H(t-2)$. **(M.U. 1998, 2004)**

Sol. : (i) Here $f(t) = 1+2t-3t^2+4t^3$ and $a = 2$.

$$\begin{aligned}\therefore \quad f(t+2) &= 1 + 2(t+2) - 3(t+2)^2 + 4(t+2)^3\\ &= 1 + 2t + 4 - 3(t^2+4t+4) + 4(t^3+6t^2+12t+8)\\ &= 4t^3 + 21t^2 + 38t + 25\end{aligned}$$

(ii) $$\begin{aligned}L[f(t+2)] &= L(4t^3+21t^2+38t+25)\\ &= 4L(t^3) + 21L(t^2) + 38L(t) + 25L(1)\\ &= 4\cdot\frac{3!}{s^4} + 21\cdot\frac{2!}{s^3} + 38\cdot\frac{1}{s^2} + 25\cdot\frac{1}{s}\end{aligned}$$

(iii) $$L[f(t)\,H(t-2)] = e^{-2s}\left[\frac{24}{s^4} + \frac{42}{s^3} + \frac{38}{s^2} + \frac{25}{s}\right].$$

Example 4 : Express the following function as Heaviside unit step function and find its Laplace transform.

$$f(t) = \begin{cases} 0, & 0 < t < 2\\ (t-2)^2, & t > 2\end{cases}$$

Sol. : Since $H(t-2) = \begin{cases} 0, & t < 2\\ 1, & t > 2\end{cases}$

$$\therefore \quad f(t) = f(t)\cdot H(t-2) = (t-2)^2\,H(t-2)$$

(i) Here $f(t) = (t-2)^2$ and $a = 2$.

$$\therefore \quad f(t+2) = (t+2-2)^2 = t^2$$

(ii) $$L[f(t+2)] = L(t^2) = \frac{2!}{s^3}$$

(iii) $$L[f(t)\cdot H(t-2)] = e^{-2s}\cdot\frac{2}{s^3}.$$

Example 5 : Find Laplace transform of $\sin t\left[H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right)\right]$. **(M.U. 1996)**

Sol. : (a) (i) Here $f(t) = \sin t$ and $a = \frac{\pi}{2}$.

$$\therefore \quad f\left(t+\frac{\pi}{2}\right) = \sin\left(t+\frac{\pi}{2}\right) = \cos t$$

(ii) $$L\left[f\left(t+\frac{\pi}{2}\right)\right] = L\cos t = \frac{s}{s^2+1}$$

(iii) $$L\left[f(t)\,H\left(t-\frac{\pi}{2}\right)\right] = e^{-\pi s/2}\cdot\frac{s}{s^2+1}$$

(b) We know that [By (23), page 2-48]

$$L\,H(t-a) = \frac{1}{s}e^{-as} \qquad \therefore \quad L\,H\left(t-\frac{3\pi}{2}\right) = \frac{1}{s}\cdot e^{-3\pi s/2}$$

Hence, $$L\left[\sin t\,H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right)\right] = e^{-\pi s/2}\cdot\frac{s}{s^2+1} - e^{-3\pi s/2}\cdot\frac{1}{s}.$$

Example 6 : Find Laplace transform of $e^{-t}\cos t\cdot H(t-\pi)$.

Sol. : (i) Here $f(t) = e^{-t}\cos t$ and $a = \pi$.

$$\begin{aligned}\therefore \quad f(t+\pi) &= e^{-(t+\pi)}\cos(t+\pi)\\ &= -e^{-(t+\pi)}\cos t = -e^{-\pi}\cdot e^{-t}\cos t\end{aligned}$$

(ii) $$\begin{aligned}L[f(t+\pi)] &= L[-e^{-\pi}\cdot e^{-t}\cos t]\\ &= -e^{-\pi}\cdot L(e^{-t}\cos t) = -e^{-\pi}\cdot\frac{s+1}{(s+1)^2+1}\end{aligned}$$

(iii) $$L[f(t+\pi)H(t-\pi)] = -e^{-\pi s}\cdot e^{-\pi}\cdot\frac{s+1}{s^2+2s+2} = -e^{-\pi(s+1)}\cdot\frac{s+1}{s^2+2s+2}.$$

Example 7 : Find Laplace transform of $(t-1)^2\cdot H(t-1)$. (M.U. 20

Sol. : (i) Here $f(t)=(t-1)^2$ and $a=1$.

$\therefore\quad f(t+1) = (t+1-1)^2 = t^2$

(ii) $$L[f(t+1)] = L(t^2) = \frac{2}{s^3}$$

(iii) $$L[f(t+1)\,H(t-1)] = e^{-s}\cdot\frac{2}{s^3}.$$

Example 8 : Find Laplace transform of $e^{-3t}H(t-2)$. (M.U. 20

Sol. : (i) Here $f(t)=e^{-3t}$ and $a=2$.

$\therefore\quad f(t+2) = e^{-3(t+2)} = e^{-6}\cdot e^{-3t}$

(ii) $$L[f(t+2)] = L(e^{-6}\cdot e^{-3t}) = e^{-6}\cdot L(e^{-3t}) = e^{-6}\cdot\frac{1}{s+3}.$$

(iii) $$L[f(t)\cdot H(t-2)] = e^{-2s}\cdot e^{-6}\cdot\frac{1}{s+3} = e^{-2(s+3)}\cdot\frac{1}{s+3}.$$

Example 9 : Find Laplace transform of $e^{4t}\cdot H(t-2)$.

Sol. : (i) Here $f(t)=e^{4t}$ and $a=2$.

$\therefore\quad f(t+2) = e^{4(t+2)} = e^{4t+8} = e^{8}\cdot e^{4t}$

(ii) $$L[f(t+2)] = L(e^{8}\cdot e^{4t}) = e^{8}\cdot L(e^{4t}) = e^{8}\cdot\frac{1}{s-4}$$

(iii) $$L[f(t)\cdot H(t-2)] = e^{-2s}\cdot e^{8}\cdot\frac{1}{s-4} = e^{-2(s-4)}\cdot\frac{1}{s-4}.$$

Example 10 : Find Laplace transform of $t\,e^{-2t}\cdot H(t-1)$.

Sol. : (i) Here $f(t)=t\,e^{-2t}$ and $a=1$.

$\therefore\quad f(t+1) = (t+1)e^{-2(t+1)} = e^{-2}\cdot e^{-2t}(t+1)$

(i) $$L[f(t+1)] = e^{-2}\cdot L[e^{-2t}(t+1)] = e^{-2}\left[L(e^{-2t}\cdot t) + L(e^{-2t})\right]$$

$$= e^{-2}\left[\frac{1}{(s+2)^2}+\frac{1}{s+2}\right]$$ [See Ex. 1(i), page 1-29]

(iii) $$L[f(t)\cdot H(t-1)] = e^{-s}\cdot e^{-2}\left[\frac{1+s+2}{(s+2)^2}\right] = e^{-(s+2)}\cdot\frac{s+3}{(s+2)^2}.$$

8. Evaluation of the Integral $\int_0^\infty e^{-at} f(t)H(t-a)\,dt$

We have see on pages 1-13, 1-22, 1-33, 1-41 and 1-60, how Laplace Transforms can be u to find certain integrals. We shall use the same technique to find definite integrals of Heaviside functions.

Example 1 : Using Laplace transform evaluate

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\,H(t-2)\,dt.$$ (M.U. 2003)

Sol. : We have proved above in Ex. 3, page 2-49 that

$$L(1+2t-3t^2+4t^3)\,H(t-2) = e^{-2s}\left[\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right]$$

By definition of Laplace transform, this means,

$$\int_0^\infty e^{-st}(1+2t-3t^2+4t^3)\,H(t-2)\,dt = e^{-2s}\left[\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right]$$

Putting $s=1$, we get

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\,H(t-2)\,dt = e^{-2}\left[\frac{25}{1}+\frac{38}{1^2}+\frac{42}{1^3}+\frac{24}{1^4}\right] = \frac{129}{e^2}.$$

Example 2 : Using Laplace Transform evaluate

$$\int_0^\infty e^{-t}(1+2t-t^2+t^3)\,H(t-1)\,dt.$$ (M.U. 2003, 07, 10)

Sol. : We have proved above in Ex. 2, page 2-49 that

$$L[(1+2t-t^2+t^3)H(t-1)] = e^{-s}\left[\frac{3}{s}+\frac{3}{s^2}+\frac{4}{s^3}+\frac{6}{s^4}\right]$$

By definition of Laplace transform this means,

$$\int_0^\infty e^{-st}(1+2t-t^2+t^3)H(t-1)\,dt = e^{-s}\left[\frac{3}{s}+\frac{3}{s^2}+\frac{4}{s^3}+\frac{6}{s^4}\right]$$

Putting $s=1$,

$$\int_0^\infty e^{-t}(1+2t-t^2+t^3)H(t-1)\,dt = e^{-1}\left[\frac{3}{1}+\frac{3}{1^2}+\frac{4}{1^3}+\frac{6}{1^4}\right] = \frac{16}{e}.$$

EXERCISE - XIV

1. Express the following functions in terms of Heaviside unit step function and hence find the Laplace transform.

(i) $$f(t) = \begin{cases}(t-3)^4, & t>3\\ 0, & 0<t<3\end{cases}$$

(ii) $$f(t) = \begin{cases}0, & 0<t<4\\ (t-4)^3, & t>4\end{cases}$$ (M.U. 2011)

(iii) $$f(x) = \begin{cases}0, & 0<t<2\\ 1-2t-t^2, & t>2\end{cases}$$

[**Ans. :** (i) $(t-3)^4 H(t-3);\ e^{-3s}\cdot\frac{4!}{s^5}$, (ii) $(t-4)^3 H(t-4);\ e^{-4s}\cdot\frac{3!}{s^4}$,

(iii) $(1-2t-t^2)H(t-2);\ e^{-2s}\left[\frac{1}{s}-\frac{2}{s^2}-\frac{2}{s^3}\right]$]

2. Find the Laplace transform of the following functions.

(i) $t\ H(t-2)$ (ii) $\sin t\ H(t-\pi)$ (M.U. 2005)

(iii) $t^4\, H(t-2)$

(iv) $(1+3t-4t^2+2t^3)\; H(t-3)$

(v) $t^2\, H(t-2)$ (M.U. 1997)

(vi) $e^{-t}\sin t\; H(t-\pi)$ (M.U. 2005)

[**Ans. :** (i) $e^{-2s}\left[\dfrac{2}{s}+\dfrac{1}{s^2}\right]$, (ii) $e^{-\pi s}\cdot\dfrac{-1}{s^2+1}$,

(iii) $e^{-4s}\left[\dfrac{16}{s}+\dfrac{32}{s^2}+\dfrac{48}{s^3}+\dfrac{48}{s^4}+\dfrac{24}{s^5}\right]$, (iv) $e^{-3s}\left[\dfrac{28}{s}+\dfrac{33}{s^2}+\dfrac{28}{s^3}+\dfrac{12}{s^4}\right]$,

(v) $e^{-2s}\left[\dfrac{4}{s}+\dfrac{4}{s^2}+\dfrac{2}{s^3}\right]$, (vi) $e^{-\pi(s+1)}\cdot\dfrac{1}{s^2+2s+2}$.]

3. Using Laplace transform evaluate the following integrals

(i) $\int_0^\infty e^{-2t}(1+t+t^2)H(t-3)\,dt$ [Ans. :

(ii) $\int_0^\infty e^{-t}\,(1+3t+t^2)H(t-2)$ [Ans. :

9. Laplace Transform of f (t − a) · H (t − a) (M.U. 1995, 200[...]

Since $f(t-a)\cdot H(t-a)=\begin{cases}0, & t<a\\ f(t-a), & t\ge a\end{cases}$ by definition of Laplace transform,

$$L[f(t-a)H(t-a)]=\int_0^\infty e^{-st}[f(t-a)\cdot H(t-a)]\,dt$$

$$=\int_0^a e^{-st}[f(t-a)\cdot H(t-a)]\,dt+\int_a^\infty e^{-st}[f(t-a)\cdot H(t-a)]\,dt$$

$$=0+\int_a^\infty e^{-st}f(t-a)\,dt \qquad \text{[Case (iv), page 2-47]}$$

Now, put $t-a=u$, $dt=du$.

When $t=a$, $u=0$; when $t=\infty$, $u=\infty$

$$L[f(t-a)\,H(t-a)]=\int_0^\infty e^{-s(a+u)}\cdot f(u)\,du=e^{-as}\int_0^\infty e^{-su}f(u)\,du$$

$$=e^{-as}\int_0^\infty e^{-st}f(t)\,dt$$

[Since a definite integral does not depend upon the variable u or t but on the function]

$\therefore$ $\boxed{L[\,f(t-a)\,H(t-a)\,]=e^{-as}\,L\,f(t)}$ ....

$\therefore$ $\boxed{L[\,f(t-a)\,H(t-a)\,]=e^{-as}\,\Phi(s)}$ where $\Phi(s)=L[\,f(t)\,]$

Cor : Taking the inverse Laplace transform, we get

$\boxed{L^{-1}[e^{-as}\,\Phi(s)]=f(t-a)\,H(t-a)}$ where $f(t)=L^{-1}\,\Phi(s)$ ....

In particular putting $a=0$, in (27), we get $L[\,f(t)\,H(t)]=L\,f(t)$.

A useful Result

If a given function is a step function having many steps then it is convenient to expres[s it in terms of] unit-step function as explained below and then write down it Laplace transform.

Suppose $f(t)=\begin{cases}f_1(t), & 0<t<a\\ f_2(t), & t>a\end{cases}$

Since, $H(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$ and $H(t-a)=\begin{cases}0, & t<a\\ 1, & t>a\end{cases}$

$H(t)-H(t-a)=1$ for $0<t<a$

[See case (v), page 2-47 with $a=0$, $b=a$]

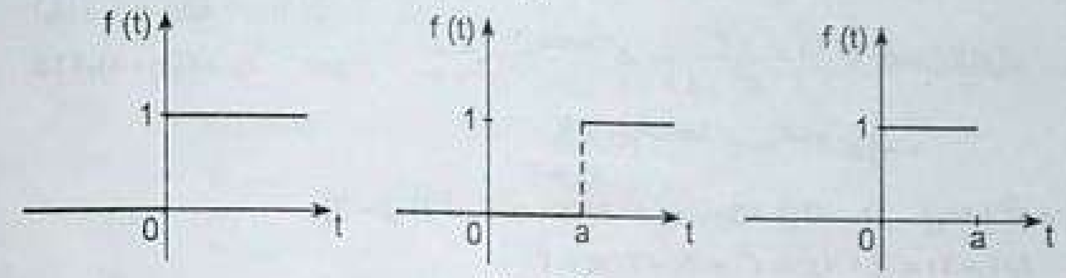


Fig. 2.15

$\therefore$ $\boxed{f(t)=f_1(t)\,[\,H(t)-H(t-a)\,]+f_2(t)\,H(t-a)}$ .... (29)

The above result can be extended further. Thus, if

$$f(t)=\begin{cases}f_1(t), & 0<t<a\\ f_2(t), & a<t<b\\ f_3(t), & b<t<c\\ f_4(t), & t>c\end{cases}$$

then following the above discussion we can write,

$$f(t)=f_1(t)\,[\,H(t)-H(t-a)\,]+f_2(t)\,[\,H(t-a)-H(t-b)\,]$$
$$+f_3(t)\,[\,H(t-b)-H(t-c)\,]+f_4(t)\,H(t-c)$$

After simplification, we get

$$\boxed{\begin{aligned}f(t)=f_1(t)\,H(t)&+[\,f_2(t)-f_1(t)\,]\,H(t-a)\\ &+[\,f_3(t)-f_2(t)\,]\,H(t-b)+[\,f_4(t)-f_3(t)\,]\,H(t-c)\end{aligned}} \quad .... (30)$$

Example 1 : Express the following functions as Heaviside's unit step functions and find their Laplace transforms.

1. $f(t)=\begin{cases}0, & 0<t<\pi/2\\ \cos t, & \pi/2<t<3\pi/2\\ 0, & t>3\pi/2\end{cases}$ **2.** $f(t)=\begin{cases}0, & 0<t<1\\ t^2, & 1<t<3\\ 0, & t>3\end{cases}$

3. $f(t)=\begin{cases}2t, & 0<t<1\\ 3t^2, & t>1\end{cases}$ **4.** $f(t)=\begin{cases}\sin t, & 0<t<\pi\\ \cos t, & t<\pi\end{cases}$

5. $f(t)=\begin{cases}e^t\sin t, & 0<t<\pi\\ e^t\cos t, & t>\pi\end{cases}$ **6.** $f(t)=\begin{cases}\cos t, & 0<t<\pi\\ \cos 2t, & \pi<t<2\pi\\ \cos 3t, & t>2\pi\end{cases}$

(M.U. 2008)

Sol. : Following the procedure given on page 2-49, we have

(1) $f(t)=\cos t\left[H\left(t-\dfrac{\pi}{2}\right)-H\left(t-\dfrac{3\pi}{2}\right)\right]=\cos t\;H\left(t-\dfrac{\pi}{2}\right)-\cos t\,H\left(t-\dfrac{3\pi}{2}\right)$

(i) $f_1(t)=\cos t$, $a=\dfrac{\pi}{2}$ $\quad\therefore\; f_1\left(t+\dfrac{\pi}{2}\right)=\cos\left(t+\dfrac{\pi}{2}\right)=-\sin t$

$$L\left[f_1\left(t+\frac{\pi}{2}\right)\right] = L(-\sin t) = -\frac{1}{s^2+1} \quad \text{and} \quad f_2(t) = \cos t, \ a = \frac{3\pi}{2}$$

$$\therefore f_2\left(t+\frac{3\pi}{2}\right) = \cos\left(t+\frac{3\pi}{2}\right) = \sin t; \quad L\left[f_2\left(t+\frac{3\pi}{2}\right)\right] = L(\sin t) = \frac{1}{s^2+1}$$

(ii) $$L[f(t)] = e^{-\pi s/2}\left(-\frac{1}{s^2+1}\right) - e^{-3\pi s/2} \cdot \frac{1}{s^2+1}$$

$$= -(e^{-\pi s/2} + e^{-3\pi s/2}) \cdot \frac{1}{s^2+1}$$

(2) $f(t) = t^2 [H(t-1) - H(t-3)] = t^2 H(t-1) - t^2 H(t-3)$

(i) $f_1(t+1) = (t+1)^2 = t^2 + 2t + 1, \ a = 1.$

$$\therefore L[f_1(t+1)] = L(t^2+2t+1) = \frac{2}{s^3} + 2 \cdot \frac{1}{s^2} + \frac{1}{s}$$

$f_2(t+3) = (t+3)^2 = t^2 + 6t + 9, \ a = 3.$

$$\therefore L[f(t+3)] = \frac{2}{s^3} + 6 \cdot \frac{1}{s^2} + 9 \cdot \frac{1}{s}$$

(ii) $$L[f(t)] = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right] + e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

(3) $f(t) = 2t H(t) + (3t^2 - 2t) H(t-1)$

(i) $f_1(t) = 2t, \ a = 0 \quad \therefore f_1(t+0) = 2(t+0) = 2t$

$$L[f_1(t)] = L(2t) = 2 \cdot \frac{1}{s^2}$$

(ii) $f_2(t) = 3t^2 - 2t, \ a = 1$

$$\therefore f_2(t+1) = 3(t+1)^2 - 2(t+1) = 3(t^2+2t+1) - 2(t+1)$$

$$= 3t^2 + 4t + 1$$

$$\therefore L[f_2(t+1)] = L(3t^2+4t+1) = 3 \cdot L(t^2) + 4 L(t) + L(1)$$

$$= 3 \cdot \frac{2}{s^3} + 4 \cdot \frac{1}{s^2} + \frac{1}{s}$$

(ii) $$L[f(t)] = \frac{2}{s^2} + e^{-s}\left[\frac{6}{s^3} + \frac{4}{s^3} + \frac{1}{s}\right]$$

(4) $f(t) = \sin t \cdot H(t) + (\cos t - \sin t) H(t-\pi)$

(i) $f_1(t) = \sin t, \ a = 0 \quad \therefore \quad L[f_1(t)] = L(\sin t) = \frac{1}{s^2+1}$

(ii) $f_2(t) = \cos t - \sin t, \quad a = \pi$

$$\therefore L[f_2(t+\pi)] = \cos(t+\pi) - \sin(t+\pi) = -\cos t + \sin t$$

$$= -\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore L[f_2(t)] = \frac{1-s}{s^2+1}$$

(iii) $$L[f(t)] = \frac{1}{s^2+1} + e^{-\pi s} \cdot \left(\frac{1-s}{s^2+1}\right).$$

(5) $f(t) = e^t \sin t \cdot H(t) + e^t (\cos t - \sin t) H(t-\pi)$

(i) $f_1(t) = e^t \sin t, \quad a = 0$

$$\therefore L[f_1(t)] = L(e^t \sin t) = \frac{s-1}{(s-1)^2+1} = \frac{s-1}{s^2-2s+2}$$

(ii) $f_2(t) = e^t \cos t - e^t \sin t, \quad a = \pi$

$$\therefore L[f_2(t+\pi)] = e^{t-\pi}\cos(t+\pi) - e^{t+\pi}\sin(t+\pi)$$

$$= L(-e^{\pi+t}\sin t) + L(e^{\pi+t}\cos t)$$

$$= -e^{\pi} \cdot L(e^t \sin t) + e^{\pi} \cdot L(e^t \cos t)$$

$$= -e^{-\pi} \cdot \frac{s-1}{(s-1)^2+1} + e^{\pi} \cdot \frac{1}{(s-1)^2+1}$$

$$= -e^{-\pi} \cdot \frac{s-2}{s^2-2s+2}$$

(iii) $$L[f(t)] = \frac{s-1}{s^2-2s+2} - e^{-\pi-\pi s} \cdot \frac{s-2}{s^2-2s+2}$$

$$\therefore L[f(t)] = \frac{s-1}{s^2-2s+2} - e^{-\pi(s+1)} \cdot \frac{s-2}{s^2-2s+2}$$

(6) $f(t) = \cos t \cdot H(t) + (\cos 2t - \cos t) H(t-\pi) + (\cos 3t - \cos 2t) H(t-2\pi)$

(i) $f_1(t) = \cos t, \quad a = 0$

$$\therefore L[f_1(t)] = L(\cos t) = \frac{s}{s^2+1}$$

(ii) $f_2(t) = \cos 2t - \cos t, \quad a = \pi$

$$\therefore f_2(t+\pi) = \cos 2(t+\pi) - \cos(t+\pi) = \cos 2t + \cos t$$

$$= L\cos 2t + L\cos t = \frac{s}{s^2+4} + \frac{s}{s^2+1}$$

(iii) $f_3(t) = \cos 3t - \cos 2t, \quad a = 2\pi.$

$$\therefore f_3(t+2\pi) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$$

$$= \cos(3t+6\pi) - \cos(2t+4\pi)$$

$$= \cos 3t - \cos 2t$$

$$\therefore L[f_3(t+2\pi)] = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

$$\therefore L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Example 2 : Express the following function in terms of unit step function and hence obtain it Laplace transform

$$f(t) = \begin{cases} 2t, & 0 < t < \pi \\ 1, & t \geq \pi \end{cases}$$

Sol. : $f(t) = 2t \cdot H(t) + (1-2t) H(t-\pi)$

(i) $f_1(t) = 2t, \quad a = 0 \quad \therefore \quad L[f_1(t)] = L(2t) = 2 \cdot \frac{1}{s^2}$

(ii) $f_2(t) = 1 - 2t, \quad a = \pi \quad \therefore \quad f_2(t+\pi) = 1 - 2(t+\pi) = 1 - 2t - 2\pi$

$$\therefore\ L[f_2(t+\pi)] = L(1) - L(2t) - L(2\pi) = \frac{1}{s} - 2\cdot\frac{1}{s^2} - \frac{2\pi}{s}$$

(iii) $$L[f(t)] = \frac{2}{s^2} + e^{-\pi s}\left(\frac{1}{s} - \frac{2}{s^2} - \frac{2\pi}{s}\right).$$

Example 3 : Express the following staircase function as Heaviside's unit step function and find its Laplace transform.

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 2, & 1 < t < 2 \\ 3, & 2 < t < 3 \\ \ldots & \ldots\ldots \\ \ldots & \ldots\ldots \end{cases}$$

Sol. : As discussed on page 2-54 [See (30)], Heaviside representation of $f(t)$ is

$$f(t) = 1\cdot H(t) + (2-1)\,H(t-1) + (3-2)\,H(t-2) + \ldots$$
$$= H(t) + H(t-1) + H(t-2) + \ldots$$
$$\therefore\ L[f(t)] = L[H(t)] + L[H(t-1)] + L[H(t-2)] + \ldots$$

But by (21) and (23), page 2-48

$$L[H(t)] = \frac{1}{s} \quad \text{and} \quad L[H(t-a)] = \frac{1}{s}e^{-as}$$

Hence, $$L[f(t)] = \frac{1}{s} + \frac{1}{s}e^{-s} + \frac{1}{s}\cdot e^{-2s} + \frac{1}{s}\cdot e^{-3s} + \ldots$$
$$= \frac{1}{s}[1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots]$$
$$= \frac{1}{s}\cdot\frac{1}{(1-e^{-s})} \quad [\text{G.P. with } r = e^{-s}]$$
$$= \frac{1}{s(1-e^{-s})}$$

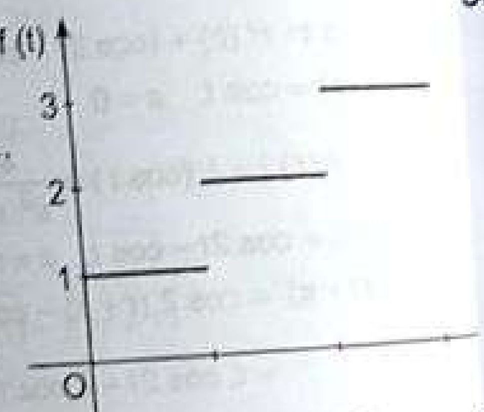


Fig. 2.16

This stair-case function is shown in Fig. 2.16.

EXERCISE - XV

Express the following functions in terms of Heaviside's unit step function and hence find Laplace transform.

1. $f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin t, & \pi < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

2. $f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

3. $f(t) = \begin{cases} t, & 0 < t < 2 \\ t^2, & t > 2 \end{cases}$

4. $f(t) = \begin{cases} \cos t, & 0 < t < \pi/2 \\ \sin t, & t > \pi/2 \end{cases}$

5. $f(t) = \begin{cases} e^t \cos t, & 0 < t < \pi \\ e^t \sin t, & t > \pi \end{cases}$

6. $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < \pi \\ \sin 3t, & t > 2\pi \end{cases}$

[**Ans. :** (1) $-(e^{-\pi s} + e^{-2\pi s})\cdot\frac{1}{s^2+1}$, (2) $(e^{-\pi s} - e^{-2\pi s})\cdot\frac{2}{s^2+4}$,

(3) $\frac{1}{s^2} + e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$, (4) $\frac{s}{s^2+1} + e^{-\pi s/2}\left[\frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$,

(5) $\frac{s-1}{s^2-2s+2} + e^{-\pi(s-1)}\cdot\frac{s-2}{s^2-2s+2}$,

(6) $\frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} + \frac{2}{s^2+4}\right]$]

10. Inverse Laplace Transforms of $e^{-as}\,\Phi(s)$

Procedure to find $L^{-1}[e^{-as}\,\Phi(s)]$

1. First find $\Phi(s)$ and a.
2. Then find $L^{-1}[\Phi(s)] = f(t)$, say.
3. Then $L^{-1}[e^{-as}\,\Phi(s)] = f(t-a)\,H(t-a)$.

Example 1 : Find $L^{-1}\left[\frac{8e^{-3s}}{s^2+4}\right]$. **(M.U. 1998)**

Sol. : (i) Here $\Phi(s) = \frac{8}{s^2+4}$ and $a = 3$.

(ii) $$L^{-1}[\Phi(s)] = L^{-1}\left[\frac{8}{s^2+4}\right] = 4L^{-1}\left[\frac{2}{s^2+2^2}\right] \qquad \therefore\ f(t) = 4\sin 2t$$

(iii) $$L^{-1}\left[\frac{8e^{-3s}}{s^2+4}\right] = f(t-a)\,H(t-a) = 4\sin 2(t-3)\,H(t-3)$$

Example 2 : Find $L^{-1}\left[\frac{e^{-5s}}{(s-2)^4}\right]$. **(M.U. 1999)**

Sol. : (i) Here $\Phi(s) = \frac{1}{(s-2)^4}$ and $a = 5$.

(ii) $$L^{-1}[\Phi(s)] = L^{-1}\left[\frac{1}{(s-2)^4}\right] = e^{2t}\cdot L^{-1}\left[\frac{1}{s^4}\right] \qquad \therefore\ f(t) = e^{2t}\cdot\frac{t^3}{3!}$$

(iii) $$L^{-1}\left[\frac{e^{-5s}}{(s-2)^4}\right] = f(t-a)\,H(t-a) = e^{2(t-5)}\cdot\frac{(t-5)^3}{3!}\,H(t-3).$$

Example 3 : Find $L^{-1}\left[\frac{e^{-3s}}{(s+4)^3}\right]$. **(M.U. 1996, 2012)**

Sol. : (i) Here $\Phi(s) = \frac{1}{(s+4)^3}$ and $a = 3$.

(ii) $$L^{-1}[\Phi(s)] = L^{-1}\left[\frac{1}{(s+4)^3}\right] = e^{-4t}\cdot L^{-1}\left[\frac{1}{s^3}\right] = e^{-4t}\cdot\frac{t^2}{2} \qquad \therefore\ f(t) = e^{-4t}\cdot\frac{t^2}{2}$$

(iii) $$L^{-1}\left[\frac{e^{-3s}}{(s+4)^3}\right] = f(t-a)\,H(t-a) = e^{-4(t-3)}\cdot\frac{(t-3)^2}{2!}\,H(t-3)$$

Example 4 : Find $L^{-1}\left[\frac{e^{-\pi s}}{s^2-2s+2}\right]$. (M.U. 1995)

Sol. : (i) Here $\Phi(s)=\frac{1}{s^2-2s+2}$ and $a=\pi$.

(ii) $L^{-1}[\Phi(s)]=L^{-1}\left[\frac{1}{(s-1)^2+1}\right]=e^t\cdot L^{-1}\left[\frac{1}{s^2+1}\right]=e^t\sin t \quad \therefore \; f(t)=e^t\sin t$

(iii) $L^{-1}\left[\frac{e^{-\pi s}}{s^2-2s+2}\right]=f(t-a)\,H(t-a)=e^{(t-\pi)}\sin(t-\pi)\;H(t-\pi)$

Example 5 : Find inverse Laplace transform of $\frac{e^{-as}}{(s+b)^{5/2}}$. (M.U. 2002)

Sol. : (i) Here $\Phi(s)=\frac{1}{(s+b)^{5/2}}$ and $a=a$.

(ii) $L^{-1}[\Phi(s)]=L^{-1}\left[\frac{1}{(s+b)^{5/2}}\right]=e^{-bt}\cdot L^{-1}\left[\frac{1}{s^{5/2}}\right]$

$$=e^{-bt}\cdot\frac{t^{3/2}}{\overline{|5/2}}=e^{-bt}\cdot\frac{t^{3/2}}{(3/2)(1/2)\overline{|1/2}}=\frac{4}{3\sqrt{\pi}}\cdot e^{-bt}\;t^{3/2}$$

$\therefore \; f(t)=\frac{4}{3\sqrt{\pi}}\cdot e^{-bt}\;t^{3/2}$

(iii) $L^{-1}\left[\frac{e^{-as}}{(s+b)^{5/2}}\right]=f(t-a)\;H(t-a)=\frac{4}{3\sqrt{\pi}}\cdot e^{-b(t-a)}\cdot(t-a)^{3/2}\;H(t-a)$.

Example 6 : Find inverse Laplace transform of $e^{-s}\left\{\frac{1-\sqrt{s}}{s^2}\right\}^2$. (M.U. 20...)

Sol. : (i) Here $\Phi(s)=\left(\frac{1-\sqrt{s}}{s^2}\right)^2$ and $a=1$.

(ii) $L^{-1}[\Phi(s)]=L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2=L^{-1}\left(\frac{1-2\sqrt{s}+s}{s^4}\right)=L^{-1}\left[\frac{1}{s^4}-\frac{2}{s^{7/2}}+\frac{1}{s^3}\right]$

$$=\frac{t^3}{3!}-2\cdot\frac{t^{5/2}}{\overline{|7/2}}+\frac{t^2}{2} \qquad \left[\overline{\left|\frac{7}{2}\right.}=\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\overline{\left|\frac{1}{2}\right.}=\frac{15}{8}\sqrt{\pi}\right]$$

$\therefore \; L^{-1}[\Phi(s)]=\frac{t^3}{6}-\frac{16}{15\sqrt{\pi}}\cdot t^{5/2}+\frac{t^2}{2}$

$\therefore \; f(t)=\frac{t^3}{6}-\frac{16}{15\sqrt{\pi}}\cdot t^{5/2}+\frac{t^2}{2}$

(iii) $L^{-1}\left[e^{-s}\left\{\frac{1-\sqrt{s}}{s^2}\right\}^2\right]=f(t-a)\;H(t-a)$

$$\therefore \; L^{-1}\left[e^{-s}\left\{\frac{1-\sqrt{s}}{s^2}\right\}^2\right]=\left[\frac{(t-1)^3}{6}-\frac{16}{15\sqrt{\pi}}(t-1)^{5/2}+\frac{(t-1)^2}{2}\right]H(t-1)$$

Example 7 : Find $L^{-1}\left[\frac{e^{4-3s}}{(s+4)^{5/2}}\right]$. (M.U. 1997, 2000, 05, 13)

Sol. : (i) We have $L^{-1}\left[\frac{e^{4-3s}}{(s+4)^{5/2}}\right]=e^4\cdot L^{-1}\left[\frac{e^{-3s}}{(s+4)^{5/2}}\right]$

Here $\Phi(s)=\frac{1}{(s+4)^{5/2}}$ and $a=3$.

(ii) $L^{-1}[\Phi(s)]=L^{-1}\left[\frac{1}{(s+4)^{5/2}}\right]=\frac{4}{3\sqrt{\pi}}\cdot e^{-4t}\;t^{3/2}$ [By Ex. 5 above]

$\therefore \; f(t)=\frac{4}{3\sqrt{\pi}}\cdot e^{-4t}\;t^{3/2}$

(iii) $L^{-1}\left[\frac{e^{4-3s}}{(s+4)^{5/2}}\right]=f(t-a)\;H(t-a)$

$$=e^4\cdot\frac{4}{3\sqrt{\pi}}\cdot e^{-4(t-3)}(t-3)^{3/2}\;H(t-3)$$

$$=\frac{4}{3\sqrt{\pi}}\cdot e^{-4(t-4)}\cdot(t-3)^{3/2}\;H(t-3) \qquad [\because e^4=e^{-4(-1)}]$$

Example 8 : Find $L^{-1}\left[\frac{e^{-2s}}{s^2+8s+25}\right]$. (M.U. 1999, 2006, 12, 15, 16)

Sol. : (i) Here $\Phi(s)=\frac{1}{s^2+8s+25}$ and $a=2$.

(ii) $L^{-1}[\Phi(s)]=L^{-1}\left[\frac{1}{s^2+8s+25}\right]=L^{-1}\left[\frac{1}{(s+4)^2+3^2}\right]$

$$=e^{-4t}\cdot L^{-1}\left[\frac{1}{s^2+3^2}\right]=e^{-4t}\cdot\frac{1}{3}\sin 3t$$

$\therefore \; f(t)=\frac{1}{3}e^{-4t}\sin 3t$

(iii) $L^{-1}\left[\frac{e^{-2s}}{s^2+8s+25}\right]=f(t-2)\;H(t-2)=\frac{1}{3}e^{-4(t-2)}\sin 3(t-2)\cdot H(t-2)$.

Example 9 : Find $L^{-1}\left[\frac{s\,e^{-as}}{s^2+3s+2}\right]$.

Sol. : (i) Here $\Phi(s)=\frac{s}{s^2+3s+2}$ and $a=a$.

(ii) $L^{-1}[\Phi(s)]=L^{-1}\left[\frac{s}{s^2+3s+2}\right]=L^{-1}\left[\frac{2}{s+2}-\frac{1}{s+1}\right]$

$$\therefore \quad L^{-1}[\Phi(s)] = L^{-1}\left[\frac{2}{s+2}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$\therefore \quad f(t) = 2e^{-2t} - e^{-t}.$$

(iii) $$L^{-1}\left[\frac{s e^{-as}}{s^2+3s+2}\right] = f(t-a)\, H(t-a) = [2e^{-2(t-a)} - e^{-(t-a)}]\, H(t-a).$$

Example 10 : Find $L^{-1}\left[\dfrac{s e^{-\pi s}}{s^2+2s+2}\right]$. (M.U. 2012, ...

Sol. : (i) Here $\Phi(s) = \dfrac{s}{s^2+2s+2}$ and $a = \pi$.

(ii) $$L^{-1}[\Phi(s)] = L^{-1}\left[\frac{s}{(s^2+2s+2)}\right] = L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+1}\right]$$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t} \cdot L^{-1}\left[\frac{s}{s^2+1}\right] - e^{-t} \cdot L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-t}\cos t - e^{-t}\sin t = e^{-t}[\cos t - \sin t]$$

$$\therefore \quad f(t) = e^{-t}[\cos t - \sin t]$$

(iii) $$L^{-1}\left[\frac{s e^{-\pi s}}{s^2+2s+2}\right] = f(t-a)\, H(t-a)$$

$$= e^{-(t-\pi)}[\cos(t-\pi) - \sin(t-\pi)] \cdot H(t-\pi)$$

Example 11 : Find $L^{-1}\left[\dfrac{(s+1)e^{-s}}{s^2+s+1}\right]$. (M.U. 1997, 2006, 10, 11, ...

Sol. : (i) Here $\Phi(s) = \dfrac{s+1}{s^2+s+1}$ and $a = 1$.

(ii) $$L^{-1}[\Phi(s)] = L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = L^{-1}\left[\frac{[s+(1/2)]+(1/2)}{[s+(1/2)]^2+(3/4)}\right]$$

$$= L^{-1}\left[\frac{s+(1/2)}{[s+(1/2)]^2+(\sqrt{3}/2)^2}\right] + L^{-1}\left[\frac{1/2}{[s+(1/2)]^2+(\sqrt{3}/2)^2}\right]$$

$$= e^{-t/2} \cdot L^{-1}\left[\frac{s}{s^2+(\sqrt{3}/2)^2}\right] + \frac{1}{2} e^{-t/2} \cdot L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]$$

$$\therefore \quad f(t) = e^{-t/2}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \cdot e^{-t/2}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$$

(iii) $$L^{-1}\left[\frac{(s+1)e^{-s}}{s^2+s+1}\right] = f(t-a)\, H(t-a)$$

$$= e^{-(t-1)/2}\left[\cos\left(\frac{\sqrt{3}(t-1)}{2}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}(t-1)}{2}\right)\right] H(t-1).$$

Example 12 : Find $L^{-1}\left[e^{-4s} \cdot \dfrac{s}{(s+4)^3}\right]$. (M.U. 2013)

Sol. : (i) Here $\Phi(s) = \dfrac{s}{(s+4)^3}$ and $a = 4$.

(ii) $$L^{-1}[\Phi(s)] = L^{-1}\left[\frac{s}{(s+4)^3}\right] = L^{-1}\left[\frac{(s+4)-4}{(s+4)^3}\right]$$

$$= L^{-1}\left[\frac{1}{(s+4)^2}\right] - 4L^{-1}\left[\frac{1}{(s+4)^3}\right]$$

$$\therefore \quad L^{-1}[\Phi(s)] = e^{-4t} \cdot L^{-1}\left[\frac{1}{s^2}\right] - 4 \cdot e^{-4t} \cdot L^{-1}\left[\frac{1}{s^3}\right]$$

$$\therefore \quad f(t) = e^{-4t} \cdot t - 4e^{-4t} \cdot \frac{t^2}{2}$$

(iii) $$L^{-1}\left[e^{-4s} \cdot \frac{s}{(s+4)^3}\right] = f(t-a) \cdot H(t-a)$$

$$= e^{-4(t-4)} \cdot (t-4) \cdot H(t-4) - 4 \cdot e^{-4(t-4)} \cdot \frac{(t-4)^2}{2} \cdot H(t-4)$$

EXERCISE - XVI

Find the inverse transform of

1. $\dfrac{e^{-s}}{(s+1)^2}$, 2. $\dfrac{e^{-\pi s}}{(s^2+9)}$, 3. $e^{-s}\dfrac{(1+\sqrt{s})}{s^3}$, 4. $\dfrac{e^{-as}}{s-b}$, 5. $\dfrac{s e^{-as}}{s^2+b^2}$,
(M.U. 2009)

6. $\dfrac{s e^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$, 7. $\dfrac{s e^{-3s}}{s^2-1}$, 8. $\dfrac{e^{-bs}}{s^2(s+a)}$, 9. $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$, 10. $\dfrac{e^{-4s}}{\sqrt{2s+7}}$
(M.U. 2003)

11. $\dfrac{s e^{-as}}{s^2+3s+2}$

[**Ans. :** (1) $e^{-(t-1)}(t-1)\,H(t-1)$, (2) $\dfrac{1}{3}\sin 3(t-\pi)\,H(t-\pi)$,

(3) $\left[\dfrac{(t-1)^2}{2} + \dfrac{4(t-1)^{3/2}}{3\sqrt{\pi}}\right] H(t-1)$, (4) $e^{b(t-a)} \cdot H(t-a)$,

(5) $\cos b(t-a) \cdot H(t-a)$, (6) $\sin \pi t\left[H\left(t-\dfrac{1}{2}\right) + H(t-1)\right]$,

(7) $\cos h(t-3) \cdot H(t-3)$, (8) $\dfrac{1}{a^2}\left[a(t-b) - 1 + e^{-a(t-b)}\right] \cdot H(t-b)$,

(9) $[(t-\pi) + \sin(t-\pi) \cdot H(t-\pi)]$, (10) $\dfrac{e^{-7(t-4)/2}}{\sqrt{2\pi(t-4)}} \cdot H(t-4)$]

(11) $[2e^{-2(t-a)} - e^{-(t-a)}] \cdot H(t-a)$]

11. Dirac-delta Function (Unit-impulse Function)

Consider the function $F(t)$ defined by,

$$F(t) = 0,\ t < a$$
$$= \frac{1}{\varepsilon},\ a \le t \le a+\varepsilon$$
$$= 0,\ t > a+\varepsilon$$

The function is represented by the adjoining figure.

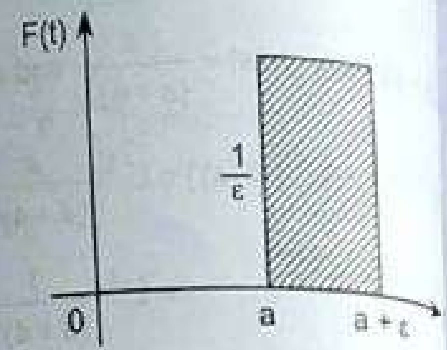


Fig. 2.17

Integrating $F(t)$, we get,

$$\int_0^\infty F(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = \frac{1}{\varepsilon}\left[t\right]_a^{a+\varepsilon}$$
$$= \frac{1}{\varepsilon}[a+\varepsilon-a] = \frac{\varepsilon}{\varepsilon} = 1 \text{ for all } \varepsilon.$$

As $\varepsilon \to 0$, the function $F(t)$ tends to infinity at a and is zero everywhere else. But the integ of $F(t)$ is unity.

If $F(t)$ represents a force acting for a short time ε at time $t = a$ then the integ $\lim\limits_{\varepsilon\to 0}\int_0^\infty F(t)\,dt\ (=1)$ represents **unit-impulse** at $t = a$. Hence, the limiting form of $F(t)$ (as $\varepsilon \to 0$) known as **unit impulse function or Dirac-delta function** and is denoted by $\delta(t-a)$.

$$\therefore\ \delta(t-a) = \lim_{\varepsilon\to 0} F(t)$$

When $a = 0$, the unit function is $\boxed{\delta(t) = \lim\limits_{\varepsilon\to 0} F(t)}$

Definition : The function $F(t)$ defined by

$$F(t) = \begin{cases} 0, & t < a \\ 1/\varepsilon, & a < t < a+\varepsilon \\ 0, & t > a+\varepsilon \end{cases}$$

where $\int_0^\infty F(t)\,dt = 1$ for all ε is called **Dirac-delta function.**

Paul Dirac (1902 - 1984)

Dirac first studied electrical engineering and gradua with first class honours Bachelor of Science degree engineering. Although he was offered a scholarship for fur studies in engineering and as it was not sufficient to live a study, he hook up an offer to study for a Bachelor of A degree in mathematics at the University of Bristol free charge. He completed his Ph.D. in 1926 with the thesis quantum mechanics from University of Cambridge. H doctoral advisers were Homi Bhabha and Harish Chan Mehta among others and later on Fred Hoyle was one of doctoral students. Dirac shared the 1933 Nobel Prize Physics with Erwin Schrodinger. After his death in 1984 Institute of Physics, the United Kingdom's professional bo for physicists established the Pual Dirac Medal in his memory for "outstanding contributio theoretical physics". Stephen Hawking (1987) was the first recipient of this medal.

Geometrical Interpretation of Dirac's δ function

As we can see from the above figure that as the height of the rectangle increases and the width of the rectangle decreases in such a way that the area of the rectangle remains unity, we get the Dirac's delta function.

12. Laplace Transform of Dirac-delta Function $L[\delta(t-a)]$

By definition of Laplace transform,

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} e^{-st}\,dt = \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon}$$
$$= -\frac{1}{\varepsilon s}\left[e^{-s(a+\varepsilon)} - e^{-as}\right] = \frac{1}{s}\cdot e^{-as}\left[\frac{1-e^{-\varepsilon s}}{\varepsilon}\right]$$
$$\therefore\ L[\delta(t-a)] = \lim_{\varepsilon\to 0} L[F(t)] = \frac{1}{s}e^{-as}\lim_{\varepsilon\to 0}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon}\right)$$
$$= \frac{1}{s}e^{-as}\lim_{\varepsilon\to 0}\frac{-e^{-\varepsilon s}(-s)}{1} \qquad \text{[By L'Hospital's Rule]}$$
$$= \frac{1}{s}e^{-as}\cdot s = e^{-as}$$

$$\therefore\ \boxed{L[\delta(t-a)] = e^{-as}} \qquad \text{.................... (31)}$$

Cor. : Putting $a = 0$, $\boxed{L[\delta(t)] = 1}$ .... (32)

13. Inverse Laplace Transform

From the above results, we get,

$$\boxed{L^{-1}(e^{-as}) = \delta(t-a) \text{ and } L^{-1}(1) = \delta(t)} \qquad \text{.................... (33)}$$

Relation between Heaviside's Unit Step Function and Dirac's Delta Function

The function $F(t)$ can be expressed in terms of unit step function as follows.

$$F(t) = \frac{1}{\varepsilon}[H(t-a) - H(t-a-\varepsilon)]$$
$$\therefore\ \lim_{\varepsilon\to 0} F(t) = \lim_{\varepsilon\to 0}\frac{H(t-a)-H(t-a-\varepsilon)}{\varepsilon}$$

Now, the left hand side is Dirac's delta function $\delta(t-a)$ and the right hand side is the derivative of Heaviside's unit step function $H(t-a)$

$$\therefore\ \boxed{\delta(t-a) = H'(t-a)} \qquad \text{.................... (34)}$$

14. Laplace Transform of $f(t)\,\delta(t-a)$ (M.U. 2011)

By definition, $L\,f(t)\,\delta(t-a) = \lim\limits_{\varepsilon\to 0}\int_0^\infty e^{-st} f(t)\cdot F(t)\,dt$

Now, $\int_0^\infty e^{-st} f(t)\cdot F(t)\,dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} e^{-st} f(t)\cdot 1\cdot dt$

$$= \frac{1}{\varepsilon}\cdot\varepsilon\cdot e^{-s(a+\theta\varepsilon)}\cdot f(a+\theta\varepsilon) \qquad \text{......}$$

[By mean value theorem for integrals,

$\int_a^{a+h} f(x)\,dx = h\,f(a+\theta h),\quad 0<\theta<1.$]

Taking the limit as $\varepsilon \to 0$, from (A), we get,

$$\boxed{L\,f(t)\,\delta(t-a) = e^{-as} f(a)} \qquad \text{......}$$

Taking inverse Laplace transform,

$$\boxed{L^{-1}[e^{-as} f(a)] = f(t)\cdot\delta(t-a)} \qquad \text{......}$$

Example 1 : Find $L^{-1}\left[\frac{s}{s+1}\right]$. (M.U. 20..)

Sol. : We have

$$L^{-1}\left[\frac{s}{s+1}\right] = L^{-1}\left[\frac{(s+1)-1}{s+1}\right] = L^{-1}(1) - L^{-1}\left[\frac{1}{s+1}\right]$$

$$= \delta t - e^{-t}.$$

Example 2 : Find the inverse Laplace transform of

(i) $\frac{s^2+6s+6}{s^2+5s+6}$ (ii) $\frac{s^2+4s+2}{s^2+3s+2}$

Sol. : (i) Since the degree of the numerator is equal to the degree of the denominator, we first divide the numerator by the denominator.

$$\therefore\quad s^2+5s+6 \,\big)\, s^2+6s+6 \,\big(\, 1$$
$$\underline{-\,s^2+5s+6}$$
$$s$$

$$\therefore\quad \Phi(s) = 1 + \frac{s}{s^2+5s+6} = 1 + \frac{s}{(s+2)(s+3)}$$

$$= 1 - \frac{2}{s+2} + \frac{3}{s+3} \qquad \text{[By partial fraction]}$$

$$\therefore\quad L^{-1}[\Phi(s)] = L^{-1}(1) - 2L^{-1}\left(\frac{1}{s+2}\right) + 3L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \delta(t) - 2e^{-2t} + 3e^{-3t}$$

(ii) Since the degree of the numerator is equal to the degree of the denominator, we first divide the numerator by the denominator.

$$\therefore\quad s^2+3s+2 \,\big)\, s^2+4s+2 \,\big(\, 1$$
$$\underline{-\,s^2+3s+2}$$
$$s$$

$$\therefore\quad \Phi(s) = 1 + \frac{s}{s^2+3s+2} = 1 + \frac{1}{s+1} - \frac{1}{s+2} \qquad \text{[By partial fraction]}$$

$$\therefore\quad L^{-1}[\Phi(s)] = L^{-1}(1) + L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \delta(t) + e^{-t} - e^{-2t}$$

Example 3 : Find $L(\sin 2t)\cdot\delta(t-2)$.

Sol. : (i) Here $f(t) = \sin 2t$ and $a = 2$.

(ii) $L[f(t)\,\delta(t-a)] = e^{-as} f(a)$

$L[\sin 2t\cdot\delta(t-2)] = e^{-2s}\sin 2(2)$

$= e^{-2s}\sin 4.$

Example 4 : Find $L^{-1}[e^{-2s}\sin 2]$.

Sol. : We know that $L[e^{-as} f(a)] = f(t)\,\delta(t-a)$ [By (31) above]

Here, $a = 2$, $f(a) = \sin 2$.

$\therefore\quad L^{-1}[e^{-2a}\sin 2] = \sin t\cdot\delta(t-2).$

Example 5 : Find Laplace transform of $t\,H(t-4) + t^2\,\delta(t-4)$. **(M.U. 2012)**

Sol. : (a) See procedure given on page 2-49.

(i) Here $f(t) = t$ and $a = 4$. $\therefore\ f(t+4) = t+4$

(ii) $L[f(t+4)] = L(t+4) = L(t) + 4L(1) = \frac{1}{s^2} + 4\cdot\frac{1}{s}$

(iii) $L[t\,H(t-4)] = e^{-4s}\left[\frac{1}{s^2} + \frac{4}{s}\right]$

(b) (i) Here $f(t) = t^2$ and $a = 4$.

(ii) $L[f(t)\,\delta(t-a)] = e^{-as} f(a) = e^{-4s}(4)^2 = 16\,e^{-as}$ [By (35), page 2-65]

Hence, $L[t\,H(t-4) + t^2\,d(t-4)] = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + 16\,e^{-4s}$

$$= \frac{e^{-4s}}{s^2}[1 + 4s + 16s^2]$$

Example 6 : Find $L[t^2 H(t-2) - \cos h\,t\,\delta(t-4)]$. **(M.U. 2011)**

Sol. : (a) (i) Here $f(t) = t^2$ and $a = 2$.

$\therefore\quad f(t+2) = (t+2)^2 = t^2 + 4t + 4$

(ii) $L[f(t+2)] = L[t^2 + 4t + 4] = \frac{2}{s^3} + 4\cdot\frac{1}{s^2} + 4\cdot\frac{1}{s}$

(iii) $L[t^2\,H(t-2)] = e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$

(b) Here, $f(t) = \cos h\,t$ and $a = 4$.

$\therefore\quad L[f(t)\,\delta(t-a)] = e^{-as} f(a) = e^{-4s}\cos h\,4$

Hence, $L[t^2\,H(t-2) - \cos h\,t\,\delta(t-4)] = e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s}\cos h\,4.$

15. Evaluation of the Integral $\int_0^\infty e^{-at} f(t)\,\delta(t-a)\,dt$

Example 1 : Evaluate $\int_0^\infty e^{-5t}\,\delta(t-3)\,dt$.

Sol. : We know that by (35), page 2-65

$$L[f(t)\cdot\delta(t-a)] = e^{-as} f(a)$$

This means, $\int_0^\infty e^{-st}\cdot f(t)\,\delta(t-a)\,dt = e^{-as} f(a)$

Comparing this with the given integral, we find that $s=5$, $f(t)=1$, $a=3$ and $f(a)=1$.

$$\therefore \int_0^\infty e^{-5t}\,\delta(t-3)\,dt = e^{-15}\cdot 1 = e^{-15}.$$

Example 2 : Evaluate $\int_0^\infty \sin 3t\,\delta\left(t-\frac{\pi}{6}\right)dt$.

Sol. : We know that by (35), page 2-65

$$L[f(t)\cdot\delta(t-a)] = e^{-as} f(a)$$

This means, $\int_0^\infty e^{-st} f(t)\cdot\delta(t-a)\,dt = e^{-as} f(a)$

Comparing this with the given integral, we find that

$$s=0,\ f(t)=\sin 3t,\ a=\frac{\pi}{6},\ f(a)=\sin 3\left(\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{2}\right)=1.$$

$$\therefore \int_0^\infty \sin 3t\cdot\delta\left(t-\frac{\pi}{6}\right)dt = e^0\cdot 1 = 1.$$

Example 3 : Evaluate $\int_0^\infty t^3 e^{-t}\sin t\cdot\delta(t-2)\,dt$.

Sol. : We know that by (35), page 2-65

$$L[f(t)\cdot\delta(t-a)] = e^{-as} f(a)$$

$$\therefore \int_0^\infty e^{-st} f(t)\,\delta(t-a)\,dt = e^{-as} f(a)$$

Comparing this with the given integral, we see that $f(t)=t^3\sin t$, $s=1$, $a=2$.

$$\therefore \int_0^\infty t^3 e^{-t}\sin t\ \delta(t-2)\,dt = e^{-2}\cdot 2^3\sin 2 = 8e^{-2}\sin 2.$$

Example 4 : Evaluate $\int_0^\infty t^\alpha(\log t)^\beta\cdot\delta(t-4)\,dt$.

Sol. : As noted above,

$$L[f(t)\cdot\delta(t-a)] = e^{-as} f(a)$$

$$\therefore \int_0^\infty e^{-st} f(t)\delta(t-a)\,dt = e^{-as} f(a)$$

Here, $f(t)=t^\alpha(\log t)^\beta$, $a=4$, $s=0$.

$$\therefore \int_0^\infty t^\alpha(\log t)^\beta\,\delta(t-4)\,dt = 4^\alpha(\log 4)^\beta$$

EXERCISE - XVII

(1) Find Laplace transform of

1. $\sin t\cdot\delta\left(t-\frac{\pi}{2}\right) - t^2\delta(t-2)$
2. $t^4 H(t-2) + t^2\delta(t-2)$.

[**Ans. :** (1) $e^{-\pi s/2} - 4e^{-2s}$, (2) $e^{-2s}\left[4+\frac{16}{s}+\frac{32}{s^2}+\frac{48}{s^3}+\frac{48}{s^4}+\frac{42}{s^5}\right]$.]

(2) Evaluate the following integrals.

1. $\int_0^\infty e^{-3t}\,\delta(t-4)\,dt$
2. $\int_0^\infty \sin 4t\cdot\delta\left(t-\frac{\pi}{8}\right)dt$
3. $\int_0^\infty e^{-2t}\cdot t^2\,\delta(t-1)\,dt$

[**Ans. :** (1) e^{-12}, (2) 1, (3) $\frac{1}{e^2}$]

(3) Find inverse Laplace transform of

1. $\frac{s}{s+a}$,
2. $e^{-as}\sin a$,
3. $e^{-3s}\cos 3$.

[**Ans. :** (1) $\delta(t) - e^{-at}$, (2) $\sin t\cdot\delta(t-a)$, (3) $\cos t\cdot\delta(t-3)$]

16. Applications of Laplace Transforms

In this article we shall see how Laplace transforms can be profitably used to solve differential equations. For the use of Laplace transforms to solve differential equations we need the following results. If Laplace transform of y i.e. $L(y)$ is denoted by $\bar{y}$ then [See (1), (2), (3) of § 14, page 1-52] we have

$$\boxed{\begin{aligned} L(y') &= s\bar{y} - y(0) \\ L(y'') &= s^2\bar{y} - sy(0) - y'(0) \\ L(y''') &= s^3\bar{y} - s^2y - sy'(0) - y''(0) \end{aligned}} \quad \ldots\ldots\ldots (37)$$

Example 1 : Solve using Laplace transforms $3\frac{dy}{dt} + 2y = e^{3t}$, $y=1$ at $t=0$. **(M.U. 2002)**

Sol. : Taking Laplace transforms of both sides,

$$3L(y') + 2L(y) = L(e^{3t})$$

$$3[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s-3}. \quad \text{But } y(0)=1$$

$$\therefore 3[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s-3}$$

$$\therefore (3s+2)\bar{y} = \frac{1}{s-3} + 3 = \frac{3s-8}{s-3}$$

$$\bar{y} = \frac{3s-8}{(s-3)(3s+2)}$$

$$\therefore \bar{y} = \frac{30}{11}\cdot\frac{1}{3s+2} + \frac{1}{11}\cdot\frac{1}{(s-3)} \quad \text{[By partial fractions]}$$

$$\therefore \quad \bar{y} = \frac{10}{11}\cdot\frac{1}{s+(2/3)} + \frac{1}{11}\cdot\frac{1}{s-3}$$

Taking inverse Laplace transforms

$$y = \frac{10}{11}L^{-1}\left[\frac{1}{s+(2/3)}\right] + \frac{1}{11}L^{-1}\left[\frac{1}{s-3}\right] = \frac{10}{11}e^{-(2/3)t} + \frac{1}{11}e^{3t}.$$

Example 2 : Solve using Laplace transforms $\frac{dy}{dx} + 3y = 2 + e^{-t}$, if $y = 1$ at $t = 0$.

Sol. : Taking Laplace transforms of both sides,

$$L(y') + 3L(y) = L(2) + L(e^{-t})$$

$$s\bar{y} - y(0) + 3\bar{y} = 2\frac{1}{s} + \frac{1}{s+1}. \quad \text{But } y(0) = 1$$

$$\therefore \quad (s+3)\bar{y} = \frac{2}{s} + \frac{1}{s+1} + 1 = \frac{s^2+4s+2}{s(s+1)}$$

$$\therefore \quad \bar{y} = \frac{s^2+4s+2}{s(s+1)(s+3)}$$

$\therefore$ By partial fractions $\quad \bar{y} = \frac{2}{3}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$

Taking inverse Laplace transforms, $\quad y = \frac{2}{3} + \frac{1}{2}\cdot e^{-t} - \frac{1}{6}e^{-3t}$

Example 3 : Solve using Laplace transforms $R\frac{dQ}{dt} + \frac{Q}{C} = V$, $Q = 0$ when $t = 0$. (M.U. 20

Sol. : Taking Laplace transforms of both sides,

$$RL(Q') + \frac{1}{C}L(Q) = VL(1)$$

$$R\left[s\bar{Q} - Q(0)\right] + \frac{1}{C}\bar{Q} = \frac{V}{s}. \quad \text{But } Q(0) = 0$$

$$\therefore \quad \left(Rs + \frac{1}{C}\right)\bar{Q} = \frac{V}{s} \quad \therefore \ \bar{Q} = \frac{V}{s}\cdot\frac{1}{[Rs+(1/C)]} = \frac{VC}{s(RCs+1)}$$

$$\therefore \quad \bar{Q} = VCL^{-1}\frac{1}{s} - VRC^2L^{-1}\frac{1}{RCs+1} = VCL^{-1}\frac{1}{s} - VCL^{-1}\left[\frac{1}{s+1/(RC)}\right]$$

Taking inverse Laplace transforms,

$$Q = VC - VCe^{-t/RC} = VC(1 - e^{-t/RC})$$

Example 4 : Solve using Laplace transforms $L\frac{dI}{dt} + RI = Ee^{-at}$, where $I(0) = 0$.

Sol. : Taking Laplace transforms of both sides,

$$LL[I'] + RL(I) = EL(e^{-at})$$

$$L[s\bar{I} - I(0)] + R\bar{I} = \frac{E}{s+a}. \qquad \text{But } I(0) = 0$$

$$\therefore \quad (Ls+R)\bar{I} = \frac{E}{s+a}$$

$$\therefore \quad \bar{I} = \frac{E}{(s+a)(Ls+R)} \qquad \text{[By partial fractions]}$$

$$= \frac{E}{R-La}\cdot\frac{1}{s+a} - \frac{EL}{R-La}\cdot\frac{1}{Ls+R}$$

Taking inverse Laplace transforms

$$I = \frac{E}{R-La}e^{-at} - \frac{E}{R-La}e^{-(R/L)t} = \frac{E}{R-La}\left[e^{-at} - e^{-(R/L)t}\right]$$

Example 5 : Using Laplace transform solve the following differential equation.

$$\frac{dx}{dt} + x = \sin\omega t, \ x(0) = 2 \qquad \textbf{(M.U. 1993)}$$

Sol. : Let $\bar{x}$ be the Laplace transform of x *i.e.* let $L(x) = \bar{x}$.

Taking Laplace transform of both sides of the given equation,

$$L(x') + L(x) = L\sin(\omega t) \qquad \text{—— (1)}$$

But $\quad L(x') = s(\bar{x}) - x(0) = s\bar{x} - 2$

Hence, the equation (1) becomes

$$s\bar{x} - 2 + \bar{x} = \frac{\omega}{s^2+\omega^2} \qquad \therefore \ (s+1)\bar{x} = 2 + \frac{\omega}{s^2+\omega^2}$$

$$\therefore \quad (s+1)\bar{x} = \frac{2s^2+2\omega^2+\omega}{s^2+\omega^2}$$

$$\therefore \quad \bar{x} = \frac{2s^2+2\omega^2+\omega}{(s^2+\omega^2)(s+1)} \qquad \text{[By partial fractions]}$$

$$= \frac{2\omega^2+\omega+2}{1+\omega^2}\cdot\frac{1}{s+1} + \frac{-\omega s+\omega}{(1+\omega^2)(s^2+\omega^2)}$$

$$= \frac{2\omega^2+\omega+2}{1+\omega^2}\cdot\frac{1}{s+1} - \frac{\omega}{1+\omega^2}\cdot\frac{s}{s^2+\omega^2} + \frac{\omega}{1+\omega^2}\cdot\frac{1}{s^2+\omega^2}$$

Taking inverse Laplace transforms,

$$x = \frac{2\omega^2+\omega+2}{1+\omega^2}L^{-1}\left(\frac{1}{s+1}\right) - \frac{\omega}{1+\omega^2}\cdot L^{-1}\left(\frac{s}{s^2+\omega^2}\right) + \frac{\omega}{1+\omega^2}L^{-1}\left(\frac{1}{s^2+\omega^2}\right)$$

$$\therefore \quad x = \frac{2\omega^2+\omega+2}{1+\omega^2}\cdot e^{-t} - \frac{\omega}{1+\omega^2}\cdot\cos\omega t + \frac{\omega}{1+\omega^2}\cdot\frac{1}{\omega}\sin\omega t$$

$$= \frac{1}{1+\omega^2}\left[(2\omega^2+\omega+2)e^{-t} - \omega\cos\omega t + \sin\omega t\right]$$

Example 6 : Solve $(D^2 - 3D + 2)y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$. **(M.U. 2004, 07, 08, 14)**

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} + 3$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} + 3s - 5$

∴ The equation becomes,

$$(s^2\bar{y} + 3s - 5) - 3(s\bar{y} + 3) + 2\bar{y} = 4\frac{1}{s-2}$$

$$(s^2 - 3s + 2)\bar{y} = \frac{4}{s-2} + 14 - 3s = \frac{-3s^2 + 20s - 24}{s-2}$$

$$\therefore \bar{y} = \frac{-3s^2 + 20s - 24}{(s-2)(s^2 - 3s + 2)} = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

By partial fractions, $\bar{y} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

Taking inverse Laplace transform,

$$y = -7L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\frac{1}{s-2} + 4L^{-1}\frac{1}{(s-2)^2}$$

$$\therefore y = -7e^t L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s^2} = -7e^t + 4e^{2t} + 4te^{2t}$$

∴ The solution is $y = -7e^t + 4e^{2t} + 4te^{2t}$.

Example 7 : Solve $(D^2 - D - 2)y = 20\sin 2t$, with $y(0) = 1$ and $y'(0) = 2$. **(M.U. 2005)**

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - L(y') - 2L(y) = 20L(\sin 2t)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 1$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s - 2$

∴ The equation becomes,

$$(s^2\bar{y} - s - 2) - (s\bar{y} - 1) - 2\bar{y} = 20\frac{2}{s^2+4}$$

$$\therefore (s^2 - s - 2)\bar{y} = \frac{40}{s^2+4} + s + 1 = \frac{s^3 + s^2 + 4s + 44}{s^2+4}$$

$$\therefore \bar{y} = \frac{s^3 + s^2 + 4s + 44}{(s^2+4)(s^2-s-2)} \qquad \therefore \bar{y} = -\frac{8}{3}\cdot\frac{1}{s+1} + \frac{8}{3}\cdot\frac{1}{s-2} + \frac{s-6}{s^2+4}$$

Taking inverse Laplace transform,

$$y = \frac{-8}{3}L^{-1}\left(\frac{1}{s+1}\right) + \frac{8}{3}L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\frac{s}{s^2+4} - 6L^{-1}\frac{1}{s^2+4}$$

$$\therefore y = -\frac{8}{3}e^{-t} + \frac{8}{3}e^{2t} + \cos 2t - 3\sin 2t.$$

Example 8 : Using Laplace Transform solve $(D^2 + 3D + 2)y = e^{-2t}\sin t$, $y(0) = 0$, $y'(0)$

Sol. : Let $L(y) = \bar{y}$. Then taking the Laplace transform of both sides

$$L(y'') + 3L(y') + 2L(y) = L(e^{-2t}\sin t)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y}$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y}$

And $L(e^{-2t}\sin t) = \frac{1}{(s+2)^2+1}$

∴ The equation becomes,

$$s^2\bar{y} + 3s\bar{y} + 2\bar{y} = \frac{1}{s^2+4s+5}$$

$$\therefore (s^2+3s+2)\bar{y} = \frac{1}{s^2+4s+5} \qquad \therefore \bar{y} = \frac{1}{(s^2+3s+2)(s^2+4s+5)}$$

Let $\frac{1}{(s^2+3s+2)(s^2+4s+5)} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{cs+d}{s^2+4s+5}$

Solving this, we get $a = \frac{1}{2},\ b = -1,\ c = \frac{1}{2},\ d = \frac{1}{2}$

$$\therefore \bar{y} = \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\left(\frac{s+1}{s^2+4s+5}\right)$$

Taking inverse Laplace transform,

$$y = \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{(s+2)-1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{s+2}{(s^2+2)^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}\cdot e^{-2t}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{2}e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}\cdot e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t$$

Example 9 : Solve $(D^2 + 3D + 2)y = 2(t^2 + t + 1)$ with $y(0) = 2$ and $y'(0) = 0$. **(M.U. 2016)**

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') + 3L(y') + 2L(y) = 2L(t^2 + t + 1)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 2$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 2s$

∴ The equation becomes,

$$(s^2\bar{y} - 2s) + 3(s\bar{y} - 2) + 2\bar{y} = 2\left(\frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}\right)$$

$$\therefore (s^2 + 3s + 2)\bar{y} = 2\frac{(2 + s + s^2)}{s^3} + 2s + 6 = \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3}$$

By partial fractions,

$$\therefore \bar{y} = \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3(s^2 + 3s + 2)} = \frac{3}{s} - \frac{2}{s^2} + \frac{2}{s^3} - \frac{1}{s+2}$$

Taking inverse Laplace transform,

$$y = 3L^{-1}\frac{1}{s} - 2L^{-1}\frac{1}{s^2} + 2L^{-1}\frac{1}{s^3} - L^{-1}\frac{1}{s+2}$$

$$y = 3 - 2t + t^2 - e^{-2t}$$

Example 10 : Use Laplace transform to solve, **(M.U. 1994, 2009, ...)**

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 1 \text{ where, } y(0) = 0,\ y'(0) = 1.$$

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of the both sides,

$$L(y'') + 4L(y') + 8L(y) = L(1)$$

Now, $L(y') = s\bar{y} - y(0) = s\bar{y}$

$$L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1 \text{ and } L(1) = \frac{1}{s}$$

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - 1 + 4s\bar{y} + 8\bar{y} = \frac{1}{s} \qquad \therefore\ \bar{y}(s^2 + 4s + 8) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$\therefore\ \bar{y} = \frac{s+1}{s(s^2+4s+8)} \qquad \therefore\ y = L^{-1}(\bar{y}) = L^{-1}\frac{s+1}{s(s^2+4s+8)}$$

We obtain the L^{-1} by partial fractions

$$\therefore\ y = L^{-1}\left[\frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s}{s^2+4s+8} + \frac{1}{2}\cdot\frac{1}{s^2+4s+8}\right]$$

$$= \frac{1}{8}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{8}L^{-1}\frac{(s+2)-2}{(s+2)^2+2^2} + \frac{1}{2}L^{-1}\frac{1}{(s+2)^2+2^2}$$

$$= \frac{1}{8}\cdot 1 - \frac{1}{8}e^{-2t}L^{-1}\frac{s}{s^2+2^2} + \frac{6}{8}e^{-2t}L^{-1}\frac{1}{s^2+2^2}$$

$$\therefore\ y = \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t.$$

Example 11 : Using Laplace transform solve $\frac{d^2y}{dt^2} + y = t,\ y(0) = 1,\ y'(0) = 0.$

(M.U. 1995, 2009, 13, 14, 15, 1...)

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y'') + L(y) = L(t)$$

Now, $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s$ and $L(t) = \frac{1}{s^2}$.

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - s + \bar{y} = \frac{1}{s^2} \qquad \therefore\ s^2\bar{y} + \bar{y} = s + \frac{1}{s^2} = \frac{s^3+1}{s^2}$$

$$\therefore\ (s^2+1)\bar{y} = \frac{s^3+1}{s^2} \qquad \therefore\ \bar{y} = \frac{s^3+1}{s^2(s^2+1)}$$

Let $\frac{s^3+1}{s^2(s^2+1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d}{s^2+1}$

$$\therefore\ s^3 + 1 = as(s^2+1) + b(s^2+1) + (cs+d)s^2$$

$$= (a+c)s^3 + (b+d)s^2 + as + b$$

Equating like powers of s,

$$a + c = 1,\ b + d = 0,\ a = 0,\ b = 1 \qquad \therefore\ a = 0,\ b = 1,\ c = 1,\ d = -1$$

$$\therefore\ \bar{y} = \frac{1}{s^2} + \frac{s-1}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform

$$y = L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s^2+1}\right) = t + \cos t - \sin t.$$

Example 12 : Solve by using Laplace transform $(D^2 + 2D + 5)y = e^{-t}\sin t$, when $y(0) = 0$, $y'(0) = 1$.

(M.U. 1995, 2003, 05, 06, 07, 11, 14, 15)

Sol. : Let $L(y) = \bar{y}$. Then taking Laplace transform of both sides,

$$L(y'') + 2L(y') + 5L(y) = L(e^{-t}\sin t)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y}$

And $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$

And $L\,e^{-t}\sin t = \frac{1}{(s+1)^2+1}$

$\therefore$ The equation becomes

$$(s^2\bar{y} - 1) + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

$$\therefore\ (s^2+2s+5)\bar{y} = 1 + \frac{1}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}$$

$$\therefore\ \bar{y} = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{as+b}{(s^2+2s+5)} + \frac{cs+d}{(s^2+2s+2)}$

After simplification, we get

$$s^2 + 2s + s = (a+c)s^3 + (2a+b+2c+d)s^2 + (2a+2b+5c+2d)s + (2b+5d)$$

Equating the coefficients of like powers of s, we get,

$$a + c = 0,\ 2a + b + 2c + d = 1,\ 2a + 2b + 5c + 2d = 2,\ 2b + 5d = 3$$

$$\therefore\ a = 0,\ b = \frac{2}{3},\ c = 0,\ d = \frac{1}{3}$$

$$\therefore \quad \bar{y} = \frac{2}{3} \cdot \frac{1}{s^2+2s+5} + \frac{1}{3} \cdot \frac{1}{s^2+2s+2}$$

$$= \frac{2}{3} \cdot \frac{1}{(s+1)^2+2^2} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1^2}$$

Taking inverse Laplace transform

$$y = \frac{2}{3} \cdot e^{-t} \cdot L^{-1}\left[\frac{1}{s^2+2^2}\right] + \frac{1}{3} e^{-t} L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t + \frac{1}{3} e^{-t} \sin t = \frac{e^{-t}}{3} (\sin 2t + \sin t).$$

Example 13 : Solve using Laplace transform $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 3\,t\,e^{-t}$ given $y\,(0) = 4$ a

(M.U. 2002, 1

$y'(0) = 2.$

Sol. : Let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y'') + 2L(y') + L(y) = L(3\,t\,e^{-t})$$

But $\quad L(y') = s\bar{y} - y(0) = s\bar{y} - 4$

and $\quad L(y'') = s^2\bar{y} - s\,y(0) - y'(0) = s^2\bar{y} - 4s - 2$

$$L(e^{-t}) = \frac{1}{s+1}$$

$$\therefore \quad L[t\,e^{-t}] = -\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - 4s - 2) + 2(s\bar{y} - 4) + \bar{y} = 3 \cdot \frac{1}{(s+1)^2}$$

$$\therefore \quad (s^2+2s+1)\bar{y} - 4s - 10 = \frac{3}{(s+1)^2}$$

$$\therefore \quad (s+1)^2\bar{y} = \frac{3}{(s+1)^2} + 4s + 10$$

$$\therefore \quad \bar{y} = \frac{3}{(s+1)^4} + \frac{4s+10}{(s+1)^2} = \frac{3}{(s+1)^4} + \frac{4s+4}{(s+1)^2} + \frac{6}{(s+1)^2}$$

$$= \frac{3}{(s+1)^4} + \frac{4}{s+1} + \frac{6}{(s+1)^2}$$

Taking the inverse Laplace transform of both sides,

$$y = L^{-1}\left[\frac{3}{(s+1)^4}\right] + 4L^{-1}\left[\frac{1}{s+1}\right] + 6L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$= 3e^{-t} L^{-1}\left[\frac{1}{s^4}\right] + 4e^{-t} L^{-1}\left[\frac{1}{s}\right] + 6e^{-t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$\therefore \quad y = 3e^{-t} \cdot \frac{t^3}{3!} + 4e^{-t} \cdot 1 + 6e^{-t} \cdot t = 3e^{-t}\left[\frac{t^3}{2} + 6t + 4\right]$$

Example 14 : Solve using Laplace transform $\dfrac{d^2y}{dt^2} + 9y = 18t$, given that $y\,(0) = 0$ and $y\,(\pi/2) = 0.$

(M.U. 1996, 98, 2004, 05, 07, 13)

Sol. : Let $L(y) = \bar{y}$. Then taking Laplace transforms of both sides

$$L(y'') + 9L(y) = 18L(t) \qquad \text{.............. (1)}$$

But $\quad L(y'') = s^2\bar{y} - sy(0) - y'(0)\,;\; L(t) = \dfrac{1}{s^2}$

Since, $y'(0)$ is not given let us assume $y'(0) = A.$ **[Note this]**

Hence, (1) becomes $\quad s^2\bar{y} - A + 9\,\bar{y} = \dfrac{18}{s^2}$

$$\therefore \quad (s^2+9)\,\bar{y} = \frac{18}{s^2} + A \qquad \therefore \quad \bar{y} = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$$

$$\therefore \quad \bar{y} = \frac{18}{9}\left[\frac{1}{s^2} - \frac{1}{s^2+9}\right] + \frac{A}{s^2+9} = \frac{2}{s^2} + \left(\frac{A-2}{s^2+9}\right)$$

$$\therefore \quad y = 2L^{-1}\left(\frac{1}{s^2}\right) + (A-2)\,L^{-1}\frac{1}{s^2+9} \qquad \therefore \quad y = 2t + \frac{A-2}{3}\sin 3t$$

To find A we put $t = \dfrac{\pi}{2}$ and use that $y\left(\dfrac{\pi}{2}\right) = 0.$

$$\therefore \quad 0 = 2 \cdot \frac{\pi}{2} + \left(\frac{A-2}{3}\right)\sin\left(\frac{3\pi}{2}\right) \qquad \therefore \quad 0 = \pi - \frac{(A-2)}{3}$$

$$\therefore \quad 0 = 3\pi - A + 2 \qquad \therefore \quad A = 3\pi + 2$$

$$\therefore \quad y = 2t + \pi \sin 3t.$$

Example 15 : Solve $(D^3 - 2D^2 + 5D)y = 0$, with $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1.$

(M.U. 2004, 05)

Sol. : Let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y''') - 2L(y'') + 5L(y') = 0$$

$$L(y') = s(\bar{y}) - y(0),\; L(y'') = s^2\bar{y} - s\,y(0) - y'(0)$$

$$L(y''') = s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)$$

From given conditions,

$$L(y') = s\bar{y},\;\; L(y'') = s^2\bar{y},\;\; L(y''') = s^3\bar{y} - 1.$$

$\therefore$ The equation becomes

$$s^3\bar{y} - 1 - 2s^2\bar{y} + 5s\,\bar{y} = 0 \qquad \therefore \quad \bar{y} = \frac{1}{s^3 - 2s^2 + 5s}$$

Taking inverse Laplace transform

$$y = L^{-1}\left[\frac{1}{s^3 - 2s^2 + 5s}\right] = L^{-1}\left[\frac{1}{s(s^2 - 2s + 5)}\right] = L^{-1}\left[\frac{1}{s[(s-1)^2 + 2^2]}\right]$$

We obtain the inverse by convolution theorem.

Let $\Phi_1(s) = \frac{1}{(s-1)^2+2^2}$ and $\Phi_2(s) = \frac{1}{s}$ $\therefore$ $\Phi(s) = \Phi_1(s)\cdot\Phi_2(s)$

$\therefore$ $f_1(t) = L^{-1}\Phi_1(s) = L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right] = e^t \cdot L^{-1}\frac{1}{s^2+2^2} = \frac{1}{2}\cdot e^t \cdot \sin 2t$

$f_2(t) = L^{-1}\Phi_2(s) = L^{-1}\left(\frac{1}{s}\right) = 1$

$\therefore$ $f_1(u) = \frac{1}{2}e^u \sin 2u$.

Then by Cor. (16A) page 2-15,

$\therefore$ $L^{-1}\Phi(s) = \int_0^t \frac{1}{2}e^u \sin 2u\,du = \frac{1}{2}\cdot\frac{1}{5}\cdot\left[e^u(\sin 2u - 2\cos 2u)\right]_0^t$

$\therefore$ $y = \frac{1}{10}\left[e^t(\sin 2t - 2\cos 2t) + 2\right]$

$\therefore$ The solution is $y = \frac{1}{5} - \frac{1}{5}e^t\cos 2t + \frac{1}{10}e^t\sin 2t$.

***Example 16 :** Solve the equation $y + \int_0^t y\,dt = 1 - e^{-t}$. (M.U. 2

Sol. : Let $L(y) = \bar{y}$. Taking the Laplace transform of both sides, we get,

$$L(y) + L\left[\int_0^t y\,dt\right] = L(1) + L(e^{-t})$$

Since, $L\left[\int_0^t y\,dt\right] = \int_0^\infty e^{-st}\int_0^t y\,dt = \left[\int_0^t y\,dt\cdot\frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\cdot y\,dt$

$$= 0 + \frac{1}{s}\int_0^\infty e^{-st}y\,dt = \frac{1}{s}L(y) = \frac{1}{s}\bar{y}$$

and $Le^{-t} = \frac{1}{s+1}$, the equation becomes $\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$.

$\therefore$ $\bar{y}\frac{(s+1)}{s} = \frac{1}{s(s+1)}$ $\therefore$ $\bar{y} = \frac{1}{(s+1)^2}$

$\therefore$ $y = L^{-1}\frac{1}{(s+1)^2} = e^{-t}L^{-1}\frac{1}{s^2} = e^{-t}\cdot t$ $\therefore$ $y = te^{-t}$.

***Example 17 :** Solve the following equation by using Laplace transform

$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$, given that $y(0) = 1$. **(M.U. 1999, 2006, 0**

Sol. : Let $L(y) = \bar{y}$. Taking Laplace transform of both sides, we get

$$L(y') + 2L(y) + L\left[\int_0^t y\,dt\right] = L(\sin t)$$

But $L(y') = sL(y) - y(0) = s\bar{y} - 1$

$L\left[\int_0^t y\,dt\right] = \frac{1}{s}L(y) = \frac{1}{s}\bar{y}$, $L(\sin t) = \frac{1}{s^2+1}$

$\therefore$ The equation becomes

$s\bar{y} - 1 + 2\bar{y} + \frac{1}{s}\bar{y} = \frac{1}{s^2+1}$ $\therefore$ $\left(s+2+\frac{1}{s}\right)\bar{y} = \frac{1}{s^2+1} + 1 = \frac{s^2+1+1}{s^2+1}$

$\therefore$ $\frac{(s^2+2s+1)}{s}\bar{y} = \frac{(s^2+2)}{s^2+1}$ $\therefore$ $\bar{y} = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$

Let $\frac{s(s^2+2)}{(s+1)^2(s^2+1)} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{cs+d}{s^2+1}$

$\therefore$ $s(s^2+2) = a(s+1)(s^2+1) + b(s^2+1) + (cs+d)(s+1)^2$

Putting $s = -1$, $-3 = 2b$ $\therefore$ $b = -3/2$

Putting $s = 0$, $0 = a + b + d$.

Equating the coefficients of s^2 and s^3.

$0 = a + b + 2c + d$ and $1 = a + c$

$\because$ $b = -3/2$, $a + d = 3/2$

and $a + 2c + d = 3/2$.

But $a + d = 3/2$ $\therefore$ $2c = 0$ $\therefore$ $c = 0$

$\because$ $1 = a + c$ and $c = 0$ $\therefore$ $a = 1$

$\because$ $a + d = 3/2$ and $a = 1$ $\therefore$ $d = 1/2$

$\therefore$ $a = 1$, $b = -3/2$, $c = 0$, $d = 1/2$

$\therefore$ $\bar{y} = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$

$\therefore$ $y = L^{-1}\left(\frac{1}{s+1}\right) - \frac{3}{2}e^{-t}L^{-1}\frac{1}{s^2} + \frac{1}{2}L^{-1}\frac{1}{s^2+1}$

$\therefore$ $y = e^{-t} - \frac{3}{2}e^{-t}\cdot t + \frac{1}{2}\sin t$.

***Example 18 :** Solve $\frac{d^2y}{dt^2} + 9y = \delta(t)$ given that $y = 0$, $\frac{dy}{dt} = 0$ at $t = 0$.

Sol. : By taking Laplace transforms of both sides.

$s^2\bar{y} - sy(0) - y'(0) + 9\bar{y} = 1$ $[\because L\delta(t) = 1]$

$\therefore$ $(s^2+9)\bar{y} = 1$ $\therefore$ $\bar{y} = \frac{1}{s^2+9}$ $\therefore$ $y = \frac{1}{3}\sin 3t$.

***Example 19 :** Solve $\frac{d^2y}{dt^2} + \frac{3dy}{dt} + 2y = t\delta(t-1)$ with the conditions $y(0) = 0$, $y'(0) = 0$.

(M.U. 2003, 08)

Sol. : Taking Laplace transforms of both sides

$$L(y'') + 3L(y') + 2L(y) = L[t\delta(t-1)]$$

But $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y}$

$L(y') = s\bar{y} - y(0) = s\bar{y}$

And $L[t\,\delta(t-1)] = e^{-s}(1)$ $\qquad \left[\because L\, f(t)\cdot\delta(t-a) = e^{-as} f(a)\right]$

$\therefore\ s^2\bar{y} + 3s\bar{y} + 2\bar{y} = e^{-s}$

$\therefore\ (s^2+3s+2)\,\bar{y} = e^{-s}$

$$\therefore\ \bar{y} = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = e^{-s}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

Taking inverse Laplace transform

$$y = e^{-(t-1)} H(t-1) - e^{-2(t-1)} H(t-1).$$

***Example 20 :** Solve $\dfrac{d^2y}{dt^2} + 4y = f(t)$, with conditions $y(0) = 0,\ y'(0) = 1$

and $f(t) = \begin{cases} 1 & \text{when } 0 < t < 1 \\ 0 & \text{when } t > 1 \end{cases}$ **(M.U. 2005, 07, 0...)**

Sol. : Taking Laplace transforms of both sides

$$L(y'') + 4L(y) = L\,f(t)$$

But $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$

$\therefore\ s^2\bar{y} - 1 + 4\bar{y} = \Phi(s)$ where, $L\,f(t) = \Phi(s)$

$\therefore\ (s^2+4)\,\bar{y} = 1 + \Phi(s)$

$$\therefore\ \bar{y} = \frac{1}{s^2+4} + \frac{\Phi(s)}{s^2+4}$$

We now express $f(t)$ as Heavisides unit step function

$\therefore\ f(t) = H(t) - H(t-1)$ $\quad\therefore\ L\,f(t) = \dfrac{1}{s} - e^{-s}\cdot\dfrac{1}{s}$.

$$\therefore\ \bar{y} = \frac{1}{s^2+4} + \frac{1}{s^2+4}\left[\frac{1}{s} - e^{-s}\cdot\frac{1}{s}\right]$$

$$= \frac{1}{s^2+4} + \frac{1}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)}$$

Taking inverse Laplace transform

$$y = \frac{1}{2}\sin 2t + \frac{1}{4}(1-\cos 2t) - \frac{1}{4}\{1 - \cos 2(t-1)\}\,H(t-1)$$

(For inverse of $\dfrac{1}{s(s^2+4)}$ see Ex. 1(iii), page 2-16 with $a = 2$. Also see the procedure given on page 2-58.)

***Example 21 :** Solve the above equation if $f(t) = H(t-2)$. **(M.U. 20...)**

Sol. : Since $f(t) = H(t-2)$, $\quad \Phi(s) = L\,f(t) = e^{-2s}\cdot\dfrac{1}{s}$

$$\therefore\ \bar{y} = \frac{1}{s^2+4} + e^{-2s}\cdot\frac{1}{s(s^2+4)} = \frac{1}{s^2+4} + \frac{e^{-2s}}{4}\cdot\left[\frac{1}{s} - \frac{1}{s^2+4}\right]$$

Taking inverse of both sides

$$y = L^{-1}\left(\frac{1}{s^2+4}\right) + L^{-1}\left[\frac{e^{-2s}}{4}\cdot\frac{1}{s}\right] - L^{-1}\left[\frac{e^{-2s}}{4}\cdot\frac{1}{s^2+4}\right]$$

$$= \frac{1}{2}\sin 2t + \frac{1}{4}\cdot 1\cdot H(t-2) - \frac{1}{4}\cos 2(t-2)\,H(t-2)$$

EXERCISE - XVIII

Using Laplace transform solve the following differential equations with the given conditions.

1. $(3D+2)\,y = e^{3t},\ y(0) = 1.$ **(M.U. 2002)**
2. $\dfrac{dy}{dt} + 2y = e^{-3t},\ y(0) = 1$
3. $\dfrac{dy}{dt} + y = e^{-2t},\ y(0) = 0$
4. $\dfrac{dy}{dt} + 2y = 5,\ y(0) = 1$
5. $\dfrac{dy}{dt} + 2y = \sin t,\ y(0) = 0$
6. $\dfrac{dy}{dt} + y = \cos 2t,\ y(0) = 1$
7. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0$; at $x = 0,\ y = 0,\ \dfrac{dy}{dx} = 4.$
8. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} = -8t;\ x(0) = 0 = x'(0).$
9. $(D^2+9)\,y = 18\,t;\ y(0) = 0,\ y'(0) = 0.$
10. $f''(t) + f'(t) = t;\ f(0) = 1,\ f'(0) = -1.$
11. $(D^2 - 3D + 2)\,y = 2e^{3t};\ y = 2,\ y' = 3$ at $t = 0.$

*12. $(D^2+1)\,y = \sin t;\ y(0) = 1,\ y'(0) = -\dfrac{1}{2}.$

13. $(D^2 - 2D + 2)\,x = 0;\ x(0) = x'(0) = 1.$
14. $y'' - 2y' + y = e^t;\ y(0) = 2,\ y'(0) = -1.$
15. $(D^2 + D - 2)\,x = 2(1 + t - t^2);\ x = 0,\ Dx = 3$ for $t = 0.$ **(M.U. 2006)**
16. $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 6y = 2;\ y(0) = 1,\ y'(0) = 0.$

*17. $(D^2 - 3D + 2)\,y = 4t + e^{3t}$ if $y = 1,\ Dy = -1$ at $t = 0.$

*18. $(D^3 + 2D^2 - D - 2)\,y = 0$ if $y = 1,\ Dy = 2,\ D^2y = 0$ at $y = 0.$

19. $(D^2 + 4D + 3)\,y = e^{-t};\ y(0) = y'(0) = 1.$
20. $(D^2 + D)\,y = t^2 + 2t$ at $t = 0,\ y = 4$ and $Dy = 2.$ **(M.U. 1997)**
21. $(D^2 - 2D + 1)\,x = e^t$ with the conditions $x = 2,\ Dx = -1$ at $t = 0.$ **(M.U. 1997, 2000)**
22. $(D+1)^2 y = 6t\,e^{-t}$ with $y(0) = 2,\ y'(0) = 5.$ **(M.U. 1994)**
23. $\dfrac{d^2y}{dx^2} + 16y = \delta(t)$ given that $y = 0,\ \dfrac{dy}{dt} = 0$ at $t = 0.$ **(M.U. 2004)**
24. $(D^2 - 3D + 2)\,y = 4e^{2t}$ at $t = 0,\ y = -3$ and $Dy = 5.$ **(M.U. 2003, 04, 13)**
25. $(D^2 - 2D - 8)\,y = 4,\ y(0) = 0$ and $y'(0) = 1.$

26. $(D^2 + D)\, y = t^2 + 2t,\ y(0) = 4,\ y'(0) = -2.$ (M.U. 20

27. $(D^2 - 4)\, y = 3e^t,\ y(0) = 0,\ y'(0) = 3.$ (M.U. 20

28. $\dfrac{d^2y}{dt^2} + y = t,\ y(0) = 1,\ y'(0) = 0.$ (M.U. 20

29. $2y'' + 5y' + 2y = e^{-2t},\ y(0) = y'(0) = 1.$ (M.U. 20

30. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = e^{3t}$, given $y(0) = 0$ and $y'(0) = 1.$ (M.U. 20

31. $\dfrac{d^2y}{dt} + 9y = \cos 2t,\ y(0) = 1$ and $y(\pi/2) = -1.$ (M.U. 20

32. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y = \sin t,\ y(0) = 0$ and $y'(0) = 0.$ (M.U. 20

[**Ans. :** (1) $y = \dfrac{1}{11}e^{3t} + \dfrac{10}{11}e^{-2t/3}$, (2) $y = 2e^{-2t} - e^{-3t}$, (3) $y = e^{-t} - e^{-2t}$, (4) $y = \dfrac{5}{2} - \dfrac{3}{2}$

(5) $y = -\dfrac{1}{5}\cos t + \dfrac{2}{5}\sin t + \dfrac{1}{5}e^{-2t}$, (6) $y = \dfrac{1}{5}\cos 2t + \dfrac{2}{5}\sin 2t + \dfrac{4}{5}e^{-t}$,

(7) $y = e^x - e^{-3x}$, (8) $x = -\dfrac{1}{8} + \dfrac{1}{2}t - t^2 + \dfrac{e^{-4t}}{8}$, (9) $y = 2\left[t - \dfrac{1}{3}\sin 3t\right]$,

(10) $f(t) = 1 - t + \dfrac{t^2}{2}$, (11) $y = 2e^t - e^{2t} + e^{3t}$, (12) $y = \left(1 - \dfrac{t}{2}\right)\cos t$,

(13) $x = e^t \cos t$, (14) $y = e^t\left(2 - 3t + \dfrac{t^2}{2}\right)$, (15) $x = t^2 + e^t - e^{-2t}$,

(16) $y = -\dfrac{1}{3} + \dfrac{8}{15}e^{3t} + \dfrac{4}{5}e^{-2t}$, (17) $y = -\dfrac{1}{2}e^t - 2e^{2t} + \dfrac{1}{2}e^{3t} + 2t + 3$,

(18) $y = \dfrac{1}{3}(5e^t + e^{-2t}) - e^{-t}$, (19) $y = \dfrac{7}{4}e^{-t} - \dfrac{3}{4}e^{-3t} + \dfrac{1}{2}te^{-t}$,

(20) $y = \dfrac{t^3}{3} + 2e^{-t} + 2$, (21) $x = 2e^t - 3e^t \cdot t + \dfrac{t^2}{2}e^t$,

(22) $y = e^{-t}(t^3 + 7t + 2)$, (23) $y = \dfrac{1}{4}\sin 4t$,

(24) $y = -7e^t + 4e^{2t} + 4te^{2t}$, (25) $y = \dfrac{1}{6}[e^{4t} + 2e^{-2t} - 3]$,

(26) $y = 2 + 2e^{-t} + \dfrac{t^3}{3}$, (27) $y = -e^t + \dfrac{3}{2}e^{2t} - \dfrac{1}{2}e^{-2t}$,

(28) $y = t - \sin t + \cos t$, (29) $y = \dfrac{20}{9}e^{-t/2} - \dfrac{11}{2}e^{-2t} - \dfrac{t}{3}e^{-2t}$,

(30) $y = \dfrac{1}{6}e^t - \dfrac{4}{15}e^{-2t} + \dfrac{e^{3t}}{10}$, (31) $y = \dfrac{1}{5}\cos 2t + \dfrac{4}{5}\cos 3t + \dfrac{4}{5}\sin 3t$.

(32) $y = -\dfrac{1}{40}e^{-3t} + \dfrac{1}{8}e^t - \dfrac{1}{10}\cos t - \dfrac{1}{5}\sin t$

EXERCISE - XIX

Theory

1. State and prove convolution theorem. (M.U. 2003)

2. State convolution theorem and deduce that

$$L^{-1}\left[\frac{1}{s}\Phi(s)\right] = \int_0^t f(u)\,du \quad \text{where, } L[f(u)] = \Phi(s).$$

3. Prove that $L\,\delta(t-a) = e^{-as}$. (M.U. 2002)

4. Prove that $L[f(t-a)H(t-a)] = e^{-as} f(s)$ where $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$. (M.U. 2003)

5. Find $L[\delta(t-a)]$ where, $\delta(t-a) \begin{cases} = 0 & \text{for } t < a \\ = 1/\varepsilon & \text{for } a < t < a+\varepsilon \\ = 0 & \text{for } t > a+\varepsilon \end{cases}$ (M.U. 1998)

6. If $f(t-a) \cdot H(t-a) = 0 \qquad t < a$

$= f(t-a) \qquad t > a$, then prove that

$$L[f(t-a) \cdot H(t-a)] = e^{-as}L[f(t)] = e^{-as}\Phi(s) \text{ where, } \Phi(s) = L[f(t)].$$ (M.U. 1996, 2004)

7. If $f(t)$ is a periodic function of period a, prove that

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt.$$ (M.U. 1996, 97, 2003, 07, 08)

8. Define Heaviside's unit step function $H(t-a)$ and obtain its Laplace transform. (M.U. 2005)

9. Define Heaviside unit step function and obtain Laplace transform of $f(t-a) \cdot H(t-a)$.

10. Define Dirac-delta function and obtain its Laplace transform. (M.U. 2004)

❀ ❀ ❀

CHAPTER

3 Fourier Series

1. Introduction

The series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots + (a_n \cos nx + b_n \sin nx) + \ldots$$

or briefly $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where, all a's and b's are constants is called a **trigonometric series.**

The function $f(x)$ represented by the above series is a periodic function of period 2π sinc[e] changing x to $2\pi + x$, we get the same series.

Any periodic function $f(x)$ of period 2π which satisfies certain conditions known as Diric[hlet's] conditions can be expressed in the form of the above series valid for all values of x in any inter[val c] to $c + 2\pi$ of length 2π. The expansion of $f(x)$ in the form of the above series is called **Fourier Se[ries]**

Joseph Fourier (1768 - 1830)

Jean Baptiste Joseph Fourier was a great Fre[nch] mathematician and physicist. He is best remembered for pione[ering] Fourier Series, Fourier Transforms. He went to Egypt with Napo[leon] Bonapararte in 1798 and was made governor of Lower Egypt [and] contributed several mathematical papers to Cairo Institute foun[ded] by Napolean. He was permanent secretary of French Academ[y of] Sciences. In 1830 he was elected to Royal Swedish Academ[y of] Sciences.

2. Dirichlet's Conditions

A function $f(x)$ defined in the interval $c_1 \le x \le c_2$ can be expressed as Fourier Series if i[n the] interval,

(i) $f(x)$ and its integrals are finite and single valued,

(ii) $f(x)$ has discontinuities, finite in number,

(iii) $f(x)$ has finite number of maxima and minima.

These conditions are known as **Dirichlet's conditions,**

For example, $\sin^{-1} x$ cannot be expressed as Fourier Series since, it is not a single v[alued] function.

Also $\tan x$ cannot be expressed as Fourier Series in $(0, 2\pi)$ since, $\tan x$ is infinite at $x = \frac{\pi}{2}\left(\text{and } \frac{3\pi}{2}\right)$ which is a point on interval.

But $\cos \alpha x$ can be expanded as a Fourier Series since, it satisfies the above conditions.

Johann Peter Gustav Lejeune Dirichlet (1805 - 1859)

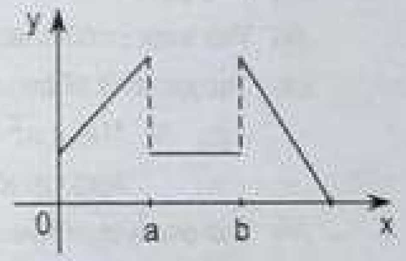
He was a great German mathematician. Poisson and Fourier were his doctoral advisers and Kronecker, Cantor, Dedekind, Riemann were among his many doctoral students. He showed a strong interest in mathematics before he was 12 and convinced his father to allow him to study mathematics. His father wanted him to be a merchant. His first original research giving a proof of Fermat's last theorem for $n = 5$ brought him immediate fame. He was then appointed as a lecturer at the French Academy of Sciences at the age of 20 in 1825, before he had a degree. He got his Ph.D. in 1827. He was in close contact with other great mathematicians, Jacobi, Liouville, etc. He died in Göttingen, Germnay. His brain is preserved in the department of Physiology at the university of Göttingen along with the brain of Gauss.

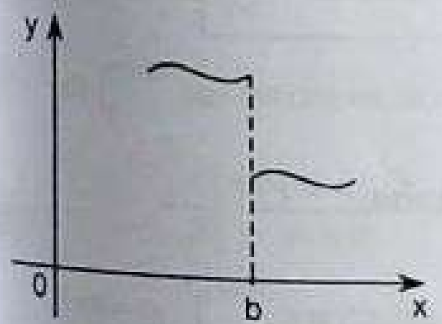
Dirichlet's main interest was Number Theory and Analysis. He was a member of several academies such as Prussian Academy of Sciences, French Academy of Sciences, Royal Swedish Academy of Sciences, Royal Belgian Academy of Sciences, etc. There are several theorems named after Dirichlet *viz.* Dirichlet's Approximation Theorem, Dirichlet's Unit Theorem, Dirichlet's Integral Theorem, Dirichlet's Convolution, Dirichlet's Series, Dirichlet's Function, Dirichlet's Pigeon hole Principle and many many others.

The Dirichlet crater on the Moon and the 11665 Dirichlet asteroid are named after him.

(a) Functions with discontinuities

A function which has finite disconti-nuities can be expanded as a Fourier Series. The graph of a function as shown in the adjoining figure may consist of a finite number of disjointed curves given by different equations.





If $x = c$ is a point of finite discontinuity both the limits of $f(x)$ as $x \to c^-$ and as $x \to c^+$ exist. At such a point Fourier Series gives the value of $f(x)$ at $x = c$ as the arithmetic mean of these two limits

i.e. $f(c) = \frac{1}{2}\left[\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\right]$

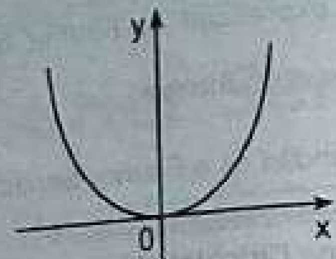
Notes ....

(i) A function $f(x)$ is said to be **periodic** with period a if $f(a + x) = f(a)$ e.g. $\sin x$, $\cos x$ are periodic with period 2π.

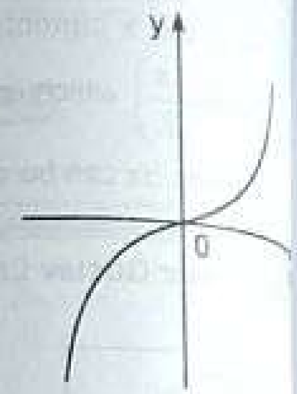
(ii) (a) A function $f(x)$ is said to be **even** if $f(-x) = f(x)$ e.g. $\cos x$, $\sec x$, x^2 are even functions. Graphically, an even function is **symmetrical about the y-axis.**

(b) A function $f(x)$ is said to be **odd** if $f(-x) = -f(x)$ e.g. $\sin x$, $\operatorname{cosec} x$, x^3 are odd functions.
Geometrically, an odd function is **symmetrical about the origin.**

(c) A function can be neither even nor odd e.g. e^x, 10^x, $x^2 - x$, are neither odd nor even.



Even Function



Odd Function

(iii) We also need the following results, where n is an integer

(i) $\sin n\pi = 0$ (ii) $\sin 2n\pi = 0$ (iii) $\cos n\pi = (-1)^n$ (iv) $\cos 2n\pi = 1$

(v) $\cos(n \pm 1)\pi = \cos n\pi \cos \pi \mp \sin n\pi \sin \pi = -\cos n\pi$

(vi) $\sin(n \pm 1)\pi = \sin n\pi \cos \pi \pm \cos n\pi \sin \pi = 0.$

(vii) $\sin(2n\pi + x) = \sin x$

(viii) $\cos(2n\pi + x) = \cos x$

(iv) We also need the following results

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$-2 \sin A \sin B = \cos(A+B) - \cos(A-B).$$

(v) **Properties of Even and Odd Functions**

We state below the properties of even and odd functions.

(i) The sum or difference of two (or more) even functions is even.

(ii) The sum or difference of two (or more) odd functions is odd.

(iii) The product of two even functions is even.

e.g., if $f(x) = x^2 + 1$ and $g(x) = x^4$, then
$f(x) \cdot g(x) = (x^2+1)\, x^4 = x^6 + x^4$ is even.

(iv) The product of two odd functions is even.

e.g., if $f(x) = x^3 + x$ and $g(x) = x$, then
$f(x) \cdot g(x) = (x^3 + x)\, x = x^4 + x^2$ is even.

(v) The product of an even function and an odd function is odd.

e.g., if $f(x) = x^4 + x^2$ and $g(x) = x$, then
$f(x) \cdot g(x) = (x^4 + x^2)\, x = x^5 + x^3$ is odd.

If $\phi(x) = f(x) \cdot g(x)$ then $\phi(x)$ will be even or odd according to the following table.

$f(x)$	E	E	O	O
$g(x)$	E	O	E	O
$\phi(x) = f(x) \cdot g(x)$	E	O	O	E

(vi) We need the following theorem

Theorem $\boxed{\begin{aligned} \int_{-a}^{a} f(x)\,dx &= 2\int_0^a f(x)\,dx \quad \text{if } f(x) \text{ is even.} \\ &= 0 \quad \text{if } f(x) \text{ is odd.} \end{aligned}}$ .... (1)

(vii) We also need the following two integrals often in Fourier series.

(a) $\boxed{\int e^{ax} \sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]}$ .... (2)

(b) $\boxed{\int e^{ax} \cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]}$ .... (3)

(viii) We also need the following results

(a) $\int_c^{c+2\pi} \cos nx\,dx = \left[\dfrac{\sin nx}{n}\right]_c^{c+2\pi} = 0 \quad (n \neq 0)$ .... (4)

(b) $\int_c^{c+2\pi} \sin nx\,dx = \left[-\dfrac{\cos nx}{n}\right]_c^{c+2\pi} = 0 \quad (n \neq 0)$ .... (5)

(c) $$\int_c^{c+2\pi} \sin mx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\sin(m+n)x + \sin(m-n)x]\,dx$$
$$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right]_c^{c+2\pi} = 0 \quad (m \neq n)$$

If $m = n$, $\int_c^{c+2\pi} \sin nx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} 2 \sin nx \cos nx\,dx$
$$= \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\,dx = \frac{1}{2}\left[-\frac{\cos 2nx}{2n}\right]_c^{c+2\pi} = 0$$

Hence, $\boxed{\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0 \text{ for all } m, n}$ .... (6)

(d) $$\int_c^{c+2\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\cos(m+n)x + \cos(m-n)x]\,dx$$
$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{(m-n)}\right]_c^{c+2\pi} = 0 \quad (m \neq n)$$

If $m = n$, $\int_c^{c+2\pi} \cos mx \cos nx\,dx = \int_c^{c+2\pi} \cos^2 nx\,dx$
$$= \int_c^{c+2\pi}\left(\frac{1+\cos 2nx}{2}\right)dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_c^{c+2\pi} = \pi$$

Hence, $\boxed{\int_c^{c+2\pi} \cos mx \cos nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}}$ .... (7)

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(e) $\int_c^{c+2\pi} \sin mx \sin nx \, dx = -\frac{1}{2}\int_c^{c+2\pi} [\cos(m+n)x - \cos(m-n)x] \, dx$

$$= -\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} - \frac{\sin(m-n)x}{m-n}\right]_c^{c+2\pi} = 0 \qquad (m \neq n)$$

If $m = n$, $\int_c^{c+2\pi} \sin mx \sin nx \, dx = \int_c^{c+2\pi} \sin^2 nx \, dx$

$$= \int_c^{c+2\pi}\left(\frac{1-\cos 2nx}{2}\right) dx = \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_c^{c+2\pi} = \pi$$

Hence, $\boxed{\int_c^{c+2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}}$ ....

(ix) Putting $c = -\pi$ in the above results, we get the following results.

(A) $\int_{-\pi}^{\pi} \cos nx \, dx = 0$, $\int_{-\pi}^{\pi} \sin nx \, dx = 0$ ....

(B) $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$ for all m, n ....

(C) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$ if $m \neq n$ ....

$= \pi$ if $m = n$

(D) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$ if $m \neq n$ ....

$= \pi$ if $m = n$

3. Determination of Fourier Coefficients (Euler's Formulae)

Let $f(x)$ be a periodic function of period 2π which can be represented in the int
$(c, c + 2\pi)$ by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{..........}$$

We assume that the above series is uniformly convergent and can be integrated term by
in the given interval.

Integrating both sides of (1) from c to $c + 2\pi$, we get

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi} a_0 \, dx + \int_c^{c+2\pi} \sum a_n \cos nx \, dx + \int_c^{c+2\pi} \sum b_n \sin nx \, dx$$

$$= a_0 [x]_c^{c+2\pi} + 0 + 0 = a_0(c + 2\pi - c) = 2a_0\pi$$

$\therefore$ $\boxed{a_0 = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx}$ ....

Leonhard Euler (1707 - 1783)

One of the great mathematicians of Switzerland. His father wanted him to become a pastor (a priest). But Bernoulli persuaded his father to allow his son to pursue mathematics. He studied under his fellow countryman, mathematician Bernoulli and had published his first paper when he was 18. The word function first suggested by Leibnitz was generalised further by Bernoulli and Euler. Euler is supposed to be the most prolific mathematical writer in history. He has written a number of text books which are known for his clarity, detail and completeness. Although he had lost his eye-sight for the last 17 years of his life, he did not allow his work to be hampered because all the formulae from trigonometry and analysis (and many poems including the entire Latin epic-Aeneid) were on the tip of his tongue.

Now, multiply (1) by $\cos nx$ and integrate from c to $c + 2\pi$.

$$\therefore \int_c^{c+2\pi} f(x)\cos nx \, dx = a_0\int_c^{c+2\pi} \cos nx \, dx + \int_c^{c+2\pi} (\Sigma a_n \cos nx)\cos nx \, dx + \int_c^{c+2\pi} (\Sigma b_n \sin x)\cos nx \, dx$$

$$= 0 + a_n\pi + 0$$

$\therefore$ $\boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx \, dx}$ .... (3)

Now, multiply (1) by $\sin nx$ and integrate from c to $c + 2\pi$.

$$\therefore \int_c^{c+2\pi} f(x)\sin nx \, dx = a_0\int_c^{c+2\pi} \sin nx \, dx + \int_c^{c+2\pi} (\Sigma a_n \cos nx)\sin nx \, dx + \int_c^{c+2\pi} (\Sigma b_n \sin nx)\sin nx \, dx$$

$$= 0 + 0 + b_n\pi$$

$\therefore$ $\boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx \, dx}$ .... (4)

Thus, we have the following results

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx \, dx \\ b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx \, dx \end{aligned}} \qquad \text{.......... (5)}$$

Cor 1 : If $c = 0$ *i.e.* if the interval is $(0, 2\pi)$, then the Fourier series of $f(x)$ is given by $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}}$$

Cor. 2 : If $c = -\pi$ *i.e.* if the interval is $(-\pi, \pi)$, then the Fourier series is given by $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}}$$

4. Parseval's Identity in $(c, c + 2l)$

If $f(x)$ converges uniformly in $(c, c + 2l)$, then

$$\boxed{\frac{1}{2l}\int_c^{c+2l}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

This is known as **Parseval's Identity** for the function $f(x)$ in the interval $(c, c + 2l)$.

Proof : We shall see later that the Fourier Series for $f(x)$ in $(c, c + 2l)$ is given by (A), page 3-43).

$$f(x) = a_0 + \Sigma a_n \frac{\cos n\pi x}{l} + \Sigma b_n \frac{\sin n\pi x}{l}$$

Now, multiplying both sides of (1) by $f(x)$ and integrating term by term from c to $c + 2l$, we

$$\int_c^{c+2l}[f(x)]^2 dx = a_0\int_c^{c+2l} f(x)\,dx + \int_c^{c+2l}\Sigma a_n f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$
$$+ \int_c^{c+2l}\Sigma b_n f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

But $\int_c^{c+2l} f(x)\,dx = 2l\,a_0$ [See (3), page 3-43]

$$\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = a_n l$$

$$\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = b_n l$$

Putting these values in (2), we get,

$$\int_c^{c+2l}[f(x)]^2 dx = 2l\,a_0^2 + \Sigma a_n^2 l + \Sigma b_n^2 l$$

$$\therefore \quad \frac{1}{2l}\int_c^{c+2l}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\Sigma\left(a_n^2 + b_n^2\right)$$

Cor 1 : If $l = \pi$ *i.e.* if the interval is $(c, c + 2\pi)$ and $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, [See (1) on the previous page], then

$$\frac{1}{2\pi}\int_c^{c+2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

(This is Parseval's identity for $f(x)$ in $(c, c + 2\pi)$. We have obtained this result independently below.)

Cor 2 : If $c = 0$ in the above corollary 1 *i.e.* if the interval is $(0, 2\pi)$

and $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$

then $$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Cor 3 : If $c = -\pi$ in the above corollary 1 *i.e.* if the interval is $(-\pi, \pi)$

and $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, then

then $$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Cor 4 : If $c = 0$ in the above identity (A) *i.e.* if the interval is $(0, 2l)$

and $f(x) = a_0 + \Sigma a_n \cos\frac{n\pi x}{l} + \Sigma b_n \sin\frac{n\pi x}{l}$

then $$\frac{1}{2l}\int_0^{2l}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Cor. 5 : If $c = -l$ in the above identity (A) *i.e.* if the interval is $(-l, l)$

and $f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right) + \Sigma b_n \sin\left(\frac{n\pi x}{l}\right)$

then $$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

(You are advised to derive this result independently.)

Cor. 6 : If the half range cosine series [See pages 3-62 and 3-63] in $(0, \pi)$ for $f(x)$ is

$f(x) = a_0 + \Sigma a_n \cos nx$,

then $$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty\right]$$

Cor. 7 : If the half range sine series in $(0, \pi)$ for $f(x)$ is $f(x) = \Sigma b_n \sin nx$

then $$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2 dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\ldots\infty\right]$$

Cor. 8 : If the half range cosine series in $(0, l)$ for $f(x)$ is

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right)$$

then $\frac{1}{l}\int_0^l [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty\right]$

(We shall derive this result independently below.)

Cor. 9 : If the half range sine series in $(0, l)$ for $f(x)$ is

$$f(x) = \Sigma b_n \sin\frac{n\pi x}{c}$$

then $\frac{1}{l}\int_0^l [f(x)]^2 dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\ldots\infty\right]$

(a) Parseval's Identity in $(c, c+2\pi)$

If $f(x)$ converges uniformly in $(c, c+2\pi)$, then

$$\boxed{\frac{1}{2\pi}\int_c^{c+2\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}$$

This is known as Parseval's Identity for the function $f(x)$ in the interval $(c, c+2\pi)$.

Proof : We know that the Fourier series for $f(x)$ in $(c, c+2\pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos x + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\ldots$$

Now, multiplying both sides of (1) by $f(x)$ and integrating term by term from c to $c+2\pi$, we

$$\int_c^{c+2\pi} [f(x)]^2 dx = a_0\int_c^{c+2\pi} f(x)\,dx + \int_c^{c+2\pi} \Sigma a_n f(x)\cos nx\,dx$$
$$+ \int_c^{c+2\pi} \Sigma b_n f(x)\sin nx\,dx \qquad \ldots\ldots\ldots$$

But $\int_c^{c+2\pi} f(x)\,dx = 2\pi a_0$ [By (2), page 3-5]

$\int_c^{c+2\pi} f(x)\cos nx\,dx = a_n\pi$ [By (3), page 3-6]

$\int_c^{c+2\pi} f(x)\sin nx\,dx = b_n\pi$ [(By (4), page 3-6]

Putting these values is (2), we get,

$$\int_c^{c+2\pi} [f(x)]^2 dx = 2\pi a_0^2 + \Sigma a_n^2\pi + \Sigma b_n^2\pi$$

$$\therefore \frac{1}{2\pi}\int_c^{c+2\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\Sigma\left(a_n^2 + b_n^2\right)$$

Cor. : Putting $c = -\pi$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\Sigma\left(a_n^2 + b_n^2\right).$$

(b) Parseval's Identity for half range cosine series in $(0, l)$ ✱

If $f(x)$ can be expanded as a half range cosine series in $(0, l)$ as

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right)$$

then $\frac{1}{l}\int_0^l [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty\right]$

Proof : We know that the half range cosine series for $f(x)$ in the interval $(0, l)$ is given by (See page 3-62).

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right) \qquad \text{[See § 13, page 3-62]}$$

Now, multiply both sides by $f(x)$ and integrate term by term from 0 to l.

$$\therefore \int_0^l [f(x)]^2 dx = a_0\int_0^l [f(x)]\,dx + \int_0^l \Sigma a_n f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

But $\int_0^l f(x)\,dx = a_0 l$ [Page 3-63]

$\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{a_n l}{2}$ [Page 3-63]

$$\therefore \int_0^l [f(x)]^2 dx = a_0^2 l + \frac{1}{2}\Sigma a_n^2 l$$

$$\therefore \frac{1}{l}\int_0^l [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty\right]$$

Cor. : Putting $l = \pi$

$$\frac{1}{\pi}\int_0^{\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty]$$

(**✱ Proofs are not expected.**)

5. Generalised Rule of Integration by Parts

If u, v are two functions (dashes denote the derivatives and suffixes denote the integrals), then

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots..$$

In words the rule states that, to find the integral of the product : Integral = first term × integral of he second – then write the derivative of the first term × integral of the second and repeat the procedure taking alternatively positive and negative signs. The rule is highly useful especially when the first term is a positive integral power of x and the second term can be easily integrated.

6. Fourier Series in $(0, 2\pi)$

Example 1 : Find a Fourier Series to represent $f(x) = x^2$ in $(0, 2\pi)$ and hence, deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots..$$ **(M.U. 2004, 09, 10, 12)**

Sol. : Let $x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ in $(0, 2\pi)$ ……………… (1)

Then, $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{4\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

(By the generalised rule of integration by parts)

$$a_n = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{4\pi^2}{n} + \frac{2}{n^3}\right\} - \left\{\frac{2}{n^3}\right\}\right] = \frac{4\pi}{n}$$

Putting these values in (1),

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$\therefore\ x^2 = \frac{4\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right] - 4\pi\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right]$$

Now, put $x = \pi$,

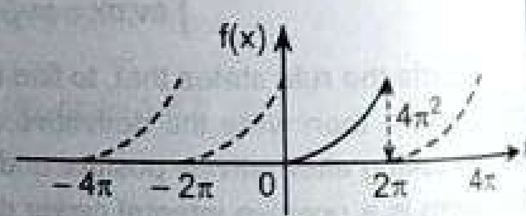
$$\therefore\ \pi^2 = \frac{4\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$\therefore\ \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Now, the given function, $y = x^2$ is a parabola with vertex at the origin and opening upwards. Further when $x = 0$, $y = 0$ and when $x = 2\pi$, $y = 4\pi^2$.

The graph of the above function is shown in the figure.



Example 2 : Find the Fourier Series of the function $f(x) = e^{-x}$, $0 < x < 2\pi$ and $f(x + 2\pi) = f(x)$. **(M.U. 20…)**

Hence, deduce the value of $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$.

Further derive the series for cosec $h\,\pi$.

Sol. : Let $\quad f(x) = e^{-x} = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$ ....

$$\therefore\ a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{2\pi}\left[-e^{-x}\right]_0^{2\pi} = \frac{1}{2\pi}\left(-e^{-2\pi}+1\right) = \frac{1-e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{-x}(-\cos nx + n\sin nx)\right]_0^{2\pi} \qquad \text{[See (3), page 3-4]}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\cos 2n\pi + n\sin 2n\pi) - e^0(-\cos 0 + n\sin 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\cdot\left[e^{-2\pi}(-1) - (-1)\right] = \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{-x}(-\sin nx - n\cos nx)\right]_0^{2\pi} \qquad \text{[See (2), page 3-4]}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\sin 2n\pi - n\cos 2n\pi) - e^0(-\sin 0 - n\cos 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-n) - 1(-n)\right] = \frac{n}{\pi(1+n^2)}(1-e^{-2\pi})$$

Putting the values of a_0, a_n, b_n in (1), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{1+n^2}\cos nx + \sum_{n=1}^{\infty}\frac{n}{1+n^2}\sin nx\right] \quad \ldots\ldots\ldots\ldots (2)$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{1+n^2}(\cos nx + n\sin nx)\right]$$

Putting $x = \pi$ in (2), we get,

$$e^{-\pi} = \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{1+n^2}\cos n\pi\right] \qquad (\because\ \sin n\pi = 0)$$

$$= \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{1}{1+n^2}(-1)^n\right]$$

$$\therefore\ e^{-\pi} = \frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} \qquad \therefore\ \frac{\pi}{e^{\pi}(1-e^{-2\pi})} = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore\ \frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} \qquad \therefore\ \frac{\pi}{2}\cdot\frac{2}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore\ \frac{\pi}{2}\cdot\frac{1}{\sinh\pi} = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore\ \text{cosec}\,h\,\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{2}{\pi}\left[\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \ldots\right]$$

Example 3 : Obtain the Fourier expansion of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \le x \le 2\pi$ an

(M.U. 2002, 04, 11, 12, 1

$f(x + 2\pi) = f(x)$.

Also deduce that

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ **(M.U. 2014)** (ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ (M.U. 20

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$ **(M.U. 2003)** (iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$ (M.U. 20

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\,dx$$

$$\therefore \quad a_0 = \frac{1}{8\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = -\frac{1}{24\pi}[-\pi^3 - \pi^3] = \frac{\pi^2}{12}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

(By generalised rule of integration by parts.)

$$= \frac{1}{4\pi}\left[\left(0 + 2\pi\frac{\cos 2n\pi}{n^2} - 0\right) - \left(0 - \frac{2\pi}{n^2} - 0\right)\right]$$

$$\therefore \quad a_n = \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2} \qquad [\because \cos 2n\pi = 1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\cdot\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right)\right]$$

$$\therefore \quad b_n = \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] = 0$$

Putting these values in (1), we get

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$= \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots$$

(i) Now, put $x = 0$ in (2).

$$\therefore \quad \frac{\pi^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \ldots\ldots(3)$$

(ii) Again, put $x = \pi$ in (2).

$$\therefore \quad 0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \qquad \ldots\ldots(4)$$

(iii) To get the result (iii) add (3) and (4).

$$\therefore \quad \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \ldots$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

(iv) To derive the last result we use Parseval's identity. We know that by Parseval's identity in $(0, 2\pi)$

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = a_0^{\,2} + \frac{1}{2}\sum(a_n^{\,2} + b_n^{\,2}) \qquad \ldots\ldots(5)$$

Now, $\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^4 dx$

$$= \frac{1}{32\pi}\int_0^{2\pi}[\pi^4 - 4\pi^3 x + 6\pi^2 x^2 - 4\pi x^3 + x^4]\,dx$$

$$= \frac{1}{32\pi}\left[\pi^4 x - 2\pi^3 x^2 + 2\pi^2 x^3 - \pi x^4 + \frac{x^5}{5}\right]_0^{2\pi}$$

$$= \frac{1}{32\pi}\left[2\pi^5 - 8\pi^5 + 16\pi^5 - 16\pi^5 + \frac{32\pi^5}{5}\right]$$

$$= \frac{1}{32\pi}\cdot\frac{2\pi^5}{5} = \frac{\pi^4}{80}$$

Hence, by (5) using (A), (B) and (C)

$$\frac{\pi^4}{80} = \frac{\pi^4}{144} + \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right)$$

$$\pi^4\left(\frac{1}{80} - \frac{1}{144}\right) = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right)$$

$$\pi^4\left(\frac{9 - 5}{720}\right) = \frac{\pi^4}{180} = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right)$$

$$\therefore \quad \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$$

Example 4 : Expand $f(x) = x \sin x$ in the interval $0 \le x \le 2\pi$. **(M.U. 2001, 02, 03, 05, 1**

Deduce that $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$.

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ....

$$\therefore \quad a_0 = \int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}(x \sin x)\cdot dx$$

$$= \frac{1}{2\pi}\left[(x)(-\cos x) - (-\sin x)(1)\right]_0^{2\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{2\pi}\left[(-2\pi + 0) - (0 + 0)\right] = -1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx \quad \text{..........}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin x \cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\cos nx \sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(2\pi)\left\{-\frac{\cos 2(n+1)\pi}{(n+1)} + \frac{\cos 2(n-1)\pi}{(n-1)}\right\} - 0\right] \qquad \text{[By § 5, page 3-10]}$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1} \quad \text{if } n \ne 1$$

If $n = 1$, the above method fails.

Putting $n = 1$ in (2), we get,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin x \cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\cdot \sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 4\pi}{2}\right) - 0\right] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\,dx \quad \text{..........}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin x \sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin nx \sin x\,dx$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n+1)x - \cos(n-1)x\right]dx \qquad \text{[By § 5, page 3-10]}$$

$$\therefore \quad b_n = -\frac{1}{2\pi}\left[x\left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\cos(n+1)x}{(n+1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi}\left[-(1)\left\{-\frac{\cos 2(n+1)\pi}{(n+1)^2} + \frac{\cos 2(n-1)\pi}{(n-1)^2}\right\} + (1)\left\{-\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right\}\right]$$

$$= -\frac{1}{2\pi}\left[\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right]$$

$$= 0 \quad \text{if } n \ne 1$$

If $n = 1$, the above method fails.

Putting $n = 1$ in (3), we gat,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi(2\pi - 0) - \left(\frac{4\pi^2}{2} + \frac{1}{4}\right)\right\} - \left(0 - \frac{1}{4}\right)\right] = \frac{1}{2\pi}\left[2\pi^2\right] = \pi$$

Putting these values in (1),

$$x \sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x$$

Deduction : Putting $x = 0$, we get $\frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

Example 5 : Find the Fourier expansion for $f(x) = \sqrt{1 - \cos x}$ in $(0, 2\pi)$. **(M.U. 2010, 13)**

Hence, deduce that $\frac{1}{2} = \sum_{1}^{\infty} \frac{1}{4n^2 - 1}$. **(M.U. 1994, 99, 2005, 06, 09)**

Sol. : Let $f(x) = \sqrt{1 - \cos x} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

Here, $f(x) = \sqrt{2}\cdot \sin(x/2)$

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{2}\cdot \sin\frac{x}{2}\,dx$$

$$= \frac{1}{\sqrt{2}\cdot\pi}\left[-2\cos\frac{x}{2}\right]_0^{2\pi} = \frac{1}{\sqrt{2}\cdot\pi}\left[-2(-1-1)\right] = \frac{4}{\sqrt{2}\cdot\pi} = \frac{2\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\cdot \sin\frac{x}{2}\cdot \cos nx\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{1}{2} + n\right)x + \sin\left(\frac{1}{2} - n\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{1+2n}{2}\right)x + \sin\left(\frac{1-2n}{2}\right)x\right]dx$$

$$\therefore\ a_n = \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{1+2n}\cos\left(\frac{1+2n}{2}\right)x - \frac{2}{1-2n}\cos\left(\frac{1-2n}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{1}{2n+1}\cos\left(\frac{2n+1}{2}\right)x + \frac{1}{2n-1}\cos\left(\frac{2n-1}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{2}{2n+1} - \frac{2}{2n-1}\right] = -\frac{4\sqrt{2}}{\pi(4n^2-1)}$$

$[\because\ \cos(2n+1)\pi = -1$ and $\cos(2n-1)\pi = ...$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\sin nx\,dx$$

$$= \frac{-\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{1}{2}+n\right)x - \cos\left(\frac{1}{2}-n\right)x\right]dx$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{2}{1+2n}\sin\left(\frac{1+2n}{2}\right)x - \frac{2}{1-2n}\sin\left(\frac{1-2n}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{1}{2n+1}\sin\left(\frac{2n+1}{2}\right)x - \frac{1}{2n-1}\sin\left(\frac{2n-1}{2}\right)x\right]_0^{2\pi}$$

$$= 0$$

$\therefore$ Putting these valued in (1)

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cdot\cos nx$$

Putting $x = 0$, $\dfrac{1}{2} = \sum_{1}^{\infty}\dfrac{1}{4n^2-1}$.

Example 6 : Find the Fourier expansion of $\cos px$ where p is not an integer in $(0, 2\pi)$. He... deduce that

$$\pi\,\mathrm{cosec}\,\pi x = \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right] \qquad \text{(M.U. 2...}$$

Also deduce that $\pi\cot 2\pi p = \dfrac{1}{2p} + p\sum_{n=1}^{\infty}\dfrac{1}{p^2-n^2}$.

Sol. : Let $\cos px = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ....

$$\therefore\ a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\cos px\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin px}{p}\right]_0^{2\pi} = \frac{1}{2\pi}\cdot\frac{\sin 2p\pi}{p} = \frac{\sin 2p\pi}{2\pi p}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos px\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2\cos px\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}[\cos(p+n)x + \cos(p-n)x]\,dx$$

$$\therefore\ a_n = \frac{1}{2\pi}\left[\frac{\sin(p+n)x}{p+n} + \frac{\sin(p-n)x}{p-n}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{\sin 2\pi(p+n)}{p+n} + \frac{\sin 2\pi(p-n)}{p-n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin 2p\pi}{p+n} + \frac{\sin 2p\pi}{p-n}\right] = \frac{1}{2\pi}\cdot\sin 2p\pi\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$$

$$= \frac{p}{\pi}\cdot\frac{\sin 2p\pi}{p^2-n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos px\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2\sin nx\cos px\,dx = \frac{1}{2\pi}\int_0^{2\pi}[\sin(n+p)x + \sin(n-p)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+p)x}{n+p} - \frac{\cos(n-p)x}{n-p}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2\pi(n+p)}{n+p} - \frac{\cos 2\pi(n-p)}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2p\pi}{n+p} - \frac{\cos 2p\pi}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-\cos 2p\pi}{n+p} - \frac{1-\cos 2p\pi}{n-p}\right]$$

$$= \frac{1-\cos 2p\pi}{2\pi}\left[\frac{1}{n+p} + \frac{1}{n-p}\right] = \frac{1-\cos 2p\pi}{2\pi}\cdot\frac{2n}{n^2-p^2}$$

$$\therefore\ b_n = -\frac{n}{\pi}\frac{(1-\cos 2p\pi)}{p^2-n^2}$$

Putting these values in (1), we get,

$$\cos px = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{p^2-n^2} \qquad \text{......... (2)}$$

To deduce the first result, put $x = \pi$ in (2).

$$\therefore\ \cos p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin n\pi}{p^2-n^2}$$

But since, n is a positive integer, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$.

$$\therefore\ \cos p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$= \frac{2\sin p\pi\cos p\pi}{2p\pi} + \frac{2p\sin p\pi\cos p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$\therefore\ \pi\,\mathrm{cosec}\,p\pi = \frac{1}{p} + 2p\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$= \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right] \qquad \text{[By partial fractions]}$$

To deduce the second result put $x = 2\pi$ in (2).

$$\therefore \quad \cos 2p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p \sin 2p\pi}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\pi}{p^2 - n^2} - \frac{(1 - \cos 2p\pi)}{\pi} \sum_{n=1}^{\infty} \frac{n \sin 2n\pi}{p^2 - n^2}$$

But, since, n is a positive integer, $\sin 2n\pi = 0$ and $\cos 2n\pi = 1$.

$$\therefore \quad \cos 2p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p \sin 2p\pi}{\pi} \sum_{n=1}^{\infty} \frac{1}{(p^2 - n^2)}$$

Dividing by $\sin 2p\pi$ throughout,

$$\therefore \quad \pi \cot 2p\pi = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{(p^2 - n^2)}$$

Example 7 : Find the Fourier series for $f(x) = \dfrac{3x^2 - 6x\pi + 2\pi^2}{12}$ in $(0, 2\pi)$.

Hence, deduce that $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\ldots$ **(M.U. 2001, 04, 07, 11, 1**

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cos nx$

$$\therefore \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi} \int_0^{2\pi} \frac{(3x^2 - 6x\pi + 2\pi^2)}{12}\, dx$$

$$= \frac{1}{24\pi} \left[x^3 - 3x^2\pi + 2\pi^2 x \right]_0^{2\pi} = \frac{1}{24\pi} \left[8\pi^3 - 12\pi^3 + 4\pi^3 \right] = 0$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{3x^2 - 6x\pi + 2\pi^2}{12} \right) \cos nx\, dx$$

$$= \frac{1}{12\pi} \left[3\left\{ x^2 \left(\frac{\sin nx}{n} \right) - (2x)\left(-\frac{\cos nx}{n^2} \right) + (2)\left(-\frac{\sin nx}{n^3} \right) \right\} \right.$$

$$\left. - 6\pi \left\{ x\left(\frac{\sin nx}{n} \right) - (1)\left(-\frac{\cos nx}{n^2} \right) \right\} + 2\pi^2 \left(\frac{\sin nx}{n} \right) \right]_0^{2\pi}$$

$$= \frac{1}{12\pi} \left[3\left\{ 0 + \frac{4\pi}{n^2} + 0 \right\} - 6\pi \left\{ 0 + \frac{1}{n^2} \right\} + 2\pi^2(0) - \left\{ 0 - 6\pi \left(\frac{1}{n^2} \right) \right\} \right]$$

$$= \frac{1}{n^2}$$

[By § 5, page 3-10]

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{3x^2 - 6x\pi + 2\pi^2}{12} \right) \sin nx\, dx$$

$$= \frac{1}{12\pi} \left[3\left\{ x^2 \left(-\frac{\cos nx}{n} \right) - (2x)\left(-\frac{\sin nx}{n^2} \right) + 2(1)\left(\frac{\cos nx}{n^3} \right) \right\} \right.$$

$$\left. - 6\pi \left\{ x\left(-\frac{\cos nx}{n} \right) - (1)\left(-\frac{\sin nx}{n^2} \right) \right\} + 2\pi^2 \left(-\frac{\cos nx}{n} \right) \right]_0^{2\pi}$$

[By § 5, page 3-10]

$$\therefore \quad b_n = \frac{1}{12\pi} \left[3\left\{ -\frac{4\pi^2}{n} + \frac{2}{n^3} \right\} - 6\pi \left\{ -\frac{2\pi}{n} \right\} - \frac{2\pi^2}{n} - \left\{ 3\left(\frac{2}{n^3} \right) - \frac{2\pi^2}{n} \right\} \right]$$

$$= 0$$

$$\therefore \quad f(x) = \sum a_n \cos nx = \sum \frac{1}{n^2} \cos nx$$

$$\therefore \quad \frac{3x^2 - 6x\pi + 2\pi^2}{12} = \frac{1}{1^2} \cos x + \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x + \ldots\ldots$$

Putting $x = 0$, we get,

$$\frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots \qquad \therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$$

Example 8 : Find Fourier Series for $f(x)$ in $(0, 2\pi)$ where

$$f(x) = \begin{cases} x, & 0 < x \le \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$$ **(M.U. 2004, 06, 11, 16)**

Hence, deduce that $\dfrac{\pi^2}{96} = \dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \ldots\ldots$

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} x\, dx + \frac{1}{2\pi} \int_{\pi}^{2\pi} (2\pi - x)\, dx = \frac{1}{2\pi} \left[\frac{x^2}{2} \right]_0^{\pi} + \frac{1}{2\pi} \left[2\pi x - \frac{x^2}{2} \right]_{\pi}^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{\pi^2}{2} \right] + \frac{1}{2\pi} \left[4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2} \right]$$

$$\therefore \quad a_0 = \frac{1}{2\pi} \cdot \pi^2 = \frac{\pi}{2} \qquad \ldots\ldots\ldots\ldots \text{(A)}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx\, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} (2\pi - x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[x\left(\frac{\sin nx}{n} \right) - \left(-\frac{\cos nx}{n^2} \right) \right]_0^{\pi} + \frac{1}{\pi} \left[(2\pi - x)\left(\frac{\sin nx}{n} \right) - (-1)\left(-\frac{\cos nx}{n^2} \right) \right]_{\pi}^{2\pi}$$

$$\therefore \quad a_n = \frac{1}{\pi \cdot n^2} \left[(-1)^n - 1 \right] - \frac{1}{\pi \cdot n^2} \left[1 - (-1)^n \right] = \frac{-2}{\pi \cdot n^2} \left[1 - (-1)^n \right]$$ [By § 5, page 3-10]

$$\therefore \quad a_1 = -\frac{4}{1^2}, \quad a_2 = 0, \quad a_3 = -\frac{4}{3^2}, \quad a_4 = 0, \quad a_5 = -\frac{4}{5^2}, \ldots\ldots \qquad \ldots\ldots\ldots\ldots \text{(B)}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin nx\, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} (2\pi - x) \sin nx\, dx$$

$$= \frac{1}{\pi} \left[x\left(-\frac{\cos nx}{n} \right) - \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi} + \frac{1}{\pi} \left[(2\pi - x)\left(-\frac{\cos nx}{n} \right) - (-1)\left(-\frac{\sin nx}{n^2} \right) \right]_{\pi}^{2\pi}$$

[By § 5, page 3-10]

$$\therefore \quad b_n = \frac{-1}{\pi n} \left[\pi (-1)^n \right] + \frac{1}{\pi n} \left[\pi (-1)^n \right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \right] = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x$$

To deduce the required result we use the Parseval's identity

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2dx = a_0^2+\frac{1}{2}\sum\left(a_n^2+b_n^2\right)$$

Now, $$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2dx = \frac{1}{2\pi}\left[\int_0^{\pi}x^2dx+\int_{\pi}^{2\pi}(2\pi-x)^2dx\right]$$

$$=\frac{1}{2\pi}\left[\int_0^{\pi}x^2dx+\int_{\pi}^{2\pi}(4\pi^2-4\pi x+x^2)\,dx\right]$$

$$=\frac{2}{2\pi}\left\{\left[\frac{x^3}{3}\right]_0^{\pi}+\left[4\pi^2x-2\pi x^2+\frac{x^3}{3}\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{2\pi}\left\{\frac{\pi^3}{3}+8\pi^3-8\pi^3+\frac{8\pi^3}{3}-4\pi^3+2\pi^3-\frac{\pi^3}{3}\right\}$$

$$=\frac{1}{2\pi}\cdot\frac{2\pi^3}{3}=\frac{\pi^2}{3}$$

Hence, by (1), using (A) and (B), we get by using the values of a_0, a_n and b_n.

$$\frac{\pi^2}{3}=\frac{\pi^2}{4}+\frac{1}{2}\cdot\frac{16}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right]$$

$$\frac{\pi^2}{12}=\frac{8}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right]$$

$$\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots$$

EXERCISE - I

Find the Fourier Series for the following functions.

1. $f(x)=e^{\alpha x}$, $(\alpha\neq 0)$ in $(0, 2\pi)$. $\left[\textbf{Ans. : } f(x)=\frac{1}{\pi}(e^{2\alpha\pi}-1)\left[\frac{1}{2\alpha}+\sum_{n=1}^{\infty}\frac{\cos nx-n\sin nx}{\alpha^2+n^2}\right.\right.$

2. $f(x)=e^{x}$ in $(0, 2\pi)$. $\left[\textbf{Ans. : } f(x)=\frac{1}{\pi}\cdot(e^{2\pi}-1)\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\cos nx-n\sin nx}{1+n^2}\right.\right.$

3. $f(x)=x$ in $(0, 2\pi)$. **(M.U. 2003, 12)** $\left[\textbf{Ans. : } f(x)=\pi-2\sum_{n=1}^{\infty}\frac{\sin nx}{n}\right.$

4. $f(x)=\frac{1}{2}(\pi-x)$ in $(0, 2\pi)$. Hence, deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

(M.U. 2007, 11, 15) $\left[\textbf{Ans. : } f(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n}.\right.$ Then put $x=\frac{\pi}{2}$

5. $f(x)=\begin{cases}\sin x, & 0\le x\le\pi\\ 0, & \pi\le x\le 2\pi\end{cases}$

Hence, deduce that $\frac{1}{2}=\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\ldots$ or $1=\sum_{n=1}^{\infty}\frac{2}{(2n-1)(2n+1)}$.

$\left[\textbf{Ans. : } f(x)=\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx+\frac{1}{2}\sin x\right]$

6. $f(x)=\begin{cases}1, & 0\le x\le\pi\\ 2-\frac{\pi}{x}, & \pi\le x\le 2\pi\end{cases}$ **(M.U. 2010)**

$\left[\textbf{Ans. : } f(x)=\frac{3}{4}-\frac{2}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ldots\right]+\frac{1}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots\right]\right]$

7. $f(x)=\begin{cases}mx, & 0\le x\le\pi\\ 2m\pi-mx, & \pi\le x\le 2\pi\end{cases}$

$\left[\textbf{Ans. : } f(x)=\frac{m\pi}{2}-\frac{4m}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right]\right]$

8. $f(x)=\begin{cases}a, & 0<x<\pi\\ -a, & \pi<x<2\pi\end{cases}$. Hence, deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

$\left[\textbf{Ans. : } f(x)=\frac{4a}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right]\right]$

(For deduction put $x=\pi/2$).

9. $f(x)=\begin{cases}x, & 0\le x\le\pi\\ 2\pi-x, & \pi\le x\le 2\pi\end{cases}$ $\left[\textbf{Ans. : } f(x)=\frac{\pi}{2}-\frac{4}{5}\left[\frac{\cos x}{1}+\frac{\cos 3x}{3}+\frac{\cos 5x}{5}+\ldots\right]\right]$

10. $f(x)=\begin{cases}(x-\pi)^2, & 0<x<\pi\\ \pi^2, & \pi<x<2\pi\end{cases}$

$\left[\textbf{Ans. : } f(x)=\frac{2}{3}\pi^2+2\left[\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\right]\right.$

$\left.-2\left[\left(\pi+\frac{2}{\pi}\right)\cos x+\left(\frac{\pi}{3}+\frac{2}{9\pi}\right)\cos 3x+\ldots\right]\right]$

11. $f(x)=\begin{cases}1, & 0<x<\pi\\ 2, & \pi<x<2\pi\end{cases}$. Hence, deduce that, $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

$\left[\textbf{Ans. : } f(x)=\frac{3}{2}-\frac{2}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right)\right]$

(For deduction put $x=\pi/2$).

12. $f(x)=\begin{cases}-\pi, & 0<x<\pi\\ x-\pi, & \pi<x<2\pi\end{cases}$

State the value of the series at $x=\pi$ and hence, show that $\frac{\pi^2}{8}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$.

$$\left[\text{Ans. : } f(x) = -\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] + \left[-3\sin x - \frac{1}{2}\sin 2x - \sin 3x + \ldots\right.\right.$$

$$\left. f(\pi)\,\frac{-\pi+0}{2} = -\frac{\pi}{2}.\ \text{Then put } x = \pi. \right]$$

13. If $f(x) = 2x,\ 0 \le x \le 2\pi$, find the Fourier series of $f(x)$. Also find a_4 and b_{10}. (M.U. 20

$$\left[\text{Ans. : } f(x) = 2\pi - 4\sum_{n=1}^{\infty}\frac{\sin nx}{n};\ a_4 = 0,\ b_{10} = -0\right.$$

7. Fourier Expansion of $f(x)$ in the Interval $(-\pi, \pi)$

Example 1 : Obtain the Fourier expansion of e^x in $-\pi < x < \pi$.

Sol. : Let $e^x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{2\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[e^{\pi} - e^{-\pi}\right] = \frac{1}{\pi}\sinh\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{1+n^2}\cdot e^x(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}\cdot e^{ax}(a\cos bx + b\sin bx)\right]$$

$$\therefore\ a_n = \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right] = \frac{2\cos n\pi}{\pi(1+n^2)}\cdot\frac{e^{\pi} - e^{-\pi}}{2}$$

$$= \frac{2\cos n\pi}{\pi(1+n^2)}\cdot\sinh\pi = \frac{(-1)^n}{\pi(1+n^2)}\cdot 2\sinh\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin x\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{1+n^2}\cdot e^x(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}\cdot e^{ax}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[-e^{\pi}\cdot n\cos n\pi + ne^{-\pi}\cos n\pi\right]$$

$$= \frac{-2\cos n\pi}{\pi(1+n^2)}\left[\frac{e^{\pi} - e^{-\pi}}{2}\right] = -\frac{2n\cos n\pi}{\pi(1+n^2)}\cdot\sinh\pi$$

$$\therefore\ b_n = -(-1)^n\cdot\frac{n}{\pi(1+n^2)}\cdot 2\sinh\pi$$

$$\therefore\ e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + (-1)^n\cdot\sum_{n=1}^{\infty}\frac{1}{(1+n^2)}\cos nx - (-1)^n\cdot\sum_{n=1}^{\infty}\frac{n}{(1+n^2)}\sin nx\right]$$

$$= \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2}\cdot\cos x + \frac{1}{5}\cdot\cos 2x - \frac{1}{10}\cdot\cos 3x + \ldots + \frac{1}{2}\cdot\sin x - \frac{2}{5}\cdot\sin 2x + \frac{3}{10}\cdot\sin 3x - \ldots\right]$$

Example 2 : Find the Fourier Series for the periodic function

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

State the value of $f(x)$ at $x = 0$ and hence, deduce that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$ (M.U. 1996, 2003, 10)

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ....(1)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\pi\cdot dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\pi\left\{x\right\}_{-\pi}^{0} + \left\{\frac{x^2}{2}\right\}_0^{\pi}\right] = \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}\right]$$ [By § 5, page 3-10]

$$= \frac{1}{\pi}\left[-\pi(0-0) + \left\{\pi(0) + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$ $[\because \cos n\pi = (-1)^n]$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\frac{(1-\cos n\pi)}{n} + \left\{\pi\left(-\frac{\cos n\pi}{n}\right) - 0\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right]$$

$$= \frac{1}{n}\left[1 - 2\cos n\pi\right] = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Putting these values in (1),

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1-2(-1)^n}{n}\sin nx$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

$$+\left[3\cdot\frac{\sin x}{1} - \frac{1}{2}\sin 2x + \sin 3x - \ldots\right] \quad \ldots\ldots$$

Now, $f(x)$ is discontinuous at $x = 0$. At a point of discontinuity $x = c$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right] \qquad \therefore \; f(0) = \frac{1}{2}[-\pi + 0] = -\frac{\pi}{2}$$

Hence, putting $x = 0$ in (2),

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\therefore \; \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Example 3 : Find the Fourier Series for $f(x) = \begin{cases}\cos x, & -\pi < x < 0\\ \sin x, & 0 < x < \pi\end{cases}$. (M.U. 201...)

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}\cos x\,dx + \int_0^{\pi}\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\{\sin x\}_{-\pi}^{0} + \{-\cos x\}_0^{\pi}\right] = \frac{1}{2\pi}[(0) + (1+1)] = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\cos nx\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 2\cos nx\cos x\,dx + \int_2^{\pi} 2\cos nx\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}[\cos(n+1)x + \cos(n-1)x]\,dx + \int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{-\pi}^{0} + \left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\{0\} + \left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right\}\right] \text{ if } n \neq 1$$

$$\therefore \; a_n = \begin{cases} 0 & \text{if } n \text{ is odd and} \neq 1\\ -\dfrac{2}{\pi(n^2-1)} & \text{if } n \text{ is even}\end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\cos x\,dx + \int_0^{\pi}\sin x\cos x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(\frac{1+\cos 2x}{2}\right)dx + \int_0^{\pi}\frac{\sin 2x}{2}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_{-\pi}^{0} + \left(-\frac{\cos 2x}{4}\right)_0^{\pi}\right]$$

$$\therefore \; a_1 = \frac{1}{\pi}\left[\frac{1}{2}\left\{\left(0 - (-\pi - 0)\right) - \left(\frac{1}{4} - \frac{1}{4}\right)\right\}\right] = \frac{1}{2}$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\sin nx\,dx + \int_0^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 2\sin nx\cos x\,dx + \int_0^{\pi} 2\sin nx\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}[\sin(n+1)x + \sin(n-1)x]\,dx - \int_0^{\pi}[\cos(n+1)x - \cos(n-1)x]\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right\}_{-\pi}^{0} - \left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right\}_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{1}{n+1} - \frac{1}{n-1} + \frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - 0\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1+\cos n\pi}{n+1} - \frac{1+\cos n\pi}{n-1}\right] = -\frac{n}{\pi}\frac{(1+\cos n\pi)}{n^2-1} \text{ if } n \neq 1$$

$$\therefore \; b_n = \begin{cases} 0 & \text{if } n \text{ is odd and} \neq 1\\ -\dfrac{2n}{\pi(n^2-1)} & \text{if } n \text{ is even}\end{cases}$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\sin x\,dx + \int_0^{\pi}\sin^2 x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{\sin 2x}{2}\,dx + \int_0^{\pi}\frac{1-\cos 2x}{2}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos 2x}{2}\right\}_{-\pi}^{0} + \frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\}_0^{\pi}\right]$$

$$\therefore \; b_1 = \frac{1}{\pi}\left[-\left(\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{2}(\pi - 0)\right] = \frac{1}{2}$$

Putting these values in

$$f(x) = a_0 + a_1 \cos x + [a_2 \cos 2x + a_4 \cos 4x + \ldots.]$$
$$+ b_1 \sin x + [b_2 \sin 2x + b_4 \sin 4x + \ldots.]$$

$$= \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \ldots.\right]$$
$$+ \frac{1}{2}\sin x - \frac{2}{\pi}\left[\frac{2\sin 2x}{3} + \frac{4\sin 4x}{15} + \ldots.\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}(\cos x + \sin x) + \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)}\cos 2nx + \sum_{n=1}^{\infty}\frac{2n}{(1-4n^2)}\sin 2nx\right]$$

EXERCISE - II

Find Fourier expansion of

1. $f(x) = e^{ax}$ in $(-\pi, \pi)$. $\left[\textbf{Ans. :}\ e^{ax} = \frac{\sin h\, a\pi}{a\pi} + \frac{2 \sin h\, a\pi}{\pi}\sum\frac{(-1)^n}{a^2+n^2}(a\cos nx - n \sin nx\right.$

2. $f(x) = e^{-ax}$ in $(-\pi, \pi)$ and hence, show that $\frac{\pi}{\sin h\, a\pi} = \frac{1}{a} + \sum\frac{(-1)^n \cdot 2a}{n^2+a^2}$.

$$\left[\textbf{Ans. :}\ f(x) = \frac{2\sin h\, a\pi}{\pi}\left[\frac{1}{2a} - \frac{a\cos x}{a^2+1} + \frac{a\cos 2x}{a^2+2^2} - \ldots.\right.\right.$$
$$\left.\left.+\frac{\sin x}{a^2+1} - \frac{\sin 2x}{a^2+2^2} + \frac{\sin 3x}{a^2+3^2} - \ldots\right]\right]$$

3. $f(x) = \begin{cases} 1/2, & -\pi < x < 0 \\ x/\pi, & 0 < x < \pi \end{cases}$. Hence deduce that $\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. (M.U. 200

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2nx}{n}\right.$$

4. $f(x) = \begin{cases} x+\pi, & 0 \le x \le \pi \\ -x-\pi, & -\pi \le x \le 0 \end{cases}$ (M.U. 200

$$\left[\textbf{Ans. :}\ f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi}{(2n-1)^2} + 4\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}\right.$$

5. $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$ and $f(x + 2\pi) = f(x)$.

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.;$ $\quad \frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

$$\left[\textbf{Ans. :}\ f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cdot\cos(2n+1)x - \sum\frac{(-1)^n}{n}\sin nx\right.$$

6. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

Hence, deduce that (i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots. = \frac{1}{2}$

(ii) $\frac{1}{4}(\pi - 2) = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots.$ (M.U. 2000, 03, 04, 09, 11)

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\left[\frac{\cos 2x}{4\cdot 1^2 - 1} + \frac{\cos 4x}{4\cdot 2^2 - 1} + \ldots.\right]\right]$$

7. $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$

$$\left[\textbf{Ans. :}\ f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]\right.$$
$$\left.-\left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right]\right]$$

8. $f(x) = \begin{cases} x-\pi, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$ (M.U. 2003)

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$

$$\left[\textbf{Ans. :}\ f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\sin 5x + \ldots\right]\right.$$
$$\left.+4\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]\right]$$

9. $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$ (M.U. 2009, 14)

$$\left[\textbf{Ans. :}\ f(x) = \frac{\pi^2}{6} - 2\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right]\right.$$
$$+\frac{2}{\pi}\left[\frac{1}{1^3}\sin x - \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x - \ldots\right]$$
$$+\pi\left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right]$$
$$\left.-\frac{1}{\pi}\left[\frac{1}{1^3}\sin x + \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x + \ldots\right]\right]$$

8. Even and Odd Functions in $(-\pi, \pi)$

(i) **Even function in $(-\pi, \pi)$:** If $f(x)$ is even in the interval $(-\pi, \pi)$, then,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{2\pi}\int_0^{\pi} f(x)\,dx \qquad [\because f(x) \text{ is even}]$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad [\because f(x)\cos nx \text{ is even}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \qquad [\because f(x)\sin nx \text{ is odd}]$$

(ii) **Odd function in $(-\pi, \pi)$:** If $f(x)$ is odd in the interval $(-\pi, \pi)$, then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \qquad [\because f(x) \text{ is odd}]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \qquad [\because f(x)\cos nx \text{ is odd}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad [\because f(x)\sin nx \text{ is even}]$$

Thus, we have,

If $f(x)$ is even in $(-\pi, \pi)$

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad b_n = 0}$$

If $f(x)$ is odd in $(-\pi, \pi)$

$$\boxed{a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx}$$

Example 1 : Find the Fourier series of $f(x) = e^{-|x|}$ in the interval $(-\pi, \pi)$.

Sol. : We have $f(x) = e^{-|x|}$.

$$\therefore\ f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$$

Hence, $f(x)$ is an even function and hence $b_n = 0$.

Further, $f(x) = \begin{cases} e^{x}, & -\pi < x < 0 \\ e^{-x}, & 0 < x < \pi \end{cases}$

Now, using (A) above,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \text{ and } b_n = 0.$$

$$\therefore\ a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-x}\,dx = \frac{1}{\pi}\left[-e^{-x}\right]_0^{\pi} = \frac{1}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{\pi} \qquad \text{[By (vii) (3), page 3-4]}$$

$$= \frac{2}{\pi(1+n^2)}\left[-e^{-\pi}\cos n\pi - (-1)\right]$$

$$\therefore\ a_n = \frac{2}{\pi(1+n^2)}\left[1 - (-1)^n e^{-\pi}\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$\therefore\ f(x) = \frac{1}{\pi}(1 - e^{-\pi}) + \frac{2}{\pi}\sum_1^{\infty}\left[\frac{1-(-1)^n e^{-\pi}}{1+n^2}\right]\cdot\cos nx$$

Example 2 : Find the Fourier expansion of $f(x) = x^2$, $-\pi \le x \le \pi$, and hence, prove that

(i) $\dfrac{\pi^2}{6} = \sum_1^{\infty}\dfrac{1}{n^2}$, (ii) $\dfrac{\pi^2}{12} = \sum_1^{\infty}\dfrac{(-1)^{n+1}}{n^2}$, (iii) $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

(M.U. 2011, 13, 14)

Sol. : Here $f(x)$ is an even function because $f(-x) = (-x)^2 = x^2 = f(x)$ $[\therefore b_n = 0]$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ .... (1)

$$\therefore\ a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 - (2\pi)\left(-\frac{\cos n\pi}{n^2}\right) + 0\right\} - \{0\}\right] = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

Putting these values in (1),

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \qquad \text{..................... (2)}$$

(i) For the first deduction put $x = \pi$ in (2),

$$\therefore\ \pi^2 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cos n\pi$$

$$\pi - \frac{\pi^2}{3} = 4\sum\frac{(-1)^n(-1)^n}{n^2} = 4\sum\frac{(-1)^{2n}}{n^2}$$

$$\therefore\ \frac{2\pi^2}{3}\cdot\frac{1}{4} = \sum\frac{(-1)^{2n}}{n^2} \qquad \therefore\ \frac{\pi^2}{6} = \sum\frac{1}{n^2}$$

(ii) For the second deduction put $x = 0$ in (1),

$$\therefore\ 0 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cdot 1 \qquad \therefore\ -\frac{\pi^2}{3} = 4\sum\frac{(-1)^n}{n^2}$$

$$\therefore\ \frac{\pi^2}{12} = (-1)\sum\frac{(-1)^n}{n^2} \qquad \therefore\ \frac{\pi^2}{12} = \sum\frac{(-1)^{n+1}}{n^2}$$

(iii) For the third deduction, add the above two results.

$$\therefore\ \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots; \qquad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$$

$$\therefore\ \frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \qquad \therefore\ \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Cor. Assuming the validity of term by term differentiation of the series for x^2, find the series for x and hence, deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Sol. : We have

$$x^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\sin 3x + \ldots\ldots\ldots\right]$$

Differentiating w.r.t. x

$$2x = 4\left[\frac{1}{1^2}\sin x - \frac{2}{2^2}\sin 2x + \frac{3}{3^2}\sin 3x - \ldots\ldots\ldots\right]$$

$$\therefore \quad x = 2\left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\ldots\ldots\right]$$

Putting $x = \frac{\pi}{2}$, we get, $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots$

Example 3 : It is given that for $-\pi < x < \pi$,

$$x^2 = \frac{\pi^2}{3} + 4\sum (-1)^n \cdot \frac{\cos nx}{n^2}.$$

By using Parseval's identiyly prove that $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$. (M.U. 2004)

Sol. : By Pareseval's identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = a_0{}^2 + \frac{1}{2}\sum (a_n{}^2 + b_n{}^2)$$

Here, $f(x) = x^2$, $a_0 = \frac{\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$, $b_n = 0$.

$$\therefore \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\, dx = \frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{\pi^4}{5}$$

$$\therefore \quad \frac{\pi^4}{5} = \frac{\pi^4}{9} + \frac{1}{2}\cdot 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \ldots\right]$$

$$\therefore \quad \pi^4\left(\frac{1}{5} - \frac{1}{9}\right) = 8\left[\frac{1}{1^4} + \frac{1}{2^4} + \ldots\right]$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$$

Example 4 : Obtain Fourier expansion of $f(x) = |\cos x|$ in $(-\pi, \pi)$. (M.U. 20..)

Sol. : Here $f(x)$ is an even function because $f(-x) = |\cos(-x)| = f(x)$. $[\therefore b_n = 0]$

Let $\quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ ....

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi}|\cos x|\, dx = \frac{1}{\pi}\left[\int_0^{\pi/2}\cos x\, dx - \int_{\pi/2}^{\pi}\cos x \cdot dx\right]$$

$[\because |\cos x| = \cos x$, for $0 < x < \pi/2$

and $|\cos x| = -\cos x$, for $\pi/2 < x < \pi$.$]$

$$= \frac{1}{\pi}\left[\{\sin x\}_0^{\pi/2} - \{\sin x\}_{\pi/2}^{\pi}\right]$$

$$\therefore \quad a_0 = \frac{1}{\pi}[(1-0) - \{0-1\}] = \frac{2}{\pi} \qquad \ldots\ldots\ldots (2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos nx\, dx - \int_{\pi/2}^{\pi}\cos x\cos nx\, dx\right] \qquad \ldots\ldots\ldots (3)$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos nx\cos x\, dx - \int_{\pi/2}^{\pi} 2\cos nx\cos x\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\{\cos(n+1)x + \cos(n-1)x\}\, dx - \int_{\pi/2}^{\pi}\{\cos(n+1)x + \cos(n-1)x\}\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_0^{\pi/2} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - 0\right\} - \left\{0 - \frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n(\pi/2)}{n+1} - \frac{\cos n(\pi/2)}{n-1}\right]$$

$$\therefore \quad a_n = \begin{cases} -\dfrac{4}{\pi(n^2-1)}\cos\left(\dfrac{n\pi}{2}\right) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd and } n \neq 1 \end{cases} \qquad \ldots\ldots\ldots (4)$$

To find a_1, we put $n = 1$ in (3)

$$\therefore \quad a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos^2 x\, dx - \int_{\pi/2}^{\pi}\cos^2 x\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x + \frac{\sin 2x}{2}\right\}_0^{\pi/2} - \left\{x + \frac{\sin 2x}{2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi}{2}\right\} - \left\{\pi - \frac{\pi}{2}\right\}\right] = 0 \qquad \ldots\ldots\ldots (5)$$

Putting these values from (2) and (4) in (1), we get,

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots\right]$$

Example 5 : Obtain Fourier Series for the function $f(x) = \begin{cases} 1+\dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

Deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ **(M.U. 1995, 97, 2001, 05, 07, 08)**

Sol. : Here $f(x)$ is an even function because

$$f(-x) = \begin{cases} 1-\dfrac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1+\dfrac{2x}{\pi}, & 0 \le -x \le \pi \end{cases} = \begin{cases} 1-\dfrac{2x}{\pi}, & \pi \ge x \ge 0 \quad i.e., \ 0 \le x \le \pi \\ 1+\dfrac{2x}{\pi}, & 0 \ge x \ge -\pi \quad i.e., \ -\pi \le x \le 0 \end{cases}$$

$$= f(x) \quad [\therefore b_n = 0]$$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$\therefore \ a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cdot dx = \frac{1}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = \frac{1}{\pi}[\pi - \pi] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{\pi}\left[\left\{0 - \frac{2\cos n\pi}{\pi n^2}\right\} - \left\{-\frac{2}{\pi n^2}\right\}\right] = \frac{4}{\pi^2 n^2}[1-\cos n\pi]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8}{\pi^2 n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \ f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

Now by putting $x = 0$ in the above result, we can get, $[\because f(0) = 1]$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 6 : Obtain the Fourier expansion of

$$f(x) = \begin{cases} \cos x, & -\pi < x < 0 \\ -\cos x, & 0 < x < \pi \end{cases} \text{ and } f(x+2\pi) = f(x).$$

Sol. : Here $f(x)$ is an odd function because

$$f(-x) = \begin{cases} \cos(-x), & -\pi < -x < 0 \\ -\cos(-x), & 0 < -x < \pi \end{cases} = \begin{cases} \cos x, & \pi > x > 0 \quad i.e., \ 0 < x < \pi \\ -\cos x, & 0 > x > -\pi \quad i.e., \ -\pi < x < 0 \end{cases}$$

$$= -f(x) \quad [\therefore a_n = 0]$$

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

$$\text{Now } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin nx\, dx \qquad \text{............... (2)}$$

$$= -\frac{1}{\pi}\int_0^{\pi} 2\sin nx\cos x\, dx = -\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(n-1)x]\, dx$$

$$= -\frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= -\frac{1}{\pi}\left[\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1}\right]$$

$$= -\frac{2n}{\pi}\,\frac{(1+\cos n\pi)}{n^2-1} \quad \text{if } n \ne 1$$

$$\therefore \ b_n = \begin{cases} 0 & \text{if } n \text{ is odd and } \ne 1 \\ -\dfrac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$$

Now putting $n = 1$ in (2)

$$b_1 = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin x\, dx = -\frac{1}{\pi}\int_0^{\pi}\sin 2x\, dx = -\frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0$$

Putting these values in (1),

$$\therefore \ f(x) = -\frac{4}{\pi}\left[\frac{2}{3}\sin 2x + \frac{4}{15}\sin 4x + \frac{6}{35}\sin 6x + \ldots\right] = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n}{4n^2-1}\sin 2nx$$

Example 7 : Obtain Fourier series for

$$f(x) = \begin{cases} x + \dfrac{\pi}{2}, & -\pi < x < 0 \\ \dfrac{\pi}{2} - x, & 0 < x < \pi \end{cases}$$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ **(M.U. 2004, 08, 09, 12)**

Also deduce that $\dfrac{\pi^4}{96} = \dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \ldots$ **(M.U. 1999, 2002, 15)**

Sol. : Clearly $f(x)$ is an even function

$$f(-x) = \begin{cases} -x + \dfrac{\pi}{2}, & -\pi < -x < 0 \\ \dfrac{\pi}{2} + x, & 0 < -x < \pi \end{cases} = \begin{cases} \dfrac{\pi}{2} - x, & \pi > x > 0 \quad i.e., \ 0 < x < \pi \\ \dfrac{\pi}{2} + x, & 0 < x < -\pi \quad i.e., \ -\pi < x < 0 \end{cases}$$

$$= f(x) \quad [\therefore b_n = 0]$$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^\pi \left(\frac{\pi}{2} - x\right) dx = \frac{1}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^\pi = 0$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi \left(\frac{\pi}{2} - x\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - (-1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi \qquad \text{[By § 5, page 3-10]}$$

$$\therefore\ a_n = \frac{2}{\pi}\left[-(-1)^n\cdot\frac{1}{n^2} + \frac{1}{n^2}\right] = \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$\therefore\ f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$

(i) Now $f(x)$ is discontinuous at $x = 0$.

But at a point of discontinuity $x = c$.

$$f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right] \qquad \therefore\ f(0) = \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{\pi}{2}$$

$\therefore$ Putting $x = 0$, in (2),

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] \qquad \therefore\ \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

(ii) We now use Parseval's identity in $(-\pi, \pi)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2)$$

$$\therefore\ \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^0\left(x + \frac{\pi}{2}\right)^2 dx + \int_0^\pi\left(\frac{\pi}{2} - x\right)^2 dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^0\left(x^2 + \pi x + \frac{\pi^2}{4}\right)dx + \int_0^\pi\left(\frac{\pi^2}{4} - \pi x + x^2\right)dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{x^3}{3} + \frac{\pi x^2}{2} + \frac{\pi^2 x}{4}\right\}_{-\pi}^0 + \left\{\frac{\pi^2 x}{4} - \frac{\pi x^2}{2} + \frac{x^3}{3}\right\}_0^\pi\right]$$

$$= \frac{1}{2\pi}\left[0 - \left\{-\frac{\pi^3}{3} + \frac{\pi^3}{2} - \frac{\pi^3}{4}\right\} + \left\{\frac{\pi^3}{4} - \frac{\pi^3}{2} + \frac{\pi^3}{3}\right\} - 0\right]$$

$$= \frac{2\pi^3}{2\pi}\left[\frac{1}{4} - \frac{1}{2} + \frac{1}{3}\right] = \frac{\pi^2}{12}$$

Hence, from (1), (2) and (3),

$$\frac{\pi^2}{12} = \frac{1}{2}\cdot\frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$$

Example 8 : Prove that $\sin h\,ax = \frac{2}{\pi}\sin h\,a\pi\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{n^2 + a^2}\sin nx\right]$ (M.U. 1998)

Sol. : Here $f(x)$ is an odd function because

$$f(-x) = \sin h(-ax) = \frac{e^{-ax} - e^{ax}}{2}$$

$$= -\frac{e^{ax} - e^{-ax}}{2} = -\sin h\,ax = -f(x) \qquad [\therefore a_n = 0]$$

$\therefore$ Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore\ b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi \sin h\,ax \sin nx\,dx$$

To find b_n, we use the result

$\cos(n - ia)\,x = \cos nx \cos i\,ax + \sin nx \sin i\,ax$

$\therefore\ \cos(n - ia)\,x = \cos nx \cos h\,ax + i\sin nx \sin h\,ax.$

$[\because\ \cos ix = \cos hx,\ \sin ix = i\sin hx]$

Thus, $\sin nx \sin h\,ax$ = Imaginary Part (I.P.) of $\cos(n - ia)\,x$

Hence, $b_n = \frac{2}{\pi}\int_0^\pi \sin nx \sin h\,ax\,dx = \frac{2}{\pi}\text{I.P.}\int_0^\pi \cos(n - ia)\,x\,dx$

$$= \frac{2}{\pi}\text{I.P.}\left[\frac{\sin(n - ia)\,x}{n - ia}\right]_0^\pi = \frac{2}{\pi}\text{I.P.}\frac{\sin(n - ia)\pi}{(n - ia)}$$

$$= \frac{2}{\pi}\text{I.P.}(n + ia)\frac{\sin(n - ia)\pi}{n^2 - i^2a^2} = \frac{2}{\pi}\text{I.P.}\frac{(n + ia)}{n^2 + a^2}\sin(n\pi - ia\pi)$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n + ia)}{(n^2 + a^2)}[\sin n\pi \cos ia\pi - \cos n\pi \sin ia\pi]$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n + ia)}{n^2 + a^2}[-i\cos n\pi \sin h\,a\pi]$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n + ia)}{n^2 + a^2}(-i)(-1)^n \sin h\,a\pi = \frac{2n}{(n^2 + a^2)\pi}(-1)^{n+1}\sin h\,a\pi$$

Alternatively $b_n = \frac{2}{\pi}\int_0^\pi\left(\frac{e^{ax} - e^{-ax}}{2}\right)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2 + n^2}(a\sin x - n\cos nx) - \frac{e^{-ax}}{a^2 + n^2}(-a\sin nx - n\cos nx)\right]_0^\pi$$

$$= \frac{1}{\pi(a^2 + n^2)}\left[\{e^{a\pi}(0 - n\cos n\pi) - e^{-a\pi}(0 - n\cos n\pi)\} - \{(0 - n) - (0 - n)\}\right]$$

$$= \frac{n}{\pi(n^2 + a^2)}\left[e^{a\pi}(-1)^{n+1} - e^{-a\pi}(-1)^{n+1}\right]$$

$$\therefore\ b_n = \frac{n}{(n^2 + a^2)\pi}\cdot(-1)^{n+1}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]$$

$$\therefore \quad b_n = \frac{2n}{(n^2+a^2)\pi} \cdot (-1)^{n+1} \sin h\, a\pi \text{ as before.}$$

$$\therefore \quad f(x) = \sum b_n \sin nx = \frac{2}{\pi} \sin h\, a\pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n}{n^2+a^2} \sin nx.$$

Example 9 : Obtain Fourier Series of $x \cos x$ in $(-\pi, \pi)$. **(M.U. 2003, 09, 15)**

Sol. : Since, $f(-x) = (-x)\cos(-x) = -x\cos x$
$= -f(x)$, $f(x)$ is an odd function $\quad [\therefore a_n = 0]$

Therefore, let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\, dx \quad \text{.......... (1)}$$

$$= \frac{1}{\pi}\int_0^{\pi} x \cdot 2\sin nx \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\, dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\, dx$$

$$= \frac{1}{\pi}\left\{\left[x \cdot \frac{-\cos(n+1)x}{n+1} - (1)\cdot\frac{-\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} + \left[x \cdot \frac{-\cos(n-1)x}{n-1} - (1)\cdot\frac{-\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n+1}\cdot\cos(n+1)\pi - \frac{\pi}{n-1}\cos(n-1)\pi\right] \quad \text{[By § 5, page 3-10]}$$

$$= \frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} \quad [\because \cos(n\pi \pm \pi) = -\cos n\pi]$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right] = (-1)^n \cdot \frac{2n}{n^2-1} \quad \text{if } n \neq 1$$

For $n = 1$, we put $n = 1$ in (1).

$$\therefore \quad b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin x\, dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\, dx$$

$$= \frac{1}{\pi}\left[x \cdot \frac{-\cos 2x}{2} - (1)\cdot\frac{-\sin 2x}{4}\right]_0^{\pi} = \frac{1}{\pi}\left[\left(-\frac{\pi}{2}\right)\right] = -\frac{1}{2}$$

Hence, $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}(-1)^n \cdot \frac{n}{n^2-1}\sin nx.$

Example 10 : Find the Fourier expansion of $f(x) = x + x^2$ when $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$ **(M.U. 2016)**

Hence, deduce that (i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ **(M.U. 2011)**

(ii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **(M.U. 1996, 2001)**

Sol. : We first note that $f(x) = x + x^2$ is the sum of the odd function $f_1(x) = x$ and the even function $f_2(x) = x^2$.

Hence, Fourier expansion of $f(x)$ is the sum of the Fourier expansions of $f_1(x)$ and $f_2(x)$.

Now, since $f_1(x) = x$ is an odd function. $\quad [\because a_n = 0]$

Let $f_1(x) = x = \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\pi(-1)\frac{(-1)^n}{n} - 0\right\} - \{0\}\right] = \frac{2(-1)^{n+1}}{n}$$

$$\therefore \quad f_1(x) = x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin nx \quad \text{.......... (2)}$$

Now, since $f_2(x) = x^2$ is an even function. $\quad [\because b_n = 0]$

Let $f_2(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} x^2\, dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cdot \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} \quad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{\pi}\left[\left\{0 + 2\pi\frac{\cos n\pi}{n^2} - 0\right\} - \{0\}\right] = 4\cdot\frac{(-1)^n}{n^2}$$

$$\therefore \quad f_2(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \quad \text{.......... (3)}$$

Form (2) and (3)

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

For deduction, we consider the series of x^2 only. We have obtained in (3).

$$x^2 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cos nx \quad \text{.......... (4)}$$

(i) Putting $x = \pi$ in (4)

$$\pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \ldots\right]$$

$$\frac{2\pi^2}{3}=4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right];\qquad \frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

(ii) Putting $x=0$ in (4), $\quad 0=\dfrac{\pi^2}{3}+4\left[-\dfrac{1}{1^2}+\dfrac{1}{2^2}-\dfrac{1}{3^2}+\ldots\right]$

$$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

(iii) Adding (5) and (6), $\quad \dfrac{\pi^2}{6}+\dfrac{\pi^2}{12}=2\left(\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\right)$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

Example 11 : Prove that $\dfrac{x}{12}(\pi-x)(2\pi-x)=\displaystyle\sum_{n=1}^{\infty}\frac{\sin nx}{n^3}$ where $0\le x\le 2\pi$.

Sol. : If we put $x=z+\pi$ then when $x=0$, $z=-\pi$ and when $x=2\pi$, $z=\pi$. Thus, the interval chang from 0 to 2π to $-\pi$ to π.

$$F(z)=\frac{(z+\pi)}{12}\cdot(\pi-z-\pi)\cdot(2\pi-z-\pi)$$

$$=\frac{(z+\pi)}{12}\cdot(-z)\cdot(\pi-z)=\frac{z}{12}\cdot(z^2-\pi^2)$$

Now $F(-z)=-\dfrac{z}{12}\cdot(z^2-\pi^2)=-F(z)$

$\therefore$ $F(z)$ is an odd function ($\therefore\ a_n=0$). Let $F(z)=\sum b_n\sin nz$

$$\therefore\ b_n=\frac{2}{\pi}\int_0^{\pi}F(z)\sin nz\,dz=\frac{2}{\pi}\int_0^{\pi}\frac{1}{12}\cdot(z^3-\pi^2 z)\sin nz\,dz$$

$$\therefore\ b_n=\frac{1}{6\pi}\left[(z^3-\pi^2 z)\left(-\frac{\cos nz}{n}\right)-(3z^2-\pi^2)\left(-\frac{\sin nz}{n^2}\right)\right.$$
$$\left.+(6z)\left(\frac{\cos nz}{n^3}\right)-6(1)\left(\frac{\sin nz}{n^4}\right)\right]_0^{\pi}\qquad \text{[By § 5, page 3-}$$

$$=\frac{1}{6\pi}\left[\left\{0-0+\frac{6\pi\cos n\pi}{n^3}-0\right\}-\{0\}\right]=\frac{\cos n\pi}{n^3}=\frac{(-1)^n}{n^3}$$

$$\therefore\ F(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nz$$

Putting $z=x-\pi$, we get

$$f(x)=\frac{x}{12}(\pi-x)(2\pi-x)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n(x-\pi)$$

$$=\sum\frac{(-1)^n}{n^3}(-1)\sin(n\pi-nx)=\sum\frac{(-1)^n}{n^3}(-1)\{\sin n\pi\cos nx-\cos n\pi\sin nx\}$$

$$=\sum\frac{(-1)^n}{n^3}(-1)\{0-(-1)^n\sin nx\}=\sum_{n=1}^{\infty}\frac{1}{n^3}\sin nx.$$

EXERCISE - III

Find the Fourier Series for $f(x)$ where

1. $f(x)=\cos px\ (-\pi,\pi)$, where p is not an integer. Hence, prove that

$$\cot p\pi=\frac{2p}{\pi}\left[\frac{1}{2p^2}-\frac{1}{p^2-1^2}+\frac{1}{p^2-2^2}-\frac{1}{p^2-3^2}+\ldots\right]\qquad\textbf{(M.U. 2011)}$$

And deduce that $\cos\theta=\dfrac{1}{\theta}-\displaystyle\sum_{n=1}^{\infty}\frac{2\theta}{n^2\pi^2-\theta^2}$.

Also deduce that $\dfrac{1}{2}-\dfrac{\pi\sqrt{3}}{18}=\dfrac{1}{9.1^2-1}+\dfrac{1}{9.2^2-1}+\dfrac{1}{9.3^2-1}+\ldots$ **(M.U. 1993, 96, 2014)**

$$\left[\textbf{Ans. :}\ \cos px=\frac{2p\sin p\pi}{\pi}\left[\frac{1}{2p^2}-\frac{\cos x}{p^2-1^2}+\frac{\cos 2x}{p^2-2^2}-\frac{\cos 3x}{p^2-3^2}+\ldots\right.\right.$$

Now put $x=\pi$, $\quad\therefore\ \cot p\pi=\dfrac{2p}{\pi}\left[\dfrac{1}{2p^2}+\dfrac{1}{p^2-1^2}+\dfrac{1}{p^2-2^2}+\ldots\right]$

Now, put $p\,\theta=\pi$ $\quad\therefore\ \cot\theta=\dfrac{1}{\theta}-2\theta\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2-\theta^2}$

Now put $p=\dfrac{1}{3}$, $\quad \dfrac{1}{\sqrt{3}}=\dfrac{2}{3\pi}\left[\dfrac{9}{2}+\dfrac{9}{1-9\cdot1^2}+\dfrac{9}{1-9\cdot2^2}+\dfrac{9}{1-9\cdot3^2}+\ldots\right]\Big]$

2. $f(x)=|\sin x|$ in $(-\pi,\pi)$. **(M.U. 2003, 04, 07, 08, 13)**

$$\left[\textbf{Ans. :}\ f(x)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\frac{\cos 6x}{35}+\ldots\right]\right]$$

3. $f(x)=\sqrt{1-\cos x}$ in $(-\pi,\pi)$ and hence, deduce that $\dfrac{1}{2}=\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$.

$$\left[\textbf{Ans. :}\ f(x)\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum\frac{1}{4n^2-1}\cdot\cos nx.\ \text{Then put } x=0\right]$$

4. $f(x)=|x|$ in $(-\pi,\pi)$. Hence, deduce that $\dfrac{\pi^2}{8}=\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\ldots$ **(M.U. 2004, 14)**

$$\left(\textbf{Hint :}\ f(x)=\begin{cases}-x, & -\pi\le x\le 0\\ x, & 0\le x\le\pi\end{cases}\right)\qquad\left[\textbf{Ans. :}\ \frac{\pi}{2}-\frac{4}{2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right]\right]$$

5. $f(x)=\sin ax$, $-\pi<x<\pi$ **(M.U. 1997, 2004, 13)**

$$\left[\textbf{Ans. :}\ f(x)=\frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\ldots\right]\right]$$

6. $f(x)=x^3$, in $(-\pi,\pi)$ **(M.U. 2009)** $\left[\textbf{Ans. :}\ f(x)=\displaystyle\sum_{n=1}^{\infty}(-1)^n\cdot\left(\frac{12}{n^3}-\frac{2\pi^2}{n}\right)\sin nx\right]$

7. $f(x) = x \sin x$ in $(-\pi, \pi)$. Hence, deduce that

$$\frac{\pi - 2}{4} = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \ldots$$ (M.U. 2000, 02,

[**Ans.** : $f(x) = 1 - \frac{1}{2}\cos x - 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - 1} \cos nx$

i.e. $f(x) = 1 - \frac{1}{2}\cos x - \frac{2}{3}\cos 2x + \frac{2}{8}\cos 3x - \frac{2}{15}\cos 4x + \ldots$

Now, put $x = \pi / 2$.]

8. $f(x) = \sin px$ in $(-\pi, \pi)$. (M.U. 1997, 20

[**Ans.** : $f(x) = \frac{2 \sin p\pi}{\pi}\left(\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n}{n^2 - p^2} \cdot \sin nx\right)$ where p is not an integ

9. $f(x) = \cos h\, px$, $-\pi < x < \pi$. [**Ans.** : $f(x) = \frac{2a \sin h\, p\pi}{\pi}\left[\frac{1}{2p^2} + \sum \frac{(-1)^n \cdot \cos n}{n^2 + a^2}\right.$

(**Hint** : Refer to solved Ex. 6. $a_n = \frac{2}{\pi} \text{R.P.} \int_0^{\pi} \cos(n + ia)\, x\, dx$, $b_n = 0$.)

10. $f(x) = x - x^2$, $-\pi < x < \pi$. Hence, deduce that, $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

(M.U. 1998, 2003) [**Ans.** : $f(x) = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin$

11. $f(x) = \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}}$, $-\pi < x < \pi$ (M.U. 20

(**Hint** : $f(x) = \frac{\sin h\, ax}{\sin h\, a\pi}$. See Ex. 6) [**Ans.** : $f(x) = \frac{2}{\pi}\sum \frac{(-1)^{n+1} \cdot n}{n^2 + a^2} \cdot \sin$

12. $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in $(-\pi, \pi)$ (M.U. 20

[**Ans.** : $f(x) = \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots$

13. (a) $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$ (M.U. 20

[**Ans.** : $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right.$

(b) $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$ (M.U. 2003

(**Hint** : Put $x = \pi + z$ then $f(x) = \begin{cases} \pi + z, & -\pi \le z \le 0 \\ \pi - z, & 0 \le z \le \pi \end{cases}$)

[**Ans.** : $\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right.$

14. $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ (M.U. 2003, 04)

[**Ans.** : $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x$]

15. $f(x) = \frac{x(\pi^2 - x^2)}{12}$, $-\pi < x < \pi$. (M.U. 2004)

[**Ans.** : $f(x) = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$]

16. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

Also deduce that $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots = \frac{1}{2}$

and $\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \ldots = \frac{\pi - 2}{4}$. (M.U. 2004, 08)

17. $f(x) = \frac{x(\pi - x)(\pi + x)}{12}$ in $(-\pi, \pi)$. Hence, find $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$ (M.U. 2004)

[**Ans.** : $f(x) = \sum \frac{(-1)^{n+1}}{n^3} \sin nx$. Then put $x = \frac{\pi}{2}$]

9. Fourier Series in $(c, c + 2l)$

In many engineering problems the period of expansion of a function is not 2π but say $2l$. To apply previous-theory to this interval we change the interval from c to $c + 2l$ to c to $c + 2\pi$ by changing the variable x to z as explained below.

Let $f(x)$ be defined in the interval c to $c + 2l$.

To transform the interval from $2l$ to 2π, we put $x = \frac{lz}{\pi}$ or $z = \frac{\pi x}{l}$.

Now when $x = c$, $z = \frac{\pi c}{l} = d$ say and when $x = c + 2l$, $z = \frac{\pi(c + 2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi$.

Thus, the function $f(x)$ of period $2l$ in the interval $(c, c + 2l)$ is transformed into the function $f\left(\frac{lz}{\pi}\right) = F(z)$ say of period 2π in the interval $(d, d + 2\pi)$.

Hence, we can write,

$$f(x) = f\left(\frac{lz}{\pi}\right) = F(z) = a_0 + \sum a_n \cos nz + \sum b_n \sin nz \quad \ldots\ldots (1)$$

where,

$$a_0 = \frac{1}{2\pi}\int_d^{d+2\pi} F(z)\, dz$$

$$a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z) \cos nz\, dz \quad \ldots\ldots (2)$$

$$b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z) \sin nz\, dz$$

Now, in these expressions we make inverse substitution $z = \frac{\pi x}{l}$, $dz = \frac{\pi}{l}dx$, when $z =$ … $x = c$ and when $z = d + 2\pi$, $x = c + 2l$ (as seen above).

Hence, (2) gives

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2l}\int_c^{c+2l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{aligned}} \quad \text{.......... (3}$$

And (1) becomes

$$\boxed{f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}} \quad \text{.......... (4}$$

Cor. 1 : If $c = 0$ *i.e.* if the interval is 0 to $2l$ then putting $c = 0$ in (3), the constants are given b

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2l}\int_0^{2l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{aligned}}$$

Cor. 2 : If $c = -l$ *i.e.* if the interval is $-l$ to l, then putting $c = -l$ in (3), the constants a given by

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2l}\int_{-l}^{l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{aligned}}$$

10. Fourier Series in the Interval $(0, 2l)$

Example 1 : Find the Fourier expansion of x^2 in $(0, a)$. **(M.U. 2004, 05,**

Hence, deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$

Sol. : Here $2l = a$ $\quad\therefore\ l = a/2$.

Let $\quad x^2 = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

i.e. $\quad x^2 = a_0 + \sum a_n \cos\frac{2n\pi x}{a} + \sum b_n \sin\frac{2n\pi x}{a}$ $\quad$ .... (1

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{a}\int_0^a x^2 dx = \frac{1}{a}\left[\frac{x^3}{3}\right]_0^a = \frac{a^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{a}\int_0^a x^2\cdot\cos\frac{2n\pi x}{a}dx$$

$$= \frac{2}{a}\left[x^2\left(\frac{a}{2n\pi}\cdot\sin\frac{2n\pi x}{a}\right) - (2x)\left(-\frac{a^2}{4n^2\pi^2}\cdot\cos\frac{2n\pi x}{a}\right) + (2)\left(-\frac{a^3}{8n^3\pi^3}\cdot\sin\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\left\{0 + (2a)\left(\frac{a^2}{4n^2\pi^2}\right) + 0\right\} - \{0\}\right] = \frac{a^2}{n^2\pi^2} \qquad \text{[By § 5, page 3-10]}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{a}\int_0^a x^2\cdot\sin\frac{2n\pi x}{a}dx$$

$$= \frac{2}{a}\left[x^2\left(-\frac{a}{2n\pi}\cdot\cos\frac{2n\pi x}{a}\right) - (2x)\left(-\frac{a^2}{4n^2\pi^2}\cdot\sin\frac{2n\pi x}{a}\right) + 2\left(\frac{a^3}{8n^3\pi^3}\cdot\cos\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\left\{a^2\left(-\frac{a}{2n\pi}\right) - 0 + 2\left(\frac{a^2}{8n^3\pi^3}\right)\right\} - \left\{0 - 0 + 2\left(\frac{a^3}{8n^3\pi^3}\right)\right\}\right]$$

$$= \frac{2}{a}\left[-\frac{a^3}{2n\pi}\right] = -\frac{a^2}{n\pi}$$

Putting these values in (1),

$$\therefore\ x^2 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{2\pi x}{a} + \frac{1}{2^2}\cos\frac{4\pi x}{a} + \ldots\right] - \frac{a^2}{\pi}\left[\frac{1}{1}\sin\frac{2\pi x}{a} + \frac{1}{2}\sin\frac{4\pi x}{a} + \ldots\right]$$

For deduction put $x = 0$ and $x = a$.

$$\therefore\ 0 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \quad i.e. \quad -\frac{1}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \quad \text{.......... (i)}$$

and $$a^2 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \quad i.e. \quad \frac{2}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \quad \text{.......... (ii)}$$

Adding (i) and (ii), we get,

$$\frac{1}{3} = \frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \qquad \therefore\ \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 2 : Find the Fourier expansion of $f(x) = 2x - x^2$, $0 \le x \le 3$ whose period is 3. Also plot the graph of the function. **(M.U. 2005, 13)**

Sol. : Here period $2l = 3$ $\quad\therefore\ l = 3/2$.

Let $\quad f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

$$= a_0 + \sum a_n \cos\frac{2n\pi x}{3} + \sum b_n \sin\frac{2n\pi x}{3} \quad \text{.......... (1)}$$

Now $\quad a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{3}\int_0^3 (2x - x^2)\,dx = \frac{1}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{1}{3}\left[9 - \frac{27}{3}\right] = 0$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{(3/2)}\int_0^3 (2x-x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\Bigg[(2x-x^2)\cdot\left(\frac{3}{2n\pi}\cdot\sin\frac{2n\pi x}{3}\right) - (2-2x)\left(-\frac{9}{4n^2\pi^2}\cdot\cos\frac{2n\pi x}{3}\right) + (-2)\left(-\frac{27}{8n^3\pi^3}\cdot\sin\frac{2n\pi x}{3}\right)\Bigg]_0^3$$

$$= \frac{2}{3}\left[\left\{0-4\cdot\frac{9}{4n^2\pi^2}\cos 2n\pi + 0\right\} - \left\{0+2\cdot\frac{9}{4n^2\pi^2}+0\right\}\right]$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}[-4-2] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{(3/2)}\int_0^3 (2x-x^2)\cdot\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\Bigg[(2x-x^2)\cdot\left(-\frac{3}{2n\pi}\cdot\cos\frac{2n\pi x}{3}\right) - (2-2x)\left(-\frac{9}{4n^2\pi^2}\cdot\sin\frac{2n\pi x}{3}\right) + (-2)\left(\frac{27}{8n^3\pi^3}\cdot\cos\frac{2n\pi x}{3}\right)\Bigg]_0^3$$

$$= \frac{2}{3}\left[\left\{\frac{9}{2n\pi} - 0 - \frac{27}{4n^3\pi^3}\right\} - \left\{0-0-\frac{27}{4n^3\pi^3}\right\}\right]$$

$$\therefore\quad b_n = \frac{2}{3}\left[\frac{9}{2n\pi}\right] = \frac{3}{n\pi}.$$

Putting these values in (1),

$$f(x) = 0 + \sum\left(-\frac{9}{n^2\pi^2}\right)\cos\frac{2n\pi x}{3} + \sum\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

$$= -\frac{9}{\pi^2}\sum\frac{1}{n^2}\cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum\frac{1}{n}\sin\frac{2n\pi x}{3}$$

For the graph we see that $y = 2x - x^2$ is a parabola.

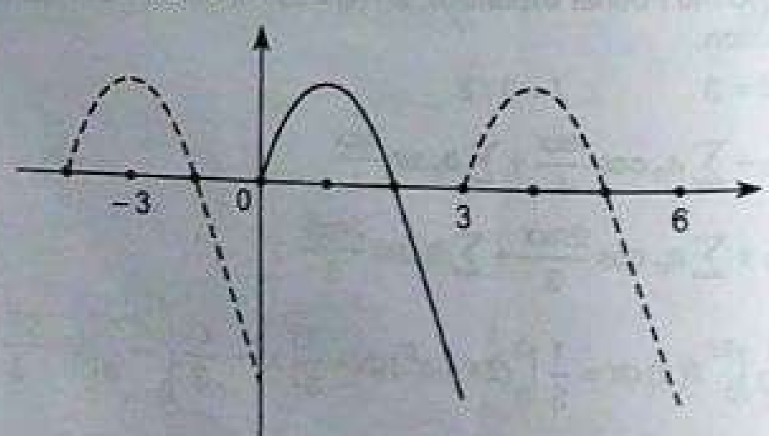
Now $y - 1 = -1 + 2x - x^2 = -(x-1)^2$

$\therefore$ $Y = -X^2$, where $Y = y-1$ and $X = x-1$. It opens downwards.

When $x = 0$, $y = 0$; when $x = 2$, $y = 0$, When $x = 3$, $y = -3$.

Since $f(x)$ is periodic with period 3, the graph repeats at 3, 6, ...

Thus, we get, the following graph.



Example 3 : Find the Fourier expansion of $f(x) = 4 - x^2$ in the interval (0, 2). Graph the function and also state the values of the series for $x = 0, 1, 2, 10, 11$. (M.U. 2002, 04)

Hence, deduce that $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots..$ (M.U. 2004, 14)

Sol. : Here $2l = 2$ $\quad\therefore\ l = 1$.

Let $f(x) = a_0 + \sum a_n\cos\dfrac{n\pi x}{l} + \sum b_n\sin\dfrac{n\pi x}{l}$

i.e. $f(x) = a_0 + \sum a_n\cos n\pi x + \sum b_n\sin n\pi x$ .... (1)

$$\therefore\quad a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\int_0^2 (4-x^2)\,dx$$

$$= \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left[8 - \frac{8}{3}\right] = \frac{8}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^2 (4-x^2)\cos n\pi x\,dx$$

$$= \left[(4-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3}\right)\right]_0^2$$

$$= \left[\left\{0 - \frac{4}{n^2\pi^2} + 0\right\} - \left\{0-0-0\right\}\right] = -\frac{4}{n^2\pi^2} \qquad \text{[By § 5, page 3-10]}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^2 (4-x^2)\sin n\pi x\,dx$$

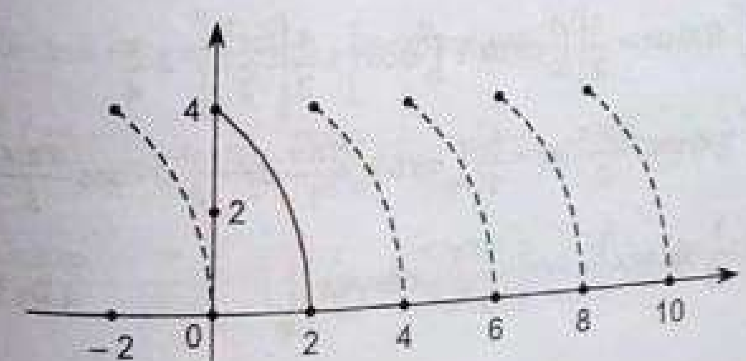
$$= \left[(4-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$\therefore\quad b_n = \left[\left\{0-0-\frac{2}{n^3\pi^3}\right\} - \left\{-\frac{4}{n\pi} - 0 - \frac{2}{n^3\pi^3}\right\}\right] = \frac{4}{n\pi}.$$

$$\therefore\quad f(x) = 4 - x^2 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \ldots.\right]$$
$$+ \frac{4}{\pi}\left[\frac{1}{1}\sin\pi x + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \ldots\right]$$

Now $y = 4 - x^2$ i.e. $y - 4 = -x^2$ i.e. $Y = -X^2$, where $Y = y - 4$ and $X = x$ is a parabola with vertex at (0, 4) and opening downwards as shown below.

When $x = 0$, $y = 4$; when $x = 2$, $y = 0$. Since $f(x)$ is periodic with period 2, the graph repeats at 2, 4, 6, ...



Since, $f(x)$ is discontinuous at $x = 0, 2, 4, 6$ ..... we find its value as follows.

$$f(c) = \frac{\lim\limits_{x \to c^-} f(x) + \lim\limits_{x \to c^+} f(x)}{2}$$

Now, for $c = 0$, from the graph

$$f(0) = \frac{\lim\limits_{x \to 0^-} f(x) + \lim\limits_{x \to 0^+} f(x)}{2} = \frac{0+4}{2} = 2$$

$$f(2) = \frac{\lim\limits_{x \to 2^-} f(x) + \lim\limits_{x \to 2^+} f(x)}{2} = \frac{4+0}{2} = 2$$

Similarly, $f(4) = f(6) = f(8) = f(10)$ ______ $= 2$, since $f(x)$ is periodic with period 2.

Now, at $x = 1$, the function is continuous $\quad \therefore f(1) = 4 - (1) = 3$

Also at $x = 11$, the function is continuous $\quad \therefore f(11) = 3$.

Thus, we have $f(1) = 3,\ f(2) = 2,\ f(10) = 2,\ f(11) = 3$.

For deduction put $x = 0$ and $x = 2$.

$$\therefore \quad 4 - 0 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

i.e. $$\frac{1}{3} = -\frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

and $$4 - 4 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

i.e. $$-\frac{2}{3} = -\frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

Adding (i) and (ii), we get,

$$-\frac{1}{3} = -\frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \qquad \therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 4 : Expand $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ period 2 into a Fourier Series.

(M.U. 1998, 2002 ...)

Sol. : Here $2l = 2 \quad \therefore l = 1$

Let $$f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 0\,dx\right] = \frac{1}{2}\left[\frac{\pi x^2}{2}\right]_0^1 = \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{1}\left[\int_0^1 \pi x\cos\frac{n\pi x}{1}\,dx + \int_1^2 0\cdot\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \left[\pi x\left(\frac{1}{n\pi}\sin n\pi x\right) - (\pi)\left(-\frac{1}{n^2\pi^2}\cos n\pi x\right)\right]_0^1$$ [By § 5, page 3-10]

$$\therefore \quad a_n = \pi\cdot\frac{1}{n^2\pi^2}\left[(-1)^n - 1\right] = -\frac{1}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{1}\left[\int_0^1 \pi x\sin\frac{n\pi x}{1}\,dx + \int_1^2 0\cdot\sin\frac{n\pi x}{1}\,dx\right]$$

$$= \left[\pi x\left(-\frac{1}{n\pi}\cos n\pi x\right) - (\pi)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1 = -\frac{1}{n}(-1)^n$$ [By § 5, page 3-10]

$$\therefore \quad f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \ldots\right]$$
$$- \left[-\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} - \frac{\sin 3\pi x}{3} + \ldots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \ldots\right]$$
$$+ \left[\frac{1}{1}\cdot\sin\pi x - \frac{1}{2}\cdot\sin 2\pi x + \frac{1}{3}\cdot\sin 3\pi x - \ldots\right]$$

Example 5 : If $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$ with period 2, show that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$$

(M.U. 2003, 04, 07, 09, 12, 14)

Sol. : Here $2l = 2 \quad \therefore l = 1$

Let $$f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$$

i.e. $$f(x) = a_0 + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x \qquad \text{.................. (1)}$$

$$\therefore \quad a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \frac{1}{2}\left[\pi\left\{\frac{x^2}{2}\right\}_0^1 + \pi\left\{2x - \frac{x^2}{2}\right\}_1^2\right]$$

$$= \frac{\pi}{2}\left[\left\{\frac{1}{2}\right\} + \left\{4 - 2 - 2 + \frac{1}{2}\right\}\right] = \frac{\pi}{2}.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 \pi x\cdot\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left[\left\{x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right\} + \left\{-\frac{1}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2}\right\}\right]$$ [By § 5, page 3-10]

$$\therefore \quad a_n = \frac{2\pi}{n^2\pi^2}[\cos n\pi - 1] = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^1 \pi x \cdot \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[\left\{x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\left\{-\frac{\cos n\pi}{n\pi}\right\} + \left\{+\frac{\cos n\pi}{n\pi}\right\}\right] = 0$$

Putting these values in (1)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$$

EXERCISE - IV

Obtain the Fourier expansions of the following functions

1. $f(x) = 2 - \dfrac{x^2}{2}$ in $0 \le x \le 2$ (M.U. 20[illegible])

$$\left[\text{Ans. : } f(x) = \frac{4}{3} - \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{2}{\pi}\cdot\sum\frac{1}{n}\cdot\sin n\pi\ldots\right.$$

2. $f(x) = \dfrac{a}{2} - x$, $0 < x < a$

$$\left[\text{Ans. : } f(x) = \frac{a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\sin\frac{2n\ldots}{a}\right.$$

3. $f(x) = l - x, \quad 0 < x < l$
$\quad = 0, \quad l < x < 2l.$

Hence, deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

$$\left[\text{Ans. : } f(x) = l\left[\frac{1}{4} + \frac{2}{\pi^2}\left(1 + \cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right)\right.\right.$$
$$\left. + \frac{1}{\pi}\left(\frac{1}{l}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\right.\right.$$

Then put $x = l/2$.]

4. $f(x) = 1, \quad 0 < x < 1$
$\quad = x, \quad 1 < x < 2$ (M.U. 2007)

$$\left[\text{Ans. : } f(x) = \frac{5}{4} - \frac{4}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \frac{1}{4^2}\cos\frac{4\pi x}{2} + \ldots\right]\right]$$

5. $f(x) = c, \quad 0 < x < a$
$\quad = 0, \quad a < x < l$

$$\left[\text{Ans. : } f(x) = \frac{ca}{l} + \frac{2c}{\pi}\left[\sin\left(\frac{\pi a}{l}\right)\cos\left(\frac{\pi x}{l}\right) + \frac{1}{2}\sin\left(\frac{2\pi a}{l}\right)\cos\left(\frac{2\pi x}{l}\right) + \ldots\right]\right]$$

6. (a) $f(x) = \begin{cases} kx, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **(M.U. 1998)** (b) $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **(M.U. 2002)**

$$\left[\text{Ans. : (a) } f(x) = \frac{k}{4} - \frac{2k}{\pi^2}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right) + \frac{k}{\pi}\left(\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \ldots\right)\right.$$

(b) In (a) put $k = \pi$.

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right) + \left(\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} + \ldots\right)\Bigg]$$

7. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1 - x, & 1 < x < 2 \end{cases}$ (M.U. 2013)

$$\left[\text{Ans. : } f(x) = -\frac{4}{\pi^2}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + \frac{2}{\pi}\left(\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + \ldots\right)\right]$$

8. $f(x) = \begin{cases} x, & 0 < x < c/2 \\ c - x, & c/2 < x < c \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{4c}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi x}{c}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{c}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi x}{c}\right) - \ldots\right]\right]$$

9. $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x-2), & 1 < x < 2 \end{cases}$

Hence, show that $\dfrac{\pi}{4} = \dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7}$. (M.U. 1997, 2002, 03, 08)

$$\left[\text{Ans. : } f(x) = \frac{\pi}{4} + 2\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\pi x. \text{ Then put } x = \frac{1}{2}\right]$$

10. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & x > 2 \end{cases}$

(**Hint :** In example 8 above put $c = 2$)

$$\left[\text{Ans. : } f(x) = \frac{8}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \ldots\right)\right]$$

11. $f(x) = \begin{cases} -x, & 0 < x < l \\ 0, & l \le x < 2l \end{cases}$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

$$\left[\textbf{Ans.}: f(x) = -\frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{1}{1}\cos\frac{\pi x}{l} + \frac{1}{3}\cos\frac{3\pi x}{l} + \ldots\right] - \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\right]\right.$$

Then put $x = 0$.]

12. $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$ (M.U. 20

Hence, deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \dfrac{\pi^2}{8}$.

$$\left[\textbf{Ans.}: f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum\left\{\frac{1}{(2n-1)^2}\cos(2n-\right.\right.$$

13. $f(x) = \begin{cases} 3kx/l, & 0 < x < (l/3) \\ 3k(l-2x)/l, & (l/3) < x < (2l/3) \\ 3k(x-l)/l, & (2l/3) < x < l \end{cases}$ (M.U.

$$\left[\textbf{Ans.}: \frac{9k}{\pi^2}\sum\frac{1}{n^2}\sin\frac{2n\pi}{3}\cdot\sin\right.$$

11. Fourier Series in the Interval $(-l, l)$

Example 1 : Find the Fourier expansion of $f(x) = e^{ax}$ in $(-l, l)$.

Sol. : Let $f(x) = a_0 + \sum a_n \cos\dfrac{n\pi x}{l} + \sum b_n \sin\dfrac{n\pi x}{l}$ ....

$$\therefore\ a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2l}\int_{-l}^{l} e^{ax}\,dx = \frac{1}{2l}\left[\frac{e^{ax}}{a}\right]_{-l}^{l}$$

$$= \frac{1}{2al}\left[e^{al} - e^{-al}\right] = \frac{1}{al}\left[\frac{e^{al} - e^{-al}}{2}\right] = \frac{\sin h\,al}{al}$$

$$\therefore\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} e^{ax}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{1}{a^2 + n^2\pi^2/l^2}\left\{e^{ax}\left(a\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right\}\right]_{-l}^{l} \quad \text{[By (3), pag}$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\left[e^{al}(a\cos n\pi + 0) - e^{-al}(a\cos n\pi + 0)\right]$$

$$= \frac{al}{a^2l^2 + n^2\pi^2}\left[e^{al}(-1)^n - e^{-al}(-1)^n\right] \quad [\because \cos n\pi$$

$$\therefore\ a_n = \frac{2al(-1)^n}{a^2l^2 + n^2\pi^2}\left[\frac{e^{al} - e^{-al}}{2}\right] = \frac{2al(-1)^n}{a^2l^2 + n^2\pi^2}\sin h\,al$$

$$\therefore\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} e^{ax}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{1}{a^2 + n^2\pi^2/l^2}\left\{e^{ax}\left(a\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right\}\right]_{-l}^{l} \quad \text{[By (2), page 3-4]}$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\left[e^{al}\left(0 - \frac{n\pi}{l}\cos n\pi\right) - e^{-al}\left(0 - \frac{n\pi}{l}\cos n\pi\right)\right]$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\left[e^{al}\left(0 - \frac{n\pi}{l}(-1)^n\right) - e^{-al}\left(0 - \frac{n\pi}{l}(-1)^n\right)\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\cdot\frac{n\pi}{l}\cdot(-1)^{n+1}(e^{al} - e^{-al})$$

$$= \frac{2n\pi(-1)^{n+1}}{a^2l^2 + n^2\pi^2}\cdot\left(\frac{e^{al} - e^{-al}}{2}\right) = \frac{2n\pi(-1)^{n+1}}{a^2l^2 + n^2\pi^2}\cdot\sin h\,al.$$

Putting these values in (1),

$$f(x) = \frac{\sin h\,al}{al} + \sum\frac{2al(-1)^n}{a^2l^2 + n^2\pi^2}\sin h\,al\cos\frac{n\pi x}{l} + \sum\frac{2n\pi(-1)^{n+1}}{a^2l^2 + n^2\pi^2}\cdot\sin h\,al\sin\frac{n\pi x}{l}$$

$$= \frac{\sin h\,al}{al} + 2al\sin h\,al\sum\frac{(-1)^n}{a^2l^2 + n^2\pi^2}\cos\frac{n\pi x}{l} + 2\pi\sin h\,al\sum\frac{(-1)^{n+1}}{a^2l^2 + n^2\pi^2}\sin\frac{n\pi x}{l}.$$

Example 2 : Find the Fourier expansion of $f(x) = \begin{cases} 2, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$ **(M.U. 2009)**

Sol. : Comparing the interval $(-2, 2)$ with $(-l, l)$ we find that here $l = 2$.

Let $f(x) = a_0 + \sum a_n \cos\dfrac{n\pi x}{l} + \sum b_n \sin\dfrac{n\pi x}{l}$

$$= a_0 + \sum a_n \cos\frac{n\pi x}{2} + \sum b_n \sin\frac{n\pi x}{2} \quad \ldots\ldots\ldots\ldots (1)$$

$$\therefore\ a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right]$$

$$= \frac{1}{4}\left[\left\{2x\right\}_{-2}^{0} + \left\{\frac{x^2}{2}\right\}_{0}^{2}\right] = \frac{1}{4}[(0+4) + (2-0)] = \frac{3}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx\right]$$

$$\therefore\ a_n = \frac{1}{2}\left[\left\{2\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right\}_{-2}^{0} + \left\{(x)\cdot\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}_0^2\right]$$

$$= \frac{1}{2}\left[\{0-0\} + \left\{0 + \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{n^2\pi^2}\{\cos n\pi - 1\} = \begin{cases} 0, & \text{when } n \text{ is even} \\ -4, & \text{when } n \text{ is odd} \end{cases} \qquad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_0^2 x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left\{2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right\}_{-2}^{0} + \left\{(x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}_0^2\right]$$

$$= \frac{1}{2}\left[\left\{-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right\} + \left\{(2)\left(-\frac{2}{n\pi}\cos n\pi\right) + 0\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{2}\left\{-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi - \frac{4}{n\pi}\cos n\pi\right\} = -\frac{2}{n\pi}$$

Putting these values in (1)

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots..\right]$$
$$- \frac{2}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots..\right]$$

Example 3 : Expand $f(x) = \begin{cases} 0, & -c < x < 0 \\ a, & 0 < x < c \end{cases}$ in a Fourier series of period $2a$. (M.U.

Sol. : Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$ and $l = c$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2c}\left[\int_{-c}^{0} 0\cdot dx + \int_0^c a\,dx\right]$$

$$= \frac{1}{2c}\left[0 + [ax]_0^c\right] = \frac{1}{2c}\cdot ac = \frac{a}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{c}\left[\int_{-c}^{0} 0\cdot\cos\frac{n\pi x}{c}\,dx + \int_0^c a\cos\frac{n\pi x}{c}\,dx\right]$$

$$= \frac{1}{c}\left[a\left(\frac{c}{n\pi}\sin\frac{n\pi x}{c}\right)\right]_0^c = \frac{a}{n\pi}[0-0] = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi x}{l}\,dx = \frac{1}{c}\left[\int_{-c}^{0} 0\cdot\sin\frac{n\pi x}{c}\,dx + \int_0^c a\sin\frac{n\pi x}{c}\,dx\right]$$

$$\therefore\ b_n = \frac{1}{c}\left[a\left(-\frac{c}{n\pi}\right)\cos\frac{n\pi x}{c}\right]_0^c$$

$$= -\frac{a}{n\pi}\{\cos n\pi - 1\} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{2a}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Putting these values in (1)

$$f(x) = \frac{a}{2} + \frac{2a}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots..\right]$$

Example 4 : A sinusoidal voltage $E \sin \omega x$ is passed through a half-wave rectifier which clips of the wave and the resulting function is given by

$$f(x) = \begin{cases} 0 & , \quad -\dfrac{\pi}{\omega} < x < 0 \\ E\sin\omega x, & \quad 0 < x < \dfrac{\pi}{\omega} \end{cases}$$

with period $\frac{2\pi}{\omega}$. Find the Fourier expansion of $f(x)$.

Sol. : Here $2l = \frac{2\pi}{\omega}$ $\quad\therefore\ l = \frac{\pi}{\omega}$

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

$$= a_0 + \sum a_n \cos n\omega x + \sum b_n \sin n\omega x \qquad \ldots\ldots\ldots\ldots (1)$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{(2\pi/\omega)}\left[\int_{-\pi/\omega}^{0} 0\cdot dx + \int_0^{\pi/\omega} E\sin\omega x\,dx\right]$$

$$= \frac{\omega}{2\pi}\left[-\frac{E\cos\omega x}{\omega}\right]_0^{\pi/\omega} = \frac{-E}{2\pi}[-1-1] = \frac{E}{\pi}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(x)\cos n\omega x\,dx$$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} 0\cdot dx + \int_0^{\pi/\omega} E\sin\omega x\cdot\cos n\omega x\,dx\right]$$

$$= \frac{E\omega}{2\pi}\int_0^{\pi/\omega}\{\sin(1+n)\omega x + \sin(1-n)\omega x\}\,dx$$

$$= \frac{E\omega}{2\pi}\left[\left\{-\frac{\cos(1+n)\omega x}{(1+n)\omega} - \frac{\cos(1-n)\omega x}{(1-n)\omega}\right\}_0^{\pi/\omega}\right]$$

$$= \frac{E\omega}{2\pi}\left[-\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{E}{2\pi}\left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] \qquad [\because \cos(1\pm n)\pi = -\cos n\pi]$$

$$\therefore \quad a_n = \frac{E}{2\pi}(1+\cos n\pi)\left[\frac{1}{1+n}+\frac{1}{1-n}\right] = \frac{E}{2\pi}\left[1+(-1)^n\right]\cdot\frac{2}{1-n^2}$$

$$= \begin{cases} \dfrac{2E}{\pi}\cdot\dfrac{1}{1-n^2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd and } n \neq 1 \end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{\pi x}{c}\,dx = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(x)\cos\omega x\,dx$$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} 0\cdot dx + \int_{0}^{\pi/\omega} E\sin\omega x\cdot\cos\omega x\,dx\right]$$

$$= \frac{\omega E}{2\pi}\int_{0}^{\pi/\omega}\sin 2\omega x\,dx = \frac{\omega E}{2\pi}\left[-\frac{\cos 2\omega x}{2\pi}\right]_{0}^{\pi/\omega} = -\frac{\omega E}{2\pi}[1-1] = 0$$

Now, $\quad b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{\omega}{\pi}\int_{-l}^{l} f(x)\sin n\omega x\,dx$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} 0\cdot dx + \int_{0}^{\pi/\omega} E\sin\omega x\sin n\omega x\,dx\right]$$

$$= -\frac{E\omega}{2\pi}\int_{0}^{\pi/\omega}\left[\cos(1+n)\omega x - \cos(1-n)\omega x\right]dx$$

$$= -\frac{E\omega}{2\pi}\left[\frac{\sin(1+n)\omega x}{(1+n)\omega} - \frac{\sin(1-n)\omega x}{(1-n)\omega}\right]_{0}^{\pi/\omega}$$

$$= -\frac{E}{2\pi}\left[\frac{\sin(1+n)\pi}{1+n} - \frac{\sin(1-n)\pi}{1-n} - 0\right]$$

$$= 0 \text{ and } n \neq 1 \qquad [\because \sin(1\pm n)\pi = 0]$$

When $n = 1$, we have

$$b_1 = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi x}{l}\,dx = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(x)\sin\omega x\,dx$$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} 0\cdot dx + \int_{0}^{\pi/\omega} E\sin\omega x\cdot\sin\omega x\,dx\right]$$

$$= \frac{\omega E}{\pi}\int_{0}^{\pi/\omega}\sin^2\omega x\,dx = \frac{\omega E}{\pi}\int_{0}^{\pi/\omega}\left(\frac{1-\cos 2\omega x}{2}\right)dx$$

$$= \frac{\omega E}{2\pi}\left[x - \frac{\sin 2\omega x}{2}\right]_{0}^{\pi/\omega} = \frac{\omega E}{2\pi}\left[\frac{\pi}{\omega}\right] = \frac{E}{2}$$

Hence, putting these values in (1),

$$f(x) = \frac{E}{\pi} - \frac{2E}{\pi}\left[\frac{1}{(2^2-1)}\cos 2\omega x + \frac{1}{(4^2-1)}\cos 4\omega x + \frac{1}{(6^2-1)}\cos 6\omega x + \ldots\right] + \frac{E}{2}\sin\omega x$$

$$= \frac{E}{\pi} - \frac{2E}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}\cos 2n\omega x + \frac{E}{2}\sin\omega x$$

EXERCISE - V

Find the Fourier Series of the following functions.

1. $f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$ period 10. (M.U. 2003)

$$\left[\textbf{Ans. : } f(x) = \frac{3}{2} + \frac{6}{\pi}\left[\frac{1}{1}\cdot\sin\frac{\pi x}{5} + \frac{1}{3}\cdot\sin\frac{3\pi x}{5} + \frac{1}{5}\cdot\sin\frac{5\pi x}{5} + \ldots\right]\right]$$

(**Hint :** See solved Ex. 3 above. Here $a = 3$, $c = 5$.)

2. $f(x) = \begin{cases} 2x, & 0 \le x < 3 \\ 0, & -3 < x < 0 \end{cases}$ period 6.

$$\left[\textbf{Ans. : } f(x) = \frac{3}{2} - \frac{12}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3} + \frac{1}{3^2}\cos\frac{3\pi x}{3} + \frac{1}{5^2}\cos\frac{5\pi x}{3} + \ldots\right]\right.$$
$$\left. + \frac{1}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{3} - \frac{1}{2}\sin\frac{2\pi x}{3} + \frac{1}{3}\sin\frac{3\pi x}{3} - \ldots\right]\right]$$

3. $f(x) = \begin{cases} x, & -1 < x < 0 \\ x+2, & 0 < x < 1 \end{cases}$ period 2. $\left[\textbf{Ans. : } f(x) = 1 + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[1-2(-1)^n\right]\sin n\pi x\right]$ (M.U. 2004)

4. $f(x) = \begin{cases} l, & -l < x < 0 \\ x, & 0 < x < l \end{cases}$

$$\left[\textbf{Ans. : } f(x) = \frac{3l}{4} - \frac{2l}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) + \ldots\right]\right.$$
$$\left. - \frac{1}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \ldots\right]\right]$$

5. $f(x) = \begin{cases} -x, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$ (M.U. 2004, 10) $\left[\textbf{Ans. : } 1 - \dfrac{4}{\pi^2}\displaystyle\sum\frac{1}{(2n-1)^2}\cos n\pi x\right]$

6. $f(x) = |x|,\ -2 < x < 2$. Hence, deduct that $\displaystyle\sum\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$. (M.U. 2001, 03, 10)

$$\left[\textbf{Ans. : } 1 - \frac{8}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos\left[\frac{(2n-1)\pi x}{2}\right]\right]$$

12. Even and Odd Functions in the Interval $(-l, l)$

(i) **Even function in $(-l, l)$:** If $f(x)$ is even in the interval $(-l, l)$ then,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{2}{2l}\int_{0}^{l} f(x)\,dx = \frac{1}{l}\int_{0}^{l} f(x)\,dx \qquad [\because f(x) \text{ is even}]$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \qquad \left[\because f(x)\cos\frac{n\pi x}{l} \text{ is even}\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0 \qquad \left[\because f(x)\sin\frac{n\pi x}{l} \text{ is odd}\right]$$

(ii) Odd function in $(-l, l)$: If $f(x)$ is odd in the interval $(-l, l)$ then,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = 0 \qquad [\because\ f(x) \text{ is odd}]$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = 0 \qquad \left[\because\ f(x)\cos\frac{n\pi x}{l} \text{ is odd}\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad \left[\because\ f(x)\sin\frac{n\pi x}{l} \text{ is even}\right]$$

Thus, we have

If $f(x)$ is even in $(-l, l)$

$$\boxed{a_0 = \frac{1}{l}\int_0^l f(x)\,dx,\ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx,\ b_n = 0}$$

If $f(x)$ is odd in $(-l, l)$

$$\boxed{a_0 = 0,\ a_n = 0,\ b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx}$$

Example 1 : Obtain the Fourier expansion of x^2 from $x = -l$ to $x = l$ and hence, deduce

(i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ (ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

(M.U. 1995, 2004, 06)

Sol. : Since, $f(x) = x^2$ is an even function, we have $b_n = 0$.

Now, $a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l x^2\,dx = \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{3}$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x^2\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{2l^3}{n^2\pi^2}(-1)^n\right] = (-1)^n\frac{4l^2}{n^2\pi^2} \qquad \text{[By § 5, page 3-10]}$$

$$\therefore\ x^2 = \frac{l^2}{3} + \sum(-1)^n\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$\therefore\ x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right]$$

Now, put $x = 0$, in (1) we get

$$\therefore\ 0 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$\therefore\ \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

(ii) Putting $x = l$, in (1) we get

$$\frac{2l^2}{3} = -\frac{4l^2}{\pi^2}\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \ldots\right]$$

$$\therefore\ \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 2 : Find the Fourier series of $f(x) = x\,|x|$ in $(-1, 1)$.

Sol. : We have $f(x) = x\,|x|$.

$\therefore\ f(-x) = -x\,|-x| = -x\,|x| = -f(x)$ $\quad\therefore\ f(x)$ is an odd function.

Now, using (B) of the previous page,

$$a_0 = 0,\quad a_n = 0 \quad\text{and}\quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Now, $f(x) = \begin{cases} -x^2, & -1 < x < 0 \\ x^2, & 0 < x < 1 \end{cases}$

$$\therefore\ b_n = \frac{2}{1}\int_0^1 x^2\sin n\pi x\,dx$$

$$= 2\left[x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1 \qquad \text{[By § 5, page 3-10]}$$

$$= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

$$= 2\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right] \qquad [\because\ \cos n\pi = (-1)^n]$$

$$\therefore\ f(x) = 2\sum\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]\sin n\pi x.$$

Example 3 : Find the Fourier Series for $f(x) = 1 - x^2$ in $(-1, 1)$. **(M.U. 2004, 10, 12)**

Sol. : We have $f(-x) = 1 - (-x^2) = 1 - x^2 = f(x)$. Hence, $f(x)$ is even.

Comparing $(-l, l)$ with $(-1, 1)$ we get $l = 1$

Now, $a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \int_0^1 (1 - x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$

$$= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = 2\int_0^1 (1 - x^2)\cos n\pi x\,dx$$

$$= 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[-\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2} \qquad \text{[By § 5, page 3-10]}$$

Hence, $f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$

Example 4 : Obtain the Fourier Series for $f(x)$, where $f(x)=\begin{cases}-c, & -a<x<0\\ c, & 0<x<a\end{cases}$

Sol. : We have $f(-x)=\begin{cases}-c, & -a<-x<0\\ c, & 0<-x<a\end{cases}=\begin{cases}-c, & a>x>0\\ c, & 0>x>-a\end{cases}$

$$=\begin{cases}c, & -a<x<0\\ -c, & 0<x<a\end{cases}=-\begin{cases}-c, & -a<x<0\\ c, & 0<x<a\end{cases}=-f(x)$$

Hence, $f(x)$ is odd function

$$\therefore\ a_0=0,\quad a_n=0$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx=\frac{2}{a}\int_0^a c\cdot\sin\left(\frac{n\pi x}{a}\right)dx$$

$$=\frac{2c}{a}\left[-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^a=\frac{2c}{a}\left(-\frac{a}{n\pi}\right)\left[(-1)^n-1\right]$$

$$=-\frac{2c}{n\pi}\left[(-1)^n-1\right]=\begin{cases}0 & \text{when } n \text{ is even}\\ \dfrac{4c}{n\pi} & \text{when } n \text{ is odd}\end{cases}$$

$$\therefore\ f(x)=\frac{4c}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{a}+\frac{1}{3}\sin\frac{3\pi x}{a}+\frac{1}{5}\sin\frac{5\pi x}{a}+.....\right]$$

Example 5 : Obtain Fourier expansion of $\sin ax$ in the interval $-l<x<l$, where a is no integer. (M.U. 20

Sol. : Here $f(-x)=\sin(-ax)=-\sin ax=-f(x)$. Hence, $f(x)$ is an odd function. [$\therefore a_0$

Let $\sin ax=\sum b_n\sin\frac{n\pi x}{l}$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}=\frac{2}{l}\int_0^l \sin ax\sin\frac{n\pi x}{l}\,dx$$

$$=-\frac{1}{l}\int_0^l\left[\cos\left(a+\frac{n\pi}{l}\right)x-\cos\left(a-\frac{n\pi}{l}\right)x\right]dx$$

$$=-\frac{1}{l}\left[\frac{l}{al+n\pi}\sin\left(a+\frac{n\pi}{l}\right)x-\frac{l}{al-n\pi}\sin\left(a-\frac{n\pi}{l}\right)x\right]_0^l$$

$$=-\left[\frac{l}{al+n\pi}\sin(al+n\pi)-\frac{l}{al-n\pi}\sin(al-n\pi)\right]$$

$$=-\left[\frac{\sin al\cos n\pi}{al+n\pi}-\frac{\sin al\cos n\pi}{al-n\pi}\right]$$

$$=\frac{(-2n\pi)}{(a^2l^2-n^2\pi^2)}\sin al\cos n\pi \qquad [\because \sin n\pi=0]$$

$$=-\frac{2n\pi(-1)^n}{n^2\pi^2-a^2l^2}\sin al \qquad [\because \cos n\pi=(-1)^n]$$

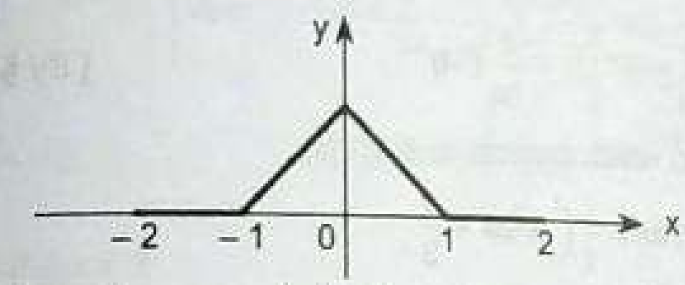
$$\therefore\ f(x)=2\pi\sin al\sum\frac{(-n)(-1)^n}{n^2\pi^2-a^2l^2}\sin\frac{n\pi x}{l}$$

$$\therefore\ f(x)=2\pi\sin al\left[\frac{1}{(1^2\pi^2-a^2l^2)}\sin\frac{\pi x}{l}-\frac{2}{(2^2\pi^2-a^2l^2)}\sin\frac{2\pi x}{l}+\frac{3}{(3^2\pi^2-a^2l^2)}\sin\frac{3\pi x}{l}-....\right]$$

Example 6 : Find the Fourier expansion of $f(x)=\begin{cases}0, & -2<x<-1\\ 1+x, & -1<x<0\\ 1-x, & 0<x<1\\ 0, & 1<x<2\end{cases}$

(M.U. 1993, 2002, 03, 13)

Sol. : The graph of the function is shown below.



Further $f(-x)=\begin{cases}0, & -2<-x<-1\\ 1-x, & -1<-x<0\\ 1+x, & 0<-x<1\\ 0, & 1<-x<2\end{cases}=\begin{cases}0, & 2>x>1\\ 1-x, & 1>x>0\\ 1+x, & 0>x>-1\\ 0, & -1>x>-2\end{cases}=\begin{cases}0, & -2<x<-1\\ 1+x, & -1<x<0\\ 1-x, & 0<x<1\\ 0, & 1<x<2\end{cases}=f(x)$

$\therefore$ $f(x)$ is an even function.

$\therefore$ $b_n=0$ and $l=2$.

$$\therefore\ f(x)=a_0+\sum a_n\cos\frac{n\pi x}{l}=a_0+\sum a_n\cos\frac{n\pi x}{2}$$

$$\therefore\ a_0=\frac{1}{l}\int_0^l f(x)\,dx=\frac{1}{2}\int_0^2 f(x)\,dx$$

$$=\frac{1}{2}\left[\int_0^1(1-x)\,dx+\int_1^2 0\cdot dx\right]=\frac{1}{2}\left[x-\frac{x^2}{2}\right]_0^1=\frac{1}{4}.$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$=\int_0^1(1-x)\cos\frac{n\pi x}{2}\,dx+\int_1^2 0\cdot dx$$

$$=\left[(1-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(-1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^1 \qquad \text{[By § 5, page 3-10]}$$

$$\therefore\ a_n=\left[\left(0-\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right)-\left(0-\frac{4}{n^2\pi^2}\right)\right]=\frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)$$

$$\therefore\ f(x)=\frac{1}{4}+\frac{4}{\pi^2}\sum\frac{1}{n^2}\left(1-\cos\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

Example 7 : Find Fourier expansion for $f(x) = x - x^2,\ -1 < x < 1.$ **(M.U. 2005, 09, 10, 14)**

Sol. : The given function is the difference between odd and even functions which can be written as

$$f(x) = f_1(x) - f_2(x)$$

where, $f_1(x) = x$, is an odd function, and

$f_2(x) = x^2$, is an even function.

Here $l = 1$.

Now, for $f_1(x) = x$ which is odd, $a_n = 0$ and

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx = \frac{2}{1}\int_0^1 x \sin n\pi x\, dx$$

$$= 2\left[x\left(-\frac{1}{n\pi}\cos n\pi x\right) - (1)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1$$

$$= 2\left[-\frac{1}{n\pi}(-1)^n\right] = -\frac{2}{n\pi}(-1)^n \qquad \text{[By § 5, page 3-10]}$$

Further, for $f_2(x) = x^2$ which is even, $b_n = 0$.

$$a_0 = \frac{1}{l}\int_0^l f(x)\, dx = \frac{1}{1}\int_0^1 x^2 dx = \frac{1}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{c}\, dx = 2\int_0^1 x^2 \cos n\pi x\, dx$$

$$= 2\left[x^2\left(\frac{1}{n\pi}\sin n\pi x\right) - 2x\left(-\frac{1}{n^2\pi^2}\cos n\pi x\right) + 2\left(-\frac{1}{n^3\pi^3}\sin n\pi x\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}(-1)^n \qquad \text{[By § 5, page 3-10]}$$

$$\therefore\ f(x) = f_1(x) - f_2(x) = -f_2(x) + f_1(x)$$

$$\therefore\ f(x) = -\frac{1}{3} - \frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum\frac{(-1)^n}{n}\sin n\pi x$$

EXERCISE - VI

Obtain Fourier Series for the following functions.

1. $f(x) = a^2 - x^2$ in $(-a, a)$ **(M.U. 2004)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{2a^2}{3} + \frac{4a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{a} - \frac{1}{2^2}\cos\frac{2\pi x}{a} + \frac{1}{3^2}\cos\frac{3\pi x}{a} - \ldots\right]\right]$$

2. $f(x) = \sin 2x$ in $(-l, l)$. (**Hint :** Put $a = 2$ in the solved ex. 4 above.)

$$\left[\textbf{Ans. :}\ f(x) = 2\pi\sin 2l\left[\frac{1}{\pi^2 - 2^2 l^2}\sin\frac{\pi x}{l} - \frac{2}{2^2\pi^2 - 2^2 l^2}\sin\frac{2\pi x}{l} + \ldots\right]\right]$$

3. $f(x) = \begin{cases} -\sin\dfrac{\pi x}{c}, & -c < x < 0 \\ \sin\dfrac{\pi x}{c}, & 0 < x < c \end{cases}$ $\left[\textbf{Ans. :}\ f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \ldots\right]\right]$

(M.U. 1999)

4. $f(x) = x^2,\ -1 < x < 1.$ (**Hint :** Put $l = 1$ in the solved ex. 1 above.)

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x - \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \ldots\right]\right]$$

5. $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ **(M.U. 2003)** [**Ans. :** See Ex. 3, page 3-58]

6. $f(x) = 9 - x^2$ in $(-3, 3)$. **(M.U. 2002, 15)**

(**Hint :** Put $a = 3$ in the above example 1 of this exercise.)

$$\left[\textbf{Ans. :}\ f(x) = 6 + \frac{36}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3} - \frac{1}{2^2}\cos\frac{2\pi x}{3} + \frac{1}{3^2}\cos\frac{3\pi x}{3} - \ldots\right]\right]$$

7. $f(x) = x + x^2$ in $(-1, 1)$ **(M.U. 1994)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum\frac{(-1)^n}{n}\sin n\pi x\right]$$

8. $f(x) = x - x^3$ in $(-1, 1)$. **(M.U. 2001)** $\left[\textbf{Ans. :}\ f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi x}{n^3}\right]$

9. $f(x) = x^2 - 2,\ -2 \le x \le 2$ **(M.U. 2000, 04, 05)**

$$\left[\textbf{Ans. :}\ f(x) = -\frac{2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2} \ldots\right]\right]$$

10. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ $\left[\textbf{Ans. :}\ f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\frac{1}{1}\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{2\pi x}{2} + \frac{1}{5}\cos\frac{5x}{2} - \ldots\right]\right]$

(M.U. 1996)

11. $f(x) = e^{-x}(-a, a)$. **(M.U. 2003)** $\left[\textbf{Ans. :}\ f(x) = \frac{\sinh a}{a} + 2a\sinh a\sum\frac{(-1)^n}{a^2 + n^2\pi^2}\cos\frac{n\pi x}{a}\right.$

$$\left. + 2\pi\sinh a\sum\frac{(-1)^{n+1}\cdot n}{a^2 + n^2\pi^2}\sin\frac{n\pi x}{a}\right]$$

12. $f(x) = \begin{cases} a(x - l), & -l < x < 0 \\ a(l + x), & 0 < x < l \end{cases}$. Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$$\left[\textbf{Ans. :}\ f(x) = \frac{2al}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right]\sin\frac{n\pi x}{l}\text{. Then put } x = \frac{l}{2}.\right]$$

13. Half-Range Series

We have seen in § 11, page 3-51 that in $(-l, l)$, $f(x)$ can be expanded in Fourier Series as

$$f(x) = a_0 + \sum a_n\cos\left(\frac{n\pi x}{l}\right) + \sum b_n\sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x)\, dx$, $a_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$ and $b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Further by the properties of even function if $f(x)$ is even in $(-l, l)$ then

$$a_0 = \frac{1}{2l}\cdot 2\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{1}{l}\cdot 2\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

and $b_n = 0$

Thus, if we imagine an even function in $(-l, l)$ *i.e.* if we define $f(x)$ in $(0, l)$ and take its reflection in the y-axis so that $f(x)$ becomes even in $(-l, l)$ then Fourier Series of $f(x)$ is given by

$$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

where, $\boxed{a_0 = \frac{1}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \ \text{ and } \ b_n = 0}$

Such a series is called **half range cosine series**. Clearly this expansion does not contain sine terms.

Similarly, if we define $f(x)$ in $(0, l)$ and take its reflection in the x-axis so that $f(x)$ becomes odd in $(-l, l)$ then by the properties of odd function, $f(x)$ is given by

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

where, $\boxed{b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \ \ a_0 = 0 \ \text{ and } \ a_n = 0}$

Such a series is called **half-range sine series**. Clearly this expansion does not contain cosine terms.

Example 1 : Find a cosine series of period 2π to represent $\sin x$ in $0 \le x \le \pi$.

(M.U. 1999, 2003, 04, 05, 10, 1

Sol. : Let $f(x) = a_0 + \sum a_n \cos nx \qquad [\because l = \pi]$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^\pi = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi \sin x\cos nx\,dx \qquad \text{.......(1)}$$

$$= \frac{1}{\pi}\int_0^\pi [\sin(1+n)x + \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$\therefore \ a_n = \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left(-\frac{2}{n^2-1}\right) - \frac{2}{n^2-1}\right] = -\frac{2}{\pi(n^2-1)}[(-1)^n + 1]$$

$= 0$ if n is odd and $n \ne 1$.

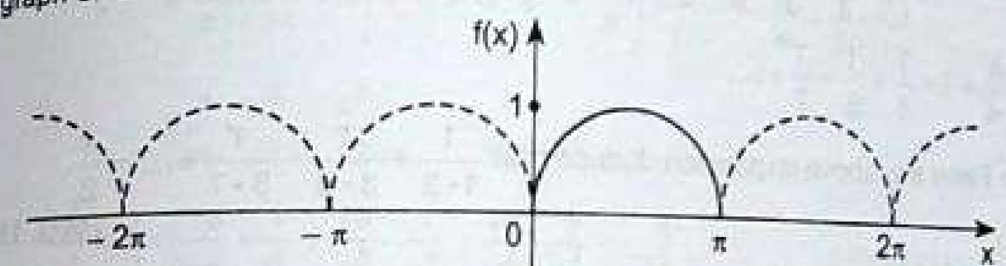
$$\therefore \ a_n = -\frac{4}{\pi(n^2-1)} \ \text{ if } n \text{ is even.}$$

$\therefore$ If $n = 1$ from (1), we get

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi \sin 2x\,dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^\pi = 0$$

$$\therefore \ f(x) = \sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \dots\right) \qquad \text{.......(2)}$$

The graph of the function is given below.



Cor. 1 : Hence, deduce that

$$\frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \frac{1}{5^2\cdot 7^2} + \dots = \frac{\pi^2 - 8}{16}$$

(M.U. 1996, 2011)

Sol. : By Paresval's identity (Cor. 6, page 3-8), we have

$$\frac{1}{\pi}\int_0^\pi [f(x)]^2\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \dots\right]$$

$$\therefore \ \frac{1}{\pi}\int_0^\pi \sin^2 x\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \dots\right]$$

Now, $\frac{1}{\pi}\int_0^\pi \sin^2 x\,dx = \frac{1}{\pi}\int_0^\pi\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2\pi}[\pi] = \frac{1}{2}$

$$\therefore \ \frac{1}{2} = \frac{1}{2}\left[2\cdot\frac{4}{\pi^2} + \frac{16}{\pi^2}\left\{\frac{1}{3^2} + \frac{1}{15^2} + \frac{1}{35^2} + \dots\right\}\right]$$

$$1 = \frac{8}{\pi^2} + \frac{16}{\pi^2}\left[\frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \frac{1}{5^2\cdot 7^2} + \dots\right]$$

$$1 - \frac{8}{\pi^2} = \frac{16}{\pi^2}\left[\frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \frac{1}{5^2\cdot 7^2} + \dots\right]$$

$$\frac{\pi^2 - 8}{16} = \frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \frac{1}{5^2\cdot 7^2} + \dots$$

Cor. 2 : From the above expansion deduce that **(M.U. 2005)**

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Sol. : The series (2) can be written as

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}\cos 2nx$$

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\cos n\pi$$

Now, putting $x = \frac{\pi}{2}$, we get

$$\therefore \quad \frac{\pi}{2} = 1 - 2\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)(-1)^n$$

$$= 1 - 2\left[\frac{1}{2}\left\{-\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)-\ldots\right\}\right]$$

$$= 1 - \left[-\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\ldots\right]$$

$$= 1 + \left[1-\frac{1}{3}-\frac{1}{3}+\frac{1}{5}+\frac{1}{5}-\frac{1}{7}-\frac{1}{7}+\ldots\right] = 2\cdot\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Cor. 3 : From the above expansion deduce that $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots = \frac{1}{2}$.

(M.U. 1997, 2

Sol. : Putting $x = 0$ in (2) we get,

$$-\frac{2}{\pi} = -\frac{4}{\pi}\left[\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\ldots\right] \qquad \therefore \quad \frac{1}{2} = \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots$$

Example 2 : Obtain half range sine series for $f(x)$ when

$$f(x) = \begin{cases} x, & 0 < x < (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$$

(M.U. 2

Hence, find the sum of $\sum_{(2n-1)}^{\infty}\frac{1}{n^4}$. **(M.U. 2002, 03, 07, 08, 11, 1**

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

Sol. : Let $f(x) = \sum b_n \sin nx \qquad [\because l = \pi]$

$$\therefore \quad b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)(1)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)(-1\right.\right.$$

$$= \frac{2}{\pi}\left[\left\{-\frac{\pi}{2}\frac{\cos(n\pi/2)}{n}+\frac{\sin(n\pi/2)}{n^2}-0-0\right\}\right.$$

$$\left. + \left\{0 - 0 + \frac{\pi}{2}\frac{\cos(n\pi/2)}{n}+\frac{\sin(n\pi/2)}{n^2}\right\}\right] \qquad \text{[By § 5, page 3-1}$$

$$= \frac{4}{\pi}\cdot\frac{\sin(n\pi/2)}{n^2}$$

$$\therefore \quad b_1 = \frac{4}{\pi}\cdot\frac{1}{1^2},\ b_2 = 0,\ b_3 = -\frac{4}{\pi}\cdot\frac{1}{3^2},\ b_4 = 0\ldots$$

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \ldots\right] \qquad \ldots\ldots (1)$$

By Parseval's identity (Cor. 7, page 3-8)

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\infty\right]$$

$$\therefore \quad \frac{1}{\pi}\left[\int_0^{\pi/2} x^2\,dx + \int_{\pi/2}^{\pi}(\pi - x)^2\,dx\right] = \frac{1}{2}\left[b_1^2 + b_2^2 + \ldots\infty\right]$$

Now, $\frac{1}{\pi}\left[\int_0^{\pi/2} x^2\,dx + \int_{\pi/2}^{\pi}(\pi^2 - 2\pi x + x^2)\,dx\right] = \frac{1}{\pi}\left[\left\{\frac{x^3}{3}\right\}_0^{\pi/2} + \left\{\pi^2 x - \pi x^2 + \frac{x^3}{3}\right\}_{\pi/2}^{\pi}\right]$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi^3}{24} - 0\right\} + \left\{\left(\pi^3 - \pi^3 + \frac{\pi^3}{3}\right) - \left(\frac{\pi^3}{2} - \frac{\pi^3}{4} + \frac{\pi^3}{24}\right)\right\}\right]$$

$$= \frac{1}{\pi}\cdot\frac{\pi^3}{12} = \frac{\pi^2}{12}$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{2}\cdot\left[\frac{16}{\pi^2}\cdot\frac{1}{1^4} + \frac{16}{\pi^2}\cdot\frac{1}{3^4} + \frac{16}{\pi^2}\cdot\frac{1}{5^4} + \ldots\right] \qquad \text{[From (1)]}$$

$$\therefore \quad \frac{\pi}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$$

For deduction put $x = \pi/2$ in (1).

$$\therefore \quad \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] \qquad \therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 3 : Find half range cosine series for $f(x) = x$, $0 < x < 2$. Using Parseval's identity, deduce that

(i) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ **(M.U. 2001, 02, 05, 06, 08, 09, 13, 16)**

(ii) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$

Sol. : Let $f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$. Here, $l = 2$.

$$\therefore \quad a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left[x\frac{\sin(n\pi x/2)}{n\pi/2} + \frac{\cos(n\pi x/2)}{n^2\pi^2/2^2}\cdot 1\right]_0^2 \qquad \text{[By § 5, page 3-10]}$$

$$\therefore\ a_n = \left[2\cdot(0) + \frac{\cos n\pi}{n^2\pi^2/2^2} - 0 - \frac{1}{n^2\pi^2/2^2}\right] = \frac{[(-1)^n - 1]}{n^2\pi^2/2^2}$$

$$= \begin{cases} -4\cdot\dfrac{2}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\therefore\ x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots.\right]$$

(i) By Parseval's identity (Cor. 8, page 3-8)

$$\frac{1}{l}\int_0^l [f(x)]^2\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + \ldots.\right]$$

$$\therefore\ \text{l.h.s.} = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

$$\therefore\ \frac{4}{3} = \frac{1}{2}\left[2 + \frac{64}{\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots.\right\}\right]$$

$$\frac{8}{3} - 2 = \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots.\right] \qquad \therefore\ \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots..$$

(ii) Let $S = \dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \dfrac{1}{4^4} + \ldots.$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots.\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots.\right)$$

$$= \left(\frac{\pi^4}{96}\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots.\right)$$

$$\therefore\ S = \frac{\pi^4}{96} + \frac{S}{16} \qquad \therefore\ S = \frac{\pi^4}{90}.$$

Example 4 : Find half range sine series for $x \sin x$ in $(0, \pi)$ and hence, deduce that

$$\frac{\pi^2}{8\sqrt{2}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \ldots.$$

(M.U. 2004

Sol. : Let $f(x) = \sum b_n \sin nx \quad [\because l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin x \sin nx\,dx$$

$$= -\frac{1}{\pi}\int_0^\pi x[\cos(n+1)x - \cos(n-1)x]\,dx$$

$$= -\frac{1}{\pi}\left[x\left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\cos(n+1)x}{(n+1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$= -\frac{1}{\pi}\left[\pi\left\{\frac{\sin(n+1)\pi}{n+1} - \frac{\sin(n-1)\pi}{n-1}\right\} - \left\{-\frac{\cos(n+1)\pi}{(n+1)^2} + \frac{\cos(n-1)\pi}{(n-1)^2}\right\}\right.$$

$$\left. - 0 - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right] \text{ if } n \neq 1$$

[By § 5, page 3-

Now, $\sin(n \pm 1)\pi = 0$ and $\cos(n \pm 1)\pi = -\cos n\pi$.

$$= -\frac{1}{\pi}\left[-\frac{\cos n\pi}{(n+1)^2} + \frac{\cos n\pi}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right]$$

$$= -\frac{1}{\pi}\begin{cases} 0 & \text{when } n \text{ is odd and } n \neq 1 \\ 2\left\{\dfrac{1}{(n-1)^2} - \dfrac{1}{(n+1)^2}\right\} & \text{when } n \text{ is even} \end{cases}$$

When $n = 1$ from (1),

$$b_1 = \frac{2}{\pi}\int_0^\pi x \sin^2 x\,dx = \frac{2}{\pi}\int_0^\pi x\,\frac{(1-\cos 2x)}{2}\,dx$$

$$= \frac{1}{\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^\pi \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{\pi}\left[\pi^2 - 0 - \frac{\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{\pi}{2}$$

$$\therefore\ f(x) = x \sin x$$

$$= \frac{\pi}{2}\sin x - \frac{2}{\pi}\left[\left(\frac{1}{1^2} - \frac{1}{3^2}\right)\sin 2x + \left(\frac{1}{3^2} - \frac{1}{5^2}\right)\sin 4x + \left(\frac{1}{5^2} - \frac{1}{7^2}\right)\sin 6x + \ldots.\right]$$

$$= \frac{\pi}{2}\sin x + \frac{2}{\pi}\left[\left(\frac{1}{3^2} - \frac{1}{1^2}\right)\sin 2x + \left(\frac{1}{5^2} - \frac{1}{3^2}\right)\sin 4x + \left(\frac{1}{7^2} - \frac{1}{5^2}\right)\sin 6x + \ldots.\right]$$

For deduction put $x = \dfrac{\pi}{4}$.

$$\therefore\ \frac{\pi}{4}\sin\frac{\pi}{4} = \frac{\pi}{2}\,\frac{1}{\sqrt{2}} + \frac{2}{\pi}\left[\left(\frac{1}{3^2} - \frac{1}{1^2}\right)\cdot 1 + \left(\frac{1}{5^2} - \frac{1}{3^2}\right)\cdot 0 + \left(\frac{1}{7^2} - \frac{1}{5^2}\right)\cdot(-1) + \ldots.\right]$$

$$\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} = \frac{\pi}{2}\cdot\frac{1}{\sqrt{2}} + \frac{2}{\pi}\left[\frac{1}{3^2} - \frac{1}{1^2} + \frac{1}{7^2} - \frac{1}{5^2} + \ldots.\right]$$

$$-\frac{\pi}{4\sqrt{2}} = \frac{2}{\pi}\left[\frac{1}{3^2} - \frac{1}{1^2} + \frac{1}{7^2} - \frac{1}{5^2} + \ldots.\right]$$

$$\frac{\pi^2}{8\sqrt{2}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots.$$

Example 5 : In $(0, \pi)$ show that $x^2 = \dfrac{2}{\pi}\left[\left(\dfrac{\pi^2}{1} - \dfrac{4}{1^3}\right)\sin x - \dfrac{\pi^2}{2}\sin 2x + \left(\dfrac{\pi^2}{3} - \dfrac{4}{3^3}\right)\sin 3x \ldots.\right]$

(M.U. 2002, 12)

Sol. : Let $f(x) = \sum b_n \sin nx \quad [\because l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin nx\,dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{n}(-\cos n\pi) - 0 + \frac{2}{n^3}(\cos n\pi) - \frac{2}{n^3}\right]$$

$$= \begin{cases} \dfrac{2}{\pi}\left[\dfrac{\pi^2}{n} - \dfrac{4}{n^3}\right] & \text{if } n \text{ is odd} \\[2ex] \dfrac{2}{\pi}\left[-\dfrac{\pi^2}{n}\right] & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x - \ldots + \ldots\right]$$

Example 6 : Expand $f(x) = \begin{cases} kx, & 0 < x < l/2 \\ k(l-x), & l/2 < x < l \end{cases}$ into half range cosine series.

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **(M.U. 1998, 2003, 1**

Sol. : Let $f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$

$$\therefore \quad a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^l k(l-x)\,dx\right]$$

$$= \frac{1}{l}\left\{\left[k\frac{x^2}{2}\right]_0^{l/2} + \left[k\,l\,x - k\frac{x^2}{2}\right]_{l/2}^l\right\}$$

$$= \frac{1}{l}\left[\frac{kl^2}{8} + kl^2 - \frac{kl^2}{2} - \frac{kl^2}{2} + \frac{kl^2}{8}\right]$$

$$= \frac{1}{l}\cdot\frac{kl^2}{4} = \frac{kl}{4}$$

$$a_n = \frac{2}{l}\left[\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^l k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{kx\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - k\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_0^{l/2}\right.$$

$$\left. + \left\{k(l-x)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (-k)\left(\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_{l/2}^l\right] \quad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{l}\left[\frac{kl}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right.$$

$$\left. - \frac{kl}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$a_1 = 0,\ a_2 = -\frac{8kl}{2^2\pi^2},\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = -\frac{8kl}{6^2\pi^2},\ \ldots\ldots a_{10} = -\frac{8kl}{10^2\pi^2}$$

$$\therefore \quad f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right]$$

Now, put $x = l/2$,

$$\therefore \quad \frac{kl}{2} - \frac{kl}{4} = \frac{8kl}{\pi^2}\left[\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \ldots\right] \qquad \therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 7 : Find half range cosine series for

$$f(x) = \begin{cases} 1, & 0 < x < (a/2) \\ -1, & (a/2) < x < a \end{cases}$$

Sol. : Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{a}$ $\quad [\because\ l = a]$

$$a_0 = \frac{1}{a}\left[\int_0^{a/2} dx - \int_{a/2}^a dx\right] = \frac{1}{a}\left[\{x\}_0^{a/2} - \{x\}_{a/2}^a\right] = \frac{1}{a}\left[\frac{a}{2} - a + \frac{a}{2}\right] = 0$$

$$= \frac{2}{a}\left[\int_0^{a/2}\cos\frac{n\pi x}{a}\,dx - \int_{a/2}^a\cos\frac{n\pi x}{a}\,dx\right]$$

$$= \frac{2}{a}\left[\left\{\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right\}_0^{a/2} - \left\{\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right\}_{a/2}^a\right]$$

$$= \frac{2}{a}\left[\frac{a}{n\pi}\sin\frac{n\pi}{2} - \frac{a}{n\pi}\sin n\pi + \frac{a}{n\pi}\sin\frac{n\pi}{2}\right] = \frac{2}{n\pi}\left[2\sin\frac{n\pi}{2}\right]$$

$$\therefore \quad a_1 = \frac{4}{\pi},\quad a_3 = -\frac{4}{3\pi},\quad a_5 = \frac{4}{5\pi}$$

$$a_2 = 0,\quad a_4 = 0,\quad a_6 = 0\ \ldots\ldots$$

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\frac{1}{1}\cos\frac{\pi x}{a} - \frac{1}{3}\cos\frac{\pi x}{a} + \frac{1}{5}\cos\frac{5\pi x}{a}\ \ldots\right]$$

Example 8 : If $f(x) = \begin{cases} \dfrac{x}{a}, & \text{for } 0 < x < a \\[1ex] \dfrac{l-x}{l-a}, & \text{for } a < x < l \end{cases}$

prove that $f(x) = \dfrac{2l^2}{a(l-a)\pi^2}\sum\dfrac{1}{n^2}\sin\dfrac{n\pi a}{l}\sin\dfrac{n\pi x}{l}$.

Sol. : We have to obtain half range sine series for $f(x)$.

Let $\quad f(x) = \sum b_n \sin\frac{n\pi x}{l}$

$$b_n = \frac{2}{l}\left[\int_0^a \frac{x}{a}\sin\frac{n\pi x}{l}\,dx + \int_a^l \frac{l-x}{l-a}\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{\frac{x}{a}\cdot\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{1}{a}\right)\cdot\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right\}_0^a\right.$$

$$\left. + \left\{\left(\frac{l-x}{l-a}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-1}{l-a}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right\}_a^l\right] \quad \text{[By § 5, page 3-10]}$$

$$\therefore\quad b_n=\frac{2}{l}\left[-\frac{l}{n\pi}\cos\frac{n\pi a}{l}+\frac{l^2}{an^2\pi^2}\sin\frac{n\pi a}{l}+\frac{l}{n\pi}\cos\frac{n\pi a}{l}+\frac{l^2}{(l-a)\,n^2\pi^2}\sin\frac{n\pi a}{l}\right]$$

$$=\frac{2l}{n^2\pi^2}\sin\frac{n\pi a}{l}\left[\frac{1}{a}+\frac{1}{l-a}\right]=\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi a}{l}\left(\frac{1}{a(l-a)}\right)$$

$$\therefore\quad b_n=\frac{2l}{n^2\pi^2}\sin\frac{n\pi a}{l}\cdot\left[\frac{1}{a}+\frac{1}{l-a}\right]=\frac{2l}{n^2\pi^2}\sin\frac{n\pi a}{l}\cdot\left(\frac{l}{a(l-a)}\right)$$

$$\therefore\quad f(x)=\frac{2l^2}{a(l-a)\pi^2}\sum\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

Example 9 : Obtain the expansion of $f(x)=x(\pi-x),\ 0<x<\pi$ as a half-range cosine series. Hence, show that

(i) $\sum_1^\infty\frac{1}{n^2}=\frac{\pi^2}{6}$, (ii) $\sum_1^\infty\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$,

(iii) $\sum_1^\infty\frac{1}{n^4}=\frac{\pi^4}{90}$ **(M.U. 1993, 96, 99, 2005, 09, 12, 14)**

Sol. : Let $f(x)=a_0+\sum a_n\cos nx$ $[\because\ l=\pi]$

$$a_0=\frac{1}{\pi}\int_0^\pi f(x)\,dx=\frac{1}{\pi}\int_0^\pi(\pi x-x^2)\,dx=\frac{1}{\pi}\left[\frac{\pi x^2}{2}-\frac{x^3}{3}\right]_0^\pi=\frac{\pi^2}{6}$$

$$a_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^\pi(\pi x-x^2)\cos nx\,dx$$

$$=\frac{2}{\pi}\left[(\pi x-x^2)\cdot\frac{\sin nx}{n}-(\pi-2x)\cdot\left(-\frac{\cos nx}{n^2}\right)+(-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$\therefore\quad a_n=-2\left(\frac{1+\cos n\pi}{n^2}\right)=\begin{cases}0 & \text{when } n \text{ is odd}\\ -4/n^2 & \text{when } n \text{ is odd}\end{cases}$$ [By § 5, page 3-10]

$$\therefore\quad x(\pi-x)=\frac{\pi^2}{6}-4\left[\frac{\cos 2x}{2^2}+\frac{\cos 4x}{4^2}+\frac{\cos 6x}{6^2}+\ldots\right]$$

$$=\frac{\pi^2}{6}-\left[\frac{1}{1^2}\cos 2x+\frac{1}{2^2}\cos 4x+\frac{1}{3^2}\cos 6x+\ldots\right]$$

(i) Now, put $x=0$,

$$\therefore\quad 0=\frac{\pi^2}{6}-\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)\qquad\therefore\quad\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

(ii) Again put $x=\frac{\pi}{2}$,

$$\therefore\quad\frac{\pi^2}{4}=\frac{\pi^2}{6}-\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\ldots\right]$$

$$\therefore\quad\frac{\pi^2}{4}-\frac{\pi^2}{6}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots\qquad\therefore\quad\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots$$

(iii) Now, by Parseval's identity (Cor. 6, page 3-8).

$$\frac{1}{\pi}\int_0^\pi f(x)^2\,dx=\frac{1}{2}\left[2a_0^2+a_1^2+a_2^2+a_3^2+\ldots\right]$$

$$\therefore\quad\frac{1}{\pi}\int_0^\pi x^2(\pi-x)^2\,dx=\frac{1}{2}\left[2a_0^2+a_1^2+a_2^2+a_3^2+\ldots\right]$$

Now, $$\frac{1}{\pi}\int_0^\pi x^2(\pi-x)^2\,dx=\frac{1}{\pi}\int_0^\pi(\pi^2x^2-2\pi x^3+x^4)\,dx$$

$$=\frac{1}{\pi}\left[\pi^2\frac{x^3}{3}-2\pi\frac{x^4}{4}+\frac{x^5}{5}\right]_0^\pi=\frac{1}{\pi}\cdot\frac{\pi^5}{30}=\frac{\pi^4}{30}$$

$$\therefore\quad\frac{\pi^4}{30}=\frac{1}{2}\left[2\cdot\frac{\pi^4}{36}+\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\right]$$

$$\frac{\pi^4}{15}-\frac{\pi^4}{18}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots$$

$$\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots$$

Example 10 : Obtain half-range sine series in $(0,\pi)$ for $x(\pi-x)$. **(M.U. 2015)**

Hence, find the value of $\sum\frac{(-1)^n}{(2n-1)^3}$. **(M.U. 1996, 2013, 14)**

Sol. : Let $f(x)=\sum b_n\sin nx$

$$\therefore\quad b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^\pi x(\pi-x)\sin nx\,dx$$

$$=\frac{2}{\pi}\left[x(\pi-x)\left(-\frac{\cos nx}{n}\right)-(\pi-2x)\left(-\frac{\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$=\frac{2}{\pi}\left[\left\{0-0-\frac{2}{n^3}\cos(n\pi)\right\}-\left\{0-0-\frac{2}{n^3}\right\}\right]$$ [By § 5, page 3-10]

$$=\frac{4}{\pi}\left[\frac{1-\cos n\pi}{n^3}\right]=\frac{4}{\pi n^3}\left[1-(-1)^n\right]$$ $[\because\ \cos n\pi=(-1)^n]$

$$\therefore\quad b_n=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{8}{\pi n^3} & \text{if } n \text{ is odd}\end{cases}$$

$$\therefore\quad x(\pi-x)=\frac{8}{\pi}\left[\frac{1}{1^3}\sin\pi x+\frac{1}{3^3}\sin 3\pi x+\frac{1}{5^3}\sin 5x+\ldots\right]$$

Now, put $x=\frac{\pi}{2}$,

$$\therefore\quad\frac{\pi}{2}\left(\pi-\frac{\pi}{2}\right)=\frac{8}{\pi}\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots\right]$$

$$\frac{\pi^3}{32}=\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots$$

Example 11 : Show that $\cos x = \frac{8}{\pi}\sum_{m=1}^{\infty} \frac{m}{4m^2-1}\sin 2mx$ if $0 < x < \pi$. **(M.U. 2002, 03, 04)**

Sol. : We have to obtain half-range sine series for $\cos x$

Let $\cos x = \sum b_n \sin nx \qquad [\because l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}\cos x \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(1+n)x - \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right] \qquad \text{..... (1)}$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos \pi x}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right] \qquad [\because \cos(n \pm 1)\pi = -\cos n\pi]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1}\right](1+\cos n\pi) = \frac{1}{\pi}\cdot\frac{2n}{n^2-1}[1+(-1)^n]$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd and } n \neq 1 \\ \frac{1}{\pi}\cdot\frac{4n}{n^2-1}, & \text{is } n \text{ is even} \end{cases}$$

When $n = 1$, from (1), we get,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \sin x\, dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\, dx$$

$$\therefore\ b_1 = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}[1-1] = 0$$

$$\therefore\ \cos x = \frac{4}{\pi}\left[\frac{2}{2^2-1}\sin 2x + \frac{4}{4^2-1}\sin 4x + \frac{6}{6^2-1}\sin 6x + \ldots\right]$$

$$= \frac{8}{\pi}\left[\frac{1}{2^2-1}\sin 2x + \frac{2}{4^2-1}\sin 4x + \frac{3}{6^2-1}\sin 6x + \ldots\right]$$

$$= \frac{8}{\pi}\sum\frac{m}{4m^2-1}\sin 2m\pi.$$

Example 12 : Expand $f(x) = lx - x^2$, $0 < x < l$ in a half-range (i) cosine series, (ii) sine series. **(M.U. 2013, 14, 15)**

Hence, from sine series deduce that

(i) $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$ **(M.U. 1994, 2003, 05, 06, 07, 08)**

(ii) $\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \ldots$ (iii) $\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$

Sol. : (i) Cosine Series

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l}$

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left[l\frac{x^2}{2} - \frac{x^3}{3}\right]_0^l = \frac{1}{l}\cdot\frac{l^3}{6} = \frac{l^2}{6}$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 - l\cdot\frac{l^2}{n^2\pi^2}\cos n\pi - 0\right\} - \left\{0 + l\cdot\frac{l^2}{n^2\pi^2} + 0\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= -\frac{2}{l}\cdot\frac{l^3}{n^2\pi^2}[\cos n\pi + 1] = -\frac{2l^2}{n^2\pi^2}[(-1)^n + 1]$$

$$\therefore\ a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\frac{4l^2}{n^2\pi^2}, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore\ f(x) = lx - x^2 = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\right]$$

(ii) Sine series

Let $f(x) = \sum b_n \sin\frac{n\pi x}{l}$

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 - 0 - \frac{2l^3}{n^3\pi^3}\cos n\pi\right\} - \left\{0 - 0 - \frac{2l^3}{n^3\pi^3}\right\}\right] = \frac{2}{l}\cdot\frac{2l^3}{n^3\pi^3}(-\cos n\pi + 1)$$

$$= \begin{cases} \frac{8l^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \qquad \text{[By § 5, page 3-10]}$$

$$\therefore\ f(x) = lx - x^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \ldots\right]$$

(i) For deduction, we put, $x = \frac{l}{2}$.

$$\therefore\ l\cdot\left(\frac{l}{2}\right) - \left(\frac{l}{2}\right)^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right] \qquad \therefore\ \frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$

(ii) By corollary 9, page 3-9

$$\frac{1}{l}\int_0^l [f(x)]^2\,dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\infty\right]$$

$$\therefore \frac{1}{l}\int_0^l [lx - x^2]^2\,dx = \frac{1}{2}\left(\frac{8l^2}{\pi^3}\right)^2\left[\left(\frac{1}{1^3}\right)^2 + \left(\frac{1}{3^3}\right)^2 + \left(\frac{1}{5^3}\right)^2 + \ldots\right]$$

$$\frac{1}{l}\int_0^l (l^2x^2 - 2lx^3 + x^4)\,dx = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right]$$

$$\frac{1}{l}\left[l^2\frac{x^3}{3} - 2l\frac{x^4}{4} + \frac{x^5}{5}\right]_0^l = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right]$$

$$\therefore \frac{1}{30}l^4 = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right] \quad \therefore \quad \frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots$$

(iii) Let $S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots$

$$= \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots\right)$$

$$\therefore S = \frac{\pi^6}{960} + \frac{1}{2^6}\left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} \ldots\right) = \frac{\pi^6}{960} + \frac{1}{64}S.$$

$$\therefore S - \frac{S}{64} = \frac{\pi^6}{960} \quad \therefore \frac{63}{64}S = \frac{\pi^6}{960} \quad \therefore S = \frac{\pi^6}{960}\cdot\frac{64}{63} = \frac{\pi^6}{945}$$

$$\therefore \frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$$

Example 13 : Find half range cosine series for $f(x) = \begin{cases} x, & 0 < x < (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$. (M.U. 202

Sol. : Let $f(x) = a_0 + \sum a_n \cos nx \qquad [\because \; l = \pi]$

$$\therefore a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{x^2}{2}\right\}_0^{\pi/2} + \left\{\pi x - \frac{x^2}{2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\cdot\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

[By § 5, page 3-1

$$a_n = \frac{2}{\pi}\left[\left\{\frac{\pi}{2}\cdot\frac{1}{n}\cdot\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right\} + \left\{-\frac{1}{n^2}\cos n\pi - \frac{\pi}{2}\cdot\frac{1}{n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right\}\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$$\therefore a_1 = 0, \; a_2 = \frac{2}{\pi\cdot 2^2}[2(-1) - (+1) - 1] = -\frac{8}{\pi\cdot 2^2}$$

$$a_3 = 0, \; a_4 = 0, \; a_5 = 0$$

$$a_6 = \frac{2}{\pi\cdot 6^2}[2(-1) - (+1) - 1] = -\frac{8}{\pi\cdot 6^2}$$

$$a_7 = 0, \; a_8 = 0, \; a_9 = 0$$

$$a_{10} = \frac{2}{\pi\cdot 10^2}[2(-1) - (+1) - 1] = -\frac{8}{\pi\cdot 10^2}$$

$$\therefore f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \ldots\right]$$

Example 14 : Obtain half range sine series to represent

$$f(x) = \begin{cases} \dfrac{2x}{3}, & 0 \le x \le \dfrac{\pi}{3} \\ \dfrac{\pi - x}{3}, & \dfrac{\pi}{3} \le x \le \pi \end{cases}$$

Sol. : Let $f(x) = \sum b_n \sin n\pi x \qquad [\because \; l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin n\pi x\,dx = \frac{2}{\pi}\left[\int_0^{\pi/3}\frac{2x}{3}\sin n\pi x\,dx + \int_{\pi/3}^{\pi}\left(\frac{\pi - x}{3}\right)\sin n\pi x\,dx\right]$$

$$= \frac{2}{3\pi}\left[\int_0^{\pi/3} 2x\sin n\pi x\,dx + \int_{\pi/3}^{\pi}(\pi - x)\sin n\pi x\,dx\right]$$

$$= \frac{2}{3\pi}\left[\left\{(2x)\left(-\frac{\cos n\pi x}{n}\right) - (2)\left(-\frac{\sin n\pi x}{n^2}\right)\right\}_0^{\pi/3}\right.$$

$$\left. + \left\{(\pi - x)\left(-\frac{\cos n\pi x}{n}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2}\right)\right\}_{\pi/3}^{\pi}\right] \quad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{3\pi}\left[\left\{\left(\frac{2\pi}{3}\right)\left(-\frac{1}{n}\cos\frac{n\pi}{3}\right) + \frac{2}{n^2}\sin\frac{n\pi}{3}\right\} + \left\{0 + \frac{2\pi}{3}\left(\frac{1}{n}\cos\frac{n\pi}{3}\right) + \frac{1}{n^2}\sin\frac{n\pi}{3}\right\}\right]$$

$$= \frac{2}{3\pi}\cdot\frac{3}{n^2}\sin\frac{n\pi}{3} = \frac{2}{\pi}\cdot\frac{1}{n^2}\sin\frac{n\pi}{3}$$

$$\therefore f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx$$

$$\therefore\ f(x) = \frac{2}{\pi}\left[\frac{1}{1^2}\cdot\frac{\sqrt{3}}{2}\sin x + \frac{1}{2^2}\frac{\sqrt{3}}{2}\cdot\cos 2x - \frac{1}{4^2}\cdot\frac{\sqrt{3}}{2}\sin 4x - \frac{1}{5^2}\cdot\frac{\sqrt{3}}{2}\sin 5x + \ldots\right]$$

$$\therefore\ f(x) = \frac{\sqrt{3}}{\pi}\left[\frac{1}{1^2}\sin x + \frac{1}{2^2}\sin 2x - \frac{1}{4^2}\sin 4x - \frac{1}{5^2}\sin 5x + \ldots\right]$$

Example 15 : Find half-range sine series for $f(x) = \frac{\pi}{4}$ in $(0, \pi)$. Hence, show that

(i) $\frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots$

(ii) $\frac{\pi}{8}x(\pi - x) = \frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots$

(iii) $\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$

Sol. : Let $f(x) = \sum b_n \sin nx$

$$\therefore\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{\pi}{4}\sin nx\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{1}{2n}[\cos n\pi - 1] = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}$$

$$\therefore\ \frac{\pi}{4} = \frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots$$

(i) Now, integrate both sides of (1) w.r.t. x from 0 to x.

$$\therefore\ \frac{\pi}{4}\int_0^x dx = \frac{1}{1}\int_0^x \sin dx + \frac{1}{3}\int_0^x \sin 3x\,dx + \ldots$$

$$\frac{\pi}{4}[x]_0^x = \frac{1}{1}[-\cos x]_0^x + \frac{1}{3}\left[-\frac{\cos 3x}{3}\right]_0^x + \ldots$$

$$\frac{\pi x}{4} = \frac{1}{1^2}(1 - \cos x) + \frac{1}{3^2}(1 - \cos 3x) + \ldots$$

Now, put $x = \frac{\pi}{2}$ in (2)

$$\therefore\ \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

Now, subtract (2) from (3)

$$\therefore\ \frac{\pi^2}{8} - \frac{\pi x}{4} = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots$$

$$\therefore\ \frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots$$

(ii) Now, integrate (4) w.r.t. x from 0 to x,

$$\frac{\pi}{4}\int_0^x\left(\frac{\pi}{2} - x\right)dx = \frac{1}{1^2}\int_0^x \cos x\,dx + \frac{1}{3^2}\int_0^x \cos 3x\,dx + \ldots$$

$$\frac{\pi}{4}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^x = \frac{1}{1^2}\left[\frac{\sin x}{1}\right]_0^x + \frac{1}{3^2}\left[\frac{\sin 3x}{3}\right]_0^x + \ldots$$

$$\therefore\ \frac{\pi}{8}x(\pi - x) = \frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots \qquad \ldots(5)$$

(iii) Now, put $x = \frac{\pi}{2}$ in (5),

$$\therefore\ \frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots \qquad \ldots(6)$$

Example 16 : Prove that in the interval $0 < x < \pi$,

$$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2 + 1} - \frac{2\sin 2x}{a^2 + 4} + \frac{3\sin 3x}{a^2 + 9} - \ldots\right]$$ **(M.U. 1996, 2003, 05, 08, 11)**

Sol. : Let $f(x) = e^{ax} - e^{-ax},\ 0 < x < \pi,\ \ [\because l = \pi]$

Since, we want $f(x)$ to be expanded in sine terms only.

Let $f(x) = \sum b_n \sin nx$

$$b_n = \int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(e^{ax} - e^{-ax})\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} e^{ax}\sin nx\,dx - \int_0^{\pi} e^{-ax}\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{e^{ax}}{a^2 + n^2}(a\sin nx - n\cos nx)\right\}_0^{\pi} - \left\{\frac{e^{-ax}}{a^2 + n^2}(-a\sin nx - n\cos nx)\right\}_0^{\pi}\right]$$

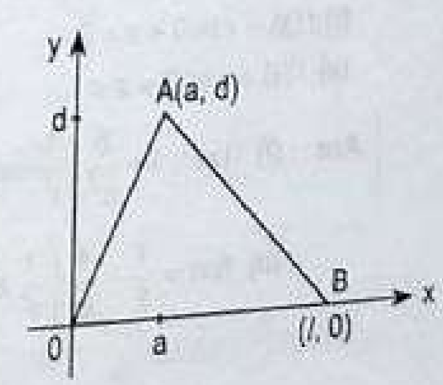
$$= \frac{2}{\pi}\left[\left\{\frac{e^{a\pi}}{a^2 + n^2}(-n\cos n\pi) + \frac{n}{a^2 + n^2}\right\} - \left\{\frac{e^{-a\pi}}{a^2 + n^2}(-n\cos n\pi) + \frac{n}{a^2 + n^2}\right\}\right]$$

$$\therefore\ b_n = \frac{2}{\pi}\cdot\frac{e^{a\pi} - e^{-a\pi}}{a^2 + n^2}(-n\cos n\pi) \qquad \text{[By (2), page 3-4]}$$

$$\therefore\ \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2 + n^2}(-n\cos n\pi)\sin nx$$

$$= \frac{2}{\pi}\left[\frac{1}{a^2 + 1}\sin x - \frac{2}{a^2 + 2^2}\sin 2x + \frac{3}{a^2 + 3^2}\sin 3x - \ldots\right]$$

Example 17 : Find half range sine series for $f(x)$ given by the given graph in the interval $(0, l)$.



Sol. : Since, A is (a, d). The equation of OA is $\frac{y - 0}{0 - d} = \frac{x - 0}{0 - a}$.

$$\therefore\ y = \frac{d}{a}\cdot x$$

Similarly, the equation of AB is $\frac{y - d}{d - 0} = \frac{x - a}{a - l}$.

$$\therefore\ y = d + \frac{d(x - a)}{a - l}$$

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$$\therefore \quad y = \frac{da - dl + dx - da}{a - l} = \frac{d}{a - l}(x - l)$$

$$\therefore \quad f(x) = \begin{cases} \frac{d}{a} \cdot x, & 0 \le x \le a \\ \frac{d}{a-l}(x-l), & a \le x \le l \end{cases}$$

Let $f(x) = \sum b_n \sin \frac{n\pi x}{l}$

Now, refer to the solved example 8 above where $d = 1$.

Following the same steps we can obtain.

$$f(x) = \frac{2dl^2}{\pi^2 \cdot a \cdot (l - a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi a}{l} \cdot \sin \frac{n\pi x}{l}$$

Example 18 : Find half-range cosine series for $f(x) = e^x$, $0 < x < 1$. **(M.U. 2013, ...)**

Sol. : Let $f(x) = a_0 + \sum a_n \cos n\pi x \quad [\because l = 1]$

$$\therefore \quad a_0 = \frac{1}{1}\int_0^1 e^x dx = \left[e^x\right]_0^1 = e - 1$$

$$= \frac{2}{1}\int_0^1 e^x \cos n\pi x \, dx = 2\left[\frac{e^x}{1 + n^2\pi^2}(\cos n\pi x + n\pi \sin n\pi x)\right]_0^1$$

$$= 2\left[\frac{e}{1 + n^2\pi^2}(-1)^n - \frac{1}{1 + n^2\pi^2}\right]_0^1 \qquad \text{[By (3), page 3-4]}$$

$$\therefore \quad e^x = (e - 1) + 2\sum_{n=1}^{\infty} \frac{1}{1 + n^2\pi^2}\left[e(-1)^n - 1\right]\cos n\pi x$$

$$= (e - 1) + 2\left[-\frac{1}{1 + 1^2 \cdot \pi^2}(e + 1) \cdot \cos \pi x + \frac{1}{1 + 2^2 \cdot \pi^2}(e - 1)\cos 2\pi x - \frac{1}{1 + 3^2 \cdot \pi^2}(e + 1)\cos 3\pi x \ldots\right.$$

EXERCISE - VII

(A) 1. Obtain half range cosine series for

(i) $f(x) = x$ in $0 < x < 2$.

(ii) $f(x) = x$ in $0 < x < 1$. **(M.U. 199...)**

$$\left[\textbf{Ans. : (i) } f(x) = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos \frac{\pi x}{2} + \frac{1}{3^2}\cos \frac{3\pi x}{2} + \frac{1}{5^2}\cos \frac{5\pi x}{2} + \ldots\right]\right.$$

$$\left.\textbf{(ii) } f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos \pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \ldots\right]\right]$$

2. Obtain cosine series for the function $(mx + c)$ in the interval $0 \le x \le p$ and hence, deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$$\left[\textbf{Ans. : } f(x) = \left(\frac{mp}{2} + c\right) - \frac{4mp}{\pi^2}\left[\frac{1}{1^2}\cos \frac{\pi x}{p} + \frac{1}{3^2}\cos \frac{3\pi x}{p} + \frac{1}{5^2}\cos \frac{5\pi x}{p} + \ldots\right]\right.$$

Then put $x = 0$.

By putting $p = 2$, $m = 1$, $c = 0$, you get ex. 1(i), by putting $p = 1$, $m = 1$, $c = 0$ you get ex. 1(ii).]

3. Obtain half range cosine series for

$$f(x) = \begin{cases} 1, & 0 < x < (c/2) \\ 1 - x, & (c/2) < x < c \end{cases}$$

$$\left[\textbf{Ans. : } f(x) = \left(1 - \frac{3c}{8}\right) + \frac{2c}{\pi^2}\left[(\pi + 1)\cos \frac{\pi x}{c} - \frac{1}{2}\cos \frac{2\pi x}{c} + \frac{1}{9}\left(1 - \frac{3\pi}{2}\right)\cos \frac{3\pi x}{c} + \ldots\right]\right]$$

4. Obtain half-range sine series for $f(x) = x(2 - x)$ in $0 < x < 2$ and hence, deduce that

$$\sum \frac{1}{n^6} = \frac{\pi^6}{945}.$$

(M.U. 2003, 14)

(Put $l = 2$ in the solved Ex. 12, page 3-73.)

5. Obtain half range sine series for $f(x) = \begin{cases} x, & 0 < x < a \\ a, & a < x < \pi - a \\ \pi - x, & \pi - a < x < \pi \end{cases}$

$$\left[\textbf{Ans. : } f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin a \sin x + \frac{1}{3^2}\sin 3a \sin 3x + \frac{1}{5^2}\sin 5a \sin 5x + \ldots\right]\right]$$

6. Obtain half range sine series for $f(x) = \begin{cases} \pi/3, & 0 \le x < (\pi/3) \\ 0, & (\pi/3) \le x < (2\pi/3) \\ -\pi/3, & (2\pi/3) \le x \le \pi \end{cases}$

$$\left[\textbf{Ans. : } f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{10}\sin 10x + \ldots\right]$$

7. Find half range cosine and sine series for $f(x)$, where

$$f(x) = \begin{cases} \pi/3, & 0 \le x \le (\pi/3) \\ 0, & (\pi/3) \le x \le (2\pi/3) \\ -\pi/3, & (2\pi/3) \le x \le \pi \end{cases}$$

$$\left[\textbf{Ans. : (i) } f(x) = \frac{2}{\sqrt{3}}\left[\frac{1}{1}\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \ldots\right]\right.$$

$$\left.\textbf{(ii) } f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{10}\sin 10x + \ldots\right]$$

8. Obtain half range cosine series for the above function.

$$\left[\textbf{Ans. : } f(x) = \frac{2}{\sqrt{3}}\left[\frac{1}{1}\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \ldots\right]\right]$$

9. Obtain half range sine series for

$$f(x) = \begin{cases} (1/4) - x, & 0 < x < (1/2) \\ x - (3/4), & (1/2) < x < 1 \end{cases}$$ (M.U. 1993, 2001)

$$\left[\textbf{Ans. : } f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \ldots \right]$$

10. Show that in the interval $0 < x < \pi$,

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{1 \cdot 3} - \frac{\cos 3x}{2 \cdot 4} + \frac{\cos 4x}{3 \cdot 5} - \ldots\right]$$ (M.U. 1995, 97, 2004)

11. Find half range cosine series for $f(x) = a\left\{1 - \frac{x}{l}\right\}, \quad 0 < x < l.$ (M.U. 2007)

$$\left[\textbf{Ans. : } f(x) = \frac{a}{2} + \frac{4a}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right]\right]$$

12. Find half range sine series for the above function.

$$\left[\textbf{Ans. : } f(x) = \frac{2a}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\right]\right]$$

13. Obtain half-range cosine series for $f(x) = x$ in $0 < x < l$.

Hence, deduce that (i) $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$

(ii) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$ (iii) $\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots = \frac{\pi^4}{1440}$ (M.U. 200…)

14. Find half range sine series for

$$f(x) = \begin{cases} x, & 0 \le x \le (l/2) \\ l - x, & (l/2) \le x \le l \end{cases}$$

$$\left[\textbf{Ans. : } f(x) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cdot\sin\frac{n\pi}{2}\right)\cdot\sin\frac{n\pi x}{l}\right]$$

15. Obtain half-range cosine series for $f(x) = \begin{cases} kx, & 0 \le x \le (l/2) \\ k(l-x), & (l/2) \le x \le 1 \end{cases}$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$ (M.U. 2001, 03)

and $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$. (M.U. 2004)

$$\left[\textbf{Ans. : } f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right]\right.$$

Then, put $x = l/2$.]

16. Find half range sine series of period $2l$ for $f(x) = \begin{cases} \frac{2x}{l}, & 0 \le x \le \frac{l}{2} \\ \frac{2}{l}(l - x), & \frac{l}{2} \le x \le l. \end{cases}$

(M.U. 1998, 2002, 04, 13) $\left[\textbf{Ans. : } f(x) = \frac{8}{\pi^2}\sum\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\right]$

17. Obtain half range sine series for $f(x) = x^2$ in $0 < x < 3$.

[See solved Ex. 1, page 3-43 and evaluate b_n for $a = 3$.] (M.U. 2013)

18. Obtain half-range sine series for $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \end{cases}$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$. (M.U. 2007)

$$\left[\textbf{Ans. : } f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} \ldots\right]\right]$$

19. Obtain half range sine series for $f(x) = \pi x - x^2$ in $(0, \pi)$ and use Parseval's identity to deduce that

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots = \frac{\pi^6}{980}.$$ (M.U. 2006)

20. Obtain half range sine series in $(0, \pi)$ for $f(x) = x(\pi - x)$ and show that $\sum\frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$.

(M.U. 2005) [**Ans. :** For the first part, see solved Ex. 10, page 3-72]

21. Find half range cosine series for $\cos \lambda x$ in $(0, \pi)$ where λ is not an integer and hence, show that

$$\pi \cot \pi\lambda = \frac{1}{\lambda} + \sum_{n=1}^{\infty}\frac{2\lambda}{\lambda^2 - n^2}.$$

$$\left[\textbf{Ans. : } \cos \lambda x = \frac{\sin \lambda\pi}{\lambda\pi} + \frac{2\lambda \sin \lambda\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\lambda^2 - n^2}\cos nx\right]$$

22. Find half range sine series for $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 4 - x, & 2 \le x \le 4 \end{cases}$

(**Hint :** Put $l = 4$ in the above example number 11.)

$$\left[\textbf{Ans. : } f(x) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\cdot\sin\frac{n\pi x}{4}\right]$$

23. Find half range cosine series for

$f(x) = 1, \quad 0 \le x \le 1$

$= x, \quad 1 \le x \le 2.$

$$\left[\textbf{Ans. : } f(x) = \frac{5}{4} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right]\right]$$

24. Obtain half range cosine series for $f(x) = x - x^2$ for $0 \le x \le 1$. (M.U. 1994)

(**Hint :** See solved example 11 above.)

$$\left[\textbf{Ans. : } f(x) = \frac{1}{6} - \frac{4}{\pi^2}\left[\frac{1}{2^2}\cos 2\pi x + \frac{1}{4^2}\cos 4\pi x + \ldots\right]\right]$$

25. Obtain half-range cosine series for $f(x) = c - x, \quad 0 < x < c.$

$$\left[\textbf{Ans. : } f(x) = \frac{c}{2} + \frac{4c}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \ldots\right]\right]$$

26. Obtain cosine series for $f(x) = \frac{1}{2}(\pi - x)\sin x$ in $(0, \pi)$.

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{2} + \frac{1}{4}\cos x - \frac{1}{3}\cos 2x + \frac{1}{8}\cos 3x - \ldots\right.$$

27. Obtain half range sine series for

$$f(x) = 1, \quad 0 < x < \frac{1}{2}$$
$$= 0, \quad \frac{1}{2} < x < 1.$$

$$\left[\textbf{Ans. :}\ f(x) = \frac{4}{\pi}\sum \frac{1}{n}\sin^2\frac{n\pi}{4}\sin n\pi x\right.$$

28. Obtain half-range cosine series for $f(x) = x \sin x$ in $(0, \pi)$. (M.U. 200

$$\left[\textbf{Ans. :}\ f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx\right.$$

29. Obtain half range cosine series for

$$f(x) = c, \quad 0 < x < a$$
$$= 0, \quad a < x < b$$

$$\left[\textbf{Ans. :}\ f(x) = \frac{ac}{b} + \frac{2c}{\pi}\left[\sin\frac{\pi a}{b}\cos\frac{\pi x}{b} + \frac{1}{2}\sin\frac{2\pi a}{b}\cos\frac{2\pi x}{b} + \ldots\right.\right.$$

30. Obtain sine series for (M.U. 200

$$f(x) = mx, \quad 0 < x \le \frac{\pi}{2}$$
$$= m(\pi - x), \quad \frac{\pi}{2} \le x < \pi$$

$$\left[\textbf{Ans. :}\ f(x) = \frac{4m}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right.\right.$$

31. Obtain half range cosine series for $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in $0 < x < l$. **(M.U. 2000, 02, 04,**

$$\left[\textbf{Ans. :}\ f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{1\cdot 3}\cos\frac{2\pi x}{l} + \frac{1}{3\cdot 5}\cos\frac{4\pi x}{l} + \ldots\right.\right.$$

32. Obtain half-range cosine series for $f(x) = (x-1)^2$ in $0 < x < 1$. **(M.U. 2002, 03,**

Hence, find $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$.

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}.\right.$$ Now, put $x = 0$ and then $x =$

33. Obtain half-range sine series for $f(x) = \pi x - x^2$ in $(0, \pi)$ and hence deduce that

(i) $\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots = \frac{\pi^2}{960}$ (ii) $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{4^6} + \ldots = \frac{\pi^2}{945}$.

$$\left[\textbf{Ans. :}\ f(x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right.\right.$$

34. Obtain half-range sine series for $f(x)$ given by

$$f(x) = \begin{cases} kx, & 0 < x < (l/2) \\ k(l-x), & (l/2) < x < l \end{cases}$$

Hence, deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$.

$$\left[\textbf{Ans. :}\ f(x) = \frac{4kl}{\pi^2}\left[\frac{\sin(\pi x/l)}{1^2} - \frac{\sin(3\pi x/l)}{3^2} + \frac{\sin(5\pi x/l)}{5^2} + \ldots\right]\right]$$

35. Obtain cosine series for $f(x) = \frac{\sin \pi x}{l}$, $0 < x < l$. (M.U. 2009)

(B) 1. If $x^2 = \frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) - \frac{4l^2}{\pi}\sum\frac{1}{n}\sin\frac{n\pi x}{l}$

in $0 < x < 2l$, find the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ (M.U. 2003)

[**Ans. :** $\pi^2/6$]

2. If $\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum\frac{1}{n^2}\cos nx$ in $0 < x < 2\pi$, find the sum of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$

[**Ans. :** $\frac{\pi^4}{90}$]

3. If $f(x) = \begin{cases} x + \pi/2, & -\pi < x < 0 \\ \pi/2 - x, & 0 < x < \pi \end{cases}$ has the Fourier expansion

$f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \ldots\right]$, prove that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

4. If the Fourier series of the function $f(x) = |x|$ in $(2, -2)$ is given by

$f(x) = 1 - \frac{8}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos\left[\frac{(2n-1)}{2}\pi x\right]$, show that $\sum\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$. (M.U. 2003)

5. If $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\cos\frac{2x}{3} + \cos\frac{4x}{15} + \cos\frac{6x}{35} + \ldots\right)$ in $0 \le x \le \pi$, find the sum of the series

$\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots$

[**Ans. :** $\frac{\pi^2 - 8}{16}$]

6. If $x - x^2 = \frac{8}{\pi^3}\left[\frac{1}{1^3}\sin\pi x + \frac{1}{3^3}\sin 3\pi x + \frac{1}{5^3}\sin 5\pi x + \ldots\right]$ in $0 < x < 1$,

find the sum of the series $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$

[**Ans. :** $\frac{\pi^6}{945}$]

7. If $x^2 = \frac{\pi}{3} + 4\sum(-1)^n \cdot \frac{\cos nx}{n^2}$ for $-\pi < x < \pi$, prove that $\sum\frac{1}{n^4} = \frac{\pi^4}{90}$. (M.U. 2003)

EXERCISE - VIII

Theory

1. Define Fourier Series and evaluate the Fourier Constants. (M.U. 1999)
2. State Parseval's identity for a function $f(x)$ in $(-l, l)$. (M.U. 1998, 2003)

3. Derive Euler's formulae. (M.U. 1997)
4. State and prove Parseval's identity for the function $f(x)$ in the interval $(c, c+2\pi)$. (M.U. 1999)
5. Define Fourier Series and State Dirichlet's Conditions. (M.U. 2004)
6. State and prove Parseval's identity for half range cosine series for $f(x)$ in the interval $(0, l)$.
7. State Parseval's identity for a function $f(x)$ in $(-\pi, \pi)$. (M.U. 2000, 04)

Short Answer Questions

1. If $f(x) = 4 - x^2$, $0 < x < 2$ and is periodic with period 2, find $f(x)$ at $x = 0, 2, 4, 6, \ldots$
2. Show that the Fourier series of (M.U. 2003)
$$f(x) = \begin{cases} 4 - x, & 3 < x < 4 \\ x - 4, & 4 < x < 5 \end{cases}$$ has no sine terms.
3. Show that the Fourier series of
$$f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$$ with period 10 has no cosine terms.
4. Show that the Fourier series of
$$f(x) = \begin{cases} -x, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$ has no sine terms.

CHAPTER 4

Complex Form of Fourier Series

1. Introduction

In this chapter we shall first derive complex from of Fourier Series. Then we shall define Fourier Integrals and solve some simple problems based on this.

2. Complex Form of Fourier Series

Let $f(x)$ be defined in the interval $(C, C+2l)$. The complex form of Fourier Series for $f(x)$ in this interval is given by

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \\ \text{where, } C_n &= \frac{1}{2l}\int_C^{C+2l} f(x)\, e^{-in\pi x/l}\, dx \end{aligned}} \quad n = 0, \pm 1, \pm 2, \ldots$$

Proof : Consider

$$f(x) = a_0 + \sum_1^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_1^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, a_0, a_n and b_n are as given in (5) on page 3-6.

$$= a_0 + \sum_1^{\infty} a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + \sum_1^{\infty} b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)$$

$$= a_0 + \sum_1^{\infty} a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) - \sum_1^{\infty} i\, b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2}\right)$$

$$= a_0 + \sum_1^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{in\pi x/l} + \sum_1^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-in\pi x/l}$$

$$= C_0 + \sum_1^{\infty} C_n e^{in\pi x/l} + \sum_1^{\infty} C_{-n}\, e^{-in\pi x/l}$$

where, $C_n = \dfrac{a_n - ib_n}{2}$, $C_{-n} = \dfrac{a_n + ib_n}{2}$

$$\therefore\ f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $C_0 = a_0 = \frac{1}{2l}\int_C^{C+2l} f(x)\,dx$

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2l}\left[\int_C^{C+2l} f(x)\cos\frac{n\pi x}{l}\,dx - i\int_C^{C+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$C_n = \frac{1}{2l}\int_C^{C+2l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx \qquad \left[\because e^{-i\theta} = \cos\theta - i\sin\theta\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{-in\pi x/l}\,dx \qquad \text{.........}$$

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2l}\left[\int_C^{C+2l} f(x)\cos\frac{n\pi x}{l}\,dx + i\int_C^{C+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx \qquad \left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{in\pi x/l}\,dx \qquad \text{.........}$$

Combining these results (1) and (2), we get

$$C_n = \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{-in\pi x/l}\,dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

Cor. 1 : If the interval is C to $C + 2\pi$ replacing l by π in the above result

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2\pi}\int_C^{C+2\pi} f(x)\,e^{-inx}\,dx}$$

Cor. 2 : If the interval is $(0, 2l)$, putting $C = 0$ in the above result

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2l}\int_0^{2l} f(x)\,e^{-in\pi x/l}\,dx}$$

Cor. 3 : If the interval is $(0, 2\pi)$, putting $l = \pi$ in the above corollary 2,

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-inx}\,dx}$$

Cor. 4 : If the interval is $(-l, l)$, putting $C = -l$ in the above result

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\,e^{-in\pi x/l}\,dx}$$

Cor. 5 : If the interval is $(-\pi, \pi)$, putting $l = \pi$ in corollary 4,

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx}$$

Example 1 : Obtain complex form of Fourier Series for $f(x) = e^{ax}$ in $(-\pi, \pi)$ where a is not an integer. **(M.U. 2001, 02, 04, 05, 11, 12, 13)**

Hence deduce that when α is a constant other than an integer

(i) $\cos\alpha x = \frac{\sin\pi\alpha}{\pi}\sum\frac{(-1)^n\,\alpha}{(\alpha^2 - n^2)}\cdot e^{inx}$ **(M.U. 2009)**

(ii) $\sin\alpha x = \frac{\sin\pi\alpha}{i\pi}\sum\frac{(-1)^n\cdot n}{(\alpha^2 - n^2)}\cdot e^{inx}$ **(M.U. 2001, 02)**

Sol. : By corollary 5 above, the complex form of $f(x) = e^{ax}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where,
$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\cdot e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a - in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}\cdot e^{-in\pi} - e^{-a\pi}\cdot e^{in\pi}\right]$$

But
$$e^{\pm in\pi} = e^{i(\pm n\pi)} = \cos(\pm n\pi) + i\sin(\pm n\pi) = (-1)^n + i(0) = (-1)^n$$

$$\therefore \quad C_n = \frac{1}{2\pi(a-in)}\left[(-1)^n e^{a\pi} - (-1)^n e^{-a\pi}\right] = \frac{(-1)^n}{\pi(a-in)}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right)$$

$$= \frac{(-1)^n}{\pi(a-in)}\sinh a\pi = \frac{(-1)^n\sinh a\pi}{\pi(a-in)}\cdot\frac{(a+in)}{(a+in)}$$

$$= \frac{(-1)^n\sinh a\pi\,(a+in)}{\pi(a^2+n^2)}$$

Hence,
$$e^{ax} = \sum_{-\infty}^{\infty}\frac{(-1)^n\sinh a\pi\cdot(a+in)}{\pi(a^2+n^2)}\,e^{inx} \qquad \text{.........}(1)$$

For deductions, replace a by $i\alpha$ in (1)

$$\therefore \quad e^{i\alpha x} = \sum\frac{(-1)^n\sinh i\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(i\alpha+in)\cdot e^{inx}$$

$$= \sum\frac{(-1)^n(i)\sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(i\alpha+in)\cdot e^{inx}$$

$$\therefore \quad e^{i\alpha x} = \sum\frac{(-1)^n\sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(-\alpha-n)\cdot e^{inx} \qquad \text{.........}(2)$$

Now, replace a by $-i\alpha$ in (1)

$$\therefore \quad e^{-i\alpha x} = \sum\frac{(-1)^n\sinh(-i\alpha\pi)}{\pi(-\alpha^2+n^2)}\cdot(-i\alpha+in)\cdot e^{inx}$$

$$= \sum\frac{(-1)^n(-i)\sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(-i\alpha+in)\cdot e^{inx}$$

$$\therefore \quad e^{-i\alpha x} = \sum\frac{(-1)^n\sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(-\alpha+n)\cdot e^{inx} \qquad \text{.........}(3)$$

Adding (2) and (3) and dividing by 2; and subtracting (3) from (2) and dividing by $2i$,

$$\therefore \quad \cos\alpha x = \frac{e^{i\alpha x}+e^{-i\alpha x}}{2} = \sum \frac{(-1)^n \sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(-\alpha)\cdot e^{inx} = \frac{\sin\alpha\pi}{\pi}\sum\frac{(-1)^n\cdot\alpha}{(\alpha^2-n^2)}\cdot e^{inx}$$

$$\therefore \quad \sin\alpha x = \frac{e^{i\alpha x}-e^{-i\alpha x}}{2i} = \sum \frac{(-1)^n \sin\alpha\pi}{\pi i(-\alpha^2+n^2)}\cdot(-n)\cdot e^{inx} = \frac{\sin\alpha\pi}{\pi i}\sum\frac{(-1)^n\cdot n}{(\alpha^2-n^2)}\cdot e^{inx}$$

Example 2 : Obtain the complex form of Fourier Series for $f(x) = e^{-ax}$ in $(-\pi, \pi)$. (M.U. 20

Sol. : Changing the sign of a, we get from the above result (1),

$$f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin h(-a\pi)\cdot(-a+in)}{\pi(a^2+n^2)}\, e^{inx}$$

But $\quad \sin h(-a\pi) = \dfrac{e^{-a\pi}-e^{a\pi}}{2} = -\dfrac{e^{a\pi}-e^{-a\pi}}{2} = -\sin h\, a\pi$

$$\therefore \quad e^{-ax} = \sum \frac{(-1)^n \sin h\, a\pi\cdot(a-in)}{\pi(a^2+n^2)}\, e^{inx}$$

Or proceeding as above obtain the result independently.

Example 3 : Obtain the complex form of Fourier Series for $f(x) = \cos h\, ax$ in $(-\pi, \pi)$ where a not an integer.

Sol. : As proved in (1) in example 1,

$$e^{ax} = \sum \frac{(-1)^n \sin h\, a\pi\cdot(a+in)}{\pi(a^2+n^2)}\, e^{inx}$$

Changing the sign of a, we get,

$$e^{-ax} = \sum \frac{(-1)^n \sin h(-a\pi)\cdot(-a+in)}{\pi(a^2+n^2)}\, e^{inx}$$

But $\sin h(-a\pi) = -\sin h\, a\pi$.

$$e^{-ax} = \sum \frac{(-1)^n \sin h\, a\pi\cdot(a-in)}{\pi(a^2+n^2)}\, e^{inx}$$

Now, $\cos h\, ax = \dfrac{e^{ax}+e^{-ax}}{2} = \dfrac{\sin h\, a\pi}{2}\left[\sum\dfrac{(-1)^n(a+in)}{(a^2+n^2)}\, e^{inx} + \sum\dfrac{(-1)^n(a-in)}{(a^2+n^2)}\, e^{inx}\right]$

$$= a\cdot\sin h\, a\pi \sum \frac{(-1)^n}{(a^2+n^2)}\, e^{in\pi x}$$

Example 4 : Obtain the complex form of Fourier Series for $f(x) = e^{ax}$ in $(-l, l)$.

(M.U. 1993, 2003, 12, 13)

Sol. : By corollary 4 above, the complex form of Fourier series of $f(x) = e^{ax}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n\, e^{in\pi x/l}$$

where, $\quad C_n = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x)\, e^{-in\pi x/l} = \dfrac{1}{2l}\int_{-l}^{l} e^{ax}\cdot e^{-in\pi x/l}\cdot dx$

$$C_n = \frac{1}{2l}\int_{-l}^{l} e^{(a-in\pi/l)x}\, dx = \frac{1}{2l}\left[\frac{e^{(a-in\pi/l)x}}{(a-in\pi/l)}\right]_{-l}^{l}$$

$$= \frac{1}{2l}\left[\frac{e^{(a-in\pi/l)l} - e^{-(a-in\pi/l)l}}{(a-in\pi/l)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{al}\cdot e^{-in\pi} - e^{-al}\cdot e^{in\pi}}{(al-in\pi)}\right]$$

Now, $\quad e^{\pm in\pi} = e^{i(\pm n\pi)} = \cos(\pm n\pi) + i\sin(\pm n\pi) = (-1)^n + i\cdot 0 = (-1)^n$

$$\therefore \quad C_n = \frac{e^{al}(-1)^n - e^{-al}(-1)^n}{2(al-in\pi)} = \frac{(-1)^n(e^{al}-e^{-al})}{2(al-in\pi)}$$

$$= \frac{(-1)^n\cdot\sin h\, al}{al-in\pi}\cdot\frac{al+in\pi}{al+in\pi} = \frac{(-1)^n\sin h\, al\,(al+in\pi)}{a^2l^2+n^2\pi^2}$$

$$\therefore \quad e^{ax} = \sum_{-\infty}^{\infty}\frac{(-1)^n\sin h\, al\cdot(al+in\pi)}{a^2l^2+n^2\pi^2}\, e^{in\pi x/l}$$

Example 5 : Obtain the complex form of Fourier Series for $f(x) = \cos h\, ax$ in $(-l, l)$.

(M.U. 2002, 05, 09, 11)

Sol. : By example (4) above

$$e^{ax} = \sum\frac{(-1)^n\sin h\, al\cdot(al+in\pi)}{a^2l^2+n^2\pi^2}\, e^{in\pi x/l}$$

Changing the sign of a, we get

$$e^{-ax} = \sum\frac{(-1)^n\sin h(-al)(-al+in\pi)}{a^2l^2+n^2\pi^2}\, e^{in\pi x/l}$$

But $\sin h(-x) = \dfrac{e^{-x}-e^{x}}{2} = -\left(\dfrac{e^{x}-e^{-x}}{2}\right) = -\sin h\, x$

$$\therefore \quad e^{-ax} = \sum\frac{(-1)^n\sin h\, al\,(al-in\pi)}{a^2l^2+n^2\pi^2}\, e^{in\pi x/l}$$

$$\therefore \cos h\, ax = \frac{e^{ax}+e^{-ax}}{2}$$

$$= \frac{\sin h\, al}{2}\left[\sum\frac{(-1)^n(al+in\pi)}{a^2l^2+n^2\pi^2}\, e^{in\pi x/l} + \sum\frac{(-1)^n(al-in\pi)}{a^2l^2+n^2\pi^2}\, e^{in\pi x/l}\right]$$

$$= al\sin h\, al\sum\frac{(-1)^n}{a^2l^2+n^2\pi^2}\, e^{in\pi x/l}$$

Example 6 : Obtain complex form of Fourier Series for $f(x) = \sin h\, ax$ in $(-l, l)$.

(M.U. 2014)

Sol. : Using the results obtained in example 5, we get

$$\sin h\, ax = \frac{e^{ax}-e^{-ax}}{2}$$

$$\therefore \sin h\,ax = \frac{\sin h\,al}{2}\left[\sum \frac{(-1)^n (al + in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l} - \sum \frac{(-1)^n (al - in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}\right]$$

$$= \sin h\,al \sum \frac{(-1)^n\, in\pi}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}$$

Example 7 : Obtain complex form of Fourier Series for $f(x) = \cos h\,ax + \sin h\,ax$ in (-l, l) ...

(M.U. 2002, 07, 08 ...)

Sol. : By using the results obtained in Examples 5 and 6,

$$f(x) = \cos h\,ax + \sin h\,ax = \sin h\,al \sum \frac{(-1)^n (al + in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}$$

Example 8 : Obtain complex form of Fourier series for $f(x) = \cos h\,2x + \sin h\,2x$ in (-2, 2) ...

(M.U. 2002 ...)

Sol. : We have by corollary (4),

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{where } C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}\, dx$$

Here, $l = 2$.

$$\therefore \quad f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/2} \quad \text{where } C_n = \frac{1}{4}\int_{-2}^{2} f(x)\, e^{-in\pi x/2}\, dx$$

But $\quad f(x) = \cos h\,2x + \sin h\,2x$

$$\therefore \quad f(x) = \frac{e^{2x} + e^{-2x}}{2} + \frac{e^{2x} - e^{-2x}}{2} = e^{2x}$$

$$\therefore \quad C_n = \frac{1}{4}\int_{-2}^{2} e^{2x} \cdot e^{-in\pi x/2}\, dx = \frac{1}{4}\int_{-2}^{2} e^{[2-(in\pi/2)]x}\, dx$$

$$= \frac{1}{4}\left[\frac{e^{[2-(in\pi/2)]x}}{2-(in\pi/2)}\right]_{-2}^{2} = \frac{1}{4}\cdot\frac{2}{4-in\pi}\left[e^{4-in\pi} - e^{-4+in\pi}\right]$$

But $\quad e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n \pm 0 = (-1)^n \qquad$ **[Note this]**

$$\therefore \quad C_n = \frac{1}{2(4-in\pi)}\left[(-1)^n e^4 - (-1)^n e^{-4}\right]$$

$$= \frac{1}{2}\cdot\frac{(4+in\pi)}{(16+n^2\pi^2)}\cdot(-1)^n\cdot[e^4 - e^{-4}]$$

$$= \frac{(4+in\pi)\cdot(-1)^n}{16+n^2\pi^2}\cdot\frac{e^4 - e^{-4}}{2}$$

$$= \sin h\,4 \cdot \frac{(-1)^n \cdot (4+in\pi)}{16+n^2\pi^2}$$

Hence, complex form of Fourier series for $f(x)$ is

$$f(x) = \sin h\,4 \sum_{-\infty}^{\infty} \frac{(-1)^n (4+in\pi)}{16+n^2\pi^2} \cdot e^{in\pi x/2}.$$

(or, put $a = 2$, $l = 2$ in Ex. 7.)

Example 9 : Obtain complex form of Fourier Series for

$f(x) = \cos h\,3x + \sin h\,3x$ in (– 3, 3).

(M.U. 2008, 14)

Sol. : We have by corollary 4

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{where } C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}\, dx.$$

Here, $l = 3$.

$$\therefore \quad f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/3} \quad \text{where } C_n = \frac{1}{6}\int_{-3}^{3} f(x)\cdot e^{-in\pi x/3}\, dx$$

But $\quad f(x) = \cos h\,3x + \sin h\,3x = \dfrac{e^{3x} + e^{-3x}}{2} + \dfrac{e^{3x} - e^{-3x}}{2} = e^{3x}$

$$\therefore \quad C_n = \frac{1}{6}\int_{-3}^{3} e^{3x}\cdot e^{-in\pi x/3}\, dx = \frac{1}{6}\int_{-3}^{3} e^{(3-in\pi/3)x}\, dx$$

$$= \frac{1}{6}\left[\frac{e^{(3-in\pi/3)x}}{3-in\pi/3}\right]_{-3}^{3} = \frac{1}{6}\cdot\frac{3}{(9-in\pi)}\left[e^{9-in\pi} - e^{-9+in\pi}\right]$$

But $\quad e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n \pm 0 = (-1)^n$

$$\therefore \quad C_n = \frac{1}{2(9-in\pi)}\left[(-1)^n \cdot e^9 - (-1)^n e^{-9}\right]$$

$$= \frac{1}{2}\cdot\frac{(9+in\pi)(-1)^n}{81+n^2\pi^2}\cdot[e^9 - e^{-9}] = \frac{(-1)^n (9+in\pi)}{81+n^2\pi^2}\sin h\,9$$

$\therefore$ Complex form of Fourier Series for

$$f(x) = \sin h\,9 \cdot \sum_{-\infty}^{\infty} \frac{(-1)^n (9+in\pi)}{81+n^2\pi^2}\cdot e^{in\pi x/3}.$$

Example 10 : Find complex form of Fourier Series for $\cos ax$, where a is not an integer in $(-\pi, \pi)$.

(M.U. 2011)

Sol. : We have $\cos ax = \dfrac{e^{aix} + e^{-aix}}{2}$

Replacing a by ai in (1) in example 1, we get

$$e^{aix} = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin h\,ai\pi \cdot (ai + in)}{\pi(-a^2 + n^2)} e^{inx} \qquad [\because -a^2 + n^2 \neq 0]$$

$$= \frac{-i \sin h\,ai\pi}{\pi} \sum \frac{(-1)^n (a+n)}{(a^2 - n^2)} e^{inx}$$

Since, $\sin h\,ix = i \sin x$, we get

$$e^{aix} = \frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx}$$

Changing the sign of a,

$$e^{-aix} = \frac{\sin(-a\pi)}{\pi} \sum \frac{(-1)^n (-a+n)}{(a^2-n^2)} e^{inx} = -\frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (-a+n)}{(a^2-n^2)} e^{inx}$$

$$= \frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}$$

$$\therefore \quad \cos ax = \frac{e^{aix} + e^{-aix}}{2}$$

$$= \frac{\sin a\pi}{2\pi}\left[\sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx} + \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}\right]$$

$$= \frac{\sin a\pi}{2\pi}\sum \frac{(-1)^n \cdot 2a}{(a^2-n^2)} e^{inx} = \frac{a \sin a\pi}{\pi}\sum \frac{(-1)^n}{(a^2-n^2)} e^{inx}.$$

Example 11 : Find complex from of Fourier Series for sin ax, where a is not an integer in $(-\pi, \pi)$. **(M.U. 2004, ...)**

Sol. : Using the results obtained in the above example, we get,

$$\sin ax = \frac{e^{aix} - e^{-aix}}{2i}$$

$$\therefore \quad \sin ax = \frac{\sin a\pi}{2\pi i}\left[\sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx} - \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}\right]$$

$$= \frac{\sin a\pi}{\pi i}\sum (-1)^n \cdot \frac{n}{(a^2-n^2)} \cdot e^{inx}$$

Example 12 : Obtain the complex form of Fourier series for $f(x) = e^{ax}$ in $(0, a)$. **(M.U. 2003, 10, 13, 14, ...)**

Sol. : By corollary (2), page 4-2, putting $2l = a$ i.e. $l = a/2$, we get

$$f(x) = \sum_{-\infty}^{\infty} C_n \cdot e^{2in\pi x/a}$$

where, $$C_n = \frac{1}{a}\int_0^a e^{ax} \cdot e^{-2in\pi x/a}\, dx$$

$$\therefore \quad C_n = \frac{1}{a}\int_0^a e^{(a-2in\pi/a)x}\, dx = \frac{1}{a}\left[\frac{e^{(a-2in\pi/a)x}}{(a-2in\pi/a)}\right]_0^a$$

$$= \frac{1}{a} \cdot \frac{a}{(a^2-2in\pi)} \cdot \left[e^{(a^2-2in\pi)} - 1\right]$$

$$= \frac{1}{(a^2-2in\pi)}\left[e^{a^2} \cdot e^{-2in\pi} - 1\right]$$

$$= \frac{1}{(a^2-2in\pi)}(e^{a^2} - 1) \quad \left[\because e^{-2in\pi} = \cos 2n\pi - i \sin 2n\pi = 1\right]$$

$$\therefore \quad e^{ax} = (e^{a^2} - 1)\sum_{-\infty}^{\infty} \frac{e^{2in\pi x/a}}{(a^2-2in\pi)}.$$

Example 13 : Obtain the complex form of Fourier Series for $f(x) = e^{2x}$ in $(0, 2)$. **(M.U. 2003, ...)**

Sol. : In the result obtained in Ex. 12 above replace a by 2 and get

$$e^{2x} = (e^4 - 1)\sum_{-\infty}^{\infty} \frac{e^{in\pi x}}{(4-2in\pi)}$$

Example 14 : Find the complex form of Fourier Series for $f(x) = \begin{cases} 0, & 0 < x < l \\ a, & l < x < 2l \end{cases}$ **(M.U. 2008)**

Sol. : By corollary 2, the complex form of Fourier Series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{where, } C_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-in\pi x/l} dx$$

$$\therefore \quad C_n = \frac{1}{2l}\left[\int_0^l 0 \cdot dx + \int_l^{2l} a e^{-in\pi x/l} dx\right] \qquad \text{.................. (1)}$$

$$= \frac{a}{2l}\int_l^{2l} e^{-in\pi x/l} dx = \frac{a}{2l}\left[\frac{e^{-in\pi x/l}}{-in\pi/l}\right]_l^{2l}$$

$$= \frac{-a}{2in\pi}\left[e^{-2in\pi} - e^{-in\pi}\right] \quad \text{except when } n = 0. \qquad \text{.................. (2)}$$

Case 1 : Where $n = 0$ from (1)

$$C_0 = \frac{1}{2l}\left[\int_0^l 0\, dx + \int_l^{2l} a e^0\, dx\right] = \frac{1}{2l}\int_l^{2l} a\, dx$$

$$= \frac{a}{2l}[2l - l] = \frac{al}{2l} = \frac{a}{2}$$

Case 2 : When $n = \pm 1, \pm 3, \ldots\ldots$ from (2)

$$C_1 = \frac{-a}{2i\pi}\left[e^{-2i\pi} - e^{-i\pi}\right]$$

$$= \frac{-a}{2i\pi}\left[\cos(-2\pi) + i\sin(-2\pi) - \cos(-\pi) - i\sin(-\pi)\right]$$

$$= \frac{-a}{2i\pi}\left[1 + i(0) - (-1) + i(0)\right] = \frac{-a}{2i\pi} \cdot 2 = \frac{ai}{\pi}$$

$$C_{-1} = \frac{a}{2i\pi}\left[\cos 2\pi + i\sin 2\pi - \cos\pi - i\sin\pi\right]$$

$$= \frac{a}{2i\pi}\left[1 + i(0) - (-1) + i0\right] = \frac{a}{2i\pi} \cdot 2 = -\frac{ai}{\pi}$$

Similarly, $C_3 = \frac{ia}{3\pi}$, $C_{-3} = -\frac{ai}{3\pi}$; $C_5 = \frac{ia}{5\pi}$, $C_{-5} = -\frac{ai}{5\pi}$.

Case 3 : When $n = \pm 2, \pm 4, \ldots\ldots$

$$C_2 = \frac{-a}{4i\pi}\left[e^{-4i\pi} - e^{-2i\pi}\right]$$

$$= \frac{-a}{4i\pi}\left[\cos(-4\pi) + i\sin(-4\pi) - \cos(-2\pi) + i\sin(-2\pi)\right]$$

$$= \frac{-a}{4i\pi}\left[1 + i(0) - (1) - i(0)\right] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = \ldots\ldots = 0$

$$\therefore \quad f(x) = \frac{a}{2} + \frac{ai}{\pi}\left[(e^u - e^{-u}) + \frac{1}{3}(e^{3u} - e^{-3u}) + \ldots\right] \text{ where } u = \frac{i\pi x}{l}$$

Example 15 : If $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$

$f(x+2) = f(x)$, find complex form of Fourier Series. (M.U. 199

Sol. : By corollary 2, the complex form of Fourier Series in $(0, 2l)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{where, } C_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-in\pi x/l}\, dx$$

Since, in this case $l = 1$, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$ ..... (1)

and $$C_n = \frac{1}{2}\int_0^2 f(x) e^{-in\pi x} dx$$

$$= \frac{1}{2}\left[\int_0^1 1 \cdot e^{-in\pi x} dx + \int_1^2 0 \cdot e^{-in\pi x} dx\right]$$

$$= \frac{1}{2}\left[\frac{e^{-in\pi x}}{-in\pi}\right]_0^1 = \frac{1}{-2in\pi}\left[e^{-in\pi} - 1\right]$$

$$= \frac{1}{2in\pi}\left[1 - e^{-in\pi}\right] \text{ except when } n = 0. \quad \text{..... (2)}$$

Case 1 : When $n = 0$, from (1)

$$C_0 = \frac{1}{2}\left[\int_0^1 1 \cdot dx + \int_1^2 0 \cdot dx\right] = \frac{1}{2}\left[x\right]_0^1 = \frac{1}{2}$$

Case 2 : When $n = \pm 1, \pm 3, \pm 5, ...$ from (2)

$$C_1 = \frac{1}{2i\pi}\left[1 - e^{-i\pi}\right] = \frac{1}{2i\pi}\left[1 - (\cos\pi - i\sin\pi)\right]$$

$$= \frac{1}{2i\pi}\left[1 - (-1)\right] = \frac{2}{2i\pi} = \frac{1}{i\pi}$$

$$C_{-1} = \frac{1}{-2i\pi}\left[1 - e^{i\pi}\right] = -\frac{1}{2i\pi}\left[1 - (\cos\pi + i\sin\pi)\right]$$

$$= -\frac{1}{2i\pi}\left[1 - (-1)\right] = -\frac{1}{i\pi}$$

Similarly, $C_3 = \frac{1}{3i\pi}$, $C_{-3} = -\frac{1}{3i\pi}$, $C_5 = \frac{1}{5i\pi}$, $C_{-5} = -\frac{1}{5i\pi}$, ...

Case 3 : When $n = \pm 2, \pm 4, ...$ from (2)

$$C_2 = \frac{1}{4i\pi}\left[1 - e^{-2i\pi}\right] = \frac{1}{4i\pi}\left[1 - (\cos 2\pi + i\sin 2\pi)\right] = \frac{1}{4i\pi}\left[1 - 1\right] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = ... = 0$

Hence, $$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left(e^{i\pi x} - e^{-i\pi x}\right) + \frac{1}{3i\pi}\left(e^{3i\pi x} - e^{-3i\pi x}\right) + ...$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2i}\right) + \frac{1}{3}\left(\frac{e^{3i\pi x} - e^{-3i\pi x}}{2i}\right) + ...\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + ...\right]$$

Example 16 : Find the complex form of the Fourier series for $f(x) = 2x$ in $(0, 2\pi)$. (M.U. 2009)

Sol. : By corollary (3), page 4-2,

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx} \quad \text{where, } C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\, dx$$

Hence, $$C_n = \frac{1}{2\pi}\int_0^{2\pi} 2x \cdot e^{-inx} dx \quad \text{..... (1)}$$

$$= \frac{1}{\pi}\int_0^{2\pi} x e^{-inx} dx$$

$$= \frac{1}{\pi}\left[x \cdot \frac{e^{-inx}}{-in} - \int \frac{e^{-inx}}{-in} \cdot 1 \cdot dx\right]_0^{2\pi} = \frac{1}{\pi}\left[-\frac{x}{in}e^{-inx} - \frac{e^{-inx}}{(in)^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi e^{-i2n\pi}}{in} + \frac{e^{-in2\pi}}{n^2} - 0 - \frac{1}{n^2}\right] \quad \text{for } n \neq 0, \; (\because i^2 = -1)$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{in} + \frac{1}{n^2} - \frac{1}{n^2}\right] \quad \text{for } n \neq 0 \quad [\because e^{-i2n\pi} = \cos(2n\pi) - i\sin(2n\pi) = 1]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{in} + \frac{1}{n^2} - \frac{1}{n^2}\right] \quad \text{for } n \neq 0$$

$$= -\frac{2}{in} = -\frac{2i}{i^2 n} = \frac{2i}{n} \quad \text{for } n \neq 0$$

For $n = 0$, we get from (1),

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} 2x \cdot e^0\, dx = \frac{1}{2\pi}\left[x^2\right]_0^{2\pi} = \frac{1}{2\pi} \cdot 4\pi^2 = 2\pi.$$

Hence, $$f(x) = 2\pi + 2i\sum_{n=-\infty}^{\infty} \frac{1}{n} \cdot e^{inx}, \quad \text{for } n \neq 0$$

Putting $n = \pm 1, \pm 2, \pm 3, ...$, we get

$$f(x) = 2\pi + 2i\left[\frac{e^{ix}}{1} + \frac{e^{2ix}}{2} + \frac{e^{3ix}}{3} + ... - \frac{e^{-ix}}{1} - \frac{e^{-2ix}}{2} - \frac{e^{-3ix}}{3} - ....\right]$$

$$= 2\pi + 2i\left[\frac{e^{ix} - e^{-ix}}{1} + \frac{1}{2}\left(\frac{e^{2ix} - e^{-2ix}}{1}\right) + \frac{1}{3}\left(\frac{e^{3ix} - e^{-3ix}}{1}\right) + ...\right]$$

$$= 2\pi + 4i^2\left[\frac{(e^{ix} - e^{-ix})}{2i} + \frac{1}{2} \cdot \frac{(e^{2ix} - e^{-2ix})}{2i} + \frac{1}{3} \cdot \frac{(e^{3ix} - e^{-3ix})}{2i} + ...\right]$$

$$= 2\pi - 4\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...\right]$$

EXERCISE - I

1. Find complex form of Fourier Series for $f(x) = e^{-x}$ in $(-1, 1)$. (M.U. 2009, 14)

$$\left[\textbf{Ans. :}\ f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)\sinh 1}{1 + n^2\pi^2} e^{in\pi x}\right]$$

2. Find the complex form of $f(x) = e^x$ in $(-\pi, \pi)$. (M.U. 1999, 2000, 0...)

$$\left[\text{Ans. : } f(x) = \sum \frac{(-1)^n \sinh \pi (1 + in)}{\pi (1^2 + n^2)} e^{in...}\right.$$

3. Obtain complex form of Fourier series for $f(x) = e^{ax}$, $-1 < x < 1$. (M.U. 1996, 200...)

$$\left[\text{Ans. : } f(x) = \sum \frac{(-1)^n \sinh a (a + in\pi)}{a^2 + n^2\pi^2} e^{in...}\right.$$

4. Find the complex form of Fourier Series for $f(x) = \sin hx$ in $(-l, l)$. (M.U. 199...)

$$\left[\text{Ans. : } f(x) = \sinh l \cdot (i\pi) \sum \frac{(-1)^n n}{l^2 + n^2\pi^2} e^{i n\pi x ...}\right.$$

5. Find the complex form of Fourier Series for $f(x) = \cos h\, 2x + \sin h\, 2x$ in $(-5, 5)$. (M.U. 1999, 2003, 1...)
(**Hint :** In solved example 7, put $a = 2$ and $l = 5$.)

6. Find the complex form of Fourier Series of $\cos h\, 3x + \sin h\, 3x$ in $(-\pi, \pi)$. (M.U. 1999, 2003, ...)

7. Find complex form of Fourier Seires of $f(x) = \cos hx + \sin hx$ in $(-\pi, \pi)$. (M.U. 2003, 0...)

8. Find the complex form of Fourier series for $f(x) = e^x$ in $(0, 2\pi)$.

$$\left[\text{Ans. : } e^x = \frac{e^{2\pi} - 1}{2\pi} \sum_{-\infty}^{\infty} \frac{1 + in}{n^2 + 1} e^{inx}\right.$$

3. Orthogonality, Orthonormality

We first define these terms and then solve some problems based on this definition.

Definition 1 : A set of functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x) \ldots$ is said to be **orthogonal** in (a, b) if

$$\int_a^b f_m(x) f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

In other words a set of functions $f_1(x), f_2(x), \ldots f_n(x) \ldots$ is orthogonal on (a, b) if

$$\boxed{\int_a^b f_m(x) f_n(x)\, dx = 0 \text{ if } m \neq n} \quad \text{and} \quad \boxed{\int_a^b [f_m(x)]^2\, dx \neq 0}$$

Definition 2 : A set of functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x) \ldots$ is said to be **orthonormal** in (a, b) if

$$\int_a^b f_m(x) f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ = 1, & \text{if } m = n \end{cases}$$

In other words a set of functions $f_1(x), f_2(x), f_3(x), \ldots f_n(x) \ldots$ is said to be **orthonormal** in (a, b) if

$$\boxed{\int_a^b f_m(x) f_n(x)\, dx = 0 \text{ if } m \neq n} \quad \text{and} \quad \boxed{\int_a^b [f_m(x)]^2\, dx = 1}$$

Note ....

Every orthonormal set of functions is orthogonal but the converse may not be true.

Example 1 : Show that the set of functions $\cos nx$, $n = 1, 2, 3, \ldots$ is orthogonal on $(0, 2\pi)$. (M.U. 1994, 2010, 14)

Sol. : We have $f_n(x) = \cos nx$.

$$\therefore \int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx = \int_0^{2\pi} \cos mx \cdot \cos nx\, dx$$

$$= \frac{1}{2} \int_0^{2\pi} [\cos (m + n) x + \cos (m - n) x]\, dx$$

$$= \frac{1}{2} \left[\frac{\sin (m + n) x}{m + n} + \frac{\sin (m - n) x}{m - n}\right]_0^{2\pi}$$

Now, two cases arise.

Case 1 : When $m \neq n$, then $\int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx = 0$

Case 2 : When $m = n$, then $\int_0^{2\pi} f_n(x) \cdot f_n(x)\, dx = \int_0^{2\pi} \cos^2 nx\, dx$

$$\therefore \int_0^{2\pi} [f_n(x)]^2 dx = \int_0^{2\pi} \left(\frac{1 + \cos 2nx}{2}\right) dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_0^{2\pi} = \pi \neq 0 \quad \text{.......... (1)}$$

$$\text{Since, } \int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

the given set of functions is orthogonal on $[0, 2\pi]$.

Example 2 : How can you construct orthonormal set of functions from the set given in the above example number 1?

Sol. : If the set of functions is to be orthonormal, we should have

$$\int_0^{2\pi} [f_n(x)]^2\, dx = 1$$

For this we divide (1) by π and write it as

$$\int_0^{2\pi} \frac{1}{\pi} [f_n(x)]^2\, dx = \pi \cdot \frac{1}{\pi} = 1 \quad \text{i.e.} \quad \int_0^{2\pi} \frac{1}{\sqrt{\pi}} f_n(x) \cdot \frac{1}{\sqrt{\pi}} f_n(x)\, dx = 1$$

This is obviously an orthonormal set, where $\phi_n(x) = \frac{1}{\sqrt{\pi}} \cos nx$.

Hence, the required orthonormal set is $\frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \cos 3x, \ldots$

Example 3 : Show that the set of functions $\{\sin x, \sin 3x, \sin 5x, \ldots\}$ i.e., $\{\sin (2n + 1) x\}$, $n = 0, 1, 2, \ldots$ is orthogonal over $[0, \pi / 2]$. Hence, construct orthonormal set of functions. (M.U. 2002, 04, 06, 08, 09, 10, 13)

Sol. : We have $f(x) = \sin (2n + 1) x$.

$$\therefore \int_0^{\pi/2} f_m(n) \cdot f_n(x)\, dx = \int_0^{\pi/2} \sin (2m + 1) x \cdot \sin (2n + 1) x\, dx$$

$$= -\frac{1}{2} \int_0^{\pi/2} [\cos (2m + 2n + 2) x - \cos (2m - 2n) x]\, dx$$

$$= -\frac{1}{2} \left[\frac{\sin (2m + 2n + 2) x}{(2m + 2n + 2)} - \frac{\sin (2m - 2n) x}{(2m - 2n)}\right]_0^{\pi/2}$$

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Now, two cases arise.

Case 1 : If $m \neq n$, then $\int_0^{\pi/2} f_m(x) \cdot f_n(x) = 0$

Case 2 : If $m = n$, then $\int_0^{\pi/2} f_n(x) \cdot f_n(x)\, dx = \int_0^{\pi/2} \sin^2(2n+1)\, x\, dx$

$$\therefore \int_0^{\pi/2} [f_n(x)]^2\, dx = \int_0^{\pi/2} \left(\frac{1 - \cos 2(2n+1)\, x}{2} \right) dx = \frac{1}{2}\left[x - \frac{\sin 2(2n+1)\, x}{2(2n+1)} \right]_0^{\pi/2}$$

$$\therefore \int_0^{\pi/2} [f_n(x)]^2\, dx = \frac{\pi}{4} \neq 0 \quad \text{..........}$$

Since, $\int_0^{\pi/2} f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$

the given set of functions is orthogonal on $[0, \pi/2]$.

Now, if the set is to be orthonormal, then we should have,

$$\int_0^{\pi/2} [f_n(x)]^2\, dx = 1$$

For this, we divide (1) by $\pi/4$ and write it as

$$\int_0^{\pi/2} \frac{4}{\pi} [f_n(x)]^2\, dx = \frac{4}{\pi} \cdot \frac{\pi}{4} = 1 \quad i.e. \quad \int_0^{\pi/2} \frac{2}{\sqrt{\pi}} f_n(x) \cdot \frac{2}{\sqrt{\pi}} f_n(x)\, dx = 1$$

This is obviously an orthonormal set, where $\phi_n(x) = \frac{2}{\sqrt{\pi}} \sin(2n+1)\, x$.

Hence, the required orthonormal set of functions is $\frac{2}{\sqrt{\pi}} \sin x,\ \frac{2}{\sqrt{\pi}} \sin 3x,\ \frac{2}{\sqrt{\pi}} \sin 5x, \ldots$

Example 4 : Is $S = \left\{ \sin\left(\frac{\pi x}{4}\right), \sin\left(\frac{3\pi x}{4}\right), \sin\left(\frac{5\pi x}{4}\right), \ldots \right\}$ orthogonal in (0, 1) ? **(M.U. 20**

Sol. : We have

$$\int_0^1 f_m(x) \cdot f_n(x)\, dx = \int_0^1 \sin\frac{(2m+1)\pi x}{4} \sin\frac{(2n+1)\pi x}{4} \cdot dx$$

$$= -\frac{1}{2}\left[\int_0^1 \left(\frac{\cos(2m+2n+2)\pi x}{4} - \frac{\cos(2m-2n)\pi x}{4} \right) dx \right]$$

$$= -\frac{1}{2}\left[\int_0^1 \left[\cos\frac{(m+n+1)\pi x}{2} - \cos\frac{(m-n)\pi x}{2} \right] dx \right]$$

$$= -\frac{1}{2}\left[\sin\frac{(m+n+1)\pi x}{2} \cdot \frac{2}{(m+n+1)\pi} - \sin\frac{(m-n)\pi x}{2} \cdot \frac{2}{(m-n)\pi} \right]_0^1$$

$$\neq 0$$

$\therefore$ S is not orthogonal.

Example 5 : Show that the set of functions

$$\sin\left(\frac{\pi x}{2L}\right), \sin\left(\frac{3\pi x}{2L}\right), \sin\left(\frac{5\pi x}{2L}\right), \ldots \text{ is orthogonal over } (0, L).$$

(M.U. 2002, 05, 06, 09, 1

Construct corresponding orthonormal set. **(M.U. 2016)**

Sol. : We have $f_n(x) = \sin\frac{(2n+1)\pi x}{2L}$, $n = 0, 1, 2, \ldots$

$$\therefore \int_0^L f_m(x) \cdot f_n(x)\, dx = \int_0^L \sin\frac{(2m+1)\pi x}{2L} \cdot \sin\frac{(2n+1)\pi x}{2L}\, dx$$

$$= -\frac{1}{2}\int_0^L \left[\cos\left(\frac{2m+2n+2}{2L}\right)\pi x - \cos\left(\frac{2m-2n}{2L}\right)\pi x \right] dx$$

$$= -\frac{1}{2}\int_0^L \left[\cos\left(\frac{m+n+1}{L}\right)\pi x - \cos\left(\frac{m-n}{L}\right) x \right] dx$$

$$= -\frac{1}{2}\left[\frac{\sin\{(m+n+1)/L\}\pi x}{(m+n+1)\pi/L} - \frac{\sin\{(m-n)/L\}\pi x}{(m-n)\pi/L} \right]_0^L$$

$$= -\frac{1}{2}\left[\frac{\sin(m+n+1)\pi}{(m+n+1)\pi/L} - \frac{\sin(m-n)\pi}{(m-n)\pi/L} \right] \quad \text{.................... (1)}$$

Now, two cases arise.

Case 1 : If $m \neq n$, then since m, n are integers from (1)

$$\int_0^L f_m(x) \cdot f_n(x)\, dx = 0$$

Case 2 : If $m = n$, then

$$\int_0^L f_n(x) \cdot f_n(x)\, dx = \int_0^L \sin^2\frac{(2n+1)\pi x}{2L}\, dx$$

$$\therefore \int_0^L [f_n(x)]^2\, dx = \int_0^L \left[\frac{1 - \cos 2\{(2n+1)/2L\}\pi x}{2} \right] dx = \frac{1}{2}\left[x - \frac{\sin 2\{(2n+1)/2L\}\pi x}{(2n+1)\pi/L} \right]_0^L$$

$$\therefore \int_0^L [f_n(x)]^2\, dx = \frac{1}{2}\left[x - \frac{\sin\{(2n+1)/L\}\pi x}{(2n+1)\pi/L} \right]_0^L = \frac{L}{2} \neq 0 \quad \text{.................... (2)}$$

Since, $\int_0^L f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$

the given set of functions is orthogonal on $[0, L]$.

Now, if the set is orthonormal then we should have

$$\int_0^L [f_n(x)]^2 dx = 1$$

For this, we divide (2) by $\frac{2}{L}$ and write it as

$$\int_0^L \frac{2}{L} \cdot [f_n(x)]^2 dx = \frac{2}{L} \cdot \frac{L}{2} = 1$$

i.e., $\int_0^{\pi/2} \sqrt{\frac{2}{L}} f_n(x) \cdot \sqrt{\frac{2}{L}} f_n(x)\, dx = 1$

This is obviously orthonormal set where

$$\Phi_n(x) = \sqrt{\frac{2}{L}} \cdot f_u(x) = \sqrt{\frac{2}{L}} \cdot \sin\left[\frac{(2n+1)\pi x}{2L}\right]$$

Hence, the required orthonormal set of function is

$$\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{2L}\right),\ \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{2L}\right),\ \sqrt{\frac{2}{L}}\sin\left(\frac{5\pi x}{2L}\right), \ldots$$

Example 6 : Show that the set of functions

$$1,\ \sin\frac{\pi x}{L},\ \cos\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\ \cos\frac{2\pi x}{L},\ \ldots$$

form an orthogonal set in $(-L, L)$ and construct an orthonormal set. **(M.U. 1998, 2002, ...)**

Sol. : Let $f_n(x) = \sin\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots$

$$g_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots$$

(a) $$\int_{-L}^{L} f_m(x)\cdot f_n(x)\,dx = \int_{-L}^{L}\sin\frac{m\pi x}{L}\cdot\sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m+n)\pi x}{L}\right) - \cos\left(\frac{(m-n)\pi x}{L}\right)\right]dx$$

$$\therefore \int_{-L}^{L} f_m(x)\cdot f_n(x)\,dx = -\frac{1}{2}\left[\frac{\sin\{(m+n)/L\}\pi x}{(m+n)\pi/L} - \frac{\sin\{(m-n)/L\}\pi x}{(m-n)\pi/L}\right]_{-L}^{L}$$

Now, two cases arise.

Case 1 : If $m \neq n$, then $\int_{-L}^{L} f_m(x)\cdot f_n(x)\,dx = 0$.

Case 2 : If $m = n$, then from (1)

$$\int_{-L}^{L} f_n(x)\,dx\cdot f_n(x)\,dx = \int_{-L}^{L}\sin^2\frac{n\pi x}{L}\,dx$$

$$\therefore \int_{-L}^{L}[f_n(x)]^2\,dx = \int_{-L}^{L}\left(\frac{1-\cos(2n\pi x/L)}{2}\right)dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2n\pi x)/L}{2n\pi/L}\right]_{-L}^{L} = L \neq 0$$

(b) $$\int_{-L}^{L} g_m(x)\cdot g_n(x)\,dx = \int_{-L}^{L}\cos\frac{m\pi x}{L}\cdot\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin\{(m+n)/L\}\pi x}{(m+n)\pi/L} + \frac{\sin\{(m-n)/L\}\pi x\}}{(m-n)\pi/L}\right]_{-L}^{L}$$

Again two cases arise.

Case 1 : When $m \neq n$, then $\int_{-L}^{L} g_m(x)\cdot g_n(x)\,dx = 0$.

Case 2 : If $m = n$, then $\int_{-L}^{L} g_n(x)\cdot g_n(x)\,dx = \int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx$

$$\therefore \int_{-L}^{L}[g_n(x)]^2\,dx = \int_{-L}^{L}\left(\frac{1+\cos(2n\pi x/L)}{2}\right)dx$$

$$\therefore \int_{-L}^{L}[g_n(x)]^2\,dx = \frac{1}{2}\left[x + \frac{\sin(2n\pi x/L)}{2n\pi/L}\right]_{-L}^{L} = L \neq 0$$

(c) $$\int_{-L}^{L} f_m(x)\cdot g_n(x)\,dx = \int_{-L}^{L}\sin\frac{m\pi x}{L}\cdot\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left[\sin\left(\frac{(m+n)\pi x}{L}\right) + \sin\left(\frac{(m-n)\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[-\frac{\cos\{(m+n)/L\}\pi x}{(m+n)/L} - \frac{\cos\{(m-n)/L\}\pi x}{(m-n)/L}\right]_{-L}^{L}$$

Again two cases arise

Case 1 : If $m \neq n$, then $\int_{-L}^{L} f_m(x)\cdot g_n(x)\,dx = 0$

Case 2 : If $m = n$, then $f_n(x)\cdot g_n(x) = [f_n(x)]^2$ or $[g_n(x)]^2$

and we have already proved above that

$$\int_{-L}^{L}[f_n(x)]^2\,dx = L \quad\text{and}\quad \int_{-L}^{L}[g_n(x)]^2\,dx = L$$

and for the first term *i.e.*, for 1, $\int_{-L}^{L} 1\,dx = 2L$.

Hence, the given sequence is orthogonal.

For orthonormality the value of the integral L must be 1. Hence, if each term is divided by $\sqrt{L}$ then the set will be orthonormal. Hence, the set $\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\sin x, \frac{1}{\sqrt{L}}\cos x, \frac{1}{\sqrt{L}}\sin 2x, \ldots$ is an orthonormal set.

Example 7 : Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = (3x^2 - 1)/2$ are orthogonal over $(-1, 1)$.

(M.U. 1997, 2002, 03, 04, 08, 14, 15)

Sol. : We have

$$\int_{-1}^{1} f_1(x)\cdot f_2(x)\,dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_1(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{1}{2}(3x^2 - 1)\,dx = \frac{1}{2}\left[x^3 - x\right]_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_2(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{x}{2}(3x^2 - 1)\,dx = \frac{1}{2}\int_{-1}^{1}(3x^3 - x)\,dx$$

$$= \frac{1}{2}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1} = 0$$

Further $$\int_{-1}^{1} f_1(x)\cdot f_1(x)\,dx = \int_{-1}^{1} 1\cdot 1\cdot dx = [x]_{-1}^{1} = 2 \neq 0$$

$$\int_{-1}^{1} f_2(x)\cdot f_2(x)\,dx = \int_{-1}^{1} x\cdot x\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \neq 0$$

$$\int_{-1}^{1} f_3(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{1}{4}(9x^4 - 6x^2 + 1)\,dx = \frac{1}{4}\left[\frac{9x^5}{5} - \frac{6x^3}{3} + x\right]_{-1}^{1}$$

$$\therefore \int_{-1}^{1} f_3(x) \cdot f_3(x)\, dx = \frac{1}{4}\left[\left(\frac{9}{5} - 2 + 1\right) - \left(-\frac{9}{5} + 2 - 1\right)\right] = \frac{1}{4}\left(\frac{18}{5} - 4 + 2\right)$$

$$= \frac{1}{4} \cdot \frac{8}{5} = \frac{2}{5} \neq 0$$

Hence, the given set is orthogonal on [−1, 1].

Example 8 : Show that the functions $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on (−1, 1). Determine the constants a and b such that the function $f_3(x) = -1 + ax + bx^2$ is orthogonal to both f_1 and f_2 on the interval. **(M.U. 2003, 05, 06, 07, 09, 12)**

Sol. : We have already proved the first part above.

Now, if $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ we should have,

(i) $\int_{-1}^{1} f_1(x) \cdot f_3(x)\, dx = 0 \quad \therefore \int_{-1}^{1} 1 \cdot (-1 + ax + bx^2)\, dx = 0$

$$\therefore \left[-x + \frac{ax^2}{2} + \frac{bx^3}{3}\right]_{-1}^{1} = 0 \quad \therefore \left(-1 + \frac{a}{2} + \frac{b}{3}\right) - \left(+1 + \frac{a}{2} - \frac{b}{3}\right) = 0$$

$$\therefore -2 + \frac{2b}{3} = 0 \quad \therefore b = 3.$$

And (ii) $\int_{-1}^{1} f_2(x) \cdot f_3(x)\, dx = 0 \quad \therefore \int_{-1}^{1} x \cdot (-1 + ax + bx^2)\, dx = 0$

$$\therefore \left[\frac{-x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4}\right]_{-1}^{1} = 0 \quad \therefore \left(-\frac{1}{2} + \frac{a}{3} + \frac{b}{4}\right) - \left(-\frac{1}{2} - \frac{a}{3} + \frac{b}{4}\right) = 0$$

$$\therefore \frac{2a}{3} = 0 \quad \therefore a = 0$$

$$\therefore f_3(x) = 3x^2 - 1$$

Now, $\int_{-1}^{1} [f_3(x)]^2\, dx = \int_{-1}^{1} (3x^2 - 1)^2\, dx$

$$= \int_{-1}^{1} (9x^4 - 6x^2 + 1)\, dx = \left[\frac{9x^5}{5} - \frac{6x^3}{3} + x\right]_{-1}^{1}$$

$$= \left(\frac{9}{5} - 2 + 1\right) - \left(-\frac{9}{5} - 12 - 1\right) = \frac{18}{5} - 4 + 2$$

$$= \frac{18}{5} - 2 = \frac{9}{5}$$

The required function $f_3(x) = 3x^2 - 1$.

Example 9 : If $f_i(x)$, $i = 1, 2, 3, \ldots$ is a set of orthogonal functions in $[a, b]$ and $g(x) = \sum_{i=1}^{\infty} a_i f_i(x)$ then find a_i. **(M.U. 2002)**

Sol. : We have $g(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + \ldots \infty$

Multiply both sides by $f_1(x)$.

$$\therefore f_1(x)\, g(x) = a_1 [f_1(x)]^2 + a_2 f_1(x) \cdot f_2(x) + a_3 f_1(x)\, f_3(x) + \ldots \infty \quad \text{.......... (1)}$$

Now, integrate both sides w.r.t. x from a to b.

$$\therefore \int_a^b f_1(x)\, g(x)\, dx = a_1 \int_a^b [f_1(x)]^2\, dx + a_2 \int_a^b f_1(x) \cdot f_2(x)\, dx + a_3 \int_a^b f_1(x) \cdot f_3(x)\, dx + \ldots \infty \quad \text{.......... (2)}$$

But by definition of orthogonal functions

$$\int_a^b f_m(x)\, f_n(x) = 0 \ \text{ if } m \neq n \quad \text{and} \quad \int_a^b [f_m(x)]^2\, dx \neq 0$$

Hence, on the r.h.s. of (2) all the integrals except the first are zero and the first not zero.

$$\therefore \int_a^b f_1(x)\, g(x)\, dx = a_1 \int_a^b [f_1(x)]^2\, dx \quad \therefore a_1 = \frac{\int_a^b f_1(x) \cdot g(x)\, dx}{\int_a^b [f_1(x)]^2\, dx}$$

Similarly, by multiplying (1) successivaly by $f_2(x)$, $f_3(x)$ ..... and integrating both sides w.r.t. x from a to b, we can obtain the values of $a_2, a_3, \ldots$

Thus, in general, we have $a_i = \dfrac{\int_a^b f_i(x)\, g(x)\, dx}{\int_a^b [f_i(x)]^2\, dx}$.

Example 10 : If $f(x) = C_1\Phi_1(x) + C_2\Phi_2(x) + C_3\Phi_3(x)$, where C_1, C_2, C_3 constants and Φ_1, Φ_2, Φ_3 are orthonormal sets on (a, b), show that

$$\int_a^b [f(x)]^2\, dx = C_1^2 + C_2^2 + C_3^2$$

(M.U. 2002, 07, 08, 10)

Sol. : Since, Φ_1, Φ_2, Φ_3 are orthonormal

$$\int_a^b [\psi_1(x)]^2\, dx = \int_a^b [\Phi_2(x)]^2\, dx = \int_a^b [\Phi_3(x)]^2\, dx = 1 \quad \text{.......... (1)}$$

and $\int_a^b \Phi_m(x)\, \Phi_n(x)\, dx = 0$ when $m \neq n$ .... (2)

Now $\int_a^b [f(x)]^2\, dx = \int_a^b [C_1\Phi_1(x) + C_2\Phi_2(x) + C_3\Phi_3(x)]^2\, dx$

$$\therefore \int_a^b [f(x)]^2\, dx = \int_a^b \Big[C_1^2(\Phi_1(x))^2 + C_2^2(\Phi_2(x))^2 + C_3^2(\Phi_3(x))^2 + 2C_1C_2\,\Phi_1(x)\Phi_2(x)$$
$$+ 2C_1C_3\,\Phi_1(x)\Phi_3(x) + 2C_2C_3\,\Phi_2(x)\Phi_3(x)\Big]\, dx$$

$$\therefore \int_a^b [f(x)]^2\, dx = C_1^2 \int_a^b [\Phi_1(x)]^2\, dx + C_2^2 \int_a^b [\Phi_2(x)]^2\, dx$$
$$+ C_3^2 \int_a^b [\Phi_3(x)]^2\, dx + 2C_1C_2 \int_a^b \Phi_1(x)\Phi_2(x)\, dx$$
$$+ 2C_1C_3 \int_a^b \Phi_1(x)\Phi_3(x)\, dx + 2C_2C_2 \int_a^b \Phi_1(x)\Phi_2(x)\, dx$$
$$= C_1^2 + C_2^2 + C_3^2 \quad \text{by (1) and (2).}$$

EXERCISE - II

1. Show that the set of functions $\frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \ldots$ form a normal set in the interval $[-\pi, \pi]$. **(M.U. 2003)**

2. Show that the set of functions $\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \ldots$ form a normal set in the interval $[-\pi, \pi]$. **(M.U. 2011)**

3. Show that the set of functions $\cos x, \cos 2x, \cos 3x, \ldots$ is a set of orthogonal functions over $[-\pi, \pi]$. Hence, construct a set of orthonormal functions. **(M.U. 1995, 98, 2005, 12)**

4. Prove that $\sin x, \sin 2x, \sin 3x, \ldots$ is orthogonal on $[0, 2\pi]$ and construct orthonormal set of functions. **(M.U. 1994, 97, 99, 2003, 05)**

5. Is the set of functions $\sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{3\pi x}{l}\right), \sin\left(\frac{5\pi x}{l}\right), \ldots$ orthogonal over $(0, l)$. **(M.U. 2003, 11)** [**Ans. :** Yes]

6. Is the set of functions $\cos x, \cos 3x, \cos 5x, \ldots$ orthogonal over $(0, \pi/2)$. **(M.U. 2003)** [**Ans. :** No]

7. Show that the set of functions $\sin x, \sin 2x, \sin 3x, \ldots$ is orthogonal on the interval $[0, \pi]$. **(M.U. 1999, 2003)**

8. Show that the set of functions $\Phi_n(x) = \sin\left(\frac{n\pi x}{l}\right)$, $n = 1, 2, 3, \ldots$ is orthogonal in $(0, l)$. **(M.U. 2010)**

9. Show that the set of functions $\cos x, \cos 2x, \cos 3x, \ldots$ is orthogonal on $[-\pi, \pi]$. **(M.U. 2003, 06)**

10. Show that the set of functions $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$ is orthogonal on $(0, 2\pi)$ but not on $(0, \pi)$.

    How can you convert the set orthonormal on $(0, 2\pi)$? Write down the orthonormal set.

    (M.U. 2003) [**Ans. :** $\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 2x, \ldots$]

11. Show that the following set of functions is orthonormal on $(0, \infty)$

$$\left\{e^{-x/2},\ e^{-x/2}(1-x),\ \frac{1}{2}e^{-x/2}(x^2-4x+2)\right\}$$ **(M.U. 2003, 04, 07)**

12. Show that the functions $\{\sin(2n-1)x\}$, $n = 1, 2, 3, \ldots, \infty$ are orthogonal on $[0, \pi/2]$. Hence construct an orthonormal set of functions from this. **(M.U. 2011)**

EXERCISE - III

Theory

1. State complex form of Fourier Series. **(M.U. 2004)**

CHAPTER

5

Complex Variables

1. Introduction

In this chapter we are going to study functions of a complex variable $z = x + iy$ where x and y are real. We shall consider how to differentiate a function of a complex variable, the meaning of analytic functions of a complex variable and the conditions of analyticity viz. Cauchy-Riemann equations.

2. Definition of A Complex Function

If by a rule or set of rules we can find one or more complex numbers w for every $z\,(= x + iy)$ in a given domain, we say that **w is a function of z** and denote it as

$$w = f(z)$$

Since, both z and w are complex quantities the function is called a complex function.

If for a given z there corresponds one and only one w then the function is called **single valued function.** For example $w = z^2$ is a single valued function. If on the other hand if for a given z there correspond two or more values of w, then the function is called **multiple valued function.** For example $w = \sqrt[6]{z}$ is a multiple valued function. We shall consider single valued functions only.

Since, $z = x + iy$, $w = f(z)$ can be put in the form $w = u(x, y) + i\,v(x, y)$ where, u and v are functions of x and y. Thus, we can write

$$\boldsymbol{w = u(x, y) + iv(x, y)}$$

For example, if $w = z^2 + 2z + 3$ then

$$w = (x + iy)^2 + 2(x + iy) + 3 = x^2 + 2ixy - y^2 + 2x + 2iy + 3$$

$$= (x^2 - y^2 + 2x + 3) + i(2xy + 2y) = u(x, y) + iv(x, y)$$

3. Z-plane and W-plane

We know that real function $y = f(x)$ can be represented by a curve on x-y plane. But $w = f(z)$ i.e. $w = u + iv$ where, x and y are real, involves four variables x, y, u and v. Hence, we cannot represent $w = f(z)$ on a single plane. However, we can have sufficient idea of a function $w = f(z)$ if we use two planes, one z-plane on which we plot a point (x, y) and another w-plane on which we plot the corresponding point (u, v).

For example, consider a simple function $w = z^2$ *i.e.* $w = (x + iy)^2 = (x^2 - y^2) + 2ixy$. Here, $u = x^2 - y^2$, $v = 2xy$. If we have a point $P\,(2, 1)$ on the z-plane then since, $u = 4 - 1 = 3$ and $v = 4$, the corresponding point on the w-plane will be $P'\,(3, 4)$. As the point P traces a curve C in the z-plane the corresponding point P' will trace another curve C' on the w-plane. Thus, a curve C in the z-plane is transformed into another curve C' in the w-plane. We shall consider in more details this type of transformation of curves in z-plane into the curves in w-plane in the next chapter.

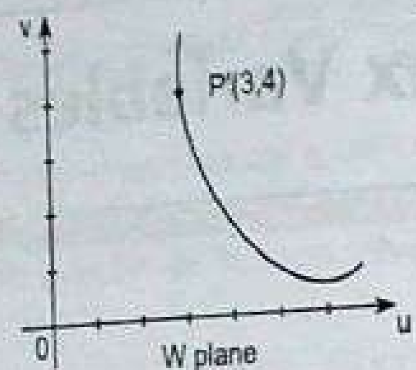

Fig. 5.1 (a)

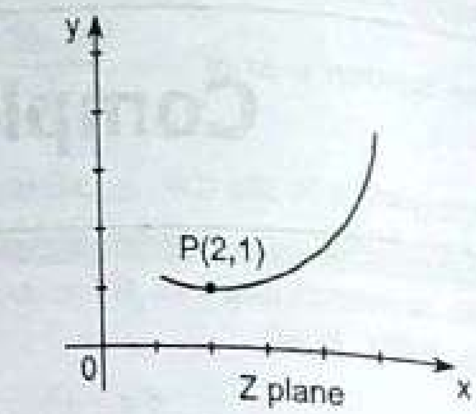

Fig. 5.1 (b)

4. Neighbourhood of A point $P(z_0)$

Consider, the inequality $|z - z_0| < \epsilon$ *i.e.* consider a circle with centre at z_0 and radius ϵ. The inequality clearly defines a region of all points lying within the circle $|z - z_0| = \epsilon$ including the point z_0 but excluding the points on the boundary of the circle. The circular region $|z - z_0| < \epsilon$ is called a **neighbourhood** of the point z_0. It is clear that as the number ϵ becomes smaller and smaller, the neighbourhood also becomes smaller and smaller.

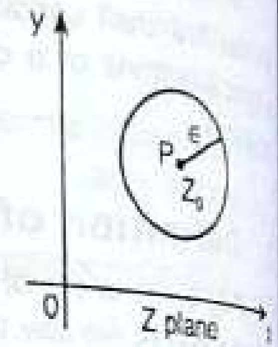

Fig. 5.2

5. Limit of A Function

The limit of a function of a complex variable $w = f(z)$ is defined on the lines of the definition of the limit of a function of a real variable.

Definition : Let $w = f(z)$ be a single valued function of z defined in a bounded and closed domain D and let z approach z_0 along any path in D. Given a positive number ϵ, however small (but not zero) if we can find another small positive number δ such that $|f(z) - w_0| < \epsilon$ for all z for which $0 < |z - z_0| < \delta$ then we say that w_0 is the limit of $f(z)$ as z tends to z_0 and denote it is

$$\lim_{z \to z_0} f(z) = w_0$$

(Note that the point z may approach z_0, along any path and the limit does not depend upon the path.)

Limits in terms of Real and Imaginary Parts : Let $f(z) = u + iv$ where, u and v are functions of real variables x, y. Let $z = x + iy$, $w_0 = u_0 + iv_0$, $z_0 = x_0 + iy_0$. Let $\lim_{z \to z_0} f(z) = w_0$.

Then by the above definition of the limit

$$|f(z) - w_0| < \epsilon \qquad \text{for } 0 < |z - z_0| < \delta$$

i.e. $|(u + iv) - (u_0 + iv_0)| < \epsilon \quad$ for $0 < |(x + iy) - (x_0 + iy_0)| < \delta$.

i.e. $|(u - u_0) + i(v - v_0)| < \epsilon \quad$ for $|(x - x_0) + i(y - y_0)| < \delta$.

This means, in terms of real functions

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u = u_0 \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v = v_0$$

$$\therefore \quad \lim_{z \to z_0} f(z) = w_0 = u_0 + iv_0 = \lim_{\substack{x \to x_0 \\ y \to y_0}} u + i \lim_{\substack{x \to x_0 \\ y \to y_0}} v$$

6. Continuity

Continuity of a function of a complex variable is defined exactly as the continuity of a function of a real variable.

Let $w = f(z)$ be a single valued function defined in a bounded and closed domain D. $w = f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

Continuity in terms of Real and Imaginary Parts : Using the results obtained in § 5, it can be easily shown that if $w = f(z)$ is continuous at $z = z_0$ then the real and imaginary parts u and v are separately continuous at $z_0 = x_0 + i\,y_0$

i.e. $$\lim_{\substack{x \to x_0 \\ y \to y_0}} u = u_0 \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v = v_0$$

Example 1 : Discuss the continuity of the following functions at the given points

(i) $\dfrac{\bar{z}}{z}$ at $z = 0$ $\qquad$ (ii) $\dfrac{z^2}{z^4 + 3z^2 + 1}$ at $z = e^{i\pi/4}$

Sol. : (i) Let $f(z) = \dfrac{\bar{z}}{z} = \dfrac{x - i\,y}{x + i\,y}$ .... (i)

$$\therefore \quad f(z) = \frac{x - iy}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x^2 - y^2 - 2i\,xy}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} - 2i\frac{xy}{x^2 + y^2}$$

Let $u = \dfrac{x^2 - y^2}{x^2 + y^2}, \quad v = -\dfrac{2xy}{x^2 + y^2}$

Let $y = kx$, $\therefore \quad u = \dfrac{1 - k^2}{1 + k^2}, \quad v = -\dfrac{2k}{1 + k^2}$

$$\lim_{\substack{x \to 0 \\ y \to 0}} u = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{1 - k^2}{1 + k^2} = \frac{1 - k^2}{1 + k^2}$$ which depends upon k *i.e.* on the path

$$\lim_{\substack{x \to 0 \\ y \to 0}} v = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{-2xy}{x^2 + y^2} = \lim_{x \to 0} -\frac{2k}{1 + k^2} = -\frac{2k}{1 + k^2}$$ which depends upon k *i.e.* on the path.

Hence, $f(z)$ is not continuous at $z = 0$.

Alternative Method

We shall find the limits along the x-axis and along the y-axis.

Along the x-axis, $z = x + iy = x + i(0) = x$ .... (ii)

$$\therefore \quad \lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{x}{x} = 1 \qquad \text{from (i)}$$

Along the y-axis, $z = x + iy = 0 + iy = iy$ .... (iii)

$$\therefore \quad \lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} -\frac{i\,y}{i\,y} = -1 \qquad \text{from (i)}$$

From (i) and (ii), we see that the two limits along two different paths are different.

Hence, the limit does not exist. Hence, $f(z)$ is not continuous at $z = 0$.

(ii) Let $f(z)=\dfrac{z^2}{z^4+3z^2+1}$

When $z=e^{i\pi/4}$, $z^2=e^{i\pi/2}=\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}=i$

$$z^4=e^{i\pi}=\cos\pi+i\sin\pi=-1$$

$\therefore$ At $z=e^{i\pi/4}$, $f(z)=\dfrac{i}{-1+3i+1}=\dfrac{i}{3i}=\dfrac{1}{3}$

Now $\lim\limits_{z\to e^{i\pi/4}}=\dfrac{i}{-1+3i+1}=\dfrac{i}{3i}=\dfrac{1}{3}$. Since, $\lim\limits_{z\to e^{i\pi/4}} f(z)=f(e^{i\pi/4})$

The function is continuous at $z=e^{i\pi/4}$.

Example 2 : Find the following limits

(i) $\lim\limits_{z\to i}\dfrac{z^2+1}{z^6+1}$ (ii) $\lim\limits_{z\to(1+i)}\left(\dfrac{z-(1+i)}{z^2-2z+z}\right)^3$

Sol. : (i) As $z\to i$, $z^2\to -1$

$$\therefore \lim_{z\to i}\frac{z^2+1}{(z^2+1)(z^4-z^2+1)}=\lim_{z^2\to -1}\frac{1}{z^4-z^2+1}=\frac{1}{3}.$$

(ii) Now, $z^2-2z+2=(z^2-2z+1)+1=(z-1)^2-(i)^2$

$$=(z-1+i)(z-1-i)$$

$$\therefore \lim_{z\to(1+i)}\left\{\frac{z-(1+i)}{[z-(1-i)][z-(1+i)]}\right\}^3=\lim_{z\to(1+i)}\left\{\frac{1}{z-(1-i)}\right\}^3$$

$$=\left\{\frac{1}{1+i-1+i}\right\}^3=\left\{\frac{1}{2i}\right\}^3=-\frac{1}{8i}.$$

EXERCISE - I

1. Find $\lim\limits_{z\to 3i}(3x+iy^2)$.
2. Find $\lim\limits_{z\to 1}\dfrac{z^3-1}{z-1}$.
3. Find $\lim\limits_{z\to i}\dfrac{z^2+1}{z-i}$.
4. Find $\lim\limits_{z\to i}\dfrac{z^3+i}{z-i}$.
5. Show that $\lim\limits_{z\to 0}\dfrac{xy}{x^2+y^2}$ does not exist.
6. Show that $\lim\limits_{z\to 0}\dfrac{x^2y}{x^4+y^2}$ does not exist. (**Hint :** Put $y=kx^2$)

[Ans. : (1) $9i$, **(2)** 3, **(3)** $2i$, (4)-

7. Differentiability

Definition : Let $w=f(z)$ be a single valued function of z defined in domain D. $f(z)$ is said to be differentiable at any point z if

$$\lim_{\delta z\to 0}\frac{\delta w}{\delta z}=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$$

is unique as $\delta z\to 0$ along any path of the domain D.

Example 1 : Find whether the following functions are differentiable.

(i) z^3 at $z=i$ (ii) $\cos z$ at $z=i$

Sol. : (i) Let $f(z)=z^3$ and $z_0=i$

$\therefore f(z_0)=i^3=-i$

$$\therefore f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to i}\frac{z^3-i^3}{z-i}=\lim_{z\to i}\frac{(z-i)(z^2+iz+i^2)}{(z-i)}$$

$$=\lim_{z\to i} z^2+iz+i^2=3i^2=-3$$

$\therefore f(z)$ is differentiable at $z=i$.

(ii) Let $f(z)=\cos z$ and $z_0=i$.

$\therefore f(z_0)=\cos i$

$$\therefore f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to i}\frac{\cos z-\cos i}{z-i}$$

$$=\lim_{z\to i}\frac{-2\sin\left(\frac{z+i}{2}\right)\sin\left(\frac{z-i}{2}\right)}{z-i}=\lim_{z\to i}-2\sin\left(\frac{z+i}{2}\right)\frac{\sin\left(\frac{z-i}{2}\right)}{\left(\frac{z-i}{2}\right)}\cdot\frac{1}{2}$$

$$=-\sin i.$$

$\therefore f(z)$ is differentiable at $z=i$.

Example 2 : Prove that the function $|z|^2$ is continuous everywhere but nowhere differentiable except at the origin.

Sol. : Let $f(z)=|z|^2=|x+iy|^2=x^2+y^2$

Since, x^2+y^2 is continuous everywhere, $f(z)$ is also continuous everywhere.

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z_0+\delta z)-f(z_0)}{\delta z}=\lim_{\delta z\to 0}\frac{|z_0+\delta z|^2-|z_0|^2}{\delta z}$$

$$=\lim_{\delta z\to 0}\frac{(z_0+\delta z)(\bar z_0+\delta\bar z)-z_0\bar z_0}{\delta z}=\lim_{\delta z\to 0}\frac{z_0\,\delta\bar z+\bar z_0\,\delta z+\delta z\,\delta\bar z}{\delta z}$$

$$=\lim_{\delta z\to 0} z_0\frac{\delta\bar z}{\delta z}+\bar z_0+\delta\bar z \qquad \text{.......... (I)}$$

(i) **When δz is real :** Then $\delta y=0$ and $\delta\bar z=\delta z=\delta x$. As $\delta z\to 0$, $\delta x\to 0$.

$$\therefore f'(z_0)=\lim_{\delta z\to 0} z_0\frac{\delta\bar z}{\delta z}+\bar z_0+\delta\bar z=\lim_{\delta x\to 0}(z_0+\bar z_0+\delta x)=z_0+\bar z_0$$

(ii) **When dz is imaginary :** Then $dx=0$ and $\delta z=i\,\delta y$, $\delta\bar z=-i\,\delta y$. As $\delta z\to 0$, $\delta y\to 0$.

$$\therefore f'(z_0)=\lim_{\delta z\to 0} z_0\frac{\delta\bar z}{\delta z}+\bar z_0+\delta\bar z=\lim_{\delta x\to 0} z_0\left(-\frac{i\,\delta y}{i\,\delta y}\right)+\bar z_0-i\,\delta y$$

$$=-z_0+\bar z_0$$

Since, the two limits are different along two different paths except at $z = 0$, $f'(z_0)$ does not exist anywhere except at $z = 0$.

Hence, $f(z)$ is not differentiable anywhere except at $z = 0$.

8. Analytic Functions

If a single valued function $w = f(z)$ is defined and differentiable at each point of a domain D then it is called analytic or regular or holomorphic function of z in the domain D.

A function is said to be analytic at a point if it has a derivative at that point and in some neighbourhood of that point.

If a function ceases to be analytic at a point of the domain then the point is called a singular point.

9. Cauchy-Riemann Equations in Cartesian Coordinates

Theorem : The necessary and sufficient conditions for a continuous one valued function

$$w = f(z) = u(x, y) + i\,v(x, y)$$

to be analytic in a region R are (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in a region R and

$$\text{(ii)} \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (i.e.\ u_x = v_y \text{ and } u_y = -v_x)}$$

at each point of R.

The conditions (ii) are known as **Cauchy-Riemann equations** or briefly **C-R equations**.

Augustin Louis (Baron de) Cauchy (1789 - 1857)

A French mathematician of great repute who contributed to various branches of mathematics. He wanted to be an engineer but because of poor health he was advised to pursue mathematics. His mathematical work began in 1811 when he gave briliant solutions to some difficult problems of that time. In the next 35 years he published 700 papers in various branches of mathematics. He is supposed to have initiated the era of modern analysis.

Bernhard Riemann (1826 - 1866)

Bernhard Riemann was a great German mathematician who made lasting contributions to analysis and differential geometry. Riemann exhibited exceptional mathematical skills at early age. He joined University of Göttingen to study mathematics and at this university he first met Carl Friedrich Gauss. In 1847 Riemann moved to Berlin where Jacobi, Dirichlet and Steiner were teaching. He founded the field of Riemannian geometry which was used by Einstein in general theory of relativity. He was the first to suggest dimensions higher than three or four. Riemann made major contributions to real analysis. He defined Riemann integral by means of Riemann sums. He introduced Riemann-zeta function. He is also known for Riemannian metric, Riemannian geometry, Riemannian curvature tensor.

Proof : (a) The conditions are necessary

Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic at every point of a region R. Then dw/dz exists uniquely at every point of R.

Let δx and δy be the increments in x, y. Let $\delta u, \delta v, \delta w$ be the corresponding increments in u, v, w respectively. Now,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\delta u + i\,\delta v}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right)$$

As $w = f(z)$ is analytic in R, the above limit exists independent of the path along which $\delta z \to 0$. Since, $\delta z = \delta x + i\,\delta y$, the limit is independent of the path along which $\delta x \to 0$ and $\delta y \to 0$.

First consider the path $(QP'P)$ first parallel to the x-axis *i.e.* $\delta z \to 0$ such that $\delta y = 0$ and $\delta z = \delta x$ and then $\delta x \to 0$.

$$\therefore\ f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{.................... (i)}$$

Now let $\delta z \to 0$ along the path $(QP''P)$ first parallel to the y-axis *i.e.* $\delta z \to 0$ such that $\delta x = 0$ and $\delta z = i\,\delta y$ and then $\delta y \to 0$.

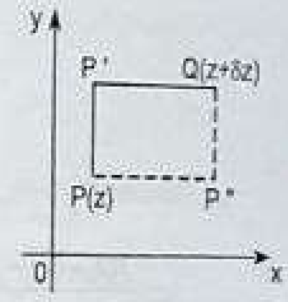

Fig. 5.3

$$\therefore\ f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\,\delta y} + i\frac{\delta v}{i\,\delta y}\right)$$

$$\therefore\ f'(z) = \frac{1}{i}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \left[\because \frac{1}{i} = \frac{i}{i^2} = -i\right]$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{.................... (ii)}$$

For existence of $f'(z)$ (i) and (ii) must be equal.

From (i) and (ii) we get,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This proves that for a function to be analytic, the **Cauchy-Riemann equations are necessary.**

(b) The conditions are sufficient

Let $f(z) = u(x, y) + i\,v(x, y)$ be a single valued function possessing continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at every point in the region R and satisfying the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We have to show that $f'(z)$ exists at every point of R.

By Taylor's theorem for functions of two variables, omitting the second and higher degree terms in x and y, we get,

$$f(z+\delta z)=u(x+\delta x,\ y+\delta y)+iv(x+\delta x,\ y+\delta y)$$

$$=\left[u(x,y)+\left(\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y\right)\right]+i\left[v(x,y)+\left(\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}\delta y\right)\right]$$

$$=[u(x,y)+iv(x,y)]+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\delta y$$

$$=f(z)+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\delta y$$

$$\therefore\ f(z+\delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\delta y$$

Using the C-R equations (in the second bracket)

$$f(z+\delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)i\,\delta y\qquad [\because i^2=-1]$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\delta x+i\,\delta y)$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta z\qquad [\because \delta x+i\,\delta y=\delta z]$$

$$\therefore\ \frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\therefore\ f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Hence, $f'(z)$ exists as $\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial x}$ exist.

$\therefore$ $f(z)$ is analytic.

$\therefore$ The conditions are sufficient.

Notes ....

1. The Cauchy-Riemann equations are only **necessary conditions** for a function to be analytic. This means even if Cauchy-Riemann equations are satisfied the function need not be analytic at that point (see Examples 10 and 11, page 5-14)
2. When $f(z)$ is analytic, its derivative is given by any one of the following expressions

$$\boxed{\begin{aligned} f'(z)&=u_x+i\,v_x\ ;\ f'(z)=v_y+i\,v_x\\ f'(z)&=u_x-i\,u_y\ ;\ f'(z)=v_y-i\,u_y\end{aligned}}$$

3. **If $f(z)$ is analytic then it can be differentiated in usual manner.**

   e.g. if $f(z)=z^2$ then $f'(z)=2z$

   if $f(z)=\sin z$ then $f'(z)=\cos z$
4. If $f(z)=f(x+iy)=u+iv$ and $f(z)$ is analytic then the functions u and v of real variables x and y are called **conjugate functions.**

Example 1 : Prove that an analytic function with its derivative zero is constant.

Sol. : Let $f(z)=u+iv$ be the given analytic function whose derivative is zero.

$\therefore\ f'(z)=u_x+iv_x=0 \qquad \therefore\ u_x=0,\ v_x=0$

But $f(z)$ is analytic. Hence, C-R equations are satisfied

$\therefore\ u_x=v_y$ and $u_y=-v_x \qquad \therefore\ v_y=0,\ u_y=0$

As $u_x=0,\ u_y=0\ \therefore\ u=$ a constant and $v_x=0,\ v_y=0\ \therefore\ v=$ a constant.

$\therefore\ f(z)=u+i\,v=$ a constant.

Example 2 : If $f(z)$ and $\overline{f(z)}$ are both analytic, prove that $f(z)$ is constant. **(M.U. 1993, 2003, 06)**

Sol. : Let $f(z)=u+iv$ then $\overline{f(z)}=u-iv=u+i(-v)$

Since, $f(z)$ is analytic $u_x=v_y$ and $u_y=-v_x$ C-R equations.

Since, $\overline{f(z)}$ is analytic $u_x=(-v_y)$ and $u_y=-(-v_x)$ C-R equation.

Adding $u_x=v_y$ and $u_x=-v_y$, we get $u_x=0$.

Adding $u_y=-v_x$ and $u_y=v_x$, we get $u_y=0$.

Since, $u_x=0$ and $u_y=0$, $u=$ a constant.

Similarly by subtraction we can prove that $v_x=0$ and $v_y=0$. $\therefore$ $v=$ a constant.

Hence, $f(z)=u+iv=$ a constant.

Example 3 : If $f(z)$ is an analytic function, show that $\frac{\partial f}{\partial \bar z}=0$. **(M.U. 1996)**

Sol. : Since, $z=x+i\,y,\ \bar z=x-i\,y$.

$$\therefore\ x=\frac{1}{2}(z+\bar z)\quad\text{and}\quad y=\frac{1}{2i}(z-\bar z).$$

Let $f(z)=u+iv$

$$\therefore\ \frac{\partial f}{\partial \bar z}=\frac{\partial}{\partial \bar z}(u+iv)=\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar z}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar z}\right)+i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar z}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar z}\right)$$

$$=\left[\frac{\partial u}{\partial x}\cdot\frac{1}{2}+\frac{\partial u}{\partial y}\left(-\frac{1}{2i}\right)\right]+i\left[\frac{\partial v}{\partial x}\cdot\frac{1}{2}+\frac{\partial v}{\partial y}\left(-\frac{1}{2i}\right)\right]$$

$$=\frac{1}{2}u_x-\frac{1}{2i}u_y+\frac{i}{2}v_x-\frac{1}{2}v_y$$

$$=\frac{1}{2}u_x+\frac{i}{2}u_y+\frac{i}{2}v_x-\frac{1}{2}v_y$$

But since, $f(z)$ is analytic, $u_x=v_y$ and $u_y=-v_x$

$$\therefore\ \frac{\partial f}{\partial \bar z}=\frac{1}{2}v_y-\frac{i}{2}v_x+\frac{i}{2}v_x-\frac{1}{2}v_y=0$$

Restatement

The above theorem can also be stated as follows.

"Every analytic function $w=u+iv$ can be expressed as a function of z only".

Notes ....

1. We have proved that if $f(z)$ is analytic then $\partial f / \partial \bar{z} = 0$. This means if $f(z)$ is analytic, f(z) is free from $\bar{z}$. In other words if $f(z)$ is analytic, it can be reconstructed in z *i.e.* in $x + iy$ only.

2. It can be easily shown that usual rules of differentiating sums, products, quotients of functions are applicable to analytic functions of complex variable.

The formulae for differentiating elementary complex functions are identical with the corresponding formulae in calculus of real variables.

For example, $\frac{d}{dz}(z^n) = nz^{n-1}$, $\frac{d}{dz}(e^z) = e^z$, $\frac{d}{dz}(\log z) = \frac{1}{z}$,

$\frac{d}{dz}(\sin z) = \cos z$, $\frac{d}{dz}(\sin hz) = \cos hz$.

In general, if the differential coefficient of $f(x)$ is $\phi(x)$, we can assume that if $f(z)$ is analytic, the differential coefficient of $f(z)$ is $\phi(z)$.

3. Since integration is inverse of differentiation a result similar to the one quoted in note 2 holds for integration also.

e.g. $\int z^n dz = \frac{z^{n+1}}{n+1}$, $\int e^z dz = e^z$ etc.

Derivatives of Elementary Functions

1. $\frac{d}{dz}(c) = 0$
2. $\frac{d}{dz}(z^n) = nz^{n-1}$
3. $\frac{d}{dz}(e^z) = e^z$
4. $\frac{d}{dz}(a^z) = a^z \log a$
5. $\frac{d}{dz}(\sin z) = \cos z$
6. $\frac{d}{dz}(\cos z) = -\sin z$
7. $\frac{d}{dz}(\tan z) = \sec^2 z$
8. $\frac{d}{dz}(\cot z) = -\mathrm{cosec}^2 z$
9. $\frac{d}{dz}(\sec z) = \sec z \tan z$
10. $\frac{d}{dz}(\mathrm{cosec}\, z) = -\mathrm{cosec}\, z \cot z$
11. $\frac{d}{dz}(\log z) = \frac{1}{z}$
12. $\frac{d}{dz}(\log_a z) = \frac{1}{z \log_e a}$
13. $\frac{d}{dz}(\sin^{-1} z) = \frac{1}{\sqrt{1-z^2}}$
14. $\frac{d}{dz}(\cos^{-1} z) = -\frac{1}{\sqrt{1-z^2}}$
15. $\frac{d}{dz}(\tan^{-1} z) = \frac{1}{1+z^2}$
16. $\frac{d}{dz}(\cot^{-1} z) = -\frac{1}{1+z^2}$
17. $\frac{d}{dz}(\sec^{-1} z) = \frac{1}{z\sqrt{z^2-1}}$
18. $\frac{d}{dz}(\mathrm{cosec}^{-1} z) = \frac{-1}{z\sqrt{z^2-1}}$
19. $\frac{d}{dz}(\sin hz) = \cos hz$
20. $\frac{d}{dz}(\cos hz) = \sin hz$
21. $\frac{d}{dz}(\tan hz) = \sec h^2 z$
22. $\frac{d}{dz}(\cot hz) = -\mathrm{cosec}\, h^2 z$
23. $\frac{d}{dz}(\sec hz) = -\sec hz \tan hz$
24. $\frac{d}{dz}(\mathrm{cosec}\, hz) = -\mathrm{cosec}\, hz \coth h$

25. $\frac{d}{dz}(\sin h^{-1} z) = \frac{1}{\sqrt{1+z^2}}$
26. $\frac{d}{dz}(\cos h^{-1} z) = \frac{1}{\sqrt{z^2-1}}$
27. $\frac{d}{dz}(\tan h^{-1} z) = \frac{1}{1-z^2}$
28. $\frac{d}{dz}(\cot h^{-1} z) = \frac{1}{z^2-1}$
29. $\frac{d}{dz}(\sec h^{-1} z) = \frac{-1}{z\sqrt{1-z^2}}$
30. $\frac{d}{dz}(\mathrm{cosec}\, h^{-1} z) = \frac{-1}{z\sqrt{z^2+1}}$

Example 4 : If $f(z)$ is an analytic and $|f(z)|$ is constant, prove that $f(z)$ is constant. **(M.U. 1999, 2002, 03, 05, 08, 09)**

Sol. : Let $f(z) = u + iv$ but $|f(z)| = C$. $\therefore \ u^2 + v^2 = C^2$

Differentiating it partially w.r.t. x, $uu_x + vv_x = 0$

Differentiating it partially w.r.t. y, $uu_y + vv_y = 0$

Since, $f(z)$ is analytic $u_x = v_y$ and $u_y = -v_x$

$\therefore \ uu_x - vu_y = 0$ and $uu_y + vu_x = 0$

Eliminating u_y, $(u^2 + v^2)u_x$ $\therefore \ C^2 u_x = 0$ $\therefore \ u_x = 0$

Similarly, we can show that $u_y = 0$, $v_x = 0$, $v_y = 0$

Since, $f(z)$ is analytic. $f'(z) = u_x + iv_x = 0$ $\therefore \ f(z) = \text{constant}$.

Restatement

(i) The above theorem can also be restated as

"If $f(z)$ is an analytic function with constant modulus then, prove that $f(z)$ is constant." **(M.U. 1994, 99)**

(ii) A regular function of constant magnitude is constant. **(M.U. 2005)**

Example 5 : If $f(z)$ is analytic and if the amplitude of $f(z)$ is constant, prove that $f(z)$ is constant. **(M.U. 2003)**

Sol. : Let $f(z) = u + iv$. Since its amplitude $= \tan^{-1}(v/u)$ is constant c say, we have

$\tan^{-1}\frac{v}{u} = c$ $\therefore \ \frac{v}{u} = \tan c$

Differentiating this w.r.t. x and y,

$\frac{uv_x - vu_x}{u^2} = 0$ and $\frac{uv_y - vu_y}{u} = 0$

$\therefore \ uv_x - vu_x = 0$ and $uv_y - vu_y = 0$

Since $f(z)$ is analytic, $u_x = v_y$ and $u_y = -v_x$.

$\therefore \ -uu_y - vu_x = 0$ .... (1) and $uu_x - vu_y = 0$ .... (2)

Multiply the first by u and second by v and add.

$\therefore \ (-u^2 - v^2)\, u_y = 0$ $\therefore \ u_y = 0$

Multiply the first by v and second by u and subtract

$\therefore \ (-v^2 - u^2)\, u_x = 0$ $\therefore \ u_x = 0$

But $u_x = v_y$ and $u_y = -v_x$ $\therefore \ v_y = 0$ and $v_x = 0$

Since, all four partial derivatives of u, v are zero, u and v are constants.

$\therefore$ $f(z)$ is constant.

Example 6 : If $f(z) = u + iv$ is an analytic function and (i) u = constant or (ii) v = constant then $f(z)$ is constant

Sol. : If u is constant $u_x = 0,\ u_y = 0$.

But $f'(z) = u_x + iv_x$

$= u_x - iu_y$ (By C-R equations)

$= 0$ (By data)

$\therefore$ $f(z)$ = constant.

Similarly we can prove the other part.

Note ....

From Examples 1, 2, 4, 5 and 6, we find that an **analytic function** $f(z)$ is constant if (i) $f'(z) = 0$ or (ii) $\overline{f(z)}$ is also analytic or (iii) modulus of $f(z)$ is constant or (iv) amplitude of $f(z)$ is constant or (v) real part is constant or (vi) imaginary part is constant.

Example 7 : Show that the following functions are analytic and find their derivatives.

(i) e^z, (ii) $\sin hz$ **(M.U. 1999, 2015)**

Sol. : (i) $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$\therefore$ $u = e^x \cos y,\ v = e^x \sin y$

$u_x = e^x \cos y,\ u_y = -e^x \sin y;\quad v_x = e^x \sin y,\ v_y = e^x \cos y$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x, v_y are continuous and Cauchy-Riemann equations are satisfied.

Hence, e^z is analytic.

Now, $f'(z) = u_x + iv_x = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y)$

$= e^x \cdot e^{iy} = e^{x+iy} = e^z.$

(ii) $f(z) = \sin h z = \sin h(x + iy)$

$= \sin h x \cos h iy + \cos h x \sin h iy$

$= \sin h x \cos y + i \cos h x \sin y$

$\therefore$ $u = \sin h x \cos y,\quad v = \cos h x \sin y$

$u_x = \cos h x \cos y,\quad u_y = -\sin h x \sin y$

$v_x = \sin h x \sin y,\quad v_y = \cos h x \cos y$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Further, u_x, u_y, v_x, v_y are continuous and Cauchy-Riemann equations are satisfied.

Hence, $\sin h z$ is analytic.

Now, $f'(z) = u_x + iv_x$

$= \cos hx \cos y + i \sin hx \sin y$

$= \cos hx \cos hiy + \sin hx \sin hiy$

$= \cos h(x + iy) = \cos hz.$

Example 8 : Show that the following functions are analytic and find their derivatives.

(i) $f(z) = z^3$ (ii) $f(z) = ze^z$ **(M.U. 2016)** (iii) $f(z) = \sin z$. **(M.U. 2003)**

Sol. : (i) We have $f(z) = z^3 = (x + iy)^3$

$\therefore$ $f(z) = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\therefore$ $u = x^3 - 3xy^2,\quad v = 3x^2y - y^3$

$\therefore$ $\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2,\ \dfrac{\partial v}{\partial x} = 6xy,\ \dfrac{\partial u}{\partial y} = -6xy,\ \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$ $\quad\therefore$ $f(z) = z^3$ is analytic and can be differentiated as usual.

$\therefore$ $f(z) = 3z^2$.

(ii) We have $f(z) = ze^z = (x + iy)\, e^{x+iy}$

$\therefore$ $f(z) = (x + iy)\, e^x (\cos y + i \sin y)$

$\therefore$ $u = e^x(x \cos y - y \sin y),\ v = e^x(x \sin y + y \cos y)$

$\dfrac{\partial u}{\partial x} = e^x(x \cos y - y \sin y) + e^x \cos y,\quad \dfrac{\partial u}{\partial y} = e^x(-x \sin y - y \cos y - \sin y),$

$\dfrac{\partial v}{\partial x} = e^x(x \sin y + y \cos y) + e^x \sin y,\quad \dfrac{\partial v}{\partial y} = e^x(x \cos y + \cos y - y \sin y).$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

$\therefore$ $f(z) = ze^z$ is analytic and can be differentiated as usual.

$\therefore$ $f'(z) = ze^z + e^z = e^z(z + 1).$

(iii) $f(z) = \sin z = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cos hy + i \cos x \sin hy$

$\therefore$ $u = \sin x \cos hy,\ v = \cos x \sin hy$

$\dfrac{\partial u}{\partial x} = \cos x \cos hy,\ \dfrac{\partial v}{\partial x} = -\sin x \sin hy,\ \dfrac{\partial u}{\partial y} = \sin x \sin hy,\ \dfrac{\partial v}{\partial y} = \cos x \cos hy$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

$\therefore$ $f(z) = \sin z$ is analytic and can be differentiated as usual.

$\therefore$ $f'(z) = \cos z.$

Example 9 : If $f(z)$ is equal to (a) $\bar{z}$ (b) $2x + ixy^2$, show that $f'(z)$ does not exist. **(M.U. 2002)**

Sol. : (a) $f(z) = \bar{z} = x - iy\quad \therefore\ u = x,\ v = -y\quad \therefore\ u_x = 1,\ u_y = 0\ ;\ v_y = -1,\ v_x = 0$

Since, $u_x \neq v_y$ Cauchy-Riemann equations are not satisfied and $f'(z)$ does not exist.

Alternatively

$$f'(z) = \lim_{\delta z \to 0} \frac{\overline{z + \delta z} - \bar{z}}{\delta z}$$

$$\therefore\ f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\overline{(x + iy + \delta x + i\delta y)} - \overline{(x + iy)}}{\delta x + i\delta y}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{x - iy + \delta x - i\delta y - x + iy}{\delta x + i\delta y} = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\delta x - i\delta y}{\delta x + i\delta y}$$

If $\delta y = 0$, the required limit $= \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = 1$

If $\delta x = 0$, the required limit $= \lim_{\delta y \to 0} -\frac{\delta y}{\delta y} = -1$

Since, the two limits are different $f'(z)$ does not exist.

(b) $f(z) = 2x + i\,xy^2 \quad \therefore\ u = 2x,\ v = xy^2$

$u_x = 2,\ u_y = 0,\ v_x = y^2,\ v_y = 2xy$

Since, $u_x \neq v_y$ and $u_y \neq v_x$. Cauchy-Riemann equations are not satisfied and hence, $f'(z)$ does not exist.

Example 10 : Show that $f(z) = z\bar{z} = |z|^2$ satisfies Cauchy-Riemann equations at $z = 0$ and yet is not analytic anywhere. **(M.U. 2004)**

Sol. : $f(z) = |z|^2 = x^2 + y^2 \quad \therefore\ u = x^2 + y^2,\ v = 0$

$$\therefore\ \frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = 2y,\ \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0$$

Hence, $u_x = v_y = 0$ and $u_y = -v_x = 0$ when $x = 0$ and $y = 0$.

Thus, C-R equations are satisfied at $z = 0$.

The partial derivatives u_x, u_y, v_x, v_y are also continuous everywhere.

Thus, $f'(z) = |z|^2$ is differentiable **only** at $z = 0$ but at no other point. There is no neighbourhood of $z = 0$ in which the conditions of analyticity are satisfied. Hence, $f(z)$ is **not analytic** anywhere.

Example 11 : Show that $f(z) = \sqrt{|xy|}$ is not analytic at the origin although Cauchy-Riemann equations are satisfied at that point. **(M.U. 2005, 08)**

Sol. : Let $f(z) = u + iv$ so that $u = \sqrt{|xy|}$ and $v = 0$

$$\text{Now,}\quad \frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \frac{0 - 0}{y} = 0$$

Clearly since, $v = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$ at zero.

$\therefore$ Cauchy-Riemann equations are satisfied at $z = 0$.

$$\text{Now}\quad f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

Let $z \to 0$ along $y = mx$

$$\therefore\ f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \lim_{x \to 0} \frac{x\sqrt{|m|}}{x(1 + im)} = \frac{\sqrt{m}}{1 + im}$$

Since, the limit depends upon m, $f'(0)$ does not exist.

Note ....

The above function satisfies Cauchy-Riemann equations and yet is not analytic at $z = 0$. This is because for analyticity in addition to C-R equations its four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

must be continuous at that point. The value of u_x at $x = 0$ is zero. We shall now find lim of u_x as $x \to 0$.

$$\text{Now, if } x > 0,\ \delta x > 0\ \frac{\partial u}{\partial x} = \frac{\sqrt{|y|}}{2\sqrt{|x|}},\ \text{let } y = mx$$

$$\therefore\ \lim_{x \to 0} u_x = \lim_{x \to 0} \frac{\sqrt{|mx|}}{2\sqrt{|x|}} = \frac{|m|}{2}$$

$\therefore$ Thus, the limit depends upon the path and hence, u_x is not continuous at $x = 0$. Similarly u_y, v_x, v_y are not continuous at $x = 0$. Hence, $f(x)$ is **not** analytic even though C-R equations are satisfied.

Example 12 : Prove that $f(z) = (x^3 - 3xy^2 + 2xy) + i(3x^2y - x^2 + y^2 - y^3)$ is analytic and find $f'(z)$ and $f(z)$ in terms of z. **(M.U. 2004)**

Sol. : We have $u = x^3 - 3xy^2 + 2xy,\ v = 3x^2y - x^2 + y^2 - y^3$

$$\therefore\ \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 2y,\quad \frac{\partial u}{\partial y} = -6xy + 2x$$

$$\frac{\partial v}{\partial x} = 6xy - 2x,\quad \frac{\partial v}{\partial y} = 3x^2 + 2y - 3y^2$$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also partial derivatives are continuous. $\therefore$ $f(z)$ is analytic.

$$\text{Now,}\ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (3x^2 - 3y^2 + 2y) + i(6xy - 2x)$$

$$\therefore\ f'(z) = 3(x^2 + 2i\,xy - y^2) - 2i(x + iy)$$

$$= 3(x^2 + 2i\,xy + i^2y^2) - 2i(x + iy) = 3z^2 - 2iz.$$

Or by Milne-Thompson method (See page 5-29), putting $x = z,\ y = 0$, we get $f'(z) = 3z^2 - 2iz$.

$\therefore$ By integration, $f(z) = z^3 - iz^2 + k$.

Example 13 : Show that $w = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$ is an analytic function and find $\frac{dw}{dz}$ in terms of z. **(M.U. 2003, 07)**

Sol. : Since, $u = \frac{x}{x^2 + y^2},\ \frac{\partial u}{\partial x} = u_x = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$

$$\frac{\partial u}{\partial y} = u_y = -\frac{x \cdot 2y}{(x^2 + y^2)^2}$$

$$v = -\frac{y}{x^2 + y^2}\quad \therefore\ \frac{\partial v}{\partial x} = v_x = +\frac{y \cdot 2x}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial y} = v_y = -\frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = -\frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x and v_y are continuous functions except at $z = x + iy = 0$ i.e. $(x = 0, y = 0)$. w is analytic everywhere except at $z = 0$.

$$\therefore \frac{dw}{dz} = u_x + iv_x = \frac{-x^2+y^2}{(x^2+y^2)^2} + i\cdot\frac{2xy}{(x^2+y^2)^2}$$

$$= -\frac{(x^2-2ixy-y^2)}{(x^2+y^2)^2} = -\frac{(x^2-2ixy+i^2y^2)}{(x^2-i^2y^2)^2}$$

$$= -\frac{(x-iy)^2}{(x-iy)^2(x+iy)^2} = -\frac{1}{(x+iy)^2} = -\frac{1}{z^2}$$

[Or to find $\frac{dw}{dz}$ in terms of z, use Milne-Thomson method (See page 5-29) and put $x=z$, $y=0$ in (i) $\therefore \frac{dw}{dz} = -\frac{1}{z^2}$.]

Example 14 : Show that $f(z) = \frac{\bar{z}}{|z|^2}$, $|z| \neq 0$ is analytic and find $f'(z)$. **(M.U. 201...)**

Sol. : We have $f(z) = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

Now, proceed as in the above example.

Example 15 : Is $f(z) = \frac{z}{\bar{z}}$ analytic? **(M.U. 201...)**

Sol. : We have

$$f(z) = \frac{z}{\bar{z}} = \frac{x+iy}{x-iy} = \frac{(x+iy)}{(x-iy)}\cdot\frac{(x+iy)}{(x+iy)}$$

$$= \frac{(x+iy)^2}{x^2+y^2} = \frac{x^2+2ixy+y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2}{x^2+y^2} + i\cdot\frac{2xy}{x^2+y^2}$$

$\therefore u = 1$ and $v = \frac{2xy}{x^2+y^2}$.

Now, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$. But $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are not zero.

Hence, $f(z)$ is not analytic.

Example 16 : Show that $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$ when $z \neq 0$

$= 0$ when $z = 0$ is not differentiable at $z=0$.

Sol. : By definition,

$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim_{z\to 0}\frac{xy^2(x+iy)-0}{(x^2+y^4)(x+iy)} = \lim_{z\to 0}\frac{xy^2}{x^2+y^4}$$

Now, let $z \to 0$ along path $y^2 = mx$

$$\therefore f'(0) = \lim_{x\to 0}\frac{x\cdot mx}{x^2+m^2x^2} = \lim_{x\to 0}\frac{x^2}{x^2}\cdot\left(\frac{m}{1+m^2}\right) = \frac{m}{1+m^2}$$

Since, the derivative depends upon the path, $f'(0)$ does not exist and hence, the function is not differentiable at $z = 0$.

Example 17 : Prove that the function defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \quad \text{when } z \neq 0$$

$$= 0 \quad \text{when } z = 0$$

is (i) continuous ; (ii) Cauchy-Riemann equations are satisfied at the origin and yet (iii) $f'(0)$ does not exist. **(M.U. 2002, 13)**

Sol. : We have $u = \frac{x^3-y^3}{x^2+y^2}$, $v = \frac{x^3+y^3}{x^2+y^2}$

(i) When $z \neq 0$, $x \neq 0$, $y \neq 0$

Hence, u, v being rational functions of x and y are continuous and consequently $f(z)$ is continuous when $z \neq 0$

To test the continuity at $z = 0$ we put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore u = r(\cos^3\theta - \sin^3\theta)$, $v = r(\cos^3\theta + \sin^3\theta)$

When $z \to 0$, $r \to 0$

$$\therefore \lim_{z\to 0} u = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

$$\lim_{z\to 0} v = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$$

$$\therefore \lim_{z\to 0} f(z) = 0 = f(0) \text{ by data}$$

$\therefore f(z)$ is continuous at $z = 0$. $\therefore f(z)$ is continuous everywhere.

(ii) Now

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{-y-0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{y-0}{y} = 1$$

$\therefore u_x = v_y = 1$ and $u_y = -v_x = -1$

Hence, C-R equations are satisfied at the origin.

(iii) Now, $f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{(x^3-y^3)+i(x^3+y^3)-0}{(x^2+y^2)(x+iy)}$

Let $z \to 0$ along the line $y = mx$

$$\therefore f'(0) = \lim_{x\to 0}\frac{x^3(1-m^3)+ix^3(1+m^3)}{x^2(1+m^2)\,x(1+im)} = \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$$

The limit depends upon the path,

$\therefore f'(0)$ does not exist.

Note ....

The above function is continuous at $z = 0$; Cauchy-Riemann equations are satisfied at $z = 0$ and yet the function is not analytic at $z = 0$. This is so because for analyticity in addition to C-R equations the four partial derivatives must be continuous at that point.

Now we have proved above that the value of u_x at $x = 0$ is one, we prove below that $\partial u / \partial x$ is **not** continuous at $z = 0$

$$u = \frac{x^3 - y^3}{x^2 + y^2},\ u_x = \frac{(x^2 + y^2)3x^2 - (x^3 - y^3)\cdot 2x}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2 + y^2)^2}$$

Now, let $y = mx$, $\lim_{x \to 0} u_x = \lim_{x \to 0} \frac{x^4 + 3m^2x^4 + 2m^3x^4}{x^4(1 + m^2)^2} = \frac{1 + 3m^2 + 2m^3}{(1 + m^2)^2}$

The limit depends upon the path chosen and hence, is not continuous. Similarly other partial derivatives can be shown to be not continuous. Hence, the function is **not** analytic even though C-R equations are satisfied.

Example 18 : Show that the following function

$$f(z) = \frac{x^2y^5(x + iy)}{x^4 + y^{10}}, \quad z \neq 0$$

$$= 0 \qquad z = 0$$

is not analytic at the origin although Cauchy-Riemann equations are satisfied.

Sol. : We have $u = \frac{x^3y^5}{x^4 + y^{10}}$, $v = \frac{x^2y^6}{x^4 + y^{10}}$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

Similarly, $\frac{\partial u}{\partial y} = 0,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 0$

Hence, Cauchy-Riemann equations are satisfied.

Now $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \left[\frac{x^2y^5(x + iy)}{x^4 + y^{10}} - 0\right] \Big/ (x + iy)$

$$= \lim_{z \to 0} \frac{x^2y^5}{x^4 + y^{10}}$$

Let $z \to 0$ along the line $y = mx$

$$\therefore f'(0) = \lim_{x \to 0} \frac{x^7m^5}{x^4[1 + m^{10}x^6]} = \lim_{x \to 0} \frac{x^3m^5}{1 + m^{10}x^6} = 0$$

Now let $z \to 0$ along $y^5 = x^2$ then from (A), we get,

$$\therefore f'(0) = \lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since, the limit depends upon the path, $f(z)$ is not analytic at $z = 0$.

Example 19 : Find k such that $\frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{kx}{y}$ is analytic. **(M.U. 2004, 07, 16)**

Sol. : Let $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{kx}{y}$

$$\therefore u = \frac{1}{2}\log(x^2 + y^2),\ v = \tan^{-1}\frac{kx}{y}$$

$$\therefore u_x = \frac{x}{x^2 + y^2},\quad u_y = \frac{y}{x^2 + y^2}$$

$$v_x = \frac{1}{1 + \frac{k^2x^2}{y^2}} \cdot \frac{k}{y} = \frac{ky}{k^2x^2 + y^2};\quad v_y = \frac{1}{1 + \frac{k^2x^2}{y^2}} \cdot \left(-\frac{kx}{y^2}\right) = -\frac{kx}{k^2x^2 + y^2}$$

Since, the function is given to be analytic C-R equations are satisfied.

$$\therefore u_x = v_y \text{ and } u_y = -v_x$$

$$\therefore \frac{x}{x^2 + y^2} = -\frac{kx}{k^2x^2 + y^2},\quad \frac{y}{x^2 + y^2} = -\frac{ky}{k^2x^2 + y^2}$$

which are satisfied when $k = -1$.

Example 20 : Determine the constants a, b, c, d if

$f(z) = x^2 + 2axy + by^2 + i(cx^2 + 2dxy + y^2)$ is analytic. **(M.U. 1998, 2010, 11, 13)**

Sol. : We have $f(z) = u + i\,v$

and $u = x^2 + 2axy + by^2$; $v = cx^2 + 2dxy + y^2$

$$\therefore u_x = 2x + 2ay,\ u_y = 2ax + 2by$$

$$v_x = 2cx + 2dy,\ v_y = 2dx + 2y$$

Since, $f(z)$ is analytic, Cauchy-Riemann Equations are satisfied.

$$\therefore u_x = v_y \text{ and } u_y = -v_x$$

$$\therefore 2x + 2ay = 2dx + 2y \quad \text{and} \quad 2ax + 2by = -2cx - 2dy$$

Equating the coefficients of x and y, we get $a = 1$, $d = 1$ and $a = -c$, $b = -d$

$$\therefore a = 1,\ b = -1,\ c = -1,\ d = 1.$$

Example 21 : Find the constants a, b, c, d, e, if

$f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic.

(M.U. 2002, 03, 07, 08, 09, 12, 13, 16)

Sol. : We have $f(z) = u + iv$

and $u = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2$

and $v = 4x^3y - exy^3 + 4xy$

$$\therefore u_x = 4ax^3 + 2bxy^2 + 2dx;\quad u_y = 2bx^2y + 4cy^3 - 4y.$$

And $v_x = 12x^2y - ey^3 + 4y$; $\quad v_y = 4x^3 - 3exy^2 + 4x.$

Since $f(z)$ is analytic.

Cauchy-Riemann equations are satisfied.

$$\therefore u_x = v_y \quad \text{and} \quad u_y = -v_x$$

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$\therefore\ 4ax^3 + 2bxy^2 + 2dx = 4x^3 - 3exy^2 + 4x$

and $\ 2bx^2y + 4cy^3 - 4y = -12x^2y + ey^3 - 4y$

$\therefore$ Equating the coefficients of like powers of x and y, we get

$4a = 4,\quad 2b = -3e \quad \therefore\ 2d = 4 \quad$ and $\quad 2b = -12,\quad 4c = e.$

Hence, we get $\quad a = 1,\ d = 2,\ b = -6$

$\therefore\ -12 = -3e \quad \therefore\ e = 4 \quad$ and $\quad 4c = 4 \quad \therefore\ c = 1.$

Thus, we have $\quad a = 1,\ b = -6,\ c = 1,\ d = 2,\ e = 4.$

Example 22 : Find the values of z for which the following functions are not analytic.

(i) $z = e^{-v}(\cos u + i\sin u)$ **(ii)** $z = \sin hu \cos v + i\cos hu \sin v.$

Sol. : (i) We have $\quad z = e^{-v}(\cos u + i\sin u) = e^{-v}e^{iu}$

$\therefore\ z = e^{-v+iu} = e^{i^2v+iu} = e^{i(u+iv)} = e^{iw}$ where $w = u + iv.$

$\therefore\ iw = \log z \quad \therefore\ w = \frac{1}{i}\log z \quad \therefore\ \frac{dw}{dz} = \frac{1}{i}\cdot\frac{1}{z}$

$\therefore$ w is not analytic at $z = 0.$

(ii) We have $\quad z = \sin hu \cos v + i\cos hu \sin v$

But $\cos v = \cos h\, iv$ and $i\sin v = \sin h\, iv$

$\therefore\ z = \sin hu \cos h\, iv + \cos hu \sin h\, iv$

$= \sin h\,(u + iv)$

$= \sin hw \quad$ where $w + iv$

$\therefore\ w = \sin h^{-1} z = \log(z + \sqrt{z^2+1})$

$$\therefore\ \frac{dw}{dz} = \frac{1}{z+\sqrt{z^2+1}}\left(1 + \frac{1}{\sqrt{z^2+1}}\right) = \frac{1}{\sqrt{z^2+1}}$$

$\therefore$ w is not analytic when $\sqrt{z^2+1} = 0$ i.e. $z^2 = -1$ i.e. $z = \pm i.$

Example 23 : If $f(z) = u + iv$ is analytic in R show that

(i) $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = |f'(z)|^2$ **(M.U. 2004, 06, 07, 08)**

(ii) $\left[\frac{\partial |f(z)|}{\partial x}\right]^2 + \left[\frac{\partial |f(z)|}{\partial y}\right]^2 = |f'(z)|^2$ **(M.U. 1995, 97, 2009, 10)**

Sol. : First we note that [see note (2) page 5-8]

if $f(z)\ u + iv, \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

(i) $\quad$ l.h.s. $= \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$

But by C-R equations $u_x = v_y,\ u_y = -v_x$

$\therefore$ l.h.s. $= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

From (i) r.h.s $= |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

Hence, the result.

(ii) Now, $|f(z)| = \sqrt{u^2+v^2}$

$$\therefore\ \frac{\partial}{\partial x}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial}{\partial y}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right)$$

$$\therefore\ \text{l.h.s.} = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial v}{\partial y}\right)^2 + 2uv\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right]$$

Using C-R equations

$$\text{l.h.s.} = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 - 2uv\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}\right]$$

$$= \frac{1}{(u^2+v^2)}(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2 = \text{r.h.s.}$$

EXERCISE - II

1. Show that the following functions are not analytic.

(i) $f(z) = \bar{z}$ **(ii)** $f(z) = z\,|\,z\,|.$

2. Determine whether the following functions are analytic and if so find their derivatives

(i) $\cos h\, z$ **(ii)** $\cos z$ **(iiii)** $\frac{1}{z}$ **(iv)** $z^2 + z$ **(v)** $z^2 - \bar{z}$

(vi) $e^x(\cos y - i\sin y)$ **(vii)** $\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

(viii) $e^{-x}(\cos y - i\sin y)$ **(ix)** $x^2 - y^2 + 2ixy$ **(M.U. 2004, 14)**

(x) $(x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$ **(M.U. 2004)**

(xi) $z\,e^{2z}$ **(M.U. 2003, 04, 05, 11)**

[**Ans. :** (i) Yes, (ii) Yes, (iii) Yes except at $z = 0$, (iv) Yes, (v) No, (vi) No, (vii) Yes ; $\log z + C$, (viii) Yes, (ix) Yes, (x) Yes, (xi) Yes.]

3. If $f(z) = \dfrac{x^3 y(y - ix)}{x^6 + y^2}$, $z \neq 0$, $f(0) = 0$, prove that $\dfrac{f(z) - f(0)}{z - 0} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in **any manner.**

(**Hint :** Putting $y = mx$ $\lim\limits_{x \to 0} \dfrac{f(z) - f(0)}{z - 0} = 0$. But putting $y = x^3$ lim is $-i/2$)

4. Prove that $f(z) = \dfrac{x^2(1+i) - y^2(1-i)}{x + y}$, $z \neq 0$

$f(0) = 0$

is not analytic although Cauchy-Riemann equations are satisfied at $z = 0$.

5. Prove that $f(z) = e^{-z^{-4}}$, $z \neq 0$

$f(0) = 0$

is not analytic although Cauchy-Riemann equations are satisfied at $z = 0$.

6. Prove that the following function

$$f(z) = \frac{x^3 y^5 (x + iy)}{x^6 + y^{10}}, \quad z \neq 0$$

$f(0) = 0$

is not analytic at the origin although Cauchy-Riemann equations are satisfied.

(**Hint :** Put $x^3 = y^5$)

7. Find the values of z for which the following function is not analytic

$z = \sin u \cos hv + i \cos u \sin hv$ [**Ans. :** $z = \pm 1$]

8. If $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}$, $z \neq 0$, $f(0) = 0$ prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in **any manner.**

(**Hint :** Put $y = mx$ and then put $y^2 = x$)

9. Prove that the function $f(z) = e^{2z}$ is analytic and find $f'(z)$.

(**Hint :** $u = e^{2x} \cos 2y$, $v = e^{2x} \sin 2y$, $f'(z) = 2e^{2z}$)

10. Show that $u + iv = \dfrac{x - iy}{x - iy + a}$, $a \neq 0$ is not analytic but $u - iv$ is analytic.

(**Hint :** Let $f(z) = u + iv = \dfrac{\bar{z}}{\bar{z} + a} = \dfrac{\bar{z} + a - a}{\bar{z} + a} = 1 - \dfrac{a}{\bar{z} + a}$

$\Phi(z) = u - iv = \dfrac{z}{z + a} = \dfrac{z + a - a}{z + a} = 1 - \dfrac{a}{z + a}$

Now try to find $f'(z)$ and $\Phi'(z)$ on the lines of alternative method of Ex. (9) (a), page 5-13)

11. If $u + iv$ is analytic prove that $v - iu$ and $-v + iu$ are analytic. Show further that $v + iu$ will be analytic if $u + iv$ is a constant.

12. Find the constants a, b, c, d, e if

(i) $f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2 y - 2y^3 + exy + y)$ **(M.U. 2006, 08)**

(ii) $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$

are analytic. **(M.U. 2002, 03, 07, 08, 09, 12, 13)**

[**Ans. :** (i) $a = 2$, $b = -6$, $c = -3$, $d = 6$, $e = 6$; (ii) $a = 1$, $b = -6$, $c = 1$, $d = 2$, $e = 4$]

13. Show that the following function is not analytic at the origin although Cauchy-Riemann equations are satisfied

$$f(z) = \frac{xy(y - ix)}{x^2 + y^2}, \quad z \neq 0$$

$f(0) = 0$ **(M.U. 2004)**

(**Hint :** $f'(0) = \dfrac{f(z) - f(0)}{z - 0} = \lim\limits_{z \to 0} \dfrac{xy(y - ix)/(x^2 + y^2) - 0}{(x + iy) - 0}$

$= \lim\limits_{z \to 0} \dfrac{xy(y - ix)}{(x^2 + y^2)(x + iy)} = \lim\limits_{z \to 0} -\dfrac{i\,xy(x + iy)}{(x^2 + y^2)(x + iy)}$

$= \lim\limits_{z \to 0} -\dfrac{i\,xy}{x^2 + y^2}$.

Now put $y = mx$.)

10. Cauchy-Riemann Equations in Polar Coordinates **(M.U. 2010)**

Let (r, θ) be the polar coordinates of a point whose Cartesian coordinates are (x, y).

$\therefore$ $x = r\cos\theta$, $y = r\sin\theta$,

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Let $f(z) = u + iv$ be the given function

$\therefore$ $f(z) = u + iv = f(re^{i\theta})$ .... (1)

Differentiating (1) partially w.r.t. r,

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \quad \text{.................. (2)}$$

Differentiating (1) partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot re^{i\theta} \cdot i = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \quad \text{[By (2)]}$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts $\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$ and $\dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r}$

or $\boxed{\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \text{ and } \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \text{ i.e. } u_r = \dfrac{1}{r}v_\theta \text{ and } u_\theta = -r\,v_r}$

Note ...

From (2), we get an important result.

$f'(re^{i\theta}) = e^{-i\theta}\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial v}{\partial r}\right)$ $\therefore$ $\boxed{f'(z) = e^{-i\theta}\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial v}{\partial r}\right)}$

Example 1 : If $w = \log z$ determine whether w is analytic and find $\dfrac{dw}{dz}$.

Sol. : Let $w = u + iv = \log(z) = \log(re^{i\theta}) = \log r + i\theta$ $\therefore u = \log r$, $v = \theta$

$\therefore \dfrac{\partial u}{\partial r} = \dfrac{1}{r}, \dfrac{\partial u}{\partial \theta} = 0 ; \dfrac{\partial v}{\partial r} = 0, \dfrac{\partial v}{\partial \theta} = 1.$

$\therefore u_r = \dfrac{1}{r} \cdot 1 = \dfrac{1}{r} v_\theta$ and $u_\theta = 0 = -r v_r$

Hence, C-R equations are satisfied when $r \neq 0$.

Partial derivatives are continuous except at $r = 0$ *i.e.* at $z = 0$

Now $f'(z) = \dfrac{d}{dz}(\log z) = e^{-i\theta}\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial u}{\partial r}\right) = e^{-i\theta}\left(\dfrac{1}{r}\right) = \dfrac{1}{r e^{i\theta}} = \dfrac{1}{z}.$

Example 2 : Find p if $f(z) = r^2 \cos 2\theta + i r^2 \sin p\theta$ is analytic. **(M.U. 1998, 08, 1[illegible])**

Sol. : Let $w = f(z) = u + iv = r^2 \cos 2\theta + i r^2 \sin p\theta$

$\therefore u = r^2 \cos 2\theta, \quad v = r^2 \sin p\theta$

$\dfrac{\partial u}{\partial r} = 2r\cos 2\theta, \quad \dfrac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta ; \quad \dfrac{\partial v}{\partial r} = 2r \sin p\theta, \quad \dfrac{\partial v}{\partial \theta} = p r^2 \cos p\theta$

Since, $f(z)$ is analytic. $\quad \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $u_\theta = -r v_r$

The first relation gives, $\quad 2r\cos 2\theta = \dfrac{1}{r} \cdot p r^2 \cos p\theta \quad \therefore p = 2$

And the second realtion also gives,

$$-2r^2 \sin 2\theta = -2r^2 \sin p\theta \qquad \therefore p = 2.$$

Hence, $p = 2$.

Example 3 : Find the value of k if $f(z) = r^3 \cos k\theta + i r^k \sin 3\theta$ is analytic.

Sol. : Left to you. **[Ans. : k=3]**

Example 4 : Is $f(z) = \dfrac{z}{\bar{z}}$ analytic ? **(M.U. 200[illegible])**

Sol. : Put $z = r e^{i\theta}$ and use C-R equations in polar form. No.

Example 5 : If $w = z^n$ find $\dfrac{dw}{dz}$.

Sol. : Let $z = r e^{i\theta} \quad \therefore z^n = r^n e^{in\theta}$

$\therefore z^n = r^n(\cos n\theta + i \sin n\theta) \qquad \therefore u = r^n \cos n\theta, \ v = r^n \sin n\theta.$

$\dfrac{\partial u}{\partial r} = n r^{n-1} \cos n\theta, \ \dfrac{\partial u}{\partial \theta} = -r^n \cdot n \cdot \sin n\theta$

$\dfrac{\partial v}{\partial r} = n r^{n-1} \sin n\theta, \ \dfrac{\partial v}{\partial \theta} = n r^n \cos n\theta$

$\therefore \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$

Also partial derivatives are continuous. Hence, w is analytic.

$\therefore \dfrac{dw}{dz} = e^{-i\theta}\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial v}{\partial r}\right) = e^{-i\theta}(n r^{n-1} \cos n\theta + i n r^{n-1} \sin n\theta)$

$\therefore \dfrac{dw}{dz} = n r^{n-1} \cdot e^{-i\theta}(\cos n\theta + i \sin n\theta) = n r^{n-1} \cdot e^{-i\theta} \cdot e^{in\theta}$

$= n r^{n-1} \cdot e^{i(n-1)\theta} = n(r e^{i\theta})^{n-1} = n z^{n-1}.$

Example 6 : Using Cauchy-Riemann equations in polar form prove that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Sol. : We know that Cauchy-Riemann equations in polar form are

$u_r = \dfrac{1}{r} v_\theta$ ...... (i)

and $u_\theta = -r v_r$ ...... (ii)

Differentiating (i) w.r.t. r, we get,

$\dfrac{\partial^2 u}{\partial r^2} = -\dfrac{1}{r^2}\dfrac{\partial v}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial^2 v}{\partial \theta \partial r}$ ...... (iii)

Differentiating (ii) w.r.t. θ, we get,

$\dfrac{\partial^2 u}{\partial \theta^2} = -r\dfrac{\partial^2 v}{\partial \theta \partial r}$ ...... (iv)

Now, using (iii) and (iv), we get,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \left(-\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}\right) + \frac{1}{r} \cdot \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{1}{r^2} \cdot r\frac{\partial^2 v}{\partial \theta \partial r}$$

$$= -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} = 0$$

Note ....

The equation $\nabla^2 \Phi = \dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial y^2} = 0$ is called Laplace's equation in **Cartesian Form** and the equation $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ is called Laplace's equation in **Polar Form.**

11. Harmonic Functions

Any function of x, y which has continuous partial derivatives of the first and second order and satisfies Laplace's equation $\nabla^2 \Phi = \dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial y^2} = 0$ is called a **Harmonic Function.**

Theorem : The real and imaginary parts u, v of an analytic function $f(z) = u + iv$ are harmonic functions. **(M.U. 2003, 13)**

Proof : Since, $f(z)$ is an analytic function in some region of the z-plane, u, v satisfy C-R equations.

$\therefore u_x = v_y$ and $u_y = -v_x$ ...... (A)

Differentiating the first w.r.t. x and second w.r.t. y, we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Assuming $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial x \partial y}$ and adding the above results we get,

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Similarly differentiating the equations in (A) with respect to y and x respectively, we can show that

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

Hence, the result.

Notes ....

1. In other words the above theorem states that if $f(z) = u + iv$ is analytic, then its real and imaginary parts u, v satisfy Laplace equation.

$$\boxed{\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0}$$

2. The above theorem states that if $f(z) = u + iv$ is analytic then u and v satisfy Laplace's equation *i.e.* u and v are harmonic functions. But, the converse is **not** true. If u and v are any two functions satisfying Laplace's equation then $u + iv$ need not be analytic. [See Ex. 7 of § 13 page 5-47].

Theorem : If $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic then the real and imaginary parts u, v are harmonic.

Proof : You can prove it easily by using Cauchy-Riemann equations in polar form. Prove it.

Note ....

3. In other words the above theorem states that if $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic then the real and imaginary parts u, v satisfy Laplace equation in polar form

$$\boxed{\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0}$$

Example 1 : Show that a harmonic function satisfies the differential equation $\dfrac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

Sol. : If u is a harmonic function then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ .... (i)

Now, $x = \dfrac{1}{2}(z + \bar{z})$ and $y = \dfrac{1}{2i}(z - \bar{z})$

$$\therefore \frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\cdot\frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial x \partial y}\cdot\frac{\partial y}{\partial z}\right] - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial y \partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial z}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2} + \frac{\partial^2 u}{\partial x \partial y}\cdot\frac{1}{2i}\right] - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial x \partial y}\cdot\frac{1}{2} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{1}{2i}\right]$$

$$\therefore \frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0 \quad \text{[By (i)]}$$

Hence, the required result.

Example 2 : If $u(x, y)$ is a harmonic function then prove that $f(z) = u_x - i\,u_y$ is an analytic function. **(M.U. 2004, 08)**

Sol. : Since u is harmonic $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ .... (1)

By data $f(z) = u_x - i\,u_y$.

Let $u_x = U$ and $-u_y = V$, so that $f(z) = U + iV$.

We have to show that $f(z)$ is analytic.

Now, $U_x = \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$ [By (1)] and $U_y = \dfrac{\partial^2 u}{\partial x \partial y}$

$V_x = -\dfrac{\partial^2 u}{\partial y \partial x}$ and $V_y = -\dfrac{\partial^2 u}{\partial y^2}$ $\therefore$ $U_x = V_y$ and $U_y = -V_x$.

$\therefore$ $f(z) = U + iV$ is analytic *i.e.* $f(z) = u_x - i\,u_y$ is analytic.

Example 3 : If u, v are harmonic conjugate functions, show that uv is a harmonic function. **(M.U. 2003)**

Sol. : Let $f(z) = u + iv$ be the analytic function. [See note (2), page 5-8]

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

And u, v are harmonic $\therefore$ $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ .... (1)

Now, $\dfrac{\partial}{\partial x}(uv) = u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial x}$

$$\therefore \frac{\partial^2}{\partial x^2}(uv) = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2}$$

$$\therefore \frac{\partial^2}{\partial x^2}(uv) = u\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} \quad \text{............ (2)}$$

Similarly, we can prove that

$$\frac{\partial^2}{\partial y^2}(uv) = u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}$$

But $u_x = v_y$ and $u_y = -v_x$

$$\therefore \frac{\partial^2}{\partial y^2}(uv) = u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} \quad \text{............ (3)}$$

Adding (2) and (3), we get

$$\frac{\partial^2}{\partial x^2}(uv) + \frac{\partial^2}{\partial y^2}(uv) = u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$= 0 \quad \text{[By (1)]}$$

$\therefore$ uv is harmonic.

Example 4 : If Φ and Ψ are functions of x and y satisfying Laplace equation and if $u = \Phi_y - \Psi_x$ and $v = \Phi_x + \Psi_y$, prove that $u + iv$ is analytic (holomorphic). (M.U.)

Sol. : Since Φ and Ψ satisfy Laplace equation, we have

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \text{.......... (1)} \quad \text{and} \quad \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

Now $\quad u_x = \Phi_{yx} - \Psi_{xx} = \Phi_{xy} + \Psi_{yy}$ [By (2)]

And $\quad u_y = \Phi_{yy} - \Psi_{xy} = -(\Phi_{xx} + \Psi_{xy})$ [By (1)]

Similarly, $\quad v_x = \Phi_{xx} + \Psi_{xy}$ and $v_y = \Phi_{xy} + \Psi_{yy}$

Hence, $u_x = v_y$ and $u_y = -v_x$

Hence, $u + iv$ is analytic.

12. To Find an Analytic Function whose Real or Imaginary Part is given

Method 1 : Let $f(z) = u + iv$ and let u be given. Since, u is given we can find u_x and u_y.

As $f(z)$ is analytic, by C-R equations $u_x = v_y$ and $u_y = -v_x$.

$\therefore \quad f'(z) = u_x + iv_x = u_x - iu_y = \Phi(z)$ say.

Hence, by mere integration $f(z)$ can be obtained. (See Ex. 1, page 5-31)

Note ....

The method can be used only when we are able to express $u_x - iu_y$ as a function of z say $\Phi(z)$.

Method 2 : Let $f(z) = u + iv$ and let u be given. This means $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are known.

Now, $\quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

But by C-R equations $u_x = v_y$ and $u_y = -v_x$

$$\therefore \quad dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

Further $\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$

Since, u is harmonic $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

i.e. $\quad -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \quad$ i.e. $\quad \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$\therefore$ R.H.S. of (1) is an exact differential.

(Recall that if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = du$)

$\therefore$ Integrating (i), $\quad v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c$

From this we get v then $\quad f(z) = u + iv$.

Notes ....

1. Recall how we obtain the solution of an exact differential equation $M\,dx + N\,dy = 0$. In this case we have $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$. Hence, to obtain v, **integrate** $\left(-\frac{\partial u}{\partial y}\right)$ **w.r.t.** x **treating** y **constant and integrate only those terms in** $\frac{\partial u}{\partial x}$ **which are free from** x**. Their sum plus a constant is equal to** v**.**

2. If $f(z) = u + iv$ is an analytic function so that u and v both are harmonic functions then u and v are called **harmonic conjugate functions**. Each is called the **harmonic conjugate function** of the other.

3. Instead of u if v is given we can reason exactly as above to find $f(z)$. In this case **integrate** $\frac{\partial v}{\partial y}$ **treating** x **constant and integrate those terms in** $-\frac{\partial v}{\partial x}$ **which are free from** x**. Their sum plus a constant is equal to** u**.**

Method 3 : Milne-Thompson's Method

Since, $\quad z = x + iy, \ \bar{z} = x - iy \quad \therefore \quad x = \frac{z + \bar{z}}{2}, \ y = \frac{z - \bar{z}}{2i}$

$$\therefore \quad f(z) = u(x, y) + iv(x, y) = u\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right] + iv\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$$

This can be regarded as an identity in two independent variables z and $\bar{z}$. We can, therefore, put $\bar{z} = z$ and get

$$f(z) = u(z, 0) + iv(z, 0)$$

Thus, **$f(z)$ can be obtained in terms of z by putting $x = z$ and $y = 0$ in $f(z) = u(x, y) + iv(x, y)$ when $f(z)$ is analytic.**

Now $\quad f'(z) = u_x + iv_x = u_x - iu_y \quad$ [$\because$ C-R equations]

Let $\quad u_x = \Phi_1(x, y)$ and $u_y = \Phi_2(x, y)$

$\therefore \quad f'(z) = \Phi_1(x, y) - i\,\Phi_2(x, y) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$

Integrating, we get,

$$\boxed{f(z) = \int \Phi_1(z, 0)\,dz - i \int \Phi_2(z, 0)\,dz + c}$$

Similarly if v is given arguing on the above lines we can show that

$$\boxed{f(z) = \int \psi_1(z, 0)\,dz + i \int \psi_2(z, 0)\,dz + c}$$

where, $\quad v_y = \psi_1(x, y), \ v_x = \psi_2(x, y)$

Louis Melville Milne-Thompson (1891-1974)

He became Wrangler in 1913 and joined Winchester College as an assistant mathematics master in 1914. He left Winchester College and joined Royal Naval College, Greenwich in 1921. His earlier research was related to tables. He published in 1931 "Standard Four Figure Mathematical Tables" and in the next year he published "Standard Table of Square Root And Jacobian Elliptic Functions". He published his first book on "The Calculus Of Finite Differences" in 1933. In 1938, he published his second book but on different subject "Theoretical Hydrodynamics". He retried from the Royal Naval College in 1956 at the age of 65 and then took various posts as Visiting professor at various institutions throughout the world. He was elected a fellow of the Royal Society of Edinburg, Royal Astronomical Society and Cambridge Philosophical Society. We know him for Milne-Thomson method for finding analytic function.

Method 4 : Let $f(z) = f(z+iy) = u(x,y) + iv(x,y)$

Then $\overline{f(z)} = \overline{f(x+iy)} = u(x,y) - iv(x,y)$ $\quad \therefore \ f(z) + \overline{f(z)} = 2u(x,y)$

We can consider $\overline{f(z)}$ as a function of $\bar{z}$ and denote it as $\bar{f}(\bar{z})$.

$$u(x,y) = \frac{1}{2}[f(z) + \bar{f}(\bar{z})] = \frac{1}{2}[f(x+iy) - \bar{f}(x-iy)]$$

This can be regarded as an identity and holds even if x and y are complex. Putting $x = \frac{z}{2}$ and $y = \frac{z}{2i}$, we get, from above

$$u(z/2, z/2i) = \frac{1}{2}\left[f\left(\frac{z}{2} + i\frac{z}{2i}\right) + \bar{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right] = \frac{1}{2}[f(z) + \bar{f}(0)]$$

$$\therefore \ f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0)$$

We can assume that $\bar{f}(0)$ is purely real and write $\bar{f}(0) = u(0,0)$

$$\therefore \ f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0)$$

We may add a purely imaginary constant ci and get

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ci}$$

Notes ....

1. Note that this method is purely algebraic and does not involve integration.
2. Of these four methods the third method *i.e.* **Milne-Thompson method** is commonly used.

Type I : To find the analytic function whose real part u is given

Example 1 : Construct an analytic function whose real part is $x^4 - 6x^2y^2 + y^4$.

Sol. : We shall first solve this example by the above four methods and then lay down the procedure to find the analytic function $f(z)$.

Method 1 : Let $u = x^4 - 6x^2y^2 + y^4$ and let $f(z) = u + iv$ be the required function.

$$\therefore \ u_x = 4x^3 - 12xy^2 \ ; \ u_y = -12x^2y + 4y^3$$

As seen above (or see note (2), page 5-8)

$$f'(z) = u_x - iu_y$$
$$= 4x^3 - 12xy^2 + 12ix^2y - 4iy^3$$
$$= 4[x^3 + 3x(iy)^2 + 3x^2(iy) + (iy)^3]$$
$$= 4(x+iy)^3 = 4z^3$$

$$\therefore \ f(z) = \int f'(z)\,dz = \int 4z^3\,dz = z^4 + c.$$

Method 2 : As before $\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$; $\frac{\partial u}{\partial y} = -12x^2y + 4y^3$.

Since, $f(z)$ is given to be analytic we can use the note 1 given on page 5-29.

$$\int -\frac{\partial u}{\partial y}\,dx = \int -(-12x^2y + 4y^3)\,dx = 4x^3y - 4y^3x$$

And $\int$ (terms in $\frac{\partial u}{\partial x}$ free from x) $dy = \int 0\,dy = 0$

$$\therefore \ v = 4x^3y - 4xy^3 + 0 = 4x^3y - 4xy^3$$

$$\therefore \ f(z) = u + iv + c = x^4 - 6x^2y^2 + y^4 + 4ix^3y - 4ixy^3 + c$$

Setting $x = z$, $y = 0$, we get $f(z) = z^4 + c$.

Method 3 : *Milne-Thompson Method* : We have as above

$$\Phi_1 = u_x = 4x^3 - 12xy^2 \ ; \ \Phi_2 = u_y = -12xy^2 + 4y^3$$

$$\therefore \ f'(z) = \Phi_1(z,0) - i\Phi_2(z,0)$$
$$= 4z^3 - i(0) \qquad [\text{Putting } x = z,\ y = 0 \text{ in } \Phi_1 \text{ and } \Phi_2]$$

$$\therefore \ f(z) = \int 4z^3\,dz = z^4 + c \text{ as before.}$$

Method 4 : We have

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{z^4}{16} - 6\left(\frac{z^2}{4}\right)\left(\frac{z^2}{4i^2}\right) + \left(\frac{z^4}{16i^4}\right) = \frac{z^4}{16} + \frac{6z^4}{16} + \frac{z^4}{16} = \frac{8z^4}{16}$$

Further $u(0,0) = 0$

$$\therefore \ f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + c_i = 2 \cdot \frac{8z^4}{16} - 0 + c_i = z^4 + c_i$$

Procedure to find $f(z)$ when real part u is given.

1. From u, first find u_x and u_y.
2. Then put $\Phi_1 = u_x$ and $\Phi_2 = u_y$.

3. Putting $x = z$ and $y = 0$, find $\Phi_1(z, 0)$ and $\Phi_2(z, 0)$.
4. Then $f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$.
5. $f(z) = \int f'(z)\,dz + c$

Example 2 : Construct an analytic function whose real part is $e^x \cos y$.

Sol. : Let $u = e^x \cos y$

$\therefore\ u_x = e^x \cos y$ and $u_y = -e^x \sin y$

$\therefore\ \Phi_1 = u_x = e^x \cos y,\ \Phi_2 = u_y = -e^x \sin y$

By Milne-Thompson method

$\therefore\ f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = e^z - i(0)$

$\therefore\ f(z) = \int e^z\,dz = e^z + c$ which is the required analytic function.

Example 3 : Find an analytic function whose real part is

$$e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\}$$

Sol. : Let $u = e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\}$

$$\therefore\ u_x = -e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\} + e^{-x}\left\{2x \cos y + 2y \sin y\right\}$$

$$= e^{-x}\left[-(x^2 - y^2)\cos y + 2x \cos y + 2y \sin y - 2xy \sin y\right]$$

$$u_y = e^{-x}\left[-(x^2 - y^2)\sin y - 2y \cos y + 2x \sin y + 2xy \cos y\right]$$

$\therefore\ \Phi_1 = u_x$ and $\Phi_2 = u_y$

By Milne-Thompson method

$$f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = e^{-z}[-z^2 + 2z]$$

$$\therefore\ f(z) = \int e^{-z}(-z^2 + 2z)\,dz$$

Integrating by parts,

$$f(z) = (-z^2 + 2z)(-e^{-z}) - \int (-e^{-z})(-2z + 2)\,dz$$

$$= e^{-z}(z^2 - 2z) + \int e^{-z}(2 - 2z)\,dz$$

Integrating by parts again,

$$\therefore\ f(z) = e^{-z}(z^2 - 2z) + (2 - 2z)(-e^{-z}) - \int (-e^{-z})(-2)\,dz$$

$$= e^{-z}(z^2 - 2z) - e^{-z}(2 - 2z) + 2e^{-z}$$

$$= z^2 e^{-z} + c.$$

Example 4 : Find the imaginary part of the analytic function whose real part is

$$e^{2x}(x \cos 2y - y \sin 2y).$$ **(M.U. 1993, 2000, 03, 14) (M.U. 2004, 05, 15)**

Also verify that v is harmonic.

Sol. : Let $u = e^{2x}(x \cos 2y - y \sin 2y)$

$$\therefore\ \Phi_1 = u_x = e^{2x} \cdot 2(x \cos 2y - y \sin 2y) + e^{2x}(\cos 2y)$$

$$= e^{2x}(2x \cos 2y - 2y \sin 2y + \cos 2y)$$

$$\Phi_2 = u_y = e^{2x}(-2x \sin 2y - \sin 2y - 2y \cos 2y)$$

By Milne-Thompson Method

$$\therefore\ f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = e^{2z}(2z + 1) - i\,e^{2z}(0) = e^{2z}(2z + 1)$$

Integrating by parts,

$$\therefore\ f(z) = \int e^{2z}(2z + 1)\,dz = (2z + 1)\frac{e^{2z}}{2} - \int \frac{e^{2z}}{2} \cdot 2 \cdot dz$$

$$= (2z + 1)\frac{e^{2z}}{2} - \int e^{2z}dz = (2z + 1)\frac{e^{2z}}{2} - \frac{e^{2z}}{2} = e^{2z} z + c.$$

Now, $f(z) = e^{2(x + iy)} \cdot (x + iy) = e^{2x} \cdot e^{2iy}(x + iy)$

$$= e^{2x}[\cos 2y + i \sin 2y](x + iy)$$

$$\therefore\ v = e^{2x}(y \cos 2y + x \sin 2y)$$

$$\therefore\ \frac{\partial v}{\partial x} = 2e^{2x}(y \cos 2y + x \sin 2y) + e^{2x}(\sin 2y)$$

$$\frac{\partial^2 v}{\partial x^2} = 4\,e^{2x}(y \cos 2y + x \sin 2y) + 2\,e^{2x} \sin 2y + 2e^{2x} \sin 2y$$

$$\frac{\partial^2 v}{\partial x^2} = 4\,e^{2x}(y \cos 2y + x \sin 2y) + 4\,e^{2x} \sin 2y$$

$$\frac{\partial v}{\partial y} = e^{2x}(\cos 2y - 2y \sin 2y + 2x \cos 2y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^{2x}(-2 \sin 2y - 2 \sin 2y - 4y \cos 2y - 4x \sin 2y)$$

$$= e^{2x}(-4 \sin 2y - 4y \cos 2y - 4x \sin 2y)$$

$$\therefore\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \therefore\ v \text{ is harmonic.}$$

Example 5 : Find an analytic function $f(z) = u + iv$ where

$$u = \frac{x}{2}\log(x^2 + y^2) - y \tan^{-1}\frac{y}{x} + \sin x \cos hy.$$ **(M.U. 2003, 08, 16)**

Sol. : We have $u = \dfrac{x}{2}\log(x^2 + y^2) - y \tan^{-1}\dfrac{y}{x} + \sin x \cos hy$

$$\therefore\ \Phi_1 = u_x = \frac{1}{2}\log(x^2 + y^2) + \frac{x^2}{x^2 + y^2} - y \cdot \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) + \cos x \cos hy$$

$$= \frac{1}{2}\log(x^2 + y^2) + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + \cos x \cos hy$$

And $$\Phi_2 = u_y = \frac{xy}{x^2 + y^2} - \tan^{-1}\left(\frac{y}{x}\right) - y \cdot \frac{1}{1 + (y^2/x^2)} \cdot \frac{1}{x} + \sin x \sin hy$$

$$= \frac{xy}{x^2 + y^2} - \tan^{-1}\frac{y}{x} - \frac{xy}{x^2 + y^2} + \sin x \sin hy$$

By Milne-Thomson method,

$$f'(z) = \Phi_1(z, 0) - i\Phi_2(z, 0)$$

$$\therefore f'(z) = \frac{1}{2}\log z^2 + \frac{z^2}{z^2} + \cos z - i(0)$$

$$= \log z + 1 + \cos z$$

$\therefore$ By integration,

$$f(z) = \int \log z\, dz + \int 1 dz + \int \cos z\, dz$$

$$= \log z \cdot z - \int 1 \cdot dz + z + \sin z \quad \text{[By Integrating by parts]}$$

$$= z \log z - z + z + \sin z$$

$$= z \log z + \sin z + c.$$

Example 6 : Find the analytic function whose real part is

$$\frac{\sin 2x}{\cos h\, 2y + \cos 2x}.$$

(M.U. 2002, 06, 10, 11, 1)

Sol. : Let $u = \dfrac{\sin 2x}{\cos h\, 2y + \cos 2x}$

$$\therefore \Phi_1 = u_x = \frac{(\cos h\, 2y + \cos 2x)(2\cos 2x) + \sin 2x \cdot 2\sin 2x}{(\cos h\, 2y + \cos 2x)^2}$$

$$= \frac{2\cos h\, 2y \cos 2x + 2}{(\cos h\, 2y + \cos 2x)^2}$$

$$\Phi_2 = u_y = -\frac{\sin 2x \cdot 2\sin h\, 2y}{(\cos h\, 2y + \cos 2x)^2}$$

By Milne-Thompson Method

$$\therefore f'(z) = \Phi_1(z, 0) - i\Phi_2(z, 0)$$

$$= \frac{2\cos 2z + 2}{(1 + \cos 2z)^2} - 0 = \frac{2}{1 + \cos 2z} = \sec^2 z$$

(See that $\sin hx = \dfrac{e^x - e^{-x}}{2}$ and $\sin h\, 0 = 0$, $\cos hx = \dfrac{e^x + e^{-x}}{2}$ and $\cos h\, 0 = 1$.)

$$\therefore f(z) = \int \sec^2 z\, dz = \tan z + c.$$

Type II : To find the analytic function whose imaginary part v is given

Procedure to find $f(z)$ when imaginary part v is given

1. From v first find v_y and v_x.
2. Then put $\Psi_1 = v_y$ and $\Psi_2 = v_x$.
3. Putting $x = z$ and $y = 0$, find $\Psi_1(z, 0)$ and $\Psi_2(z, 0)$.
4. Then $f'(z) = \Psi_1(z, 0) + i\Psi_2(z, 0)$.
5. $f(z) = \int f'(z)\, dz + c$

Example 1 : Find an analytic function whose imaginary part is

$$(x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$$

Sol. : We have $v = (x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$

$$\therefore v_y = \psi_1(x, y) = -12x^2y + 4y^3 - 2y + 2x$$

$$v_x = \psi_2(x, y) = 4x^3 - 12xy^2 + 2x + 2y$$

$$\therefore \psi_1(z, 0) = 2z, \quad \psi_2(z, 0) = 4z^3 + 2z$$

By Milne-Thompson method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore f(z) = \int \psi_1(z, 0)\, dz + i\int \psi_2(z, 0)\, dz$$

$$= \int 2z\, dz + i\int (4z^3 + 2z)\, dz$$

$$= z^2 + i(z^4 + z^2) + c.$$

Example 2 : Find an analytic function whose imaginary part is $e^{-x}(y\cos y - x\sin y)$.

Sol. : We have $v = e^{-x}(y\cos y - x\sin y)$

$$\therefore v_y = \psi_1(x, y) = e^{-x}(\cos y - y\sin y - x\cos y)$$

$$v_x = \psi_2(x, y) = -e^{-x}(y\cos y - x\sin y) + e^{-x}(-\sin y)$$

$$= e^{-x}(-\sin y - y\cos y + x\sin y)$$

$$\therefore \psi_1(z, 0) = e^{-z}(1 - z), \quad \psi_2(z) = 0$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore f(z) = \int \psi_1(z, 0)\, dz + i\int \psi_2(z, 0)\, dz = \int (1 - z)e^{-z} dz$$

$$= (1 - z)(-e^{-z}) - \int (-e^{-z})(-1)\, dz$$

$$= -e^{-z} + ze^{-z} + e^{-z} = ze^{-z} + c.$$

Example 3 : Find an analytic function $f(z)$ whose imaginary part is $e^{-x}(y\sin y + x\cos y)$.

(M.U. 1995, 2007, 09)

Sol. : We have $v = e^{-x}(y\sin y + x\cos y)$

$$\therefore v_y = \psi_1(x, y) = e^{-x}(\sin y + y\cos y - x\sin y)$$

$$v_x = \psi_2(x, y) = -e^{-x}(y\sin y + x\cos y) + e^{-x}(\cos y)$$

$$\therefore v_x = \psi_2(x, y) = e^{-x}(\cos y - y\sin y - x\cos y)$$

$$\therefore \psi_1(z, 0) = 0, \quad \psi_2(z, 0) = e^{-z}(1 - z)$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore f(z) = \int \psi_1(z, 0)\, dz + i\int \psi_2(z, 0)\, dz = i\int e^{-z}(1 - z)\, dz$$

$$= i\left[(1 - z)(-e^{-z}) - \int -e^{-z}(-1)\, dz\right] = i[(1 - z)(-e^{-z}) + e^{-z}]$$

$$= i\,e^{-z} z + c.$$

Example 4 : Find the analytic function whose imaginary part is $\tan^{-1}\dfrac{y}{x}$.

Sol : We have $v = \tan^{-1}\dfrac{y}{x}$.

$$\therefore \quad v_y = \psi_1(x,y) = \frac{1}{1+(y^2/x^2)}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$v_x = \psi_2(x,y) = \frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\therefore \quad \psi_1(z,0) = \frac{z}{z^2} = \frac{1}{z}; \quad \psi_2(z,0) = 0$$

By Milne-Thompson method

$$f'(z) = \Psi_1(z,0) + i\Psi_2(z,0) = \frac{1}{z} \qquad \therefore \quad f(z) = \int \frac{1}{z}\,dz = \log z + c.$$

Example 5 : If the imaginary part of the analytic function $w = f(z)$ is $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$ show that the real part $u = -2xy + \dfrac{y}{x^2+y^2} + c$. **(M.U. 2007, 08, 12, 1...)**

Sol. : We have $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

$$\therefore \quad v_y = \psi_1(x,y) = -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$v_x = \psi_2(x,y) = 2x - \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\psi_1(z,0) = 0, \quad \psi_2(z,0) = 2z - \frac{1}{z^2}$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z,0) + i\psi_2(z,0)$$

$$\therefore \quad f'(z) = i\left(2z - \frac{1}{z^2}\right) = i\int\left(2z - \frac{1}{z^2}\right)dz = i\left(z^2 + \frac{1}{z}\right)$$

$$= i(x^2 + 2i\,xy - y^2) + i\,\frac{(x-iy)}{x^2+y^2}$$

$$= \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c$$

$$\therefore \quad u = -2xy + \frac{y}{x^2+y^2} + c.$$

Type III : To find the analytic function when $u + v$ or $u - v$ is given

Procedure to find $f(z)$ when $u + v$ is given

1. Let $V = u + v$ (given).
2. Find $\dfrac{\partial V}{\partial x} = \Psi_1(x, y)$ and $\dfrac{\partial V}{\partial y} = \Psi_2(x, y)$.
3. Find $\Psi_2(z, 0)$ and $\Psi_1(z, 0)$.
4. Then $(1+i)\,f'(z) = \Psi_2(z,0) + i\,\Psi_1(z,0)$.
5. By integrating both sides w.r.t. z, we get $f(z)$.

Example 1 : If $f(z) = u + iv$ is analytic and $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, find $f(z)$.
(M.U. 1998, 2003, 04, 06, 07, 15, 16)

Sol : $f(z) = u + iv \quad \therefore \quad i\,f(z) = iu - v$

$$\therefore \quad (1+i)\,f(z) = (u-v) + i(u+v) = U + iV \text{ say.}$$

$$\therefore \quad (1+i)\,f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial y} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} \qquad [\because U_x = V_y]$$

(When $u + v$ is given we use $\dfrac{\partial V}{\partial y} + i\dfrac{\partial V}{\partial x}$)

But $\quad V = u + v = \dfrac{\sin 2x}{\{(e^{2y} + e^{-2y})/2\} - \cos 2x} = \dfrac{\sin 2x}{\cos h2y - \cos 2x}$

$$\therefore \quad \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(u+v) = \frac{(\cos h2y - \cos 2x)(2\cos 2x) - 2\sin 2x \sin 2x}{(\cos h2y - \cos 2x)^2}$$

$$= [2\cos h2y\cos 2x - 2]\,/\,[\cos h2y - \cos 2x]^2 = \psi_1(x,y)$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(u+v) = -\frac{2\sin h y \sin 2x}{[\cos h2y - \cos 2x]^2} = \psi_2(x,y)$$

By Milne-Thompson method

$$\therefore \quad (1+i)\,f'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} = \psi_2(z,0) + i\,\psi_1(z,0)$$

$$\therefore \quad (1+i)\,f(z) = \int 0 + i\,\frac{2(\cos 2z - 1)}{(1-\cos 2z)^2}\cdot dz = -2i\int\frac{dz}{1-\cos 2z} = -2i\int\frac{dz}{2\sin^2 z}$$

$$\therefore \quad (1+i)\,f(z) = -i\int \operatorname{cosec}^2 z\,dz = i\cot z + c'$$

$$\therefore \quad f(z) = \frac{i}{1+i}\cdot\cot z + c.$$

Example 2 : Find an analytic function $f(z) = u + iv$ where $u + v = e^x(\cos y + \sin y)$.
(M.U. 2004, 05, 15)

Sol. : We have $\quad f(z) = u + iv \quad \therefore \quad i\,f(z) = iu - v$

$$\therefore \quad (1+i)\,f(z) = (u-v) + i(u+v) = U + iV, \text{ say}$$

$$\therefore \quad \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(u+v) = e^x(\cos y + \sin y) = \psi_1(x,y)$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(u+v) = e^x(-\sin y + \cos y) = \psi_2(x,y)$$

By Milne-Thompson method

$$(1+i)\,f'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} = \psi_2(z,0) + i\,\psi_1(z,0)$$

$$= e^z(0+1) + i\,e^z(1+0) = (1+i)\,e^z$$

$$\therefore \quad f'(z) = e^z \qquad \therefore \quad f(z) = \int e^z\,dz = e^z + c.$$

Procedure to find $f(z)$ when $u-v$ is given

1. Let $U = u - v$ (given).
2. Find $\dfrac{\partial U}{\partial x} = \Phi_1(x, y)$ and $\dfrac{\partial U}{\partial y} = \Phi_2(x, y)$.
3. Find $\Phi_1(z, 0)$ and $\Phi_2(z, 0)$.
4. Then $(1+i)\, f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$.
5. Integrating both sides w.r.t. z, we get $f(z)$.

Example 3 : Find the analytic function $f(z) = u + iv$ such that

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}} \text{ when } f\left(\frac{\pi}{2}\right) = 0.$$

(M.U. 2003, 06, 07, 09, 13)

Sol. : We have $f(z) = u + iv \quad \therefore\ i f(z) = iu - v$

$\therefore (1+i) f(z) = (u - v) + i(u + v) = U + iV$, say. ......(A)

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(u - v) = \frac{(2\cos x - e^{y} - e^{-y})(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{(2\cos x - e^{y} - e^{-y})^2}$$

$$= \frac{2 + e^{y}\sin x - e^{-y}\sin x - e^{y}\cos x - e^{-y}\cos x}{(2\cos x - e^{y} - e^{-y})^2}$$

$$= \Phi_1(x, y)$$

And $$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(u - v) = \frac{(2\cos x - e^{y} - e^{-y})(e^{-y}) - (\cos x + \sin x - e^{-y})(-e^{y} + e^{-y})}{(2\cos x - e^{y} - e^{-y})^2}$$

$$\frac{\partial U}{\partial y} = \frac{\cos x \cdot e^{-y} + \cos x\, e^{y} + \sin x\, e^{y} - \sin x \cdot e^{-y} - 2}{(2\cos x - e^{y} - e^{-y})^2} = \Phi_2(x, y)$$

(When $u - v$ is given we used $\dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y}$.)

From (A), $\quad (1+i) f'(z) = \dfrac{\partial U}{\partial x} + i\dfrac{\partial V}{\partial x} = \dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y} \qquad [\because U_y = -V_x]$

$$(1+i) f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$$

$$= \frac{2 + \sin z - \sin z - \cos z - \cos z}{(2\cos z - 2)^2} - i \cdot \frac{\cos z + \cos z + \sin z - \sin z - 2}{(2\cos z - 2)^2}$$

$$= \frac{2(1 - \cos z)}{4(\cos z - 1)^2} - i\frac{2(\cos z - 1)}{4(\cos z - 1)^2} = -\frac{1}{2(\cos z - 1)} - i \cdot \frac{1}{2(\cos z - 1)}$$

$$= (1+i) \cdot \frac{1}{2(1 - \cos z)} = (1+i) \cdot \frac{1}{4\sin^2(z/2)} = \frac{(1+i)}{4} \cdot \operatorname{cosec}^2 \frac{z}{2}.$$

$$\therefore f'(z) = \frac{1}{4}\operatorname{cosec}^2 \frac{z}{2}$$

$$\therefore f(z) = \frac{1}{4}\int \operatorname{cosec}^2(z/2)\, dz = -\frac{1}{2}\cot\frac{z}{2} + c$$

But when $z = \dfrac{\pi}{2}$, $f(z) = 0 \quad \therefore\ 0 = -\dfrac{1}{2} + c \quad \therefore\ c = \dfrac{1}{2}$

$$\therefore f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 4 : Find the analytic function $f(z) = u + iv$ in terms of z if $u - v = (x - y)(x^2 + 4xy + y^2)$.

(M.U. 2003, 04, 07, 08, 12, 15)

Sol. : We have $f(z) = u + iv$

$\therefore\ i f(z) = iu - v$

$\therefore\ (1+i) f(z) = (u - v) + i(u + v) = U + iV$ say

We have

$$U = u - v$$
$$= (x - y)(x^2 + 4xy + y^2)$$
$$= x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$$
$$= x^3 + 3x^2y - 3xy^2 - y^3$$

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial(u - v)}{\partial x} = 3x^2 + 6xy - 3y^2 = \Phi_1(x, y)$$

$$\frac{\partial U}{\partial y} = \frac{\partial(u - v)}{\partial y} = 3x^2 - 6xy - 3y^2 = \Phi_2(x, y)$$

$\therefore\ \Phi_1(z, 0) = 3z^2$ and $\Phi_2(z, 0) = 3z^2$.

$$\therefore (1+i) f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$$
$$= 3z^2 - i \cdot 3z^2$$
$$= -3i^2 z^2 - i \cdot 3z^2 \qquad [\because i^2 = -1]$$

By integration,

$$(1+i) f(z) = -3i^2 \int z^2\, dz - 3i\int z^2\, dz$$
$$= -i^2 z^3 - i z^3 = -i z^3 (1 + i)$$

$$\therefore f(z) = -i z^3 + c.$$

Example 5 : If $f(z) = u + iv$ is analytic and $u - v = e^{x}(\cos y - \sin y)$, find $f(z)$ in terms of z.

(M.U. 2003, 05, 14)

Sol. : $\because\ f(z) = u + iv, \quad i f(z) = iu - v.$

$\therefore\ (1+i) f(z) = (u - v) + i(u + v) = U + iV$ say.

Now $$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(u - v) = e^{x}(\cos y - \sin y) = \Phi_1(x, y)$$

$$\frac{\partial U}{\partial y} = e^{x}(-\sin y - \cos y) = \Phi_2(x, y)$$

$$\therefore (1+i) f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$$

$$\therefore (1+i) f(z) = \int [e^{z} + i e^{z}]\, dz = (1+i)\int e^{z}\, dz = (1+i) e^{z} + c'$$

$$\therefore f(z) = e^{z} + c.$$

Example 6 : Find the analytic function $f(z) = u + iv$ if $3u + 2v = y^2 - x^2 + 16xy$. **(M.U. 2002, 08, 13)**

Sol. : Differentiating the given relation w.r.t. x and y

$$3\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} = -2x + 16y \qquad \text{.......... (i)}$$

and $$3\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} = 2y + 16x \qquad \text{.......... (ii)}$$

But $u_x = v_y$ and $u_y = -v_x$. Hence, from (ii), we get,

$$-3\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial x} = 2y + 16x \qquad \text{.......... (iii)}$$

Now, multiply (i) by (3) and (iii) by (2) and add.

$$\therefore\ 13\frac{\partial u}{\partial x} = 26x + 52y \quad i.e.\ \frac{\partial u}{\partial x} = 2x + 4y = \Phi_1(x, y)$$

Again, multiply (i) by (– 2) and (iii) by (3) and add.

$$\therefore\ -13\frac{\partial v}{\partial x} = 52x - 26y \quad i.e.\ \frac{\partial v}{\partial x} = -4x + 2y = \Phi_2(x, y)$$

But $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \Phi_1(x, y) + i\,\Phi_2(x, y) = \Phi_1(z, 0) + i\,\Phi_2(z, 0)$$

$$\therefore\ f'(z) = 2z - i \cdot 4z$$

$$\therefore\ f(z) = \int 2z\,dz - 2i\int 2z\,dz = z^2 - 2i\,z^2 + c = (1 - 2i)\,z^2 + c$$

Example 7 : State true or false with proper justification "There does not exist an analytic function whose real part is $x^3 - 3x^2y - y^3$". **(M.U. 1995, 2004, 14)**

Sol. : We shall use the theorem (page 5-26) to check whether $u = x^3 - 3x^3 - y^3$ is a real part of some analytic function. By the result referred to above $u = x^3 - 3x^2y - y^3$ must satisfy Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ if it is a real part of some analytic function.

Now $$\frac{\partial u}{\partial x} = 3x^2 - 6xy,\quad \frac{\partial^2 u}{\partial x^2} = 6x - 6y; \qquad \frac{\partial u}{\partial y} = -3x^2 - 3y^2,\quad \frac{\partial^2 u}{\partial y^2} = -6y$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 12y \neq 0$$

$\therefore$ There does not exist an analytic function whose real part is $u = x^3 - 3x^2y - y^3$.

Type IV : To find the analytic function whose real part u is given in polar form

Example 1 : State Laplace's equation in polar form and verify it for $u = r^2\cos 2\theta$ and also find v and $f(z)$.

Sol. : Laplace's equation in polar form is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$

$$\because\ u = r^2\cos 2\theta$$

$$\therefore\ \frac{\partial u}{\partial r} = 2r\cos 2\theta,\quad \frac{\partial^2 u}{\partial r^2} = 2\cos 2\theta; \qquad \frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta,\quad \frac{\partial^2 u}{\partial \theta^2} = -4r^2\cos 2\theta$$

$$\therefore\ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta)$$

$$= 4\cos 2\theta - 4\cos 2\theta = 0$$

$\therefore$ Laplace's equation is satisfied.

By Cauchy-Riemann equations in polar form $u_r = \dfrac{1}{r}v_\theta \quad \therefore\ \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$$\therefore\ \frac{\partial v}{\partial \theta} = r(2r\cos 2\theta) = 2r^2\cos 2\theta$$

Integrating w.r.t. θ, $\quad v = r^2\sin 2\theta + c$

Hence, $f(z) = u + iv$

$$\therefore\ f(z) = r^2\cos 2\theta + i\,r^2\sin 2\theta + c = r^2(\cos 2\theta + i\sin 2\theta) + c$$

$$= r^2 e^{i2\theta} = (r\,e^{i\theta})^2 + c = z^2 + c$$

Example 2 : Verify Laplace's equation for $u = \left(r + \dfrac{a^2}{r}\right)\cos\theta$. Also find v and $f(z)$. **(M.U. 2004, 14)**

Sol. : $\because\ u = \left(r + \dfrac{a^2}{r}\right)\cos\theta$

$$\therefore\ \frac{\partial u}{\partial r} = \left(1 - \frac{a^2}{r^2}\right)\cos\theta,\quad \frac{\partial^2 u}{\partial r^2} = \frac{2a^2}{r^3}\cos\theta$$

$$\frac{\partial u}{\partial \theta} = -\left(r + \frac{a^2}{r}\right)\sin\theta,\quad \frac{\partial^2 u}{\partial \theta^2} = -\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$\therefore\ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{2a^2}{r^3}\cos\theta + \frac{1}{r}\cdot\left(1 - \frac{a^2}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$= 0.$$

$\therefore$ Laplace's equation is satisfied.

By Cuachy - Riemann equations in polar form $u_r = \dfrac{1}{r}v_\theta \quad \therefore\ \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$$\therefore\ \left(1 - \frac{a^2}{r^2}\right)\cos\theta = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \qquad \therefore\ \frac{\partial v}{\partial \theta} = \left(r - \frac{a^2}{r}\right)\cos\theta$$

Integrating w.r.t. θ, $\quad v = \left(r - \dfrac{a^2}{r}\right)\sin\theta + c$.

Hence, $$f(z) = u + iv = \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta + c$$

$$= r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta) + c = z + \frac{a^2}{z} + c.$$

Alternatively we can express u in terms of x and y and use cartesian form of Laplace's equation. However, it may be noted that this method is rather tedius.

Example 3 : If $u = k(1+\cos\theta)$, find v so that $u + iv$ is analytical. (M.U. 2007)

Sol. : Since, $u = k + k\cos\theta$, $\frac{\partial u}{\partial r} = 0$ and $\frac{\partial u}{\partial \theta} = -k\sin\theta$.

But by C-R equations in polar coordinates.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \therefore \frac{\partial v}{\partial \theta} = 0, \ \frac{\partial v}{\partial r} = -\frac{1}{r}(-k\sin\theta)$$

Integrating the first equation partially w.r.t θ,

$v = f(r)$ where $f(r)$ is an arbitrary function.

(If $v = f(r)$ then $\frac{\partial v}{\partial \theta} = 0$)

$$\therefore \frac{\partial v}{\partial r} = f'(r) = \frac{k\sin\theta}{r} \qquad \therefore v = k\sin\theta \log r + c$$

Hence, the analytic function is

$$f(z) = u + iv = k(1+\cos\theta) + ik\sin\theta \log r + c$$

Example 4 : If $u = -r^3 \sin 3\theta$, find the analytic function $f(z)$ whose real part is u. (M.U. 2004, 05, 1

Sol. : We have $\frac{\partial u}{\partial r} = -3r^2\sin\theta$ and $\frac{\partial u}{\partial \theta} = -3r^3\cos 3\theta$

By Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad \therefore \frac{\partial v}{\partial \theta} = -3r^3\sin 3\theta.$$

Integrating w.r.t. θ, $v = r^3\cos 3\theta$.

$$\therefore f(z) = u + iv = -r^3\sin 3\theta + ir^3\cos 3\theta$$
$$= ir^3(\cos 3\theta + i\sin 3\theta) = ir^3e^{i3\theta} = iz^3 + c.$$

Example 5 : Show that $u = r - \frac{a^2}{r}\sin\theta$ cannot be the real part of an analytic functio $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$. (M.U. 2003, 0

Sol. : Left to you. Show that the Cauchy equation in polar form, page 5-23 is not satisfied by u.

EXERCISE - III

1. Prove that $u(x, y) = x^2 - y^2$ and $v(x, y) = -y/(x^2+y^2)$ are both harmonic functions bu $u + iv$ is not analytic. (M.U. 200

2. Show that there does not exist an analytic function whose real part is (M.U. 200
   (i) $3x^2 + \sin x + y^2 + 5y + 4$ (M.U. 2002) (ii) $3x^2 - 2x^2y + y^2$

3. Show that the following functions are harmonic (M.U. 200
   (i) $e^x\cos y + x^3 - 3xy^2$ (M.U. 2003) (ii) $e^{2x}(x\cos 2y - y\sin 2y)$
   (iii) $\log\sqrt{x^2+y^2}$ (M.U. 2003)

4. Find the analytic function whose real part is

1. $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ (M.U. 2011, 13)
2. $u = (x-1)^3 - 3xy^2 + 3y^2$ 3. $u = x^2 - y^2 - 5x + y + 2$
4. $u = \log\sqrt{x^2+y^2}$ 5. $u = \sin x \cosh y$
6. $u = e^{2x}(x\cos 2y - y\sin 2y)$ 7. $u = e^x(x\cos y - y\sin y)$ (M.U. 2014)
8. $u = e^{-x}(x\sin y - y\cos y)$ 9. $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$ (M.U. 1995)
10. $u = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\left(\frac{y}{x}\right) + \sin x\cosh y$ (M.U. 2003, 08)

[Ans.: (1) $z^3 + 3z^2 + c$, (2) $(z-1)^3 + c$, (3) $z^2 - 5z - iz + c$, (4) $\log z + c$, (5) $\sin z + c$, (6) $ze^{2z} + c$, (7) $ze^z + c$, (8) $ize^{-z} + c$, (9) $\cot z + c$, (10) $z\log z + \sin z + c$.]

5. Find the analytic function whose imaginary part is

1. $v = \log(x^2+y^2) + x - 2y$ 2. $v = \frac{x-y}{x^2+y^2}$ 3. $v = \cos x \cosh y$
4. $v = \sinh x \cos y$ 5. $v = e^x(x\sin y + y\cos y)$ (M.U. 2005)
6. $v = e^{-x}(x\sin y - y\cos y)$ 7. $v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$
8. $v = \sinh x \sin y$ 9. $v = e^{-x}\left[2xy\cos y + (y^2 - x^2)\sin y\right]$ (M.U. 2003)
10. $\frac{x}{x^2+y^2} + \cosh x\cos y$ (M.U. 2002, 09, 14) 11. $v = \frac{y}{x^2+y^2}$

[Ans.: (1) $(i-2)z + i\log z + c$, (2) $(1+i)\frac{1}{z}$, (3) $i\cos z + 1$, (4) $i\sin hz$, (5) $ze^z + c$, (6) $-ze^{-z} + c$, (7) $\tan z + c$, (8) $\sin hz + c$, (9) $e^{-z}\cdot z^2 + c$, (10) $i\left(\frac{1}{z} + \cos hz\right) + c$, (11) $-\frac{1}{z} + c$.]

6. Find the analytic function $f(z) = u + iv$ in terms of z if

(i) $u - v = (x-y)(x^2 + 4xy + y^2)$ (M.U. 2003, 04, 07, 08, 12) (ii) $u + v = \frac{x}{x^2+y^2}$ (M.U. 2003, 15)

(iii) $u - v = \frac{\sin x + \sinh y}{\cosh y - \cos x}$ (iv) $u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

(v) $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ and $f(\pi/2) = 0$ (M.U. 1996)

(vi) $u - v = x^3 + x^2 - 3xy^2 - y^2 - 3x^2y + y^3 - 2xy$

(vii) $u + v = e^x(\cos y + \sin y) + \frac{x-y}{x^2+y^2}$ (M.U. 2005)

[**Ans. :** (i) $f(z)=-iz^3+c$, (ii) $\frac{1}{1+i}\left(\frac{i}{z}\right)+c$, (iii) $\cot\frac{z}{2}+c$,

(iv) $\frac{(1+i)}{2}\cot z+c$, (v) $\cot\left(\frac{z}{2}\right)-1$, (vi) z^3+z^2+c, (vii) $e^z+\frac{1}{z}+c$.]

7. Find the analytic function $f(z)$ whose real part is

(i) $r^2\cos 2\theta - r\sin\theta + 2$ **(M.U. 2014)** (ii) $r^n\cos n\theta$ [**Ans. :** (i) z^2+z+c, (ii) z^n+c]

13. To Find an Analytic Function when Harmonic Function is Given

Example 1 : Show that $u=y^3-3x^2y$ is a harmonic function. Find its harmonic conjugate and the corresponding analytic function. **(M.U. 2003)**

Sol. : Method 1 : Since, $u=y^3-3x^2y$

$$u_x=-6xy,\ u_{xx}=-6y\ ;\ u_y=3y^2-3x^2,\ u_{yy}=6y$$

$$\therefore\ \nabla^2u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=-6y+6y=0$$

$\therefore$ $u=x^3-3x^2y$ is a harmonic function.

Now by the note 1 after § 12, page 5-29.

$$\int -\frac{\partial u}{\partial y}\,dx=\int -(3y^2-3x^2)\,dx=-3xy^2+x^3$$

and $\int$(terms in $\frac{\partial u}{\partial x}$ free from x) $dy=\int 0\,dy=0$

$$\therefore\ v=x^3-3xy^2+c.$$

$$\therefore\ f(z)=u+iv=(y^3-3x^2y)+i(x^3-3xy^2)+c.$$

Putting $x=z,\ y=0$ $\quad\therefore\ f(z)=z^3i+c$

Method 2 : Since, $u=y^3-3x^2y$ by Milne-Thompson method given on page 5-29.

$$u_x=\Phi_1=-6xy,\quad u_y=\Phi_2=3y^2-3x^2$$

$$\therefore\ f'(z)=\Phi_1(z,0)-i\,\Phi_2(z,0)=0+3iz^2$$

$\therefore$ $f(z)=\int 3iz^2\,dz=iz^3+c$ as above is the required analytic function.

Now, $f(z)=i(x+iy)^3=i(x^3+3ix^2y-3xy^2-iy^3)$

$$\therefore\ u+iv=-3x^2y+y^3+i(x^3-3xy^2)$$

$\therefore$ $v=x^3-3xy^2$ is the harmonic conjugate.

Note ....

This method is more convenient.

Method 3 : $u=y^3-3x^2y$

$$\therefore\ u\left(\frac{z}{2},\frac{z}{2i}\right)=\frac{z^3}{8i^3}-3\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right)=-\frac{z^3}{8i}-\frac{3z^3}{8i}=-\frac{z^3}{2i}=\frac{z^3}{2}i$$

$$u(0,0)=0$$

$$\therefore\ f(x)=2u\left(\frac{z}{2},\frac{z}{2i}\right)-u(0,0)+ci=z^3i+ci.$$

Example 2 : Show that the function $u=\sin x\cos hy+2\cos x\sin hy+x^2-y^2+4xy$ satisfies Laplace's equation and find it corresponding analytic function $f(z)=u+iv$.

(M.U. 1998, 2005, 09, 13)

Sol. : We have $\quad\frac{\partial u}{\partial x}=\cos x\cos hy-2\sin x\sin hy+2x+4y$

$$\frac{\partial^2u}{\partial x^2}=-\sin x\cos hy-2\cos x\sin hy+2$$

$$\frac{\partial u}{\partial y}=\sin x\sin hy+2\cos x\cos hy-2y+4x$$

$$\frac{\partial^2u}{\partial y^2}=\sin x\cos hy+2\cos x\sin hy-2$$

$$\therefore\ \frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0$$

Hence, u satisfies Laplace's equation.

Now $u_x=\Phi_1(x,y)=\cos x\cos hy-2\sin x\sin hy+2x+4y$

$$\Phi_1(z,0)=\cos z+2z$$

$$u_y=\Phi_2(x,y)=\sin x\sin hy+2\cos x\cos hy-2y+4x$$

$$\Phi_2(z,0)=2\cos z+4z$$

Now we use Milne-Thompson Method given on page 5-29.

(Read the procedure given on page 5-31)

$$\therefore\ f'(z)=\Phi_1(z,0)-i\,\Phi_2(z,0)=(\cos z+2z)-i(2\cos z+4z)$$

$$\therefore\ f(z)=\int[(\cos z+2z)-i(2\cos z+4z)]\,dz=\sin z+z^2-i(2\sin z+2z^2)+c$$

which is the required analytic function.

Example 3 : If $v=e^x\sin y$, prove that v is a harmonic function. Also find the corresponding harmonic conjugate function and analytic function. **(M.U. 2011)**

Sol. : We have $\quad\frac{\partial v}{\partial x}=e^x\sin y,\quad\frac{\partial^2v}{\partial x^2}=e^x\sin y\ ;\quad\frac{\partial v}{\partial y}=e^x\cos y,\quad\frac{\partial^2v}{\partial y^2}=-e^x\sin y.$

$$\therefore\ \frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial y^2}=0.\qquad\therefore\ v\text{ satisfies Laplace's equation.}$$

Now, we use, Milne-Thompson Method given on page 5-29.

(Read the procedure given on page 5-34)

$v_x = e^x \sin y \quad \therefore \ \psi_2(z,0) = 0; \qquad v_y = e^x \cos y \quad \therefore \ \psi_1(z,0) = e^z$

$\therefore \ f'(z) = \psi_1(z,0) + i\,\psi_2(z,0) = e^z + 0$

$\therefore \ f(z) = e^z + c$

Now, $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$\therefore \ u = e^x \cos y$

Example 4 : Show that the function $v = e^x\,(x \sin y + y \cos y)$ satisfies Laplace equation and find its corresponding analytic function and its harmonic conjugate. **(M.U. 2013)**

Sol. : We have $\dfrac{\partial v}{\partial x} = e^x\,(x \sin y + y \cos y) + e^x \sin y$

$$\therefore \ \frac{\partial^2 v}{\partial x^2} = e^x\,(x \sin y + y \cos y) + e^x\,(\sin y) + e^x \sin y$$

$$\frac{\partial v}{\partial y} = e^x\,(x \cos y + \cos y - y \sin y)$$

$$\therefore \ \frac{\partial^2 v}{\partial y^2} = e^x\,(-x \sin y - \sin y - \sin y - y \cos y)$$

$$\therefore \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad \therefore \ v \text{ satisfies Laplace equation.}$$

Now, $v_x = e^x\,(x \sin y + y \cos y + \sin y)$

$\therefore \ \Psi_2(z, 0) = 0$

$v_y = e^x\,(x \cos y + \cos y - y \sin y)$

$\therefore \ \Psi_1(z, 0) = e^z\,(z + 1)$

$\therefore \ f'(z) = \Psi_1(z, 0) + i\,\Psi_2(z, 0) = e^z\,(z + 1) + 0$

$\therefore \ f(z) = \int e^z\,(z+1)\,dz = z\,e^z$ which is the required analytic function.

Now, $f(z) = (x + iy)\,(e^{x+iy}) = (x + iy)\,(e^x \cdot e^{iy})$

$= (x + iy)\,e^x\,(\cos y + i \sin y)$

$\therefore \ u = e^x\,(x \cos y - y \sin y)$ which is the required harmonic conjugate.

Example 5 : If $v = 3x^2y + 6xy - y^3$, show that v is harmonic and find the corresponding analytic function. **(M.U. 2003, 07, 13, 14, 1**

Sol. : We have

$$\frac{\partial v}{\partial x} = 6xy + 6y, \quad \frac{\partial^2 v}{\partial x^2} = 6y; \quad \frac{\partial v}{\partial y} = 3x^2 + 6x - 3y^2, \quad \frac{\partial^2 v}{\partial y^2} = -6y$$

$$\therefore \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y - 6y = 0. \qquad \therefore \ v \text{ is satisfies Laplace's equation.}$$

Now we use Milne-Thompson Method given on page 5-29.

$\therefore \ v_x = 6xy + 6y, \ \psi_2(z,0) = 0$

$v_y = 3x^2 + 6x - 3y^2, \ \psi_1(z,0) = 3z^2 + 6z$

$\therefore \ f'(z) = \psi_1(z,0) + i\,\psi_2(z,0) = (3z^2 + 6z) + 0$

$\therefore \ f(z) = \int (3z^2 + 6z)\,dz = (z^3 + 3z^2) + c$

Example 6 : Show that $u = \cos x \cosh y$ is a harmonic function. Find its harmonic conjugate and corresponding analytic function. **(M.U. 2005)**

Sol. : Since, $u = \cos x \cosh y$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y, \quad \frac{\partial^2 u}{\partial y^2} = \cos x \cosh y$$

$$\therefore \ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \therefore \ u = \cos x \cosh y \text{ is a harmonic functions.}$$

Now, we use Milne-Thompson Method.

Now, $u_x = \Phi_1(x, y) = -\sin x \cosh y \qquad \therefore \ \Phi_1(z, 0) = -\sin z$

$u_y = \Phi_2(x, y) = \cos x \sinh y \qquad \therefore \ \Phi_2(z, 0) = 0$

$\therefore \ f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = -\sin z$

$\therefore \ f(z) = \int -\sin z\,dz = \cos z + c$ is the required analytic function.

Now, $f(z) = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy$

$\therefore \ u + iv = \cos x \cosh y - i \sin x \sinh y$

$\therefore \ v = -\sin x \sinh y$ is the required harmonic conjugate.

Example 7 : Prove that $u = x^2 - y^2$, $v = -\dfrac{y}{x^2 + y^2}$ both u and v satisfy Laplace's equation, but that $u + iv$ is not an analytic function of z. **(M.U. 1996, 2003, 09)**

Sol. : $u_x = 2x,\ u_{xx} = 2;\ u_y = -2y,\ u_{yy} = -2$

$$v_x = \frac{2xy}{(x^2 + y^2)^2}$$

$$\therefore \ v_{xx} = 2y\left[\frac{(x^2+y^2)^2 \cdot 1 - x \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4}\right]$$

$$= \frac{2y(x^2+y^2)[x^2 + y^2 - 4x^2]}{(x^2+y^2)^4} = 2y\,\frac{(y^2 - 3x^2)}{(x^2+y^2)^3}$$

$$v_y = -\left[\frac{(x^2+y^2) \cdot 1 - y \cdot 2y}{(x^2+y^2)^2}\right] = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$v_{yy} = \frac{(x^2+y^2)^2 \cdot 2y - (y^2 - x^2) \cdot 2(x^2+y^2) \cdot 2y}{(x^2+y^2)^4}$$

$$= 2y(x^2+y^2)\frac{[x^2 + y^2 - 2y^2 + 2x^2]}{(x^2+y^2)^4} = 2y\,\frac{(3x^2 - y^2)}{(x^2+y^2)^3} = -2y\,\frac{(y^2 - 3x^2)}{(x^2+y^2)^3}$$

$\therefore \ u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

Hence, u, v satisfy Laplace's Equations.

But Cauchy-Riemann equations are not satisfied as $u_x \neq v_y$ and $u_y \neq -v_x$.

Hence, $u + iv$ is not analytic.

Example 8 : Show that the following function satisfies Laplace's equation and find the corresponding analytic function and the harmonic conjugate, $u = \frac{1}{2}\log(x^2+y^2)$.

(M.U. 2002, 05, 06, 08, ...)

Sol. : We have $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$ $\therefore \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x \cdot 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

Similarly, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$ $\therefore \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-x^2+y^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

$\therefore$ u satisfies Laplace's equation.

Now $u_x = \Phi_1(x,y) = \frac{x}{x^2+y^2}$ $\therefore \Phi_1(z,0) = \frac{z}{z^2+0} = \frac{1}{z}$

$u_y = \Phi_2(x,y) = \frac{y}{x^2+y^2}$ $\therefore \Phi_2(z,0) = 0$

By Milne-Thompson method,

$$\therefore f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0) = \frac{1}{z} - i\,0 = \frac{1}{z}$$

$$\therefore f(z) = \int \frac{1}{z}\,dz = \log z + c = \log(x+iy) + c$$

$$\therefore u + iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} + c$$ **[Note this]**

(See Applied Mathematics I by the same author.)

$\therefore$ $v = \tan^{-1}\frac{y}{x} + c$ is the corresponding harmonic conjugate.

Example 9 : Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$.

Sol. : We have $z = x + iy$, $\bar{z} = x - iy$

$$\therefore x = \frac{1}{2}(z+\bar{z}), \quad y = \frac{1}{2i}(z-\bar{z})$$

Treating z and $\bar{z}$ as independent variables

$$\therefore \frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = \frac{1}{2i}, \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}.$$

Now, $\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{1}{2} + \frac{\partial}{\partial y}\cdot\frac{1}{2i} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

And $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{1}{2} + \frac{\partial}{\partial y}\cdot\left(-\frac{1}{2i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$\therefore \frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\cdot\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Hence, the result.

Example 10 : If $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 - y^2$, find u.

Sol. : Let $z = x + iy$ $\therefore \bar{z} = x - iy$ $\therefore x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$\therefore x^2 - y^2 = \frac{1}{4}[(z+\bar{z})^2 + (z-\bar{z})^2] = \frac{1}{2}(z^2+\bar{z}^2) \quad \text{.......... (1)}$$

By the above result $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial z\,\partial \bar{z}}$ .... (2)

$$\therefore 4\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = x^2 - y^2 \quad \text{[By data]}$$

$$\therefore 4\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{2}(z^2+\bar{z}^2) \quad \text{[By (1)]}$$

$$\therefore \frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{8}(z^2+\bar{z}^2)$$

$\therefore$ By integration w.r.t. $\bar{z}$ $\frac{\partial u}{\partial z} = \frac{1}{8}\cdot z^2\bar{z} + \frac{\bar{z}^3}{24} + f_1(\bar{z})$

where $f_1(\bar{z})$ if a constant of integration and f_1 is an arbitrary function.

Integrating again, w.r.t. z,

$u = \frac{z^3}{24}\cdot\bar{z} + \frac{\bar{z}^3\cdot z}{24} + f_2(z) + f_1(\bar{z})$ where $f_2(z)$ is an arbitrary function of z.

$$\therefore u = \frac{z\bar{z}}{24}(z^2+\bar{z}^2) + f_1(\bar{z}) + f_2(z)$$

$$= \frac{(x+iy)(x-iy)}{24}[(x+iy)^2 + (x-iy)^2] + f_1(x-iy) + f_2(x+iy)$$

$$= \frac{(x^2-y^2)}{12}\cdot(x^2+y^2) + f_1(x-iy) + f_2(x+iy)$$

$$= \frac{x^4-y^4}{12} + f_1(x-iy) + f_2(x+iy).$$

Example 11 : If $f(z)$ is analytic function, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

(M.U. 1993, 96, 98, 2003, 06, 07)

Sol. : We first note that if $f(z) = u + iv$, $f(\bar{z}) = u - iv$ .... (1)

$$\therefore |f(z)|^2 = u^2 + v^2 = (u+iv)(u-iv) = f(z)\cdot f(\bar{z})$$

Also, $f'(z) = u_x + iv_x$, $f'(\bar{z}) = u_x - iv_x$ .... (2)

$$\therefore |f'(z)|^2 = {u_x}^2 + {v_x}^2 = (u_x+iv_x)(u_x-iv_x) = f'(z)\cdot f'(\bar{z})$$

Now, by the result obtained in Ex. 9 above.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\, f(z)\cdot f(\bar{z})$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}} f(z)\cdot f(\bar{z})\right] = 4\frac{\partial}{\partial z}\left[f(z)\cdot\frac{\partial}{\partial \bar{z}} f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z)\cdot f'(\bar{z})\right] = 4\cdot f'(\bar{z})\frac{\partial}{\partial z} f(z) = 4\cdot f'(\bar{z})\cdot f'(z)$$

$$= 4|f'(z)|^2 \qquad \text{[By (2)]}$$

Restatement

Since $|f(z)|^2 = u^2+v^2$ and $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$, the above example can be restated in the following way.

"If $f(z)$ is an anylatic function, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]."$$

Example 12 : If $f(z)$ is an analytic function, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2|f'(z)|^{n-2}\cdot|f'(z)|^2.$$ **(M.U. 1994, 200[illegible])**

Sol. : We first note that

$$|f(z)| = \left\{|f(z)|^2\right\}^{1/2} = \left[f(z)\cdot f(\bar{z})\right]^{1/2} \quad \therefore \quad |f(z)|^n = \left[f(z)\cdot f(\bar{z})\right]^{n/2} \qquad \ldots\ldots(1)$$

Now $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\cdot f(\bar{z})\right]^{n/2}$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}[f(z)]^{n/2}\cdot[f(\bar{z})]^{n/2}\right] = 4\left[\frac{\partial}{\partial z}\left\{[f(z)]^{n/2}\cdot\frac{\partial}{\partial \bar{z}}[f(\bar{z})]^{n/2}\right\}\right]$$

$$= 4\left[\frac{\partial}{\partial z}[f(z)]^{n/2}\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\right]$$

$$= 4\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\cdot\frac{\partial}{\partial z}\cdot[f(z)]^{n/2}$$

$$= 4\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\cdot\frac{n}{2}\cdot[f(z)]^{(n/2)-1}\cdot f'(z)$$

$$= n^2\left[f(z)\cdot f(\bar{z})\right]^{(n/2)-1}\cdot\left[f'(z)\cdot f'(\bar{z})\right]$$

But $f(z)\cdot f(\bar{z}) = |f(z)|^2$ and $f'(z)\cdot f'(\bar{z}) = |f'(z)|^2$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2\left\{|f(z)|^2\right\}^{(n/2)-1}\cdot|f'(z)|^2$$

$$= n^2|f(z)|^{n-2}\cdot|f'(z)|^2$$

Example 13 : If $f(z) = u + iv$ is an analytic function, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$. **(M.U. 2005)**

Sol. : We first note that if $f(z) = u+iv$, $f(\bar{z}) = u - iv$

$$\therefore |Rf(z)|^2 = \left|\frac{1}{2}\left[f(z)+f(\bar{z})\right]\right|^2 = \frac{1}{4}|f(z)+f(\bar{z})|^2$$

And $$|f(z)|^2 = u^2+v^2 = (u+iv)(u-iv) = f(z)\cdot f(\bar{z}) \qquad \ldots\ldots(1)$$

$$\therefore |Rf(z)|^2 = \frac{1}{4}\left[f(z)+f(\bar{z})\right]\left[f(\bar{z})+f(\bar{z})\right] = \frac{1}{4}\left[f(z)+f(\bar{z})\right]\cdot\left[f(\bar{z})+f(z)\right] \qquad \ldots\ldots(2)$$

$$= \frac{1}{4}\left[f(z)+f(\bar{z})\right]^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|Rf(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left\{\frac{1}{4}\left[f(z)+f(\bar{z})\right]^2\right\}$$

$$= \frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)+f(\bar{z})\right]^2 = \frac{\partial}{\partial z}2\left[f(z)+f(\bar{z})\right]\cdot f'(\bar{z})$$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2f'(\bar{z})\cdot\frac{\partial}{\partial z}\left[f(z)+f(\bar{z})\right] = 2f'(\bar{z})\cdot f'(z)$$

$$= 2|f'(z)|^2 \qquad \text{[By (2)]}$$

Example 14 : If $f(z)$ is an analytic function, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^n = n(n-1)|Rf(z)|^{n-2}\cdot|f'(z)|^2.$$

Sol. : We know that

$$|Rf(z)|^2 = \left|\frac{1}{2}\left[f(z)+f(\bar{z})\right]\right|^2 = \frac{1}{4}|f(z)+f(\bar{z})|^2 \qquad \ldots\ldots(1)$$

But $$|f(z)|^2 = f(z)\cdot f(\bar{z}) \qquad \ldots\ldots(2)$$

$$\therefore |Rf(z)|^2 = \frac{1}{4}|f(z)+f(\bar{z})|^2 = \frac{1}{4}\left[f(z)+f(\bar{z})\right]\cdot\left[f(\bar{z})+f(\bar{z})\right]$$

$$= \frac{1}{4}\left[f(z)+f(\bar{z})\right]\cdot\left[f(\bar{z})+f(z)\right] = \frac{1}{4}\left[f(z)+f(\bar{z})\right]^2 \qquad \ldots\ldots(3)$$

$$\therefore |Rf(z)|^n = \left[|Rf(z)|^2\right]^{n/2} = \left\{\frac{1}{4}\left[f(z)+f(\bar{z})\right]^2\right\}^{n/2} = \frac{1}{2^n}\left[f(z)+f(\bar{z})\right]^n \qquad \ldots\ldots(4)$$

Now, $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|Rf(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left\{\frac{1}{2^n}\left[f(z)+f(\bar{z})\right]^n\right\}$$

$$= 4\cdot\frac{1}{2^n}\frac{\partial}{\partial z}\left\{n\left[f(z)+f(\bar{z})\right]^{n-1}\cdot f'(\bar{z})\right\} = \frac{n}{2^{n-2}}f'(\bar{z})\cdot\frac{\partial}{\partial z}\left[f(z)+f(\bar{z})\right]^{n-1}$$

$$= \frac{n}{2^{n-2}}f'(\bar{z})\cdot(n-1)\left[f(z)+f(\bar{z})\right]^{n-2}\cdot f'(z)$$

$$\therefore\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^n=\frac{n(n-1)}{2^{n-2}}\left[f(z)+f(\bar{z})\right]^{n-2}\cdot f'(z)\cdot f'(\bar{z})$$

By (2), $f'(z)\cdot f'(\bar{z})=|f'(z)|^2$

By (4), $\frac{1}{2^{n-2}}\left[f(z)+f(\bar{z})\right]^{n-2}=|Rf(z)|^{n-2}$

$$\therefore\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^n=n(n-1)|Rf(z)|^{n-2}\cdot|f'(z)|^2$$

Example 15 : If $f(z)$ is an analytic function, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0.$ **(M.U. 200**

Sol. : We first note that $\log|f'(z)|=\frac{1}{2}\log|f'(z)|^2$

But $|f'(z)|^2=f'(z)\cdot f'(\bar{z})$

$$\therefore\ \log|f'(z)|=\frac{1}{2}\log\left[f'(z)\cdot f'(\bar{z})\right]=\frac{1}{2}\log f'(z)+\frac{1}{2}\log f'(\bar{z})$$

Now, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=4\frac{\partial^2}{\partial z\,\partial\bar{z}}\log|f'(z)|$

$$=4\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[\frac{1}{2}\log f'(z)+\frac{1}{2}\log f'(\bar{z})\right]=2\frac{\partial^2}{\partial z\,\partial\bar{z}}\log f'(z)+2\frac{\partial^2}{\partial z\,\partial\bar{z}}\log f'(\bar{z})$$

$$=2\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\bar{z}}\log f'(z)\right]+2\frac{\partial}{\partial\bar{z}}\left[\frac{\partial}{\partial z}\log f'(\bar{z})\right]=2\frac{\partial}{\partial z}(0)+2\frac{\partial}{\partial z}(0)=0$$

Example 16 : If Φ and Ψ are functions satisfying Laplace equation, then show that $s+i$ holomorphic (analytic) where $s=\frac{\partial\Phi}{\partial y}-\frac{\partial\Psi}{\partial x}$ and $t=\frac{\partial\Phi}{\partial x}+\frac{\partial\Psi}{\partial y}$. **(M.U. 1999, 2003**

Sol.: Since Φ and Ψ satisfy Laplace equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0\ \text{and}\ \frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=0 \quad \text{.........}$$

Now, $\frac{\partial s}{\partial x}=\frac{\partial^2\Phi}{\partial y\,\partial x}-\frac{\partial^2\Psi}{\partial x^2}=\frac{\partial^2\Phi}{\partial y\,\partial x}+\frac{\partial^2\Psi}{\partial y^2}$ [By (1)] ....

$\frac{\partial s}{\partial y}=\frac{\partial^2\Phi}{\partial y^2}-\frac{\partial^2\Psi}{\partial x\,\partial y}=-\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Psi}{\partial x\,\partial y}$ [By (1)] ....

Also, $\frac{\partial t}{\partial x}=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y\,\partial x}$ ....

$\frac{\partial t}{\partial y}=\frac{\partial^2\Phi}{\partial x\,\partial y}+\frac{\partial^2\Psi}{\partial y^2}$ ....

From (2) and (5), we have $\frac{\partial s}{\partial x}=\frac{\partial t}{\partial y}$

From (3) and (4), we have $\frac{\partial s}{\partial y}=-\frac{\partial t}{\partial x}$

Since, $s+it$ satisfies Cauchy-Riemann equations it is analytic.

EXERCISE - IV

1. Prove that $u=e^x\cos y+x^3-3xy^2$ is harmonic. **(M.U. 2003)**
2. Check whether $u=x+e^{xy}+y+e^{-xy}$ is harmonic. **(M.U. 2004) [Ans. : No]**
3. Prove that $u=e^x\cos y$ is harmonic. Determine its harmonic conjugate v and the analytic function $f(z)$. **[Ans. :** $f(z)=e^z+c,\ v=e^x\sin y$ **]**
4. Prove that $u=x^2-y^2-2xy-2x+3y$ is harmonic. Find the corresponding analytic function $f(z)$. Also find the conjugate function. **[Ans. :** $f(z)=(1+i)z^2-(z+3i)z,\ v=x^2+2xy-y^2-3x-2y$ **]**
5. Show that $f(z)=e^x(\cos y+i\sin y)$ is analytic, it also satisfies Laplace's equation. Find its derivative. **[Ans. :** $f(z)=e^z+c$ **]**
6. Show that the function $u=3x^2y+2x^2-y^3-2y^2$ is harmonic. Find the harmonic conjugate function v and express $u+iv$ as an analytic function of z. **(M.U. 2006)** **[Ans. :** $v=4xy-x^3+3xy^2;\ f(z)=-iz^3+2z^2+c$ **]**
7. Show that $u=e^x(x\cos y-y\sin y)$ is harmonic. Find the harmonic conjugate v and the analytic function $f(z)$. **(M.U. 2009) [Ans. :** $v=e^x(x\sin y+y\cos y);\ f(z)=ze^z+c$ **]**
8. Show that the following functions are harmonic and find the corres-ponding analytic function $f(z)=u+iv$

   (i) $v=e^{-x}(x\cos y+y\sin y)$, (ii) $v=e^{2x}(y\cos 2y+x\sin 2y)$.

   (iii) $u=e^x\cos y-x^2+y^2$ (iv) $u=x^4-6x^2y^2+y^4$ **(M.U. 2013)**

   [Ans. : (i) $ize^{-z}+c$, **(ii)** $ze^{2z}+c$, **(iii)** e^z-z^2+c, **(iv)** z^4+c. **]**
9. Show that the following functions are harmonic and find their corres-ponding analytic functions $f(z)=u+iv$

   (i) $u=x^2-y^2$, (ii) $u=2x(1-y)$ (iii) $u=3x^2y-y^3$,

   (iv) $u=x^3-3xy^2+3x^2-3y^2+1$ (v) $u=(x-1)^3-3xy^2+3y^2$ **(M.U. 2004)**

   [Ans. : (i) z^2+c, **(ii)** $2z+iz^2=c$, **(iiii)** z^3+c, **(iv)** z^3+3z^2+c, **(v)** $(z-1)^3+c$. **]**
10. Find the analytic function $f(z)=u+iv$ where $u=2a\,xy+b(y^2-x^2)$. Also verify that u is harmonic. **(M.U. 2004) [Ans. :** $f(z)=(a-bi)z^2$ **]**
11. Show that $u=e^{-2xy}\sin(x^2-y^2)$ is harmonic. Find the corresponding analytic function $f(z)$ and also find the conjugate function v. **[Ans. :** $f(z)=-ie^{iz^2}+c;\ v=-e^{-2xy}\cos(x^2-y^2)+c$ **]**
12. Show that $u=\frac{\sin 2x}{\cosh 2y+\cos 2x}$ is harmonic. Hence, find its harmonic conjugate v and corresponding analytic function $f(z)=u+iv$. **(M.U. 1997, 2006) [Ans. :** $f(z)=\tan z+c$ **]**

13. If u and v are harmonic functions of x and y and $s = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ and $t = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, prove that $s + it$ is an analytic function of $z = x + iy$.

14. If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function of $z = x + iy$. (M.U. 2003)

Remark ....

The solved Example No. 16 above and this example are the same with the difference in wording them.

14. Orthogonal Curves - Orthogonal Trajectories

(a) If $f(z) = u(x, y) + iv(x, y)$ is an analytic function then the curves $u = c_1$ and $v = c_2$ intersect orthogonally. (M.U. 2003, 16)

Proof : Let $u = f(x, y) = c_1$ and $v = \Phi(x, y) = c_2$

Then $\left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\partial u/\partial x}{\partial u/\partial y}$ and $\left(\frac{dy}{dx}\right)_{v=c_2} = -\frac{\partial \Phi/\partial x}{\partial \Phi/\partial y} = -\frac{\partial v/\partial x}{\partial v/\partial y}$

Since, $f(z)$ is analytic C-R equations give $u_x = v_y$ and $u_y = -v_x$

$$\therefore \left(\frac{\partial y}{\partial x}\right)_{u=c_1} \times \left(\frac{dy}{dx}\right)_{v=c_2} = \frac{\partial u/\partial x}{\partial u/\partial y} \times \frac{\partial v/\partial x}{\partial v/\partial y} = \frac{v_y}{-v_x} \cdot \frac{v_x}{v_y} = -1.$$

Hence, $u = c_1$ and $v = c_2$ intersect orthogonally.

(b) Orthogonal Trajectories : By an **orthogonal trajectory** of a family of curves we mean a curve which cuts every member of the given family at right angles. For example, consider a family of straight lines passing through the origin given by $y = mx$, where m is an arbitrary constant.

It is easy to see that these straight lines are cut by a circle with centre at the origin at right angles at every point of intersection. Its equation is of the form $x^2 + y^2 = a^2$ where a is a parameter.

Thus, the family of circles $x^2 + y^2 = a^2$ represents the family of orthogonal trajectories to the family of straight lines given by $y = mx$.

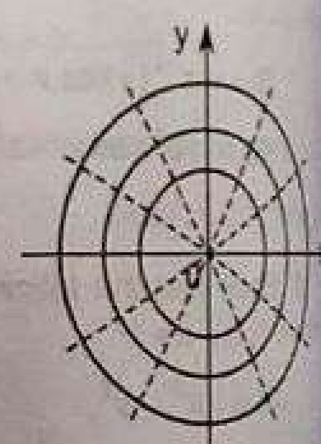


Fig. 5.4

(c) Orthogonal trajectories of the family of curves given by $u = c$.

We have seen that if $f(z) = u + iv$ is an analytic function then the curves $u = c_1$ and $v = c_2$ intersect orthogonally *i.e.* $v = c_2$ is the family of orthogonal trajectories of the family of curves given by $u = c_1$.

Hence, to find the orthogonal trajectory of $u = c_1$ (or $v = c_2$) we find the harmonic conjugate $v = c_2$ (or $u = c_1$) of u (or v).

Example 1 : Find the orthogonal trajectory of the family of curves $x^3y - xy^3 = c$. **(M.U. 2003, 04, 05, 06, 07, 11, 1...)**

Sol. : As seen above we have to find the harmonic conjugate of u.

$\therefore u = x^3y - xy^3 \qquad \therefore u_x = 3x^2y - y^3$ and $u_y = x^3 - 3xy^2$

$\therefore f'(z) = u_x + iv_x = u_x - iu_y$ [By C-R equations]

$= (3x^2y - y^3) - i(x^3 - xy^2)$

By Milne-Thompson's method, we put $x = z,\ y = 0$

$\therefore f'(z) = -iz^3$

$\therefore f(z) = -\int iz^3 dz = -i\frac{z^4}{4} + c = -\frac{i}{4}(x + iy)^4 + c$

$= -\frac{i}{4}(x^4 + 4x^3iy - 6x^2y^2 - 4xiy^3 + y^4) + c$

$\therefore$ Imaginary part $v = -\frac{1}{4}(x^4 - 6x^2y^2 + y^4)$

Hence, the required orthogonal trajectories are $x^4 - 6x^2y^2 + y^4 = c'$.

Example 2 : Find the orthogonal trajectories of the family of curves $e^{-x}\cos y + xy = \alpha$, where α is the real constant in the xy-plane. **(M.U. 2004, 05, 13, 16)**

Sol. : We have seen above that the orthogonal trajectories of $u = c_1$ are given by $v = c_2$ where v is the harmonic conjugate of u.

$\therefore u = e^{-x}\cos y + xy \qquad \therefore u_x = -e^{-x}\cos y + y$ and $u_y = -e^{x}\sin y + x$

Also $f'(z) = u_x + iv_x = u_x - iu_y$ [By C-R equations]

$= (-e^{-x}\cos y + y) - i(-e^{-x}\sin y + x)$

By Milne-Thompson's method, we replace x by z and y by zero.

$\therefore f'(z) = -e^{-z} - iz$

By integration $f(z) = e^{-z} - i\frac{z^2}{2} + c$

$\therefore f(z) = e^{-(x+iy)} - i\frac{(x + iy)^2}{2} + c$

$= e^{-x}(\cos y - i\sin y) - \frac{i}{2}(x^2 + 2i\,xy - y^2) + c$

$\therefore$ Imaginary part, $\quad v = -e^{-x}\sin y - \frac{1}{2}(x^2 - y^2)$

Hence, the required orthogonal trajectories are $e^{-x}\sin y + \frac{1}{2}(x^2 - y^2) = c_2$.

Example 3 : Find the orthogonal trajectory of the family of curves given by $2x - x^3 + 3xy^2 = a$. **(M.U. 2002, 04, 16)**

Sol. : Let $u = 2x - x^3 + 3xy^2$

$\therefore u_x = 2 - 3x^2 + 3y^2,\ u_y = 6xy$

$\therefore f'(z) = u_x + i\,v_x = u_x - i\,u_y$ [By C-R equations]

$= 2 - 3x^2 + 3y^2 - i \cdot 6xy$

By Milne-Thomson method, we put $x = z,\ y = 0$.

$\therefore f'(z) = 2 - 3z^2$

Integrating w.r.t. z, we get

$$f(z) = 2z - z^3 + c = 2(x+iy) - (x+iy)^3 + c$$
$$= 2x + 2iy - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$$

$\therefore$ Imaginary part is $v = 2y - 3x^2y + y^3$.

$\therefore$ The required orthogonal trajectory is $2y - 3x^2y + y^3 = c$.

Example 4 : For the function $f(z) = z^3$, verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally where c_1 and c_2 are constants and $f(z) = u + iv$. **(M.U. 2004)**

Sol. : $f(z) = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\therefore$ $u = x^3 - 3xy^2,\ v = 3x^2y - y^3;\quad u_x = 3x^2 - 3y^2,\ u_y = -6xy$

$v_x = 6xy,\ v_y = 3x^2 - 3y^2$

$$\therefore\ m_1 = \left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{u_x}{u_y} = -\frac{3(x^2-y^2)}{-6xy}$$

$$m_2 = \left(\frac{dy}{dx}\right)_{v=c_2} = -\frac{v_x}{v_y} = -\frac{6xy}{3(x^2-y^2)}$$

$$\therefore\ m_1 \times m_2 = \frac{3(x^2-y^2)}{6xy} \cdot \left(-\frac{6xy}{3(x^2-y^2)}\right) = -1$$

Hence, the families cut orthogonally.

EXERCISE - V

Find the orthogonal trajectories of the family of curves

1. $3x^2y - y^3 = c$ **(M.U. 2002, 04, 07, 15)**
2. $x^2 - y^2 + x = c$
3. $e^{-x}(x \sin y - y \cos y) = c$
4. $x^2 - y^2 - 2xy + 2x - 3y = c$ **(M.U. 2004, 10)**
5. $e^x \cos y - xy = c$ **(M.U. 2007, 08)**
6. $3x^2y + 2x^2 - y^3 - 2y^2 = c$ **(M.U. 2003, 12, 14)**

[**Ans. :** (1) $3xy^2 - x^3 = c$, (2) $2xy + y = c$, (3) $e^{-x}(x \cos y + y \sin y) = c$
(4) $2xy + 2y + x^2 - y^2 + 3y = c$ (5) $e^x \sin y + (x^2 - y^2)/2 = c$,
(6) $4y - 3x^2 + 3xy^2 = c$.]

(d) Angle of Intersection of Two Curves in Polar Coordinates : If the equation of a curve is given in polar coordinates by $r = f(\theta)$ and if Φ is the angle between the radius vector at P and the tangent at P, then $\tan\Phi = r\dfrac{d\theta}{dr}$. (We accept this result without proof.)

If two curves given by polar equations $r = f(\theta)$ and $r = F(\theta)$ intersect at P and if Φ_1 and Φ_2 are the angles between the common radius vector and the tangents to the two curves then it is clear from the figure that the angle between the curves i.e. the angle between the tangents is equal to $\Phi_1 \sim \Phi_2$ i.e. the difference between Φ_1 and Φ_2.

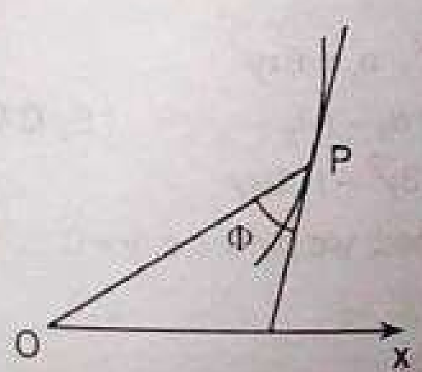


Fig. 5.5 (a)

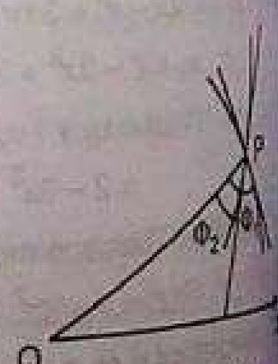


Fig. 5.5 (b)

The two curves intersect orthogonally if the angle between the tangents to the two curves at the common point is $\pi/2$ i.e. if $\Phi_1 - \Phi_2 = \pi/2$.

Example 1 : Prove that the curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ intersect orthogonally. **(M.U. 2005)**

Sol. : Let $P(r, \theta)$ be the point of intersection.

For the first curve $\quad n \log r = \log a + \log \sec n\theta$

$$\therefore\ \frac{n}{r} \cdot \frac{dr}{d\theta} = \frac{1}{\sec n\theta} \cdot \sec n\theta \cdot \tan n\theta \cdot n \quad \therefore\ \frac{dr}{d\theta} = r \tan n\theta$$

$$\therefore\ \tan\Phi_1 = r\frac{d\theta}{dr} = \frac{r}{r\tan n\theta} = \cot n\theta \quad \therefore\ \Phi_1 = \frac{\pi}{2} - n\theta.$$

For the second curve, $\quad n \log r = \log b + \log \operatorname{cosec} n\theta$

$$\therefore\ \frac{n}{r}\frac{dr}{d\theta} = -\frac{1}{\operatorname{cosec} n\theta} \cdot \operatorname{cosec} n\theta \cdot \cot n\theta \cdot n \quad \therefore\ \frac{dr}{d\theta} = -r\cot n\theta$$

$$\therefore\ \tan\Phi_2 = r\frac{d\theta}{dr} = -\frac{r}{r\cot n\theta} = -\tan n\theta \quad \therefore\ \Phi_2 = -n\theta$$

$$\therefore\ \Phi_1 - \Phi_2 = \frac{\pi}{2} - n\theta - (-n\theta) = \frac{\pi}{2}.$$

$\therefore$ The curves intersect orthogonally.

Example 2 : Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.

Sol. : If the two curves intersect at $P(r, \theta)$, then by eliminating r, we have $\sin\theta + \cos\theta = 2\sin\theta$.

$\therefore\ \sin\theta = \cos\theta \quad \therefore\ \tan\theta = 1 \quad \therefore\ \theta = \dfrac{\pi}{4}$ at the point of intersection.

Now, for the curve $r = \sin\theta + \cos\theta$

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta \quad \therefore\ \tan\Phi = r\frac{d\theta}{dr} \quad \therefore\ \tan\Phi = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

$$\therefore\ \text{At } P,\ i.e. \text{ at } \theta = \frac{\pi}{4},\ \tan\Phi = \frac{(1/\sqrt{2}) + (1/\sqrt{2})}{(1/\sqrt{2}) - (1/\sqrt{2})} = \infty.$$

$$\therefore\ \text{At } P,\ \Phi_1 = \frac{\pi}{2}$$

Also for the curve $r = 2\sin\theta$.

$$\frac{dr}{d\theta} = 2\cos\theta \quad \therefore\ \tan\Phi = r\frac{d\theta}{dr} = \frac{2\sin\theta}{2\cos\theta}$$

$\therefore\ \tan\Phi = \tan\theta \quad \therefore\ \Phi = \theta$

$$\therefore\ \text{At } P,\ i.e., \text{ at } \theta = \frac{\pi}{4},\ \Phi_2 = \frac{\pi}{4}.$$

$$\therefore\ \text{Angle of intersection } = \Phi_1 - \Phi_2 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Example 3 : Prove that the curves $r^2 = a^2 \cos 2\theta$ and $r = a(1 + \cos\theta)$ intersect at an angle $3\sin^{-1}(3/4)^{1/4}$.

Sol. : Let us first find θ at the point of intersection of the two curves.

Eliminating r, $a^2 \cos 2\theta = a^2(1+\cos\theta)^2$ $\therefore$ $2\cos^2\theta - 1 = 1 + 2\cos\theta + \cos^2\theta$

$\therefore$ $\cos^2\theta - 2\cos\theta - 2 = 0$ $\therefore$ $\cos\theta = \dfrac{2 \pm \sqrt{4-4(1)(-2)}}{2} = 1 \pm \sqrt{3}$.

Since, $\cos\theta$ lies between -1 and $+1$.

$$\cos\theta = 1-\sqrt{3} \quad \therefore \quad 1 - 2\sin^2\frac{\theta}{2} = 1 - \sqrt{3}$$

$$\therefore \quad \sin^2\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} = \left[\left(\frac{\sqrt{3}}{2}\right)^2\right]^{1/2} = \left(\frac{3}{4}\right)^{1/2} \quad \therefore \quad \sin\left(\frac{\theta}{2}\right) = \left(\frac{3}{4}\right)^{1/4} \quad \therefore \quad \frac{\theta}{2} = \sin^{-1}\left(\frac{3}{4}\right)^{1/4}$$

Now, for the curve $r^2 = a^2 \cos 2\theta$.

$$2r\frac{dr}{d\theta} = a^2 \cdot (-2)\sin 2\theta \quad \therefore \quad \frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

$$\therefore \quad \tan\Phi = r\frac{d\theta}{dr} = r\left(\frac{-r}{a^2\sin 2\theta}\right) = -\frac{r^2}{a^2 \sin 2\theta}$$

$$= -\frac{a^2\cos 2\theta}{a^2 \sin 2\theta} = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\therefore \quad \Phi_1 = \frac{\pi}{2} + 2\theta$$

Further for the second curve, $r = a\,(1+\cos\theta)$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \quad \tan\Phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \quad \Phi_2 = \frac{\pi}{2} + \frac{\theta}{2}$$

$\therefore$ The angle of intersection $= \Phi_1 - \Phi_2 = \dfrac{\pi}{2} + 2\theta - \dfrac{\pi}{2} - \dfrac{\theta}{2} = 3\left(\dfrac{\theta}{2}\right) = 3\sin^{-1}\left(\dfrac{3}{4}\right)^{1/4}$.

EXERCISE - VI

Prove that the following curves in each case intersect orthogonally.

1. $r = 2\sin\theta$, $r = 2\cos\theta$.
2. $r = a\,(1+\cos\theta)$, $r = b\,(1-\cos\theta)$
3. $r^2 \sin 2\theta = a^2$, $r^2\cos 2\theta = b^2$

15. Applications of Analytic Functions

Analytic functions are useful in the study of 'field problems'. For example, in the study of flow of a fluid or of heat or of electricity in a given region analytic functions can be used.

In physical applications of analytic functions, the complex function w is denoted by

$$w = u\,(x, y) + iv\,(x, y) = \Phi(x, y) + i\,\psi(x, y)$$

(a) In the flow of fluids $\Phi(x, y)$ is called velocity potential and $\psi(x, y)$ is called stream functions. The function $w = \Phi(x, y) + i\,\psi(x, y)$ is called potential function $\Phi(x, y) = c_1$ gives family of equipotential curves and $\psi(x, y) = c_2$ gives family of stream lines. $\Phi(x, y) = c_1$ are orthogonal to $\psi(x, y) = c_2$. Stream lines are the paths of fluid particles.

(b) In electrostatic fields $\Phi(x, y)$ is called potential function and $\psi(x, y)$ is known as stream function. $\Phi(x, y) = c_1$ gives family of equipotential curves and $\psi(x, y) = c_2$ gives family of lines of force of electrostatic field. As before $\Phi(x, y) = c_1$ and $\psi(x, y) = c_2$ are orthogonal.

(c) In the study of heat flow, $\Phi(x, y)$ is known as temperature function and $\psi(x, y)$ is known as stream function $\Phi(x, y) = c_1$ gives a family of isothermals and $\psi(x, y) = c_2$ gives a family of heat flow lines.

Example 1 : If $w = \Phi + i\,\psi$ represents complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, determine Φ. Also show that ψ is harmonic. (M.U. 2003, 06)

Sol. : See Ex. 5 on page 5-36.

Example 2 : The potential function of an electrostatic field is given by $\Phi = 3x^2y - y^3$ find the corresponding stream function.

Sol. : See Ex. 1, page 5-44.

EXERCISE - VII

1. In aerodynamics and fluid mechanics the functions Φ and ψ in $f(z) = \Phi + i\,\psi$ where $f(z)$ is analytic are called potential and stream functions. If potential function $\Phi = x^2 + 4x - y^2 + 2y$, find the stream function. [**Ans. :** $\psi = 2xy + 4y - 2x$]

2. If the potential function is $\log(x^2+y^2)$, find the flux function and the complex potential function. [**Ans. :** $2\tan^{-1}\dfrac{y}{x}$, $2\log z + c$]

3. In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\dfrac{y}{x}\right)$ find the velocity potential. [**Ans. :** $\dfrac{1}{2}\log(x^2+y^2)$]

4. An electrostatic field in the x-y plane is given by the potential function $\Phi = x^2 - y^2$ find the stream function. [**Ans. :** $\psi = 2xy + c$]

EXERCISE - VIII

Theory

1. Define analytic function and harmonic function. (M.U. 2007)
2. State and prove the conditions for a function $w = f(z)$ to be analytic. (M.U. 1997, 2003, 04)
3. State and prove Cauchy - Riemann equations. (M.U. 1999, 2002, 03)
4. State and prove Cauchy - Riemann equations in polar form. (M.U. 2002, 04, 08, 11)
5. If $f(z)$ and $\overline{f(z)}$ are both analytic then show that $f(z)$ is constant. (M.U. 1993)

6. If $f(z)$ is analytic and $|f(z)|$ is constant, prove that $f(z)$ is constant. **(M.U. 1994, 1997, 2003)**

7. Prove that real and imaginary parts of an analytic function satisfy Laplace Equation. **(M.U. 1993)**

8. If $f(z)$ is analytic, show that $\frac{\partial f}{\partial \bar{z}} = 0$.

9. If $f(z)$ is analytic, show that $f(z)$ is independent of $\bar{z}$.

10. If $f(z)$ is analytic, show that x and y can occur in $f(z)$ in the combination of $x + iy$ only.

11. Show that real and imaginary parts of an analytic function are harmonic. **(M.U. 1996, 2005, 06, 07)**

12. State true or false with proper justification.

    (i) There does not exist an analytic function whose real part is $x^3 - 3x^2 y - y^3$. **(M.U. 2013)**

    (ii) If $f(z)$ and $\overline{f(z)}$ are both analytic then $f(z)$ is a constant function. **(M.U. 1995)**

13. Show that an analytic function with constant modulus is constant. **(M.U. 1994, 99, 2003)**

14. If $f(z) = u + iv$ is analytic in a region R then show that

    (i) u and v are harmonic functions and

    (ii) u = constant and v = constant intersect orthogonally. **(M.U. 1994, 97, 98, 2003)**

15. If $f(z) = u + iv$ is analytic in a region R show that

    (i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

    (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ **(M.U. 1995)**

16. If $f(z) = u + iv$ is analytic and $z = re^{i\theta}$, show that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$ **(M.U. 1996, 2002, 03)**

17. If $f(z) = u + iv$ is an analytic function, prove that

    (i) u = constant and v constant are orthogonal trajectories.

    (ii) u and v are harmonic functions.

    (iii) $f(z)$ is constnat if $f'(z) = 0$.

    (iv) $u - iv$ is also analytic. **(M.U. 2004)**

18. If u and v are conjugate harmonic functions, prove that uv is also harmonic. **(M.U. 2003)**

❀ ❀ ❀

CHAPTER

6 Conformal Mapping

1. Mapping of A Complex Function

We know that a real continuous function $y = f(x)$ can be represented by a smooth curve on x-y plane. We also know that a real continuous function $z = f(x, y)$ can be represented by a surface in space. But, how can we represent the complex function $w = f(z)$ i.e. $u + iv = f(x + iy)$ graphically ? Here, we have two dependent variables u, v and two independent variable x, y i.e. in all four variables and to represent such a function graphically we will need four diamensional space. Since this is impractical we take two complex planes - one z-plane and another w-plane. In the z-plane we plot the point $z = x + iy$ and in w-plane we plot the corresponding point $w = u + iv$. When the point z describes a curve C in z-plane the corresponding point w will describe a curve C' in w-plane, if the correspondence is one-one. Thus, **the complex function $w = f(z)$ defines a mapping or transformation of the z-plane into the w-plane.** The corresponding points, curves or regions of the two planes are called the **images** of each other.

Example 1 : Determine the region D' in w-plane corresponding to the region D in the z-plane given by $x = 0$, $y = 0$, $x = 1$, $y = 1$ under the transformation $w = z + (2 - i)$.

Sol. : Since $w = z + (2 - i)$, we have

$$u + iv = x + iy + 2 - i = (x + 2) + i(y - 1)$$

$$\therefore \quad u = x + 2, \ v = y - 1.$$

When $x = 0$, $u = 2$, when $x = 1$, $u = 3$ and when $y = 0$, $v = -1$, when $y = 1$, $v = 0$.

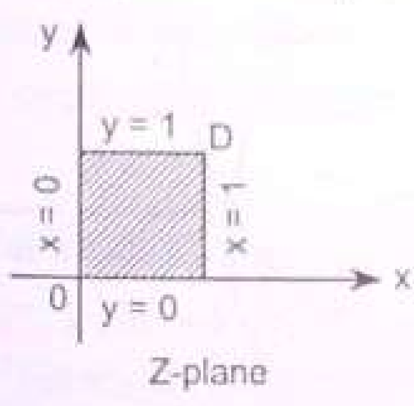


Fig. 6.1 (a)

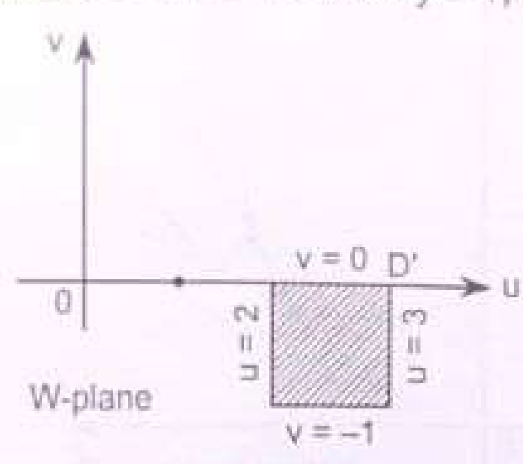


Fig. 6.1 (b)

The region D in the z-plane, is transformed into the region D' in the w-plane.

Example 2 : Determine the region in the w-plane into which the region bounded by the lines $x = 0$, $y = 0$, $x = 1$, $y = 1$ in the z-plane is transformed under the transformation

$$w = (1 + i)\, z - (1 - i).$$

Sol. : Since $w = (1 + i)\, z - (1 - i)$

We have $u + iv = (1 + i)(x + iy) - (1 - i)$

$$\therefore \quad u + iv = (x - y - 1) + i(x + y + 1) \qquad \therefore \quad u = x - y - 1, \ \ v = x + y + 1$$

When $x = 0$, $u = -y - 1$ and $v = y + 1$. Adding $u + v = 0$ which is a straight line.
When $y = 0$, $u = x - 1$, $v = x + 1$. Subtracting $u - v = -2$, which is a straight line.
When $x = 1$, $u = 1 - y - 1$, $v = 1 + y + 1$. Adding $u + v = 2$ which is a straight line.
When $y = 1$, $u = x - 1 - 1$, $v = x + 1 + 1$. Subtracting $u - v = -4$ which is a straight line.

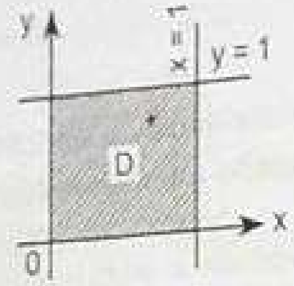


Fig. 6.2 (a)

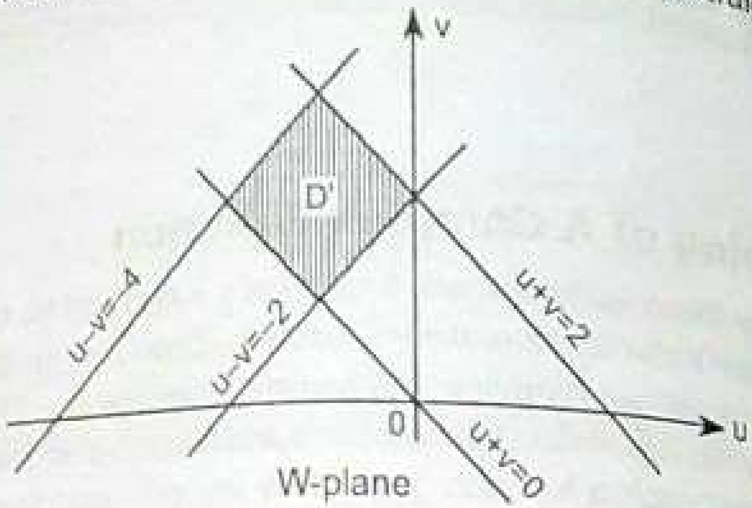


Fig. 6.2 (b)

The region D in the z-plane is transformed into the region D' in the w-plane.

2. Conformal Mapping

Let two curves C_1 and C_2 in the z-plane intersect at the point P and let the corresponding curves C'_1 and C'_2 in the w-plane intersect at the point P' under the mapping $w = f(z)$.

If the angle of intersection of the curves C_1 and C_2 at P is equal to the angle of intersection of the curves C'_1 and C'_2 at P' both in magnitude and sense then the mapping is said to be CONFORMAL at P.

If the angle between C_1, C_2 at P is equal to the angle between C'_1, C'_2 at P' in magnitude only then the mapping is said to be **isogonal.**

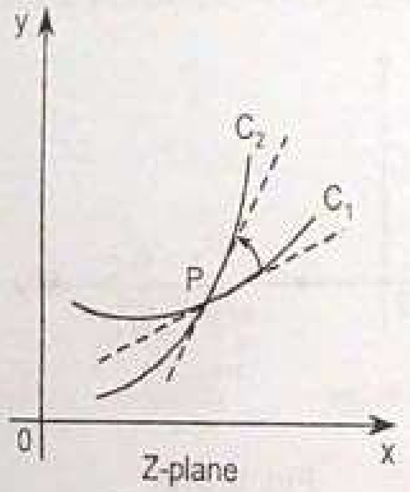


Fig. 6.3 (a)

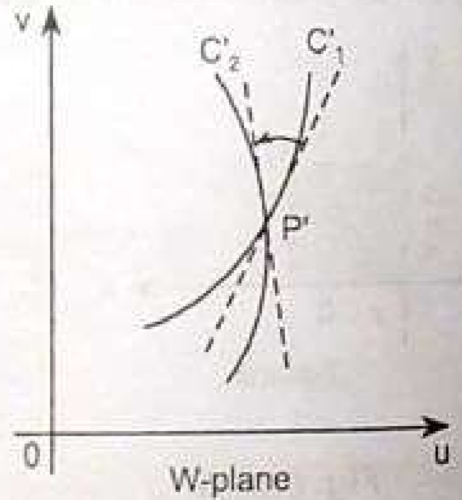


Fig. 6.3 (b)

3. Conformal Property of Analytic Functions

Theorem : If $w = f(z)$ is an analytic function and $f'(z) \neq 0$ in a region R of z-plane then the mapping $w = f(z)$ is conformal at all points of R.

Proof : Let $P(z)$ be a point in the region R of z-plane and $P'(w)$ be the corresponding point of the region R' of w-plane. Let $Q(z + \delta z)$ be a neighbouring point on a curve C in R through P and $Q'(w + \delta w)$ be the corresponding neighbouring point on the curve C' in R.

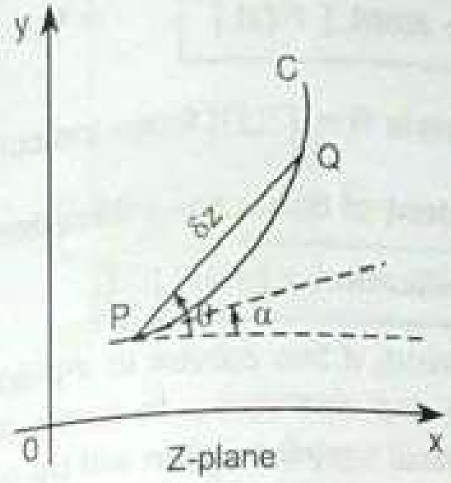


Fig. 6.4 (a)

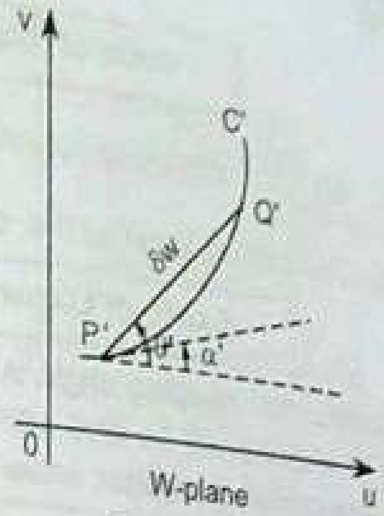


Fig. 6.4 (b)

Let the complex numbers δz and δw be expressed in exponential form as

$$\delta z = re^{i\theta} \text{ and } \delta w = r'e^{i\theta'}. \quad \therefore \quad \frac{\delta w}{\delta z} = \frac{r'}{r} e^{i(\theta' - \theta)}$$

Let the tangents at P and P' make angles α and α' with the x and u axis respectively. Hence, as $\delta z \to 0$, $\theta \to \alpha$, $\delta w \to 0$ and $\theta' \to \alpha'$.

$$\therefore \quad f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta z \to 0} \frac{r'}{r} e^{i(\theta' - \theta)}$$

Since $f(z)$ is analytic $f'(z) \neq 0$. Let $f'(z) = Re^{i\Phi}$. Then $R = |f'(z)|$ and Φ = amplitude of $f'(z)$.

$$\therefore \quad Re^{i\Phi} = \lim_{\delta z \to 0} \frac{r'}{r} e^{i(\theta' - \theta)} \quad \therefore \quad R = \lim_{\delta z \to 0} \frac{r'}{r} \text{ and } \Phi = \lim_{\delta z \to 0} (\theta' - \theta) = \alpha' - \alpha$$

Let us consider another curve C_1 through P and let C'_1 be the corresponding curve in the w-plane through P'. If the tangent to the curve C_1 makes an angle β with the x-axis and the tangent to the curve C'_1 makes an angle β' with the u-axis then as above.

$$\Phi = \beta' - \beta \quad \therefore \quad \alpha' - \alpha = \beta' - \beta \quad \therefore \beta - \alpha = \beta' - \alpha' = \gamma \text{ say.}$$

Thus, the angle between the curves C, C_1, in z-plane is equal to the angle between the curves C', C'_1 in the w-plane in magnitude and sense.

Hence, the mapping by an analytic function $w = f(z)$ is conformal at each points where $f'(z) \neq 0$.

Notes ....

1. A point at which $f'(z) = 0$ is called a **critical point** of the transformation $w = f(z)$.
2. Since $\alpha' = \alpha + \Phi$ the curve C' at P' is rotated through an angle Φ = amp. $[f'(z)]$ under the transformation. This angle is called **the angle of rotation.**

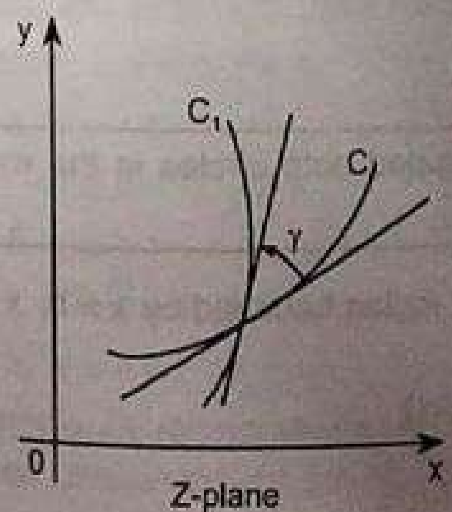


Fig. 6.5 (a)

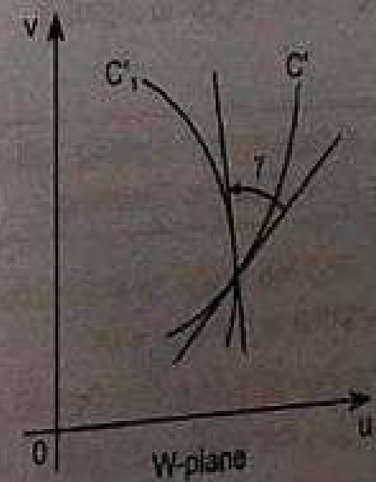


Fig. 6.5 (b)

$$\therefore \text{ angle of rotation} = \text{ampl.} [\ f'(z)\ ]$$

3. Since $R = \lim_{\delta z \to 0} \frac{r'}{r}$ a small segment in w-plane is $R = |f'(z)|$ times the corresponding small segment in z-plane. The ratio R is called the **coefficient of linear magnification.**

$$\therefore \text{ Coeff. of Linear Magnification} = |\ f'(z)\ |$$

4. Since in conformal mapping angle is preserved, if two curves in z-plane are orthogonal then their images in w-plane will be also orthogonal and vice versa. In particular since the lines $x = \lambda$, $y = \mu$ are orthogonal their images under conformal transformation will be orthogonal.

Example 1 : Find the angle of rotation and the coefficient of magnification at $z = 2 + i$ under the transformation $w = z^2$.

Sol. : $\because\ w = z^2,\ \frac{dw}{dz} = 2z$

The transformation is conformal except at $z = 0$.

Hence, the transformation is conformed at $z = 2 + i$

$\therefore\ f'(z_0) = 2z_0 = 2(2 + i) = 4 + 2i$

Angle of rotation = ampl. $[\ f'(z_0)\ ]$

$$= \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{4} = \tan^{-1} 0.5.$$

Coefficient of magnification $= |f'(z_0)| = \sqrt{16 + 4} = 2\sqrt{5}$.

Example 2 : Find the angle of rotation and the coefficient of magnification at $z = 1 + i$ under the transformation $w = z^2$. **[Ans. :** $\pi/4$; $2\sqrt{2}$]

4. Some Standard Transformations

(i) **Translation** $\boxed{w = z + c}$ where c is a complex constant.

Let $z = x + iy$, $c = a + ib$ and $w = u + iv$

Now, $u + iv = (x + iy) + (a + ib)$

$= (x + a) + i(y + b)$

$\therefore\ u = x + a,\ v = y + b.$

Thus, the transformation $w = z + c$ is simply a **translation** of the axes and as such preserves **the shape and size** of the region in z-plane.

Remark ....

In particular, circles in the z-plane will be mapped onto circles in the w-plane under the transformation $w = z + c$ (See Ex. 2 below).

Example 1 : Find the image of the rectangular region bounded by $x = 0$, $x = 3$, $y = 0$, $y = 2$ under the transformation $w = z + (1 + i)$. **(M.U. 2015)**

Sol. : Since $u + iv = (x + iy) + (1 + i) = (x + 1) + i(y + 1)$

$\therefore\ u = x + 1,\quad v = y + 1$

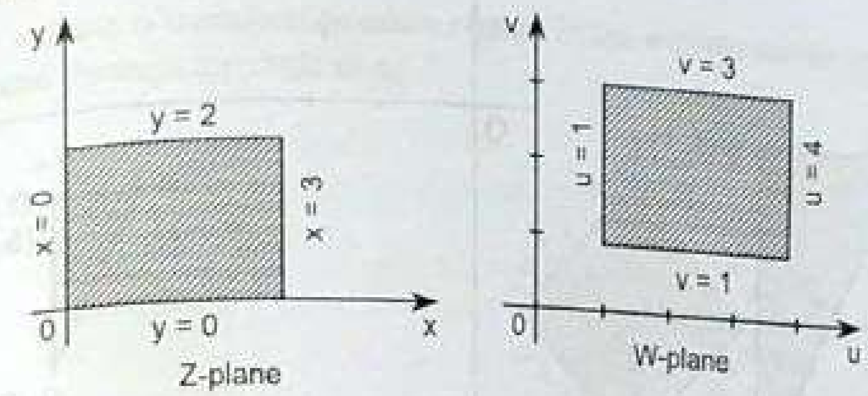


Fig. 6.6 (a) Fig. 6.6 (b)

When $x = 0$, $u = 1$ and when $x = 3$, $u = 4$.

When $y = 0$, $v = 1$ and when $y = 2$, $v = 3$.

Thus, the rectangular region of sides 3 and 2 in the z-plane is transformed into a rectangular region of sides 3 and 2 in the w-plane. Only the rectangle is shifted in the position in the w-plane preserving its size and shape.

Example 2 : Find the image of the circle $|z| = 2$ under the transformation $w = z + 3 + 2i$. **(M.U. 2002)**

Sol. : We have $w = z + 3 + 2i$.

$\therefore\ u + iv = (x + iy) + (3 + 2i)$

$= (x + 3) + i(y + 2)$

$\therefore\ u = x + 3,\ v = y + 2.$

But $|z| = 2$ means $\sqrt{x^2 + y^2} = 2\ \therefore\ x^2 + y^2 = 4$.

Hence, $(u - 3)^2 + (v - 2)^2 = 4$.

Thus, the circle $x^2 + y^2 = 4$ will be transformed to the circle $(u - 3)^2 + (v - 2)^2 = 4$.

Note ....

Note that the circle is transformed into a circle with the same radius but with translation of the centre from (0, 0) to (3, 2).

Example 3 : Find the image of the triangular region whose vertices are i, $1 + i$, $1 - i$ under the transformation $w = z + 4 - 2i$. Draw the sketch. **(M.U. 2015)**

Sol. : We have $w = z + - 2i$

$\therefore\ u + iv = x + iy + 4 - 2i = (x + 4) + i(y - 2)$

Equating real and imaginary parts $u = x + 4$, $v = y - 2$.

Now, the point i is $x = 0$, $y = 1$.

$\therefore\ u = 4,\ v = -1$

The point $1 + i$ is $x = 1$, $y = 1$.

$\therefore\ u = 5,\ v = -1$

The point $1 - i$ is $x = 1$, $y = -1$.

$\therefore\ u = 5,\ v = -3$

Thus, the triangular region whose vertices are i, $1 + i$, $1 - i$ in the x-y plane is transformed into a triangular region whose vertices are $(4, -1)$, $(5, -1)$ and $(5, -3)$ in the u-v plane.

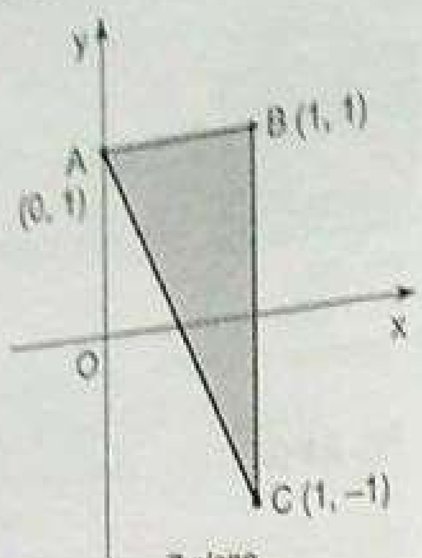


Fig. 6.7 (a)

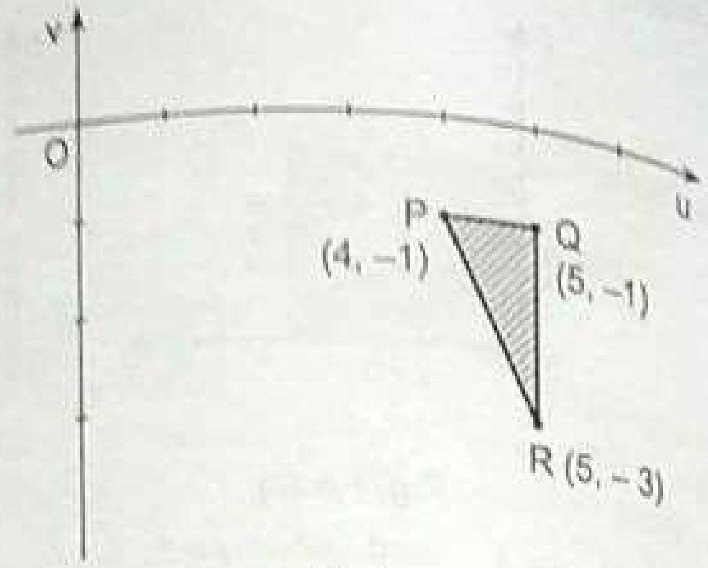


Fig. 6.7 (b)

EXERCISE - I

1. Find the image of the rectangular region bounded by $x = 0,\ y = 0,\ x = 1,\ y = 2$ under the transformation $w = z + (2 - i)$. Draw the sketch.

[**Ans. :** A rectangular region bounded by $u = 2,\ u = 3,\ v = -1,\ v = 1$]

2. Find the image of the triangular region whose vertices are $i,\ 1 + i,\ 1 - i$ under the transformation $w = z + 4 - 2i$. **(M.U. 2015)**

[**Ans. :** The triangular region whose vertices are $4 - i,\ 5 - i,\ 5 - 3i$]

3. Find the map of the circle $|z| = k$ under the transformation $w = z + 4 + 3i$.

[**Ans. :** $u + iv = (x + 4) + i(y + 3)\quad \therefore\ x = u - 4,\ y = v - 3$

$\therefore\ x^2 + y^2 = k^2$ becomes $(u - 4)^2 + (v - 3)^2 = k^2$]

(ii) Rotation And Magnification $\boxed{w = cz}$ where c is a complex number.

Let $w = Re^{i\Phi},\ z = re^{i\theta},\ c = \rho e^{i\alpha}$

Now $Re^{i\Phi} = re^{i\theta} \cdot \rho e^{i\alpha} = r\rho e^{i(\theta + \alpha)} \quad \therefore\ R = r\rho,\ \Phi = \theta + \alpha$

Thus, the transformation maps a point $P(r, \theta)$ in the z-plane into a point $P'(r\rho, \theta + \alpha)$ in the w-plane. The radius vector r of the point P is **magnified by $\rho = |c|$ and is rotated through an angle α = amp. (c)**. Hence, it maps figures in the z-plane into **geometrically similar figures** in the w-plane. But the figures are magnified and rotated.

Remark ....

In particular circles in the z-plane will be mapped onto circles in the w-plane under the transformation $w = cz$.

Example 1 : Find the image of the region bounded by $x = 0,\ x = 2,\ y = 0,\ y = 2$ in the z-plane under transformation $w = (1 + i)z$. **(M.U. 2005, 13, 14)**

Sol. : Since $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} \cdot e^{i\pi/4}$

$\therefore\ w = cz$ becomes $Re^{i\Phi} = \sqrt{2} \cdot e^{i\pi/4} \cdot re^{i\theta} = \sqrt{2} \cdot r\, e^{i(\theta + \pi/4)}$

The square in z-plane is transformed into a square in the w-plane but its sides are magnified by $\sqrt{2}$ and it is rotated through an angle of $\pi / 4$.

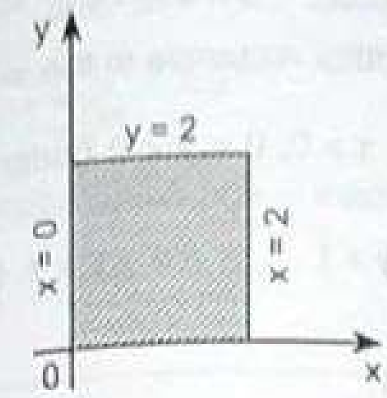


Fig. 6.8 (a)

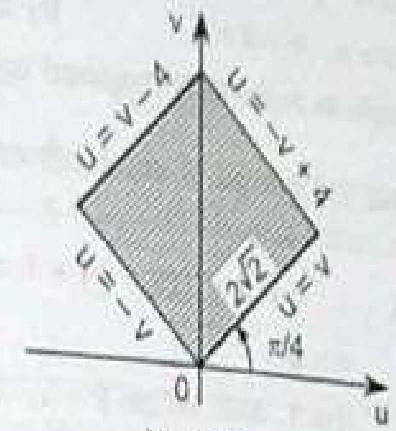


Fig. 6.8 (b)

Alternatively We may write $z = x + iy$.

Then $u + iv = (1 + i)(x + iy) = (x - y) + i(x + y) \quad \therefore\ u = x - y,\ v = x + y$

When $x = 0,\ u = -y,\ v = y \quad \therefore\ u = -v$

When $x = 2,\ u = 2 - y,\ v = 2 + y \quad \therefore\ u + v = 4$

When $y = 0,\ u = x,\ v = x \quad \therefore\ u = v$

When $y = 2,\ u = x - 2,\ v = x + 2 \quad \therefore\ u - v = -4$

Thus, we get the same square as above.

Example 2 : Consider the transformation $w = (1 + i)z + (2 - i)$ and determine the region in the w-plane into which the rectangular region bounded by $x = 0,\ y = 0,\ x = 1,\ y = 2$ in the z-plane is mapped under this transformation. Sketch the regions. **(M.U. 2000)**

Sol. : We have $u + iv = (1 + i)(x + iy) + 2 - i$

$= x + ix + iy - y + 2 - i$

$= (x - y + 2) + i(x + y - 1)$

$\therefore\ u = x - y + 2,\quad v = x + y - 1$

When $x = 0,\quad u = -y + 2,\ v = y - 1,$

by addition $\quad u + v = 1.$

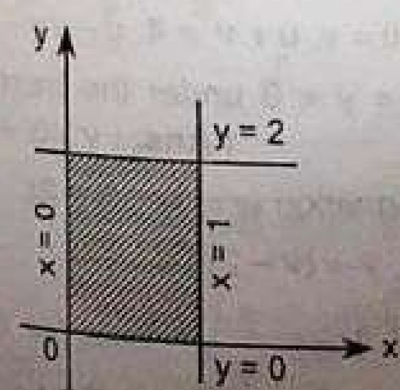


Fig. 6.9 (a)

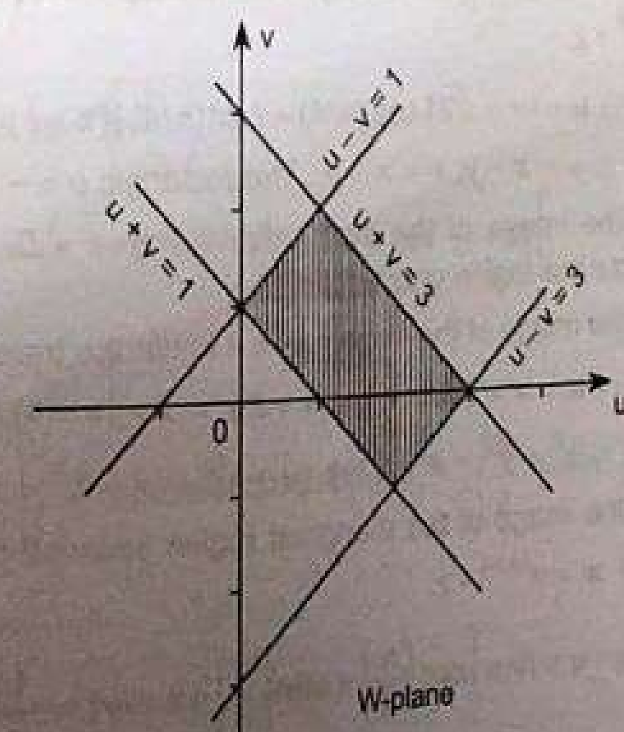


Fig. 6.9 (b)

When $y = 0$ $u = x + 2,\ v = x - 1,$ by subtraction, $u - v = 3$

When $x = 1,$ $u = -y + 3,\ v = y,$ by addition, $u + v = 3$

When $y = 2,$ $u = x,\ v = x + 1,$ by subtraction, $u - v = -1$

Thus, the rectangle in z-plane is mapped onto another rectangle in the w-plane but rotated.

Example 3 : Find the image of semi-infinite strip $x > 0,\ 0 < y < 2$ under the transformation $w = iz + 1$.

Sol. : $\because\ w = iz + 1 \quad \therefore\ u + iv = i(x + iy) + 1 = ix + y + 1 \quad \therefore\ u = -y + 1$ and $v = x$

When $x > 0,$ $v > 0$

When $y = 0,$ $u = 1$ and when $y = 2,$ $u = -1.$

When $0 < y < 2,$ $-1 < u < 1.$

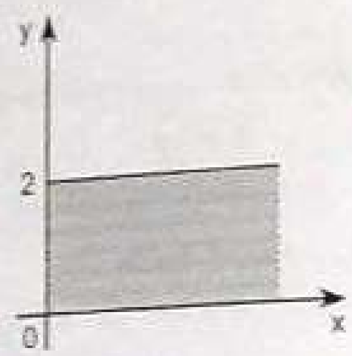


Fig. 6.10 (a)

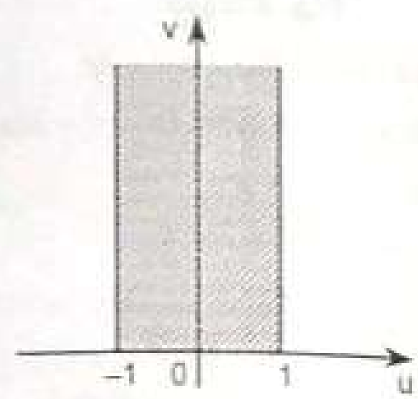


Fig. 6.10 (b)

Thus, the strip $x > 0,\ 0 < y < 2$ in the z-plane s mapped on to the strip $v > 0,\ -1 < u < 1.$

EXERCISE - II

1. Find the image of the triangle whose vertices are (0, 1), (2, 2) (2, 0) under the transformation $w = 4z$. [**Ans. :** Use $u + iv = 4x + 4iy$. The triangle whose vertices are (0, 4), (8, 8), (8, 0)]
2. Find the image of the triangle whose vertices $i,\ 1 + i,\ 1 - i$ under the transformation $w = 3z + 4 - 2i$. [**Ans. :** The triangle whose vertices are (4, 1), (7, 1), (7, −5)]
3. Find the image of the rectangle bounded by $x = 0,\ y = 0,\ x = 2,\ y = 3$ under the transformation $w = \sqrt{2} \cdot e^{i\pi/4} \cdot z$ (M.U. 1999)

   [**Ans. :** $u + iv = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)](x + iy) = (1 + i)(x + iy)$

   $\therefore\ u = x - y,\ v = x + y$. The rectangle $u = -v,\ u = v,\ u + v = 4,\ u - v = -6$]
4. Find the image of the semi-infinite strip $x > 0,\ 1 < y < 3$ under the transformation $w = iz + 2$. Show the region graphically. [**Ans. :** $v > 0,\ -1 < u < 1$]
5. Find the image of the circle $|\,z\,| = k$ under the transformation $w = 3z + 4 + 2i$.

   [**Ans. :** $u + iv = 3(x + iy) + (4 + 2i)\ \therefore\ x = (u - 4)/3,\ y = (v - 2)/3$

   $\therefore\ x^2 + y^2 = k^2$ gives $(u - 4)^2 + (v - 2)^2 = 9k^2$]
6. Find the image of the triangular region bounded by $x = 0,\ y = 0,\ x + y = \sqrt{2}$ under the transformation $w = e^{i\pi/4} \cdot z$

   [**Ans. :** $u + iv = \left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right](x + iy) = \frac{1}{\sqrt{2}}(x - y) + \frac{i}{\sqrt{2}}(x + y)$

   The triangular region bounded by $u = v,\ u = -v,\ v = 1.$]

(iii) Inversion and Reflection $\boxed{w = \dfrac{1}{z}}$

Let $z = re^{i\theta}$ and $w = Re^{i\Phi}$

$$\therefore\ Re^{i\Phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \qquad \therefore\ R = \frac{1}{r},\ \Phi = -\theta$$

Thus, the transformation $w = \frac{1}{z}$ maps the point $P(r, \theta)$ in the z-plane onto the point $P'\left(\frac{1}{r}, -\theta\right)$ in the w-plane.

Remark ....

If $r = k$, then $R = \frac{1}{k} = k'$, hence, the circles in the z-plane will be mapped onto circles in the w-plane under the transformation $w = \frac{1}{z}$.

Example 1 : Find the images of the following under the transformation $w = \frac{1}{z}$.

(i) $z = \frac{\sqrt{5}}{2} + i$ (ii) $z = \frac{2\sqrt{5}}{9} + \frac{4}{9}i$ (iii) $|\,z\,| = 1$

(iv) $|\,z\,| < 1$ (v) $|\,z\,| > 1$ (vi) $|\,z\,| = k.$

Sol. : Since $w = \frac{1}{z}$ as above $Re^{i\Phi} = \frac{1}{r} \cdot e^{-i\theta}\ \therefore\ R = \frac{1}{r},\ \Phi = -\theta$

(i) Now, $w = \frac{\sqrt{5}}{2} + i = \frac{3}{2}e^{i\theta}$ where $\theta = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$$\left[\because\ x + iy = \sqrt{x^2 + y^2}\,e^{i\theta} \text{ where } \theta = \tan^{-1}(y/x)\right]$$

$$\therefore\ R = \frac{1}{r} = \frac{2}{3} \text{ and } \Phi = -\theta = -\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

Thus, the point $P_1\left(\frac{3}{2}, \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ is mapped onto the point $P'_1\left(\frac{2}{3}, -\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$.

(ii) $z = \frac{2\sqrt{5}}{9} + \frac{4}{9}i = \frac{2}{3}e^{i\theta}$ where $\theta = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$$\left[\because\ x + iy = \sqrt{x^2 + y^2}\,e^{i\theta} \text{ where } \theta = \tan^{-1}(y/x)\right]$$

$$\therefore\ R = \frac{1}{r} = \frac{3}{2} \text{ and } \Phi = -\theta = -\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

Thus, the point $P_2\left(\frac{2}{3}, \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ is mapped onto the point $P'_2\left(\frac{3}{2}, -\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$.

(iii) $|z| = 1$ is a circle $r = 1 \quad \therefore\ R = \frac{1}{r} = 1 \quad \therefore\ |w| = 1$

(iv) $|z| < 1$ is the interior of the circle $r = 1$ *i.e.* the region for which $r < 1$.

$\therefore\ R = \frac{1}{r} > 1$ *i.e.* the exterior of the circle $|\,w\,| = 1.$

(v) $|z| > 1$ is the exterior of the circle $r = 1$ *i.e.* the region for which $r > 1$.

$\therefore R = \frac{1}{r} < 1$ *i.e.* the interior of the circle $|w| = 1$.

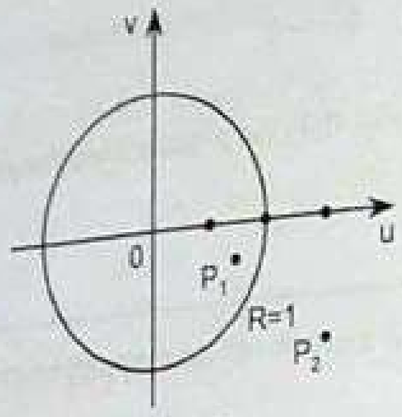


Fig. 6.11 (a)

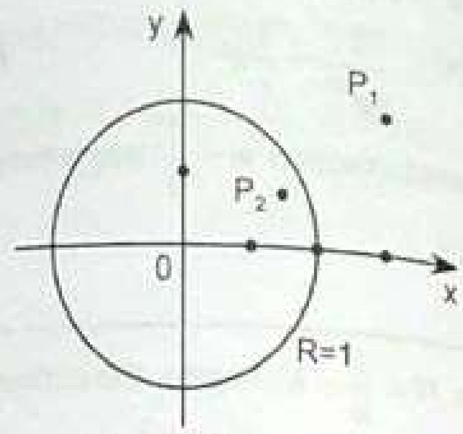


Fig. 6.11 (b)

Thus, the unit circle $|z| = 1$ of the *z*-plane is mapped onto the unit circle $|w| = 1$ of the *w*-plane, the interior of the unit circle of *z*-plane is mapped onto the exterior of the unit circle of the *w*-plane and the exterior of the unit circle of the *z*-plane is mapped onto the interior of the unit circle of the *w*-plane.

(vi) $|z| = k$ is a circle $r = k$ $\therefore R = \frac{1}{r} = \frac{1}{k}$ $\therefore |w| = \frac{1}{k}$.

Thus, a circle with centre at the origin in the *z*-plane is mapped onto the circle with centre at the origin in the *w*-plane.

If the circle with centre at the origin in the *z*-plane is inside (outside) the unit circle $|z| = 1$ then its image in the *w*-plane is a circle with centre at the origin outside (inside) the unit circle $|w| = 1$.

Remark ....

Let us superimpose *w*-plane on *z*-plane. Let P be (r, θ) and P_1 be $\left(\frac{1}{r}, \theta\right)$. Then as $OP \cdot OP_1 = r \cdot \frac{1}{r} = 1$, the point P_1 is the **inverse** of P w.r.t. the unit circle. If P' is $\left(\frac{1}{r}, -\theta\right)$, then P' is the **reflection** of P_1 in the *x*-axis.

Thus, the transformation $w = \frac{1}{z}$ is an inversion of z w.r.t. the unit circle and reflection of the inverse in the real axis thereafter.

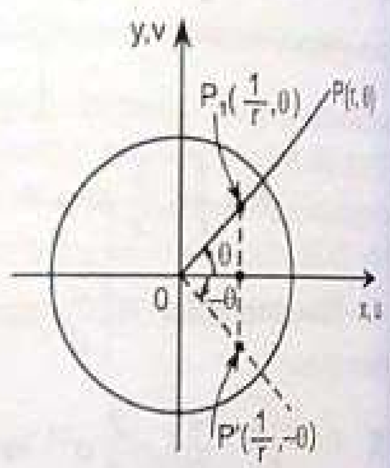


Fig. 6.12

EXERCISE - III

1. Find the images of the following under the transformation $w = \frac{1}{z}$.

(i) The points (1, 0), (0, 1) (–1, 0), (0, –1)

(ii) The points (2, 2), $\left(\frac{1}{2}, \frac{1}{2}\right)$ (iii) $|z| = 1$ (iv) $|z| = 3$ (v) $|z| = \frac{1}{3}$

[**Ans. :** (i) (1, 0), (0, – 1), (– 1, 0), (0, 1) (ii) $\left(\frac{1}{4}, -\frac{1}{4}\right)$, (1, – 1),

(iii) $|w| = 1$, (iv) $|w| = \frac{1}{3}$ (v) $|w| = 3$]

2. Show that under the transformation $w = \frac{1}{z}$ the interior (exterior) of the circle $|z| = 1$ is mapped onto the exterior (interior) of the circle $|w| = 1$.

(iv) Mapping under $w = \frac{k}{z}$ where, k is real

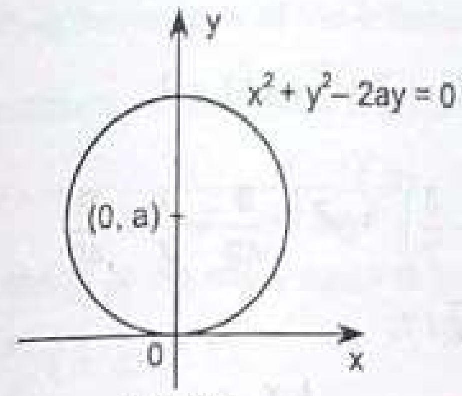
Example 1 : Find the image of $|z - ai| = a$ under the transformation $w = 1/z$.

(M.U. 1999, 2003, 04, 07, 08)

Sol. : The given curve is $|z - ai| = a$

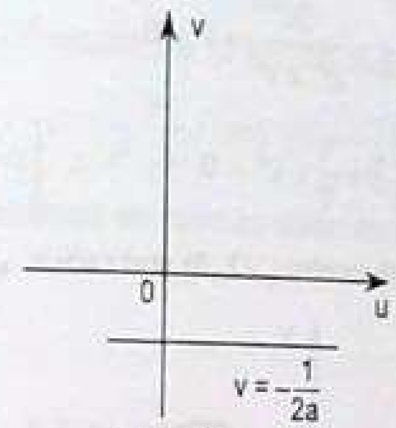
i.e. $|(x + iy) - ai| = a$ *i.e.* $|x + i(y - a)| = a$

$$\therefore x^2 + (y - a)^2 = a^2 \text{ i.e. } x^2 + y^2 - 2ay = 0 \quad \text{.............. (i)}$$



Z-plane

Fig. 6.13 (a)



W-plane

Fig. 6.13 (b)

This is a circle with centre at (0, *a*) and radius *a*.

Now, $w = \frac{1}{z}$ *i.e.* $z = \frac{1}{w}$ $\therefore x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$

$$\therefore x = \frac{u}{u^2 + v^2} \text{ and } y = -\frac{v}{u^2 + v^2} \quad \text{.............. (ii)}$$

Putting the values of x and y from (i) in (ii)

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{2av}{(u^2 + v^2)} = 0$$

$$\therefore \frac{u^2 + v^2}{(u^2 + v^2)^2} + \frac{2av}{u^2 + v^2} = 0 \text{ i.e. } 1 + 2av = 0 \text{ i.e. } v = -\frac{1}{2a} \quad \text{.............. (iii)}$$

which is a straight line.

Thus, the given circle in *z*-plane is mapped onto a straight line in *w*-plane.

Remark ....

$w = 1/z$ transforms a circle passing through the origin in the *z*-plane to a straight line not passing through the origin in the *w*-plane. **(M.U. 2003)**

Example 2 : Find the image of the circle $(x-3)^2 + y^2 = 2$ under the transfor-mation $w = 1/z$. **(M.U. 1996, 2006, 07)**

Sol. : The given circle has centre at (3, 0) and radius $\sqrt{2}$.

Now $w = \dfrac{1}{z}$ $\therefore$ $z = \dfrac{1}{w}$ $\therefore$ $x + iy = \dfrac{1}{u+iv} = \dfrac{u - iv}{u^2+v^2}$

$$\therefore \quad x = \frac{u}{u^2+v^2} \text{ and } y = -\frac{v}{u^2+v^2} \qquad \ldots\ldots\ldots (i)$$

The equation of the circle is $x^2 + y^2 - 6x + 7 = 0$ ……… (ii)

Eliminating x and y from (i) and (ii) we get

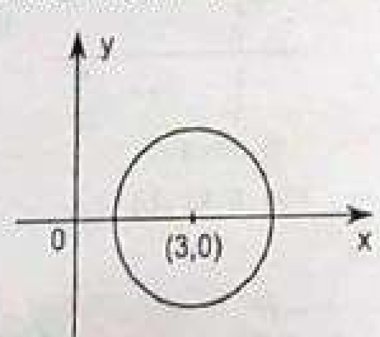
$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - \frac{6u}{(u^2+v^2)} + 7 = 0$$

$$\therefore \quad \frac{u^2+v^2}{(u^2+v^2)^2} - \frac{6u}{(u^2+v^2)} + 7 = 0$$

$$\therefore \quad \frac{1}{u^2+v^2} - \frac{6u}{u^2+v^2} + 7 = 0 \qquad 1 - 6u + 7(u^2+v^2) = 0$$

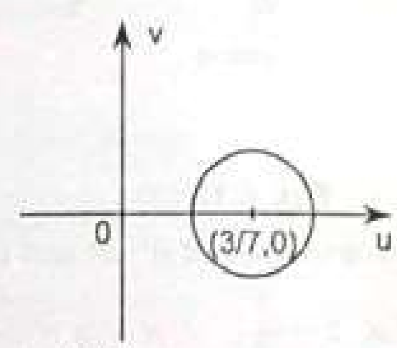
$$\therefore \quad u^2 - \frac{6}{7}u + \frac{1}{7} + v^2 = 0 \qquad \therefore \quad \left(u - \frac{3}{7}\right)^2 + v^2 = \frac{9}{49} - \frac{1}{7} = \frac{2}{49}$$

which is a circle with centre (3/7, 0) and radius $\sqrt{2}/7$.



Z-plane

Fig. 6.14 (a)



W-plane

Fig. 6.14 (b)

Example 3 : Show that the function $w = 4/z$ transforms the straight lines $x = c$ in the z-plane into circles in the w-plane. **(M.U. 1993, 2016)**

Sol. : We have $w = \dfrac{4}{z} = \dfrac{4}{x+iy}$

$$\therefore \quad w = u + iv = \frac{4}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{4(x-iy)}{x^2+y^2}$$

$$u = 4 \cdot \frac{x}{x^2+y^2}, \quad v = -4 \cdot \frac{y}{x^2+y^2}$$

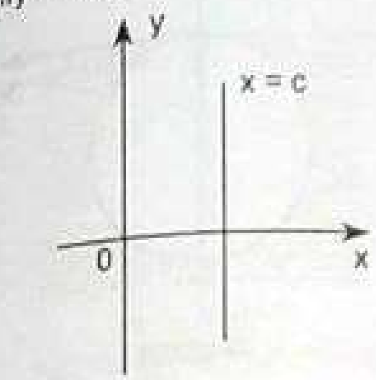
When $x = c$, $u = 4\dfrac{c}{c^2+y^2}$ and $v = -\dfrac{4y}{c^2+y^2}$

We have to eliminate y between these two. Now, squaring and adding, we get,

$$u^2 + v^2 = 16\frac{(c^2+y^2)}{(c^2+y^2)^2} = \frac{16}{c^2+y^2} = 16 \cdot \frac{u}{4c} = \frac{4u}{c}$$

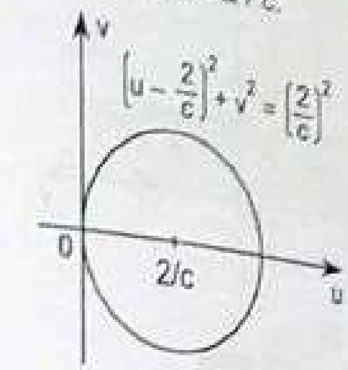
$$\therefore \quad u^2 + v^2 - \frac{4u}{c} = 0 \quad \therefore \quad \left(u - \frac{2}{c}\right)^2 + v^2 = \frac{4}{c^2} = \left(\frac{2}{c}\right)^2$$

This is a family of circles with centre at (2 / c, 0) and radius 2 / c.



Z-plane

Fig. 6.15 (a)



W-plane

Fig. 6.15 (b)

Example 4 : Find the images of the infinite strips (i) $\dfrac{1}{2b} < y < \dfrac{1}{2a}$, (ii) $0 < y < \dfrac{1}{2a}$ under the transformation $w = 1/z$. Show the regions graphically ($a > 0$, $b > 0$, $a < b$).

Sol. : We have $u + iv = \dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2}$ $\therefore$ $u = \dfrac{x}{x^2+y^2}$ and $v = -\dfrac{y}{x^2+y^2}$

To obtain the image of the line y equal to a constant we have to eliminate x from above.

Now $\dfrac{u}{v} = -\dfrac{x}{y}$ i.e. $x = -\dfrac{uy}{v}$

$$\therefore \quad v = -\frac{y}{\dfrac{u^2y^2}{v^2} + y^2} = -\frac{v^2}{u^2+v^2} \cdot \frac{1}{y} \qquad \therefore \quad y = -\frac{v}{u^2+v^2}$$

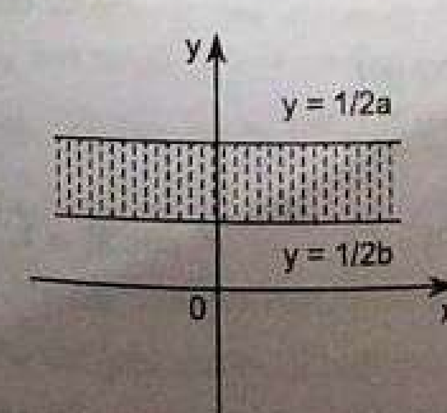
If $y = \dfrac{1}{2b}$, $-\dfrac{v}{u^2+v^2} = \dfrac{1}{2b}$ $\therefore$ $u^2 + v^2 + 2bv = 0$ $\therefore$ $u^2 + (v+b)^2 = b^2$

The line $y = 1/2b$ is transformed into a circle with centre $(0, -b)$ and radius b

If $y = \dfrac{1}{2a}$, $-\dfrac{v}{u^2+v^2} = \dfrac{1}{2a}$ $\therefore$ $u^2 + v^2 + 2av = 0$ $\therefore$ $u^2 + (u+a)^2 = a^2$

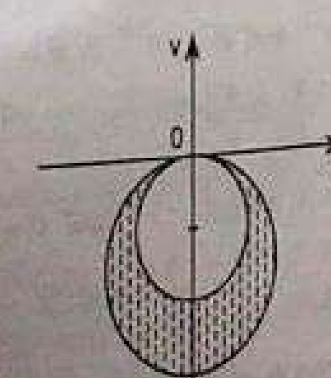
The line $y = 1/2a$ is transformed into a circle with centre $(0, -a)$ and radius a.

Hence, the region $\dfrac{1}{2b} < y < \dfrac{1}{2a}$ is mapped into the region between the above two circles as shown in the figure below.



Z-plane

Fig. 6.16 (a)



W-plane

Fig. 6.16 (b)

(iii) As b becomes larger and larger the line $y = 1/2b$ comes closer and closer to the x-axis and the circle with centre b becomes larger and larger.

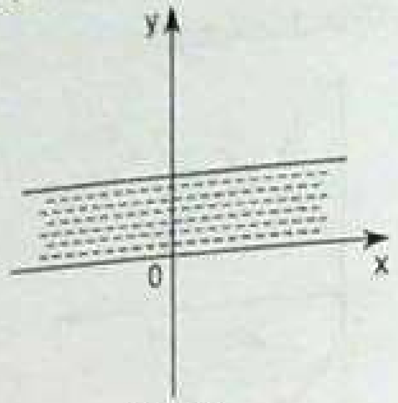


Fig. 6.17 (a)

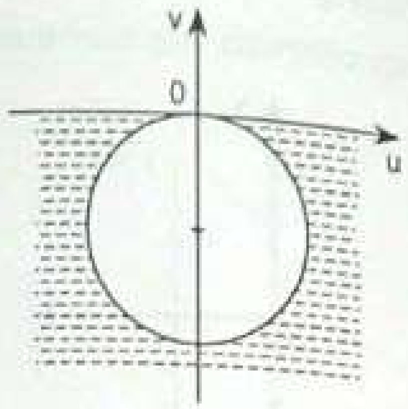


Fig. 6.17 (b)

Hence, the strip $0 < y < \dfrac{1}{2a}$ is mapped onto the region outside the circle $u^2 + (v + a)^2 = a^2$ below the real line.

Example 5 : Prove that the circle $|z - 2| = 3$ is mapped into the circle $\left|w + \dfrac{2}{5}\right| = \dfrac{9}{25}$ under the transformation $w = 1/z$.

Sol. : We have $|z - 2| = 3$

$$\therefore \left|\frac{1}{w} - 2\right| = 3 \quad \therefore \left|\frac{1-2w}{w}\right| = 3$$

$$\therefore |1 - 2w| = 3|w| \quad \therefore |1 - 2(u + iv)| = 3|u + iv|$$

$$\therefore (1 - 2u)^2 + 4v^2 = 9(u^2 + v^2) \quad \therefore 5u^2 + 5v^2 + 4u - 1 = 0$$

$$\therefore u^2 + v^2 + \frac{4u}{5} - \frac{1}{5} = 0$$

$$\left(u + \frac{2}{5}\right)^2 + v^2 = \frac{1}{5} + \frac{4}{25} = \frac{9}{25} \qquad \therefore \left|\left(u + \frac{2}{5}\right) + iv\right| = \frac{9}{25}$$

$$\therefore \left|(u + iv) + \frac{2}{5}\right| = \frac{9}{25} \qquad \therefore \left|w + \frac{2}{5}\right| = \frac{9}{25}$$

(Alternatively you can use the method of example 1.)

Example 6 : Show that the image of the rectangular hyperbola $x^2 - y^2 = 1$ under the transformation $w = 1/z$ is the lemniscate $\rho^2 = \cos 2\Phi$. Also draw the sketches.

(M.U. 1997, 2005, 04, 08)

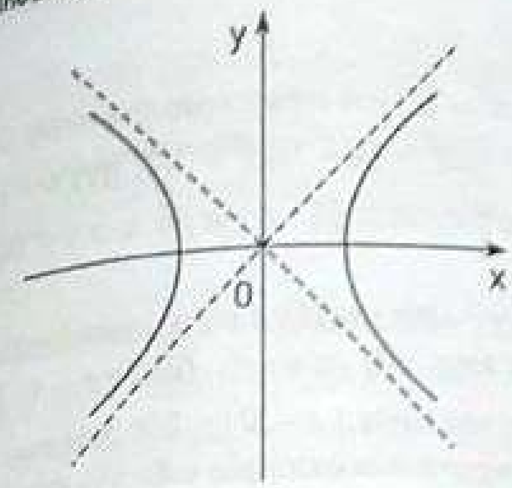
Sol. : Putting $z = re^{i\theta}$ and $w = \rho e^{i\Phi}$ in $w = 1/z$, we get

$$\rho e^{i\Phi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} \qquad \therefore \rho = \frac{1}{r}, \quad \Phi = -\theta.$$

Since $x^2 - y^2 = 1$, we get $\quad r^2 (\cos^2 \theta - \sin^2 \theta) = 1 \qquad \therefore r^2 \cos 2\theta = 1.$

$$\therefore r^2 = \frac{1}{\cos 2\theta} \qquad \therefore \frac{1}{r^2} = \cos 2\theta$$

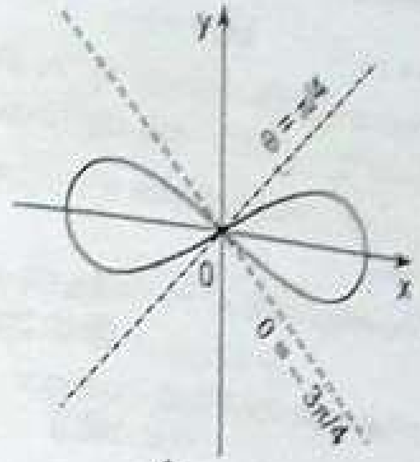
Putting $\rho = \dfrac{1}{r}$ and $\Phi = -\theta$, we get $\rho^2 = \cos(-2\Phi) = \cos 2\Phi$, which is a lemniscate.



$x^2 - y^2 = 1$

Rectangular Hyperbola

Fig. 6.18 (a)



$\rho^2 = \cos 2\Phi$

Lemniscate

Fig. 6.18 (b)

EXERCISE - VI

1. Find the image of the circle with centre at (0, 3) and radius 3 *i.e.* of $|z - 3i| = 3$ in the z-plane into the w-plane under the transformation $w = 1/z$. **(M.U. 2003, 05, 07)**

   [**Ans. :** Circle is $x^2 + y^2 - 6y = 0$; $v = -1/6$. Put $a = 3$ in Ex. 1, page 6-11.]

2. Prove that under the transformation $w = 1/z$ in general circles in the z-plane are transformed into circles in the w-plane. **(M.U. 2003)**

3. Find the image of the strip $2 \le x \le 4$ in the z-plane under the trans-formation $w = 2/z$.

   [**Ans. :** Region bounded by the circles $u^2 + v^2 = u$, and $u^2 + v^2 = u/2$.]

4. Find the image of the infinite strip (i) $\dfrac{1}{6} \le y < \dfrac{1}{4}$, (ii) $0 \le y < \dfrac{1}{4}$ under the transformation $w = 1/z$. Show the regions graphically.

   [**Ans. :** **(i)** Region between the two circles $u^2 + v^2 + 6v = 0$, $u^2 + v^2 + 4v = 0$,

   (ii) Region outside the circle $u^2 + v^2 + 4v = 0$.]

5. Prove that the circle $|z - 3| = 5$ is mapped onto the circle $\left|w + \dfrac{3}{16}\right| = \dfrac{5}{16}$ under the transformation $w = 1/z$. **(M.U. 1996)**

6. Find the image of the line $y - x + 1 = 0$ under the transformation $w = 1/z$. Also find the image of $y - x = 0$. Draw rough sketches. **(M.U. 1995, 2003, 06)**

   [**Ans. :** The circle with centre (1/2, 1/2) and radius $1/\sqrt{2}$.]

7. Find the images of the following curves under the transformation $w = 1/z$. (i) The line $2y - 2x = 1$, (ii) The circle $|z - 4| = 6$.

   [**Ans. : (i)** The circle $u^2 + v^2 - 2u - 2v = 0$, **(ii)** The circle $20u^2 + 20v^2 + 8u - 1 = 0$.]

8. Find the image of the strip $\dfrac{1}{4} \le y \le \dfrac{1}{2}$ under the transformation $w = 1/z$. Draw the rough sketches.

   [**Ans. :** Region bounded by the circles $u^2 + v^2 + 4v = 0$ and $u^2 + v^2 + 2v = 0$.]

9. Find the images of the following under the transformation $w = 1/z$.

   (i) $|z| = 2$ **(ii)** $|z - 1| = 1$, $|z - i| = 1$ **(M.U. 2003)**

(iii) $x + y = 1$ (iv) $x = y$

[**Ans.** : See solved Ex. 1 above. **(i)** The circle $x^2 + y^2 = 4$ maps onto the circle $u^2 + v^2 = 1/4$, **(ii)** The lines $u = 1/2$, $v = -1/2$, **(iii)** The circle $(u - 1/2)^2 + (v + 1/2)^2 = 1/2$, **(iv)** $u + v = 0$, a line.]

10. Show that the transformation $w = 1/z$ maps the circle **(i)** $|z - a| = a$, $a > 0$ **(ii)** $|z - ai| = a$ into a straight line. **(M.U. 1995)**

11. Find the image of (i) $x^2 + y^2 = 2x$, **(ii)** $x^2 + y^2 = 2y$ and **(ii)** $x = 2y$ under the transformation $w = 1/z$. **(M.U. 2004)** [**Ans.** : **(i)** $2u = -1$, **(ii)** $2v = -1$, **(iii)** $u = -2v$]

12. Show that under the transformation $w = 1/z$, the circle $|z - 3i| = 3$ is mapped on to the line $6u + 1 = 0$. **(M.U. 2005)** (Compare this example with the solved Ex. No. 1.)

13. Find the image of $y = x + (1/2)$ under the transformation $w = 1/z$.

[**Ans.** : $(u + 1)^2 + (v + 1)^2 = 2$]

5. Bilinear Transformation $\mathbf{w = \dfrac{az + b}{cz + d}}$

The transformation $w = \dfrac{az + b}{cz + d}$ .... (1)

where a, b, c, d are complex constants and $ad - bc \neq 0$ is called **bilinear** or **Mobius transformation**.

Since $\dfrac{dw}{dz} = \dfrac{(cz + d)\,a - (az + b)\,c}{(cz + d)^2} = \dfrac{ad - bc}{(cz + d)^2} \neq 0$ the transformation is conformal.

From (1) we get $wcz + wd = az + b$

i.e. $z(cw - a) = b - wd$ i.e. $z = \dfrac{-wd + b}{cw - a}$ .... (2)

This inverse mapping is also bilinear.

Notes ....

1. The transformation (1) can be written as $cwz + wd - az - b = 0$

   It is linear in both w and d and hence, the name bilinear transformation.

2. From (1) we see that except $z = -d/c$ to every point of z-plane there is a unique point on the w-plane. From (2) We see that except $w = a/c$ to every point of w-plane there is a unique point of z-plane. Considering the two exceptional points at points at infinity we can say that **there is one to one correspondence between the points of the two planes under a bilinear transformation.**

3. **"A bilinear transformation is a combination of three basic trans-formations viz. (i) translation, (ii) rotation and magnification and (iii) inversion and reflection."**

For, we can write $w = \dfrac{az + b}{cz + d}$ as $w = \dfrac{a}{c} + \dfrac{bc - ad}{c^2} \cdot \dfrac{1}{z + d/c}$ [By actual division]

Now consider the transformations

$w_1 = z + \dfrac{d}{c}$ which is a translation

$w_2 = \dfrac{1}{w_1}$ which is inversion and reflection

$w_3 = \dfrac{bc - ad}{c^2} \cdot w_2$ which is rotation and magnification

$w = \dfrac{a}{c} + w_3$ which is again translation.

Thus, by these transformations we successively go from z-plane to w_1-plane from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane. Hence, the above result.

Remark ....

(i) $ad - bc$ is called the **determinant** of the transformation.

(ii) If $ad - bc = 1$ then the transformation is said to be **normalised.**

(iii) A bilinear transformation is also known as **Mobius Transformation** who first studied it.

Theorem : A bilinear transformation, in general, maps circles into circles. **(M.U. 1996, 2003, 04)**

Proof : Let the bilinear transformation be

$$w = \frac{az + b}{cz + d}$$

As above, by actual division, we can write it as

$$w = \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{z + d/c}.$$

Now, consider the transformation $w_1 = z + \dfrac{d}{c}$ which a translation.

Since in the transformation $w = z + k$ size and shape is preserved, circles in z-plane will be transformed into circles in w_1-plane.

Now, consider the transformation $w_2 = 1/w_1$, which is inversion and reflection.

Since in the transformation $w = 1/z$ circles are mapped onto circles, circles in w_1-plane will be mapped onto circles in w_2-plane.

Now, consider the transformation $w_3 = \dfrac{bc - ad}{c^2} \cdot w_2$, which is a rotation and magnification.

Since in the transformation $w = kz$ figures are only rotated and magnified, circles in w_2-plane will be transformed into circles in w_3-plane.

Now, consider the transformation $w = \dfrac{a}{c} + w_3$, which again is a translation.

Hence, circles in w_3-plane will be transformed into circles in the w-plane.

Hence, circles in z-plane will be transformed into circles in w-plane.

Note ....

The above theorem is true **in general**. But there are exceptions.

6. Cross-Ratio

If z_1, z_2, z_3, z_4 are any four points then their cross ratio **in this order is** defined as $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$ and is denoted by (z_1, z_2, z_3, z_4).

Notes ....

1. The cross-ratio can be easily remembered as follows. We start with z_1, and write the elements in the given order $z_1 - z_2,\ z_2 - z_3,\ z_3 - z_4\ z_4 - z_1$ in cyclic order. We then put the first element in the numerator, second in the denominator, third in the numerator and fourth in the denominator.

Fig. 6.19

2. If z_1, z_2, z_3, z_4 are the given four points they can be ordered 4 at a time amongst themselves in $4! = 24$ ways. This means from 4 points we can have 24 cross-ratios. But, it can be seen that there are only 6 different cross ratios. For, we have

$$(z_1, z_2, z_3, z_4) = (z_2, z_1, z_4, z_3) = (z_4, z_3, z_2, z_1) = (z_3, z_4, z_1, z_2)$$
$$(z_1, z_2, z_4, z_3) = (z_2, z_1, z_3, z_4) = (z_3, z_4, z_2, z_1) = (z_4, z_3, z_1, z_2) \quad \text{.......... (1)}$$
$$(z_1, z_3, z_2, z_4) = (z_3, z_1, z_4, z_2) = (z_4, z_2, z_3, z_1) = (z_2, z_4, z_1, z_3) \quad \text{.......... (2)}$$
$$(z_1, z_3, z_4, z_2) = (z_3, z_1, z_2, z_4) = (z_2, z_4, z_3, z_1) = (z_4, z_2, z_1, z_3) \quad \text{.......... (3)}$$
$$(z_1, z_4, z_2, z_3) = (z_4, z_1, z_3, z_2) = (z_3, z_2, z_4, z_1) = (z_2, z_3, z_1, z_4) \quad \text{.......... (4)}$$
$$(z_1, z_4, z_3, z_2) = (z_4, z_1, z_2, z_3) = (z_2, z_3, z_4, z_1) = (z_3, z_2, z_1, z_4) \quad \text{.......... (5)}$$

To obtain the second, third and fourth cross-ratio in any row, first inter-change first and second elements and third and forth elements (z_2, z_1, z_4, z_3) then we interchange the first pair with the second pair (z_4, z_3, z_2, z_1) ; and again interchange the first and second elements and third and fourth elements in the previous ratio. The cross-ratios of the first row can be seen to be,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_1 - z_4)(z_3 - z_2)}$$
$$= \frac{(z_4 - z_3)(z_2 - z_1)}{(z_3 - z_2)(z_1 - z_4)} = \frac{(z_3 - z_4)(z_1 - z_2)}{(z_4 - z_1)(z_2 - z_3)}$$

(a) Cross-ratio preservation property

If w_1, w_2, w_3, w_4 are the images in the w-plane of the four distinct points z_1, z_2, z_3, z_4 in the z-plane under a bilinear transformation

$$w = \frac{az + b}{cz + d} \ (ad - bc \neq 0)$$

then the cross-ratio is preserved *i.e.* the cross-ratio of w_1, w_2, w_3, w_4 is equal to the cross-ratio of z_1, z_2, z_3, z_4. **(M.U. 1994, 97, 2002, 03, 04)**

i.e. $$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

Proof : We have $w_i = \dfrac{az_i + b}{cz_i + d},\quad i = 1, 2, 3, 4$

$$\therefore\ w_1 - w_2 = \frac{az_1 + b}{cz_1 + d} - \frac{az_2 + b}{cz_2 + d}$$
$$= \frac{ac z_1 z_2 + ad z_1 + bc z_2 + bd - ac z_1 z_2 - ad z_2 - bc z_1 - bd}{(cz_1 + d)(cz_2 + d)}$$
$$\therefore\ w_1 - w_2 = \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

Similarly, $$w_2 - w_3 = \frac{az_2 + b}{cz_2 + d} - \frac{az_3 + b}{cz_3 + d} = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}$$

$$w_3 - w_4 = \frac{az_3 + b}{cz_3 + d} - \frac{az_4 + b}{cz_4 + d} = \frac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)}$$
$$w_4 - w_1 = \frac{az_4 + b}{cz_4 + d} - \frac{az_1 + b}{cz_1 + d} = \frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)}$$
$$\therefore\ \frac{w_1 - w_2}{w_2 - w_3} \cdot \frac{w_3 - w_4}{w_4 - w_1} = \frac{(ab - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \cdot \frac{(cz_2 + d)(cz_3 + d)}{(ad - bc)(z_2 - z_3)}$$
$$\times \frac{(ad - bc)(z_3 - z_4)}{(cz_3 + d)(cz_4 + d)} \cdot \frac{(cz_4 + d)(cz_1 + d)}{(ad - bc)(z_4 - z_1)}$$
$$= \frac{(z_1 - z_2)}{(z_2 - z_3)} \cdot \frac{(z_3 - z_4)}{(z_4 - z_1)}$$
$$\therefore\ \frac{w_1 - w_2}{w_2 - w_3} \cdot \frac{w_3 - w_4}{w_4 - w_1} = \frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_3 - z_4}{z_4 - z_1}.$$

Hence, the result.

Remark ....

By taking the reciprocal of the above equality and by replacing w_4 by general point w and also z_4 by z we get

$$\boxed{\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}}$$

The above equality is easy to remember and can be conveniently used to obtain bilinear transformations.

Example 1 : Find the bilinear transformation which maps the points $2, i, -2$ onto the points $1, i, -1$ by using cross-ratio property. **(M.U. 1993, 2000, 02, 03, 05, 07, 16)**

Sol. : We have $$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

Putting $z_1 = 2,\ z_2 = i,\ z_3 = -2$ and $w_1 = 1,\ w_2 = i,\ w_3 = -1$, we get

$$\frac{(w - 1)(i + 1)}{(1 - i)(-1 - w)} = \frac{(z - 2)(i + 2)}{(2 - i)(-2 - z)}$$
$$\therefore\ \frac{(w - 1)(1 + i)(1 + i)}{(-1 - w)(1 - i)(1 + i)} = \frac{(z - 2)(2 + i)(2 + i)}{(-z - 2)(2 - i)(2 + i)}$$
$$\frac{(w - 1)(1 + i)^2}{(w + 1)(1 - i^2)} = \frac{(z - 2)(2 + i)^2}{(z + 2)(4 - i^2)}$$
$$\frac{(w - 1) \cdot (2i)}{(w + 1) \cdot 2} = \frac{(z - 2)(3 + 4i)}{(z + 2) \cdot 5}$$
$$\frac{w - 1}{w + 1} = \frac{(z - 2)(3 + 4i)}{(z + 2) \cdot 5i} = \frac{(z - 2)(4 - 3i)}{(z + 2) \cdot 5}$$

By componendo and dividendo,

$$\frac{2w}{2} = \frac{(z - 2)(4 - 3i) + (z + 2) \cdot 5}{(z + 2) \cdot 5 - (z - 2)(4 - 3i)}$$
$$\therefore\ w = \frac{4z - 8 - 3iz + 6i + 5z + 10}{5z + 10 - 4z + 8 + 3iz - 6i} = \frac{9z - 3iz + 2 + 6i}{z + 3iz + 18 - 6i}$$
$$= \frac{3z(3 - i) + 2(1 + 3i)}{z(1 + 3i) + 6(3 - i)} = \frac{3z + 2\{(1 + 3i)/(3 - i)\}}{z\{(1 + 3i)/(3 - i)\} + 6}$$

Now $\frac{1+3i}{3-i} = \frac{1+3i}{3-i} \cdot \frac{3+i}{3+i} = \frac{3+9i+i-3}{9-i^2} = \frac{10i}{10} = i$

$\therefore \quad w = \frac{3z+2i}{zi+6}$ which is the required bilinear transformation.

Note : We shall learn another method of finding bilinear transformation on page 6-29.

Example 2 : Find the bilinear transformation which maps the points $z = \infty, i, 0$ onto the points $0, i, \infty$ by using cross-ratio property. **(M.U. 2003, 04, 05, 09)**

Sol. : We have the transformation

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Since, $z_1 = \infty$ and $w_3 = \infty$, we divide the N and D of l.h.s. by w_3 and N and D of r.h.s. by z_1.

$$\therefore \quad \frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\left(1-\frac{w}{w_3}\right)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{\left(1-\frac{z_2}{z_1}\right)(z_3-z)}$$

Since, $w_3 = \infty$, $w_2 / w_3 = 0$, $w / w_3 = 0$.

Also since $z_1 = \infty$, $z / z_1 = 0$, $z_2 / z_1 = 0$

$$\therefore \quad \frac{(w-w_1)(-1)}{(w_1-w_2)} = \frac{(-1)(z_2-z_3)}{(z_3-z)}$$

Now put $z_2 = i$, $z_3 = 0$ and $w_1 = 0$ and $w_2 = i$.

$$\therefore \quad \frac{w-0}{0-i} = \frac{i-0}{0-z} \qquad \therefore \quad \frac{w}{-i} = \frac{i}{-z} \qquad \therefore \quad w = \frac{-i^2}{-z} = -\frac{1}{z}$$

7. Fixed points of a Bilinear Transformation

Definition : The points which coincide with their transforms under a bilinear transformation are called **fixed points** of the bilinear transform.

Consider a bilinear transform

$$w = \frac{az+b}{cz+d} \qquad (ad-bc \neq 0)$$

The fixed points of this transformation are obtained by putting $w = z$ *i.e.* by

$$z = \frac{az+b}{cz+d} \quad \text{i.e. by the quadratic equation.}$$

$$cz^2 + (d-a)z - b = 0 \qquad \text{..........(1)}$$

In other words the roots of this equation are the fixed points.

Case I : If $c \neq 0$, the roots of the equation (1) are given by

$$z = \frac{-(d-a) \pm \sqrt{(d-a)^2+4bc}}{2c}$$

The roots are distinct *i.e.* the points are distinct if $(d-a)^2 + 4bc \neq 0$.

The roots are identical *i.e.* the points are coincident (one point) if $(d-a)^2 + 4bc = 0$.

Case II : If $c = 0$ but $d \neq 0$ then from (1) again, we get, $w = \frac{a}{d}z + \frac{b}{d}$.

If z is infinity then obviously w is infinity. So one fixed point is infinity. The other fixed point is given by

$$z = \frac{a}{d}z + \frac{b}{d} \quad i.e. \ (a-d)z + b = 0 \quad i.e. \quad z = -\frac{b}{a-d} \ \text{if } a \neq d$$

If $a = d$ there is only one fixed point and it is infinity.

Thus, by the bilinear transformation, we have

(i) two finite fixed points if $c \neq 0$ and $(d-a)^2 + 4bc \neq 0$.

(ii) only one finite fixed point if $c \neq 0$ and $(d-a)^2 + 4bc = 0$. This bilinear transformation is called **parabolic**.

(iii) one finite fixed point and other infinite fixed point if $c = 0$ and $a - d \neq 0$. .... (A)

(iv) only one infinite fixed point if $c = 0$ and $a - d = 0$. .... (B)

Theorem 1 : Every bilinear transformation with two finite fixed points α, β can be put in the form

$$\frac{w-\alpha}{w-\beta} = \lambda \cdot \frac{z-\alpha}{z-\beta}$$

(M.U. 2002, 06)

Proof : Let $w = \frac{az+b}{cz+d}$ be the given bilinear transformation with α, β as fixed points.

For fixed points, we put, $w = z \quad \therefore \quad z = \frac{az+b}{cz+d} \quad \therefore \ cz^2 + (d-a)z - b = 0$

Since α, β are fixed points

$$c\alpha^2 + (d-a)\alpha - b = 0 \quad i.e. \quad b - d\alpha = c\alpha^2 - a\alpha$$

and $\quad c\beta^2 + (d-a)\beta - b = 0 \quad i.e. \quad b - d\beta = c\beta^2 - a\beta$ .... (1)

$$\text{Now, } \frac{w-\alpha}{w-\beta} = \frac{\frac{az+b}{cz+d} - \alpha}{\frac{az+b}{cz+d} - \beta} = \frac{(a-c\alpha)z + (b-d\alpha)}{(a-c\beta)z + (b-d\beta)}$$

By using (1), we get,

$$\frac{w-\alpha}{w-\beta} = \frac{(a-c\alpha)z + (c\alpha^2 - a\alpha)}{(a-c\beta)z + (c\beta^2 - a\beta)} = \frac{(a-c\alpha)(z-\alpha)}{(a-c\beta)(z-\beta)}$$

$$= \lambda \cdot \frac{z-\alpha}{z-\beta} \quad \text{where} \quad \lambda = \frac{a-c\alpha}{a-c\beta}$$

Theorem 2 : Every bilinear transformation which has only one fixed point α can be put in the form called **normal form.**

$$\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k$$

(M.U. 1996, 2005, 08)

Proof : Let $w = \frac{az+b}{cz+d}$ be the given bilinear transformation. .... (A)

For fixed point we put $w = z \quad \therefore \quad z = \frac{az+b}{cz+d} \quad \therefore \ cz^2 + (d-a)z - b = 0$

Since α is the only one fixed point

$$cz^2 + (d-a)z - b \equiv c(z-\alpha)^2 = c(z^2 - 2z\alpha + \alpha^2)$$

Equating the coefficients of like powers of z

$$d - a = -2\alpha c \quad \text{and} \quad -b = c\alpha^2$$

i.e. $\quad d = a - 2\alpha c \quad$ and $\quad b = -c\alpha^2 \qquad \ldots\ldots (1)$

Now, $\quad w - \alpha = \dfrac{az+b}{cz+d} - \alpha = \dfrac{(a-c\alpha)z + (b - d\alpha)}{cz+d}$

$$\therefore \frac{1}{w-\alpha} = \frac{cz+d}{(a-c\alpha)z + (b-d\alpha)}$$

$$= \frac{cz + a - 2c\alpha}{(a-c\alpha)z - c\alpha^2 - a\alpha + 2c\alpha^2} \qquad [\text{By } (1)]$$

$$= \frac{cz + a - 2c\alpha}{(a-c\alpha)z - \alpha(a - c\alpha)} = \frac{(a-c\alpha) + c(z-\alpha)}{(a-c\alpha)(z-\alpha)} = \frac{1}{z-\alpha} + \frac{c}{a-c\alpha}$$

$$= \frac{1}{z-\alpha} + k \quad \text{where} \quad k = \frac{c}{a-c\alpha}.$$

Example : Find the fixed points of $w = \dfrac{3z-4}{z-1}$. Also express it in the normal form $\dfrac{1}{w-\alpha} = \dfrac{1}{z-\alpha} + \lambda$, where λ is a constant and α is the fixed point. Is this transformation parabolic?

(M.U. 2003, 15)

Sol. : Fixed points are given by

$$z = \frac{3z-4}{z-1} \qquad \text{i.e.} \qquad z^2 - z = 3z - 4$$

$$\therefore z^2 - 4z + 4 = 0 \qquad \therefore (z-2)^2 = 0 \qquad \therefore z = 2, 2.$$

Hence, $z = 2$ is the only fixed point.

Now subtracting the fixed point 2 from both sides of the equation $w = \dfrac{3z-4}{z-1}$, we get

$$w - 2 = \frac{3z-4}{z-1} - 2 = \frac{3z - 4 - 2z + 2}{z-1} = \frac{z-2}{z-1}$$

$$\therefore \frac{1}{w-2} = \frac{z-1}{z-2} = \frac{(z-2)+1}{z-2} = \frac{1}{z-2} + 1 \text{ which is the required normal form.}$$

Since $z = 2$ is the only one fixed point, the transformation is parabolic.

Theorem 3 : Every bilinear transformation which has one finite fixed point α and the other fixed point ∞ can be put in the form $w - \alpha = \lambda(z - \alpha)$.

Proof : Let $w = \dfrac{az+b}{cz+d}$ be the given bilinear transformation.

Since it has one finite fixed point and the other infinite fixed point from (A), in case (ii), page 6-20, it follows that $c = 0$ and $a - d \neq 0$ and it takes the form

$$w = \frac{a}{d}z + \frac{b}{d}.$$

Since α is a fixed point $\quad \alpha = \dfrac{a}{d}\alpha + \dfrac{b}{d}$.

Subtracting we get,

$$w - \alpha = \frac{a}{d}(z-\alpha) = \lambda(z-\alpha) \quad \text{where} \quad \lambda = \frac{a}{d}.$$

Remark ....

If ∞ is the only fixed point of the transformation then $c = 0$, $a = d$ and $w = \dfrac{az+b}{cz+d}$ [from (B), page 6-21] takes the form $w = z + \dfrac{b}{d}$ which is a translation.

Example : Find the fixed points of the bilinear transformation

(i) $w = \dfrac{1+3iz}{i+z}$ **(M.U. 1993, 2002)** $\qquad$ (ii) $w = \dfrac{z-1}{z+1}$. Is this transformation parabolic?

(iii) $w = \dfrac{z-4}{2z-5}$ **(M.U. 1995, 2002)** $\qquad$ (iv) $w = \dfrac{2z+1}{z+2}$. Is the transformation conformal?

(M.U. 2002)

Sol. : (i) The fixed points are given by $z = \dfrac{1+3iz}{i+z}$

$$\therefore iz + z^2 = 1 + 3iz \qquad \therefore z^2 - 2iz - 1 = 0$$

$$\therefore z^2 - 2iz + i^2 = 0 \qquad \therefore (z-i)^2 = 0$$

$\therefore z = i$ is the only one fixed point of the transformation.

(ii) The fixed points are given by

$$z = \frac{z-1}{z+1} \qquad \therefore z^2 + z = z - 1$$

$$\therefore z^2 + 1 = 0 \qquad \therefore z^2 - i^2 = 0$$

$\therefore (z+i)(z-i) = 0 \quad \therefore z = i, -i$ are the finite fixed points.

Since there are two distinct fixed points, the mapping is not parabolic.

(iii) The fixed points are given by

$$z = \frac{z-4}{2z-5} \qquad \therefore 2z^2 - 5z = z - 4$$

$$\therefore 2z^2 - 6z + 4 = 0 \qquad \therefore 2(z^2 - 3z + 2) = 0$$

$$\therefore z^2 - 3z + 2 = 0 \qquad \therefore (z-2)(z-1) = 0$$

$\therefore z = 1, 2$ are the fixed points.

(iv) The fixed points are given by

$$z = \frac{2z+1}{z+2} \qquad \therefore z^2 = 1 \qquad \therefore z = \pm 1$$

The mapping is conformal (See theorem 4 below).

Theorem 4 : Every bilinear transformation represents a one to one conformal mapping. (M.U. 1995)

Proof : We know that the transformation $w = \dfrac{az+b}{cz+d}$

where a, b, c, d are complex constants and $ad - bc \neq 0$ is called a bilinear transformation.

Now, differentiating this equation w.r.t. z, we get

$$\frac{dw}{dz} = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

If $z = -\dfrac{d}{c}$, $\dfrac{dw}{dz} = \infty$ and if $z = \infty$, $\dfrac{dw}{dz} = 0$.

Thus the bilinear transformation $w = \dfrac{az+b}{cz+d}$ is conformal except at $z = -\dfrac{d}{c}$ and $z = \infty$ which are critical points of the transformation. However by suitably defining the conformal mapping at these points we can overcome this difficulty (unfortunately we do not go into details of this aspect in this book).

Hence, **every bilinear transformation is conformal.**

Example : Find the invariant points of the bilinear transformation

$$w = -\frac{(2z+4i)}{(iz+1)}$$

and prove that these two points together with any point z and its image form a set of points whose cross-ratio is constant. (M.U. 2003, 05)

Sol. : The fixed points are given by $\quad z = -\dfrac{2z+4i}{iz+1}$

$$\therefore\ iz^2 + z + 2z + 4i = 0 \qquad \therefore\ iz^2 + 3z + 4i = 0$$

$$\therefore\ i^2z^2 + 3iz + 4i^2 = 0 \qquad \therefore\ -z^2 + 3iz - 4 = 0$$

$$\therefore\ z^2 - 3iz + 4 = 0 \qquad \therefore\ (z+i)(z-4i) = 0 \qquad \therefore\ z = -i,\ 4i.$$

Now, consider the cross-ratio of $z_1 = z$, $z_2 = -i$, $z_3 = 4i$,

$$z_4 = -\frac{(2z+4i)}{(iz+1)} \quad \text{(image of } z\text{)}$$

$$\text{Cross-ratio} = \frac{(z+i)\left(4i + \dfrac{2z+4i}{iz+1}\right)}{(-i-4i)\left(-\dfrac{2z+4i}{iz+1} - z\right)} = \frac{(z+i)(-2z+8i)}{5i(2z+4i+iz^2+z)}$$

$$= \frac{-2(z+i)(z-4i)}{-5(z^2-3iz+4)} = \frac{2}{5}\cdot\frac{(z+i)(z-4i)}{(z+i)(z-4i)} = \frac{2}{5}, \text{ a constant.}$$

EXERCISE - V

1. Find the fixed points of bilinear transformation $w = \dfrac{-1+(2+i)z}{i+2+z}$. [Ans. : $z = \pm$

2. Find the fixed points of bilinear transformation
(i) $\dfrac{1+3iz}{3i+z}$ (ii) $\dfrac{2z-2+iz}{i+z}$ (M.U. 1998) (iii) $\dfrac{2z-5}{z+4}$ (M.U. 1997) (iv) $-\dfrac{2z+4i}{iz+1}$ (M.U. 2002) (v) $-\dfrac{z+2}{2z+3}$

[**Ans. :** (i) $z = \pm 1$, (ii) $z = 1 \pm i$, (iii) $z = -1 \pm 2i$, (iv) $z = -i, 4i$, (v) $z = -1, -1$]

3. Find the fixed points of (i) $w = \dfrac{4z-9}{z-2}$ (ii) $w = \dfrac{3z-2}{2z-1}$.
Also express it in the normal form.

[**Ans. :** (i) $z = 3, 3$; $\dfrac{1}{w-3} = \dfrac{1}{z-3} + 1$, (ii) $z = 1, 1$; $\dfrac{1}{w-1} = \dfrac{1}{z-1} + 2$]

4. Find the fixed point of
(i) $w = \dfrac{3z-5i}{iz-1}$ (M.U. 2003) (ii) $w = \dfrac{2z+6}{z+7}$ (M.U. 2003)
(iii) $w = \dfrac{3z-5i}{iz-1}$. Is this mapping conformal ? (M.U. 2003)

[**Ans. :** (i) $5/i$, i, (ii) -6, 1, (iii) $-2i \pm 1$; Yes.]

Mapping Under Bilinear Transformation $w = \dfrac{az+b}{cz+d}$

Example 1 : Find the image of the circle $|z| = k$ where k is real under the bilinear transformation

$$w = \frac{5-4z}{4z-3}.$$

(M.U. 2002, 05, 12, 13)

Sol. : Since $w = \dfrac{5-4z}{4z-3}$, we have $4wz - 3w = 5 - 4z$.

$$\therefore\ z(4w+4) = 5 + 3w \qquad \therefore\ z = \frac{5+3w}{4(w+1)}$$

When $|z| = k$, $\left|\dfrac{5+3w}{4(w+1)}\right| = k \quad \therefore\ \left|\dfrac{5+3(u+iv)}{(u+iv)+1}\right| = 4k \quad \therefore\ \left|\dfrac{(5+3u)+i3v}{(u+1)+iv}\right| = 4k$

$$\therefore\ (5+3u)^2 + 9v^2 = 16k^2[(u+1)^2 + v^2]$$

$$\therefore\ 25 + 30u + 9u^2 + 9v^2 = 16k^2(u^2 + 2u + 1 + v^2)$$

$$\therefore\ (16k^2 - 9)(u^2 + v^2) + (32k^2 - 30)u + (16k^2 - 25) = 0 \text{ which is a circle.}$$

Remark ....

We have already proved that under a bilinear transformation, in general, circles in the z-plane are mapped onto the circles in the w-plane.

Example 2 : Under the transformation $w = \dfrac{z-1}{z+1}$, show that the map of the straight line $y = x$ is a circle and find its centre and radius. (M.U. 1994, 2002)

Sol. : We have $w = \dfrac{z-1}{z+1}$ $\quad\therefore$ We express z in terms of w as follows (Note this).

By componendo and dividendo

$$\frac{w+1}{1-w}=z \qquad \therefore \frac{(u+iv)+1}{1-(u+iv)}=x+iy$$

$$\therefore \frac{(u+1)+iv}{(1-u)-iv}\cdot\frac{(1-u)+iv}{(1-u)+iv}=x+iy$$

$$\therefore \frac{(1-u^2)+iv(1-u)+iv(1+u)-v^2}{(1-u)^2+v^2}=x+iy$$

$$\therefore \frac{(1-u^2-v^2)+2iv}{1+u^2+v^2-2u}=x+iy$$

$\therefore$ Equating real and imaginary parts

$$x=\frac{1-u^2-v^2}{1+u^2+v^2-2u}, \quad y=\frac{2v}{1+u^2+v^2-2u}$$

Since $y=x$, we get, $1-u^2-v^2=2v$

$$\therefore u^2+v^2+2v=1 \qquad \therefore u^2+(v+1)^2=2$$

Hence, the map of $y=x$ is a circle with centre at $(0,-1)$ and radius $\sqrt{2}$.

Example 3 : Show that under the transformation $w=\dfrac{z-i}{z+i}$, real axis in the z-plane is mapped onto the circle $|w|=1$. **(M.U. 1997)**

Sol. : We have $w=\dfrac{z-i}{z+i}$ $\quad\therefore\ wz+wi=z-i$

$$\therefore wi+i=z-wz \qquad \therefore i(w+1)=z(1-w)$$

$$\therefore z=i\frac{(1+w)}{(1-w)} \qquad \therefore x+iy=i\cdot\frac{[1+(u+iv)]}{1-(u+iv)}$$

$$\therefore x+iy=i\left[\frac{(1+u)+iv}{(1-u)-iv}\cdot\frac{(1-u)+iv}{(1-u)+iv}\right]=i\left[\frac{(1-u^2)+iv(1+u)+iv(1-u)-v^2}{(1-u)^2+v^2}\right]$$

$$\therefore x=-\frac{2v}{(1-u)^2+v^2}, \quad y=\frac{1-u^2-v^2}{(1-u)^2+v^2}$$

For the real axis $y=0$.

$$\therefore 1-u^2-v^2=0 \qquad \therefore u^2+v^2=1 \qquad \therefore |w|=1$$

$\therefore$ The real axis $y=0$ in the z-plane is mapped onto the unit circle $|w|=1$ in the w-plane.

Example 4 : Show that the map of the real axis of the z-plane is a circle under the transformation $w=\dfrac{2}{z+i}$. Find its centre and the radius. **(M.U. 2014)**

Sol. : We have $w=\dfrac{2}{z+i}$ $\quad\therefore\ z+i=\dfrac{2}{w}$

$$\therefore (x+iy)+i=\frac{2}{u+iv}=\frac{2}{(u+iv)}\cdot\frac{(u-iv)}{(u-iv)} \qquad \therefore x+i(y+1)=\frac{2(u-iv)}{u^2+v^2}$$

$$\therefore x=\frac{2u}{u^2+v^2} \text{ and } y+1=-\frac{2v}{u^2+v^2}$$

For the real axis i.e. the x-axis, $y=0$.

$$\therefore 1=-\frac{2v}{u^2+v^2} \qquad \therefore u^2+v^2+2v=0 \qquad \therefore u^2+(v+1)^2=1$$

The map is a circle with centre at $(0,-1)$ and radius 1.

Example 5 : Prove that the transformation $w=\dfrac{1}{z+i}$ transforms the real axis of the z-plane into a circle in the w-plane. **(M.U. 2005, 13, 15)**

Sol. : We have $w=\dfrac{1}{z+i}$

$$\therefore z+i=\frac{1}{w}=\frac{1}{u+iv}=\frac{1}{u+iv}\cdot\frac{u-iv}{u-iv}=\frac{u-iv}{u^2+v^2}$$

$$\therefore x+iy+i=\frac{u-iv}{u^2+v^2}$$

$$x+i(y+1)=\frac{u}{u^2+v^2}-i\cdot\frac{v}{u^2+v^2}$$

Equating real and imaginary parts, $x=\dfrac{u}{u^2+v^2}$ and $y+1=-\dfrac{v}{u^2+v^2}$.

Now, the real axis in the z-plane is given by $y=0$.

Putting $y=0$ in the second part,

$$1=-\frac{v}{u^2+v^2} \qquad \therefore u^2+v^2+v=0 \qquad \therefore u^2+\left(v+\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$$

This is a circle with centre $\left(0,-\dfrac{1}{2}\right)$ and radius $\dfrac{1}{2}$ in the w-plane.

Hence, the result.

Example 6 : Find the mapping of the y-axis under the transformation $w=\dfrac{1}{kz+1}$, where k is real. **(M.U. 2006)**

Sol. : We have $w=\dfrac{1}{kz+1}$ $\quad\therefore\ kz+1=\dfrac{1}{w}$

$$\therefore k(x+iy)+1=\frac{1}{u+iv}=\frac{1}{u+iv}\cdot\frac{u-iv}{u-iv}$$

$$\therefore (kx+1)+iky=\frac{u}{u^2+v^2}-\frac{iv}{u^2+v^2}$$

$$\therefore kx+1=\frac{u}{u^2+v^2} \text{ and } ky=-\frac{v}{u^2+v^2}$$

For the y-axis, $x=0$

$$\therefore 1=\frac{u}{u^2+v^2} \qquad \therefore u^2+v^2-u=0$$

$$\therefore \left(u^2-u+\frac{1}{4}\right)+v^2=\frac{1}{4} \qquad \therefore \left(u-\frac{1}{2}\right)^2+v^2=\left(\frac{1}{2}\right)^2$$

$\therefore$ The map of the y-axis is a circle with centre at $(1/2, 0)$ and radius $= 1/2$.

Example 7 : Show that $w = i\left(\frac{1-z}{1+z}\right)$ transforms the circle $|z| = 1$ onto the real axis of the w-plane and the interior of the circle $|z| < 1$ onto the upper half of the w-plane. **(M.U. 2003, 05)**

Sol. : We have $w = i\left(\frac{1-z}{1+z}\right)$. $\therefore\ w + wz = i - iz \quad \therefore\ z(w+i) = i - w$

$$\therefore\ z = \frac{i-w}{i+w} = \frac{i-(u+iv)}{i+(u+iv)} = \frac{i(-v+1)-u}{i(v+1)+u}$$

Since $|z| = 1$, we get $\left|\frac{i(-v+1)-u}{i(v+1)+u}\right| = 1$

$$\therefore\ |i(-v+1)-u| = |i(v+1)+u| \qquad \therefore\ (1-v)^2 + u^2 = (1+v)^2 + u^2$$

$$\therefore\ 1 - 2v + v^2 + u^2 = 1 + 2v + v^2 + u^2 \qquad \therefore\ -4v = 0 \qquad \text{......... (A)}$$

$$\therefore\ v = 0$$

$\therefore$ The circle $|z| = 1$ is mapped onto the real axis in w-plane.

Now when $|z| < 1$ we have from the above relation (A).

$-4v < 0$ *i.e.* $-v < 0$ *i.e.* $v > 0$ *i.e.* the upper half of the w-plane.

Example 8 : Prove that $w = i\left(\frac{z-i}{z+i}\right)$ maps upper half of the z-plane into the interior of the unit circle in the w-plane.

Sol. : Since $w = i\left(\frac{z-i}{z+i}\right)$, we get $zw + iw = iz + 1 \quad \therefore\ z(w-i) = 1 - iw$

$$\therefore\ z = \frac{1-iw}{w-i} \qquad \therefore\ x + iy = \frac{1-i(u+iv)}{u+iv-i}$$

$$\therefore\ x + iy = \frac{(1+v)-iu}{u+i(v-1)} \cdot \frac{u-i(v-1)}{u-i(v-1)} = \frac{(1+v)u - u(v-1)}{u^2+(v-1)^2} - i \cdot \frac{u^2+v^2-1}{u^2+(v-1)^2}$$

$$\therefore\ y = -\frac{u^2+v^2-1}{u^2+(v-1)^2}$$

For the upper half of the z-plane $y > 0$,

$$\therefore\ -\frac{(u^2+v^2-1)}{u^2+(v-1)^2} > 0 \qquad \therefore -(u^2+v^2-1) > 0$$

$$\therefore\ u^2+v^2-1 < 0 \qquad \therefore\ u^2+v^2 < 1 \qquad \therefore\ |w| < 1$$

$\therefore$ The upper of the z-plane is mapped onto the interior of the unit circle with centre at the origin.

EXERCISE - VI

1. Find the image of the real axis in the z-plane onto the w-plane under the transformation $w = \frac{1}{z+i}$. **(M.U. 2005, 13)**

[**Ans. :** $z + i = \frac{1}{w} = \frac{u-iv}{u^2+v^2}$, $y + 1 = -\frac{v}{u^2+v^2}$ when $y = 0$, $u^2+v^2+v = 0$.

The circle withe centre $(0, -1/2)$ and radius $1/2$.]

2. Show that under the transformation $w = \frac{i-z}{i+z}$, the circle $|z| = 1$ is mapped onto the ...ary axis of the w-plane.

[**Ans. :** $u + iv = \frac{-(x^2+y^2)+1+2ix}{x^2+y^2+2y+1}$ when $x^2+y^2 = 1$, $u = 0$

i.e. the imaginary axis.]

3. Show that under the transformation $w = \frac{2z+3}{z-4}$, the circle $x^2+y^2 = 4x$ in the z-plane is ...med into the straight line $4u+3 = 0$ in the w-plane. **(M.U. 2003, 07)**

[**Ans. :** $u + iv = \frac{2(x^2+y^2)-5x-12-11iy}{x^2+y^2-8x+16}$

when $x^2+y^2 = 4x$, $u + iv = \frac{(3x-12)-11iy}{16-4x}$ $\therefore\ 4u+3 = 0$.]

4. Show that under the transformation $w = \frac{5-4z}{4z-2}$ the circle $|z| = 1$ in the z-plane is ...med into a circle of unity in the w-plane. Also find the centre of the circle. **(M.U. 1998)**

[**Ans. :** $z = \frac{2w+5}{4(w+1)}$ $\therefore |z| = 1$ gives $\left|\frac{2w+5}{w+1}\right| = 4$

$$\therefore\ (2u+5)^2 + 4v^2 = 16\left[(u+1)^2+v^2\right]$$

$$\therefore\ u^2+v^2+u-3/4 = 0.\ \text{Centre}\left(-\frac{1}{2}, 0\right),\ r = 1.\ ]$$

5. Show that the transformation $w = \frac{iz+2}{4z+i}$ maps the real axis in the z-plane into a circle in ...plane. Find the centre and radius of the circle. Also find the points on the z-plane which is ...ed on to the centre of the circle. **(M.U. 2000, 02, 08)**

[**Ans. :** Centre $(0, 7/8)$, radius $= 9/8$; $(0, -1/4)$]

6. Find the image of the circle $x^2+y^2 = 1$, under the transformation $w = \frac{5-4z}{4z-2}$.

(M.U. 2002, 03, 12) [**Ans. :** Circle with centre $(-1/2, 0)$ and radius 1.]

7. Find the image of $|z| < 1$ under the transformation $w = \frac{1+iz}{1-iz}$. **(M.U. 2004)**

[**Ans. :** $u > 0$]

To Find Bilinear Transformation $w = \frac{az+b}{cz+d}$

Example 1 : Find the bilinear transformation which maps $z = 2, 1, 0$ onto $w = 1, 0, i$. **(M.U. 1998, 2002)**

...Let the transformation be $\quad w = \frac{az+b}{cz+d}$ .... (1)

Putting the given values of z and w we get,

$$1 = \frac{2a+b}{2c+d}, \quad 0 = \frac{a+b}{c+d}, \quad i = \frac{b}{d}$$

From the third we get $b = di$.

From the second we get $a = -b = -di$

Putting these values in the first, $2c + d = 2a + b$, we get $2c + d = -2di + di$

$\therefore \; 2c = -di - d \quad \therefore \; c = -(i+1)\,d/2$

Hence, from (1), we get,

$$w = \frac{-diz + di}{\{-(i+1)\,dz/2\} + d} = \frac{2(-iz+i)}{-(i+1)z+2} = \frac{2(z-1)}{(1-i)z+2i}.$$

Note ....

You can verify the transformation by putting the values of z.

Example 2 : Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. **(M.U. 1996, 2002, 03, 13, 15)**

Hence, find the fixed points of the transformation and the image of $|z| < 1$. **(M.U. 2004, 06, 07, 08, 10)**

Sol. : Let the transformation be $\quad w = \dfrac{az+b}{cz+d}$ .... (1)

Putting the given values of z and w, we get,

$$i = \frac{a+b}{c+d}, \quad 0 = \frac{ai+b}{ci+d}, \quad -i = \frac{-a+b}{-c+d}$$

From these equalities, we get,

$(a+b) - i(c+d) = 0$ .... (2) ; $\qquad b + ia = 0$ .... (3)

$(-a+b) + i(-c+d) = 0$ .... (4)

Adding (2) and (4), we get, $2b - 2ic = 0 \quad \therefore \; c = \dfrac{b}{i}$.

But from (3), $\quad b = -ia \quad \therefore \; c = -a$.

Substracting (4) from (2), we get, $2a - 2id = 0 \quad \therefore \; d = \dfrac{a}{i} = -ia$.

Putting the values $b = -ia$, $c = -a$ and $d = -ia$ in (1) we get

$$w = \frac{az - ia}{-az - ia} = \frac{z-i}{-z-i}$$

$\therefore \; w = \dfrac{i-z}{i+z}$ is the required bilinear transformation.

Now, the fixed points of the transformation are given by

$$\frac{i-z}{i+z} = z \quad \therefore \; i - z = iz + z^2 \quad \therefore \; z^2 + (1+i)z - i = 0$$

$$\therefore \; z = \frac{-(1+i) \pm \sqrt{(1+i)^2 + 4i}}{2} = \frac{-(1+i) \pm \sqrt{6i}}{2}$$

For the second part refer to the next Ex. 3.

Example 3 : Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the $w = i, 0, -i$. Hence, find the image of $|z| < 1$ onto the w-plane. **(M.U. 1996, 2003, 04, 09)**

As proved in the above example,

$$w = \frac{az - ai}{-az - ai} = \frac{i-z}{i+z}$$

Now, from (1) we get $\quad wi + wz = i - z$. .... (1)

$\therefore \; wi - i = -z(1+w) \qquad \therefore \; z = i \cdot \dfrac{1-w}{1+w}$

Further, $|z| < 1$ is mapped onto the region

$$\left| i\frac{(1-w)}{(1+w)} \right| < 1 \quad \therefore \; |i|\frac{|1-w|}{|1+w|} < 1$$

$\therefore \; |1-w| < |1+w| \qquad [\because |i| = 1]$

$\therefore \; |(1-u) - iv| < |(1+u) + iv|$

$\therefore \; (1-u)^2 + v^2 < (1+u)^2 + v^2 \quad \therefore \; -2u < 2u$

$\therefore \; -4u < 0 \quad \therefore \; -u < 0 \quad \therefore \; u > 0$

Hence, the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped onto entire half of the w-plane to the right of imaginary axis.

The regions in the z-plane and w-plane are shown below.

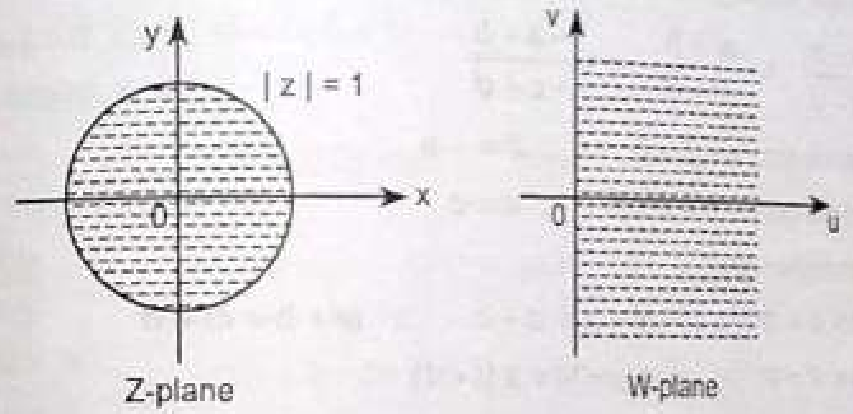


Fig. 6.20 (a) $\qquad$ **Fig. 6.20 (b)**

Example 4 : Find the bilinear transformation which maps the points $1, -i, 2$ on z-plane onto $0, 2, -i$ respectively of w-plane. **(M.U. 1993)**

Sol. : Let the transformation be $\quad w = \dfrac{az+b}{cz+d}$ .... (1)

Putting the given values, $\quad 0 = \dfrac{a+b}{c+d}, \quad 2 = \dfrac{-ai+b}{-ci+d}, \quad -i = \dfrac{2a+b}{2c+d}$ .... (2)

From these equations, we get, $\quad a + b = 0$ .... (3)

$(a - 2c)\,i + (2d - b) = 0$ .... (4)

$(2c + d)\,i + (2a + b) = 0$

From (2) we get $b = -a$.

Putting this value of b in (3) and (4), we get

$(a - 2c)\,i + (2d + a) = 0$ .... (5)

$(2c + d)\,i + a = 0$ .... (6)

Adding (5) and (6), we get,

$$(a+d)\,i+2\,(a+d)=0 \qquad \therefore\ (a+d)\,(i+2)=0$$

$$\therefore\ d=-a. \qquad [\because\ i\neq -2]$$

Putting the values of d and b in $2=\dfrac{-ai+b}{-ci+d}$, we get $2=\dfrac{-ai-a}{-ci-a}=\dfrac{a(1+i)}{ci+a}$.

$$\therefore\ 2ci+2a=a+ai \qquad \therefore\ 2ci=-a+ai \qquad \therefore\ 2ci=ai^2+ai=ai\,(1+i)$$

$$\therefore\ 2c=(1+i)\,a \quad \therefore\ c=\left(\frac{1+i}{2}\right)a$$

Putting the values of b, c, d in (1),

$$w=\frac{az-a}{\left(\frac{1+i}{2}\right)az-a}=\frac{z-1}{\left(\frac{1+i}{2}\right)z-1} \qquad \therefore\ w=\frac{2(z-1)}{(1+i)\,z-2}$$

Example 5 : Find the bilinear transformation under which 1, i, -1 from the z-plane are mapped onto 0, 1, ∞ of w-plane.

Further show that under this transformation the unit circle in w-plane is mapped onto a straight line in the z-plane. Write the name of this line. **(M.U. 1997, 2007, 14)**

Sol. : Let the transformation be $w=\dfrac{az+b}{cz+d}$

Putting the given values of z and w, we get,

$$0=\frac{a+b}{c+d},\quad 1=\frac{ai+b}{ci+d},\quad \infty=\frac{-a+b}{-c+d}$$

From the first we get, $a+b=0 \quad \therefore\ b=-a$

From the last we get, $-c+d=0 \quad \therefore\ d=c$

From the second we get,

$$ai+b=ci+d \qquad \therefore\ ai-a=ci+c \qquad \therefore\ ai+b=ci+d$$

$$\therefore\ ai-a=ci+c \qquad \therefore\ a(i-1)=c\,(i+1)$$

$$\therefore\ c=a\frac{(i-1)}{(i+1)} \qquad \therefore\ c=a\frac{(i-1)}{(i+1)}\cdot\frac{(i-1)}{(i-1)}=a\frac{(i^2-2i+1)}{i^2-1}=ai$$

$$\therefore\ d=c=ai \qquad \therefore\ w=\frac{az-a}{a\,i\,z+ai}=\frac{z-1}{i(z+1)}=-i\frac{(z-1)}{(z+1)}$$

Now, when $|w|=1$, $\left|-i\cdot\dfrac{(z-1)}{(z+1)}\right|=1$

$$\therefore\ |z-1|=|z+1| \qquad [\because |i|=1]$$

$$\therefore\ |(x-1)-iy|=|(x+1)+iy|$$

$$\therefore\ (x-1)^2+y^2=(x+1)^2+y^2$$

$$-2x=2x \quad \therefore\ 4x=0 \quad \therefore\ x=0$$

Hence, the map is the y-axis.

Example 6 : Find bilinear transformation that maps 0, 1, ∞ of the z-plane onto -5, -1, 3 of w-plane. **(M.U. 2004, 13, 15)**

Sol. : Let the transformation be $w=\dfrac{az+b}{cz+d}$.

Putting the given values of z and w.

(i) When $z=0$, $w=-5$, we get $\quad -5=\dfrac{b}{d} \qquad \therefore\ b=-5d$ ——— (1)

(ii) When $z=1$, $w=-1$, we get $\quad -1=\dfrac{a+b}{c+d} \qquad \therefore\ a+b=-c-d$ ——— (2)

(iii) When $z=\infty$, $w=3$.

Now, $w=\dfrac{a+(b/z)}{c+(d/z)} \qquad \therefore\ 3=\dfrac{a}{c} \qquad \therefore\ a=3c$ ——— (3)

From (2), we have

$$a=-b-c-d=5d-c-d=-c+4d$$

From (3), we have

$$3c=a=-c+4d \qquad \therefore\ 4c=4d \qquad \therefore\ c=d.$$

But, $a=3c=3d. \qquad \therefore\ w=\dfrac{az+b}{cz+d}=\dfrac{3dz-5d}{dz+d}=\dfrac{3z-5}{z+1}$.

Example 7 : Find the bilinear transformation which maps the points 2, i, -2 onto the points 1, i, -1. **(M.U. 2014)**

Sol. : Let the required transformation be $w=\dfrac{az+b}{cz+d}$

By data, when $z=2, i, -2,\ w=1, i, -1$.

Putting these values, we get

$$1=\frac{2a+b}{2c+d} \qquad \therefore\ 2a+b=2c+d \qquad \therefore\ 2a+b-2c-d=0 \qquad \text{——— (1)}$$

$$i=\frac{ai+b}{ci+d} \qquad \therefore\ ai+b=-c+di \qquad \therefore\ ai+b+c-di=0 \qquad \text{——— (2)}$$

$$-1=\frac{-2a+b}{-2c+d} \qquad \therefore\ -2a+b=2c-d \qquad \therefore\ -2a+b-2c+d=0 \qquad \text{——— (3)}$$

From (1) and (3), we get $2b-4c=0 \quad \therefore\ b=2c$

Multiply (1) by $i \quad \therefore\ 2ai+bi-2ci-di=0$ ——— (4)

Subtract (2) from (4) $\quad ai+b\,(i-1)-c\,(2i+1)=0$

But $b=2c$, $\quad \therefore\ ai+2c\,(i-1)-c\,(2i+1)=0$

$$\therefore\ ai-3c=0 \qquad \therefore\ a=3c/i=-3ci$$

Putting the values of a and b in (1), we get

$$-6ci+2c-2c-d=0 \qquad \therefore\ d=-6ci$$

Putting the values of a, b, d

$$w=\frac{-3ciz+2c}{cz-6ci}=\frac{-3zi+2}{z-6i}$$

Multiply by i in the numerator and denominator of r.h.s.

$$w=\frac{-3z(i)^2+2i}{zi-6(i)^2}=\frac{3z+2i}{zi+6}.$$

EXERCISE - VII

1. Find the bilinear transformation which maps the points $z = 1, -i, -1$ onto the points $w = i, 0, -i$.

[**Ans.** : $w = \frac{z+1}{z-1}$

2. Find the bilinear transformation which maps the points
   (i) $z = 0, -i, -1$ onto the points $w = i, 1, 0$. Is the transformation parabolic ? (M.U. 2008)
   (ii) $z = 1, -1, \infty$ onto the points $w = 1 + i, 1 - i, 1$.

[**Ans.** : (i) $w = \frac{i(1+z)}{1-z}$; No, (ii) $w = \frac{z+i}{z}$

3. Find the bilinear transformation which maps the points
   (i) $z = -1, 1, \infty$ onto the points $w = -i, -1, i$. **(M.U. 2003, 08, 14)** [**Ans.** : $w = \frac{iz-2+i}{z+1-2i}$
   (ii) $z = 2, 1, 0$ onto the points $w = 1, 0, i$. **(M.U. 2002)** [**Ans.** : $w = \frac{2i(z-1)}{z(1+i)-2i}$
   (iii) $z = -1, 0, 1$ onto the points $w = -1, -i, 1$. **(M.U. 2003)** [**Ans.** : $w = \frac{z-i}{1-iz}$
   (iv) $z = -i, 0, i$ onto the points $w = -1, i, 1$. **(M.U. 2005)** [**Ans.** : $w = \left(\frac{1-z}{1+z}\right) \cdot i$
   (v) $z = i, -1, 1$ onto the points $w = 0, 1, \infty$. **(M.U. 2003)** [**Ans.** : $w = \frac{2(z-i)}{(1+i)(z-1)}$
   (vi) $z = 0, 1, \infty$ onto the points $w = -5, -1, 3$. **(M.U. 2004, 13, 15)** [**Ans.** : $w = \frac{3z-5}{z+1}$
   (vii) $z = -2, i, 2$ onto the points $w = 0, i, -i$. **(M.U. 2004)** [**Ans.** : $w = \frac{z+2}{i(3z-2)}$

4. Find the bilinear transformation which maps $z = \infty, i, 0$ onto the points $w = 0, i, \infty$.

(M.U. 1998, 2006, 09) [**Ans.** : $w = -1/z$

5. Find the bilinear transformation which maps the points $0, i, -2i$ of z-plane onto the points $-4i$, ∞, 0 respectively of w-plane. Also obtain fixed points of the transformation.

(M.U. 1995, 2006, 16) [**Ans.** : $w = -\frac{2z+4i}{iz+1}$, $z = -\frac{4}{i}, -1$

6. Find the bilinear transformation which maps the points $z = 0, i, -1$ onto $w = i, 1, 0$.

(M.U. 1999) [**Ans.** : $w = \frac{z+1}{(2-i)z-i}$

EXERCISE - VIII

Theory

1. Define conformal mapping and show that the mapping defined by an analytic function is conformal at all points where $f'(z) \neq 0$. (M.U. 1998)
2. State and prove that cross-ratio preservation property of a bilinear transformation. Hence, find the bilinear transformation which maps the points $z = 2, i, -2$ onto the points $w = 1, i, -1$.

(M.U. 1993, 2002) [**Ans.** : $w = \frac{-3iz+2}{z-6i}$

3. Define the cross-ratio of the numbers z_1, z_2, z_3 and z_4. Prove that the cross ratio remains invariant under a bilinear transformation. **(M.U. 1994, 97, 2002)**
4. If w_1, w_2, w_3, w_4 are distinct images of z_1, z_2, z_3, z_4 (all different) under the transformation $w = \frac{az+b}{cz+d}$, $(ad - bc \neq 0)$, then show that $\frac{(z_1 - z_2)}{(z_2 - z_3)} \cdot \frac{(z_3 - z_4)}{(z_4 - z_1)} = \frac{(w_1 - w_2)}{(w_2 - w_3)} \cdot \frac{(w_3 - w_4)}{(w_4 - w_1)}$. **(M.U. 1995)**
5. Prove that, in general, a bilinear transformation maps a circle into a circle. **(M.U. 1996)**
6. Prove that a Bilinear transformation with single fixed point α can be put in the form $\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k$ where k is a constant. **(M.U. 1996)**
7. Define bilinear transformation and prove that under this transformation the cross-ratio is invariant. **(M.U. 1998, 2002, 03)**
8. Out of the following two statements one statement is true. Prove that statement.
   (i) Every bilinear transformation is conformal.
   (ii) If u and v are harmonic functions then $f(z) = u + iv$ is analytic function. **(M.U. 1995)**
9. If $f(z)$ is analytic and $f'(z) \neq 0$ in R, prove that $w = f(z)$ is conformal in R. **(M.U. 1997)**
10. Define bilinear transformation. Show that every bilinear trans-formation is a combination of (i) translation, (ii) Rotation and Magnification, (iii) Inversion. **(M.U. 1999)**

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CHAPTER 7

Z - Transforms

1. Introduction

We have already studied Laplace Transforms and Fourier Transforms. Both these transforms are continuous functions. These transforms are not useful for studying discrete systems. Linear systems in which the input signals are in the form of discrete pulses of short duration are called 'Linear Time Invariant' (LTI) systems. For the analysis of such systems we need Z-transforms. In this chapter we shall first get acquainted with sequences, then study Z-transforms and then inverse Z-transforms. After studying Laplace Transforms and Z-transforms, you will find that Z-transform is the discrete analogue of Laplace Transform. For every operational rule and application of Laplace transform there corresponds an operational rule or application of Z-transform. For example, you will find Linearity Property, Shifting Theorem, Convolution Theorem etc. in both Laplace Transforms and Z-transforms.

2. Sequences

If objects are arranged according to a certain rule, this arrangement is called a sequence. We are particularly interested in sequences whose members are real or complex numbers. So we define a sequence as follows.

Definition : An ordered set of real or complex numbers is called a **sequence**.

We shall denote a sequence by $\{ f(k) \}$ and k-th term of the sequence by $f(k)$. For example, we have a sequence

$$\{2^0, 2^1, 2^2, 2^3, \ldots\ldots 2^k, \ldots\ldots\}$$

For $k = 0$, $f(k) = 2^0$; for $k = 1$, $f(k) = 2^1$ ..... Thus, in a sequence we have to take into account the order of a term k, and the term of k-th order $f(k)$. The set of all such ordered terms $\{ f(0), f(1), \ldots\ldots, f(k), \ldots\ldots \}$ is called a sequence.

1. The most elementary way to denote a sequence is to list all the members of the sequence. For example,

$$\{f(k)\} = \{12, 10, \underset{\uparrow}{8}, 5, 3, 6, 9\}$$

The arrow ↑ indicates the element corresponding to $k = 0$. The elements on the left of the arrow correspond to $k = -1, -2, -3, \ldots\ldots$ and those on the right correspond to $k = 1, 2, 3, \ldots\ldots$

2. Another way of denoting a sequence is to give the general term in terms of k which varies from $-\infty$ to ∞ taking integral values.

For example $\{f(k)\} = 2^k$ (where k is an integer). This sequence is

$$\{\ldots\ldots, 2^{-3}, 2^{-2}, 2^{-1}, 2^0, 2^2, 2^3, \ldots\ldots\}$$

As illustrations we can have the following sequences and can have many more.

$$\{1, 1, 1, 1, \ldots\ldots\}$$
$$\{1, 2, 3, \ldots\ldots, k, \ldots\ldots\}$$
$$\{1^2, 2^2, 3^2, \ldots\ldots, k^2, \ldots\ldots\}$$
$$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots\}$$
$$\{1 \cdot 2^1, 2 \cdot 2^2, 3 \cdot 2^3, \ldots\ldots, k \cdot 2^k, \ldots\ldots\}$$
$$\{1 \cdot \alpha^1, 2 \cdot \alpha^2, 3 \cdot \alpha^3, \ldots\ldots, k \cdot \alpha^k, \ldots\ldots\}$$
$$\left\{\frac{\alpha^1}{1}, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \ldots\ldots, \frac{\alpha^k}{k}, \ldots\ldots\right\}$$
$$\left\{\frac{5^1}{1!}, \frac{5^2}{2!}, \frac{5^3}{3!}, \ldots\ldots, \frac{5^k}{k!}, \ldots\ldots\right\}$$
$$\{e^{1 \cdot \alpha}, e^{2 \cdot \alpha}, e^{3 \cdot \alpha}, \ldots\ldots, e^{k \cdot \alpha}, \ldots\ldots\}$$

3. Basic Operations On Sequences

We shall see below some properties of sequences through examples.

1. **Addition :** The sum (or difference) of two sequences is obtained by adding (or subtracting) the corresponding terms of the two sequences.

For example, if
$$\{f(k)\} = 1^3, 2^3, 3^3, 4^3, \ldots\ldots$$
$$\{g(k)\} = 1^2, 2^2, 3^2, 4^2, \ldots\ldots$$
then
$$\{f(k)\} + \{g(k)\} = \{(1^3 + 1^2), (2^3 + 2^2), (3^3 + 3^2), \ldots\ldots, (k^3 + k^2) \ldots\ldots\}$$
$$= \{1^2 \cdot 2, 2^2 \cdot 3, 3^2 \cdot 4, \ldots\ldots, k^2 (k+1) \ldots\ldots\}$$
$$\{f(k)\} - \{g(k)\} = \{(1^3 - 1^2), (2^3 - 2^2), (3^3 - 3^2), \ldots\ldots, (k^3 - k^2), \ldots\ldots\}$$
$$= \{1^2 \cdot 0, 2^2 \cdot 1, 3^2 \cdot 2, \ldots\ldots, k^2 \cdot (k-1), \ldots\ldots\}$$

2. **Scalar Multiplication :** If α is a scalar then from a given sequence $\{f(k)\}$ we can obtain another sequence $\alpha \{f(k)\}$ by multiplying each term $f(k)$ of the sequence $\{f(k)\}$ by α.

For example, if $\{f(k)\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots$, then
$$3 \cdot \{f(k)\} = 3, \frac{3}{2}, \frac{3}{3}, \ldots\ldots, \frac{3}{k}, \ldots\ldots$$
If $\{f(k)\} = \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\ldots, \sqrt{k}, \ldots\ldots$, then
$$2 \cdot \{f(k)\} = 2\sqrt{1}, 2\sqrt{2}, 2\sqrt{3}, \ldots\ldots, 2\sqrt{k}, \ldots\ldots$$

3. **Linearity :** If α and β are two scalars then from two sequences $\{f(k)\}$ and $\{g(k)\}$, we can obtain another sequence by multiplying the terms of the two sequences by α and β as above and adding the corresponding terms.

i.e. $\alpha \{f(k)\} + \beta \{g(k)\} = \{\alpha \cdot f(k) + \beta \cdot g(k)\}$

For example, if $\{f(k)\} = 1, 2, 3, \ldots\ldots, k, \ldots\ldots$

and $\{g(k)\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots$

then, $2 \cdot \{f(k)\} + 3 \cdot \{g(k)\} = \{2 \cdot f(k) + 3 \cdot g(k)\}$

$$= 2(1) + 3\left(\frac{1}{1}\right),\ 2 \cdot 2 + 3\left(\frac{1}{2}\right),\ 2(3) + 3\left(\frac{1}{3}\right),\ \ldots\ldots,\ 2 \cdot k + 3\left(\frac{1}{k}\right) + \ldots$$

$$= 2 + \frac{3}{1},\ 4 + \frac{3}{2},\ 6 + \frac{3}{3},\ \ldots\ldots,\ 2k + \frac{3}{k},\ \ldots\ldots$$

4. **Convergence And Divergence :** Consider the following sequence

$$\frac{1+1}{1},\ \frac{2+1}{2},\ \frac{3+1}{3},\ \frac{4+1}{4},\ \ldots\ldots,\ \frac{k+1}{k},\ \ldots\ldots$$

i.e. $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\ldots, 1 + \frac{1}{k}, \ldots\ldots$ *i.e.*, 2, 1·5, 1·33, 1·25, ...

It is easy to see that as the number of terms become infinite the sequence goes on decreasing and ultimately takes the value 1. Such a sequence $\{f(k)\}$ is called a convergent sequence.

Definition : If $\{f(k)\}$ is a given sequence and if $f(k)$ tends to a (finite) real number L as k tends to infinity then $\{f(k)\}$ is called a **convergent sequence**.

The following sequences are convergent.

(i) $a, a, a, \ldots\ldots, a, \ldots\ldots, a, \ldots\ldots$ converges to a

(ii) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots$ converges to 0

(iii) $1 + \frac{1}{2^0}, 1 + \frac{1}{2^1}, 1 + \frac{1}{2^2}, \ldots\ldots, 1 + \frac{1}{2^k}, \ldots\ldots$ converges to 1.

Definition : A sequence which is not convergent *i.e.* which does not tend to a (finite) real number is called a **divergent** sequence.

The following are divergent sequences.

(i) $1, 2, 3, \ldots\ldots, k, \ldots\ldots$ diverges to ∞

(ii) $-1, -2, -3, \ldots\ldots, -k, \ldots\ldots$ diverges to $-\infty$

(iii) $1, 2, 1, 2, 1, 2, \ldots\ldots$ oscillates between 1 and 2

(iv) $0, 1, 0, 1, 0, 1, \ldots\ldots$ oscillates between 0 and 1.

EXERCISE - I

1. Write down the term corresponding to $k = 3$ of the following sequence
$\{-6, -3, -1, \underset{\uparrow}{0}, 2, 4, 6, 8, 10\}$ [Ans. : 8]

2. Write down the term corresponding to $k = -3$ of the following sequence.
$\{-12, -10, -9, \underset{\uparrow}{-7}, -5, -3, 1, 4, 6, 10\}$ [Ans. : -10]

3. Write down the sequence if k-th term is 3^k for $-2 \le k \le 4$.
[Ans. : $\frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, 81$]

4. Write down the sequence whose k-th term is 2^k for $-\infty < k < \infty$.
[Ans. : $\left\{\ldots\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\right\}$]

5. Write down the sequence whose k-th term $= \begin{cases} 4^k, & k < 0 \\ 3^k, & k \ge 0 \end{cases}$
[Ans. : $\left\{\ldots\ldots, \frac{1}{64}, \frac{1}{16}, \frac{1}{4}, 1, 3, 9, 27, \ldots\right\}$]

6. Write down the sequence whose k-th term $= \begin{cases} a^k, & k < 0 \\ b^k, & k \ge 0 \end{cases}$
[Ans. : $\left\{\ldots\ldots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a}, 1, b, b^2, b^3, \ldots\right\}$]

4. Z - transforms

We shall now define Z-transform of a sequence.

Definition : Let $\{f(k)\} = \{\ldots\ldots f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \ldots\ldots\}$ be a sequence of terms where k varies from $-\infty$ to ∞.

Let $z = x + iy$ be a complex number then

$$Z\{f(k)\} = \ldots + f(-3)\,z^3 + f(-2)\,z^2 + f(-1)\,z^1 + f(0)\,z^0 + f(1)\,z^{-1} + f(2)\,z^{-2} + f(3)\,z^{-3} + \ldots$$

$$= \sum_{k=-\infty}^{\infty} f(k)\,z^{-k} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k}$$

is called the ***Z*-transform of the sequence $\{f(k)\}$**.

In words, the sum of the product of k-th term of the sequence $f(k)$ with z^{-k} taken from $-\infty$ to ∞ is called the ***Z*-transform of the sequence $\{f(k)\}$**.

Thus,
$$\boxed{Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\,z^{-k}}$$

Notation : Unfortunately there is no unanimity in the notations used in the case of Z-transform. Some use $x_k, y_k, \ldots\ldots$ to denote sequences and $x_0, x_1, \ldots\ldots, y_0, y_1, \ldots\ldots$ to denote the terms of the sequences. We shall denote the sequences by $\{f(k)\}, \{g(k)\}, \ldots\ldots$ the terms of the sequences by $f(1), f(2), \ldots\ldots, g(1), g(2), \ldots\ldots$ and Z-transforms by $Z\{f(k)\}, Z\{g(k)\}, \ldots\ldots$ or by $F(z), G(z), \ldots\ldots$ etc.

Notes ....

1. It is necessary to know which is the zeroth term, first term, second term .... minus first term, minus second term ..... *i.e.* we must know the order of each term. To obtain Z-transform of a sequence we multiply each term by **negative power of z of the order of that term** and take the sum.
2. $Z\{f(k)\}$ is a function of a complex variable z and is defined only if the sum is finite *i.e.* **if the infinite series $\sum f(k)\,z^{-k}$ is absolutely convergent.** We shall denote the Z-transform of the sequence $\{f(k)\}$ by $Z\{f(k)\}$ or by $F(z)$.
3. Wherever necessary we shall denote the sequences by $\{f(k)\}, \{g(k)\}$ etc.
4. If Z-transform of $\{f(k)\}$ is $F(z)$ we call **$\{f(k)\}$ the inverse Z transform of $F(z)$ and denote** it by $Z^{-1}[F(z)]$.

Example 1 : If $\{f(k)\} = \{-6, -3, \underset{\uparrow}{0}, 2, 4\}$,

find $Z\{f(k)\}$ where $\uparrow$ denotes the element corresponding to $k = 0$.

Sol. : $Z\{f(k)\} = \Sigma f(k)\, z^{-k}$

To obtain $Z\{f(k)\}$, we multiply each term $f(k)$ of the sequence by z^{-k}, and take the sum.

Multiply the 0^{th} term 0 by z^0, the first term 2 by z^{-1}, the second term 4 by z^{-2}, $(-1)^{st}$ term (-3) by $(z^{-1})^{-1}$ i.e., z, $(-2)^{nd}$ term (-6) by $(z^{-1})^{-2}$ i.e., z^2 and take the sum of these products.

$$\therefore \quad Z\{f(k)\} = f(-2)\,(z^{-1})^{-2} + f(-1)(z^{-1})^{-1} + f(0)\, z^0 + f(1)\, z^{-1} + f(2)\, z^{-2}$$

where, $f(-2) = -6,\ f(-1) = -3,\ f(0) = 0,\ f(1) = 2,\ f(2) = 4.$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-2}^{2} f(k)\, z^{-k} = (-6)\, z^2 + (-3)\, z^1 + 0 \cdot z^0 + 2z^{-1} + 4z^{-2}$$

$$= -6z^2 + 3z + 0 + \frac{2}{z} + \frac{4}{z^2}$$

Example 2 : If $\{f(k)\} = \{9, \underset{\uparrow}{6}, 3, 0, -3, -6, -9\}$, find $Z\{f(k)\}$.

Sol. : Since 3 is the term corresponding to $k = 0$. We have

$$f(-2) = 9,\ f(-1) = 6,\ f(0) = 3,\ f(1) = 0,\ f(2) = -3,\ f(3) = -6,\ f(4) = -9.$$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-2}^{4} f(k)\, z^{-k} = 9\, z^2 + 6\, z^1 + 3\, z^0 + 0\, z^{-1} - 3\, z^{-2} - 6\, z^{-3} - 9\, z^{-4}$$

$$= 9z^2 + 6z + 3 + 0 - \frac{3}{z^2} - \frac{6}{z^3} - \frac{9}{z^4}.$$

Example 3 : If $\{f(k)\} = \{2^0, 2^1, 2^2, 2^3, ...\}$ find $Z\{f(k)\}$.

Sol. : Z-transform of the sequence is

$$\therefore \quad Z\{f(k)\} = \sum_{k=0}^{k=\infty} f(k)\, z^{-k} = 2^0 z^0 + 2^1 z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + ...$$

$$= 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + ...$$

$$= \frac{1}{1-(2/z)} = \frac{z}{z-2} \quad \text{if } \left|\frac{2}{z}\right| < 1 \qquad \left(\because\ s_\infty = a + ar + ar^2 + ... = \frac{a}{1-r},\ \text{if } |r| < 1\right)$$

Example 4 : If $\{f(k)\} = \begin{cases} 4^k, & \text{for } k < 0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$, find $Z\{f(k)\}$.

Sol. : The sequence is $\{f(k)\} = \{....., 4^{-4}, 4^{-3}, 4^{-2}, 4^{-1}, 3^0, 3^1, 3^2, 3^3, ...\}$

And Z-transform of $f(k)$ is

$$\therefore \quad Z\{f(k)\} = \{....., 4^{-4} z^4 + 4^{-3} z^3 + 4^{-2} z^2 + 4^{-1} z + 3^0 z^0 + 3^1 z^{-1} + 3^2 z^{-2} + 3^3 z^{-3} + ...$$

We write positive powers of z in reverse order.

$$\therefore \quad Z\{f(k)\} = \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + ...\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + ...\right]$$

$$= \frac{z}{4}\left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + ...\right] + \left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + ...\right]$$

$$\therefore \quad Z\{f(k)\} = \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(3/z)} \quad \text{if } \left|\frac{z}{4}\right| < 1,\ \left|\frac{3}{z}\right| < 1$$

$$= \frac{z}{4} \cdot \frac{4}{4-z} + \frac{z}{z-3} = \frac{z}{4-z} + \frac{z}{z-3}$$

$$= \frac{z}{(4-z)(z-3)} \quad \text{if } 3 < |z| < 4.$$

Note ....

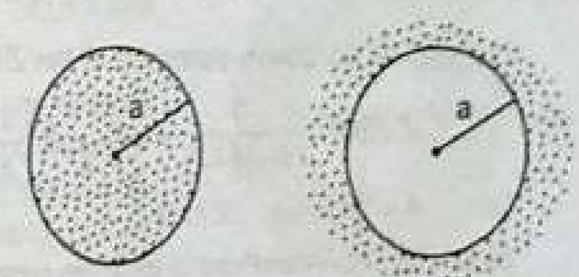
We shall require the following results in finding Z-transforms.

1. $1 + r + r^2 + r^3 + ... = \dfrac{1}{1-r}$ if $|r| < 1$.
2. $1 - x + x^2 - x^3 + ... = (1+x)^{-1}$ if $|x| < 1$.
3. $1 + x + x^2 + x^3 + ... = \sum_{k=0}^{\infty} x^k = (1-x)^{-1}$ if $|x| < 1$.
4. $1 + nx + \dfrac{n(n-1)}{2!} x^2 + \dfrac{n(n-1)(x-2)}{3!} x^3 + ... = (1+x)^n$
5. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... = e^x$
6. If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$.
7. **The set $|z| < a$.** Since $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$.

$\therefore \quad |z| < a$ means $\sqrt{x^2 + y^2} = a$

i.e. $x^2 + y^2 < a^2$.

This means, $|z| < a$ is the set of points inside the circle of radius a and centre at the origin. By the same reasoning $|z| > a$ is the set of points outside the circle with radius a and centre at the origin.



|z|<a **|z|>a**

EXERCISE - II

1. Write down the Z-transforms of the following sequences

(i) $\{f(k)\} = \{8, 6, 4, 2, \underset{\uparrow}{0}, 1, 3, 5, 7\}$ (ii) $\{f(k)\} = \{-6, -4, -2, \underset{\uparrow}{1}, 2, 4, 6\}$

[**Ans. :** (i) $8z^6 + 6z^5 + 4z^4 + 2z^3 + 0 + 1z + 3 + \dfrac{5}{z} + \dfrac{7}{z^2}$

(ii) $-6z^3 - 4z^2 - 2z + 1 + \dfrac{2}{z} + \dfrac{4}{z^2} + \dfrac{6}{z^3}$]

2. Write down the Z-transform of the following sequences

(i) $\{f(k)\} = 3^k,\ k \geq 0$ (ii) $\{f(k)\} = 5^k,\ k \geq 0$

[**Ans. :** (i) $\dfrac{z}{z-3}$, $\left|\dfrac{3}{z}\right| < 1$, (ii) $\dfrac{z}{z-5}$, $\left|\dfrac{5}{z}\right| < 1$.]

Inverse Z-Transform

Definition : If $F(z)$ is the Z-transform of the sequence $\{f(k)\}$ then the sequence $\{f(k)\}$ is called the inverse Z-transform of $F(z)$ and is denoted as

$$\boxed{\{f(k)\} = Z^{-1}[F(z)]}$$

Thus, we have if $Z\{f(k)\} = F(z)$, then $\{f(k)\} = Z^{-1}[f(z)]$ and vice versa.

5. Region of Convergence (ROC)

We shall try to understand this important concept in relation to Z-transforms through two examples.

Example 1 : Consider the sequence $f(k) = \begin{cases} 0 & \text{for } k < 0 \\ 4^k & \text{for } k \geq 0 \end{cases}$

i.e., the sequence $\{f(k)\} = \{4^0, 4^1, 4^2, 4^3, \ldots, 4^k, \ldots\}$.

Its Z-transform by definition is

$$Z\{f(k)\} = \Sigma f(k)\, z^{-k} = \sum_{k=0}^{\infty} 4^k z^{-k} = 4^0 z^0 + 4z^{-1} + 4^2 z^{-2} + 4^3 z^{-3} + \ldots$$

$$= 1 + \frac{4}{z} + \frac{16}{z^2} + \frac{64}{z^3} + \ldots$$

Notice that $Z\{f(k)\}$ is a Geometric Progression with common ratio $4/z$. We know that the sum S of infinite terms of a G.P. with first term 1 and common ratio r is given by $S = \frac{1}{1-r}$ if $|r| < 1$.

The sum of the above series *i.e.* of the Z-transform is

$$Z\{f(k)\} = \frac{1}{1-(4/z)} = \frac{z}{z-4}, \text{ if } \left|\frac{4}{z}\right| < 1$$

i.e. $4 < |z|$ *i.e.* $|z| > 4$.

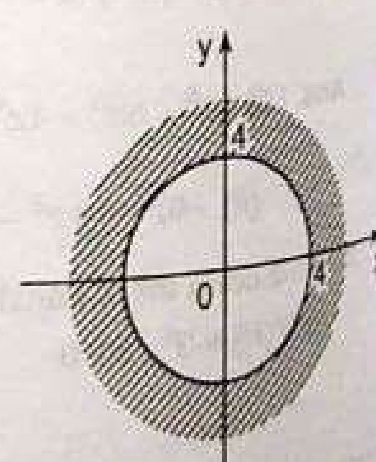
Since, for the Z-transform to exist the corresponding series must be convergent. The above Z-transform is defined only if $|z| > 4$.

Note ....

Note that a G.P. $a, ar, ar^2, \ldots, ar^n \ldots$ is convergent if $|r| < 1$ and its sum

$$S = \frac{a}{1-r} \text{ where } |r| < 1.$$

But $|z| = 4$ is a circle with centre at the origin and radius $= 4$. Hence, the above Z-transform is defined if $z > 4$ *i.e.* if z is on the exterior of the circle $|z| = 4$.



In this case, we say that the region of convergence of this Z-transform is the exterior of the circle $|z| = 4$.

The region for which $\Sigma f(k)\, z^{-k}$ is convergent is called the region of convergence denoted in short by R.O.C.

Example 2 : Find the Z-transform and the region of convergence of $f(k) = \begin{cases} 5^k & \text{for } k < 0 \\ 3^k & \text{for } k \geq 0 \end{cases}$

Sol. : By definition $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

Putting $k = -n$ in the first series, we get

$$Z\{f(k)\} = \sum_{n=1}^{\infty} 5^{-n} z^n + \sum_{k=0}^{\infty} 3^k z^{-k}$$

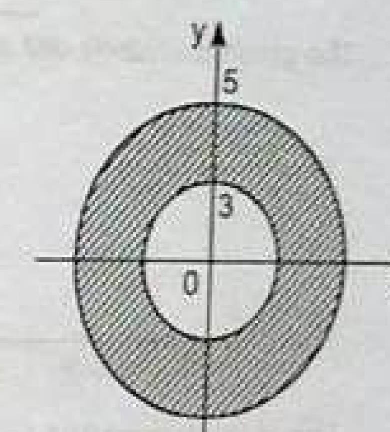
$$Z\{f(k)\} = \left[\frac{z}{5} + \frac{z^2}{5^2} + \frac{z^3}{5^3} + \ldots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right]$$

$$= \frac{z}{5}\left[1 + \frac{z}{5} + \frac{z^2}{5^2} + \ldots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right]$$

$$= \frac{z}{5} \cdot \frac{1}{1-(z/5)} + \frac{1}{1-(3/z)} = \frac{z}{5-z} + \frac{z}{z-3}$$

$$= \frac{2z}{(5-z)(z-3)}$$

Now, $Z\{f(k)\}$ is the sum of two Geometric Progressions with the common ratios $(z/5)$ and $(3/z)$ respectively. The series will be convergent if $|z/5| < 1$ and $|3/z| < 1$. *i.e.* $|z| < 5$ and $3 < |z|$ *i.e.* $3 < |z| < 5$.



But $|z| = 3$ is a circle with centre at the origin and radius 3 and $|z| = 5$ is a circle with centre at the origin and radius 5. Hence, $Z\{f(k)\}$ is convergent if z lies between the annulus as shown in the figure. This is the region of convergence of $Z\{f(k)\}$ which is shown by shaded area.

$\therefore$ ROC is $3 < |z| < 5$.

6. Z-Transforms of Some Standard Functions

Example 1 : Find the Z-transform of *Unit Impulse function*

$$\delta(k) = 1 \quad \text{for } k = 0$$
$$= 0 \quad \text{otherwise}$$

Sol. :

$$Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \{\ldots 0 + 0 + 0 + 1 \cdot z^0 + 0 + 0 + 0 \ldots\}$$

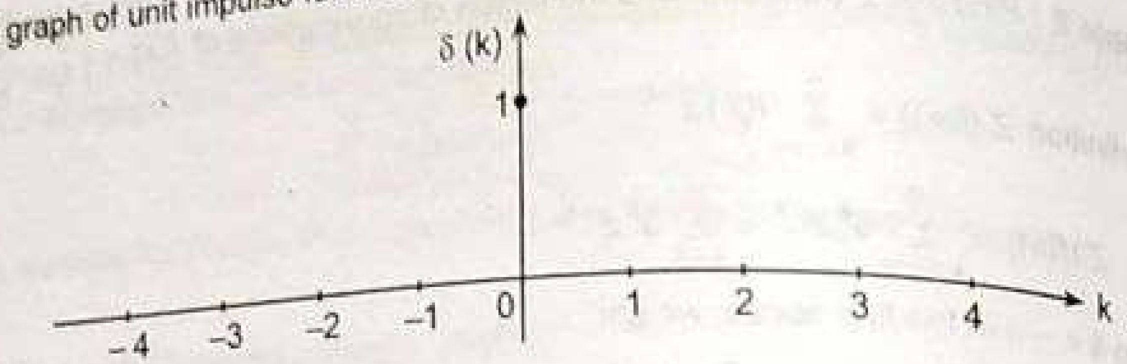
$$= 1 \text{ for all } z$$

$$\therefore \quad \boxed{Z\{\delta(k)\} = 1}$$

This is convergent for all z.

$\therefore$ ROC is whole of z-plane.

The graph of unit impulse function is,



Example 2 : Find the Z-transform of *Discrete Unit Step function*

$U(k) = 1 \quad$ for $k \geq 0$

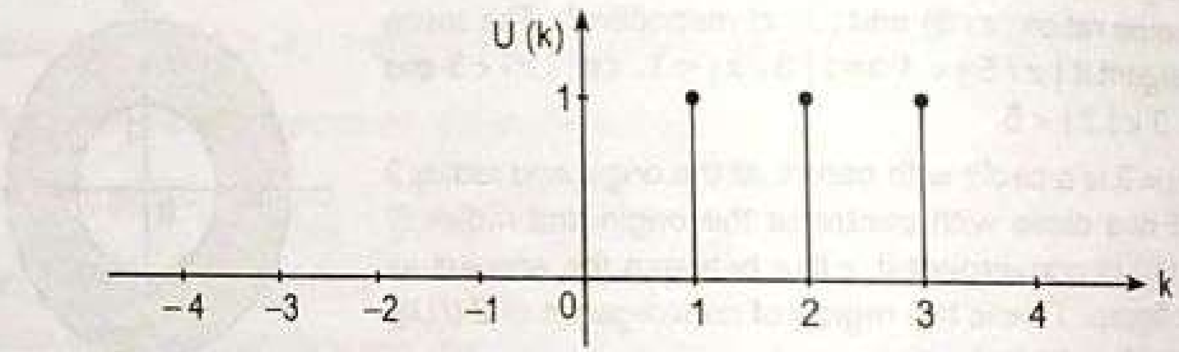
$\quad\;\; = 0 \quad$ for $k < 0$

Sol. : $$Z\{U(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} 1 \cdot z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

$$= \left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right] = \frac{1}{1-(1/z)} = \frac{z}{z-1}$$

$$\therefore \quad \boxed{Z\{U(k)\} = \frac{z}{z-1},} \qquad k \geq 0$$

The graph of discrete unit step function is,



This is convergent if $| 1/z | < 1$ *i.e.* $1 < | z |$ *i.e.* $| z | > 1$

$\therefore$ ROC is $| z | > 1$.

Since, $Z\{U(k)\} = \frac{z}{z-1}$, $Z^{-1}\left[\frac{z}{z-1}\right] = \{U(k)\}$ where Z^{-1} denotes inverse Z-transform.

Example 3 : Find the z-transform of $f(k) = \alpha^k$, $\alpha > 0$, $k \geq 1$.

Sol. : We have $$Z\{f(k)\} = \sum_{k=1}^{\infty} \alpha^k z^{-k} = \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + \frac{\alpha^3}{z^3} + \ldots$$

$$= \frac{\alpha}{z}\left(1 + \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + \ldots\right) \qquad [\text{G.P.}]$$

$$= \frac{\alpha}{z} \cdot \frac{1}{1-(\alpha/z)} = \frac{\alpha}{z-\alpha}, \; |\alpha| < |z|.$$

$$\therefore \quad \boxed{Z\{\alpha^k\} = \frac{\alpha}{z-\alpha},} \qquad k \geq 1$$

Cor. : Putting $\alpha = 1$, $$\boxed{Z\{1\} = \frac{1}{z-1},} \qquad k \geq 1$$

Example 4 : Find the Z-transform of $f(k) = k\alpha^k$, $k \geq 0$.

Sol. : Assuming $f(k) = 0$ for $k < 0$,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^k + \sum_{k=0}^{\infty} k\alpha^k z^{-k}$$

$$= 0 + 1 \cdot \frac{\alpha}{z} + 2 \cdot \frac{\alpha^2}{z^2} + 3 \cdot \frac{\alpha^3}{z^3} + \ldots \qquad \ldots\ldots (1)$$

$$= \frac{\alpha}{z}\left(1 + 2\frac{\alpha}{z} + 3\frac{\alpha^2}{z^2} + \ldots\right)$$

$$= \frac{\alpha}{z}\left(1 - \frac{\alpha}{z}\right)^{-2} = \frac{\alpha}{z} \cdot \frac{1}{[1-(\alpha/z)]^2} = \frac{\alpha z}{(z-\alpha)^2}$$

$$\therefore \quad \boxed{Z\{k\alpha^k\} = \frac{\alpha z}{(z-\alpha)^2}}$$

Cor. : Putting $\alpha = 1$, $$\boxed{Z\{k\} = \frac{z}{(z-1)^2}}$$

Applying D'Alembert's radio test to (1), we find that the series is convergent if $| \alpha / z | < 1$ *i.e.,* $|\alpha| < | z |$ *i.e.,* $| z | > | \alpha |$.

$\therefore$ ROC is $| z | > | \alpha |$

Particular Cases : **(i)** Find the Z-transform of $f(k) = k2^k$, $k \geq 0$.

(ii) Find the z-transform of $f(k) = k\,2^k + k\,3^k$.

Sol. : Put (i) $\alpha = 2$ and (ii) $\alpha = 2$, $\alpha = 3$, in the above example.

$$\left[\textbf{Ans. : (i)} \; \frac{2z}{(z-2)^2}, |z| > 2, \; \textbf{(ii)} \; \frac{2z}{(z-2)^2} + \frac{3z}{(z-3)^2}, |z| > 3\right]$$

Example 5 : Find the Z-transform of $f(k) = \frac{\alpha^k}{k}$, $k \geq 1$.

Sol. : Assuming $f(k) = 0$ for $k \leq 0$,

$$Z\{f(k)\} = \sum_{k=-\infty}^{0} 0 \cdot z^{-k} + \sum_{k=1}^{\infty} \frac{\alpha^k}{k} z^{-k}$$

$$= \frac{\alpha}{z} + \frac{\alpha^2}{2z^2} + \frac{\alpha^3}{3z^3} + \frac{\alpha^4}{4z^4} + \ldots \qquad \ldots\ldots (1)$$

$$Z\{f(k)\} = -\log\left(1 - \frac{\alpha}{z}\right)$$

$$\therefore \quad \boxed{Z\left(\frac{\alpha^k}{k}\right) = -\log\left(1 - \frac{\alpha}{z}\right)}$$

Applying D'Alembert's Ratio Test to (1), we find that the series is convergent if $| \alpha / z | < 1$ *i.e.,* $|\alpha| < | z |$ *i.e.,* $| z | > | \alpha |$.

$\therefore$ ROC is $| z | > | \alpha |$.

Particular cases : (i) Find Z-transform of $f(k) = \frac{1}{k}$, $k \ge 1$.

(ii) Find the Z-transform of $f(k) = \frac{2^k}{k}$, $k \ge 1$.

Sol. : Put (i) $\alpha = 1$, and (ii) $\alpha = 2$ in the above example.

[**Ans. : (i)** $-\log\left(1-\frac{1}{z}\right)$, $|z| > 1$; **(ii)** $-\log\left(1-\frac{2}{z}\right)$, $|z| > 2$.]

Example 6 : Find the Z-transform of $f(k) = a^k$, $k \ge 0$. **(M.U. 2009, 16)**

Sol. : Assuming that $f(k) = 0$ when $k < 0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} a^k z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots\ldots = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

$$\therefore \quad \boxed{Z(a^k) = \frac{z}{z-a}}$$

The series being G.P. is convergent if $1 > |a/z|$ i.e. $|z| > |a|$.

$\therefore$ ROC is $|z| > |a|$.

Since, $Z\{a^k\} = \frac{z}{z-a}$, $Z^{-1}\left[\frac{z}{z-a}\right] = a^k$, $k \ge 0$.

Example 7 : Find the Z-transform of $f(k) = b^k$, $k < 0$. **(M.U. 2008, 09)**

Sol. : Assuming that $f(k) = 0$ when $k \ge 0$.

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{-1} b^k z^{-k} = \sum_{n=1}^{\infty} b^{-n} z^n \text{ where } n = -k$$

(Note the substitution $n = -k$)

$$= \frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} + \ldots = \frac{z}{b}\left(1 + \frac{z}{b} + \frac{z^2}{b^2} + \ldots\ldots\right)$$

$$= \frac{z}{b}\,\frac{1}{1-(z/b)} = \frac{z}{b-z}$$

$$\therefore \quad \boxed{Z(b^k) = \frac{z}{b-z}}$$

The series being G.P. is convergent if $1 > |z/b|$ i.e. $|b| > |z|$.

$\therefore$ ROC is $|z| < |b|$.

Since, $Z\{b^k\} = \frac{z}{b-z}$, $Z^{-1}\left[\frac{z}{b-z}\right] = b^k$, $k < 0$.

Example 8 : Find Z-transform of $f(k) = \begin{cases} b^k, & k < 0 \\ a^k, & k \ge 0 \end{cases}$

Sol. : By Examples 7 and 6, we get

$$Z\{f(k)\} = \frac{z}{b-z} + \frac{z}{z-a} = \frac{z^2 - az + bz - z^2}{(z-a)(b-z)}$$

$$\therefore \quad Z\{f(k)\} = \frac{-z(a-b)}{-(z-a)(z-b)} = \frac{(a-b)\,z}{(z-a)(z-b)}$$

if $|z| > a$ and $|z| < b$ i.e., $a < |z| < b$.

Note ....

In general the Z-transform of $\{f(k)\} = b^k$, for $k < 0$ and $\{f(k)\} = a^k$ for $k \ge 0$ is $\frac{(a-b)\,z}{(z-a)(z-b)}$ and $a < |z| < b$. This Z-transform exists only if $a < b$.

Example 9 : Find the Z-transform of $f(k) = {}^nC_k$, $0 \le k \le n$.

Sol. : $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^nC_k z^{-k} = \sum_{k=0}^{n} {}^nC_k z^{-k}$$

$$= {}^nC_0 + {}^nC_1 \frac{1}{z} + {}^nC_2 \frac{1}{z^2} + \ldots\ldots + {}^nC_n \frac{1}{z^n} = \left(1 + \frac{1}{z}\right)^n$$

The series being finite is obviously convergent if $z \ne 0$.

$$\therefore \quad \boxed{Z({}^nC_k) = \left(1 + \frac{1}{z}\right)^n}$$

$\therefore$ ROC is all of z-plane except the origin.

Example 10 : Find the Z-transform of $f(k) = {}^kC_n$, $k \ge n$.

Sol. : $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^kC_n z^{-k} = \sum_{k=n}^{\infty} {}^kC_n z^{-k}$$

To find the sum we change the dummy index k by $k = n + r$.

$$\therefore \quad Z\{f(k)\} = \sum_{r=0}^{\infty} {}^{n+r}C_n z^{-(n+r)} = \sum_{r=0}^{\infty} {}^{n+r}C_n z^{-n} z^{-r}$$

$$= z^{-n} \sum_{r=0}^{\infty} {}^{n+r}C_r z^{-r} \quad [\because {}^nC_r = {}^nC_{n-r}, \text{ we get } {}^{n+r}C_n = {}^{n+r}C_{n+r-n} = {}^{n+r}C_r]$$

$$= z^{-n}\left[1 + {}^{n+1}C_1 z^{-1} + {}^{n+1}C_2 z^{-2} + \ldots\right] = z^{-n}\left(1 - \frac{1}{z}\right)^{-(n+1)}$$

$$\therefore \quad \boxed{Z({}^kC_n) = z^{-n}\left(1 - \frac{1}{z}\right)^{-(n+1)}}$$

$\therefore$ ROC is $|z| > 1$.

Example 11 : Find the Z-transform of $f(k) = {}^{k+n}C_n$.

Sol. : $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^{k+n}C_n z^{-k}$$

But ${}^{k+n}C_n = 0$ if $k + n < n$ i.e. if $k < 0$.

$$\therefore \quad Z\{f(k)\} = \sum_{k=0}^{\infty} {}^{k+n}C_n z^{-k} = \sum_{k=0}^{\infty} {}^{k+n}C_k z^{-k} \quad \text{(as in the previous example)}$$

$$= 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + \ldots\ldots$$

$$\therefore \quad Z\{f(k)\} = \left(1-\frac{1}{z}\right)^{-(n+1)}$$

$$\therefore \quad \boxed{Z\left({}^{k+n}C_n\right) = \left(1-\frac{1}{z}\right)^{-(n+1)}}$$

As above ROC is $|z| > 1$.

Example 12 : Find the Z-transform of $f(k) = \frac{a^k}{k!}$, $k \geq 0$. (M.U. 2009)

Sol. :
$$Z\{f(k)\} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{(a/z)^k}{k!}$$
$$= 1 + \frac{a}{z} + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \ldots\ldots$$
$$= e^{a/z} \text{ ROC is all of } z\text{-plane.}$$

Remark ....

You are advised to memorise the Z-transforms of these standard functions.

EXERCISE - III

Find the Z-transform and its ROC of each of the following sequences $\{f(k)\}$ where $f(k)$ is given by

1. $f(k) = 3^k, k \geq 0$ 2. $f(k) = 4^k, k \geq 0$ 3. $f(k) = (1/6)^k, k \geq 0$
4. $f(k) = 2, k \geq 0$ 5. $f(k) = 4, k \geq 0$ 6. $f(k) = 5, k \geq 0$
7. $f(k) = 2^k, k < 0$ 8. $f(k) = 4^k, k < 0$ 9. $f(k) = (1/3)^k, k < 0$
10. $f(k) = 3^k, k < 0$; $= 2^k, k \geq 0$ 11. $f(k) = 4^k, k < 0$; $= 3^k, k \geq 0$ 12. $f(k) = a^k, k < 0$; $= b^k, k \geq 0$ $(a, b > 0, a > b)$
13. $f(k) = k3^k, k \geq 0$ 14. $f(k) = k\,5^k, k \geq 0$ 15. $f(k) = ka^k, k \geq 0$ $(a > 0)$
16. $f(k) = \frac{3^k}{k}, k > 1$ 17. $f(k) = \frac{2^k}{k}, k > 1$ 18. $f(k) = \frac{a^k}{k}, k > 1, a > 0$
19. $f(k) = (1/2)^{|k|}$, for all k 20. $f(k) = (1/4)^{|k|}$, for all k
21. $f(k) = a^k$ for all k $(0 < a < 1)$ 22. $f(k) = (3^k / k!), k \geq 0$
23. $f(k) = (5^k / k!), k \geq 0$ 24. $f(k) = e^{k\alpha}, k \geq 0$
25. $f(k) = \begin{cases} 2^k, & k \leq -1 \ldots\ldots \\ (1/2)^k, & k = 0, 2, 4 \ldots\ldots \\ (1/3)^k, & k = 1, 3, 5 \ldots\ldots \end{cases}$

[**Ans. :** (1) $\frac{1}{1-(3/z)}$; $|z| > 3$, (2) $\frac{1}{1-(4/z)}$; $|z| > 4$, (3) $\frac{1}{1-(1/6z)}$; $|z| > \frac{1}{6}$,

(4) $2 \cdot \frac{1}{1-(1/z)}$; $|z| > 1$, (5) $4 \cdot \frac{1}{1-(1/z)}$; $|z| > 1$, (6) $5 \cdot \frac{1}{1-(1/z)}$; $|z| > 1$,

(7) $\frac{z}{2} \cdot \frac{1}{1-(z/2)}$; $|z| < 2$, (8) $\frac{z}{4} \cdot \frac{1}{1-(z/4)}$; $|z| < 4$, (9) $3z \frac{1}{1-3z}$; $|z| < \frac{1}{3}$,

(10) $\frac{2z}{(3-z)(z-2)}$; $2 < |z| < 3$, (11) $\frac{3z}{(4-z)(z-3)}$; $3 < |z| < 4$,

(12) $\frac{(a-b)z}{(a-z)(z-b)}$; $b < |z| < a$, (13) $\frac{3z}{(z-3)^2}$; $|z| > 3$,

(14) $\frac{5z}{(z-5)^2}$; $|z| > 5$, (15) $\frac{az}{(z-a)^2}$; $|z| > a$, (16) $-\log\left(1-\frac{3}{z}\right)$; $|z| > 3$,

(17) $-\log\left(1-\frac{2}{z}\right)$; $|z| > 2$, (18) $-\log\left(1-\frac{a}{z}\right)$; $|z| > a$,

(19) $\frac{1}{2} \cdot \frac{z}{1-(z/2)} + \frac{1}{1-2z}$; $\frac{1}{2} < |z| < 2$, (20) $\frac{1}{4} \cdot \frac{z}{1-(z/4)} + \frac{1}{1-(1/4z)}$; $\frac{1}{4} < |z| < 4$,

(21) $\frac{az}{1-az} + \frac{1}{1-(a/z)}$; $|a| < |z| < \frac{1}{|a|}$, (22) $e^{3/z}$, ROC z plane,

(23) $e^{5/z}$, ROC z plane, (24) $\left(1-\frac{e^{\alpha}}{z}\right)^{-1}$; $|z| > |e^{\alpha}|$,

(25) $Z\{f(k)\} = \sum_{-1}^{-\infty} 2^k z^{-k} + \sum_{k=0}^{2n} \left(\frac{1}{2}\right)^k z^{-k} + \sum_{k=0}^{2n-1} \left(\frac{1}{3}\right)^k z^{-k}$ as $n \to \infty$

$$= \sum_{1}^{\infty} \left(\frac{z}{2}\right)^k + \sum_{k=0}^{2n} \left(\frac{1}{2z}\right)^k + \sum_{k=1}^{2n-1} \left(\frac{1}{3z}\right)^k$$

$$= \frac{z}{2-z} + \frac{4z^2}{4z^2-1} + \frac{3z}{9z^2-1} \;;\; \frac{1}{2} < |z| < 2 \,]$$

Properties of Z-transforms

As in the Laplace transforms we have the following properties of Z-transforms. We shall prove how these properties and use them to solve some problems.

Linearity

If a and b are constants and $\{f(k)\}$ and $\{g(k)\}$ are two sequences which can be added then,

$$\boxed{Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + b\,Z\{g(k)\}}$$

Proof : We have by definition,

$$Z\{a\,f(k) + b\,g(k)\} = \sum_{k=-\infty}^{\infty} [a\,f(k) + b\,g(k)]\,z^{-k}$$
$$= \sum [a\,f(k)\,z^{-k} + b\,g(k)\,z^{-k}] = a \sum f(k)\,z^{-k} + b \sum g(k)\,z^{-k}$$
$$= a\,Z\{f(k)\} + b\,Z\{g(k)\}$$

Corollary : If $a = b = 1$, we have $Z\{f(k) + g(k)\} = Z\{f(k)\} + Z\{g(k)\}$

where, $\{f(k)\}$ and $\{g(k)\}$ can be added.

In words, this means, the Z-transform of the sum (or difference when $b = -1$) of two sequences which can be added (or subtracted) is equal to the sum (or difference) of the Z-transforms of the two sequences.

We shall now use this property to solve some problems.

Example 1 : Find $Z\{a^{|k|}\}$. (M.U. 2008, 12)

Sol. : We have

$$Z\{a^{|k|}\}=\sum_{k=-\infty}^{\infty} a^{|k|}z^{-k}=\sum_{k=-\infty}^{-1} a^{-k}z^{-k}+\sum_{k=0}^{\infty} a^{k}z^{-k}$$

$$=[.....+a^3z^3+a^2z^2+az]+[1+az^{-1}+a^2z^{-2}+a^3z^{-3}+.....]$$

$$=az(1+az+az^2+a^3z^3+.....)+\left(1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+.....\right)$$

$$=az\cdot\frac{1}{1-az}+1\cdot\frac{1}{1-(a/z)}=\frac{az}{1-az}+\frac{z}{z-a}$$

$$=\frac{az^2-a^2z+z-az^2}{(1-az)(z-a)}=\frac{z(1-a^2)}{(1-az)(z-a)}.$$

$$\boxed{Z\{a^{|k|}\}=\frac{z(1-a^2)}{(1-az)(z-a)}}$$

The series in G.P. are convergent if $1>|az|$ and $|z|>a$ *i.e.* $\frac{1}{a}>|z|$ and $|z|>a$.

$\therefore$ The ROC is $(1/a)>|z|>a$.

Example 2 : Find the Z-transform of $\left\{\left(\frac{1}{3}\right)^{|k|}\right\}$. (M.U. 2014)

Sol. : We have

$$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\}=\Sigma\left(\frac{1}{3}\right)^{|k|}\cdot z^{-k}=\sum_{k=-\infty}^{-1}\left(\frac{1}{3}\right)^{-k}z^{-k}+\sum_{k=\infty}^{\infty}\left(\frac{1}{3}\right)^{k}z^{-k}$$

$$=\left[.....+\left(\frac{1}{3}\right)^3z^3+\left(\frac{1}{3}\right)^2z^2+\left(\frac{1}{3}\right)z\right]+\left[1+\frac{1}{3}\cdot z^{-1}+\left(\frac{1}{3}\right)^2z^{-2}+\left(\frac{1}{3}\right)^3z^{-3}+.....\right]$$

$$=\left[\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+....\right]+\left[1+\frac{1}{3z}+\frac{1}{(3z)^2}+\frac{1}{(3z)^3}+....\right]$$

$$=\frac{z}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+.....\right]\left[1+\left(\frac{1}{3z}\right)+\left(\frac{1}{3z}\right)^2+.....\right]$$

$$=\frac{z}{3}\cdot\frac{1}{1-(z/3)}+1\cdot\frac{1}{1-[1/(3z)]},\quad\left|\frac{z}{3}\right|<1,\ \left|\frac{1}{3z}\right|<1$$

$$=\frac{z}{3}\cdot\frac{3}{3-z}+\frac{3z}{3z-1}=\frac{z}{3-z}+\frac{3z}{3z-1},\quad |z|<3,\ \frac{1}{3}<|z|$$

$$=\frac{3z^2-z+9z-3z^2}{(3-z)(3z-1)}=\frac{8z}{(3-z)(3z-1)},\quad \frac{1}{3}<|z|<3.$$

Remark ....

The above Ex. 2 is a particular case of Ex. 1 where $a=1/3$.

Example 3 : Find the Z-transform $\{3^{|k|}\}$.

Sol. : Putting $a=3$ in Ex. 1 or proceeding independently, we get

$$Z\{3^{|k|}\}=\frac{-8z}{(1-3z)(z-3)}$$

Example 4 : Find the Z-transform of $f(k)=c^k\cos\alpha k,\ k\geq 0$, where α is real.

Sol. : Assuming $f(k)=0$ for $k<0$

$$Z\{f(k)\}=\sum_{k=-\infty}^{\infty}f(k)\,z^{-k}=\sum_{k=-\infty}^{-1}0\cdot z^{-k}+\sum_{k=0}^{\infty}c^k\cos\alpha k\,z^{-k}$$

$$=\sum_{k=0}^{\infty}c^k\left[\frac{e^{i\alpha k}+e^{-i\alpha k}}{2}\right]z^{-k}$$

$$=\sum_{k=0}^{\infty}c^k\cdot\frac{e^{i\alpha k}}{2}z^{-k}+\sum_{k=0}^{\infty}c^k\cdot e^{-i\alpha k}z^{-k}$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{c\,e^{i\alpha}}{z}\right)^k+\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{ce^{-i\alpha}}{z}\right)^k$$

$$=\frac{1}{2}\cdot\left[\frac{1}{1-(ce^{i\alpha}/z)}\right]+\frac{1}{2}\left[\frac{1}{1-(ce^{-i\alpha}/z)}\right]\qquad\text{.......... (1)}$$

[See note (3), page 7-6]

$$=\frac{1}{2}\left[\frac{z}{z-c\,e^{i\alpha}}+\frac{z}{z-ce^{-i\alpha}}\right]=\frac{z}{2}\left[\frac{z-ce^{-i\alpha}+z-ce^{i\alpha}}{z^2-zc(e^{i\alpha}+e^{-i\alpha})+c^2}\right]$$

$$=\frac{z}{2}\left[\frac{2z-2c\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right)}{z^2-2zc\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right)+c^2}\right]=\frac{z}{2}\cdot 2\left[\frac{z-c\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right)}{z^2-2zc\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right)+c^2}\right]$$

$$=\frac{z(z-c\cos\alpha)}{z^2-2zc\cos\alpha+c^2}\qquad\text{.......... (2)}$$

$$\therefore\quad\boxed{Z(c^k\cos\alpha k)=\frac{z(z-c\cos\alpha)}{z^2-2zc\cos\alpha+c^2}}$$

From (1), we find that the series being in G.P. are convergent if $|z|>|ce^{i\alpha}|$ and $|z|>|ce^{-i\alpha}|$ i.e. if $|z|>|c(\cos\alpha\pm i\sin\alpha)|$ *i.e.* if $|z|>|c|$.

Example 5 : Find the Z-transform of

(i) $f(k)=\cos\alpha k,\ k>0$ where α is real. (ii) $f(k)=\cos\frac{k\pi}{3}$.

Sol. : Put $c=1$ in the above example.

$$\boxed{Z\{\cos\alpha k\}=\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1};\ |z|>1}$$

Putting $\alpha=\frac{\pi}{3}$, $Z\left\{\cos\frac{k\pi}{3}\right\}=\frac{z(z-1/2)}{z^2-z+1};\ |z|>1$

Example 6 : Find the Z-transform of $f(k) = c^k \sin \alpha k,\ k \ge 0.$

Sol. : Following the above lines we find that

$$\boxed{Z\{c^k \sin \alpha k\} = \frac{c\, z \sin \alpha}{z^2 - 2cz\cos\alpha + c^2},\ |z| > |c|}$$

Example 7 : Find the Z-transform of

(i) $f(k) = \sin \alpha k,\ k \ge 0$ where α is real. (ii) $f(k) = \sin \dfrac{k\pi}{3}$.

Sol. : Put $c = 1$ in the above example.

$$\boxed{Z\{\sin \alpha k\} = \frac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1};\ |z| > 1}$$

Putting $\alpha = \dfrac{\pi}{3}$, $Z\left\{\sin\dfrac{k\pi}{3}\right\} = \dfrac{\sqrt{3}\,z/2}{z^2 - z + 1}$.

Example 8 : Find the Z-transform of $f(k) = c^k \cos h\, \alpha\, k,\ k \ge 0.$

Sol. : By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} c^k \cos h\,\alpha k \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \left(\frac{e^{\alpha k} + e^{-\alpha k}}{2}\right) \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{c^k e^{\alpha k}}{2} \cdot z^{-k} + \sum_{k=0}^{\infty} \frac{c^k e^{-\alpha k}}{2} \cdot z^{-k}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{c e^{\alpha}}{z}\right)^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{c e^{-\alpha}}{z}\right)^k$$

$$= \frac{1}{2}\left[\frac{1}{1-(ce^{\alpha}/z)}\right] + \frac{1}{2}\left[\frac{1}{1-(ce^{-\alpha}/z)}\right] \qquad \text{[By note (3), page 7-6]}$$

$$= \frac{1}{2}\left[\frac{z}{z - ce^{\alpha}} + \frac{z}{z - ce^{-\alpha}}\right] = \frac{z}{2}\left[\frac{1}{z - ce^{\alpha}} + \frac{1}{z - ce^{-\alpha}}\right]$$

$$= \frac{z}{2}\left[\frac{z - ce^{-\alpha} + z - ce^{\alpha}}{z^2 - cz(e^{\alpha} + e^{-\alpha}) + c^2}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2c\left(\dfrac{e^{\alpha}+e^{-\alpha}}{2}\right)}{z^2 - 2cz\left(\dfrac{e^{\alpha}+e^{-\alpha}}{2}\right) + c^2}\right] = \frac{z}{2}\cdot 2\left[\frac{z - c\left(\dfrac{e^{\alpha}+e^{-\alpha}}{2}\right)}{z^2 - 2cz\left(\dfrac{e^{\alpha}+e^{-\alpha}}{2}\right) + c^2}\right]$$

$$= \frac{z(z - c\cos h\alpha)}{z^2 - 2cz\cos h\alpha + c^2} \qquad |z| > \max(|c e^{\alpha}|, |c e^{-\alpha}|)$$

Corollary : Putting $c = 1$, we get if $f(k) = \cos h\, \alpha\, k$, then

$$Z\{f(k)\} = \frac{z(z - \cos h\alpha)}{z^2 - 2z\cos h\alpha + c^2} \qquad |z| > \max(|e^{\alpha}|, |e^{-\alpha}|)$$

Example 9 : Find the Z-transform of $f(k) = c^k \sin h\, \alpha\, k,\ k \ge 0.$

Sol. : Following the above lines, we find that,

$$Z\{f(k)\} = \frac{cz\sin h\alpha}{z^2 - 2cz\cos h\alpha + c^2} \qquad |z| > \max(|c e^{\alpha}|, |c e^{-\alpha}|)$$

Corollary : Putting $c = 1$, we get, if $f(k) = \sin h\, \alpha\, k$, then

$$Z\{f(k)\} = \frac{z\sin h\alpha}{z^2 - 2z\cos h\alpha + 1} \qquad |z| > \max(|e^{\alpha}|, |e^{-\alpha}|)$$

Example 10 : Find the Z-transform of $\left\{\sin\left(\dfrac{k\pi}{3} + \alpha\right)\right\},\ k \ge 0.$

Sol. : We have

$$Z\{f(k)\} = Z\left\{\sin\left(\frac{k\pi}{3} + \alpha\right)\right\} = Z\left\{\sin\frac{k\pi}{3}\cos\alpha + \cos\frac{k\pi}{3}\sin\alpha\right\}$$

$$= \cos\alpha \cdot Z\left\{\sin\frac{k\pi}{3}\right\} + \sin\alpha \cdot Z\left\{\cos\frac{k\pi}{3}\right\}$$

$$= \cos\alpha \cdot \frac{z\sin(\pi/3)}{z^2 - 2z\cos(\pi/3) + 1} + \sin\alpha \cdot \frac{z^2 - z\cos(\pi/3)}{z^2 - 2z\cos(\pi/3) + 1}$$

[By Ex. 7 and 5 above]

$$= \frac{z\{z\sin\alpha - \sin\alpha\cos(\pi/3) + \cos\alpha\sin(\pi/3)\}}{z^2 - 2z\cos(\pi/3) + 1}$$

$$= \frac{z\{z\sin\alpha + \sin[(\pi/3) - \alpha]\}}{z^2 - 2z\cos(\pi/3) + 1} = \frac{z[z\sin\alpha + \sin(\pi/3 - \alpha)]}{z^2 - z + 1}.$$

Example 11 : Find the Z-transform of $\left\{\cos\left(\dfrac{k\pi}{3} + \alpha\right)\right\},\ k \ge 0.$ **(M.U. 2009)**

Sol. : We have

$$Z\{f(k)\} = Z\left\{\cos\left(\frac{k\pi}{3} + \alpha\right)\right\} = Z\left\{\cos\frac{k\pi}{3}\cdot\cos\alpha - \sin\frac{k\pi}{3}\cdot\sin\alpha\right\}$$

$$= \cos\alpha \cdot Z\left\{\cos\frac{k\pi}{3}\right\} - \sin\alpha \cdot Z\left\{\sin\frac{k\pi}{3}\right\}$$

$$\therefore\ F(z) = \cos\alpha \cdot \frac{z^2 - z\cos(\pi/3)}{z^2 - 2z\cdot\cos(\pi/3) + 1} - \sin\alpha \cdot \frac{z\sin(\pi/3)}{z^2 - 2z\cdot\cos(\pi/3) + 1}$$

$$= \frac{z\{z\cos\alpha - [\cos(\pi/3)\cdot\cos\alpha + \sin(\pi/3)\sin\alpha]\}}{z^2 - 2\cos(\pi/3)\cdot z + 1}$$

$$= \frac{z\{z\cos\alpha - \cos[(\pi/3) - \alpha]\}}{z^2 - 2\cos(\pi/3)\cdot z + 1} = \frac{z[z\cos\alpha - \cos(\pi/3 - \alpha)]}{z^2 - z + 1}$$

(M.U. 2008)

Example 12 : Find the Z-transform of $\{\sin(ak + b)\},\ k \ge 0.$

Sol. : We have $\sin(ak + b) = \sin ak\cos b + \cos ak\sin b$

$$\therefore\ Z\{\sin(ak + b)\} = \cos b \cdot Z\{\sin ak\} + \sin b \cdot Z\{\cos ak\}$$

$$= \cos b \cdot \frac{z\sin a}{z^2 - 2z\cos a + 1} + \sin b \cdot \frac{z(z - \cos a)}{z^2 - 2z\cos a + 1}$$

$$\therefore \quad Z\{\sin(ak+b)\} = \frac{z[\sin a \cos b - \cos a \sin b + z \sin b]}{z^2 - 2z\cos a + 1}$$

$$= \frac{z[\sin(a-b) + z\sin b]}{z^2 - 2z\cos a + 1}.$$

Example 13 : Find the Z-transform of $\{\cos(ak+b)\}$, $k \geq 0$.

Sol. : We have $\cos(ak+b) = \cos ak \cos b - \sin ak \sin b$

$$\therefore \quad Z\{\cos(ak+b)\} = \cos b \cdot Z\{\cos ak\} - \sin b \cdot Z\{\sin ak\}$$

$$= \cos b \cdot \frac{z(z-\cos a)}{z^2 - 2z\cos a + 1} - \sin b \cdot \frac{z \sin a}{z^2 - 2z\cos a + 1}$$

$$= \frac{z[z\cos b - (\cos a \cos b + \sin a \sin b)]}{z^2 - 2z\cos a + 1}$$

$$= \frac{z[z\cos b - \cos(a-b)]}{z^2 - 2z\cos a + 1}.$$

EXERCISE - IV

Find the Z-transforms of $\{f(k)\}$ where $f(k)$ is given by $(k \geq 0)$.

1. $2^{|k|}$ 2. $\left(\frac{1}{2}\right)^k$ 3. $\cos k$ 4. $\cos 2k$ 5. $\sin k$

6. $\sin 2k$ 7. $\cos hk$ 8. $\cos h\, 2k$ 9. $\sin h\, k$ 10. $\sin h\, 2k$

11. $\sin(k+1)$ 12. $2^k \cos k$ 13. $\sin(3k+2)$ 14. $\cos(3k+2)$

15. $\sin\left(\alpha k + \frac{\pi}{2}\right)$ 16. $\cos\left(\alpha k + \frac{\pi}{2}\right)$ 17. $\sin\left(\frac{k\pi}{4} + a\right)$ **(M.U. 2008)**

[**Ans. :** (1) $-\frac{3z}{(1-2z)(z-2)}$, (2) $\frac{3z}{(2-z)(2z-1)}$, (3) $\frac{z(z-\cos 1)}{z^2-2z\cos 1+1}$, (4) $\frac{z(z-\cos 2)}{z^2-2z\cos 2+1}$,

(5) $\frac{z\sin 1}{z^2-2\cos 1+1}$, (6) $\frac{z\sin 2}{z^2-2\cos 2+1}$, (7) $\frac{z(z-\cos h1)}{z^2-2z\cos h1+1}$, (8) $\frac{z(z-\cos h2)}{z^2-2z\cos h2+1}$

(9) $\frac{\sin h1}{z^2-2z\cos h1+1}$, (10) $\frac{\sin h2}{z^2-2z\cos h2+1}$, (11) $\frac{z^2\sin 1}{z^2-2z\cos 1+1}$,

(12) $\frac{z^2-2\cos 1\cdot z}{z^2-4\cos 1\cdot z+4}$, (13) $\frac{z(\sin 1+z\sin 2)}{z^2-2z\cos 3+1}$, (14) $\frac{z(z\cos 2-\cos 1)}{z^2-2z\cos 3+1}$,

(15) $\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$, (16) $\frac{-z\cos\alpha}{z^2-2z\cos\alpha+1}$, (17) $\frac{z\sin[(\pi/4)-a]+z^2\sin a}{z^2-\sqrt{2}\cdot z+1}$.]

(2) Change of Scale

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{a^k f(k)\} = F(z/a)$ and if ROC of $Z\{f(k)\}$ is $R_1 < |z| < R_2$ then ROC of $Z\{a^k f(k)\}$ is $|a|R_1 < |z| < |a|R_2$.

Proof : By definition, $F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

Replacing z by z/a, we get, $F\left(\frac{z}{a}\right) = \sum f(k)\left(\frac{z}{a}\right)^{-k}$ .... (1)

But by definition again, $Z\{a^k f(k)\} = \sum_{k=-\infty}^{\infty} a^k f(k) z^{-k} = \sum f(z)\left(\frac{z}{a}\right)^{-k}$ .... (2)

From (1) and (2), we get $Z\left\{a^k f(k)\right\} = F\left(\frac{z}{a}\right)$

Further, if ROC of $Z\{f(k)\}$ is $R_1 < |z| < R_2$, then ROC of $Z\{a^k f(k)\}$ from (2) will be $R_1 < |z/a| < R_2$ i.e. $|a|R_1 < |z| < |a|R_2$.

Example 1 : Obtain $Z\{1\}$ and hence deduce $Z\{a^k\}$, $k \geq 0$.

Sol. : By definition

$$Z\{1\} = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

$$= \frac{1}{1-(1/z)} = \frac{z}{z-1}.$$

Now, $a^k = a^k \cdot 1$. Hence, by change of scale property,

$$Z\{a^k\} = Z\{a^k \cdot 1\} = \frac{z/a}{(z/a)-1} = \frac{z}{z-a}.$$

Example 2 : Find $Z\{c^k \sin \alpha k\}$ from $Z\{\sin \alpha k\}$

Sol. : We know that, $Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$

By change of scale property,

$$Z\{c^k \sin\alpha k\} = \frac{(z/c)\sin\alpha}{(z/c)^2 - 2(z/c)\cos\alpha + 1} = \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}.$$

Example 3 : Find $Z\{c^k \cos\alpha k\}$ from $Z\{\cos\alpha k\}$.

Sol. : We know that $Z\{\cos\alpha k\} = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$

By change of scale property,

$$Z\{c^k \cos\alpha k\} = \frac{\frac{z}{c}\left\{\frac{z}{c} - \cos\alpha\right\}}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1} = \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}.$$

(3) Shifting Property

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{f(k+n)\} = z^n F(z)$ and $Z\{f(k-n)\} = z^{-n} F(z)$.

Proof : We have $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = F(z)$

$$\therefore \quad Z\{f(k+n)\} = \sum f(k+n) z^{-k} = \sum f(k+n) z^{-(k+n)} \cdot z^n$$

$$= z^n \sum_{k=-\infty}^{\infty} f(k+n) z^{-(k+n)}$$

If we put $k + n = m$ when $k = -\infty$, $m = -\infty$ and when $k = +\infty$, $m = +\infty$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=-\infty}^{\infty} f(m) z^{-m} = z^n F(z)$$

Changing the sign of n or proceeding as above $Z\{f(k-n)\} = z^{-n} F(z)$.

Example : Find the Z-transform of $\frac{1}{k+1}$, $k \geq 1$. Indicate the region of convergence.

(M.U. 2009)

Sol. : By Ex. 5, page 7-10, putting $\alpha = 1$, we have

$$Z\left(\frac{1}{k}\right) = -\log\left(1-\frac{1}{z}\right), \quad |z| > 1$$

By Shifting property, $Z\{f(k-n)\} = z^{-n} F(z)$.

Hence, $Z\left[\frac{1}{k+1}\right] = z \cdot \left\{-\log\left(1-\frac{1}{z}\right)\right\} = -z\log\left(1-\frac{1}{z}\right), \quad |z| > 1.$ [Put $n = -1$]

R.O.C. is $|z| > 1$.

Unilateral or one sided or causal sequence : If a sequence $\{f(k)\}$ is defined for the right side only *i.e.* if $\{f(k)\}$ extends to infinity on the right only *i.e.* if k varies from 0 to $+\infty$, the sequence is called **unilateral** or **one sided** or **causal sequence**.

For example, $f(k) = \begin{cases} 0, & k < 0 \\ 2^k, & k \geq 0 \end{cases}$ is a causal sequences.

Theorem : If $\{f(k)\}$ is one sided and if $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$,

then $Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}$

and $Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^{n} f(-r) z^{-n+r}$

Proof : Since $\{f(k)\}$ is one sided sequence, by data,

$$F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} \qquad \text{.......... (1)}$$

$$\therefore \quad Z\{f(k+n)\} = \sum_{k=0}^{\infty} f(k+n) z^{-k} = \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)} \cdot z^n$$

$$= z^n \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)}$$

Put $k + n = m$. When $k = 0$, $n = m$; when $k = \infty$, $m = \infty$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=n}^{\infty} f(m) z^{-m}$$

The interval from n to ∞ can be changed to the interval from 0 to ∞ by subtracting from it, the terms in the interval 0 to $n - 1$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=0}^{\infty} f(m) z^{-m} - z^n \sum_{m=0}^{n-1} f(m) z^{-m}$$

$$= z^n \cdot F(z) - z^n \sum_{m=0}^{n-1} f(m) z^{-m}$$

Taking z^n inside the summation,

$$\therefore \quad Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}.$$

Further, from (1) again,

$$Z\{f(k-n)\} = \sum_{k=0}^{\infty} f(k-n) z^{-k} = \sum_{k=0}^{\infty} f(k-n) z^{-(k-n)} \cdot z^{-n}$$

$$= z^{-n} \sum_{k=0}^{\infty} f(k-n) z^{-(k-n)}$$

Put $k - n = m$. When $k = 0$, $m = -n$ and when $k = \infty$, $m = \infty$.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} \sum_{k=-n}^{\infty} f(m) z^{-m}$$

As before we split the interval from $-n$ to ∞ into two intervals, $-n$ to -1 and 0 to ∞.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} \sum_{m=-n}^{-1} f(m) z^{-m} + z^{-n} \sum_{m=0}^{\infty} f(m) z^{-m}$$

$$= z^{-n} F(z) + z^{-n} \sum_{m=-n}^{-1} f(m) z^{-m}$$

Taking z^{-n} inside the summation,

$$\therefore \quad Z\{f(k-n)\} = z^{-n} F(z) + \sum_{m=-n}^{-1} f(m) z^{-(m+n)}$$

Putting $m = -r$, when $m = -1$, $r = 1$ and when $m = -n$, $r = n$.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^{n} f(-r) z^{-n+r}.$$

Corollary 1 : For one sided sequence ($k \geq 0$), we have

$$Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^{n} f(-r) z^{-n+r}$$

Since for causal sequence $f(-1) = f(-2) = \ldots\ldots f(-n) = 0$, the second term is zero.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} F(z).$$

Corollary 2 : Since for one sided sequence ($k \geq 0$), we have

$$Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}$$

Putting $n = 1$,

$$Z\{f(k+1)\} = z^1 F(z) - \sum_{m=0}^{0} f(0) z^{1-0} = z F(z) - z f(0) \qquad \text{.......... (A)}$$

Putting $n = 2$,

$$Z\{f(k+2)\} = z^2 F(z) - \sum_{m=0}^{2-1} f(m) z^{2-m} = z^2 F(z) - \left[f(0) z^{2-0} + f(1) z^{2-1}\right]$$

$$= z^2 F(z) - z^2 f(0) - z f(1).$$

(4) **Multiplication by k**

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{k f(k)\} = -z \frac{d}{dz} F(z)$.

Proof : We have, by definition,

$$Z\{k f(k)\} = \sum_{k=-\infty}^{\infty} k f(k) z^{-k} = \sum k f(k) z^{-k-1} \cdot z$$

$$= -z \sum -k f(k) z^{-k-1} = -z \sum f(k) \frac{d}{dz}(z^{-k})$$

$$= -z \cdot \frac{d}{dz} \sum f(k) z^{-k} = -z \frac{d}{dz} F(z)$$

In general, $Z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$

Note that $\left(-z\frac{d}{dz}\right)^2 \neq z^2 \frac{d^2}{dz^2}$, but it is equal to repeated operations $\left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)$.

Corollary 1 : $Z\{k\} = \frac{z}{(z-1)^2}$, $|z| > 1$

Proof : By definition,

$$Z\{1\} = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + z^{-1} + z^{-2} + \ldots..$$

$$\therefore \quad Z\{1\} = \frac{1}{1-z^{-1}} = \frac{1}{1-(1/z)}, \quad |z^{-1}| < 1$$

$$= \frac{z}{z-1}, \quad |z| > 1$$

Now, by the above theorem,

$$Z\{k\} = Z\{k \cdot 1\} = \left(-z\frac{d}{dz}\right)[Z\{1\}]$$

$$= \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = -z\left[\frac{(z-1)1 - z \cdot 1}{(z-1)^2}\right]$$

$$= -z\left[\frac{-1}{(z-1)^2}\right] = \frac{z}{(z-1)^2}, \quad |z| > 1.$$

Corollary 2 : $Z\{k^2\} = \frac{z(z+1)}{(z-1)^3}$.

Proof : We have proved above that $Z\{1\} = \frac{z}{z-1}$.

By the above theorem,

$$Z\{k^2\} = Z\{k^2 \cdot 1\} = \left(-z\frac{d}{dz}\right)^2 Z\{1\}$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = \left(-z\frac{d}{dz}\right)\left[\frac{z}{(z-1)^2}\right] \quad \text{[As above]}$$

$$= -z\left[\frac{(z-1)^2 \cdot 1 - z \cdot 2(z-1) \cdot 1}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1) - 2z}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}.$$

Initial Value

Theorem : If $Z\{f(k)\} = F(z)$, $k \geq 0$, then $f(0) = \lim_{z\to\infty} F(z)$.

Proof : By definition $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$

$$\therefore \quad f(0) z^0 + f(1) z^{-1} + f(2) z^{-2} + \ldots = F(z)$$

$\therefore$ Taking the limit as $z \to \infty$ of both sides of

$$f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots.. = F(z) \qquad \therefore \quad f(0) = \lim_{z\to\infty} F(z).$$

Final Value

Theorem : $\lim_{k\to\infty} f(k) = \lim_{z\to 1} (z-1) F(z)$

Proof : By definition,

$$Z\{f(k+1) - f(k)\} = \sum_{k=0}^{\infty} [f(k+1) - f(k)] z^{-k}$$

$$\therefore \quad Z\{f(k+1)\} - Z\{f(k)\} = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)] z^{-k}$$

$$\therefore \quad zF(z) - zf(0) - F(z) = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)] z^{-k} \qquad \text{[By (A), page 7-22]}$$

$$\therefore \quad (z-1) F(z) = zf(0) + \lim_{n\to\infty} \sum [f(k+1) - f(k)] z^{-k}$$

Taking the limits of both sides as $z \to 1$,

$$\therefore \quad \lim_{z\to 1} (z-1) F(z) = f(0) + \lim_{z\to 1} \lim_{n\to\infty} \sum [f(k+1) - f(k)] z^{-k}$$

Changing the order of limits,

$$\lim_{z\to 1} (z-1) F(z) = f(0) + \lim_{n\to\infty} \cdot \lim_{z\to 1} \sum_{k=0}^{n} [f(k+1) - f(k)] \cdot z^{-k}$$

$$= f(0) + \lim_{n\to\infty} \sum [f(k+1) - f(k)] \qquad \left[\because \lim_{z\to 1} z^{-k} = 1\right]$$

$$= \lim_{n\to\infty} [f(0) + f(1) - f(0) + f(2) - f(1) + \ldots.. + f(n+1) - f(n)]$$

$$= \lim_{n\to\infty} f(n+1) = \lim_{n\to\infty} f(n) = \lim_{k\to\infty} f(k).$$

Convolution

If $\{f(k)\}$ and $\{g(k)\}$ are two sequences then their convolution $\{f(k)\} * \{g(k)\}$ is defined by $\{h(k)\} = \{f(k)\} * \{g(k)\}$

where, $\{h(k)\} = \sum_{m=-\infty}^{\infty} f(m)\, g(k-m) = \sum_{m=-\infty}^{\infty} g(m)\, f(k-m) = \{g(k)\} * \{h(k)\}$

Theorem : If $\{h(k)\}$ is the convolution of two sequences $\{f(k)\}$ and $\{g(k)\}$ then (M.U. 2008)

$$Z\{h(k)\} = Z\{f(k)\}\, Z\{g(k)\} \quad \text{i.e.} \quad H(z) = F(z)\, G(z).$$

Proof : By definition, $H(z) = Z\{h(k)\} = Z\left[\{f(k)\} * \{g(k)\}\right]$

$$\therefore \quad H(z) = Z\left[\sum_{m=-\infty}^{\infty} f(m)\, g(k-m)\right] = \sum_{k=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} f(m)\, g(k-m)\right] z^{-k}.$$

Since the power series coverges absolutely, it converges uniformly also within ROC. Hence, we can interchange the order of summation.

$$H(z) = \sum_{m=-\infty}^{\infty}\ \sum_{k=-\infty}^{\infty} f(m)\ g(k-m)\, z^{-k}$$

$$= \sum_{m=-\infty}^{\infty}\ \sum_{k=-\infty}^{\infty} f(m)\cdot g(k-m)\, z^{-k+m-m}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \cdot \sum_{k=-\infty}^{\infty} g(k-m)\, z^{-(k-m)}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \cdot \sum_{p=-\infty}^{\infty} g(p)\, z^{-p} \quad \text{where, } p = k-m.$$

(When $k = -\infty,\ p = -\infty$ and when $k = \infty,\ p = \infty$)

$\therefore \quad H(z) = F(z)\ G(z).$

Example 1 : If $f(k) = U(k)$ and $g(k) = 2^k\, U(k)$, find Z-transform of $\{f(k) * g(k)\}$. **(M.U. 2008)**

Sol. : We know that $\{f(k)\} = U(k) = \{1, 1, 1, 1, \ldots\}$

$$\therefore \quad Z\{f(k)\} = \Sigma 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\ldots$$

[See Ex. 2, page 7-9]

$$\therefore \quad F(z) = \frac{1}{1-(1/z)} = \frac{z}{z-1}, \quad \left|\frac{1}{z}\right| < 1.$$

By the change of scale property (page 7-19),

$$Z\{g(k)\} = Z\{2^k U(k)\} = \frac{z/2}{(z/2)-1}$$

$$G(z) = \frac{z}{z-2}, \quad \left|\frac{2}{z}\right| < 1$$

By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z)\ G(z) = \frac{z}{z-1} \cdot \frac{z}{z-2} = \frac{z^2}{(z-1)(z-2)}, \quad |z| > 2.$$

Example 2 : If $f(k) = 4^k\, U(k)$ and $g(k) = 5^k\, U(k)$, then find the Z-transform of $\{f(k) * g(k)\}$. **(M.U. 2009, 14)**

Sol. : As above, $\{f(k)\} = \{4^0, 4^1, 4^2, \ldots\}$

$$\therefore \quad Z\{f(k)\} = \Sigma f(k)\, z^{-k} = 4^0 z^0 + 4z^{-1} + 4^2 z^{-2} + \ldots\ldots$$

$$\therefore \quad Z\{f(k)\} = 1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \left(\frac{4}{z}\right)^3 + \ldots\ldots = \frac{1}{1-(4/z)} = \frac{z}{z-4}, \quad \left|\frac{4}{z}\right| < 1$$

$\therefore \quad \{g(k)\} = \{5^0, 5^1, 5^2, \ldots\}$

$$\therefore \quad Z\{g(k)\} = \Sigma g(k)\, z^{-k} = 5^0 z^0 + 5z^{-1} + 5^2 z^{-2} + \ldots\ldots$$

$$= 1 + \left(\frac{5}{z}\right) + \left(\frac{5}{z}\right)^2 + \ldots\ldots = \frac{1}{1-(5/z)} = \frac{z}{z-5}, \quad \left|\frac{5}{z}\right| < 1.$$

By convolution theorem

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \frac{z}{(z-4)} \cdot \frac{z}{(z-5)} = \frac{z^2}{(z-4)(z-5)}, \quad |z| > 5.$$

Alternatively : If $\{h(k)\} = U(k) = \{1, 1, 1, \ldots\}$, then

$$Z\{h(k)\} = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots = \frac{1}{1-(1/z)} = \frac{z}{z-1}$$

$$\therefore \quad F(z) = \frac{z}{z-1}, \quad \left|\frac{1}{z}\right| < 1.$$

By change of scale property,

$$Z\{f(k)\} = Z\{4^k\, U(k)\} \quad \therefore \quad F(z) = \frac{z/4}{(z/4)-1} = \frac{z}{z-4}$$

$$Z\{g(k)\} = Z\{5^k\, U(k)\} \quad \therefore \quad G(z) = \frac{z/5}{(z/5)-1} = \frac{z}{z-5}$$

$\therefore$ By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \frac{z}{z-4} \cdot \frac{z}{z-5} = \frac{z^2}{(z-4)(z-5)}.$$

Example 3 : Find $Z\{f(k)\}$ where $f(k) = \dfrac{1}{2^k} * \dfrac{1}{3^k}$.

Sol. :

$$Z\left\{\frac{1}{2^k}\right\} = \sum_{k=0}^{\infty} \frac{1}{2^k}\, z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(2z)^k} = 1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \ldots\ldots$$

$$= \frac{1}{1-[1/(2z)]} = \frac{2z}{2z-1}, \quad |2z| > 1 \text{ i.e. } |z| > \frac{1}{2}.$$

Similarly, $Z\left\{\dfrac{1}{3^k}\right\} = \dfrac{3z}{3z-1}, \quad |z| > \dfrac{1}{3}.$

By convolution theorem,

$$Z\{f(k)\} = \left(\frac{2z}{2z-1}\right)\left(\frac{3z}{3z-1}\right), \quad |z| > \frac{1}{2}.$$

Example 4 : Find $Z\{f(k) * g(k)\}$ if $f(k) = \dfrac{1}{3^k}$ and $g(k) = \dfrac{1}{5^k}$. **(M.U. 2015)**

Sol. : We have $Z\left\{\dfrac{1}{3^k}\right\} = \displaystyle\sum_{k=0}^{\infty} \frac{1}{3^k} \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(3z)^k}$

$$= \frac{1}{1} + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \ldots$$

$$= \frac{1}{1-(1/3z)} \qquad \left[\because S_\infty = \frac{a}{1-r} \text{ and } |3z| > 1\right]$$

$$= \frac{3z}{3z-1} \qquad \left[|z| > \frac{1}{3}\right] \qquad \left[\text{Or by Ex. 6, page 7-11, } Z\left\{\frac{1}{3^k}\right\} = \frac{z}{z-(1/3)}\right]$$

Similarly, $Z\left\{\dfrac{1}{5^k}\right\} = \dfrac{5z}{5z-1} \qquad \left[|5z| > 1 \text{ i.e. } |z| > \dfrac{1}{5}\right]$

$\therefore$ By convolution theorem,

$$Z\{f(k)*g(k)\} = F(z)\cdot G(z) = \left(\frac{3z}{3z-1}\right)\left(\frac{5z}{5z-1}\right), \quad |z|>\frac{1}{3}.$$

Example 5 : Find $Z\{f(k)*g(k)\}$ if $f(k)=\dfrac{1}{5^k}$ and $g(k)=\dfrac{1}{7^k}$.

Sol. : We have $Z\left\{\dfrac{1}{5^k}\right\} = \sum_{k=0}^{\infty}\dfrac{1}{5^k}\cdot z^{-k} = \sum_{k=0}^{\infty}\dfrac{1}{(5z)^k}$

$$= \frac{1}{1}+\frac{1}{5z}+\frac{1}{(5z)^2}+\frac{1}{(5z)^3}+\ldots.$$

$$= \frac{1}{1-(1/5z)} \quad \left[\because S_\infty = \frac{a}{1-r} \text{ and } |5z|>1\right]$$

$$= \frac{5z}{5z-1} \quad \left[|z|>\frac{1}{5}\right] \quad \left[\text{ Or by Ex. 6, page 7-11, } Z\left\{\frac{1}{5^k}\right\} = \frac{z}{z-(1/5)}\right]$$

Similarly, $Z\left\{\dfrac{1}{7^k}\right\} = \dfrac{7z}{7z-1} \quad \left[|7z|>1 \text{ i.e., } |z|>\dfrac{1}{7}\right]$

$\therefore$ By convolution theorem,

$$Z\{f(k)*g(k)\} = F(z)\cdot G(z) = \left(\frac{5z}{5z-1}\right)\left(\frac{7z}{7z-1}\right), \quad |z|>\frac{1}{5}.$$

Theorem : If $Z\{f(k)\}=F(z)$, then $Z\{e^{-ak}f(k)\}=F(e^a z)$.

Proof : By definition,

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\,z^{-k}$$

$$\therefore \quad Z\{e^{-ak}f(k)\} = \sum e^{-ak}f(k)\,z^{-k} = \sum f(k)(e^a z)^{-k} = F(e^a z).$$

Example 1 : Find the Z-transform of $\{k\,e^{-ak}\}$, $k\geq 0$.

Sol. : We know that if $U(k)=1$, for $k>0$, then $Z\{U(k)\} = \dfrac{z}{z-1}$.

By the above theorem, $\quad Z[e^{-ak}U(k)] = Z(e^{-ak}) = \dfrac{e^a z}{e^a z-1}$.

Now, by (4) (property of multiplication by k), page 7-22

$$Z[k\,e^{-ak}] = -z\frac{d}{dz}\left(\frac{e^a z}{e^a z-1}\right)$$

$$= -z e^a\cdot\left[\frac{(e^a\cdot z-1)1 - z(e^a)}{(e^a\cdot z-1)^2}\right]$$

$$= -z\cdot e^a\cdot\frac{(-1)}{(e^a\cdot z-1)^2} = \frac{e^a\cdot z}{(e^a z-1)^2}.$$

Example 2 : Find the Z transform of $\{k^2 e^{-ak}\}$, $k\geq 0$. (M.U. 2009)

Sol. : As in Ex. 1 above $\quad Z[k\,e^{-ak}] = \dfrac{e^a\cdot z}{(e^a z-1)^2}$

Now, by (4) (property of multiplication by k), page 7-22

$$Z[k^2 e^{-ak}] = \left(-z\frac{d}{dz}\right)F(z)$$

$$= -z\frac{d}{dz}\left[\frac{e^a z}{(e^a z-1)^2}\right] = -z\cdot e^a\cdot\frac{d}{dz}\left[\frac{z}{(e^a\cdot z-1)^2}\right]$$

$$= -z e^a\cdot\left[\frac{(e^a\cdot z-1)^2\cdot 1 - z\cdot 2(e^a\cdot z-1)\cdot e^a}{(e^a\cdot z-1)^4}\right]$$

$$= -z\cdot e^a\left[\frac{e^a\cdot z-1-2\cdot z\cdot e^a}{(e^a\cdot z-1)^3}\right] = z\cdot e^a\,\frac{(z\cdot e^a+1)}{(z\cdot e^a-1)^3}.$$

Example 3 : Find $Z\{e^{-ak}\cos bk\}$.

Sol. : We have already obtained (Ex. 5, page 7-16)

$$Z\{\cos bk\} = \frac{z(z-\cos b)}{z^2-2z\cos b+1}$$

Now, by the above result,

$$Z\{e^{-ak}\cos bk\} = \frac{e^a z(e^a z-\cos b)}{(e^a\cdot z)^2 - 2(e^a z)\cos b+1}$$

Multiply in the numerator and denominator by e^{-2a}.

$$Z\{e^{-ak}\cos bk\} = \frac{z(z-e^{-a}\cos b)}{z^2-2e^{-a}z\cos b+e^{-2a}}.$$

Example 4 : Find $Z\{e^{-ak}\sin bk\}$.

Sol. : We have already proved that (Ex. 7, page 7-17)

$$Z\{\sin bk\} = \frac{z\sin b}{z^2-2z\cos b+1}$$

$\therefore$ By the above property,

$$Z\{e^{-ak}\sin bk\} = \frac{(e^a z)\sin b}{(e^a z)^2-2(e^a z)\cos b+1}$$

$$= \frac{e^{-a}z\cdot\sin b}{z^2-2e^{-a}\cdot z\cos b+e^{-2a}}.$$

We give below the list of Z-transforms obtained above.

Table of Z-transforms

1. $Z[\delta(k)]=1$,	for all z	2. $Z\{U(k)\}=\dfrac{z}{z-1}$,	$\lvert z\rvert>1$
3. $Z\{1\}=\dfrac{z}{z-1}$,	$\lvert z\rvert>1$	4. $Z\{k\}=\dfrac{z}{(z-1)^2}$,	$\lvert z\rvert>1$
5. $Z\{a^k\}=\dfrac{z}{z-a}$,	$k\geq 0$, $\lvert z\rvert>\lvert a\rvert$.	6. $Z\{a^k\}=\dfrac{z}{a-z}$,	$k<0$, $\lvert z\rvert<\lvert a\rvert$.
7. $Z\{k\,a^k\}=\dfrac{az}{(z-a)^2}$,	$k\geq 0$, $\lvert z\rvert>\lvert a\rvert$.	8. $Z\{{}^nC_k\}=\left(1+\dfrac{1}{z}\right)^n$,	$0\leq k\leq n$, $\lvert z\rvert>0$

9. $Z\{{}^kC_n\} = z^{-n}\left(1-\frac{1}{z}\right)^{-(n+1)}$, $|z| > 1$

10. $Z\left\{\frac{a^k}{k!}\right\} = e^{a/z}$, $k \geq 0$ for all z.

11. $Z\{a^{|k|}\} = \frac{az}{1-az} + \frac{z}{z-a}$, $|a| < |z| < \frac{1}{|a|}$

12. $Z\{c^k \cos \alpha k\} = \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}$, $|z| > |c|$ **(M.U. 2008)**

13. $Z\{c^k \sin \alpha k\} = \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}$, $|z| > |c|$.

14. $Z\{c^k \cos h\alpha k\} = \frac{z(z - c\cos h\alpha)}{z^2 - 2cz\cos h\alpha + c^2}$, $k \geq 0$ $|z| > \max(|ce^{\alpha}|, |ce^{-\alpha}|)$

15. $Z\{c^k \sin h\alpha k\} = \frac{cz\sin h\alpha}{z^2 - 2cz\cos h\alpha + c^2}$, $k \geq 0$ $|z| > \max(|ce^{\alpha}|, |ce^{-\alpha}|)$.

Miscellaneous Examples

Example 1 : Find Z-transform of $\{a\cos k\alpha + b\sin k\alpha\}$, $k \geq 0$.

Sol. : $Z\{a\cos k\alpha + b\sin k\alpha\} = aZ\{\cos k\alpha\} + bZ\{\sin k\alpha\}$ [By linearity property]

$$= a\cdot\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1} + b\cdot\frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}, \quad |z| > 1$$

$$= \frac{az^2 + z(b\sin\alpha - a\cos\alpha)}{z^2 - 2z\cos\alpha + 1}.$$

Example 2 : Find $Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\}$, $k \geq 0$. **(M.U. 2008)**

Sol. : Since $\sin\left(\frac{k\pi}{4} + a\right) = \sin\frac{k\pi}{4}\cos a + \cos\frac{k\pi}{4}\sin a$

$$Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\} = Z\left\{\sin\frac{k\pi}{4}\cos a + \cos\frac{k\pi}{4}\sin a\right\}$$

$$= \cos a\, Z\cdot\left\{\sin\left(\frac{\pi}{4}k\right)\right\} + \sin a\, Z\cdot\left\{\cos\left(\frac{\pi}{4}\cdot k\right)\right\}$$

$$= \cos a\,\frac{z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1} + \sin a\,\frac{z[z - \cos(\pi/4)]}{z^2 - 2z\cos(\pi/4) + 1}$$

$$= \frac{\cos a\cdot(z/\sqrt{2})}{z^2 - (2z/\sqrt{2}) + 1} + \frac{\sin a\, z\cdot[z - (1/\sqrt{2})]}{z^2 - (2z/\sqrt{2}) + 1}$$

$$= \frac{z}{\sqrt{2}}\cdot\frac{[\cos a + \sin a\cdot(\sqrt{2}\,z - 1)]}{z^2 - \sqrt{2}\cdot z + 1}.$$

Example 3 : Find $Z\{f(k)\}$ where $f(k) = \cos\left(\frac{k\pi}{4} + a\right)$, $k \geq 0$.

Sol. : We have $\cos\left(\frac{k\pi}{4} + a\right) = \cos\frac{k\pi}{4}\cos a - \sin\frac{k\pi}{4}\sin a$

$$\therefore\ Z\left\{\cos\left(\frac{k\pi}{4} + a\right)\right\} = Z\left\{\cos\frac{k\pi}{4}\cos a - \sin\frac{k\pi}{4}\sin a\right\}$$

$$= \cos a\cdot Z\left\{\cos\frac{k\pi}{4}\right\} - \sin a\cdot Z\left\{\sin\frac{k\pi}{4}\right\}$$

$$= \cos a\cdot\frac{z[z - \cos(\pi/4)]}{z^2 - 2z\cos(\pi/4) + 1} + \frac{\sin a\cdot z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1}$$

$$= \frac{\cos a\cdot z[z - (1/\sqrt{2})]}{z^2 - \sqrt{2}\cdot z + 1} - \frac{\sin a\cdot z\cdot 1\sqrt{2}}{z^2 - \sqrt{2}\cdot z + 1}$$

$$= \frac{z}{\sqrt{2}}\cdot\left[\frac{\cos a(\sqrt{2}z - 1) - \sin a}{z^2 - \sqrt{2}\cdot z + 1}\right].$$

Example 4 : Find $Z\{2^k\cos(3k + 2)\}$, $k \geq 0$. **(M.U. 2015)**

Sol. : We have $\cos(3k + 2) = \cos 3k\cos 2 - \sin 3k\sin 2$

$$\therefore\ Z\{\cos(3k + 2)\} = \cos 2\cdot Z\{\cos 3k\} - \sin 2\cdot Z\{\sin 3k\}$$

$$= \cos 2\cdot\frac{z(z - \cos 3)}{z^2 - 2z\cos 3 + 1} - \frac{\sin 2\cdot z\sin 3}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[z\cos 2 - (\cos 3\cos 2 + \sin 3\sin 2)]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z\cdot[z\cos 2 - \cos 1]}{z^2 - 2z\cos 3 + 1}$$

By change of scale property,

If $Z\{f(k)\} = F(z)$, then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$.

$$\therefore\ Z\{2^k\cos(3k + 2)\} = \frac{\frac{z}{2}\left[\frac{z}{2}\cdot\cos 2 - \cos 1\right]}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1} = \frac{z[z\cos 2 - 2\cos 1]}{z^2 - 4z\cos 3 + 4}.$$

Example 5 : Find $Z\{2^k\sin(3k + 2)\}$, $k \geq 0$. **(M.U. 2009)**

Sol. : We have $\sin(3k + 2) = \sin 3k\cos 2 + \cos 3k\sin 2$

$$\therefore\ Z\{\sin(3k + 2)\} = \cos 2\cdot Z\{\sin 3k\} + \sin 2\cdot Z\{\cos 3k\}$$

$$= \cos 2\cdot\frac{z\sin 3}{z^2 - 2z\cos 3 + 1} + \frac{\sin 2\cdot z(z - \cos 3)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin 3\cos 2 - \cos 3\sin 2 + z\sin 2]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin(3-2) + z\sin 2]}{z^2 - 2z\cos 3 + 1} = \frac{z[\sin 1 + z\sin 2]}{z^2 - 2z\cos 3 + 1}.$$

Now, by change of scale property as above,

$$\therefore\ Z\{2^k\sin(3k + 2)\} = \frac{\frac{z}{2}\left[\sin 1 + \frac{z}{2}\sin 2\right]}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1} = \frac{z[2\sin 1 + z\sin 2]}{z^2 - 4z\cos 3 + 4}.$$

Example 6 : Find $Z\{3^k \sin h\,\alpha\,k\},\ k \geq 0$.

Sol. : We have $Z\{\sin h\,\alpha\,k\} = \dfrac{z \sin h\,\alpha}{z^2 - 2z\cos h\,\alpha + 1}$

By change of scale property,

$$Z\{3^k \sin h\,\alpha\} = \frac{\frac{z}{3}\sin h\,\alpha}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cos h\,\alpha + 1} = \frac{3z \sin h\,\alpha}{z^2 - 6z\cos h\,\alpha + 9}.$$

Example 7 : Find $Z\{3^k \cos h\,\alpha\,k\},\ k \geq 0$.

Sol. : We have $Z\{\cos h\,\alpha\,k\} = \dfrac{z(z - \cos h\,\alpha)}{z^2 - 2z\cos h\,\alpha + 1}$

By change of scale property,

$$\therefore\ Z\{3^k \cos h\,\alpha\,k\} = \frac{\frac{z}{3}\left(\frac{z}{3} - \cos h\,\alpha\right)}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cos h\,\alpha + 1} = \frac{z(z - 3\cos h\,\alpha)}{z^2 - 6z\cos h\,\alpha + 9}.$$

Example 8 : Find $Z\{(k+1)\,a^k\},\ k \geq 0$.

Sol. : We have $Z\{(k+1)a^k\} = Z\{k\,a^k\} + Z\{a^k\}$

But we know that $Z\{a^k\} = \dfrac{z}{z-a}$.

Now, by (4) (property of multiplication by k), page 7-22

$$\therefore\ Z\{k a^k\} = -z\frac{d}{dz}\left\{\frac{z}{z-a}\right\} = -z\left[\frac{(z-a)1 - z \cdot 1}{(z-a)^2}\right] = -z \cdot \frac{(-a)}{(z-a)^2} = \frac{az}{(z-a)^2}$$

$$\therefore\ Z\{(k+1)a^k\} = \frac{az}{(z-a)^2} + \frac{z}{(z-a)} = \frac{az + z^2 - az}{(z-a)^2} = \frac{z^2}{(z-a)^2}.$$

Example 9 : Find Z-transform of $\{k^2 a^{k-1}\},\ k \geq 0$.

Sol. : We know that, $Z\{f(k-n)\} = z^{-n} \cdot F(z)$

$$\therefore\ Z\{a^{k-1}\} = z^{-1}F(z) \text{ where } F(z) = Z\{a^k\} = \frac{z}{z-a}$$

$$\therefore\ Z\{a^{k-1}\} = z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}$$

By the property of multiplication by k,

$$\therefore\ Z\{k \cdot a^{k-1}\} = -z\frac{d}{dz} \cdot \left(\frac{1}{z-a}\right) = -z \cdot \frac{(-1)}{(z-a)^2} = \frac{z}{(z-a)^2}.$$

By the property of multiplication by k,

$$\therefore\ Z\{k^2 \cdot a^{k-1}\} = Z\{k \cdot (k a^{k-1})\} = -z\frac{d}{dz}\left[\frac{z}{(z-a)^2}\right] = -z\left[\frac{(z-a)^2 \cdot 1 - z \cdot 2(z-a) \cdot 1}{(z-a)^4}\right]$$

$$= -z\left[\frac{(z-a) - 2z}{(z-a)^3}\right] = \frac{z(z+a)}{(z-a)^3},\quad |z| > |a|.$$

Example 11 : Find $Z\{k^2\, a^{k-1}\, U(k-1)\}$. (M.U. 2016)

Sol. : We know that, $Z\{U(k)\} = \dfrac{z}{z-1}$.

By change of scale property,

$$Z\{a^k\,U(k)\} = \frac{z/a}{(z/a) - 1} = \frac{z}{z-a}$$

$$\because\ Z\{f(k-n)\} = z^{-n} \cdot Z\{f(k)\}$$

$$Z\{a^k\,U(k-1)\} = z^{-1} \cdot Z\{a^k\,U(k)\} = z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}$$

$$\therefore\ Z\{k^2 \cdot a^2\,U(k-1)\} = \left(-z\frac{d}{dz}\right)^2\left(\frac{1}{z-a}\right) = \frac{z(z+a)}{(z-a)^3} \text{ as above.}$$

EXERCISE - V

Find the Z-transforms of the following sequences.

1. $\{3^k + 5^k\},\ k < 0$
2. $\{\alpha^k + \beta^k\},\ k < 0$
3. $\{3^k + 5^k\},\ k \geq 0$
4. $\{\alpha^k + \beta^k\},\ k \geq 0$
5. $\{k2^k + k3^k\},\ k \geq 0$
6. $\{k\alpha^k + k\beta^k\},\ k \geq 0$
7. $\left\{\dfrac{2^k}{k} + \dfrac{3^k}{k}\right\},\ k \geq 1$
8. $\left\{\dfrac{\alpha^k}{k} + \dfrac{\beta^k}{k}\right\},\ k \geq 1$
9. $\left\{3^k + \dfrac{1}{3^k}\right\},\ k \geq 0$
10. $\left\{\alpha^k + \dfrac{1}{\alpha^k}\right\},\ k \geq 0$
11. Find $Z\{f(k) * g(k)\}$ where $f(k) = 4^k$ and $g(k) = 7^k$.
12. Find $Z\{f(k) * g(k)\}$ where $f(k) = 5^k$ and $g(k) = 7^k$.
13. Find $Z\{f(k) * g(k)\}$ where $f(k) = \dfrac{1}{4^k}$ and $g(k) = \dfrac{1}{6^k}$.
14. Find $Z\{f(k) * g(k)\}$ where $f(k) = \dfrac{1}{3^k}$ and $g(k) = \dfrac{1}{7^k}$.
15. $\{\sin 5k\},\ k \geq 0$
16. $\{\cos 5k\},\ k \geq 0$
17. $\left\{\sin\left(\dfrac{k\pi}{2} + a\right)\right\},\ k \geq 0$
18. $\left\{\cos\left(\dfrac{k\pi}{2} + a\right)\right\},\ k \geq 0$
19. $\left\{\sin\left(\dfrac{k\pi}{3} + a\right)\right\},\ k \geq 0$
20. $\left\{\cos\left(\dfrac{k\pi}{3} + a\right)\right\},\ k \geq 0$
21. $\left\{\sin\left(\dfrac{k\pi}{6} + a\right)\right\},\ k \geq 0$
22. $\left\{\cos\left(\dfrac{k\pi}{6} + a\right)\right\},\ k \geq 0$
23. $\left\{3^k \cos\left(\dfrac{k\pi}{2} + \dfrac{\pi}{4}\right)\right\},\ k \geq 0$
24. $\left\{3^k \sin\left(\dfrac{k\pi}{2} + \dfrac{\pi}{4}\right)\right\},\ k \geq 0$
25. $\{e^{-2k}\cos 3k\},\ k \geq 0$
26. $\{e^{-3k}\sin 2k\},\ k \geq 0$

Ans. : (1) $\dfrac{8z - 2z^2}{(3-z)(5-z)},\ |z| < 3$ (2) $\dfrac{(\alpha+\beta)z - 2z^2}{(\alpha - z)(\beta - z)},\ |z| <$ min. of $|\alpha|, |\beta|$

(3) $\dfrac{2z^2 - 8z}{(z-3)(z-5)},\ |z| > 5$ (4) $\dfrac{2z^2 - (\alpha+\beta)z}{(z-\alpha)(z-\beta)},\ |z| >$ max of $|\alpha|, |\beta|$

(5) $\dfrac{2z}{(z-2)^2} + \dfrac{3z}{(z-3)^2},\ |z| > 3$ (6) $\dfrac{\alpha z}{(z-\alpha)^2} + \dfrac{\beta z}{(z-\beta)^2},\ |z| >$ max of $|\alpha|, |\beta|$

(7) $-\log\left[\dfrac{(z-2)(z-3)}{z^2}\right], |z|>3$

(8) $-\log\left[\dfrac{(z-\alpha)(z-\beta)}{z^2}\right], |z|>$ max. of $|\alpha|, |\beta|$

(9) $\dfrac{z}{z-3}+\dfrac{3z}{3z-1}, |z|>3$

(10) $\dfrac{z}{z-\alpha}+\dfrac{\alpha z}{\alpha z-1}, |z|>|\alpha|$

(11) $\left(\dfrac{z}{z-4}\right)\left(\dfrac{z}{z-7}\right), |z|>7$

(12) $\left(\dfrac{z}{z-5}\right)\left(\dfrac{z}{z-7}\right), |z|>7$

(13) $\left(\dfrac{4z}{4z-1}\right)\left(\dfrac{6z}{6z-1}\right), |z|>\dfrac{1}{4}$

(14) $\left(\dfrac{3z}{3z-1}\right)\left(\dfrac{7z}{7z-1}\right), |z|>\dfrac{1}{3}$

(15) $\dfrac{z\sin 5}{z^2-2z\cos 5+1}$

(16) $\dfrac{z(z-\cos 5)}{z^2-2z\cos 5+1}$

(17) $\dfrac{z^2\sin a+z\cos a}{z^2+1}, |z|>1$

(18) $\dfrac{z^2\cos a-z\sin a}{z^2+1}, |z|>1$

(19) $\dfrac{z}{2}\left[\dfrac{\cos a\cdot\sqrt{3}+\sin a(2z-1)}{z^2-z+1}\right]$

(20) $\dfrac{z}{2}\left[\dfrac{\cos a(2z-1)-\sin a\cdot\sqrt{3}}{z^2-z+1}\right]$

(21) $\dfrac{z}{2}\left[\dfrac{\cos a\cdot 1+\sin a(2z-\sqrt{3})}{z^2-\sqrt{3}\cdot z+1}\right]$

(22) $\dfrac{z}{2}\left[\dfrac{\cos a(2z-\sqrt{3})-\sin a\cdot 1}{z^2-\sqrt{3}\cdot z+1}\right]$

(23) $\dfrac{1}{\sqrt{2}}\cdot\dfrac{z^2-3z}{z^2+9}, |z|>3$

(24) $\dfrac{1}{\sqrt{2}}\cdot\dfrac{z^2+3z}{z^2+9}, |z|>3$

(25) $\dfrac{z(z-e^{-2}\cos 3)}{z^2-2e^{-2}z\cos 3+e^{-4}}$

(26) $\dfrac{e^{-3}z\sin 2}{z^2-2e^{-3}z\cos 2+e^{-6}}$.]

8. Inverse Z-transforms

We shall now consider the reverse problem *i.e.* given the Z-transform $Z\{f(k)\}=F(z)$ of a sequence to find the original sequence denoted by $\{f(k)\}$ or $Z^{-1}[F(z)]$. We shall consider Z-transforms which are rational functions of z *i.e.* of the form $F(z)=\dfrac{P(z)}{Q(z)}$ and $P(z)$ and $Q(z)$ are algebraic polynomials in z. It should be noted that to find the inverse Z-transform **we should know its region of convergence *i.e.* ROC**. We shall consider the following three methods.

(a) Direct Division, (b) Binomial Expansion, (c) Partial Fraction.

(a) Direct Division

In this method, we divide the numerator by the denominator and obtain a power series *i.e.* if $F(z)=\dfrac{P(z)}{Q(z)}$, we actually divide $P(z)$ by $Q(z)$. However, this method, now, is of academic interest only.

Example 1 : Find the inverse Z-transform of $\dfrac{1}{z-a}$ if (i) $|z|>|a|$, (ii) $|z|<|a|$.

Sol. : (i) If $|z|>|a|$, $\left|\dfrac{z}{a}\right|>1$ *i.e.* $\left|\dfrac{a}{z}\right|<1$, we consider $F(z)=\dfrac{1}{z-a}$

By actual division, we obtain a series expansion of $\dfrac{1}{z-a}$

$$z-a\,\big)\,1\,\left(\dfrac{1}{z}+\dfrac{a}{z^2}+\dfrac{a^2}{z^3}\right.$$

$$1-\dfrac{a}{z}$$

$$\dfrac{a}{z}$$

$$\dfrac{a}{z}-\dfrac{a^2}{z^2}$$

$$\dfrac{a^2}{z^2}$$

$$\therefore\quad \frac{1}{z-a}=\frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\dots\dots=z^{-1}+az^{-2}+a^2z^{-3}+\dots$$

The coefficient of $z^{-k}=a^{k-1}$. $\therefore\ Z^{-1}[F(z)]=\{a^{k-1}\}$, $k\ge 1$.

(ii) If $|z|<|a|$ *i.e.*, $\left|\dfrac{z}{a}\right|<1$, we consider $F(z)=\dfrac{1}{-a+z}$.

$$-a+z\,\big)\,1\,\left(-\dfrac{1}{a}-\dfrac{z}{a^2}-\dfrac{z^2}{a^3}\right.$$

$$1-\dfrac{z}{a}$$

$$\dfrac{z}{a}$$

$$\dfrac{z}{a}-\dfrac{z^2}{a^2}$$

$$\dfrac{z^2}{a^2}$$

$$\dfrac{z^2}{a^2}-\dfrac{z^3}{a^3}$$

$$\dfrac{z^3}{a^3}$$

$$\therefore\quad \frac{1}{-a+z}=-\frac{1}{a}-\frac{z}{a^2}-\frac{z^2}{a^3}=-\left\{\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\dots\right\}$$

$$=-\left\{\frac{z^k}{a^{k+1}}\right\},\quad k\ge 0$$

$$\therefore \frac{1}{-a+z} = -\left\{a^{-k-1}\right\} z^{k}, \quad k \geq 0$$

$$= -\left\{a^{k-1}\right\} z^{-k}, \quad k \leq 0$$

$$\therefore Z^{-1}[F(z)] = -\{a^{k-1}\}, \quad k \leq 0$$

(b) Binomial Expansion

To apply Binomial Expansion method we take a suitable factor common depending upon ROC from the denominator so that the denominator is of the form $1 - r$ where $|r| < 1$ and then use Binomial Theorem.

Example 1 : Find the inverse Z-transform of $F(z) = \dfrac{1}{z-a}$ when (i) $|z| < |a|$, (ii) $|z| > |a|$.

Sol. : (i) If $|z| < |a|$ *i.e.* $|z/a| < 1$ we take the larger term 'a' outside and write

$$F(z) = Z\{f(k)\} = \frac{1}{z-a} = \frac{1}{a[(z/a)-1]} = -\frac{1}{a} \cdot \frac{1}{1-(z/a)} = -\frac{1}{a}\left(1-\frac{z}{a}\right)^{-1}$$

$$F(z) = -\frac{1}{a}\left[1+\frac{z}{a}+\frac{z^2}{a^2}+\ldots\ldots+\frac{z^k}{a^k}+\ldots\ldots\right]$$

$$= -\left[\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\ldots\ldots+\frac{z^k}{a^{k+1}}+\ldots\ldots\right]$$

$$= -[a^{-1}+a^{-2}z+a^{-3}z^2+\ldots\ldots+a^{-k-1}z^k+\ldots\ldots]$$

$\therefore$ Coefficient of $z^k = -a^{-k-1}$, $k \geq 0$

$\therefore$ Coefficient of $z^{-k} = -a^{k-1}$, $k \leq 0$

$\therefore Z^{-1}[F(z)] = \{f(k)\} = \{-a^{k-1}\}$, $k \leq 0$

(Find $Z\{f(k)\} = -\sum\limits_{k=0}^{-\infty} a^{k-1} z^{-k}$ when $f(k) = -a^{k-1}$, $k \leq 0$ and verify the result.)

(ii) If $|z| > |a|$, $\left|\dfrac{z}{a}\right| > 1$ *i.e.* $\left|\dfrac{a}{z}\right| < 1$, we take the larger term 'z' outside and write

$$F(z) = Z\{f(k)\} = \frac{1}{z-a} = \frac{1}{z[1-(a/z)]} = \frac{1}{z}\left(1-\frac{a}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{a}{z}+\frac{a^2}{z^2}+\ldots\ldots+\frac{a^{k-1}}{z^{k-1}}+\ldots\ldots\right)$$

$$= \frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\ldots\ldots+\frac{a^{k-1}}{z^k}+\ldots\ldots$$

$\therefore$ Coefficient of $z^{-k} = a^{k-1}$, $k \geq 1$

$\therefore Z^{-1}[F(z)] = \{f(k)\} = \{a^{k-1}\}$, $k \geq 1$

Example 2 : Find the inverse Z-transform of $\dfrac{z}{z-a}$, $|z| > a$.

Sol. : We have $F(z) = \dfrac{z}{z-a} = \dfrac{z}{z[1-(a/z)]} = \left(1-\dfrac{a}{z}\right)^{-1}$

$$= 1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\ldots\ldots+\left(\frac{a}{z}\right)^k+\ldots\ldots$$

$\therefore$ Coefficient of $z^{-k} = a^k$.

$\therefore Z^{-1}[F(z)] = \{a^k\}$; $k \geq 0$, $|z| > a$

Example 3 : Find the inverse Z-transform of $\dfrac{z}{z+a}$, $|z| > a$.

Sol. : Changing the sign of a or proceeding as above we get

$$Z^{-1}F(z) = \{(-a)^k\}$$

Note ....

The following inverse Z-transforms may be remembered.

(i) $Z^{-1}\left[\dfrac{1}{z-a}\right] = \{a^{k-1}\}$, $k \geq 1$, $|z| > a$

(ii) $Z^{-1}\left[\dfrac{1}{z+a}\right] = \{(-a)^{k-1}\}$, $k \geq 1$, $|z| > a$

(iii) $Z^{-1}\left[\dfrac{z}{z-a}\right] = \{a^k\}$, $k \geq 0$, $|z| > a$

(iv) $Z^{-1}\left[\dfrac{z}{z+a}\right] = \{(-a)^k\}$, $k \geq 0$, $|z| \geq a$

Example 4 : Find the inverse Z-transform of $\dfrac{1}{(z-a)^2}$ (i) $|z| < a$, (ii) $|z| > a$.

Sol. : (i) If $|z| < a$,

$$F(z) = \frac{1}{a^2[1-(z/a)]^2} = \frac{1}{a^2}\left[1-\frac{z}{a}\right]^{-2}$$

$$= \frac{1}{a^2}\left[1+2\cdot\frac{z}{a}+3\cdot\frac{z^2}{a^2}+\ldots\ldots+(n+1)\frac{z^n}{a^n}+\ldots\ldots\right]$$

$$= \frac{1}{a^2}+2\cdot\frac{z}{a^3}+3\cdot\frac{z^2}{a^4}+\ldots\ldots+(n+1)\frac{z^n}{a^{n+2}}+\ldots\ldots$$

Coefficient of $z^n = \dfrac{n+1}{a^{n+2}}$, $n \geq 0$

$\therefore$ Coefficient of $z^{-k} = \dfrac{-k+1}{a^{-k+2}}$, $-k \geq 0$ *i.e.* $k \leq 0$

$\therefore Z^{-1}[F(z)] = \left\{\dfrac{-k+1}{a^{-k+2}}\right\}$, $k \leq 0$

(ii) If $|z| > a$

$$F(z) = \frac{1}{z^2[1-(a/z)]^2} = \frac{1}{z^2}\left(1-\frac{a}{z}\right)^{-2}$$

$$= \frac{1}{z^2}\left[1+2\cdot\frac{a}{z}+3\cdot\frac{a^2}{z^2}+\ldots..+(n-1)\frac{a^{n-2}}{z^{n-2}}+\ldots..\right]$$

$$= \frac{1}{z^2}+2\cdot\frac{a}{z^3}+3\cdot\frac{a^2}{z^4}+\ldots..+(n-1)\frac{a^{n-2}}{z^n}+\ldots..$$

Coefficient of $z^{-k} = (k-1)\,a^{k-2}$

$\therefore \quad Z^{-1}[F(z)] = \{(k-1)a^{k-2}\},\ k \geq 2.$

Example 5 : Find the inverse Z-transform of $\frac{1}{(z-5)^3}$, $|z| > 5$. **(M.U. 2009)**

Sol. : Since, $|z| > 5$, $\left|\frac{z}{5}\right| > 1$ i.e. $\frac{5}{|z|} < 1$ hence, we take out 'z' and write.

$$F(z) = \frac{1}{(z-5)^3} = \frac{1}{z^3}\cdot\frac{1}{[1-(5/z)]^3} = \frac{1}{z^3}\left(1-\frac{5}{z}\right)^{-3}$$

$$= \frac{1}{z^3}\left[1-(-3)\frac{5}{z}+\frac{(-3)(-4)}{2!}\cdot\left(\frac{5}{z}\right)^2-\frac{(-3)(-4)(-5)}{3!}\cdot\left(\frac{5}{z}\right)^3+\ldots..\right]$$

$$= \frac{1}{z^3}\left[1+3\cdot 5z^{-1}+6\cdot 5^2z^{-2}+10\cdot 5^3z^{-3}+\ldots..+\frac{(n+1)(n+2)}{2}5^n\cdot z^{-n}+\ldots..\right]$$

$$= z^{-3}+3\cdot 5\cdot z^{-4}+6\cdot 5^2z^{-5}+10\cdot 5^3z^{-6}+\ldots..+\frac{(n+1)(n+2)}{2}5^n\cdot z^{-n-3}$$

Coefficient of $z^{-n-3} = \frac{(n+1)(n+2)}{2}5^n$ [Put $n+3=k$]

$\therefore$ Coefficient of $z^{-k} = \frac{(k-3+1)(k-3+2)}{2}5^{k-3}$

$$= \frac{(k-2)(k-1)}{2}\cdot 5^{k-3}$$

$$Z^{-1}[F(z)] = \left\{\frac{(k-2)(k-1)}{2}\cdot 5^{k-3}\right\},\ k \geq 3.$$

Example 6 : Find the Z-transform of $\frac{1}{(z-5)^3}$ if $|z| < 5$. **(M.U. 2016)**

Sol. : Since $|z| < 5$ we take out 5 and write

$$F(z) = \frac{1}{5^3[(z/5)-1]^3} = -\frac{1}{5^3}\left(1-\frac{z}{5}\right)^{-3}$$

$$= -\frac{1}{5^3}\left[1+3\cdot\frac{z}{5}+6\cdot\frac{z^2}{5^2}+\ldots..+\frac{(n+1)(n+2)}{2}\frac{z^n}{5^n}+\ldots..\right]$$

Coefficient of $z^n = -\frac{(n+1)(n+2)}{2}\cdot\frac{1}{5^{n+3}},\ n \geq 0$ [Put $n=-k$]

$\therefore$ Coefficient of $z^{-k} = -\frac{(-k+1)(-k+2)}{2}\cdot\frac{1}{5^{-k+3}},\ k \leq 0$

$$\therefore \quad Z^{-1}[F(z)] = \left\{-\frac{(-k+1)(-k+2)}{2}\cdot\frac{1}{5^{-k+3}}\right\},\ k \leq 0$$

(e) **Method of Partial Fractions**

If $F(z)$ can be factorised into partial fractions, linear, quadratic or repeated we express $F(z) = \frac{P(z)}{Q(z)}$ as the sum such factors, find the constants and then use the method of Binomial Expansion. This is illustrated in the following problems. If the degree of $P(z)$ is greater than or equal to that of $Q(z)$ we write $\frac{F(z)}{z} = \frac{P(z)}{Q(z)}$ as in Ex. 3, page 7-43. We now discuss the three cases separately.

(i) ***Linear non-repeated factors***

Let the linear non-repeated factor be $\frac{z}{z-a}$. Then

$$Z^{-1}\left(\frac{z}{z-a}\right) = Z^{-1}\left[\frac{1}{1-(a/z)}\right] = Z^{-1}\left[1-\left(\frac{a}{z}\right)\right]^{-1},\ \text{if } |z| > |a|$$

$$= Z^{-1}\left[1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\ldots..\right]$$

$$= Z^{-1}\left[1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\ldots..+a^kz^{-k}+\ldots..\right]$$

$$= \{a^k\}$$

Also, $$Z^{-1}\left(\frac{z}{z-a}\right) = Z^{-1}\left[\frac{z/a}{(z/a)-1}\right] = -Z^{-1}\left[\frac{z/a}{1-(z/a)}\right],\ \text{if } |a| > |z|$$

$$= -Z^{-1}\left[\left(\frac{z}{a}\right)\left(1-\frac{z}{a}\right)^{-1}\right] = -Z^{-1}\left[\frac{z}{a}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\ldots..\right)\right]$$

$$= -Z^{-1}\left[\left(\frac{z}{a}\right)+\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)^3+\ldots..\right] = -Z^{-1}\left\{\frac{z^k}{a^k}\right\},\ k \geq 0$$

$$= -Z^{-1}\{a^{-k}z^k\},\ k \geq 0$$

$$= -Z^{-1}\{a^kz^{-k}\},\ k \leq 0$$

$$\therefore \quad Z^{-1}\left(\frac{z}{z-a}\right) = -\{a^k\},\ k \leq 0$$

Example 1 : Find inverse Z-transform of $F(z)=\dfrac{z}{(z-1)(z-2)}$, $|z|>2$. **(M.U. 2016)**

Sol. : We have $F(z)=\dfrac{1}{(z-1)(z-2)}=\dfrac{2}{z-2}-\dfrac{1}{z-1}$

Since $|z|>2$ clearly $|z|>1$.

$\therefore\ |z/2|>1$ and $|z|>1$.

$\therefore\ |2/z|<1$ and $|1/z|<1$. $\therefore$ We take z common.

$$\therefore\ F(z)=\frac{2}{z[1-(2/z)]}-\frac{1}{z[1-(1/z)]}=\frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=\frac{2}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^{k-1}}+\ldots\right)$$

$$=\left(\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\ldots+\frac{2^k}{z^k}+\ldots\right)-\left(\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^k}+\ldots\right)$$

Coefficient of $z^{-k}=2^k-1,\ k\geq 1$

$\therefore\ Z^{-1}[F(z)]=\{2^k-1\},\ k\geq 1.$

Example 2 : Find the inverse Z-transform of $F(z)=\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$, $3<z<4$.

Sol. : We have (by partial fractions) $F(z)=\dfrac{1}{z-2}+\dfrac{1}{z-3}+\dfrac{1}{z-4}$

Since $z>3$, $z>2$, hence we take z out from the first two terms. Since $4>z$, we take out 4 from the last bracket.

$$\therefore\ F(z)=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}+\frac{1}{z}\left(1-\frac{3}{z}\right)^{-1}-\frac{1}{4}\left(1-\frac{z}{4}\right)^{-1}$$

$$\therefore\ F(z)=\frac{1}{z}\left(1+\frac{2}{z}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right)+\frac{1}{z}\left(1+\frac{3}{z}+\ldots+\frac{3^{k-1}}{z^{k-1}}+\ldots\right)$$
$$-\frac{1}{4}\left(1+\frac{z}{4}+\ldots+\frac{z^k}{4^k}+\ldots\right)$$

From the first series the coefficient of $z^{-k}=2^{k-1},\ k\geq 0.$

From the second series the coefficient of $z^{-k}=3^{k-1},\ k\geq 0.$

From the third series the coefficient of $z^k=-\dfrac{1}{4^{k+1}},\ k\geq 0.$

$\therefore$ The coefficient of $z^{-k}=-4^{k-1},\ k\leq 0$

$\therefore\ Z^{-1}[F(z)]=\{2^{k-1}+3^{k-1}\},\ k\geq 0$

$\qquad=\{-4^{k+1},\},\ k\leq 0$

Example 3 : Find the inverse Z-transform of $F(z)=\dfrac{1}{(z-3)(z-2)}$ if ROC is (i) $|z|<2$, $2<|z|<3$, (iii) $|z|>3$. **(M.U. 2008, 09, 13)**

Sol. : We have $F(z)=\dfrac{1}{(z-3)(z-2)}=\dfrac{1}{z-3}-\dfrac{1}{z-2}$

If $|z|<2$, clearly $|z|<3$ $\therefore\ \left|\dfrac{z}{2}\right|<1$ and $\left|\dfrac{z}{3}\right|<1$

Hence, we take out 3 and 2 from the fractions and write.

$$F(z)=\frac{1}{3[(z/3)-1]}-\frac{1}{2[(z/2)-1]}=-\frac{1}{3[1-(z/3)]}+\frac{1}{2[1-(z/2)]}$$

$$=-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=-\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\ldots+\frac{z^k}{3^k}+\ldots\right)+\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\ldots+\frac{z^k}{2^k}+\ldots\right)$$

$$=-\left(\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+\ldots+\frac{z^k}{3^{k+1}}+\ldots\right)+\left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\ldots+\frac{z^k}{2^{k+1}}+\ldots\right)$$

$$=-(3^{-1}+3^{-2}z+3^{-3}z^2+\ldots+3^{-k-1}z^k+\ldots)$$
$$+(2^{-1}+2^{-2}z+2^{-3}z^2+\ldots+2^{-k-1}z^k+\ldots)$$

From the first series we find that the coefficient of $z^k=-3^{-k-1},\ k\geq 0.$

$\therefore$ The coefficient of $z^{-k}=-3^{k-1},\ k\leq 0.$

From the second series, we find that the coefficient of $z^k=2^{-k-1},\ k\geq 0.$

$\therefore$ The coefficient of $z^{-k}=2^{k-1},\ k\leq 0$

$\therefore\ Z^{-1}[F(z)]=\{-3^{k-1}+2^{k-1}\},\ k\leq 0$

(ii) If $2<|z|<3$ i.e. $2<|z|$ $\therefore\ |2/z|<1$ and $|z|<3$ i.e. $|z/3|<1$.

Hence, we take out 3 from the first fraction and z from the second fraction.

$$\therefore\ F(z)=\frac{1}{3[(z/3)-1]}-\frac{1}{z[1-(2/z)]}=-\frac{1}{3}\cdot\frac{1}{[1-(z/3)]}-\frac{1}{z}\frac{1}{[1-(2/z)]}$$

$$=-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=-\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\ldots+\frac{z^k}{3^k}+\ldots\right)-\frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right)$$

$$=-\left(\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+\ldots+\frac{z^k}{3^{k+1}}+\ldots\right)-\left(\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\ldots+\frac{2^{k-1}}{z^k}+\ldots\right)$$

$$=-[3^{-1}+3^{-2}z+3^{-3}z^2+\ldots+3^{-k-1}z^k+\ldots]-\left[\frac{1}{z}+\frac{2}{z^2}+\ldots+\frac{2^{k-1}}{z^k}+\ldots\right.$$

From the first series we find that the coefficient of $z^k = -3^{-k-1}$, $k \geq 0$.

$\therefore$ The coefficient of $z^{-k} = -3^{k-1}$, $k \leq 0$.

For the second series the coefficient of $z^{-k} = -2^{k-1}$, $k \geq 1$.

$$\therefore \quad Z^{-1}[F(z)] = \{-3^{k-1}\}, \ k \leq 0$$
$$= \{-2^{k-1}\}, \ k \geq 1$$

(iii) If $|z| > 3$, clearly $|z| > 2$ i.e. $|z/3| > 1$ and $|z/2| > 1$ i.e. $|3/z| < 1$ and $|2/z| < 1$. Hence we take out z from both fractions.

$$\therefore \quad F(z) = \frac{1}{z[1-(3/z)]} - \frac{1}{z[1-(2/z)]} = \frac{1}{z}\left(1 - \frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots + \frac{3^{k-1}}{z^{k-1}} + \ldots\right) - \frac{1}{z}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \ldots + \frac{2^{k-1}}{z^{k-1}} + \ldots\right)$$

$$= \left(\frac{1}{z} + \frac{3}{z^2} + \frac{3^{k-1}}{z^k} + \ldots\right) - \left(\frac{1}{z} + \frac{2}{z^2} + \ldots + \frac{2^{k-1}}{z^k} + \ldots\right)$$

$\therefore$ Coefficient of $z^{-k} = 3^{k-1} - 2^{k-1}$, $k \geq 1$.

$\therefore$ $Z^{-1}[F(z)] = \{3^{k-1} - 2^{k-1}\}$, $k \geq 1$

(ii) *Linear repeated factors*

When the linear factors are repeated, we use the above technique and expand $\frac{1}{(z-a)^2}$, $\frac{1}{(z-a)^3}$ by Binomial Theorem as illustrated in the following examples.

Example 1 : Find the inverse Z-transform of $F(z) = \dfrac{z+2}{z^2 - 2z + 1}$, $|z| > 1$. **(M.U. 2008, 14, 15)**

Sol. : We have $F(z) = \dfrac{z+2}{z^2 - 2z + 1} = \dfrac{z+2}{(z-1)^2} = \dfrac{3}{(z-1)^2} + \dfrac{1}{z-1}$

Since, $|z| > 1$, $\dfrac{1}{|z|} < 1$. $\quad \therefore$ We take out z.

$$\therefore \quad F(z) = \frac{3}{z^2[1-(1/z)]^2} + \frac{1}{z[1-(1/z)]} = \frac{3}{z^2}\left(1 - \frac{1}{z}\right)^{-2} + \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{3}{z^2}\left(1 - (-2)\cdot\frac{1}{z} + \frac{(-2)(-3)}{2!}\cdot\frac{1}{z^2} - \frac{(-2)(-3)(-4)}{3!}\cdot\frac{1}{z^3} + \ldots\right) + \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

$$= \frac{3}{z^2}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \ldots + \frac{k-1}{z^{k-2}} + \ldots\right) + \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots + \frac{1}{z^{k-1}} + \ldots\right)$$

$$= 3\left(\frac{1}{z^2} + \frac{2}{z^3} + \ldots + \frac{k-1}{z^k} + \ldots\right) + \left(\frac{1}{z} + \frac{1}{z^2} + \ldots + \frac{1}{z^k}\right)$$

$$= \frac{1}{z} + \frac{3+1}{z^2} + \ldots + \frac{3k-3+1}{z^k} + \ldots$$

$$\therefore \quad F(z) = \frac{1}{z} + \frac{4}{z^2} + \frac{7}{z^3} + \ldots + \frac{3k-2}{z^k} + \ldots$$

$\therefore$ Coefficient of $z^{-k} = 3k - 2$, $k \geq 1$.

$\therefore$ $Z^{-1}[F(z)] = \{3k - 2\}$, $k \geq 1$.

Example 2 : Find the inverse Z-transform of $\dfrac{2z^2 - 10z + 13}{(z-3)^2(z-2)}$, $2 < |z| < 3$.

Sol. : We have $F(z) = \dfrac{2z^2 - 10z + 13}{(z-3)^2(z-2)} = \dfrac{1}{z-2} + \dfrac{1}{z-3} + \dfrac{1}{(z-3)^2}$

Since, $2 < |z|$, $|2/z| < 1$ and since $|z| < 3$, $|z/3| < 1$.

$\therefore$ We take out z from the first bracket and 3 out from the last two bracket.

$$\therefore \quad F(z) = \frac{1}{z}\cdot\frac{1}{[1-(2/z)]} + \frac{1}{3}\cdot\frac{1}{[(z/3)-1]} + \frac{1}{9}\cdot\frac{1}{[(z/3)-1]^2}$$

$$= \frac{1}{z}\cdot\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z}\cdot\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \ldots + \frac{2^{k-1}}{z^{k-1}} + \ldots\right) - \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \ldots + \frac{z^k}{3^k} + \ldots\right)$$
$$+ \frac{1}{9}\left(1 + 2\cdot\frac{z}{3} + 3\cdot\frac{z^2}{3^2} + \ldots + (k+1)\frac{z^k}{3^k} + \ldots\right)$$

$\therefore$ From the first series we find that the coefficient of z^{-k} is 2^{k-1}, from the second series, we find that the coefficient of z^k is $-\dfrac{1}{3^{k+1}}$ and from the third series, we find that the coefficient of z^k is $\dfrac{k+1}{3^{k+2}}$.

$\therefore$ From the first series, coefficient of $z^{-k} = 2^{k-1}$, $k \geq 1$.

From second and third series,

$$\text{Coefficient of } z^k = \frac{k+1}{3^{k+2}} - \frac{1}{3^{k+1}} = \frac{k+1}{3^{k+2}} - \frac{3}{3^{k+2}}$$
$$= \frac{k-2}{3^{k+2}}, \ k \geq 0$$

$\therefore$ Coefficient of $z^{-k} = \dfrac{-k-2}{3^{-k+2}}$, $k \leq 0$.

Hence, $Z^{-1}[F(z)] = \{2^{k-1}\}$, $k \geq 1$

$$= \left\{\frac{-k-2}{3^{-k+2}}\right\}, \ k \leq 0.$$

Example 3 : Find the inverse Z-transform of $F(z) = \dfrac{z^2}{[z-(1/4)][z-(1/5)]}$

(i) $\dfrac{1}{5} < |z| < \dfrac{1}{4}$, (ii) $|z| < \dfrac{1}{5}$.

Sol. : (i) Since, the degree of the numerator is equal to the degree of the numerator, we write,

$$\frac{F(z)}{z} = \frac{z}{[z-(1/4)][z-(1/5)]} = \frac{5}{z-(1/4)} - \frac{4}{z-(1/5)}$$

Now, $\dfrac{1}{5} < |z| \quad \therefore \dfrac{1}{|5z|} < 1$; $|z| < \dfrac{1}{4} \quad \therefore |4z| < 1$.

$$\therefore \frac{F(z)}{z} = \frac{5 \cdot 4}{4z-1} - \frac{4}{z[1-(1/5z)]} = -\frac{20}{1-4z} - \frac{4}{z[1-(1/5z)]}$$

$$= -20(1-4z)^{-1} - \frac{4}{z}\left(1-\frac{1}{5z}\right)^{-1}$$

$$= -20(1+4z+(4z)^2+\ldots\ldots) - \frac{4}{z}\left(1+\frac{1}{5z}+\frac{1}{(5z)^2}+\ldots\ldots\right)$$

$$= -5(4+4^2z+\ldots\ldots+4^k z^{k-1}+\ldots\ldots) - \frac{4}{z}\left(1+\frac{1}{5z}+\ldots\ldots+\frac{1}{5^k}\cdot\frac{1}{z^k}+\ldots\ldots\right)$$

$$\therefore F(z) = -5(4z+4^2z^2+\ldots\ldots+4^k z^k+\ldots\ldots) - 4\left(1+\frac{1}{5z}+\ldots\ldots+\frac{1}{5^k}\cdot\frac{1}{z^k}+\ldots\ldots\right)$$

In the first series the coefficient of $z^k = -5 \cdot 4^k, \quad k \geq 1$.

$\therefore$ The coefficient of $z^{-k} = -5 \cdot 4^{k}, \quad k \leq -1$

$$= -5 \cdot \left(\frac{1}{4}\right)^k, \quad k < 0.$$

In the second series the coefficient of $z^{-k} = -4 \cdot \left(\dfrac{1}{5}\right)^k, \quad k \geq 0$.

$$\therefore Z^{-1}[F(z)] = \left\{ \underset{k<0}{-5\cdot\left(\frac{1}{4}\right)^k} \quad \underset{k \geq 0}{-4\left(\frac{1}{5}\right)^k} \right\}$$

(ii) Since, $|z| < 1/5$, clearly $|z| < 1/4$

$\therefore |5z| < 1$ and $|4z| < 1$.

$$\therefore \frac{F(z)}{z} = \frac{5}{z-(1/4)} - \frac{4}{z-(1/5)} = \frac{20}{4z-1} - \frac{20}{5z-1}$$

$$= -\frac{20}{1-4z} + \frac{20}{1-5z} = 20\left[\frac{1}{1-5z} - \frac{1}{1-4z}\right]$$

$$= 20[(1-5z)^{-1} - (1-4z)^{-1}]$$

$$= 20[(1+5z+\ldots+5^{k-1}z^{k-1}+\ldots) - (1+4z+\ldots+4^{k-1}z^{k-1}+\ldots)]$$

$$= 4\cdot(5+5^2\cdot z+\ldots+5^k z^{k-1}+\ldots) - 5(4+4^2\cdot z+\ldots+4^k\cdot z^{k-1}+\ldots)$$

$$\therefore F(z) = 4(5z+5^2z^2+\ldots+5^k z^k+\ldots) - 5(4z+4^2z^2+\ldots+4^k z^k+\ldots)$$

In the first series the coefficient of $z^k = 4 \cdot 5^k, \quad k \geq 1$.

$\therefore$ The coefficient of $z^{-k} = 4 \cdot 5^{-k}, \quad k \leq -1$

$$= 4\left(\frac{1}{5}\right)^k, \quad k < 0$$

In the second series the coefficient of $z^k = -5 \cdot 4^k, \quad k \geq 1$

$\therefore$ The coefficient of $z^{-k} = -5 \cdot 4^{-k}, \quad k \leq 1$

$$= -5 \cdot \left(\frac{1}{4}\right)^k, \quad k < 0$$

$$\therefore Z^{-1}[F(z)] = \left\{ \underset{k<0}{4\left(\frac{1}{5}\right)^k} \quad \underset{k<0}{-5\left(\frac{1}{4}\right)^k} \right\}$$

EXERCISE - VI

Find the inverse Z-transforms of the following :

1. $\dfrac{1}{z-1}$, $|z| < 1$, $|z| > 1$
2. $\dfrac{1}{z-3}$, $|z| < 3$, $|z| > 3$
3. $\dfrac{z}{z-1}$, $|z| < 1$, $|z| > 1$
4. $\dfrac{z}{z-a}$, $|z| < a$, $|z| > a$, $a > 0$
5. $\dfrac{1}{(z-1)^2}$, $|z| < 1$, $|z| > 1$
6. $\dfrac{1}{(z-5)^2}$, $|z| < 5$, $|z| > 5$
7. $\dfrac{1}{(z-3)^3}$, $|z| < 3$, $|z| > 3$
8. $\dfrac{1}{(z-1)^3}$, $|z| < 1$, $|z| > 1$
9. $\dfrac{1}{z^2-3z+2}$, $|z| > 2$
10. $\dfrac{z}{[z-(1/4)][z-(1/5)]}$, $\dfrac{1}{5} < |z| < \dfrac{1}{4}$
11. $\dfrac{z}{(z-2)(z-3)}$, $|z| < 2$, $2 < |z| < 3$, $|z| > 3$
12. $\dfrac{1}{[z-(1/2)][z-(1/3)]}$ (i) $\dfrac{1}{3} < |z| < \dfrac{1}{2}$, (ii) $\dfrac{1}{2} < |z|$ **(M.U. 2015)**
13. $\dfrac{3z^2+2z}{z^2-3z+2}$, $1 < |z| < 2$
14. $\dfrac{z^3}{(z-1)(z-2)^2}$, $|z| > 2$
15. $\dfrac{z^3}{(z-3)(z-2)^2}$, $|z| > 3$ **(M.U. 2014)**
16. $\dfrac{z^2}{(z-1)[z-(1/2)]}$; $|z| < \dfrac{1}{2}$, $\dfrac{1}{2} < |z| < 1$, $|z| > 1$

[**Ans. :** **(1)** (a) -1, $k \leq 0$; (b) 1, $k \geq 1$; **(2)** (a) -3^{k-1}, $k \leq 0$; (b) 3^{k-1}, $k \geq 1$;
(3) (a) -1, $k < 0$; (b) 1, $k \geq 0$; **(4)** (a) $-a^k$, $k < 0$; (b) a^k, $k \geq 0$;
(5) (a) $-k+1$, $k \leq 0$; (b) $k-1$, $k \geq 2$,

(6) (a) $\dfrac{-k+1}{5^{-k+2}}$, $k \le 0$; (b) $(k-1)\,5^{k-2}$, $k \ge 2$.

(7) (a) $-\dfrac{(-k+1)(-k+2)}{2} \cdot \dfrac{1}{3^{-k+3}}$, $k \le 0$; (b) $\dfrac{(k-2)(k-1)}{2} \cdot 3^{k-3}$, $k \ge 3$,

(8) (a) $-\dfrac{(-k+1)(-k+2)}{2}$, $k \le 0$; (b) $\dfrac{(k-2)(k-1)}{2}$, $k \ge 3$

(9) $2^{k-1} - 1$, $k \ge 1$,

(10) $5\left(\dfrac{1}{4}\right)^k + 4\left(\dfrac{1}{5}\right)^k$,

$(k \le 0)$, $(k \ge 0)$

(11) (i) $2^k - 3^k$, $k \le 0$, (ii) -2^k, $(k > 0)$, -3^k $(k \le 0)$, (iii) $3^k - 2^k$, $k \ge 0$,

(12) (a) $f(k) = -\dfrac{6}{3^{k-1}}$, $k > 0$; (b) $f(k) = 6\left[\dfrac{1}{2^{k-1}} - \dfrac{1}{3^{k-1}}\right]$, $k \ge 1$

$= -12 \cdot 2^{-k}$, $k \le 0$

(13) $-8 \cdot 2^k$ $-k$

$k < 0$ $k > 0$

(14) $f(k) = 1 + k \cdot 2^{k+1}$, $k \ge 0$

(15) $f(k) = 3^{k+2} - 2^{k+3} - k \cdot 2^{k+1}$, $k \ge 0$

(16) **Hint :** $\dfrac{F(z)}{z} = \dfrac{z}{z-1} - \dfrac{1}{z-(1/2)}$

(i) $2 - \left(\dfrac{1}{2}\right)^k$, $k \ge 0$; (ii) $-2 + \dfrac{1}{2^k}$, $k < 0$; (iii) -2 when $k < 0$ and $-\dfrac{1}{2^k}$ when $k^3 0$.]

EXERCISE - VII

Theory

1. Define Z-transform.
2. State convolution theorem for Z-transform. **(M.U. 2008)**

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CHAPTER 8

Correlation

1. Introduction

In Statistics so far we have studied problems involving a single variable. Many a time, we come across problems which involve two or more than two variables. Data relating to two variates are called **Bivariate Data**. If we study bivariate data carefully, we may find some relation between the two variables. For example, if a car-owner maintains the record of petrol consumption and mileage, he will find that there is some relation between the two variables. On the other hand, if we compare the figures of rainfall with the production of cars, we may find that there is no relation between the two variables. If there is any relation between two variables *i.e.* if as one variable changes the other also changes in the same or opposite direction, we say that they are correlated. Thus, correlation means "**the study of existence and the magnitude and direction of variation between two or more variables**".

2. Types of Correlation

Correlation may be classified as :

(1) Positive and negative,

(2) Linear and non-linear.

(1) Positive and Negative Correlation

The distinction between the positive and negative correlation depends upon the direction of change of two variables. If both the variables change in the same direction *i.e.* if, as one variable increases, the other also increases and as one variable decreases, the other also decreases, the correlation is called positive (*e.g.* advertising and sales). If, on the other hand, the variables change in opposite direction *i.e.* if, as one variable increases, the other decreases and vice-versa, then the correlation is called negative (*e.g.* T.V. registrations and cinema attendance).

(2) Linear or Non-linear Correlation

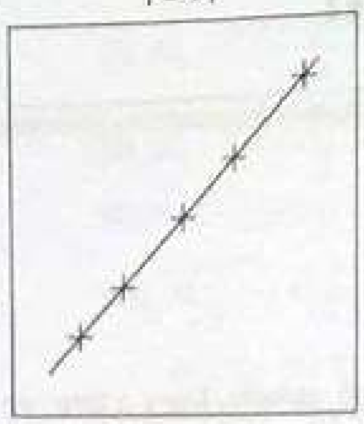
This distinction is based upon the nature of the graph of the relation between the variables. If the graph is a straight line the correlation is called linear and if the graph is not a straight line but a curve it is called non-linear or curvi-linear correlation.

We shall consider the following commonly used methods of studying correlation. (1) Scatter Diagram, (2) Karl Pearson's Coefficient of Correlation, (3) Spearman's Rank-Correlation Coefficient.

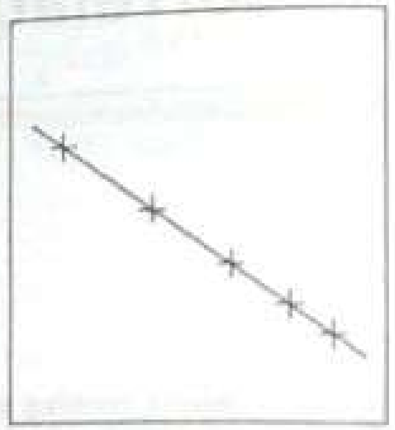
3. Scatter Diagram

One of the most simple methods of studying correlation between two variables is to construct a scatter diagram.

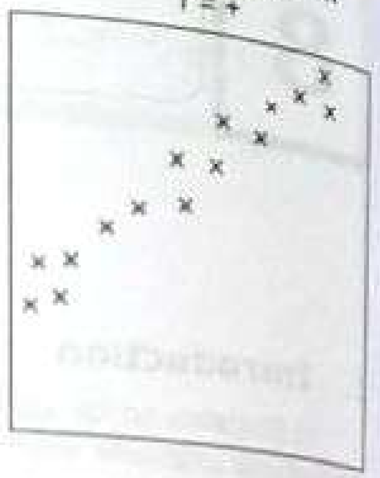
Perfect Positive Correlation
$r = +1$



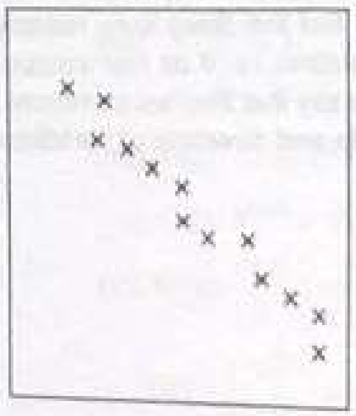
Perfect Negative Correlation
$r = -1$



High Degree of Positive Correlation
$r = +$



High Degree of Negative Correlation
$r = -$

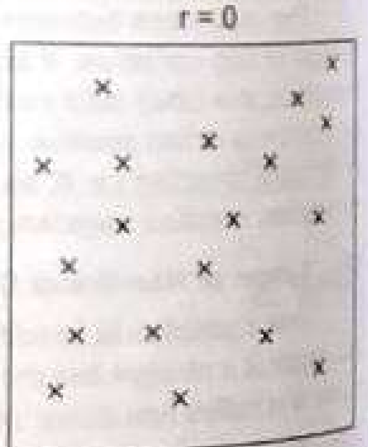


Low Degree of Positive Correlation
$r = +$

Low Degree of Negative Correlation
$r = -$

To obtain a scatter diagram, one variable is plotted along the x-axis and the other along the y-axis, on a graph paper. By plotting data in this way, we got points which are generally scattered but which show a pattern. The way in which the points are scattered indicates the degree and direction of correlation. If the points are close to each other we infer that the variables are correlated. If they are spread away from each other, we infer that the variables are not correlated. Moreover, if the points lie in a narrow strip rising from left-hand bottom to the right-hand top, we say that there is positive correlation of high order. If the points lie in a narrow strip, falling from the left-hand top to the right-hand bottom, we say that there is negative correlation of high order. If the points are all spread over, we say that there is zero correlation.

No Correlation
$r = 0$



4. Karl Pearson's Coefficient of Correlation

The method of scatter diagram is descriptive in nature and gives only a general idea of correlation. The most commonly used method which gives a mathematical expression for correlation is the one suggested by Karl Pearson (1867-1936) a British Biometrician.

Karl Pearson (1857 - 1936)

Born in London, he went to King's College, Cambridge in 1876 to study mathematics graduating in 1879 as Third Wrangler in the Mathematical Tripos. He then went to Germany to study Physics at the University of Heidelberg. Other subjects he studied in Germany include metaphysics, physiology, Roman Law, German Literature and Socialism. He studied so many subjects because he believed that there was no subject in the universe unworthy of study. Then he returned to London to study Law, although he never practised. In 1881 he returned to mathematics and was appointed as professor of mathematics at University College, London. In 1891 he met Walter Weldon, a zoologist and worked with him in biometry and evolutionary theory. He was introduced to Galton, Darwins cousin and became Galtons statistical heir. He was the first holder of the Galton's Chair of Eugenics. In 1911 he founded the world's first university statistics department at University College, London. He remained with the department until his retirement in 1933 and continued to work until his death in 1936. He thus established the new discipline of mathematical statistics.

His famous book **"The Grammar of Science"** covers several themes that were later to become part of the theories of Einstein and other scientists. He speculated that an observer who travelled at the speed of light would see an eternal now and an observer who travelled faster than light would see time reversal. He also discussed antimatter, fourth dimension and wrinkles in time.

Karl Pearson was awarded many medals including The Darwin Medal, a DSc from university of London. His commitment to socialism and his ideals led him to refuse the honours of being an OBE (Officer of the Order of the British Empire) and knighthood in 1935.

Karl Pearson is known for Karl Pearson's coefficient of correlation, methods of moments, Pearson's system of continuous curves, Chi-distance, Statistical hypothesis testing theory, Statistical decision theory, Pearson's chi-square test, etc.

Just as $\sigma_x^2 = \frac{1}{N}\Sigma(x-\bar{x})^2$ gives us a measure of variation in x and $\sigma_y^2 = \frac{1}{N}\Sigma(y-\bar{y})^2$ gives a measure of variation in y we may expect $\frac{1}{N}\Sigma(x-\bar{x})(y-\bar{y})$ to give the measure of simultaneous variation in x and y. But this will depend upon the units of x and y. To find a ratio which is independent of those units, we divide it by the quantities of the same order that is by $\sigma_x \cdot \sigma_y$. With this view in mind Karl Pearson suggested in 1890 the following coefficient of correlation to measure correlation between x and y. It is denoted by r.

Thus,
$$\boxed{r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{N\sigma_x\sigma_y}} \quad \text{.................... (1)}$$

But $\frac{1}{N}\Sigma(x-\bar{x})(y-\bar{y})$ is called the **covariance** between x and y. Hence, from (1), we have

$$\boxed{r = \frac{\text{cov.}(x, y)}{\sigma_x \cdot \sigma_y}} \quad \text{.................... (2)}$$

If we put $\sigma_x = \sqrt{\dfrac{\Sigma(x-\bar{x})^2}{N}}$, $\sigma_y = \sqrt{\dfrac{\Sigma(y-\bar{y})^2}{N}}$, then

$$\boxed{r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2\,\Sigma(y-\bar{y})^2}}} \quad \text{.................. (3)}$$

If we write $x - \bar{x} = x'$, $y - \bar{y} = y'$, then

$$\boxed{r = \frac{\Sigma x'y'}{\sqrt{\Sigma x'^2 . \Sigma y'^2}}} \quad \text{.................. (4)}$$

The Karl Pearson's coefficient of correlation is also called the **product moment coefficient** correlation.

Further, we can expand (3) and write

$$r = \frac{\Sigma(xy - x\bar{y} - \bar{x}y + \bar{x}\,\bar{y})}{\sqrt{\Sigma(x^2 - 2x\bar{x} + \bar{x}^2)\cdot\Sigma(y^2 - 2y\bar{y} + \bar{y}^2)}}$$

$$= \frac{\Sigma xy - \bar{y}\Sigma x - \bar{x}\Sigma y + \bar{x}\,\bar{y}\,\Sigma 1}{\sqrt{(\Sigma x^2 - 2\bar{x}\Sigma x + \bar{x}^2\Sigma 1)\cdot(\Sigma y^2 - 2\bar{y}\Sigma y + \bar{y}^2\Sigma 1)}}$$

But $\Sigma x = N\bar{x}$, $\Sigma y = N\bar{y}$ and $\Sigma 1 = N$

$$\therefore \quad \boxed{r = \frac{\Sigma xy - N\bar{x}\,\bar{y}}{\sqrt{(\Sigma x^2 - N\bar{x}^2)\cdot(\Sigma y^2 - N\bar{y}^2)}}} \quad \text{.................. (5)}$$

If $\bar{x}$, $\bar{y}$ are integers we take deviations of x and y from them and use the formula (3). If we have to find r from direct values we use the formula (5). This is the most *commonly used formula.*

(i) Limits for r : $-1 \le r \le 1$ **(M.U. 2004)**

Proof : If we write $E(X) = \mu_X$, $E(Y) = \mu_Y$, then

$$E\left[\left(\frac{X-\mu_X}{\sigma_X}\right) \pm \left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]^2 \ge 0$$

$$\therefore \quad E\left(\frac{X-\mu_X}{\sigma_X}\right)^2 + E\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2 \pm 2\,\frac{E\left[(X-\mu_X)^2(Y-\mu_Y)\right]}{\sigma_X\,\Sigma_Y} \ge 0$$

$\therefore \ 1 + 1 \pm 2r \ge 0 \qquad \therefore \ 1 \pm r \ge 0$

$\therefore \ 1 + r \ge 0 \qquad$ or $\qquad 1 - r \ge 0$

$\therefore \ 1 \ge -r \qquad$ or $\qquad 1 \ge r$

$\therefore \ -1 \le r,\ r \le 1 \qquad \therefore \ -1 \le r \le 1 \qquad [\because E(X-\mu_X)^2 = \sigma_X^2]$

(ii) Theorems on correlation

Theorem 1 : If x, y are independent variables they are not correlated.

We accept this theorem without proof.

Theorem 2 : Correlation coefficient is independent of change of origin and change of scale. **(M.U. 2002, 07)**

This means if we write $u_i = \dfrac{x-a}{h}$, $v_i = \dfrac{y-b}{k}$, then

$$\boxed{r_{xy} = r_{uv}} \quad \text{.................. (6)}$$

i.e. the correlation between x and y is equal to the correlation between u and v.

We accept this theorem without proof.

Remark ....

The above theorem can also be stated as :-

"If $x = au + b$, $y = cv + d$ where a, b, c, d are constants then $r_{xy} = r_{uv}$."

Example : Discuss the statement : "If the coefficient of correlation between x and y is negative then the coefficient of correlation between $(-x)$ and $(-y)$ is positive. **(M.U. 1998)**

Sol. : If we write $a = 0$ and $b = 0$, $h = -1$ and $k = -1$ in (A), then by the above theorem since $r_{xy} = r_{uv}$ the coefficient of correlation between $-x$ and $-y$ will be also the same in magnitude and sign as the coefficient of correlation between x and y.

Or since the coefficient of correlation does not change under change of scale and since $-x$ and $-y$ mean the change of scale, the coefficient of correlation between $-x$ and $-y$ will be also negative.

Theorem 3 : *If d_x and d_y denote the deviations of x and y from the assumed means A and B then*

$$\boxed{r = \frac{\Sigma d_x d_y - \dfrac{(\Sigma d_x)(\Sigma d_y)}{N}}{\sqrt{\Sigma d_x^2 - \dfrac{(\Sigma d_x)^2}{N}}\sqrt{\Sigma d_y^2 - \dfrac{(\Sigma d_y)^2}{N}}}} \quad \text{.................. (7)}$$

We accept this result without proof.

5. Interpretation of the Coefficient of Correlation

1. **$r > 0.95$:** If r is greater than 0·95, it indicates high degree of correlation and the value of one variable can be estimated from a known value of the other fairly accurately.

2. **$r > 0.75$ but < 0.85 :** If r is greater than 0·75 but less than 0·85, there is probably a definite relationship between the variables and the value of one variable can be roughly estimated from a known value of the other.

3. **$r > 0.40$ but < 0.60 :** If r is greater than 0·40 but less than 0·60 there may be some relationship between the two variables. But the value of one variable calculated from a known value of the other cannot be reliable.

4. **$r < 0.35$:** If r is less than 0·35 the correlation is poor and one variable cannot be estimated from the other.

5. **r nearly zero :** If r is nearly equal to zero, it indicates that there is probably no relation between the two variables *i.e.* they are independent of each other.

6. Computation of Coefficient of Correlation : *(Ungrouped Data)*

There are three methods of calculating r.

(1) Actual mean method, (2) Step deviation method,

(3) Assumed mean method.

(1) Actual Mean Method

The formula to be used is,
$$r = \frac{\sum xy}{\sqrt{\sum x^2 \cdot \sum y^2}}$$

Steps :

(i) Calculate mean $\overline{X}$ and then take deviation x of X from $\overline{X}$ *i.e.*, calculate $x = X - \overline{X}$.

(ii) Calculate mean $\overline{Y}$ and then take deviation y of Y from $\overline{Y}$ *i.e.*, calculate $y = Y - \overline{Y}$.

(iii) Multiply x by y and prepare the column of xy.

(iv) Take the squares of x and prepare the column of x^2.

(v) Take the squares of y and prepare the column of y^2.

(vi) Apply the above formula.

Example 1 : Find from the following values of the demand and the corresponding price of a commodity, the degree of correlation between the demand and price by computing Karl Pearson's coefficient of correlation.

Demand in quintals : 65, 66, 67, 67, 68, 69, 70, 72.

Price in Paise per k.g. : 67, 68, 65, 68, 72, 72, 69, 71.

Sol. : Let X denote the demand in Quintals and Y denote the price in paise per kg.

Calculation of r between demand and price

Sr. No.	Demand in Qnt. $X-\overline{X}$, $\overline{X}=68$			Price per Kg. $Y-\overline{Y}$, $\overline{Y}=69$			Product
	X	x	x^2	Y	y	y^2	xy
1	65	−3	9	67	−2	4	6
2	66	−2	4	68	−1	1	2
3	67	−1	1	65	−4	16	4
4	67	−1	1	68	−1	1	1
5	68	0	0	72	+3	9	0
6	69	+1	1	72	+3	9	3
7	70	+2	4	69	0	0	0
8	72	+4	16	71	+2	4	8
$N=8$	$\sum X = 544$		$\sum x^2 = 36$	$\sum Y = 552$		$\sum y^2 = 44$	$\sum xy = +24$

Now, $\overline{X} = \frac{544}{8} = 68$ and $\overline{Y} = \frac{552}{8} = 69$.

$$\therefore \quad r = \frac{\sum xy}{\sqrt{\sum x^2 \cdot \sum y^2}}$$

But $\sum xy = 24$, $\sum x^2 = 36$, $\sum y^2 = 44$.

$$\therefore \quad r = \frac{24}{\sqrt{36 \times 44}} = \frac{24}{39{\cdot}80} = 0{\cdot}6030.$$

Example 2 : Calculate Karl Pearson's coefficient of correlation for the following bivariate series.

X : 28, 45, 40, 38, 35, 33, 40, 32, 36, 33

Y : 23, 34, 33, 34, 30, 26, 28, 31, 36, 35 **(M.U. 2015)**

Sol. :

Calculation of r between X and Y

Sr. No.	$X-\overline{X}$, $\overline{X}=36$			$Y-\overline{Y}$, $\overline{Y}=31$			Product
	X	x	x^2	Y	y	y^2	xy
1	28	−8	64	23	−8	64	+64
2	45	+9	81	34	+3	9	+27
3	40	+4	16	33	+2	4	+8
4	38	+2	4	34	+3	9	+6
5	35	−1	1	30	−1	1	+1
6	33	−3	9	26	−5	25	−15
7	40	+4	16	28	−3	9	−12
8	32	−4	16	31	0	0	0
9	36	0	0	36	+5	25	0
10	33	−3	9	35	+4	16	−12
$N=10$	$\sum X = 360$		$\sum x^2 = 216$	$\sum Y = 310$		$\sum y^2 = 162$	$\sum xy = +97$

Now, $\overline{X} = \frac{360}{10} = 36$ and $\overline{Y} = \frac{310}{10} = 31$.

$$\therefore \quad r = \frac{\sum xy}{\sqrt{\sum x^2 \cdot \sum y^2}}$$

But $\sum xy = 97$, $\sum x^2 = 216$, $\sum y^2 = 162$.

$$\therefore \quad r = \frac{97}{\sqrt{216 \times 162}} = \frac{97}{187{\cdot}1} = 0{\cdot}5186.$$

) Step-deviation Method

As in the case of mean and standard deviation, to simplify calculations we can use step-iation method whenever possible for calculating r. But it should be noted that *the result is not to multiplied by the constant in the final stage.* The reason is the coefficient of correlation is dependent of change of origin and change of scale. (See Theorem 2 of r, page 8-4)

Example 1 : Calculate the co-efficient of correlation from the following data.

X : 100, 200, 300, 400, 500.

Y : 30, 40, 50, 60, 70. **(M.U. 2015)**

Sol. :

Calculations of r between X and Y

Sr. No.	$\overline{X}=300$, $X-\overline{X}$		$\frac{X-\overline{X}}{100}$		$\overline{Y}=50$, $Y-\overline{Y}$		$\frac{Y-\overline{Y}}{10}$		Product
	X	$X-300$	x	x^2	Y	$Y-50$	y	y^2	xy
1	100	−200	−2	4	30	−20	−2	4	4
2	200	−100	−1	1	40	−10	−1	1	1
3	300	0	0	0	50	0	0	0	0
4	400	100	1	1	60	10	1	1	1
5	500	200	2	4	70	20	2	4	4
$N=5$	$\sum X = 1500$			$\sum x^2 = 10$	$\sum Y = 250$			$\sum y^2 = 10$	$\sum xy = 10$

Now $\bar{X} = \frac{1500}{5} = 300$ and $\bar{Y} = \frac{250}{5} = 50$ $\quad \therefore \quad r = \frac{\sum xy}{\sqrt{\sum x^2 \cdot \sum y^2}}$

But $\sum xy = 10,\ \sum x^2 = 10,\ \sum y^2 = 10.$ $\quad \therefore \quad r = \frac{10}{\sqrt{10 \times 10}} = \frac{10}{10} = +1.$

(3) Assumed Mean Method

Since in the calculation of r, deviations are to be squared the calculations will be tedious if the means are not integers but data are in integers. In such cases, we take deviations from an assumed mean conveniently chosen. The corresponding formula is

$$r = \frac{\sum d_x d_y - \frac{(\sum d_x)(\sum d_y)}{N}}{\sqrt{\sum d_x^2 - \frac{(\sum d_x)^2}{N}}\sqrt{\sum d_y^2 - \frac{(\sum d_y)^2}{N}}}$$

where, d_x = deviations of X from an assumed mean, $(X - A)$,

d_y = deviations of Y from an assumed mean, $(Y - B)$,

N = Number of pairs of observations.

Steps :

(i) Assume any mean A for X and calculate deviations d_x of X from A i.e., $d_x = X - A$.

(ii) Assume any mean B for Y and calculate deviations d_y of Y from B i.e., $d_y = Y - B$.

(iii) Take the squares of d_x.

(iv) Take the squares of d_y.

(v) Take the products of d_x and d_y.

(vi) Apply the formula.

Example 4 : Find the co-efficient of correlation for the prices (in Rs.) and sales units.

Price in Rs. : 100, 98, 85, 92, 90, 84, 88, 90, 93, 95.

Sales Units : 500, 610, 700, 630, 670, 800, 800, 750, 700, 690.

Sol. : Let us assume 92 and 670 to be the means of X and Y respectively.

Calculations of r between price and sale

Sr. No.	Price in Rs. $(X-92)$			Sales Units $(Y-670)$			Product
	X	d_x	d_x^2	Y	d_y	d_y^2	$d_x d_y$
1	100	+8	64	500	−170	28900	−1360
2	98	+6	36	610	−60	3600	−360
3	85	−7	49	700	+30	900	−210
4	92	0	0	630	−40	1600	0
5	90	−2	4	670	0	0	0
6	84	−8	64	800	+130	16900	−1040
7	88	−4	16	800	+130	16900	−520
8	90	−2	4	750	+80	6400	−160
9	93	+1	1	700	+30	900	+30
10	95	+3	9	690	+20	400	+40
$N = 10$		$\Sigma d_x = -5$	$\Sigma d_x^2 = 247$		$\Sigma d_y = 150$	$\Sigma d_y^2 = 76500$	$\Sigma d_x d_y = -3560$

Now, $$r = \frac{\sum d_x d_y - \frac{(\sum d_x)(\sum d_y)}{N}}{\sqrt{\sum d_x^2 - \frac{(\sum d_x)^2}{N}}\sqrt{\sum d_y^2 - \frac{(\sum d_y)^2}{N}}}$$

But, $\sum d_x d_y = -3560,\ \sum d_x = -5,\ \sum d_y = 150$

$N = 10,\ \sum d_x^2 = 247,\ \sum d_y^2 = 76500.$

$$\therefore \quad r = \frac{-3560 - \frac{(-5)\times(150)}{10}}{\sqrt{247 - \frac{25}{10}}\sqrt{76500 - \frac{22500}{100}}}$$

$$= \frac{-3560 + 75}{\sqrt{247 - 2{\cdot}5}\sqrt{76500 - 2250}}$$

$$= \frac{-3485}{\sqrt{244{\cdot}5}\sqrt{74250}}$$

$$= \frac{3485}{4261} = -0{\cdot}8179$$

Example 5 : Calculate the correlation coefficient from the following data.

X : 23, 27, 28, 29, 30, 31, 33, 35, 36, 39.

Y : 18, 22, 23, 24, 25, 26, 28, 29, 30, 32. **(M.U. 2004, 14)**

Sol. : Let us assume 30 and 25 to be the means of x and y respectively.

Calculation of r between X and Y

Sr. No.	$(X-30)$			$(Y-25)$			Product
	X	d_x	d_x^2	Y	d_y	d_y^2	$d_x d_y$
1	23	−7	49	18	−7	49	49
2	27	−3	9	22	−3	9	9
3	28	−2	4	23	−2	4	4
4	29	−1	1	24	−1	1	1
5	30	0	0	25	0	0	0
6	31	+1	1	26	+1	1	1
7	33	+3	9	28	+3	9	9
8	35	+5	25	29	+4	16	20
9	36	+6	36	30	+5	25	30
10	39	+9	81	32	+7	49	63
$N = 10$		$\Sigma d_x = +11$	$\Sigma d_x^2 = 215$		$\Sigma d_y = +7$	$\Sigma d_y^2 = 163$	$\Sigma d_x d_y = 186$

Now, $$r = \frac{\sum d_x d_y - \frac{(\sum d_x)(\sum d_y)}{N}}{\sqrt{\sum d_x^2 - \frac{(\sum d_x)^2}{N}}\sqrt{\sum d_y^2 - \frac{(\sum d_y)^2}{N}}}$$

But $\sum d_x d_y = 186$, $\sum d_x = 11$, $\sum d_y = 7$

$N = 10$, $\sum d_x^2 = 215$, $\sum d_y^2 = 163$

$$\therefore \quad r = \frac{186 - \frac{11 \times 7}{10}}{\sqrt{215 - \frac{(11)^2}{10}}\sqrt{163 - \frac{(7)^2}{10}}}$$

$$= \frac{186 - 7{\cdot}7}{\sqrt{215 - 12{\cdot}1}\sqrt{163 - 4{\cdot}9}}$$

$$= \frac{178{\cdot}3}{\sqrt{202{\cdot}9}\sqrt{158{\cdot}1}} = 0{\cdot}9948$$

7. Direct Method of Calculating Coefficient of Correlation

We can find the coefficient of correlation directly without taking the deviations of x and of y from their respective means. In such cases the following formula is used.

$$\boxed{r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\left(\sum x^2 - \frac{(\sum x)^2}{n}\right)\left(\sum y^2 - \frac{(\sum y)^2}{n}\right)}}} \quad \text{.................... (8)}$$

where, x and y are the observed values of the variables and $\bar{x}, \bar{y}$ are their respective means.

The formula can also be written as,

$$r = \frac{\sum xy - n\bar{x}\bar{y}}{n \cdot \sqrt{\left(\frac{\sum x^2}{n} - \bar{x}^2\right)\left(\frac{\sum y^2}{n} - \bar{y}^2\right)}}$$

$$\boxed{r = \frac{\sum xy - n\bar{x}\bar{y}}{n\sigma_x\sigma_y}} \quad \text{.................... (9)}$$

Example 1 : Calculate the coefficient of correlation between X and Y from the following data.

X : 3, 5, 4, 6, 2.

Y : 3, 4, 5, 2, 6.

Sol. :

Calculations of r

x	x^2	y	y^2	xy
3	9	3	9	9
5	25	4	16	20
4	16	5	25	20
6	36	2	4	12
2	4	6	36	12
$\sum x = 20$	$\sum x^2 = 90$	$\sum y = 20$	$\sum y^2 = 90$	$\sum xy = 73$

Since, $\bar{x} = 4$, $\bar{y} = 4$ putting these values in equation (8).

$$\therefore \quad r = \frac{73 - \frac{20^2}{5}}{\sqrt{90 - \frac{20^2}{5}}\sqrt{90 - \frac{20^2}{5}}} = \frac{73 - 80}{10} = -0{\cdot}7.$$

EXERCISE - I

1. Calculate the coefficient of correlation from the following data. Is there any marked correlation between the production and price of tea ?

   Production in crores (kgs) : 34, 27, 31, 38, 38, 36, 39, 40

   Price in Rs. per kg : 3·75, 4·62, 4·25, 4·12, 4·28, 4·32, 4·21, 4·05

   [**Ans.** : $r = -0{\cdot}48$]

2. Compute the coefficient of correlation between X and Y from their values given below.

   X : 30, 33, 25, 10, 33, 75, 40, 85, 90, 95.

   Y : 68, 65, 80, 85, 70, 30, 55, 18, 15, 10. **(M.U. 2015)** [**Ans.** : $r = -0{\cdot}7069$]

3. The following data give the hardness (X) and tensile strength (Y) for some specimens of a material in certain units in a factory. Find the correlation coefficient and interpret your result.

   X : 23·3, 17·5, 17·8, 20·7, 18·1, 20·9, 22·9, 20·8.

   Y : 4·2, 3·8, 4·6, 3·2, 5·2, 4·7, 4·4, 5·6.

   [**Ans.** : $r = -0{\cdot}072$. No correlation]

4. Calculate the product moment coefficient of correlation between the indices of business activity (X) and employment (Y) from the following data.

   X : 100, 102, 108, 111, 115, 116, 118.

   Y : 110, 100, 104, 108, 112, 116, 120. [**Ans.** : $r = 0{\cdot}75$]

5. Find Karl Pearson's coefficient of correlation between X and Y.

   X : 10, 12, 14, 15, 16, 17, 18, 10, 14, 15

   Y : 17, 16, 15, 12, 10, 9, 8, 15, 13, 12 [**Ans.** : $r = -0{\cdot}93$]

6. Compute a coefficient of correlation between X and Y.

   X : 3, 6, 4, 5, 7

   Y : 2, 4, 5, 3, 6 [**Ans.** : $r = 0{\cdot}7$]

7. Calculate the coefficient of correlation between price and demand by direct method.

   Price : 2, 3, 4, 7, 4

   Demand : 8, 7, 3, 1, 1 [**Ans.** : $-0{\cdot}81$]

8. Calculate the coefficient of correlation between the X and Y by direct method.

   X : 8, 8, 7, 5, 6, 2

   Y : 3, 4, 10, 13, 22, 8 [**Ans.** : 0·2646]

9. Soil temperature (x) and Germination interval (y) for winter wheat in 12 places are as follows.

   x (in °F) : 57, 42, 38, 42, 45, 42, 44, 40, 46, 44, 43, 40

   y (days) : 10, 26, 41, 29, 27, 27, 19, 18, 19, 31, 29, 33

   Calculate the coefficient of correlation between x and y. **(M.U. 2015)** [**Ans.** : $r = -0{\cdot}74$]

10. Find the Karl Pearson's coefficient of correlation between X and Y from the following.

X : 31, 34, 38, 59, 65, 60, 70

Y : 38, 44, 33, 36, 33, 23, 13

[Ans. : $r = -0{\cdot}7977$]

8. Spearman's Rank Correlation (M.U. 2009)

The method developed by Spearman is simpler than Karl Pearson's method since, it depends upon ranks of the items and actual values of the items are not required. Hence, this can be used to study correlation even when actual values are not known. For instance we can study correlation between intelligence and honesty by this method.

Let x_i, y_i be the ranks in the two characteristics of the i-th member where $i = 1, 2, \ldots, n$. We assume that no two members have the same rank either for x or for y. Thus, x and y take all integral values between 1 and n.

$$\therefore \quad \bar{x} = \frac{1}{2}(1+2+3+\ldots+n) = \frac{n+1}{2}$$

Similarly, $\bar{y} = \frac{1}{2}(1+2+3+\ldots+n) = \frac{n+1}{2} \qquad \therefore \ \bar{x} = \bar{y}$

$$\therefore \quad \Sigma(x_i - \bar{x})^2 = \Sigma(x_i^2 - 2x_i\bar{x} + \bar{x}^2) = \Sigma x_i^2 - 2\bar{x}\,\Sigma x_i + \bar{x}^2\Sigma 1$$

$$= \Sigma x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \Sigma x_i^2 - n\bar{x}^2$$

$$= (1^2 + 2^2 + \ldots + n^2) - n\left(\frac{n+1}{2}\right)^2$$

$$\therefore \quad \Sigma(x_i - \bar{x})^2 = \frac{n}{6}(n+1)(2n+1) - \frac{n(n+1)^2}{4} = \frac{1}{12}(n^3 - n)$$

Similarly, $\Sigma(y_i - \bar{y}) = \frac{1}{12}(n^3 - n)$.

If d_i denotes the difference between the ranks of i-th member in the two variables, we have

$$d_i = (x_i - y_i) = (x_i - \bar{x}) - (y_i - \bar{y}) = x_i' - y_i' \ (\text{since, } \bar{x}, \bar{y} \text{ are equal})$$

where, x_i', y_i' denote the deviations of x_i, y_i from their means $\bar{x}, \bar{y}$ respectively.

$$\therefore \quad \Sigma d_i^2 = \Sigma x_i'^2 + \Sigma y_i'^2 - 2\Sigma x_i' y_i'$$

$$\therefore \quad \Sigma d_i^2 = \frac{1}{12}(n^3 - n) + \frac{1}{12}(n^3 - n) - 2\Sigma x_i' y_i'$$

$$\therefore \quad \Sigma x_i' y_i' = \frac{1}{2}\left[\frac{n^3 - n}{6} - \Sigma d_i^2\right]$$

But the coefficient of correlation

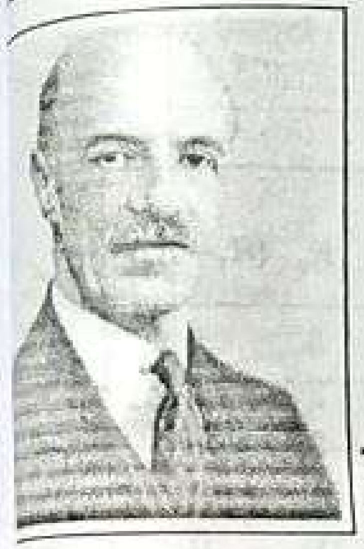
$$= \frac{\Sigma x_i' y_i'}{\sqrt{\Sigma x_i'^2 \cdot \Sigma y_i'^2}} = \frac{\frac{1}{2}\left[\frac{n^3 - n}{6} - \Sigma d_i^2\right]}{\frac{1}{12}(n^3 - n)} = 1 - \frac{6\Sigma d_i^2}{n^3 - n}$$

This coefficient is denoted by R

$$\therefore \quad \boxed{R = 1 - \frac{6\Sigma d_i^2}{n^3 - n}} \qquad \ldots\ldots (10)$$

The value of R, as of r, lies between +1 and −1. If $R = +1$, there is perfect positive correlation there is complete agreement in the same direction. If $R = -1$, there is perfect negative correlation there is complete agreement but in opposite direction. Generally, the value of R is neither +1 nor but lies somewhere in between. If $R = 0$, there is no correlation between X and Y.

Charles Edward Spearman (1863 - 1945)



Spearman was an English psychologist known for his work in statistics as a pioneer in factor analysis and for Spearman's rank correlation coefficient.

After serving for fifteen years as an officer in the British Army he went to Leipzig, Germany to study experimental psychology and obtained his Ph.D. degree in 1906. He was elected to Royal Society of London in 1924. In 1928 he became Professor of Psychology at University College, London. His many published papers cover a wide field, but he especially did pioneering work in the application of mathematical methods to the analysis of human mind. He discovered the general factor in human intelligence and developed a theory of 'g'.

He was greatly influenced by the work of Galton. He did pioneering work in psychology and developed correlation coefficient known by his name.

Interpretation of $R = +1$ and $R = -1$

Two values of R need special attention. They are + 1 and − 1. $R = +1$, when the scatter diagram is a straight line rising to the right. In this case the ranks of the value of X are the same as the ranks of the values of Y. $R = -1$, when the scatter diagram is a straight line falling to right. In this case, when the ranks of the values of X go on increasing in order, the ranks of the corresponding values of Y go on decreasing in the same order.

For example, consider the following data.

$R = +1$			
X	Y	R_1	R_2
8	115	1	1
11	120	2	2
14	125	3	3
17	130	4	4
20	135	5	5

$R = -1$			
X	Y	R_1	R_2
8	135	1	5
11	130	2	4
14	125	3	3
17	120	4	2
20	115	5	1

Note ....

When the scatter diagram is a straight line $r = +1$ or − 1 and also $R = +1$ or − 1.

Relation between Spearman's Rank Correlation Coefficient R and Karl Pearson's Correlation Coefficient r.

Generally, for a given distribution the values of Spearman's rank correlation coefficient and Karl Pearson's correlation coefficient are different. Although both of them lie between + 1 and − 1, the actual values of the two coefficients for a given distribution are different. However, if the data are such that if the values of the two variables x and y are arranged in either ascending or descending order and if they are found to increase or decrease by the equal amount *i.e.* if the difference between two values of x and the difference between the corresponding two values of y is constant, then the two values of R and r are equal. This is illustrated by the following example.

Example 1 : Calculate R and r from the following data.

X : 12 17 22 27 32.

Y : 113 119 117 115 121.

Interpret your result.

Sol. : Calculation of R and r

Sr. No.	$X-\bar{X}$			$Y-\bar{Y}$			xy	R_1	R_2	$(R_1-R_2)^2$
	X	x	x^2	Y	y	y^2				
1	12	−10	100	117	−4	16	40	5	5	0
2	17	−5	25	119	2	4	−10	4	2	4
3	22	0	0	117	0	0	0	3	3	0
4	27	5	25	115	−2	4	−10	2	4	4
5	32	10	100	121	4	16	40	1	1	0
$N=5$	110		250	585		40	60			8

$$R = 1-\frac{6\Sigma D^2}{N^3-N} = 1-\frac{6\times 8}{125-5} = 0\cdot 6; \qquad \bar{X} = \frac{110}{5} = 22, \ \bar{Y} = \frac{585}{5} = 117$$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}} = \frac{60}{\sqrt{250\times 40}} = 0\cdot 6.$$

Thus, the values of R and r are equal. It should be noted that the values of X increase by 5 and the values of Y, when arranged in ascending order also increase by the same amount 2 every time.

In general, if the values of x, when arranged in ascending order increase (or decrease) by a fixed amount and if the values of y, when arranged in ascending order increase (or decrease) by another (or the same) fixed amount, then the values of r and R come out to be equal.

(c) Computation of Correlation

There are two types of problems.

(i) When ranks of items are given.

(ii) When the actual values of the items are given.

(i) When ranks are given :

Steps : (i) Calculate the difference $D = R_1 - R_2$.

(ii) Calculate : D^2.

(iii) Apply the formula, $R = 1-\dfrac{6\Sigma D^2}{N^3-N}$.

(ii) When the actual vales are given : We first ascertain the ranks of all items and follow the above procedure.

Example 1 : Compute Spearman's rank correlation coefficient from the following data.

X : 18 20 34 52 12

Y : 39 23 35 18 46

(M.U. 2016)

Sol. : First we give ranks to the data in descending order and then calculate $D^2 = (R_1-R_2)^2$.

Calculation of R between X and Y

Serial No.	X	R_1	Y	R_2	D^2 $(R_1-R_2)^2$
1	18	4	39	2	4
2	20	3	23	4	1
3	34	2	35	3	1
4	52	1	18	5	16
5	12	5	46	1	16
$N=5$					$\Sigma D^2 = 38$

$\therefore \quad R = 1-\dfrac{6\Sigma D^2}{N^3-D}$. Here, $\Sigma D^2 = 38$, $N=5$.

$$\therefore \quad R = 1-\frac{6\times 38}{125-5} = 1-\frac{228}{120} = 1-1\cdot 9 = -0\cdot 9$$

Example 2 : Calculate the rank correlation coefficient from the following data, relating to the ranks of 10 students in English and Mathematics.

Student No.	: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Rank in English	: 1, 3, 7, 5, 4, 6, 2, 10, 9, 8.
Rank in Mathematics	: 3, 1, 4, 5, 6, 9, 7, 8, 10, 2.

Sol. : Calculation of R between English and Mathematics

Student No.	Rank in English R_1	Rank in Mathematics R_2	D^2 $(R_1-R_2)^2$
1	1	3	4
2	3	1	4
3	7	4	9
4	5	5	0
5	4	6	4
6	6	9	9
7	2	7	25
8	10	8	4
9	9	10	1
10	8	2	36
$N=10$			$\Sigma D^2 = 96$

Now, $\quad R = 1-\dfrac{6\Sigma D^2}{N^3-N} \qquad \because \Sigma D^2 = 96, \ N=10$

$$\therefore \quad R = 1-\frac{6\times 96}{990} = 1-\frac{96}{165} = 1-0\cdot 5819 = 0\cdot 4181$$

Example 3 : Calculate Spearman's coefficient of rank correlation from the data on height and weight of eight students.

Height (in inces) : 60 62 64 66 68 70 72 74

Weight (in lbs.) : 92 83 101 110 128 119 137 146 **(M.U. 2016)**

Sol. :

Calculation of R between Height and Weight

Serial No.	Height	Rank R_1	Weight	Rank R_2	D^2 $(R_1 - R_2)^2$
1	60	1	92	2	1
2	62	2	83	1	1
3	64	3	101	3	0
4	66	4	110	4	0
5	68	5	128	6	1
6	70	6	119	5	1
7	72	7	137	7	0
8	74	8	146	8	0
$N = 8$					$\Sigma D^2 = 4$

Now, $R = 1 - \frac{6 \Sigma D^2}{N^3 - N}$ $\quad \because \Sigma D^2 = 4, \; N = 8$

$\therefore \; R = 1 - \frac{6 \times 4}{512 - 8} = 1 - \frac{24}{504} = 1 - 0{\cdot}048 = 0{\cdot}952$

(d) Equal Ranks

In some cases it may happen that there is a tie between two or more members *i.e.*, they have equal values and hence equal ranks. In such cases we divide the rank among equal members. For instance, if two items have 4th rank we divide the 4th and the next rank 5th between them equally and give $\frac{4+5}{2} = 4{\cdot}5$th rank to each of them. If three items have the same 4th rank, we give each of them $\frac{4+5+6}{3} = 5$th rank.

After assigning ranks in this way an adjustment is necessary. If m is the number of items having equal ranks then the factor $\frac{1}{12}(m^3 - m)$ is added to Σd_i^2. If there are more than one cases of this type this factor is added corresponding to each case. Then,

$$R = 1 - \frac{6\left[\Sigma d_i^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \ldots\ldots\right]}{n^3 - n}$$

Example 1 : Obtain the rank correlation coefficient from the following data.

X : 10, 12, 18, 18, 15, 40.

Y : 12, 18, 25, 25, 50, 25. **(M.U. 2004, 05, 10, 14)**

Calculation of R

X	Rank R_1	Y	Rank R_2	D^2 $(R_1 - R_2)^2$
10	1	12	1	0·00
12	2	18	2	0·00
18	4·5	25	4	0·25
18	4·5	25	4	0·25
15	3	50	6	9·00
40	6	25	4	4·00
$N = 6$				$\Sigma D^2 = 13{\cdot}50$

There are two items in X series having equal values at the rank 4. Each is given the rank 5 $= 4{\cdot}5$. Similarly, there are three items in Y series at the rank 3. Each of them is given the $\frac{3+4+5}{3} = 4$.

$$\therefore \; R = 1 - \frac{6\left[\Sigma d_i^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\right]}{n^3 - n}$$

Since, $\Sigma D^2 = 13{\cdot}50, \; m_1 = 2, \; m_2 = 3, \; N = 6.$

$$R = 1 - \frac{6\left[13{\cdot}50 + \frac{1}{12}(8-2) + \frac{1}{12}(27-3)\right]}{216 - 6} = 1 - 0{\cdot}4571 = 0{\cdot}5429.$$

Example 2 : Calculate the value of rank correlation coefficient from the following data regarding s of 6 students in statistics and accountancy in a test :

Marks in Statistics : 40, 42, 45, 35, 36, 39.

Marks in Accountancy : 46, 43, 44, 39, 40, 43. **(M.U. 2014)**

Calculation of R

X	R_1	Y	R_2	D^2 $(R_1 - R_2)^2$
40	3	46	1	4·00
42	2	43	3·5	2·25
45	1	44	2	1·00
35	6	39	6	0·00
36	5	40	5	0·00
39	4	43	3·5	0·25
$N = 6$				$\Sigma D^2 = 7{\cdot}50$

$$R = 1 - \frac{6\left\{\Sigma D^2 + \frac{1}{12}(2^3 - 2)\right\}}{N^3 - N} = 1 - \frac{6(7{\cdot}5 + 0{\cdot}5)}{216 - 6}$$

$$= 1 - \frac{48}{210} = 1 - 0{\cdot}229 = 0{\cdot}771$$

Example 3 : From the following data calculate the coefficient of rank correlation between X and Y.

X : 32, 55, 49, 60, 43, 37, 43, 49, 10, 20.

Y : 40, 30, 70, 20, 30, 50, 72, 60, 45, 25. **(M.U. 2016)**

Sol. : Calculation of R between X and Y

X	R_1	Y	R_2	D^2 $(R_1 - R_2)^2$
32	3	40	5	4·00
55	9	30	3·5	30·25
49	7·5	70	9	2·25
60	10	20	1	81·00
43	5·5	30	3·5	4·00
37	4	50	7	9·00
43	5·5	72	10	20·25
49	7·5	60	8	0·25
10	1	45	6	25·00
20	2	25	2	0
$N = 10$				$\Sigma D^2 = 176$

Since there are two items in the X series having equal values at the rank 5 and two at the rank 7 they are given rank 5·5 and 7·5 each respectively. Similarly, in the Y series two items at the rank 3 are given the rank 3·5 each. There are three cases where there is a tie each having 2 times.

$$\therefore \quad R = 1 - \frac{6\left\{\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3)\right\}}{N^3 - N}$$

But, $\Sigma D^2 = 176$, $m_1 = m_2 = m_3 = 2$, $N = 10$

$$\therefore \quad R = 1 - \frac{6\left\{176 + \frac{1}{12}(8-2) + \frac{1}{12}(8-2) + \frac{1}{12}(8-2)\right\}}{1000 - 10}$$

$$= 1 - \frac{6(177 \cdot 5)}{990} = 1 - 1 \cdot 076 = -0 \cdot 076.$$

EXERCISE - II

1. Sixteen industries of the State have been ranked as follows according to profits earned is 1980 - 81 and the working capital for the year. Calculate the rank correlation coefficient.

Industry :	A,	B,	C,	D,	E,	F,	G,	H,	I,	J,	K,	L,	M,	N,	O,	P.
Rank (Profit) :	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11	12,	13,	14,	15,	16.
Rank (Capital) :	13,	16,	14,	15,	10,	12,	4,	11,	5,	9,	8,	3,	1,	6,	7,	2.

[**Ans. :** $R = -0 \cdot 8176$]

2. Distribution of marks in Economics and Mathematics for ten students in a certain test are given below :

Student No. :	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Marks in Eco. :	25,	28,	32,	36,	38,	40,	39,	42,	41,	45.
Marks in Maths. :	70,	80,	85,	75,	59,	65,	48,	50,	54,	66.

Calculate the value of Spearman's Rank correlation coefficient. [**Ans. :** – 0·6364]

3. Calculate Spearman's coefficient of rank correlation for the following data.

X : 53, 98, 95, 81, 75, 61, 59, 55.

Y : 47, 25, 32, 37, 30, 40, 39, 45. [**Ans. :** – 0·9048]

4. Calculate Spearman's coefficient of rank correlation from the following data.

X : 35, 38, 43, 30, 54, 68, 70, 92, 44, 56.

Y : 51, 37, 48, 62, 93, 73, 56, 72, 70, 92. [**Ans. :** 0·59]

5. Calculate Spearman's coefficient of rank correlation for the following data of scores in psychological tests (X) and arithmetical ability (Y) of 10 children.

Child : A, B, C, D, E, F, G, H, I, J.

X : 105, 104, 102, 101, 100, 99, 98, 96, 93, 92.

Y : 101, 103, 100, 98, 95, 96, 104, 92, 97, 94. [**Ans. :** $R = 0 \cdot 782$]

6. Find the rank correlation coefficient between poverty and over crowding of cities from the following data.

Town :	A,	B,	C,	D,	E,	F,	G,	H,	I,	J.
No. of poor families :	17,	13,	15,	16,	6,	11,	14,	9,	7,	12.
Population (over crowding) :	30,	46,	35,	24,	12,	18,	27,	22,	46,	8.

[**Ans. :** $R = 0 \cdot 73$]

7. Calculate the rank coefficient of correlation from the following data.

X : 105, 110, 112, 108, 111, 116, 120, 104, 115, 125.

Y : 39, 41, 45, 38, 48, 58, 60, 35, 54, 69. [**Ans. :** $R = 0 \cdot 9636$]

8. Two judges X, Y ranked 8 candidates as follows. Find the correlation coefficient.

Candidates :	A,	B,	C,	D,	E,	F,	G,	H.
First judge X :	5,	2,	8,	1,	4,	6,	3,	7.
Second judge Y :	4,	5,	7,	3,	2,	8,	1,	6.

[**Ans. :** $R = 0 \cdot 67$]

9. Calculate the rank correlation coefficient from the following data.

Marks in Paper I : 52, 63, 45, 36, 72, 65, 45, 25.

Marks in Paper II : 62, 53, 51, 25, 79, 43, 60, 33. [**Ans. :** $R = 0 \cdot 648$]

Miscellaneous Examples

Example 1 : State true or false with proper justification. **(M.U. 1998)**

If coefficient of correlation between x and y is negative then the coefficient of correlation between $-x$ and $-y$ is positive.

Sol. : Coefficient of correlation between x and y is given by

$$r = \frac{\Sigma xy - \Sigma x \cdot \Sigma y / N}{\sqrt{\Sigma x^2 - (\Sigma x)^2 / N}\sqrt{\Sigma y^2 - (\Sigma y)^2 / N}}$$

If we change the signs of x and y both, since the product and square terms occur, the sign of r will remain the same.

$\therefore$ The statement is false.

Example 2 : If the arithmetic mean of regression coefficients is p and their difference is $2q$, find the correlation coefficient. **(M.U. 1998)**

Sol. : Let the coefficients of regression be b_1 and b_2.

Now by data $\dfrac{b_1+b_2}{2} = p$ and $b_1 - b_2 = 2q$

$\therefore \quad b_1 + b_2 = 2p$ and $b_1 - b_2 = 2q$

$\therefore \quad b_1 = p + q$ and $b_2 = p - q$

$\therefore$ Coefficient of correlation $= r = \sqrt{b_1 b_2} = \sqrt{p^2 - q^2}$

Example 3 : The coefficient of rank correlation of the marks obtained by 10 students in Physics and Chemistry was found to be 0·5. It was later discovered that the difference in ranks in the two subjects obtained by one of the students was wrongly taken as 3 instead of 7. Find the correct coefficient of rank correlation. **(M.U. 2005)**

Sol. : Since $R = 1 - \dfrac{6\Sigma d_i^2}{n^3 - n}$ and $R = 0{\cdot}5$, $n = 10$.

$$0{\cdot}5 = 1 - \frac{6\Sigma d_i^2}{1000-10} \quad \therefore \Sigma d_i^2 = \frac{495}{6}$$

$\therefore$ Correct Σd_i^2 = Incorrect Σd_i^2 – (Incorrect rank diff.)^2 + (Correct rank diff.)^2

$$= \frac{495}{6} - 3^2 + 7^2 = \frac{735}{6}$$

$$\therefore \text{Correct } R = 1 - \frac{6 \times (735/6)}{990} = 1 - \frac{735}{990} = 0{\cdot}26.$$

Example 4 : (a) Let $r_{xy} = 0{\cdot}4$, Cov. $(x, y) = 1{\cdot}6$, $\sigma_y^2 = 25$. Find σ_x.

(b) If $R_{x,\,y} = 0{\cdot}143$ and the sum of the squares of the differences between the ranks is 48, find N. **(M.U. 2005)**

Sol. : (a) We have $r = \dfrac{\text{cov}(x, y)}{\sigma_x \cdot \sigma_y}$. But $r = 0{\cdot}4$, cov. $(x, y) = 1{\cdot}6$, $\sigma_y = 5$.

$$\therefore 0{\cdot}4 = \frac{1{\cdot}6}{5\sigma_x} \quad \therefore \sigma_x = \frac{1{\cdot}6}{5 \times 0{\cdot}4} = 0{\cdot}8$$

(b) We have, $R = 1 - \dfrac{6\Sigma D^2}{N^3 - N}$. By data, $R = 0{\cdot}143$, $\Sigma D^2 = 48$.

$$\therefore 0{\cdot}143 = 1 - \frac{6 \times 48}{N^3 - N} = 1 - \frac{288}{N^3 - N} \quad \therefore \frac{288}{N^3 - N} = 1 - 0{\cdot}143 = 0{\cdot}857$$

$$\therefore N^3 - N = \frac{288}{0{\cdot}857} = 336 \quad \therefore N^3 - N - 336 = 0$$

$$\therefore N^3 - 7N^2 + 7N^2 - 49N + 48N - 336 = 0$$

$\therefore (N-7)(N^2 + 7N + 48) = 0 \quad \therefore N = 7$, other roots of N are imaginary.

Example 5 : Calculate the correlation coefficient between x and y from the following data.

$N = 10$, $\Sigma x = 140$, $\Sigma y = 150$, $\Sigma (x - 10)^2 = 180$,

$\Sigma (y - 15)^2 = 215$, $\Sigma (x - 10)(y - 15) = 60$. **(M.U. 1997, 99)**

Sol. : With usual notation $\Sigma d_x^2 = 180$, $\Sigma d_y^2 = 215$, $\Sigma d_x d_y = 60$.

Now, $\bar{x} = A + \dfrac{\Sigma dx}{N} \quad \therefore 14 = 10 + \dfrac{\Sigma d_x}{10} \quad \therefore \Sigma d_x = 40$

Similarly, $\bar{y} = B + \dfrac{\Sigma d_y}{N} \quad \therefore 15 = 15 + \dfrac{\Sigma d_y}{10} \quad \therefore \Sigma d_y = 10$

Now,
$$r = \frac{\Sigma d_x d_y - \dfrac{\Sigma d_x \cdot \Sigma d_y}{N}}{\sqrt{\Sigma d_x^2 - \dfrac{(\Sigma d_x)^2}{N}}\sqrt{\Sigma d_y^2 - \dfrac{(\Sigma d_y)^2}{N}}}$$

$$= \frac{60 - \dfrac{40 \times 0}{10}}{\sqrt{180 - \dfrac{(40)^2}{10}}\sqrt{215 - \dfrac{(0)^2}{10}}} = \frac{60}{\sqrt{20}\sqrt{215}} = 0{\cdot}915.$$

EXERCISE - III

Type I

1. Compute Spearman's rank correlation coefficient from the following data.

   X : 85, 74, 85, 50, 65, 78, 74, 60, 74, 90

   Y : 78, 91, 78, 58, 60, 72, 80, 55, 68, 70. **(M.U. 2007)** [**Ans. :** $R = 0{\cdot}45$]

2. Compute Spearman's rank correlation coefficient from the following data.

   X : 18, 20, 34, 52, 12.

   Y : 39, 23, 35, 18, 46. **(M.U. 2004)** [**Ans. :** – 0·824]

3. From the following data calculate Spearman's rank correlation between x and y.

   x : 36, 56, 20, 42, 33, 44, 50, 15, 60.

   y : 50, 35, 70, 58, 75, 60, 45, 80, 38. **(M.U. 2010)** [**Ans. :** $R = 0{\cdot}92$]

4. Find the coefficient of correlation (r) between x and y for the following data.

   x : 62 64 65 69 70 71 72 74

   y : 126 125 139 145 165 152 180 208 **(M.U. 2003, 04)** [**Ans. :** 0·9032]

5. Find Karl Pearson's coefficient of correlation and also, the Spear-man's rank coefficient of correlation for the following data.

   x : 12 17 22 27 32

   y : 113 119 117 115 121 [**Ans. :** $r = R = 0{\cdot}6$]

   Also interpret your result.

6. The following data gave the growth of employment in lakhs in organised sector in India between 1988 and 1995.

   Year : 1988, 89, 90, 91, 92, 93, 94, 95.

   Public Sector : 98, 101, 104, 107, 113, 120, 125, 128.

   Private Sector : 65, 65, 67, 68, 68, 69, 68, 68.

   Find the correlation coefficient (r) between the employment in public and private sectors and give your comments. **(M.U. 1998)** [**Ans. :** $r = 0{\cdot}98$]

7. Calculate the coefficient of correlation from the following figures. Is there any marked correlation between the production and price of tea ?

Production in crores of lbs.	:	44,	37,	31,	38,	36,	35,	40.
Price in Rs. per lbs.	:	2·75,	3·62,	4·25,	4·12,	4·28,	4·32,	4·05.

8. Draw a scatter diagram to represent the following data.

X	:	2,	4,	5,	6,	8,	11.
Y	:	18,	12,	10,	8,	7,	5.

Calculate the coefficient of correlation between X and Y for the above data. **(M.U. 1998)**

[**Ans. :** $r = -0{\cdot}92$]

9. Find the coefficient of correlation between height of father and height of son from the following data.

Height of father	:	65,	66,	67,	67,	68,	69,	71,	73.
Height of son	:	67,	68,	64,	68,	72,	70,	69,	70.

[**Ans. :** $r = +0{\cdot}55$]

10. Calculate Spearman's coefficient of rank correlation and Pearson's coefficient of correlation from the data on height and weight of eight students. Why the two values are same ?

Height *(in inches)*	:	60,	62,	64,	66,	68,	70,	72,	74.
Weight *(in lbs.)*	:	92,	83,	101,	110,	128,	119,	137,	146.

[**Ans. :** $r = R = 0{\cdot}93$; For both the series, the difference between consecutive terms remains constant if arranged in order.]

11. The following table shows the marks obtained by 10 students in Accountancy and Statistics. Find the Spearman's coefficient of rank correlation.

Student No.	:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Accountancy	:	45,	70,	65,	30,	90,	40,	50,	57,	85,	60.
Statistics	:	35,	90,	70,	40,	95,	40,	60,	80,	80,	50.

Will the result change if the marks in the two subjects of all the students are increased by 5 and 10 respectively ? Will the result change if marks in the two subjects of all the students are halved ?

[**Ans. :** $r = 0{\cdot}8606$; No. : No.]

Type II

1. The coefficient of rank correlation between marks in Physics and Chemistry obtained by a group of students is 0·8. If the sum of the squares of differences in ranks in 33, find the number pairs students.

[**Ans. :** $N = 10$]

2. Find the number of pairs of observations from the following data.

$r = 0{\cdot}4,\ \Sigma xy = 108,\ \sigma_y = 3,\ \Sigma x^2 = 900.$

where x, y are the deviations of x, y from there respective means. [**Ans. :** $N = 10$]

3. Coefficient of correlation between two variables is 0·4. Their covariance is 12. The variance of x is 25. Find the standard deviation of y. [**Ans. :** $\sigma_y = 6$]

4. A computer while calculating the correlation coefficient between two variables x and y, from 25 observations obtained the following results

$N = 25,\ \Sigma x = 125,\ \Sigma y = 100,\ \Sigma x^2 = 650,\ \Sigma y^2 = 960,\ \Sigma xy = 508$

where x, y denote the actual values of the variables. Find the value of r. [**Ans. :** 0·067]

5. A sample of 25 pairs of values of x and y lead to the following results.

$\Sigma x = 127,\ \Sigma y = 100,\ \Sigma x^2 = 760,\ \Sigma y^2 = 449,\ \Sigma xy = 500.$

...ater on it was found that two pairs of values were taken as (8, 14) and (8, 6) instead of correct ...s (6, 12) and (6, 8).
...ind corrected correlation coefficient between x and y. **(M.U. 2004)** [**Ans. :** $r = -0{\cdot}31$]

6. Given : Number of pairs of observations = 10

X series standard deviation = 22·70. Y series standard deviation = 9·592

...mation of the products of corresponding deviations of X and Y from their respective actual ...s = − 1439. Find r.

[**Ans. :** $r = -0{\cdot}66$]

EXERCISE - IV

...ory

1. What is meant by correlation ? Describe scatter diagram and interpret.

2. Define - (i) Karl Pearson's coefficient of correlation, (ii) Spearman's rank correlation ...cient, (iii) Coefficient of variation. **(M.U. 2005)**

3. Two variables x and y are connected by the relation $ax + by + c = 0$. Show that the coefficient ...relation is either + 1 or − 1.

4. Define product moment correlation coefficient and show that it is always numerically less ...or equal to unity.

5. Define - (i) Covariance between two variables x and y, (ii) Karl Pearson's Product moment ...ation coefficient.

6. Prove that Spearson's rank correlation coefficient R is given by

$$R = 1 - \frac{6 \Sigma D^2}{N^3 - N}$$

(M.U. 1996, 99, 2002, 03, 04, 05, 06, 09)

7. What is scatter diagram ? How does it help in studying the correlation between two variables. Draw scatter diagrams for $r = +1$, $r = -1$ and $r = 0$.

8. Define the Karl Pearson's coefficient of correlation r between two variables x and y. What ...purious Correlation" ?

...nterpret the cases $r = +1$, $r = -1$, $r = 0$. Also draw the scatter diagrams corresponding to these ...es.

❀ ❀ ❀

CHAPTER 9

Regression

1. Introduction

We have seen in the previous chapter how to examine and measure in magnitude and direction correlation between two variables. After establishing correlation, it is natural to search for a method which will help us to estimate the value of one variable when that of the other is known. This is achieved by the analysis of regression. Regression can be defined as **'a method of estimating the value of one variable when that of the other is known and when the variables are correlated'**.

The term, regression was first used by Galton. He found that although tall fathers have tall sons, and short fathers have short sons, the average height of sons of tall fathers is less than the average height of their fathers and the average height of sons of short fathers is more than the average height of their fathers. In other words the average height of sons of tall fathers or short fathers will regress or go back to the general average height. This phenomenon was described by him as 'regression.'

2. Lines of Regression

We have seen in the previous chapter that if the variables which are highly correlated are plotted on a graph then the points lie in a narrow strip. If the strip is nearly straight, we may draw a line such that all the points are close to it from both the sides. Such a line can be taken as the representative of the ideal variation. It is called the line of best fit. It is a line such that the sum of the distances of the points from the line is minimum. It is also called **'the line of regression'**. But we do not measure the distance by dropping a perpendicular from a point to the line. We measure, the deviations (i) vertically and (ii) horizontally, and get one line when distances are minimised vertically and second line when distances are minimised horizontally. Thus, we get **two lines of regression.**

(i) Line of regression of Y on X

If we minimise the deviations of the points from the line measured along y-axis we get a line which is called **the line of regression of Y on X**. Its equation is written in the from $Y = a + bX$. **This line is used for estimating the value of Y for a given value of X.** [See Fig. 9.1]

The equation of the line of regression of y on x must be written with y on the left hand side and x and the constant term on the right hand side.

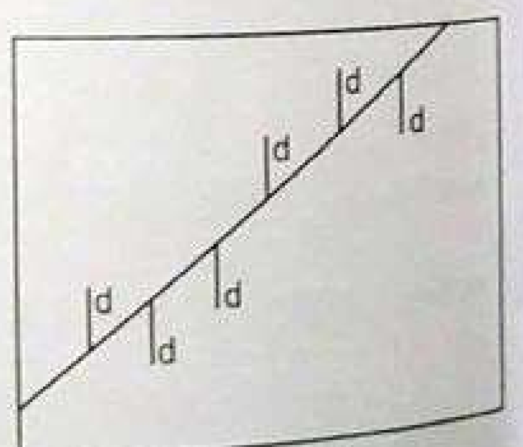


Fig. 9.1
For the Line of Regression of Y on X the distances d are minimised.

(ii) Line of regression of X on Y

If we minimise the deviations of the points from the line measured along x-axis we get a line which is called **the line of regression of X on Y**. Its equation is written in the form $X = a + bY$. **This line is used for estimating the value of X for a given value of Y.** [See Fig. 9.2]

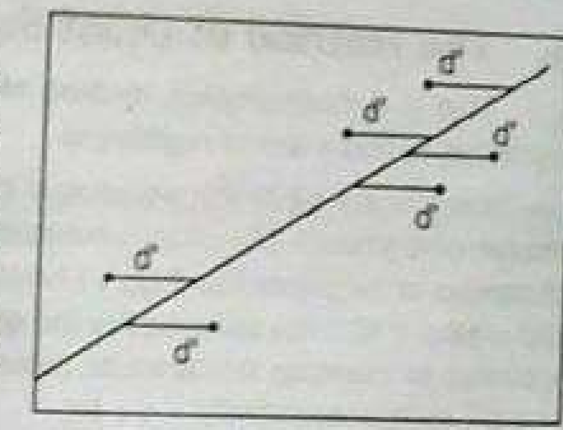


Fig. 9.2
For the line of regression of X on Y the distances d' are minimised.

The equation of the line of regression of x on y must be written with x on the left hand side and y and constant term on the right hand side.

There are two methods of obtaining the lines of regression. The first is graphical, the other is mathematical. They are :

1. The method of Scatter Diagram,
2. The Method of Least Squares.

3. The Method of Scatter Diagram

It is the simplest method of obtaining the lines of regression. The data are plotted on a graph paper by taking the independent variable on x-axis and the dependent variable on y-axis. We thus get points which are generally scattered. If the correlation is perfect *i.e.* if r is equal to one, positive or negative, the points will lie on a line, which is the line of regression. And there is only one line of regression and not two in such cases. However, in practice we rarely come across problems wherein we have perfect correlation. Usually, the points are scattered in a narrow straight strip and we have to find a line which will best represent all the points of the scatter diagram. We draw a line which will be close to all the points as far as possible.

Example : Given the following pairs of values of X and Y.

X	:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12.
Y	:	5,	6,	5,	6,	6,	8,	7,	9,	8,	9,	10,	11.

Plot the points on a graph and draw a line of regression.

Sol. :

Scatter Diagram and Line of Regression

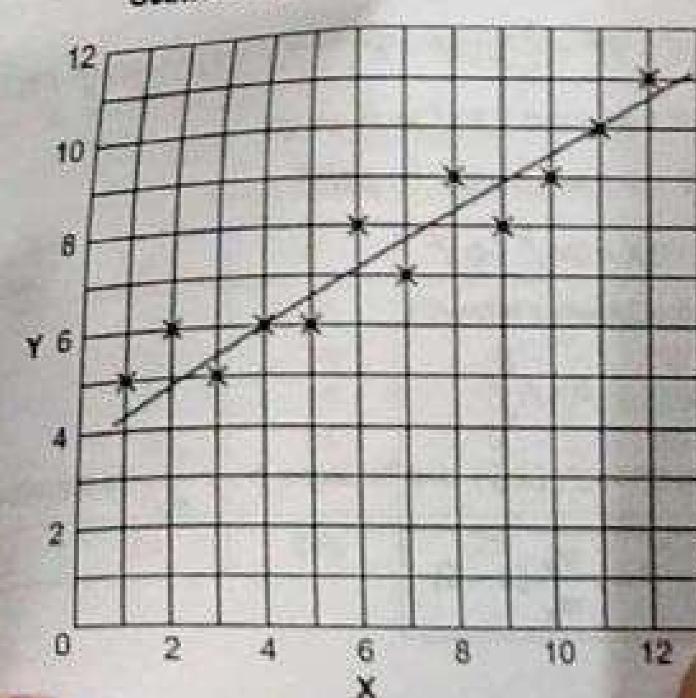


Fig. 9.3

4. The Method of Least Square

This is a mathematical method which gives an objective treatment to find a line of regression.

Let $y = a + bx$ be the equation of the line required. To find the line of regression of y on x we minimise the sum of the absolute distances of the points like $P(x_i, y_i)$ from the line measured along the y-axis. If Q is the point on the line corresponding to $P(x_i, y_i)$ we have to minimise the absolute distance PQ. Since Q lies on $y = a + bx$ its y-coordinate $= a + bx_i$

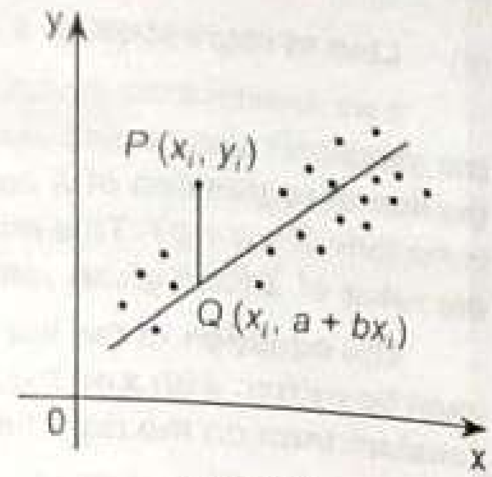


Fig. 9.4

$$\therefore \quad |PQ| = |y_i - a - bx_i|$$

For minimising $|PQ|$ we minimise its squares. Hence, if S denotes the sum of the squares of these distances,

$$S = \Sigma f_i (y_i - a - bx_i)^2 \quad \text{where } f_i \text{ is the frequency of } (x_i, y_i).$$

We have to find a and b such that S is minimum, the conditions for which are

$$\frac{\partial S}{\partial a} = 2\Sigma f_i (y_i - a - bx_i) = 0 \quad \text{and} \quad \frac{\partial S}{\partial b} = 2\Sigma f_i (y_i - a - bx_i)\, x_i = 0$$

$$\Sigma f_i (y_i - a - bx_i) = 0 \qquad \text{.......... (A)}$$

$$\Sigma x_i f_i (y_i - a - bx_i) = 0 \qquad \text{.......... (B)}$$

From (A) we get, $\Sigma f_i y_i - a\Sigma f_i - b\Sigma f_i x_i = 0$

$$\therefore \quad N\bar{y} - aN - bN\bar{x} = 0 \quad \therefore \quad \bar{y} = a + b\bar{x} \qquad \text{.......... (C)}$$

which shows that **the line of regression passes through** $(\bar{x}, \bar{y})$.

From (B) we get, $\Sigma f_i x_i y_i - a\Sigma f_i x_i - b\Sigma f_i x_i^2 = 0$ .... (D)

We now find the values of these expressions in terms of r, σ_x, σ_y.

But since, $r = \dfrac{\frac{1}{N}\Sigma f_i (x_i - \bar{x})(y_i - \bar{y})}{\sigma_x \sigma_y}$

$$r = \frac{\Sigma f_i x_i y_i - N\bar{x}\bar{y}}{N\sigma_x \sigma_y} \quad \therefore \quad \Sigma f_i x_i y_i = Nr\sigma_x \sigma_y + N\bar{x}\bar{y}$$

and $\quad \sigma_x^2 = \dfrac{1}{N}\Sigma f_i (x_i - \bar{x})^2 = \dfrac{1}{N}\Sigma f_i x_i^2 - \bar{x}^2 \quad \therefore \quad \Sigma f_i x_i^2 = N\sigma_x^2 + N\bar{x}^2$

Putting the values of $\Sigma f_i x_i y_i$ and $\Sigma f_i x_i^2$ in D, we get

$$Nr\sigma_x\sigma_y + N\bar{x}\bar{y} = aN\bar{x} + bN\sigma_x^2 + bN\bar{x}^2$$

i.e. $\quad r\sigma_x\sigma_y + \bar{x}\bar{y} = a\bar{x} + b\sigma_x^2 + b\bar{x}^2$ .... (E)

Multiply (C) by $\bar{x}$ and subtract it form (E)

$$r\sigma_x\sigma_y = b\sigma_x^2 \quad \therefore \quad b = r\frac{\sigma_y}{\sigma_x}$$

Since, the line passes through $(\bar{x}, \bar{y})$ and its slope $b = r\dfrac{\sigma_y}{\sigma_x}$ its equation is

$$\boxed{y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})} \qquad \text{.......... (F)}$$

Similarly, the equation of the line of regression of x on y can be shown to be

$$\boxed{x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})} \qquad \text{.......... (G)}$$

Alternative Method : Instead of calculating $\bar{x}, \bar{y}, \sigma_x, \sigma_y$ and r we may use the following method.

(i) **The line of regression of y on x**

Let the equation of the line of regression of y on x be $y = a + bx$.

Then as before we have to minimise

$$S = \Sigma f_i (y_i - a - bx_i)^2 \quad \text{the conditions of which are}$$

$$\frac{\partial S}{\partial a} = -2\Sigma f_i (y_i - a - bx_i) = 0 \quad \text{and} \quad \frac{\partial S}{\partial b} = -2\Sigma f_i (y_i - a - bx_i)\, x_i = 0$$

i.e. $\quad \Sigma f_i (y_i - a - bx_i) = 0 \quad$ and $\quad \Sigma f_i x_i (y_i - a - bx_i) = 0$

i.e. $\quad \Sigma f_i y_i - a\Sigma f_i - b\Sigma f_i x_i = 0 \quad$ and $\quad \Sigma f_i x_i y_i - a\Sigma f_i x_i - b\Sigma f_i x_i^2 = 0.$

If the required sums are known, by solving the above two equations simultaneously for a and b we get the required equation.

In particular if all values occur only once i.e. if $f_i = 1$ for all i then the above equations take the form

$$\boxed{\Sigma y = aN + b\Sigma x} \quad \text{and} \quad \boxed{\Sigma xy = a\Sigma x + b\Sigma x^2}$$

from which a and b can be calculated.

The above equations are called **normal equations**.

(ii) **The line of regression of x on y**

Let the equation of the line be $x = a + by$.

Proceeding as above we get the normal equations as

$$\boxed{\Sigma x = aN + b\Sigma y} \quad \text{and} \quad \boxed{\Sigma xy = a\Sigma y + b\Sigma y^2}$$

from which we can find the values of a and b.

Example : Find the equations of the lines of regression from the following data.

x: 5 6 7 8 9

y: 2 4 5 6 8 $\quad$ Also find r.

Sol. :

Calculations of regression

Sr. No.	x	x^2	y	y^2	xy
1	5	25	2	4	10
2	6	36	4	16	24
3	7	49	5	25	35
4	8	64	6	36	48
5	9	81	8	64	72
N = 5	35	255	25	145	189

The equation of the line of regression of y on x is $y = a + bx$ where a, b are given by

$$\sum y = aN + b\sum x \quad \text{and} \quad \sum xy = a\sum x + b\sum x^2$$

Putting the values of $\sum x, \sum x^2, \sum xy$, we get

$$25 = 5a + 35b \quad \text{.......... (i) and} \quad 189 = 35a + 255b \quad \text{.......... (ii)}$$

Multiply the first by 7 and subtract it from the second.

$$\begin{aligned} 189 &= 35a + 255b \\ 175 &= 35a + 245b \\ \hline 14 &= \quad\quad 10b \qquad \therefore\ b = 1{\cdot}4 \end{aligned}$$

Putting this value of b in (i), we get

$$25 = 5a + 35\,(1{\cdot}4) \quad \therefore\ 5a = 25 - 49 = -24 \quad \therefore\ a = -4{\cdot}8$$

$\therefore$ The equation of the line of regression of y on x is

$$y = -4{\cdot}8 + 1{\cdot}4x$$

The equation of the line of regression of x on y is $x = a + by$ where a, b are given by

$$\sum x = aN + b\sum y \quad \text{and} \quad \sum xy = a\sum y + b\sum y^2$$

Putting the values of $\sum x, \sum y, \sum xy, \sum y^2$, we get

$$35 = 5a + 25b \quad \text{.......... (iii) and} \quad 189 = 25a + 145b \quad \text{.......... (iv)}$$

Multiply (iii) by 5 and subtract it from (iv)

$$\begin{aligned} 189 &= 25a + 145b \\ 175 &= 25a + 120b \\ \hline 14 &= \quad\quad 25b \qquad \therefore\ b = 0{\cdot}56 \end{aligned}$$

Putting this value of b in (iii), we get

$$35 = 5a + 25\,(0{\cdot}56) \quad \therefore\ 5a = 35 - 14 = 11 \quad \therefore\ a = 2{\cdot}2$$

$\therefore$ The equation of the line of regression of x on y is

$$x = 2{\cdot}2 + 0{\cdot}56y$$

Now, $r = \sqrt{b_1 \times b_2} = \sqrt{1{\cdot}4 \times 0{\cdot}56} = 0{\cdot}88$.

5. Calculations of the Equations of the Lines of Regression

There are various methods of calculating the equations of the lines of regression. The choice is yours. We state them below.

(a) By calculating the coefficient of correlation r and standard deviation σ_x and σ_y

The equation of the line of regression of Y on X.

The equation of the line of regression of Y on X is given by,

$$Y - \overline{Y} = b_{yx}(X - \overline{X})$$

where, $\overline{X}$ = the mean of X, $\overline{Y}$ = the mean of Y and $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$. [See § 6]

$\therefore$ The equation, therefore can be written as

$$\boxed{Y - \overline{Y} = r\frac{\sigma_y}{\sigma_x}(X - \overline{X})} \qquad \text{.............. (1)}$$

This equation is to be used for calculating the most probable value of Y for a given value of X.

The equation of the line of regression of X on Y.

The equation of the line of regression of X on Y is given by,

$$X - \overline{X} = b_{xy}(Y - \overline{Y})$$

here, $\overline{X}$ = the mean of X, $\overline{Y}$ = the mean of Y and $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$.

$\therefore$ The equation, therefore can be written as

$$\boxed{X - \overline{X} = r\frac{\sigma_x}{\sigma_y}(Y - \overline{Y})} \qquad \text{.............. (2)}$$

This equation is to be used for calculating the most probable value of X for a given value of Y.

When the values of X and Y are known, we can as usual calculate r, σ_x and σ_y and then obtain equations of the lines of regression of X and Y.

(b) By calculating a and b directly

Instead of calculating $\overline{X}, \overline{Y}, \sigma_x, \sigma_y$ and r we may calculate a and b directly as explained below.

The line of regression of Y on X : The constants a and b of the equation of the line of regression of Y on X i.e. of,

$$Y = a + bX$$

can be obtained by solving the following simultaneous equations.

$$\boxed{\sum Y = aN + \sum X} \quad \text{and} \quad \boxed{\sum YX = a\sum X + b\sum X^2} \qquad \text{.............. (3)}$$

The line of regression of X on Y : The constants a and b of the equation of the line of regression of X on Y i.e. of,

$$X = a + bY$$

can be obtained by solving the following simultaneous equations,

$$\boxed{\sum X = aN + b\sum Y} \quad \text{and} \quad \boxed{\sum XY = a\sum Y + b\sum y^2} \qquad \text{.............. (4)}$$

The equations, are called 'Normal equations'.

The formulae (3) and (4) are convenient when $\sum X, \sum Y, \sum XY, \sum X^2, \sum Y^2$ are known.

(c) By taking deviations from the means X and Y

If x and y denote the deviations of X and Y from their means, we know that,

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{\sum xy}{N\sigma_x\sigma_y}\cdot\frac{\sigma_y}{\sigma_x} = \frac{\sum xy}{N\sigma_x^2} = \frac{\sum xy}{N \cdot \sum x^2/N} = \frac{\sum xy}{\sum x^2}$$

$$\therefore\ \boxed{b_{yx} = \frac{\sum xy}{\sum x^2}} \qquad \text{.............. (5)}$$

Similarly, we can show that,

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{\sum xy}{\sum y^2} \quad \therefore \quad \boxed{b_{xy} = \frac{\sum xy}{\sum y^2}} \quad \text{.............. (6)}$$

where, $x = X - \bar{X},\ y = Y - \bar{Y}$.

Hence, the equations of the lines of regression become,

$$Y - \bar{Y} = \frac{\sum xy}{\sum x^2}(X - \bar{X}) \quad \text{.............. (7)}$$

and

$$X - \bar{X} = \frac{\sum xy}{\sum y^2}(Y - \bar{Y}) \quad \text{.............. (8)}$$

(d) **By taking deviations from assumed means**

If the deviations of X and Y are taken from assumed means *i.e.* if $d_x = X - A$ and $d_y = Y - B$ then the coefficients b_{yx} and b_{xy} are given by,

$$b_{yx} = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{N}}{\sum d_x^2 - \frac{(\sum d_x)^2}{N}} \quad \text{.............. (9)}$$

$$b_{xy} = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{N}}{\sum d_y^2 - \frac{(\sum d_y)^2}{N}} \quad \text{.............. (10)}$$

(e) **By using actual values directly**

If X, Y are actual values of the two variates then it can be shown that

$$b_{yx} = \frac{\sum XY - \frac{\sum X \cdot \sum Y}{N}}{\sum X^2 - \frac{(\sum X)^2}{N}} \quad \text{.............. (11)}$$

$$b_{xy} = \frac{\sum XY - \frac{\sum X \cdot \sum Y}{N}}{\sum Y^2 - \frac{(\sum Y)^2}{N}} \quad \text{.............. (12)}$$

6. Regression Coefficients

The slope b of the line of regression of y on x *i.e.* b of the equation $y = a + bx$ is called the **coefficient of regression of y on x**. It represents the increment in y for unit change in the value of x. It is denoted by b_{yx}.

$\therefore$ b_{yx} = Coefficient of Regression of y on x.

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

Similarly, the slope b of the line of regression x on y *i.e.* b of the equation $x = a + by$ is called the **coefficient of regression of x on y**. It represents the increment in x for unit change in y. It is denoted by b_{xy}.

$\therefore$ b_{xy} = Coefficient of Regression of x on y.

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

Putting b_{yx} and b_{xy} which are the slopes of the lines of regression in (F) and (G), we can write the equations of lines of regression as

$$\boxed{y - \bar{y} = b_{yx}(x - \bar{x})} \quad \text{_____ (H)}$$

and

$$\boxed{x - \bar{x} = b_{xy}(y - \bar{y})} \quad \text{_____ (I)}$$

By putting the values of r and σ_y, σ_x in terms of actual values of x and y or by taking deviations from actual means or assumed means, we get the following formulae for b_{yx} and b_{xy}. (We repeat the first two.)

$$\boxed{b_{yx} = r\frac{\sigma_y}{\sigma_x}} \quad \text{_____ (1)}$$

$$\boxed{b_{yx} = \frac{\sum xy}{\sum x^2}} \quad \text{_____ (2)}$$

where $x = X - \bar{x},\ y = Y - \bar{y}$ i.e. x, y are deviations of X, Y from actual means.

$$b_{yx} = \frac{\sum d_x d_y - \frac{\sum d_x \cdot \sum d_y}{N}}{\sum d_x^2 - \frac{(\sum d_x)^2}{N}}$$

where $d_x = X - A$, $d_y = Y - B$ i.e. d_x, d_y are deviations of X, Y from assumed means A and B.

$$b_{yx} = \frac{\sum XY - \frac{\sum X \cdot \sum Y}{N}}{\sum X^2 - \frac{(\sum X)^2}{N}} \quad \text{_____ (4)}$$

where X, Y are the actual values of the variables.

Also,

$$\boxed{b_{xy} = r\frac{\sigma_x}{\sigma_y}} \quad \text{_____ (1')}$$

$$\boxed{b_{xy} = \frac{\sum xy}{\sum y^2}} \quad \text{_____ (2')}$$

where $x = X - \bar{x},\ y = Y - \bar{y}$ i.e. x, y are the deviations of X, Y from actual means

$$b_{xy} = \frac{\sum d_x d_y - \frac{\sum d_x \cdot \sum d_y}{N}}{\sum d_y^2 - \frac{(\sum d_y)^2}{N}} \quad \text{_____ (3')}$$

where $d_x = X - A$, $d_y = Y - B$ i.e. d_x, d_y are the deviations of X, Y from assumed means A and B.

$$\boxed{b_{xy} = \frac{\sum XY - \frac{\sum X \cdot \sum Y}{N}}{\sum Y^2 - \frac{(\sum Y)^2}{N}}} \qquad \text{.............. (4')}$$

where X, Y are the actual values of the variables.

7. Properties of Coefficients of Regression

1. Coefficient of correlation is the geometric mean between the coefficients of regression.

Proof : From the above results we have

$$b_{yx} \cdot b_{xy} = r\frac{\sigma_y}{\sigma_x} \cdot r\frac{\sigma_x}{\sigma_y}$$

$$\therefore \quad \boxed{b_{yx} \cdot b_{xy} = r^2} \qquad \text{Hence, the result.}$$

Remark ....

Since the product of b_{yx} and b_{xy} is positive, if one of them is negative, the other also must be negative. In other words both the coefficients of regression are positive or both the coefficients of regression are negative together.

2. If one coefficient of regression is greater than one, the other must be less than one.

Proof : Since $-1 \le r \le 1$, $r^2 \le 1$.

Hence, from the above result,

$$b_{yx} \cdot b_{xy} \le 1 \quad \therefore \; b_{yx} \le \frac{1}{b_{xy}} \qquad \therefore \text{ If } b_{yx} < 1, \; b_{xy} > 1.$$

[See the values of b_{yx} and b_{xy} in Ex. 4, page 9-16. See that b_{yx} is less than one and b_{xy} is greater than one.]

3. Arithmetic mean of the coefficients of regression is greater than or equal to the coefficient of correlation.

Proof : We have to show that $\dfrac{b_{yx} + b_{xy}}{2} \ge r$

i.e. $\dfrac{1}{2}\left(r\dfrac{\sigma_y}{\sigma_x} + r\dfrac{\sigma_x}{\sigma_y}\right) \ge r$ i.e. $\dfrac{\sigma_y}{\sigma_x} + \dfrac{\sigma_x}{\sigma_y} \ge 2$

i.e. $\sigma_x^2 + \sigma_y^2 \ge 2\sigma_x\sigma_y$ i.e. $\sigma_x^2 - 2\sigma_x\sigma_y + \sigma_y^2 \ge 0$

i.e. $(\sigma_x - \sigma_y)^2 \ge 0$ which is obviously true.

Remark ....

In other words this means **the sum of the two coefficients of regression is greater than or equal to 2r.** We shall verify this in Ex. 2, page 9-19; Ex. 5, page 9-17 and Ex. 6, page 9-22 below. We further note that equality will hold when $\sigma_x = \sigma_y$.

4. Coefficients of regression are independent of change of origin but not of change of scale.

If $u = ax + h$ and $v = by + k$, then

$$\boxed{b_{uv} = \frac{a}{b} \cdot b_{xy} = \frac{\text{Coefficient of } x}{\text{Coefficient of } y} \cdot b_{xy}}$$

We accept this result without proof.

5. If the correlation is perfect then the two coefficients of regression are reciprocals of each other.

Proof : We have $r = \pm 1$ and $r = \sqrt{b_{yx} \cdot b_{xy}}$ $\quad \therefore \; \pm 1 = \sqrt{b_{yx} \cdot b_{xy}}$.

Squaring $1 = b_{yx} \cdot b_{xy}$ $\quad \therefore \; b_{yx} = \dfrac{1}{b_{xy}}$.

e.g., if one coefficient of regression is 0·5 and if the correlation is perfect, then the other coefficient of regression is 2.

Example 1 : State whether the following statement is true or false with reasoning : "The regression coefficients between $2x$ and $2y$ are the same as those between x and y". **(M.U. 1997)**

Sol. : As seen above if $u = ax + h$ and $v = by + k$, $\quad b_{uv} = \dfrac{a}{b} \cdot b_{xy}$.

But by data $u = 2x$ i.e. $a = 2$ and $v = 2y$ i.e. $b = 2$.

$$\therefore \; b_{vu} = \frac{2}{2} \cdot b_{xy} = b_{xy}. \qquad \text{Hence, the statement is true.}$$

Example 2 : State whether the following statement is true or false : "The lines of regression between x and y are parallel to the lines of regression between $2x$ and $2y$.

Sol. : True. Explanation is left to you.

Example 3 : State whether the following statement is true or false : "The coefficients of regression between x and y are the same as the coefficients of regression between $2x + 5$ and $2y - 7$.

Sol. : True. Explanation is left to you.

Example 4 : If the arithmetic mean of regression coefficients is p and their difference is $2q$, find the correlation coefficient. **(M.U. 1998)**

Sol. : Let the coefficients of regression be b_1 and b_2.

Now by data $\dfrac{b_1 + b_2}{2} = p$ and $b_1 - b_2 = 2q$

$\therefore \; b_1 + b_2 = 2p$ and $b_1 - b_2 = 2q$

$\therefore \; b_1 = p + q$ and $b_2 = p - q$

$\therefore$ Coefficient of correlation $= r = \sqrt{b_1 b_2} = \sqrt{p^2 - q^2}$.

Example 5 : State true of false with reasoning : "$2x + y = 3$ and $x = 2y + 3$ cannot be the lines of regression." **(M.U. 2004)**

Sol. : If the first line is the line of regression of y on x it must written as $y = -2x + 3$ and if the [illegible] line is the line of regression of x on y, then it must be written as $x = 2y + 3$.

Hence, the coefficients of regression are $b_{yx} = -2$ and $b_{xy} = 2$ which is not possible as one of the them is negative and the other is positive and both are greater 1 numerically.

Now, we consider the lines in other way round. Let the first line be the line of regression of x on y and let the second line be the line of regression of y on x.

$$\therefore \quad x = -\frac{1}{2}y + \frac{3}{2} \quad \text{and} \quad y = \frac{1}{2}x - \frac{3}{2}.$$

Hence, the coefficients of regression are

$$b_{yx} = -\frac{1}{2} \quad \text{and} \quad b_{xy} = \frac{1}{2}$$

which is again not possible because one is positive and the other is negative.

Hence, the statement is true.

Example 6 : State true or false with justification. If two lines of regression are $x + 3y - 5 = 0$ and $4x + 3y - 8 = 0$ then the correlation coefficient is $+ 0{\cdot}5$. **(M.U. 2003, 14)**

Sol. : Let the line $x + 3y - 5 = 0$ be the line of regression of x on y. Writing it as $x = -3y + 5$, we get

$$b_{xy} = -3.$$

Let the line $4x + 3y - 8 = 0$ be the line of regression of y on x. Writing it as $3y = -4x + 8$

i.e., as $y = -\frac{4}{3}x + 2$, we get $b_{yx} = -\frac{4}{3}$.

$$\therefore \quad r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{(-3)(-4/3)} = \sqrt{4} = 2$$

But r cannot be greater than 1.

Hence, our suppositions are wrong.

Now, let the line $x + 3y - 5 = 0$ be the line of regression of y on x. Writing it as

$$3y = -x + 5 \qquad i.e., \quad y = -\frac{1}{3}x + \frac{6}{3}, \text{ we get } \quad b_{yx} = -\frac{1}{3}.$$

Let the line $4x + 3y - 8 = 0$ be the line of regression of x on y. Writing it as

$$4x = -3y + 8 \quad i.e., \quad x = -\frac{3}{4}y + 2, \text{ we get } \quad b_{xy} = -\frac{3}{4}.$$

Now, $r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{\left(-\frac{1}{3}\right)\left(-\frac{3}{4}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2} = 0{\cdot}5$

Hence, the statement is true.

6. Angle between the lines of regression

The equation of the lines of regression of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}). \qquad \text{Hence, its slope } m_1 = r\frac{\sigma_y}{\sigma_x}$$

The equation of the line of regression of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$i.e., \quad y - \bar{y} = \frac{\sigma_y}{r \cdot \sigma_x}(x - \bar{x}) \qquad \text{Hence, its slope } m_2 = \frac{\sigma_y}{r \cdot \sigma_x}$$

If θ is the angle between the lines of regression

$$\therefore \quad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{r\frac{\sigma_y}{\sigma_x} - \frac{\sigma_y}{r\sigma_x}}{1 + \frac{r\sigma_y}{\sigma_x} \cdot \frac{\sigma_y}{r\sigma_x}} = \frac{\dfrac{(r^2\sigma_y\sigma_x - \sigma_y\sigma_x)}{r\sigma_x^2}}{\dfrac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}}$$

$$= \frac{(r^2 - 1)}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\right) = \frac{1 - r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$

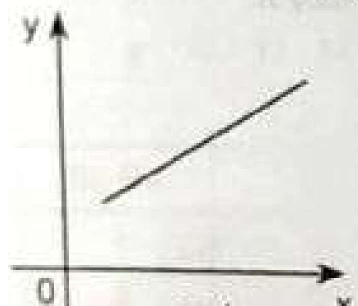
Corollary 1 : If $r = 0$, $\tan\theta = \infty \therefore \theta = \pi/2$.

The lines of regression are perpendicular to each other.

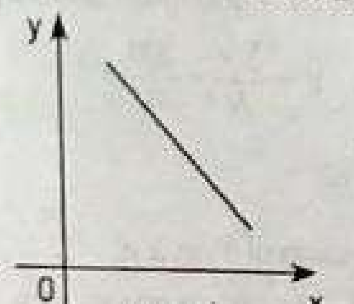
Corollary 2 : If $r = \pm 1$, $\tan\theta = 0 \therefore \theta = 0$.

The lines of regression are coincident.

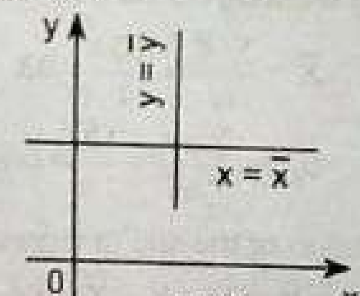
Various Cases Are Shown Below.



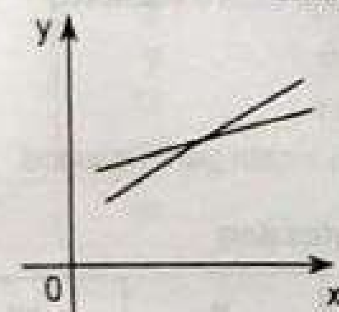
$r = 1$
Lines coincide



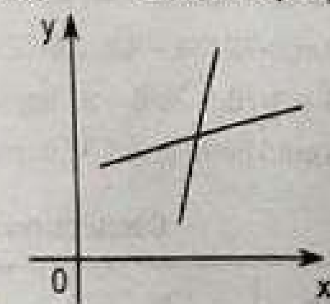
$r = -1$
Lines coincide



$r = 0$
Lines are perpendicular



θ small, r nearer to 1
High degree of correlation



θ large, r nearer to 0
Low degree of correlation

Illustrative Examples

Example 1 : A panel of two judges A and B graded dramatic performances by independently awarding marks as follows :

Performance No.	:	1,	2,	3,	4,	5,	6,	7.
Marks by A	:	36,	32,	34,	31,	32,	32,	34.
Marks by B	:	35,	33,	31,	30,	34,	32,	36.

The eighth performance, however, which judge B could not attend, got 38 marks by judge A. If judge B had also been present, how many marks would he be expected to have awarded to the eighth performance ?

Sol. :

Calculations of coefficient of correlation etc.

Sr. No.	$X-\overline{X}$			$Y-\overline{Y}$			Product
	X	x	x^2	Y	y	y^2	xy
1	36	3	9	35	2	4	6
2	32	−1	1	33	0	0	0
3	34	1	1	31	−2	4	−2
4	31	−2	4	30	−3	9	6
5	32	−1	1	34	1	1	−1
6	32	−1	1	32	−1	1	1
7	35	1	1	36	3	9	3
$N=7$	$\Sigma X=231$		$\Sigma x^2=18$	$\Sigma Y=231$		$\Sigma y^2=28$	$\Sigma xy=13$

We have to find the marks that would have been awarded by the judge B. Therefore, let the marks given by the judge B be denoted by Y and those given by A by X.

$$\therefore \quad \overline{X}=\frac{\Sigma X}{N}=\frac{231}{7}=33, \quad \overline{Y}=\frac{\Sigma Y}{N}=\frac{231}{7}=33$$

$$\therefore \quad b_{yx}=\frac{\Sigma xy}{\Sigma x^2}=\frac{13}{18}=0{\cdot}72$$

The equation of the line of regression of Y on X is,

$$Y-\overline{Y}=b_{yx}(X-\overline{X}) \quad i.e. \quad Y-33=0{\cdot}72(X-33)$$

To find the value of Y when $X=38$, put $X=38$ in the above equation.

$$\therefore \quad Y-33=0{\cdot}72(38-43)=0{\cdot}72\times 5=3{\cdot}6$$

$$\therefore \quad Y=33+3{\cdot}76=36{\cdot}6=37 \text{ approximately.}$$

$\therefore$ The judge B would have given 37 marks to the eighth performannce.

Alternatively :

Calculations of regression

Sr. No.	x	x^2	y	xy
1	36	1296	35	1260
2	32	1024	33	1056
3	34	1156	31	1054
4	31	961	30	930
5	32	1024	34	1088
6	32	1024	32	1024
7	34	1156	36	1224
$N=7$	231	7641	231	7636

Let the marks given by A be x and those given by B be y.

Then the line of regression of y on x is $y=a+bx$.

And the normal equations are

$$\Sigma y = Na+b\Sigma x \qquad \therefore \quad 231=7a+231b \qquad \text{.............. (i)}$$

$$\Sigma xy = a\Sigma x+b\Sigma x^2 \qquad \therefore \quad 7636=231a+7641b \qquad \text{.............. (ii)}$$

Multiply the first by 33 and subtract it from the second.

$$\therefore \quad 7636 = 231a+7641b$$
$$7623 = 231a+7626b$$
$$\overline{\quad 13 = \qquad 18b \quad} \qquad \therefore \quad b=13/18=0{\cdot}72$$

Putting this value in (i), we get,

$$231=7a+231\,(0{\cdot}72) \qquad \therefore \quad 7a=231-231\,(0{\cdot}72)=64{\cdot}68$$

$$\therefore \quad a=9{\cdot}24$$

$\therefore$ The equation of the line of regression of y on x is

$$y=9{\cdot}24+0{\cdot}72x$$

To estimate y when $x=38$, we put $x=38$ in the above equation

$$\therefore \quad y=9{\cdot}24+0{\cdot}72\,(38)=36{\cdot}6=37 \text{ approximately.}$$

Example 2 : Find the equations of the lines of regression for the following data.

x :	5	6	7	8	9	10	11
y :	11	14	14	15	12	17	16

(M.U. 2003, 05, 16)

Sol. :

Calculations of regression

Sr. No.	x	x^2	y	y^2	xy
1	5	25	11	121	55
2	6	36	14	196	84
3	7	49	14	196	98
4	8	64	15	225	120
5	9	81	12	144	108
6	10	100	17	289	170
7	11	121	16	256	176
$N=7$	56	476	99	1427	811

Now, the line of regression of y on x is $y=a+bx$.

The normal equations are

$$\Sigma y = Na+b\Sigma x \qquad \therefore \quad 99=7a+56b \qquad \text{.............. (1)}$$

$$\Sigma xy = a\Sigma x+b\Sigma x^2 \qquad \therefore \quad 811=56a+476b \qquad \text{.............. (2)}$$

Multiply the first equation by 56 and the second by 7 and subtract

$$\therefore \quad 5544 = 392a+3136b$$
$$5677 = 392a+3332b$$
$$\overline{\quad -133 = \qquad -196b \quad} \qquad \therefore \quad b=\frac{133}{196}=0{\cdot}6789$$

Putting this value of b in (1), we get

$$99=7a+56\times\frac{133}{196} \qquad \therefore \quad 7a=99-38 \qquad \therefore \quad 7a=61 \qquad \therefore \quad a=8{\cdot}7143$$

$\therefore$ The equation of the line of regression of y on x is

$$y=8{\cdot}7143+0{\cdot}6786x$$

Now, the equation of the line of regression of x on y is $x=a+by$.

The normal equations are

$$\sum x = Na + b\sum y \qquad \therefore\ 56 = 7a + 99b \qquad \text{———— (3)}$$

$$\sum xy = a\sum x + b\sum y^2 \qquad \therefore\ 811 = 99a + 1427b \qquad \text{———— (4)}$$

Multiply the third equation by 99 and the fourth by 7 and subtract

$$\therefore\ 5544 = 693a + 9801b$$

$$5677 = 693a + 9989b$$

$$\overline{133 = \qquad 188b} \qquad \therefore\ b = \frac{133}{188} = 0{\cdot}7074$$

Putting this value of b in (1), we get

$$56 = 7a + 99 \times \frac{133}{188} \qquad \therefore\ 7a = 56 - 70{\cdot}0372 = -14{\cdot}0372 \qquad \therefore\ a = -2{\cdot}0053$$

$\therefore$ The equation of the line of regression of x on y is

$$x = -2{\cdot}0053 + 0{\cdot}7074y$$

(Further the coefficient of correlation is given by

$$r = \sqrt{b_1 b_2} = \sqrt{0{\cdot}6786 \times 0{\cdot}7074} = 0{\cdot}6928\text{.)}$$

Example 3 : From the following table showing age of cars of a certain make and annual maintenance costs, obtain the regression equation for costs related to age.

Age of Cars (years)	:	2, 4, 6, 7, 8, 10, 12.
Annual maintenance Cost (Rs.)	:	1,600, 1,500, 1,800, 1,900, 1,700, 2,100, 2,000.

Find the approximate cost of maintaining a 3 years old car of the same make.

Sol. :

Calculations of regression

Sr. No.	$X - \bar{X}$			$Y - \bar{Y}$			Product
	X	x	x^2	Y	y	y^2	xy
1	2	−5	25	1,600	−200	40,000	1000
2	4	−3	9	1,500	−300	90,000	900
3	6	−1	1	1,800	0	0	0
4	7	0	0	1,900	100	10,000	0
5	8	1	1	1,700	−100	10,000	−100
6	10	3	9	2,100	300	90,000	900
7	12	5	25	2,000	200	40,000	1000
$N = 7$	$\Sigma X = 49$		$\Sigma x^2 = 70$	$\Sigma Y =$ 12,600		$\Sigma y^2 =$ 2,80,000	$\Sigma xy = 3{,}700$

Let X denote age in years, Y denote cost in Rs.

Now $\bar{X} = \dfrac{\Sigma X}{N} = \dfrac{49}{7} = 7,\quad \bar{Y} = \dfrac{12600}{7} = 1{,}800$

and $b_{yx} = \dfrac{\Sigma xy}{\Sigma x^2} = \dfrac{3700}{40} = 52{\cdot}86$

The equation of the line of regression of Y on X is,

$$Y - \bar{Y} = b_{yx}(X - \bar{X}) \qquad \therefore\ Y - 1800 = 52{\cdot}86\,(X - 7)$$

To find the value of Y when $X = 3$ put this value in the above equation

$$\therefore\ Y - 1800 = 52{\cdot}86\,(3 - 7) = -211{\cdot}44$$

$$\therefore\ Y = 1800 - 212.64 = 1588{\cdot}56$$

$\therefore$ The cost of maintenance of 3 years old car = ₹ 1588.56

Example 4 : Find the coefficients of regression and hence the equations of the lines of regression for the following data.

X : 78, 36, 98, 25, 75, 82, 90, 62, 65, 39

Y : 84, 51, 91, 60, 68, 62, 86, 58, 53, 47

Draw the lines of regression from your equations on the graph. Estimate the value of Y when $X = 50$ and the value of X when $Y = 90$ from the graph.

What is the significance of the point of intersection of the two lines?

Sol. :

Calculations of coefficients of regression

Sr. No.	$X - \bar{X}$			$Y - \bar{Y}$			Product
	X	x	x^2	Y	y	y^2	xy
1	78	+13	169	84	+18	324	234
2	36	−29	841	51	−15	225	435
3	98	+33	1089	91	+25	625	825
4	25	−40	1600	60	−6	36	240
5	75	+10	100	68	+2	4	20
6	82	+17	289	62	−4	16	−68
7	90	+25	625	86	+20	400	500
8	62	−3	9	58	−8	64	24
9	65	0	0	53	−13	169	0
10	39	−26	676	47	−19	361	494
$N = 10$	$\Sigma X = 650$		$\Sigma x^2 = 5398$	$\Sigma Y = 660$		$\Sigma y^2 = 2224$	$\Sigma xy = 2704$

Now $\bar{X} = \dfrac{\Sigma X}{N} = \dfrac{650}{10} = 65,\quad \bar{Y} = \dfrac{\Sigma Y}{N} = \dfrac{660}{10} = 66.$

Coefficient of regression of Y on X is,

$$b_{yx} = \frac{\Sigma xy}{\Sigma x^2} = \frac{2704}{5398} = 0{\cdot}5009$$

Coefficient of regression of X on Y is,

$$b_{xy} = \frac{\Sigma xy}{\Sigma y^2} = \frac{2704}{2224} = 1{\cdot}215$$

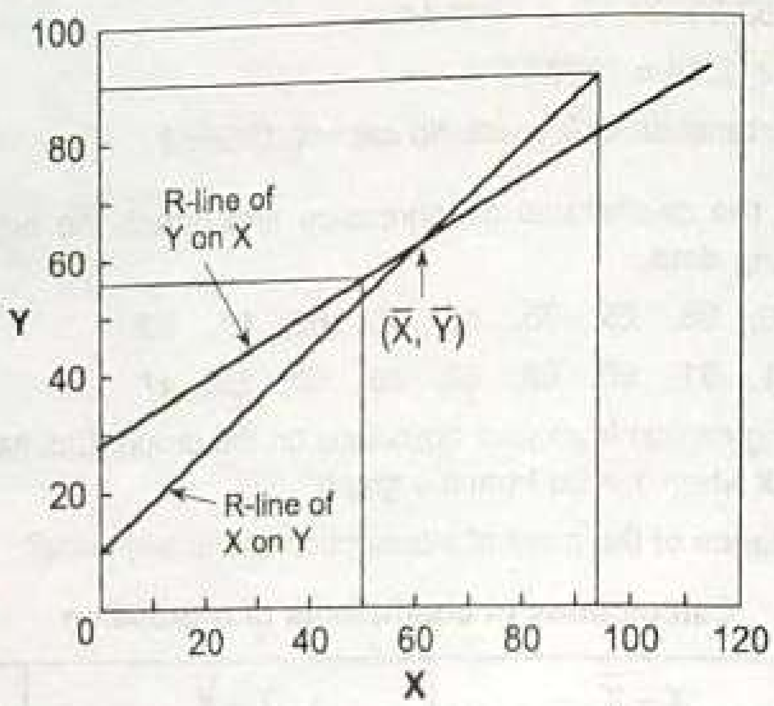
The equation of the line of regression of Y on X is,

$$Y - \bar{Y} = b_{yx}(X - \bar{X}) \qquad \therefore\ Y - 66 = 0{\cdot}5(X - 65)$$

The equation of the line of regression of X on Y is,

$$X - \bar{X} = b_{xy}(Y - \bar{Y}) \qquad \therefore\ X - 65 = 1{\cdot}2(Y - 66)$$

(ii) **Graph of the two lines of regression**



To draw the lines of regression we take two points on each and connect them by a straight line.

From $Y - 66 = 0{\cdot}5\,(X - 65)$

we find that when $X = 65,\ \ Y = 66$

and when $X = 0,\ \ Y = 27{\cdot}5$

From $X - 65 = 1{\cdot}2\,(Y - 66)$

we find that when $X = 65,\ \ Y = 66$

and when $X = 0,\ \ Y = 12.$

To estimate Y when $X = 50$ we draw a line at $X = 50$, parallel to y-axis to meet the line of regression of Y on X and read the corresponding value of Y. It is 58. To estimate X when $Y = 90$, we draw a line at $Y = 90$, parallel to x-axis, to meet the line of regression of X on Y and read the corresponding value of X. It is 94.

The two lines intersect at the point for which $X = \overline{X} = 65$ and $Y = \overline{Y} = 66$.

Example 5 : A chemical engineer is investigating the effect of process operating temperature X on product yield Y. The study results in the following data.

X : 100, 110, 120, 130, 140, 150, 160, 170, 180, 190.

Y : 45, 51, 54, 61, 66, 70, 74, 78, 85, 89.

Find the equation of the least square line which will enable to predict yield on the basis of temperature. Find also the degree of relationship between the temperature and the yield.

(M.U. 2004, 16)

Also verify that the sum of the coefficients of regression is greater than $2r$.

Calculations of b_{xy}, b_{yx} etc.

Sr. No.	dx: X	$X - 150$	d_x^2	dy: Y	$X - 70$	d_y^2	$d_x d_y$
1	100	− 50	2500	45	− 25	625	1250
2	110	− 40	1600	51	− 19	361	760
3	120	− 30	900	54	− 16	256	480
4	130	− 20	400	61	− 9	81	180
5	140	− 10	100	66	− 4	16	40
6	150	00	000	70	0	0	00
7	160	10	100	74	4	16	40
8	170	20	400	78	8	64	160
9	180	30	900	85	15	225	450
10	190	40	1600	89	19	361	760
$N = 10$		− 50	8500		− 27	2005	4120

$$\overline{X} = A + \Sigma\frac{dx}{N} = 150 - \frac{50}{10} = 145; \qquad \overline{Y} = B + \Sigma\frac{dy}{N} = 70 - \frac{27}{10} = 67{\cdot}3$$

$$b_{yx} = \frac{\Sigma d_x d_y - \dfrac{\Sigma d_x \Sigma d_y}{N}}{\Sigma d_x^{\,2} - \dfrac{(\Sigma d_x)^2}{N}} = \frac{4120 - \dfrac{(-50)(-27)}{10}}{8500 - \dfrac{(-50)^2}{10}} = \frac{4120 - 135}{8500 - 250} = \frac{3985}{8250} = 0{\cdot}483$$

$$b_{xy} = \frac{\Sigma d_x d_y - \dfrac{\Sigma d_x \Sigma d_y}{N}}{\Sigma d_y^{\,2} - \dfrac{(\Sigma d_y)^2}{N}} = \frac{4120 - \dfrac{(-50)(-27)}{10}}{2005 - \dfrac{(-27)^2}{10}} = \frac{4120 - 135}{2005 - 72{\cdot}9} = \frac{3985}{1932{\cdot}1} = 2{\cdot}06$$

The line of regression of Y on X is

$$Y - \overline{Y} = b_{yx}\,(X - \overline{X})$$

$$\therefore\ Y - 67{\cdot}3 = 0{\cdot}483\,(X - 145) \qquad \therefore\ Y = 0{\cdot}483X - 2{\cdot}735$$

The coefficient of correlation

$$r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{0{\cdot}483 \times 2{\cdot}06} = 0{\cdot}9975$$

Now, $b_{xy} + b_{yx} = 2{\cdot}060 + 0{\cdot}483 = 2{\cdot}543$ and $2r = 2 \times 0{\cdot}9975 = 1{\cdot}995$

Hence, we see that $b_{yx} + b_{xy} > 2r$.

Miscellaneous Examples

Example 1 : Find the angle between the lines of regression using the following data. **(M.U. 1998)**

$$n = 10,\ \Sigma x = 270,\ \Sigma y = 630,\ \sigma_x = 4,\ \sigma_y = 5,\ r_{xy} = 0{\cdot}6.$$

Sol.: The angle between the lines of regression is given by

$$\tan\theta = \left(\frac{1 - r^2}{r}\right)\left(\frac{\sigma_x \cdot \sigma_y}{\sigma_x^{\,2} + \sigma_y^{\,2}}\right)$$

Putting the given values

$$\tan\theta = \left(\frac{1-0.6^2}{0.6}\right)\left(\frac{4\times 5}{16+25}\right) = 0.52$$

Example 2 : Discuss the statement "The sum of the two coefficients of regression is always greater than $2r$ where r is the coefficient of correlation". **(M.U. 1998)**

Sol. : We have proved above (§ 7, 3, page 9-9) that

$$b_{yx} + b_{xy} \geq 2r.$$

Because this statement leads us to

$$(\sigma_x - \sigma_y)^2 \geq 0 \quad \text{which is always true.} \qquad \text{.......................} \text{(A)}$$

This means the above given statement is partially true. The sum of the two coefficients of regression is greater than $2r$ but not always. The sum can be also equal to $2r$.

The condition (A) shows that if $\sigma_x = \sigma_y$, then the sign of equality will hold and then $b_{yx} + b_{xy}$ will be equal to $2r$.

This is also clear otherwise. From (1) and (1') (page 9-8), we have

$$b_{yx} + b_{xy} = r\frac{\sigma_y}{\sigma_x} + r\frac{\sigma_x}{\sigma_y}$$

If $\sigma_x = \sigma_y$, we will get $b_{yx} + b_{xy} = 2r$.

Example 3 : Given the following results of weights X and heights Y of 1000 men

$\bar{X} = 150$ lbs. $\quad \sigma_x = 20$ lbs.

$\bar{Y} = 68$ inches, $\quad \sigma_y = 2{\cdot}5$ inches, $\quad r = 0{\cdot}6$.

where $\bar{X}$ and $\bar{Y}$ are means of X and Y, σ_x and σ_y are standard deviations of X and Y and r is the correlation coefficient between X and Y.

John weighs 200 lbs., Smith is five feet tall. Estimate the height of John and weight of Smith.

From the value of height of John estimate his weight. Why is it different from 200 ?

Sol. : With the given notation the line of regression of Y on X is

$$Y - \bar{Y} = r\frac{\sigma_y}{\sigma_x}(X - \bar{X})$$

Substituting the given values,

$$Y - 68 = 0{\cdot}6 \times \frac{2{\cdot}5}{20}(X - 150) = \frac{15}{200}(X - 150).$$

Put $X = 200$,

$$\therefore \quad Y - 68 = \frac{15}{200}(200 - 150) = \frac{15}{4} = 3{\cdot}75$$

$$\therefore \quad Y = 68 + 3{\cdot}75 = 71{\cdot}75 \text{ inches.}$$

Now the line of regression of X of Y is

$$X - \bar{X} = r\frac{\sigma_x}{\sigma_y}(Y - \bar{Y})$$

Substituting the given values,

$$X - 150 = 0{\cdot}6 \times \frac{20}{2{\cdot}5}(Y - 68) = \frac{24}{5}(Y - 68)$$

Put $Y = 5$ feet $= 60$ inches.

$$\therefore \quad X - 150 = \frac{24}{5}(60 - 68) = -\frac{192}{5}$$

$$\therefore \quad X = 150 - \frac{192}{5} = 111{\cdot}6 \text{ lbs.}$$

Hence, height of John $= 71{\cdot}75$ inches and weight of Smith $= 111{\cdot}6$ lbs.

To estimate the weight of John from his height 71·25 we have to use the equation of line of regression of X on Y (and not Y on X).

i.e., $\quad X - 150 = \dfrac{24}{5}(Y - 68)$

Putting $Y = 71{\cdot}75$, we get

$$X - 150 = \frac{24}{5}(3.75) = 18 \quad \therefore \; X = 168.$$

The difference is due to the fact that for estimating Y we use one equation and for estimating X we use another equation.

Example 4 : Given $6Y = 5X + 90$, $15X = 8Y + 130$, $\sigma_x^2 = 16$.

Find (i) $\bar{X}$ and $\bar{Y}$, (ii) r and (iii) σ_y^2. **(M.U. 2009, 10)**

Sol. : (i) To find $\bar{X}$ and $\bar{Y}$: We solve the given equations simultaneously. Multiply the first equation by 3.

$$\therefore \quad -15X + 18Y = 270 \text{ and add } 15X - 8Y = 130$$

$$\therefore \quad 10Y = 40 \qquad \therefore \; \bar{Y} = 40$$

Putting this value in any of the given equations.

$$6 \times 40 = 5X + 90 \quad \therefore \; X = 30 \quad \therefore \; \bar{X} = 30.$$

(ii) To find r : Suppose the first equation represents, the line of regression of X on Y.

Writing it as $X = \dfrac{6}{5}Y - 18$, we find $b_{xy} = \dfrac{6}{5}$.

Suppose the second equation represents the line of regression of Y on X.

Writing it as $Y = \dfrac{15}{8}X - \dfrac{130}{8}$, we find $b_{yx} = \dfrac{15}{8}$.

$$\therefore \quad r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{\frac{6}{5} \times \frac{15}{8}} = \sqrt{\frac{9}{4}} = \sqrt{2{\cdot}25} = 1{\cdot}5.$$

But the value of r can never be greater than 1 numerically. Hence, our supposition is wrong.

Now treating the first equation as representing the line of regression of Y on X, we write it as,

$$Y = \frac{5}{6}X + 15 \quad \therefore \; b_{yx} = \frac{5}{6}.$$

Treating the second equation as representing the line of regression of X on Y, we write it as,

$$X = \frac{8}{15}Y + \frac{130}{15} \quad \therefore \; b_{xy} = \frac{8}{15}$$

$\therefore\ r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{\dfrac{8}{15} \times \dfrac{5}{6}} = \sqrt{\dfrac{4}{9}} = \dfrac{2}{3} = 0{\cdot}667.$

(iii) To find σ_y. Consider, $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$ $\therefore\ \dfrac{5}{6} = \dfrac{2}{3} \times \dfrac{\sigma_y}{4}$ $\therefore\ \sigma_y = 5.$

Example 5 : The equations of the two regression lines are $3x + 2y = 26$ and $6x + y = 31$. Find : (i) the means of x and y, (ii) coefficient of correlation between x and y, (iii) σ_y if $\sigma_x = 3$. **(M.U. 2007, 16)**

Sol. : (i) To find $\bar{X}$ and $\bar{Y}$

We solve the equations simultaneously. Multiply the second by 2 and subtract from the first.

$$\begin{aligned} \therefore\quad 3x + 2y &= 26 \\ 12x + 2y &= 62 \\ \hline 9x \quad &= 36 \end{aligned} \qquad \therefore\ x = 4.$$

Putting this value of x in the second equation, we get $24 + y = 31$ $\therefore\ y = 7$.

$\therefore\ \bar{X} = 4,\ \bar{Y} = 7.$

(ii) To find r : Suppose the first equation represents the line of regression of X on Y.

Writing it as $3x = -2y + 26$. $\therefore\ x = -\dfrac{2}{3}y + \dfrac{26}{3}$

$\therefore$ We find that $b_{xy} = -\dfrac{2}{3}$.

Suppose the second equation represents the line of regression of Y on X.

Writing it as $y = -6x + 31$ $\therefore\ b_{yx} = -6.$

$\therefore\ r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{(-2/3)(-6)} = \sqrt{4} = 2$

But the value of r can never be greater than 1. Hence, our supposition is wrong.

Now, treating the first equation as representing the line of regression on of Y on X, we write it as

$$2y = -3x + 26 \quad \therefore\ y = -\frac{3}{2}x + 13 \quad \therefore\ b_{yx} = -\frac{3}{2}.$$

Treating the second equation as representing the line of regression of X on Y, we write it as

$$6x = -y + 31 \quad \therefore\ x = -\frac{1}{6}y + \frac{31}{6} \quad \therefore\ b_{xy} = -\frac{1}{6}.$$

$$\therefore\ r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{\left(-\frac{3}{2}\right)\left(-\frac{1}{6}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2} = 0{\cdot}5$$

Since, both b_{xy} and b_{yx} are negative r is negative. $\therefore\ r = -0{\cdot}5.$

(iii) To find σ_y : Consider $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$.

But $b_{yx} = -\dfrac{3}{2},\ r = -\dfrac{1}{2}$, and $\sigma_x = 3.$ $\therefore\ \sigma_y = b_{yx} \cdot \dfrac{\sigma_x}{r} = \left(-\dfrac{3}{2}\right) \cdot \dfrac{3}{(-1/2)} = 9$

Example 6 : The regression lines of a sample are $x + 6y = 6$, and $3x + 2y = 10$. Find (i) sample means $\bar{x}$ and $\bar{y}$, (ii) coefficient of correlation between x and y. Also estimate y when $x = 12$. **(M.U. 2004, 14, 15)**

Also verify that the sum of the coefficients of regressions is greater than $2r$.

Sol. : (i) Mean $\bar{x}$ and $\bar{y}$ are obtained by solving the two given equations.

$$\begin{aligned} 3x + 18y &= 18 \qquad \therefore\ y = 1/2 \\ 3x + 2y &= 10 \qquad \therefore\ x = 3 \\ \hline 16y &= 8 \end{aligned}$$

(ii) If the line $x + 6y = 6$ is the line of regression of y on x, then

$$6y = -x + 6 \ \text{ i.e. }\ y = -\frac{1}{6}x + 1 \quad \therefore\ b_{yx} = -\frac{1}{6}$$

If the line $3x + 2y = 10$ is the line of regression of x on y, then

$$3x = -2y + 10 \ \text{ i.e. }\ x = -\frac{2}{3}y + \frac{10}{3} \quad \therefore\ b_{xy} = -\frac{2}{3}$$

$$\therefore\ r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{\left(-\frac{1}{6}\right) \times \left(-\frac{2}{3}\right)} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Since b_{yx} and b_{xy} are both negative, r is negative $\therefore\ r = -1/3$.

Since $b_{yx} + b_{xy} = \dfrac{1}{6} + \dfrac{2}{3} = \dfrac{5}{6}$ (Numerically)

and $2r = \dfrac{2}{3}$, we see that $b_{yx} + b_{xy} > 2r$.

(iii) To estimate y when $x = 12$, we use the line of regression of y on x i.e. $y = -\dfrac{1}{6}x + 1$, when $x = 12$, $y = -2 + 1 = -1$.

Example 7 : If the tangent of the angle made by the line of regression of y on x is 0.6 and $\sigma_y = 2\,\sigma_x$, find the correlation coefficient between x and y. **(M.U. 2004, 09, 10, 15)**

Sol. : If the equation of the line of regression of y on x is $y - \bar{y} = b_{yx}(x - \bar{x})$ then we know that b_{yx} is the slope of the line of regression. We are thus, given $b_{yx} = 0{\cdot}6$.

But $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$ and $\sigma_y = 2\sigma_x$

Putting these value, $0{\cdot}6 = r \cdot \dfrac{2\sigma_x}{\sigma_x} = 2r$ $\therefore\ r = \dfrac{0{\cdot}6}{2} = 0{\cdot}3$

Example 8 : If $\sigma_x = \sigma_y = \sigma$ and the angle between the lines of regression is $\tan^{-1} 3$, find the coefficient of correlation. **(M.U. 2004)**

Sol. : We have

$$\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^{\,2} + \sigma_y^{\,2}}\right) \qquad \therefore\ 3 = \frac{1-r^2}{r} \cdot \left(\frac{\sigma^2}{\sigma^2 + \sigma^2}\right) = \frac{1-r^2}{2r}$$

$$\therefore\ \frac{1-r^2}{r} = 6 \quad \therefore\ r^2 + 6r - 1 = 0 \quad \therefore\ r = \frac{-6 \pm \sqrt{36-4}}{2} = -3 \pm 2\sqrt{2}$$

Since r cannot be numerically greater than 1, $r = -3 + 2\sqrt{2} = -0{\cdot}17$.

Example 9 : The following data regarding the heights (y) and weights (x) of 100 college students are given

$$\Sigma x = 15000,\ \Sigma x^2 = 2272500,\ \Sigma y = 6800,\ \Sigma y^2 = 463025,\ \Sigma xy = 1022250.$$

Find the coefficient of correlation between height and weight and also the equation of regression of height and weight. **(M.U. 1996)**

Sol. : The coefficients of regression are given by

$$b_{yx} = \frac{\Sigma xy - \dfrac{\Sigma x \cdot \Sigma y}{N}}{\Sigma x^2 - \dfrac{(\Sigma x)^2}{N}} = \frac{1022250 - \dfrac{15000 \times 6800}{100}}{2272500 - \dfrac{15000^2}{100}} = \frac{2250}{22500} = 0{\cdot}1.$$

$$b_{xy} = \frac{\Sigma xy - \dfrac{\Sigma x \cdot \Sigma y}{N}}{\Sigma y^2 - \dfrac{(\Sigma y)^2}{N}} = \frac{1022250 - \dfrac{15000 \times 6800}{100}}{463025 - \dfrac{6800^2}{100}} = \frac{2250}{625} = 3{\cdot}6.$$

$$\therefore \quad r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{0{\cdot}1 \times 3{\cdot}6} = 0{\cdot}6.$$

The equation of the lines of regression of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x}) \quad \therefore \quad y - 68 = 0{\cdot}1(x - 1500) \quad \therefore \quad y = 0{\cdot}1x - 82.$$

Example 10 : It is given that the means of x and y are 5 and 10. If the line of regression of y on x is parallel to the line $20y = 9x + 40$, estimate the value of y for $x = 30$. **(M.U. 1998, 2015)**

Sol. : The line of regression of y on x is $y - \bar{y} = b_{yx}(x - \bar{x})$.

Its slope is b_{yx}. But this line is parallel to $20y = 9x + 40$

i.e. $\quad y = \dfrac{9}{20}x + 2$ whose slope is $\dfrac{9}{20}$. $\quad \therefore \quad b_{yx} = \dfrac{9}{20}$.

But by data $\bar{x} = 5$ and $\bar{y} = 10$. Hence, the equation of the line of regression of y on x is

$$y - 10 = \frac{9}{20}(x - 5) \quad i.e. \quad y = \frac{9}{20}x + \frac{155}{20}$$

When $x = 30$, $\quad y = \dfrac{270}{20} + \dfrac{155}{20} = \dfrac{425}{20} = 21{\cdot}25.$

EXERCISE - I

Type I

State true or false with proper reasoning.

1. If $r = 0$, the lines of regression are parallel to each other. **[Ans. :** False **]**
2. The values of r and R can never be equal. **[Ans. :** False **]**
3. In a regression analysis it was found that $b_{yx} = 0{\cdot}87$, $b_{xy} = 1{\cdot}55$. These values are not consistent. **[Ans. :** True **]**
4. The two regression coefficients are both positive or both negative. **[Ans. :** True **]**
5. $3x + y = 5$ and $2x - 3y = 7$ cannot be lines of regression for any set of values of x and y. **[Ans. :** True **]**

Type II

1. The following table gives the age of car of a certain make and annual maintenance cost. Obtain the equation of the line of regression of cost on age.

Age of a car	:	2	4	6	8
Maintenance	:	1	2	2·5	3

(M.U. 1998, 2014) **[Ans. :** $y = 0{\cdot}325x + 0{\cdot}5$ **]**

2. Obtain the equation of the line of regression of y on x from the following data and estimate y for $x = 73$.

x	:	70,	72,	74,	76,	78,	80.
y	:	163,	170,	179,	188,	196,	220.

(M.U. 1997)
[Ans. : $y = 5{\cdot}31x - 212{\cdot}57$; $y = 175{\cdot}37$ **]**

3. The heights in cms of fathers (x) and of the eldest sons (y) are given below.

x	:	165	160	170	163	173	158	178	168	173	170	175	180
y	:	173	168	173	165	175	168	173	165	180	170	173	178

Estimate the height of the eldest son if the height of the father is 172 cms. and the height of the father if the height of the eldest son is 173 cm.

Also find the coefficient of correlation between the heights of fathers and sons. **(M.U. 2002, 05)**

[Ans. : (i) $y = 1{\cdot}016x - 5{\cdot}123$, **(ii)** $x = 0{\cdot}476y + 98{\cdot}98$, **(iii)** 169·97, 173·45, **(iv)** $r = 0{\cdot}696$ **]**

4. Find (i) the lines of regression, (ii) coefficient of correlation for the following data.

x	:	65	66	67	67	68	69	70	72
y	:	67	68	65	66	72	72	69	71

(M.U. 2002, 14)
[Ans. : $y = 19{\cdot}79 + 0{\cdot}72x$, **(ii)** $x = 33{\cdot}29 + 0{\cdot}5y$; $r = 0{\cdot}6$ **]**

5. Find the line of regression for the following data and estimate y corresponding to $x = 15{\cdot}5$.

x	:	10	12	13	16	17	20	25
y	:	19	22	24	27	29	33	37

(M.U. 2004) [Ans. : $y = 0{\cdot}8x + 13{\cdot}23$; 25·63 **]**

Type III

1. Given

	x series	y series
Mean	18	100
S.D.	14	20

$r = 0{\cdot}8$.

Find the most probable value of y when $x = 70$ and most probable value of x when $y = 90$.
[Ans. : $y = 159{\cdot}3$, $x = 12{\cdot}4$ **]**

2. Given the following information about marks of 60 students.

	Mathematics	English
Mean	80	50
S.D.	15	10

Coefficient of correlation $r = 0{\cdot}4$. Estimate the marks of the student in mathematics who scored 60 marks in English. **(M.U. 2006) [Ans. :** 86 **]**

3. You are supplied with the following information. The equation of the lines regression are $4x + 3y + 8 = 0$ and $x + 2y - 5 = 0$.
Find the means of x and y and the coefficient of correlation between them. **(M.U. 1997)**
[Ans. : $\bar{x} = -31$, $\bar{y} = 18$, $r = -0{\cdot}87$ **]**

4. From 8 observations the following results were obtained :
$\Sigma x = 59$, $\Sigma y = 40$, $\Sigma x^2 = 524$, $\Sigma y^2 = 256$, $\Sigma xy = 364$.
Find the equation of the line of regression of x on y and the coefficient of correlation.
[Ans. : $x = 1{\cdot}5y - 0{\cdot}5$, $r = 0{\cdot}98$ **]**

5. The equations of the two lines of regression are $x = 19{\cdot}13 - 0{\cdot}87y$ and $y = 11{\cdot}64 - 0{\cdot}50x$. Find (i) the means of x and y, (ii) the coefficient of correlation between x and y.

(M.U. 2004) [**Ans. : (i)** $\bar{x} = 15{\cdot}79$, $\bar{y} = 3{\cdot}74$; **(ii)** $r = -0{\cdot}66$, $b_{yx} = -0{\cdot}50$, $b_{xy} = -0{\cdot}87$]

6. Out of the two equations given below which can be a line of regression of x on y and why ?

$x + 2y - 6 = 0$ and $2x + 3y - 8 = 0$. **(M.U. 2003)** [**Ans. :** $2x + 3y - 8 = 0$]

7. In a partially destroyed laboratory record of analysis of correlation data the following results are legible. Variance of $x = 9$, equations of the lines of regression

$$4x - 5y + 33 = 0,\ 20x - 9y - 107 = 0.$$

Find (i) the mean values of x and y, (ii) the standard deviation of y, (iii) coefficient of correlation between x and y. **(M.U. 1999, 2003)** [**Ans. : (i)** $\bar{x} = 13$, $\bar{y} = 17$, **(ii)** $\sigma_y = 4$, **(iii)** $r = 0{\cdot}6$]

8. Given : var $(x) = 25$. The equations of the two lines of regression are $5x - y = 22$ and $64x - 45y = 24$.

Find (i) $\bar{x}$ and $\bar{y}$, (ii) r, (iii) σ_y. **(M.U. 1998)**

[**Ans. : (i)** $\bar{x} = 6$, $\bar{y} = 8$, **(ii)** $r = 1{\cdot}87$, **(iii)** $\sigma_y = 1/5$]

9. Find the regression coefficients and the coefficient of correlation from the following data where x, y denote the actual values.

$$N = 12,\ \Sigma x = 120,\ \Sigma y = 432,\ \Sigma xy = 4992,\ \Sigma x^2 = 1392,\ \Sigma y^2 = 18252.$$

[**Ans. :** $b_{yx} = 3{\cdot}5$, $b_{xy} = 0{\cdot}249$, $r = 0{\cdot}93$]

EXERCISE - II

Theory

1. Distinguish between correlation and regression.
2. Explain "the line of regression". Why there are two lines of regression ? **(M.U. 2007)**
3. Explain what you understand by regression. What are lines of regression ? Why are there in general two lines of regression ? When do they coincide, when are they perpendicular ? **(M.U. 2004)**
4. Explain the method of scatter diagram to obtain a line of regression.
5. Obtain the equations of lines of regression. **(M.U. 2002, 03)**
6. Prove that the sum of the coefficients of regression is greater than or equal to $2r$ where r is the coefficient of correlation. [See 3, page 9-9] **(M.U. 1998)**
7. Find the expression for the acute angle between the lines of regression. **(M.U. 2004, 05)**
8. With usual notation prove that **(i)** $r = \sqrt{b_{yx} \cdot b_{xy}}$, **(ii)** $b_{xy} + b_{yx} \geq 2r$.
9. Examine whether the following statement is correct

$b_{xy} = 3{\cdot}2$, $b_{yx} = 0{\cdot}7$. [**Ans. : No**]

10. If θ is the angle between the two lines of regression, prove that

$$\tan\theta = \left(\frac{1 - r^2}{r}\right)\left(\frac{\sigma_x - \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$ **(M.U. 1996, 2004, 07)**

❀ ❀ ❀

CHAPTER 10

Curve Fitting

1. Introduction

In many social, economical, engineering and physical problems we have a set of values of x and y although we do not know the functional relationship between them. Fitting of a curve to a given set of values means finding a functional relationship between x and y whose curve is the closest possible curve to the given values. The curve so obtained does not pass through all the given points but is close to them to the maximum extent. Finding such a curve for a given set of values is called **Curve Fitting**. The relation in general is assumed to be a linear function $y = a + bx$ or a parabolic function $y = a + bx + cx^2$ or even exponential or logarithmic. The method used for fitting the curve is based on the **principle of least squares**.

2. Fitting a Straight Line by the Method of Least Squares

Suppose we have a set of values (x_i, y_i). Suppose further that we want to fit a straight line $y = a + bx$ to these values. The straight line must be close to the given points to the maximum extent. **The principle of least square states that the straight line should be such that the distances of the given points from the straight line measured along the *y*-axis must be minimum.** The line obtained in this way is called **the line of best fit**.

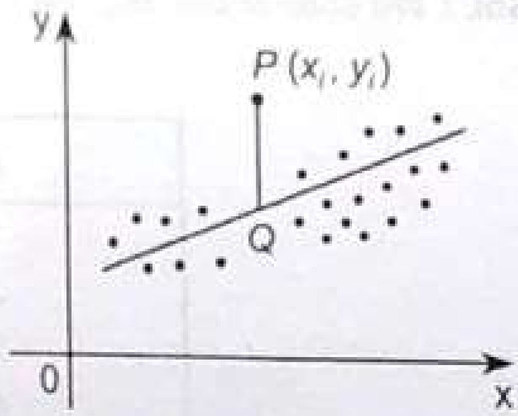


Fig. 10.1

Suppose $P(x_i, y_i)$ is a given point and suppose the equation of the line of best fit be $y = a + bx$. Suppose further that the line parallel to the y-axis through P intersects the line in Q. Now, the coordinates of Q are $(x_i, a + bx_i)$. We want to find a, b such that the distance $|PQ|$ is minimum. But the distance $|PQ|$ is minimum when the square of the distance $(y_i - a - bx_i)^2$ is minimum. This must be true for all points. This means we should have $S = \sum (y_i - a - bx_i)^2$ minimum. Since we find a, b such that the sum of the squares S is least, the method is known as the method of least squares.

Now, for S to be minimum the conditions are

$$\frac{\partial S}{\partial a} = 0 \quad \text{and} \quad \frac{\partial s}{\partial b} = 0$$

$$\therefore \quad \sum (y_i - a - bx_i) = 0 \quad \text{and} \quad \sum (y_i - a - bx_i)\, x_i = 0 \qquad \therefore \quad \sum y_i = \sum a - b \sum x_i$$

$$\therefore \quad \sum y_i = Na + b \sum x_i \quad \text{and} \quad \sum y_i x_i = a \sum x_i + b \sum x_i^2$$

We shall drop the suffix i and write the equations as

$$\boxed{\begin{aligned} \sum y &= Na + b \sum x \\ \sum xy &= a \sum x + b \sum x^2 \end{aligned}}$$

These equations are known as **Normal Equations**.

Methods of finding the constants a

1. Direct Method : If we find $\sum x$, $\sum y$, etc. and substitute their values in the above equations, we get two equations in two unknowns a and b. Solving these two equations we can find the constants a and b.

2. Assumed Mean Method : We may assume a certain mean of x and of y, *i.e.*, we may put $X = x - m$ and $Y = y - n$ and find $\sum X$, $\sum Y$, etc. put these values in the above two equations.

If necessary, we may divide the deviations by a common factor, if there is any, *i.e.*, we may put $X = \dfrac{x-m}{h}$ and $Y = \dfrac{y-n}{k}$.

Solving these equations, we get a and b and resubstituting the values of X and Y, we get the equation of the line.

3. Actual Mean Method : If actual means of x and y are round figures, we put $X = x - \bar{x}$ and $Y = y - \bar{y}$ $\left(\text{or } X = \dfrac{x-\bar{x}}{h},\ Y = \dfrac{y-\bar{y}}{k}\right)$, find $\sum X$, $\sum Y$, etc. and put these values in the above two equations.

Solving these equations as above and resubstituting the values of X and Y, we get the equation of the line.

Of these three methods we shall see below that the last method is the simplest one.

Example 1 : Fit a straight line to the following data :

(x, y) : (1, 1), (2, 5), (3, 11), (4, 8), (5, 14)

Sol. : We shall obtain the line from the above normal equations.

Calculations of $\sum x$, $\sum x^2$ etc.

x	y	x^2	xy
1	1	1	1
2	5	4	10
3	11	9	33
4	8	16	32
5	14	25	70
$N = 5$, 15	39	55	146

The normal equations are

$\sum y = Na + b\sum x$ $\quad\therefore\ 39 = 5a + 15b$ .... (1)

and $\sum xy = a\sum x + b\sum x^2$ $\quad\therefore\ 146 = 15a + 55b$ .... (2)

Multiply (1) by 3,

$\therefore\ 117 = 15a + 45b$ and $146 = 15a + 55b$

By subtraction $10b = 29$ $\quad\therefore\ b = 2{\cdot}9$.

Putting this value of b in (1),

$5a = 39 - 15 \times 2{\cdot}9 = -4{\cdot}5$

$\therefore\ a = -\dfrac{4{\cdot}5}{5} = -0{\cdot}9$.

Hence, the equation of the line is $y = -0{\cdot}9 + 2{\cdot}9x$.

Example 2 : Fit a straight line $y = a + bx$ to the following data.

x	:	0	5	10	15	20	25
y	:	12	15	17	22	24	30

Sol. : (1) Direct Method

Calculations of $\sum x$, $\sum x^2$ etc.

x	y	x^2	xy
0	12	0	0
5	15	25	75
10	17	100	170
15	22	225	330
20	24	400	480
25	30	625	750
$N = 6$, $\sum x = 75$	$\sum y = 120$	$\sum x^2 = 1375$	$\sum xy = 1805$

Let the equation be $y = a + bx$.

The normal equations are

$\sum y = Na + b\sum x$ $\quad\therefore\ 120 = 6a + 75b$ .... (1)

and $\sum xy = a\sum x + b\sum x^2$ $\quad\therefore\ 1805 = 75a + 1375b$ .... (2)

Now, to find a, b multiply (1) by 50 and (2) by 4 and subtract.

$$6000 = 300a + 3750b$$
$$7220 = 300a + 5500b$$
$$1220 = 1750b$$

$\therefore\ b = \dfrac{1220}{1750} = 0{\cdot}6971 = 0{\cdot}7$

Putting this value of b in (1), we get

$120 = 6a + 75(0{\cdot}7)$ $\quad\therefore\ 6a = 120 - 52{\cdot}5 = 67{\cdot}5$ $\quad\therefore\ a = 11{\cdot}25$.

$\therefore$ The equation of the line is $y = 11{\cdot}25 + 0{\cdot}7x$.

(2) Assumed Mean Method : We shall assume 10 and 17 as means of x and y, *i.e.*, we shall put $X = x - 10$ and $Y = y - 17$.

Calculations of $\sum X$, $\sum X^2$ etc.

x	y	$X = x - 10$	$Y = y - 17$	X^2	XY
0	12	-10	-5	100	50
5	15	-5	-2	25	10
10	17	0	0	0	0
15	22	5	5	25	25
20	24	10	7	100	70
25	30	15	13	225	195
$N = 6$		15	18	475	350

Let the equation be $Y = a + bX$.

The normal equations are

$\sum Y = Na + b\sum X$ $\quad\therefore\ 18 = 6a + 15b$ .... (1)

and $\sum XY = a\sum X + b\sum X^2$ $\quad\therefore\ 350 = 15a + 475$ .... (2)

Multiply (1) by 5 and (2) by 2 and subtract.

$$\therefore\ 90 = 30a + 75b$$
$$700 = 30a + 950b$$
$$\overline{610 = 875b}$$

$\therefore\ b = \dfrac{610}{875} = 0{\cdot}7$

Putting this value of b in (1), we get

$18 = 6a + 15(0{\cdot}7)$ $\quad\therefore\ 6a = 18 - 10{\cdot}5$ $\quad\therefore\ a = 1{\cdot}25.$

$\therefore$ The equation is $Y = 1{\cdot}25 + 0{\cdot}7X$.

Resubstituting $Y = y - 17$ and $X = x - 10$, we get

$y - 17 = 1{\cdot}25 + 0{\cdot}7(x - 10)$ $\quad\therefore\ y = 0{\cdot}7x + 11{\cdot}25.$

Example 3 : Fit a straight line $y = a + bx$ to the following data.

x :	1	2	3	4	5	6
y :	49	54	60	73	80	86

(M.U. 2014)

Sol. : We shall solve this example by all the three methods.

(a) Direct method :

Calculations of $\sum x$, $\sum x^2$ etc.

x	y	x^2	xy
1	49	1	49
2	54	4	108
3	60	9	180
4	73	16	292
5	80	25	400
6	86	36	516
$N = 6$, 21	402	91	1545

Now, $y = a + bx$

$\sum y = Na + b\sum x$ $\quad\therefore\ 402 = 6a + 21b$ .... (1)

and $\sum xy = a\sum x + b\sum x^2$ $\quad\therefore\ 1545 = 21a + 91b$ .... (2)

Multiply (1) by 7 and (2) by 2 and subtract

$$\therefore\ 2814 = 42a + 147b$$
$$3090 = 42a + 182b$$
$$\overline{276 = 35b}$$

$\therefore\ b = 7{\cdot}89$

Putting this value of b in (1), we get

$402 = 6a + 21(7{\cdot}89)$ $\quad\therefore\ 6a = 402 - 165{\cdot}69 = 236{\cdot}31$ $\quad\therefore\ a = 39{\cdot}38$

$\therefore\ y = 39{\cdot}38 + 7{\cdot}89x.$

(b) Assumed Mean Method : We shall assume 3 and 60 as means of x and y and find the deviations.

Calculations of $\sum X$, $\sum X^2$ etc.

x	y	$X = x - 3$	$Y = y - 60$	X^2	XY
1	49	−2	−11	4	22
2	54	−1	−6	1	6
3	60	0	0	0	0
4	73	1	13	1	13
5	80	2	20	4	40
6	86	3	26	9	78
$N = 6$		3	42	19	159

Now, $Y = a + bX$

$\sum Y = Na + b\sum X$ $\quad\therefore\ 42 = 6a + 3b$ .... (1)

and $\sum XY = a\sum X + b\sum X^2$ $\quad\therefore\ 159 = 3a + 19b$ .... (2)

Multiply (2) by 2 and subtract (1) from the result.

$$\therefore\ 318 = 6a + 38b$$
$$42 = 6a + 3b$$
$$\overline{276 = 35b}$$

$\therefore\ b = 7{\cdot}89$

Putting this value of b in (1),

$42 = 6a + 3(7{\cdot}89)$ $\quad\therefore\ 6a = 18{\cdot}33$ $\quad\therefore\ a = 3{\cdot}05$

$\therefore\ Y = 3.05 + 7.89X.$

Putting $X = x - 3$, $Y = y - 60$, we get

$y - 60 = 3{\cdot}05 + 7{\cdot}89(x - 3)$ $\quad\therefore\ y = 39{\cdot}38 + 7{\cdot}89x.$

(c) Actual Mean method : We shall take deviations from actual means of x and y.

Now, $\bar{x} = \dfrac{21}{6} = 3{\cdot}5$, $\bar{y} = \dfrac{402}{6} = 67.$

Calculations of $\sum X$, $\sum X^2$ etc.

x	y	$X = x - 3{\cdot}5$	$Y = y - 67$	X^2	XY
1	49	−2·5	−18	6·25	45·0
2	54	−1·5	−13	2·25	19·5
3	60	−0·5	−7	0·25	3·5
4	73	0·5	6	0·25	3·0
5	80	1·5	13	2·25	19·5
6	86	2·5	19	6·25	47·5
$N = 6$		0	0	17·5	138·0

Now, $Y = a + bX$ and $\sum XY = a\sum X + b\sum X^2$

$\therefore\ \sum Y = Na + b\sum X$

Putting the above values

$0 = 6a + b\,(0)$ $\therefore\ a = 0$

and $138 = a\,(0) + 17{\cdot}5\,b$ $\therefore\ b = 7{\cdot}89$

Hence, the equation is $Y = 7{\cdot}89X$.

Putting the values of X, Y, we get

$y - 67 = 7{\cdot}89\,(x - 3{\cdot}5)$ $\therefore\ y = 39{\cdot}38 + 7{\cdot}89x$.

Note ....

Thus, we see that if we take deviations from $\bar{x}$ and $\bar{y}$, we get $a = 0$ and we get the value of b directly. In this way the last method is more convenient.

Example 4 : Fit a straight line to the following data.

Year x	:	1951	1961	1971	1981	1991
Production y (000 tons)	:	10	12	8	10	15

Also estimate the production in 1987. **(M.U. 2013)**

Sol. : Now the means of x and y are $\bar{x} = 1971$ and $\bar{y} = 11$.

Since $\bar{x}$ and $\bar{y}$ are round figures, we shall take deviations from 1971 and 11.

Calculations of $\sum X$, $\sum Y$ etc.

x	y	$X = x - 1971$	$Y = y - 11$	X^2	XY
1951	10	−20	−1	400	20
1961	12	−10	1	100	−10
1971	8	0	−3	0	0
1981	10	10	−1	100	−10
1991	15	20	4	400	80
$N = 5$		0	0	1000	80

Now, $Y = a + bX$

$\therefore\ \sum Y = Na + b \sum X$ and $\sum XY = a \sum X + b \sum X^2$

Putting the above values

$0 = 5a + 0$ $\therefore\ a = 0$

and $80 = 0 + 1000\,b$ $\therefore\ b = 0{\cdot}08$

Hence, the equation is $Y = 0{\cdot}08X$.

Putting the values of X, Y, we get

$y - 11 = 0.08\,(x - 1971)$ $\therefore\ y = -146{\cdot}68 + 0{\cdot}08x$.

Putting $x = 1987$, $y = -146{\cdot}68 + 0{\cdot}08\,(1987) = 12{\cdot}28$.

$\therefore$ The production in 1987 is 12·28.

Example 5 : Fit a straight line of the form $y = a + bx$ to the following data and estimate the value of y for $x = 3{\cdot}5$.

x	:	0	1	2	3	4
y	:	1	1·8	3·3	4·5	6·3

Sol. : We shall solve this problem by both the methods *i.e.*, by direct method and also by deviation method.

(a) Direct method :

Calculations of $\sum x$, $\sum x^2$ etc.

x	y	x^2	xy
0	1·0	0	0·0
1	1·8	1	1·8
2	3·3	4	6·6
3	4·5	9	13·5
4	6·3	16	25·2
$N = 5$, 10	16·9	30	47·1

Let the equation of the line be $y = a + bx$.

Then the normal equations are

$\sum y = Na + b \sum x$ $\therefore\ 16{\cdot}9 = 5a + 10b$ .... (1)

and $\sum xy = a \sum x + b \sum x^2$ $\therefore\ 47{\cdot}1 = 10a + 30b$ .... (2)

Now, multiply (1) by 2 and subtract the result from (2).

$\therefore\ 47{\cdot}1 = 10\,a + 30\,b$

$33{\cdot}8 = 10\,a + 20\,b$

$13{\cdot}3 = 10\,b$ $\therefore\ b = 1{\cdot}33$

Putting this value of b in (1), we get

$16{\cdot}9 = 5a + 13{\cdot}3$ $\therefore\ 5a = 3{\cdot}6$ $\therefore\ a = 0{\cdot}72$

Hence, the equation of the line is

$y = 0{\cdot}72 + 1{\cdot}33x$.

(b) Deviation method : We shall take deviations of x from the mid-point of the values of x.

x	$X = x - 2$	$Y = y$	X^2	XY
0	−2	1.0	4	−2·0
1	−1	1.8	1	−1·8
2	0	3.3	0	0·0
3	1	4.5	1	4·5
4	2	6.3	4	12·6
$N = 5$	0	16.9	10	13·3

Let the equation of the line be $Y = a + bX$.

The normal equations are

$\sum Y = Na + b \sum X$ and $\sum XY = a \sum X + b \sum X^2$

Putting the values from the table, we get,

$16{\cdot}9 = 5a + b\,(0)$ $\therefore\ a = \frac{16{\cdot}9}{5} = 3{\cdot}38$.

And $13{\cdot}3 = a\,(0) + 10\,b$ $\therefore\ b = 1{\cdot}33$.

Hence, the equation of the line is $Y = 3{\cdot}38 + 1{\cdot}33X.$

Putting $X = x - 2$ and $Y = y$ we get

$$y = 3{\cdot}38 + 1{\cdot}33\,(x - 2) = 3{\cdot}38 - 2{\cdot}66 + 1{\cdot}33\,x$$

$$\therefore \quad y = 0{\cdot}72 + 1{\cdot}33\,x$$

Thus, the equation of the line is $y = 0{\cdot}72 + 1{\cdot}33x.$

Putting $x = 3{\cdot}5,\quad y = 0{\cdot}72 + 4{\cdot}655 = 5{\cdot}375.$

Hence, when $x = 3{\cdot}5$, $y = 5{\cdot}375$.

Example 6 : Fit a straight line to the following data, with x as independent variable.

x : 1965 1966 1967 1968 1969

y : 125 140 165 195 200 **(M.U. 2014)**

Sol. : The means of x and y are $\bar{x} = 1967$, $\bar{y} = 165$.

Since $\bar{x}$ and $\bar{y}$ are round figures, we shall take deviations from 1967 and 165.

Calculations of $\sum X, \sum Y$ etc.

x	y	$X = x - 1967$	$Y = y - 165$	X^2	XY
1965	125	-2	-40	4	80
1966	140	-1	-25	1	25
1967	165	0	0	0	0
1968	195	1	30	1	1
1969	200	2	35	4	70
$N = 5$		0	0	10	176

Now, $\quad Y = a + bX$

$\therefore \quad \sum Y = Na + b\sum X \qquad$ and $\qquad \sum XY = a\sum X + b\sum X^2$

Putting the above values

$0 = 5a + 0 \qquad \therefore\ a = 0$

and $\quad 176 = 0 + 10\,b \qquad \therefore\ b = 17{\cdot}6$

Hence, the equation of line is $\quad Y = 17{\cdot}6X.$

Putting the values of X and Y, we get

$$y - 165 = 17{\cdot}6\,(x - 1967)$$
$$= 165 + 17{\cdot}6\,x - 34619{\cdot}2$$
$$\therefore \quad y = 17.6x - 34454{\cdot}2.$$

Example 7 : Fit a straight line to the following data.

x : 1 2 3 4 5 6

y : 83 92 71 90 160 191

Sol. : Now, the mean of $x = 3{\cdot}5$. Since the deviations will be in decimals, we multiply them by 2, *i.e.*, we put $2\,(x - 35) = X$.

The mean of $y = 114{\cdot}5$.

Since, the deviations will be in decimals, we multiply them by 2, *i.e.*, we put $2\,(y - 114{\cdot}5) = Y.$

Calculations of $\sum X, \sum Y$ etc.

x	y	$X = 2\,(x - 3{\cdot}5)$	$Y = 2\,(y - 114.5)$	X^2	XY
1	83	-5	-63	25	315
2	92	-3	-45	9	135
3	71	-1	-87	1	87
4	90	1	-49	1	-49
5	160	3	91	9	273
6	191	5	153	25	765
$N = 6$		0	0	70	1526

Now, $\quad Y = a + bX$

$\therefore \quad \sum Y = Na + b\sum X \qquad$ and $\qquad \sum XY = a\sum X + b\sum X^2$

Putting the above values

$0 = 6a + 0 \qquad \therefore\ a = 0$

and $\quad 1526 = 0 + 70\,b \qquad \therefore\ b = 21{\cdot}8$

Hence, the equation of line is $\quad Y = 21{\cdot}8X.$

Putting the values of X and Y, we get

$$2\,(y - 114{\cdot}5) = 21{\cdot}8\,(2)\,(x - 3{\cdot}5)$$
$$\therefore \quad y - 114{\cdot}5 = 21{\cdot}8\,(x - 3{\cdot}5) \qquad \therefore\ y = 38{\cdot}2 + 21{\cdot}8x$$

Note ....

We shall consider fitting of a second degree curve and an exponential curve on page 10-10 and 10-16.

EXERCISE - I

Class (a) : 4 Marks

1. Fit a straight line to the following data :

   x : 0 1 2 3 4 5

   y : 1 2 3 4·5 6 7·5 [**Ans. :** $y = 0{\cdot}70 + 1{\cdot}32x$]

2. Fit a straight line to the following data :

   x : 100 120 140 160 180 200

   y : 0·45 0·55 0·60 0·70 0·80 0·85 [**Ans. :** $y = 0{\cdot}041 + 0{\cdot}0041x$]

3. Fit a first degree curve to the following data and estimate the value of y when $x = 73$.

   x : 10 20 30 40 50 60 70 80

   y : 1 3 5 10 6 4 2 1

   [**Ans. :** $y = 4 - 0{\cdot}071u$ where $u = (x - 45)/5$, $y = 3{\cdot}596$, when $x = 73$]

4. Fit a straight line to the following data and estimate y when $x = 12$

   x : 1 2 3 4 5 6 7 8 9 10

   y : 52·5 58·7 65·0 70·2 75·4 81·1 87·2 95·5 102·2 108·4

   (M.U. 2005) [**Ans. :** $Y = 79{\cdot}62 + 3{\cdot}08X$ where $X = (x - 5{\cdot}5)2$, when $x = 12$, $y = 119{\cdot}66$]

5. Fit a straight line to the following data.

$(x, y) = (-1, -5), (1, 1), (2, 4), (3, 7), (4, 10)$. Estimate y when $x = 7$. **(M.U. 2015)**

[**Ans. :** $y = -2 + 3x$; 19]

3. Fitting A Parabola

Suppose we have a set of values (x_i, y_i). Suppose further that we want to fit a parabola $y = a + bx + cx^2$ to these values. The parabola must be close to the given points as much as possible. The principle of least squares states that the parabola should be such that the distances of the given points from the parabola measured along the y axis must be minimum.

Suppose $P(x_i, y_i)$ is a given point and a line through P parallel to the y axis intersects the curve $y = a + bx + cx^2$ in Q. Then Q is $(x_i, a + bx_i + cx_i^2)$. We find a, b, c such that the distance $|PQ|$ is minimum. But distance $|PQ|$ is minimum when the square of the distance $(y_i - a - bx_i - cx_i^2)^2$ is minimum. This must be true for all points. This means we should have $s = \sum (y_i - a - bx_i - cx_i^2)^2$ minimum. Since we find a, b, c such that the sum of the squares S is least the method as you know is knowm as the **method of least squares**.

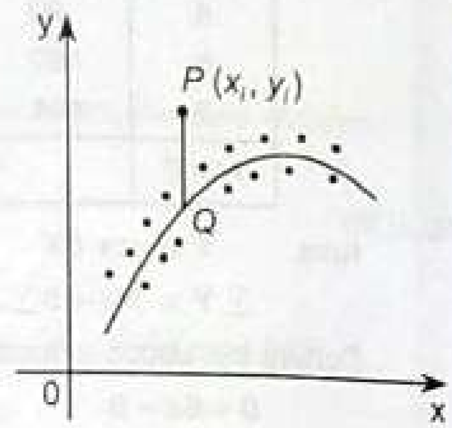


Fig. 10.2

Now for s to be minimum, the conditions are

$$\frac{\partial s}{\partial a} = 0, \quad \frac{\partial s}{\partial b} = 0, \quad \frac{\partial s}{\partial c} = 0$$

Differentiating s partially w.r.t. a, b, c, we get

$$\therefore \quad \sum (y_i - a - bx_i - cx_i^2) = 0 ; \qquad \sum (y_i - a - bx_i - cx_i^2)\, x_i = 0 ;$$

$$\sum (y_i - a - bx_i - cx_i^2)\, x_i^2 = 0.$$

i.e., $\sum y_i = Na + b \sum x_i + c \sum x_i^2$; $\quad \sum x_i y_i = a \sum x_i + b \sum x_i^2 + c \sum x_i^3$;

$$\sum x_i^2 y_i = a \sum x_i^2 + b \sum x_i^3 + c \sum x_i^4.$$

where N is the number of observations.

Dropping the suffix, i, we get the equations as

$$\boxed{\begin{aligned} \sum y &= Na + b \sum x + c \sum x^2 \\ \sum xy &= a \sum x + b \sum x^2 + c \sum x^3 \\ \sum x^2 y &= a \sum x^2 + b \sum x^3 + c \sum x^4 \end{aligned}}$$

These equations are called **normal equations**.

As in the case of straight line for fitting of parabola also we have two methods. (i) Direct Method and (ii) Deviation Method (or Change of Origin Method).

In direct method we use the given values of x and y and find the constants a, b, c.

If the values of x (or y) are evenly spaced we change the origin to the mid-point of x and use the deviations as in Ex. 3 or 4 below.

Example 1 : By the method of least squares find the best values of a, b, c in the law $y = a + bx + cx^2$ to fit the following data.

x : -2 $\quad -1$ $\quad 0$ $\quad 1$ $\quad 2$

y : $-3{\cdot}150$ $\quad -1{\cdot}390$ $\quad 0{\cdot}620$ $\quad 2{\cdot}880$ $\quad 5{\cdot}378$

Sol. :

Calculations of $\sum x, \sum y, \sum x^2$

Sr. No.	x	y	x^2	x^3	x^4	xy	x^2y
1	-2	$-3{\cdot}150$	4	-8	16	6·300	$-12{\cdot}600$
2	-1	$-1{\cdot}390$	1	-1	1	1·390	$-1{\cdot}390$
3	0	0·620	0	0	0	0·000	0·000
4	1	2·880	1	1	1	2·880	2·880
5	2	5·378	4	8	16	10·756	21·512
$N = 5$	0	4·338	10	0	34	21·326	10·402

The normal equations are

$$\sum y = Na + b \sum x + c \sum x^2 \qquad \therefore \quad 4{\cdot}338 = 5a + 10c$$

$$\sum xy = a \sum x + b \sum x^2 + c \sum x^3 \qquad \therefore \quad 21{\cdot}325 = 10b$$

$$\sum x^2 y = a \sum x^2 + b \sum x^3 + c \sum x^4 \qquad \therefore \quad 10{\cdot}402 = 10a + 34c$$

Solving these three equations, we get

$$a = 0{\cdot}621, \quad b = 2{\cdot}1326, \quad c = 0{\cdot}1233$$

Hence, the law is

$$y = 0{\cdot}621 + 2{\cdot}1326x + 0{\cdot}1233x^2.$$

Example 2 : Fit a second degree curve *i.e.,* parabola to the following data.

x : 0 $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4

y : 1·0 $\quad$ 1·8 $\quad$ 1·3 $\quad$ 2·5 $\quad$ 6·3

Sol. : Let the equation of the parabola be $y = a + bx + cx^2$.

Calculations of $\sum x, \sum x^2$, etc.

Sr. No.	x	y	x^2	x^3	x^4	xy	x^2y
1	0	1·0	0	0	0	0	0
2	1	1·8	1	1	1	1·8	1·8
3	2	1·3	4	8	16	2·6	5·2
4	3	2·5	9	27	81	7·5	22·5
5	4	6·3	16	64	256	25·2	100·8
$N = 5$	10	12·9	30	100	354	37·1	130·3

The normal equations are

$$\sum y = Na + b \sum x + c \sum x^2$$

$$\sum xy = a \sum x + b \sum x^2 + c \sum x^3$$

$$\sum x^2 y = a \sum x^2 + b \sum x^3 + c \sum x^4$$

Putting the values from the table,

$$12{\cdot}9 = 5a + 10b + 30c \quad \ldots\ldots\ldots (1)$$
$$37{\cdot}1 = 10a + 30b + 100c \quad \ldots\ldots\ldots (2)$$
$$130{\cdot}3 = 30a + 100b + 354c \quad \ldots\ldots\ldots (3)$$

We first eliminate a. For this multiply (1) by 2 and subtract the result from the second.

$$\therefore \quad 37{\cdot}1 = 10a + 30b + 100c$$
$$25{\cdot}8 = 10a + 20b + 60c$$
$$\overline{11{\cdot}3 = \quad 10b + 40c} \quad \ldots\ldots\ldots (4)$$

Now, multiply (1) by 6 and subtract the result from (3).

$$\therefore \quad 130{\cdot}3 = 30a + 100b + 354c$$
$$77{\cdot}4 = 30a + 60b + 180c$$
$$\overline{52{\cdot}9 = \quad 40b + 174c} \quad \ldots\ldots\ldots (5)$$

Now, multiply (4) by 4 and subtract the result from (5).

$$\therefore \quad 52{\cdot}9 = 40b + 174c$$
$$45{\cdot}2 = 40b + 160c$$
$$\overline{7{\cdot}7 = \quad 14c} \qquad \therefore \ c = \frac{7{\cdot}7}{14} = 0{\cdot}55.$$

Putting this value of c in (4), we get,

$$11{\cdot}3 = 10b + 40\,(0{\cdot}55)$$
$$\therefore \ 10b = 11{\cdot}3 - 22 = -10{\cdot}7 \qquad \therefore \ b = -1{\cdot}07$$

Putting the values of b and c in (1), we get

$$12{\cdot}9 = 5a + 10\,(-1{\cdot}07) + 30\,(0{\cdot}55)$$
$$\therefore \ 5a = 12{\cdot}9 + 10{\cdot}7 - 16{\cdot}5 = 7{\cdot}1 \qquad \therefore \ a = 1{\cdot}42$$

Hence, the equation of the parabola is

$$y = 1{\cdot}42 - 1{\cdot}07x + 0{\cdot}55x^2.$$

Example 3 : Fit a non-linear trend of the form $y = a + bx + cx^2$ to the following data.

x :	0	1	2	3	4
y :	1·0	1·5	1·5	2·5	3·5

Sol. : Calculations of $\Sigma X, \Sigma Y, \Sigma X^2$

Sr. No.	x	$Y = y$	Deviations from mid point $X = x - 2$	X^2	X^3	X^4	XY	X^2Y
1	0	1·0	−2	4	−8	16	−2·0	4·0
2	1	1·5	−1	1	−1	1	−1·5	1·5
3	2	1·5	0	0	0	0	0·0	0·0
4	3	2·5	1	1	1	1	2·5	2·5
5	4	3·5	2	4	8	16	7·0	14·0
$N = 5$	10	10·0	0	10	0	34	6·0	22·0

Thus, we have

$$N = 5,\ \Sigma X = 0,\ \Sigma Y = 10,\ \Sigma X^2 = 10,\ \Sigma X^3 = 0,\ \Sigma X^4 = 34,\ \Sigma XY = 6,\ \Sigma X^2 Y = 22.$$

Since, we have taken mid-point of X as origin $\Sigma X = 0$ and $\Sigma X^3 = 0$. The normal equations reduce to

$$\Sigma Y = Na + b\Sigma X^2$$
$$\Sigma XY = a\Sigma X^2$$
$$\Sigma X^2 Y = a\Sigma X^2 + c\Sigma X^4$$

Putting the above values, we get $\quad 10 = 5a + 10c$

$$\therefore \quad 22 = 10a + 34c \quad \text{and} \quad 6 = 10b \qquad \therefore \ b = \frac{6}{10}$$
$$20 = 10a + 20c$$
$$\overline{2 = \quad 14c} \qquad \therefore \ c = \frac{2}{14} = \frac{1}{7}$$

$$\therefore \ 10 = 5a + \frac{10}{7} \qquad \therefore \ a = \frac{12}{7}. \qquad \therefore \text{ The equation is } Y = \frac{12}{7} + \frac{6}{10}X + \frac{1}{7}X^2.$$

Now putting $Y = y$ and $X = x - 2$, we get,

$$y = \frac{12}{7} + \frac{6}{10}(x-2) + \frac{1}{7}(x-2)^2 = \frac{12}{7} + \frac{6}{10}x - \frac{12}{10} + \frac{1}{7}x^2 - \frac{4}{7}x + \frac{4}{7}$$

$$\therefore \ y = \frac{38}{35} + \frac{1}{35}x + \frac{1}{7}x^2.$$

Example 4 : Fit a second degree parabolic curve to the following data.

x :	1,	2,	3,	4,	5,	6,	7,	8,	9.
y :	2,	6,	7,	8,	10,	11,	11,	10,	9.

(M.U. 2004, 07, 12)

Sol. : Since the values of x are odd and are equally spaced we change x to X by the relation $X = x - 5$ and put $y = Y$.

Let the equation of the parabola be $Y = a + bX + cX^2$. Then the normal equations are

$$\Sigma Y = Na + b\Sigma X + c\Sigma X^2$$
$$\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$
$$\Sigma X^2 Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

Calculations of $\Sigma X, \Sigma X^2$ etc.

Sr. No.	x	$X = x - 5$	$Y = y$	X^2	X^3	X^4	XY	X^2Y
1	1	−4	2	16	−64	256	−8	32
2	2	−3	6	9	−27	81	−18	54
3	3	−2	7	4	−8	16	−14	28
4	4	−1	8	1	−1	1	−8	8
5	5	0	10	0	0	0	0	0
6	6	1	11	1	1	1	11	11
7	7	2	11	4	8	16	22	44
8	8	3	10	9	27	81	30	90
9	9	4	9	16	64	256	36	144
$N = 9$		0	74	60	0	708	51	411

Now the normal equations become

$$74 = 9a + 60c \quad \text{.........(1)}$$
$$51 = 60b \qquad \therefore\ b = 0{\cdot}85 \quad \text{.........(2)}$$
$$411 = 60a + 708c \quad \text{.........(3)}$$

Now to find a and c multiply (1) by 60 and (3) by 9.

$$\therefore\ 4440 = 540a + 3600c$$
$$3699 = 540a + 6372c$$
$$741 = \quad -2772c \qquad \therefore\ c = -0{\cdot}2673$$

Now, $9a = 74 - 60c = 74 + 16{\cdot}038 = 90{\cdot}038$

$\therefore\ a = 10{\cdot}0042 = 10$

$\therefore$ The equation of the parabola is $Y = 10 + 0{\cdot}85X - 0{\cdot}27X^2$.

i.e. $y = 10 + 0{\cdot}85\,(x-5) - 0{\cdot}27\,(x-5)^2 \qquad \therefore\ y = -1 + 3{\cdot}55x - 0{\cdot}27x^2$.

Example 5 : Fit a second degree parabolic curve to the following data and estimate the production in 1982.

Year (x) :	1974	75	76	77	78	79	80	91
Production (y) : (in tons)	12	14	26	42	40	50	52	53

Sol. : Since the values of X though not odd are equispaced we change x to X by the relation $X = (x - 1977{\cdot}5)2$ and we put $y = Y$.

Let the equation of the parabola be $Y = a + bX + cX^2$. Then the normal equations are

$$\Sigma Y = Na + b\Sigma X + c\Sigma X^2$$
$$\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$
$$\Sigma X^2Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

Calculations of ΣX, ΣX^2 etc.

Sr. No.	x	X	Y	X^2	X^3	X^4	XY	X^2Y
1	1974	−7	12	49	−343	2401	−84	588
2	1975	−5	14	25	−125	625	−70	350
3	1976	−3	26	9	−27	81	−78	234
4	1977	−1	42	1	−1	1	−42	42
5	1978	1	40	1	1	1	40	40
6	1979	3	50	9	27	81	150	450
7	1980	5	52	25	125	625	260	1300
8	1981	7	53	49	343	2401	371	2597
$N = 9$		0	289	168	0	6216	547	5601

Now, the normal equations become

$$289 = 8a + 168c \quad \text{.........(1)}$$
$$547 = 168b \qquad \therefore\ b = 3{\cdot}2559 \quad \text{.........(2)}$$
$$5601 = 168a + 6216c \quad \text{.........(3)}$$

Multiply (1) by 21 and subtract (3) from the result.

$$6069 = 168a + 3528c$$
$$5601 = 168a + 6216c$$
$$468 = \quad -2688c \qquad \therefore\ c = -0{\cdot}1741$$

Now $289 = 8a + 168c$

$\therefore\ 289 = 8a + (168)(-0{\cdot}1741) \qquad \therefore\ a = 39{\cdot}7811$

The equation to the curve is $Y = 39{\cdot}78 + 3{\cdot}26X - 0{\cdot}17X^2$.

Putting $Y = y$ and $X = (x - 1977{\cdot}5)2$, the equation of the parabola is

$$y = 39{\cdot}78 + 3{\cdot}26\,(x - 1977{\cdot}5)2 - 0{\cdot}17\,(x - 1977{\cdot}5)^2 \cdot 4$$
$$y = -2671997{\cdot}77 + 2695{\cdot}92x - 0{\cdot}68x^2.$$

Putting $x = 1982$, we get $y = 55{\cdot}35$.

EXERCISE - II

1. Fit a parabola to the following data and estimate the value of y for $x = 6$.

x : 1 2 3 4 5
y : 25 28 33 39 46 (M.U. 2006)

[**Ans. :** $Y = -0{\cdot}086 + 5{\cdot}3X + 0{\cdot}643X^2$ where $X = x - 3$, $Y = y - 33$, $y = 54{\cdot}6$]

2. Fit a second parabola to the following data taking x as the independent variable and shifting the origin to 2 for x.

x : 0 1 2 3 4
y : 1 1·8 1·3 2·5 6·3

Find the difference between the actual value and estimated value of y when $x = 2$.

[**Ans. :** $Y = 1{\cdot}48 + 1{\cdot}13X + 0{\cdot}55X^2$ where $X = x - 2$. Difference $= -0{\cdot}18$]

3. Fit a parabola to the following data :

x : −2 −1 0 1 2
y : 1·0 1·8 1·3 2·5 6·3 (M.U. 2006) [**Ans. :** $y = 1{\cdot}48 + 1{\cdot}13x + 0{\cdot}55x^2$]

4. Fit a parabola to the following data :

x : −2 −1 0 1 2
y : −3·150 −1·390 0·620 2·880 5·378

[**Ans. :** $y = 0{\cdot}621 + 2{\cdot}1326x + 0{\cdot}1233x^2$]

5. Fit a second degree curve to the following data and estimate the value of y when $x = 80$

x : 10 20 30 40 50 60 70
y : 20 60 70 80 90 100 100

[**Ans. :** Put $X = (x - 40)/10$, $Y = y/10$, $Y = 8{\cdot}381 + 1{\cdot}2143X - 0{\cdot}2381X^2$ when $x = 80$, $y = 94{\cdot}286$]

6. Fit a parabola to the following data and estimate y when $x = 10$.

(M.U. 2004)

x : 1 2 3 4 5 6 7 8 9
y : 2 6 7 8 10 11 11 10 9

[**Ans. :** $Y = 3 + 0{\cdot}85X - 0{\cdot}27X^2$ where $X = x - 5$, $Y = y - 7$, $y = 7{\cdot}5$]

7. Fit a second degree curve to the following data and estimate the production in 1975.

Year :	1921,	31,	41,	51,	61,	71,	81.
Production (in tons) :	3,	5,	9,	10,	12,	14,	15.

[**Ans.** : $Y = 10{\cdot}33 + 2{\cdot}0357X - 0{\cdot}1547X^2$ where $X = (x - 1951) / 10$. Production = 14·32]

8. Fit a second degree curve to the following data

Year :	1965,	66,	67,	68,	69,	70,	71,	72.
Profit (in Crores Rs.) :	125,	140,	165,	195,	200,	215,	220,	230.

Also estimate the profit in 1973.

[**Ans.** : $Y = 194{\cdot}68 + 7{\cdot}68X - 0{\cdot}40X^2$ where $X = (x - 1968{\cdot}5) / 2$.
And in 1973 profit = 231·4 crores of Rs.]

4. Fitting Exponential Curve

If we have to fit an exponential curve of the form $y = ax^b$ or $y = ab^x$ or $y = ac^x$, we take logarithm of both sides. Then the given exponential equation transforms into a linear equation. We then use the above technique.

Suppose the law as $y = ab^x$.

Taking logarithm of both sides to the base 10, we get

$$\log_{10} y = \log_{10} a + x \log_{10} b$$

Putting $\log_{10} y = Y$, $\log_{10} a = A$, $x = X$, $\log_{10} b = B$, we get

$$Y = A + BX$$

This is, a linear equation in X and Y and we can obtain the constants A and B by the method explained in § 2 and resubstituting for A and B, we can obtain the equation $y = ab^x$.

If the law is $y = ax^b$ taking logarithm of both sides, we get

$$\log y = \log a + b \log x.$$

Putting $\log y = Y$, $\log a = A$, $b = B$ and $\log x = X$, we get

$$Y = A + BX.$$

Again using the technique of § 2, we can find A and B and then a and b.

Example 1 : Fit a curve $y = ab^x$ to the following data, using the method of least squares.

x :	2	3	4	5	6
y :	144	172·8	207·4	248·8	298·5

Sol. : Taking logarithms of $y = ab^x$, we get $\log y = \log a + (\log b)\, x$. Let us put $\log y = Y$, $\log a = A$, $\log b = B$ and $x = X$. Then the law becomes $Y = A + BX$.

Calculations of $\sum X$, $\sum Y$, etc.

Sr. No.	x	y	X	Y	X^2	XY
1	2	144·0	2	2·1584	4	4·3168
2	3	172·8	3	2·2375	9	6·7125
3	4	207·4	4	2·3168	16	9·2672
4	5	248·8	5	2·3959	25	11·9795
5	6	298·5	6	2·4749	36	14·8494
$N = 5$			20	11·5835	90	47·1254

Now, we have

$$\sum Y = NA + B \sum X \qquad \therefore\ 11{\cdot}5835 = 5A + 20B$$

$$\sum XY = A \sum X + B \sum X^2 \qquad \therefore\ 47{\cdot}1254 = 20A + 90B$$

Solving these equations, we get,

$$B = 0{\cdot}07914 \quad \text{and} \quad A = 2{\cdot}0001$$

Hence, b = antilog 0·07914 = 1·199, a = antilog 2·0001 = 100.

$\therefore$ The law is $y = 100\,(1{\cdot}199)^x$.

Example 2 : Find the law of the form $y = ab^x$ to the following data. **(M.U. 2005)**

x :	1	2	3	4	5	6	7	8
y :	1	1·2	1·8	2·5	3·6	4·7	6·6	9·1

Sol. : Taking logarithm of both sides of $y = ab^x$, we get

$$\log y = \log a + x \log b.$$

Putting $\log y = Y$, $\log a = A$, $x = X$, $\log b = B$, we get $Y = A + BX$.

We, now, form the following table.

Calculations of $\sum X$, $\sum Y$, etc.

Sr. No.	$X = x$	y	$Y = \log y$	XY	X^2
1	1	1·0	0·0000	0·0000	1
2	2	1·2	0·0792	0·1584	4
3	3	1·8	0·2553	0·7689	9
4	4	2·5	0·3979	1·5916	16
5	5	3·6	0·5563	2·7815	25
6	6	4·7	0·6721	4·0326	36
7	7	6·6	0·8195	5·7365	49
8	8	9·1	0·9590	7·6720	64
$N = 8$	36		3·7393	22·7385	204

Now, $\quad Y = A + BX$

$$\therefore\ \sum Y = NA + B \sum X$$

$$\sum XY = A \sum X + B \sum X^2$$

Putting $N = 8$, $\sum X = 36$, $\sum Y = 3{\cdot}7393$, $\sum XY = 22{\cdot}7385$, $\sum X^2 = 204$, we get

$$3{\cdot}7393 = 8A + 36B$$

$$22{\cdot}7385 = 36A + 204B$$

Solving the two equations, we get,

$$A = -\,0{\cdot}1657, \quad B = 0{\cdot}1407$$

$\therefore$ a = anti-log (– 0·1657) = 0·6828

b = anti-log (0·1407) = 1·3828

Hence, the required law correct to two places of decimals is

$$y = 0{\cdot}68\,(1{\cdot}38)^x.$$

Example 3 : Fit a curve of the form $y = ax^b$ to the following data.

x :	1	2	3	4	5	6
y :	120	90	60	20	11	5

Sol. : Taking logarithms of both sides of $y = ax^b$, we get

$$\log y = \log a + b \log x.$$

Putting $\log y = Y$, $\log a = A$, $b = B$, $\log x = X$, we get

$$Y = A + BX.$$

Calculations of $\sum X$, $\sum Y$, etc.

Sr. No.	x	y	$X = \log x$	$Y = \log y$	XY	X^2
1	1	120	0·0000	2·0792	0·0000	0·0000
2	2	90	0·3010	1·9542	0·5882	0·0906
3	3	60	0·4771	1·7782	0·8484	0·2276
4	4	20	0·6021	1·3010	0·7833	0·3625
5	5	11	0·6990	1·0414	0·7279	0·4886
6	6	5	0·7781	0·6990	0·5439	0·6054
$N = 6$			2·8573	8·8530	3·4917	1·7747

Now, the normal equations are

$$\sum Y = NA + B\sum X$$
$$\sum XY = A\sum X + B\sum X^2$$

Putting the values from the table

$$8{\cdot}8530 = 6A + 2{\cdot}8573B \quad \ldots\ldots\ldots\ldots (1)$$
$$3{\cdot}4917 = 2{\cdot}8573A + 1{\cdot}7747B \quad \ldots\ldots\ldots\ldots (2)$$

Multiply (1) by 2·8573 and (2) by 6 and subtract.

$$25{\cdot}2957 = 17{\cdot}1438A + 8{\cdot}1642B$$
$$20{\cdot}9502 = 17{\cdot}1438A + 10{\cdot}6482B$$
$$4.3455 = \quad - 2.4840B$$

$\therefore \quad B = -1{\cdot}7494$

Putting this value of B in (1), we get

$6A = 8{\cdot}8530 + 4{\cdot}9991 \qquad \therefore \ A = 2{\cdot}3087$

$\therefore \quad a = \text{antilog } A = \text{antilog } 2{\cdot}3087 = 203{\cdot}56$

and $\quad b = B = -1{\cdot}7494$

Hence, we get $y = 203{\cdot}86x^{-1{\cdot}494}$.

Example 4 : Fit a curve of the form $y = ab^x$ to the following data.

x :	1	2	3	4	5	6
y :	151	100	61	50	20	8

Sol. : Let the equation of the curve be $y = ab^x$.

Taking logarithms of both sides, we get

$$\log y = \log a + x \log b$$

Let $\log y = Y$, $\log a = A$, $\log b = B$, $X = x$.

$\therefore \quad Y = A + BX.$

Calculations of $\sum X$, $\sum Y$, etc.

Sr. No.	$X = x$	y	$Y = \log y$	X^2	XY
1	1	151	2·1790	1	2·1790
2	2	100	2·0000	4	4·0000
3	3	61	1·7853	9	5·3559
4	4	50	1·6990	16	6·7960
5	5	20	1·3010	25	6·5050
6	6	8	0·9031	36	5·4186
$N = 6$	21		9·8674	91	30·2545

Now, the normal equations are

$$\sum Y = NA + B\sum X$$
$$\sum XY = A\sum X + B\sum X^2$$

Putting the above values, we get,

$$9{\cdot}8674 = 6A + 21B \quad \ldots\ldots\ldots\ldots (1)$$
$$30{\cdot}2545 = 21A + 91B \quad \ldots\ldots\ldots\ldots (2)$$

Multiply (1) by 7 and (2) by 2 and subtract.

$$\therefore \quad 69{\cdot}0718 = 42A + 147B$$
$$60{\cdot}5090 = 42A + 182B$$
$$8{\cdot}5628 = \quad -35B \qquad \therefore \ B = -0{\cdot}2447$$

Putting the value of B in (1), we get

$9{\cdot}8674 = 6A + 21(-0{\cdot}2447)$

$\therefore \quad 6A = 9{\cdot}8674 + 5{\cdot}1387$

$6A = 15{\cdot}0061 \qquad \therefore \ A = 2{\cdot}5010$

$\therefore \quad a = \text{antilog } A = 316{\cdot}96$

$b = \text{antilog } B = 0{\cdot}5692$

$\therefore \quad y = 316{\cdot}96\,(0{\cdot}5692)^x.$

EXERCISE - III

Class (c) : 8 Marks

1. Fit a curve of the form $y = ab^x$ to the following data. **[Ans. : $y = 4{\cdot}68\,(1{\cdot}58)^x$]**

x :	0	2	4
y :	5.012	10.000	31.620

2. Fit a curve of the form $y = ax^b$ to the following data by the method of least square. **[Ans. : $y = 2{\cdot}22\,x^{2{\cdot}09}$]**

x :	1	2	3	4
y :	2.50	8.00	19.00	50.00

3. Fit a curve of the form $y = ax^b$ to the following data by the method of least squares.

x :	1	2	3	4
y :	76	210	390	605

[**Ans.** : $y = 75{\cdot}2\, x^{1{\cdot}51}$]

4. Fit a curve of the form $y = ab^x$ to the following data using the method of least squares.

x :	2	3	4	5	6
y :	144	172·8	207·4	248·8	298·5

[**Ans.** : $y = 100\,(1{\cdot}999)^x$]

5. Fit a curve of the form $y = ab^x$ to the following data using the method of least squares.

x :	2	3	4	5	6
y :	34.385	79.0855	181.90	418.36	962.23

(M.U. 2007)

[**Ans.** : $y = 6{\cdot}5\,(2{\cdot}3)^x$]

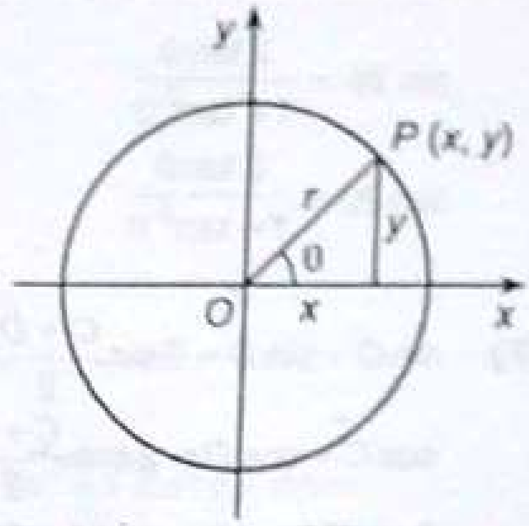
❀ ❀ ❀

APPENDIX

List of Formulae

1. Trigonometic Formulae

1. Definitions of Trigonometric Ratios



$\sin\theta = \dfrac{y}{r}$; $\cos\theta = \dfrac{x}{r}$; $\tan\theta = \dfrac{y}{x}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$; $\sec\theta = \dfrac{1}{\cos\theta}$; $\cot\theta = \dfrac{1}{\tan\theta}$

2. Fundamental Trigonometic Identities

(A) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

$\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$, $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$,

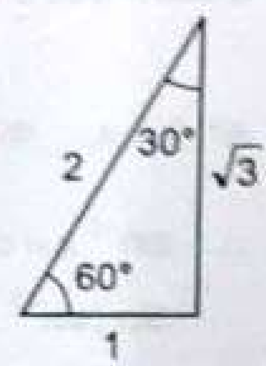
$\sin\left(\dfrac{\pi}{2} + \theta\right) = \cos\theta$, $\cos\left(\dfrac{\pi}{2} + \theta\right) = -\sin\theta$,

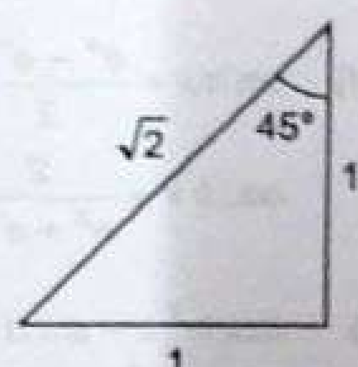
$\sin(\pi - \theta) = \sin\theta$, $\cos(\pi - \theta) = -\cos\theta$,

$\sin(\pi + \theta) = -\sin\theta$, $\cos(\pi + \theta) = -\cos\theta$,

$\sin\left(\dfrac{3\pi}{2} - \theta\right) = -\cos\theta$, $\cos\left(\dfrac{3\pi}{2} - \theta\right) = -\sin\theta$,

$\sin\left(\dfrac{3\pi}{2} + \theta\right) = -\cos\theta$, $\cos\left(\dfrac{3\pi}{2} + \theta\right) = \sin\theta$.

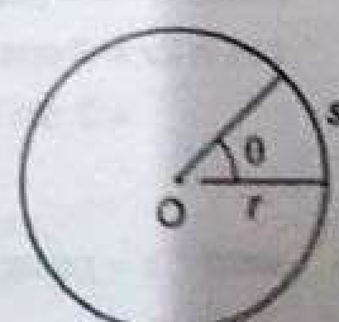




(B)

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$
I	+	+	+
II	+	−	−
III	−	−	+
IV	−	+	−

θ	$\dfrac{0^\circ}{360^\circ}$	30°	45°	60°	90°	180°	$\dfrac{270^\circ}{-90^\circ}$
$\sin\theta$	0	$\dfrac{1}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{2}$	0	-1	0



$s = r\theta$, $180^\circ = \pi^c$

(C) $\sin^2\theta + \cos^2\theta = 1,\qquad \sec^2\theta = 1 + \tan^2\theta,\qquad \text{cosec}^2\,\theta = 1 + \cot^2\theta.$

(D) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

(E) $\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \qquad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} \qquad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \qquad \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

(F) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2};\qquad \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2};$

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2};\qquad \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2};$

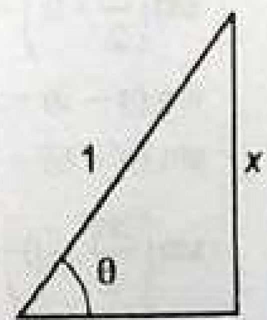
$2\sin A\cos B = \sin(A+B) + \sin(A-B);\qquad 2\cos A\sin B = \sin(A+B) - \sin(A-B);$

$2\cos A\cos B = \cos(A+B) + \cos(A-B);\qquad -2\sin A\sin B = \cos(A+B) - \cos(A-B).$

(G) $\sin\theta = x,\quad \text{cosec}\,\theta = \frac{1}{x}$

$\therefore\ \sin^{-1}x = \text{cosec}^{-1}\frac{1}{x},\qquad \cos^{-1}x = \sec^{-1}\frac{1}{x},\qquad \tan^{-1}x = \cot^{-1}\frac{1}{x},$

$\sin(\sin^{-1}x) = x,\qquad \cos(\cos^{-1}x) = x,\qquad \tan(\tan^{-1}x) = x.$



(H) $\sin h\,x = \frac{e^x - e^{-x}}{2},\qquad \cos h\,x = \frac{e^x + e^{-x}}{2},\qquad \tan h\,x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$

$\sec h\,x = \frac{2}{e^x + e^{-x}},\qquad \text{cosec}\,h\,x = \frac{2}{e^x - e^{-x}},\qquad \cot h\,x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$

(I) $\cos h^2 x - \sin h^2 x = 1,\qquad \sec h^2 x = 1 - \tan h^2 x$

$\text{cosec}\,h^2 x = \cot h^2 x - 1,\qquad \sin h\,2x = 2\sin hx\cos hx$

$\cos h\,2x = 2\cos h^2 x - 1$

$= 2\sin h^2 x + 1.$

(J) $\sin h^{-1}x = \text{cosec}\,h^{-1}\frac{1}{x},\qquad \cos h^{-1}x = \sec h^{-1}\frac{1}{x},\qquad \tan h^{-1}x = \cot h^{-1}\frac{1}{x}.$

2. Algebraic Formulae

(A) $a^m \times a^n = a^{m+n}$ $\quad \dfrac{a^m}{a^n} = a^{m-n}$ $\quad a^m b^m = (ab)^m$

$(a^m)^n = a^{mn}$ $\quad a^0 = 1,\ a^{-r} = \dfrac{1}{a^r}$ $\quad a^{p/q} = \sqrt[q]{a^p}$

(B) $a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^4 - b^4 = (a^2 + b^2)(a+b)(a-b)$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3 = (a^3 + b^3) + 3ab(a+b)$

$(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3 = (a^3 - b^3) - 3ab(a-b)$

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3b^2c + 3bc^2 + 3c^2a + 3a^2c + 6abc$

$= (a^3 + b^3 + c^3) + 3(a+b+c)\cdot(ab+bc+ca) - 3abc.$

$a^2 + b^2 = (a+b)^2 - 2ab = (a-b)^2 + 2ab$

$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

$a^3 + b^3 + c^3 = (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc$

$= 3abc$ if $a + b + c = 0$

(C) $\log mn = \log m + \log n,$ $\quad \log \dfrac{m}{n} = \log m - \log n$

$\log m^n = n \log m$ $\quad \log_b m = \dfrac{\log_a m}{\log_a b}$

(D) If $ax^2 + bx + c = 0,$ $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Sum of the roots $= -\dfrac{b}{a},$ $\quad$ Product of the roots $= \dfrac{c}{a}$

(E) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$

(a) If two rows or columns are identical then the determinant is zero.

(b) If rows and columns are interchanged the determinant is not changed.

(c) If two rows or columns are interchanged the determinant changes its sign.

(d) If each element of a row or column is multiplied by a constant then the determinant is multiplied by that constant.

(e) The value of a determinant is unchanged if equimultiples of a row or a column are added to the corresponding elements of any other row or column.

Cramer's Rule

If $a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$,

then $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, $z = \dfrac{D_z}{D}$ where, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and D_x, D_y, D_z are obtained by replacing the coefficients of x, y, z respectively by d_1, d_2, d_3.

3. Differentiation Formulae

(A) $\lim\limits_{\theta \to 0} \left(\dfrac{\sin \theta}{\theta} \right) = 1$, $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n} \right)^n = e$,

$\lim\limits_{y \to 0} (1 + y)^{1/y} = e$, $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$.

(B) 1. If $y = x^n$, $\dfrac{dy}{dx} = n x^{n-1}$

2. If $y = \sin x$, $\dfrac{dy}{dx} = \cos x$

3. If $y = \cos x$, $\dfrac{dy}{dx} = -\sin x$

4. If $y = \tan x$, $\dfrac{dy}{dx} = \sec^2 x$

5. If $y = \operatorname{cosec} x$, $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$

6. If $y = \sec x$, $\dfrac{dy}{dx} = \sec x \tan x$

7. If $y = \cot x$, $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x$

8. If $y = e^x$, $\dfrac{dy}{dx} = e^x$

9. If $y = a^x$, $\dfrac{dy}{dx} = a^x \log a$

10. If $y = \log_e x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$

11. If $y = \log_a x$, $\dfrac{dy}{dx} = \dfrac{1}{x \log a}$

12. If $y = \sin^{-1} x$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$

13. If $y = \cos^{-1} x$, $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1 - x^2}}$

14. If $y = \tan^{-1} x$, $\quad \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

15. If $y = \sec^{-1} x$, $\quad \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$

16. If $y = \operatorname{cosec}^{-1} x$, $\quad \dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{x^2-1}}$

17. If $y = \cot^{-1} x$, $\quad \dfrac{dy}{dx} = -\dfrac{1}{1+x^2}$

18. If $y = \sin hx$, $\quad \dfrac{dy}{dx} = \cos h\, x$

19. If $y = \cos hx$, $\quad \dfrac{dy}{dx} = \sin h\, x$

20. If $y = \tan hx$, $\quad \dfrac{dy}{dx} = \sec h^2\, x$

21. If $y = \operatorname{cosec} hx$, $\quad \dfrac{dy}{dx} = -\operatorname{cosec} h\, x \cot x$

22. If $y = \sec hx$, $\quad \dfrac{dy}{dx} = \sec h\, x \tan h\, x$

23. If $y = \cot hx$, $\quad \dfrac{dy}{dx} = -\operatorname{cosec} h^2\, x$

24. If $y = \sin h^{-1} x$, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$

25. If $y = \cos h^{-1} x$, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2-1}}$

26. If $y = \tan h^{-1} x$, $\quad \dfrac{dy}{dx} = \dfrac{1}{1-x^2}$

27. If $y = \sec h^{-1} x$, $\quad \dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{1-x^2}}$

28. If $y = \operatorname{cosec} h^{-1} x$, $\quad \dfrac{dy}{dx} = -\dfrac{1}{|x|\sqrt{1+x^2}}$

29. If $y = \cot h^{-1} x$, $\quad \dfrac{dy}{dx} = \dfrac{1}{1-x^2}$

(C) 1. If $y = u \pm v$, $\quad \dfrac{dy}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

2. If $y = uv$, $\quad \dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

3. If $y = \dfrac{u}{v}$, $\quad \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

4. If $y = x^x$, $\quad \dfrac{dy}{dx} = x^x(1+\log x)$

5. If $x = f(t)$, $y = \Phi(t)$, $\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

4. Integration Formulae

$$\int \mathrm{I} \cdot \mathrm{II} \cdot dx = \mathrm{I} \cdot \int \mathrm{II}\ dx - \int \left[\int \mathrm{II}\ dx\right] \cdot \frac{d\mathrm{I}}{dx} \cdot dx; \qquad \int e^x [f(x) + f'(x)]\, dx = e^x f(x)$$

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1}$ if $n \neq -1$
2. $\int \frac{dx}{x} = \log x$
3. $\int \sin x\, dx = -\cos x$
4. $\int \cos x\, dx = \sin x$
5. $\int \sec^2 x\, dx = \tan x$
6. $\int \operatorname{cosec}^2 x\, dx = -\cot x$
7. $\int \sec x \tan x\, dx = \sec x$
8. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$
9. $\int \tan x\, dx = \log \sec x$
10. $\int \cot x\, dx = -\log \operatorname{cosec} x = \log \sin x$
11. $\int \sec x\, dx = \log\left\{\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\} = \log(\sec x + \tan x)$
12. $\int \operatorname{cosec} x\, dx = \log\left(\tan \frac{x}{2}\right) = \log(\operatorname{cosec} x - \cot x)$
13. $\int e^x\, dx = e^x$
14. $\int a^x dx = \frac{a^x}{\log a}$
15. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$
16. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right)$
17. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right)$
18. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$
19. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left(\frac{x - a}{x + a}\right)$
20. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left(\frac{a + x}{a - x}\right)$
21. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x$
22. $\int e^{ax} \sin bx\, dx = \frac{1}{a^2 + b^2} \cdot e^{ax}(a \sin bx - b \cos bx)$
23. $\int e^{ax} \cos bx\, dx = \frac{1}{a^2 + b^2} \cdot e^{ax}(a \cos bx + b \sin bx)$
24. $\int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$
25. $\int \sqrt{x^2 + a^2}\ dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2 + a^2}\right)$
26. $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right)$
27. $\int \sin h\, x\, dx = \cos h\, x$
28. $\int \cos h\, x\, dx = \sin h\, x$
29. $\int \tan h\, x\, dx = \log(\cos h\, x)$
30. $\int \sec h\, x\, dx = \sin^{-1}(\tan h\, x)$
31. $\int \operatorname{cosec} h\, x\, dx = \tan \left| \tan h \frac{x}{2} \right|$
32. $\int \cot h\, x\, dx = \log |\sin h\, x|$

Definite Integrals

1. $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$
2. $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

Chapterwise Distribution of Examples from Mumbai University Examinations, November 2017 and May 2018 with solutions

Chapter 1 : Laplace Transforms - I

Example 1 : Find the Laplace transform of $\frac{1}{t}e^{-t}\sin t$. **(Nov. 2017 - 5 marks)**

Sol. : See solved Ex. 1 (ii), page 1-39.

Example 2 : Use Laplace transform to prove $\int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t}dt = \frac{1}{4}\log 5$. **(Nov. 2017 - 6 marks)**

Sol. : See solved Ex. 7, page 1-45.

Example 3 : Find the Laplace transform of $e^{-2t}\,t\cos t$. **(May 2018 - 5 marks)**

Sol. : Similar to Ex. B (6), page 1-34.

Example 4 : Evaluate $\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{te^t}\right)dt = \frac{3\pi}{4}$. **(May 2018 - 6 marks)**

Sol. : See Ex. 8, page 1-52.

Chapter 2 : Laplace Transforms - II

Example 1 : Find the inverse Laplace transform of $\frac{1}{\sqrt{2s+1}}$. **(Nov. 2017 - 5 marks)**

Sol. : See solved Ex. 2, page 2-5.

Example 2 : Find the inverse Laplace transform by using convolution theorem $\frac{1}{(s-a)(s+a)^2}$. **(Nov. 2017 - 6 marks)**

Sol. : See solved Ex. 2 (iii), page 2-17.

Example 3 : Using Laplace Transform evaluate

$$\int_0^\infty e^{-t}(1+2t-t^2+t^3)\,H(t-1)\,dt.$$ **(Nov. 2017 - 6 marks)**

Sol. : See solved Ex. 2, page 2-52.

Example 4 : Solve using Laplace transform

$(D^3 - 2D^2 + 5D)\,y = 0$, with $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$. **(Nov. 2017 - 8 marks)**

Sol. : See solved Ex. 15, page 2-76.

Example 5 : Find the inverse Laplace transform of $\frac{3s+7}{s^2-2s-3}$. **(May 2018 - 5 marks)**

Sol. : See solved Ex. 4 (iii), page 2-6.

Example 6 : Find the inverse Laplace Transform by using convolution theorem.

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)}$$ **(May 2018 - 6 marks)**

Sol. : See solved Ex. 3 (ii), page 2-19.

Example 7 : Using Laplace Transform, evaluate $\int_0^\infty e^{-1}(1+3t+t^2)H(t-2)\,dt$.

(May 2018 - 6 marks)

Sol. : See Ex. 3 (ii), page 2-53. Similar to solved Ex. 1, page 2-52.

Example 8 : Solve using Laplace transform $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$, $y = 2$, $y' = 3$ at $x = 0$.

(May 2018 - 8 marks)

Sol. : See Ex. 11, page 2-80. Similar to solved Ex. 6, page 2-70.

Chapter 3 : Fourier Series

Example 1 : Find the Fourier series for $f(x) = x$ in $(0, 2\pi)$. **(Nov. 2017 - 5 marks)**

Sol. : See Ex. 3, page 3-21. Similar to solved Ex. 1, page 3-10.

Example 2 : Obtain half range sine series for $f(x) = \begin{cases} x, & 0 < x < (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$.

Hence, find the sum of $\sum_{(2n-1)}^{\infty} \frac{1}{n^4}$. **(Nov. 2017 - 8 marks)**

Sol. : See solved Ex. 2, page 3-65.

Example 3 : Find the Fourier series for $f(x) = x^2$ in the interval $(-\pi, \pi)$.

(May 2018 - 5 marks)

Sol. : See solved Ex. 2, page 3-30.

Example 4 : Obtain half-range sine series for $f(x) = (x-1)^2$ in $0 < x < 1$.

Hence, find $\sum_{n=1}^{\infty} \frac{1}{n^2}$. **(May 2018 - 8 marks)**

Sol. : See Ex. 33, page 3-83.

Chapter 4 : Complex Form of Fourier Series

Example 1 : Show that the set of functions $\cos x, \cos 2x, \cos 3x, \ldots$ is a set of orthogonal functions over $[-\pi, \pi]$. Hence, construct a set of orthonormal functions. **(Nov. 2017 - 6 marks)**

Sol. : See Ex. 9, page 4-20. Similar to solved Ex. 1, page 4-13.

Example 2 : Find the complex form of the Fourier series for $f(x) = 2x$ in $(0, 2\pi)$.

(Nov. 2017 - 6 marks)

Sol. : See solved Ex. 16, page 4-11.

Example 3 : Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = \frac{3x^2 - 1}{2}$ are orthogonal over $(-1, 1)$.

(May 2018 - 6 marks)

Sol. : See solved Ex. 7, page 4-17.

Example 4 : Find the complex form of Fourier series for $f(x) = e^x$, $(-\pi, \pi)$. **(May 2018 - 6 marks)**

Sol. : See solved Ex. 1, page 4-3 with $a = 1$.

Chapter 5 : Complex Variables

Example 1 : Show that the function $f(z) = \sin hz$ is analytic and find $f'(z)$ in terms of z. **(Nov. 2017 - 5 marks)**

Sol. : See solved Ex. 7 (ii), page 5-12.

Example 2 : If $f(z)$ and $\overline{f(z)}$ are both analytic, prove that $f(z)$ is constant. **(Nov. 2017 - 6 marks)**

Sol. : See solved Ex. 2, page 5-9.

Example 3 : Show that the function $u = \cos x \cos hy$ is a harmonic function. Find its harmonic conjugate and corresponding analytic function. **(Nov. 2017 - 8 marks)**

Sol. : See solved Ex. 6, page 5-47.

Example 4 : Determine whether the function $f(z) = (x^3 + 3xy^2 - 3x) + i(3x^2y - y^3 + 3y)$ is analytic and if so find its derivative. **(May 2018 - 5 marks)**

Sol. : See Ex. 2 (x), page 5-21. Similar to solved Ex. 12, page 5-15.

Example 5 : Show that the function $v = e^x (x \sin y + y \cos y)$ is a harmonic function. Find its harmonic conjugate and corresponding analytic function. **(May 2018 - 8 marks)**

Sol. : See solved Ex. 4, page 5-46.

Example 6 : If u, v are harmonic conjugate functions, show that uv is a harmonic function. **(May 2018 - 6 marks)**

Sol. : See solved Ex. 3, page 6-27.

Chapter 6 : Conformal Mapping

Example 1 : Find the bilinear transformation which maps the points $1, -i, 2$ on z-plane onto $0, 2, -i$ respectively of w-plane. **(Nov. 2017 - 6 marks)**

Sol. : See solved Ex. 4, page 6-31.

Example 2 : Find the bilinear transformation which maps the points $z = -1, 0, 1$ onto the points $w = -1, -i, 1$. **(May 2018 - 6 marks)**

Sol. : See Ex. 3 (iii), page 6-34. Similar to solved Ex. 2, page 6-30.

Chapter 7 : Z - Transforms

Example 1 : If $f(k) = \begin{cases} 4^k, & k < 0 \\ 3^k, & k \geq 0 \end{cases}$, find $Z[f(k)]$. **(Nov. 2017 - 6 marks)**

Sol. : See solved Ex. 8, page 7-11 with $b = 4$ and $a = 3$.

Example 2 : Find the inverse Z-transform of

(i) $\dfrac{1}{(z-a)^2}$, $|z| < a$, (ii) $\dfrac{1}{(z-3)(z-2)}$, $|z| > 3$. **(Nov. 2017 - 8 marks)**

Sol. : See solved Ex. 4 (i), page 7-36 and solved Ex. 3 (iii), page 7-40.

Example 3 : Find the *Z*-transform of $\{(1/4)^{|k|}\}$. **(May 2018 - 6 marks)**

Sol. : See solved Ex. 1, page 7-15 with $a = 1/4$.

Example 4 : Find the inverse *Z*-transform for the following :

(i) $\dfrac{1}{(z-5)^2}$, $|z| < 5$, (ii) $\dfrac{z}{(z-2)(z-3)}$, $|z| > 3$. **(May 2018 - 8 marks)**

Sol. : See solved Ex. 4, page 7-36 with $a = 5$ and solved Ex. 3 (iii), page 7-40.

Chapter 8 : Correlation

Example 1 : Calculate the coefficient of correlation between *X* and *Y* from the following data.

X	:	8	8	7	5	6	2
Y	:	3	4	10	13	22	8

(Nov. 2017 - 6 marks)

Sol. : See Ex. 8, page 8-11. Similar to solved Ex. 1, page 8-10.

Example 2 : Compute Spearman's rank correlation coefficient for the following data.

X	:	85	74	85	50	65	78	74	60	74	90
Y	:	78	91	78	58	60	72	80	55	68	70

(May 2018 - 6 marks)

Sol. : See Ex. 1, page 8-21. Similar to solved Ex. 3, page 8-18.

Chapter 9 : Regression

Example 1 : Find the equations of the lines of *Y* on *X* for the following data.

x	:	5	6	7	8	9	10	11
y	:	11	14	14	15	12	17	16

(Nov. 2017 - 6 marks)

Sol. : See solved Ex. 2, page 9-14.

Example 2 : From 8 observations the following results were obtained :

$\Sigma x = 59$, $\Sigma y = 40$, $\Sigma x^2 = 524$, $\Sigma y^2 = 256$, $\Sigma xy = 364$.

Find the equation of the line of regression of *x* on *y* and the coefficient of correlation. **(May 2018 - 6 marks)**

Sol. : See Ex. 4, page 9-24. Similar to solved Ex. 9, page 9-22.

Chapter 10 : Curve Fitting

Example 1 : Fit a curve of the form $y = ab^x$ to the following data.

x	:	1	2	3	4	5	6
y	:	151	100	61	50	20	8

(Nov. 2017 - 8 marks)

Sol. : See solved Ex. 4, page 10-18.

Example 2 : Fit a straight line of the form $y = a + bx$ to the following data and estimate the value of *y* for $x = 3{\cdot}5$.

x	:	0	1	2	3	4
y	:	1	1·8	3·3	4·5	6·3

(May 2018 - 8 marks)

Sol. : See solved Ex. 5, page 10-6.

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