EIGEN VALUES & EIGEN VECTORS

Friday, December 10, 2021 2:11 PM

DEFINITION

Let A be a square matrix of order n

$A - \lambda I$ is a matrix known as characteristic matrix

$|A - \lambda I| \rightarrow$ polynomial in λ

↳ characteristic polynomial

$|A - \lambda I| = 0 \rightarrow$ Characteristic Equation

The Roots of characteristic Equation are known as the characteristic roots, Latent roots or propervalues or Eigen Values.

$(-1)^n$

$(-1)^2 \rightarrow \lambda^2 - 2\lambda - 3 = 0$

trace of A

$|A|$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3$$

$$|A - \lambda I| = 0$$

$$\boxed{\lambda^2 - 2\lambda - 3 = 0}$$

$$\lambda = -1, 3$$

PROPERTIES OF THE CHARACTERISTIC POLYNOMIAL:

(1) The characteristic polynomial $|A - \lambda I|$ of a matrix A is an ordinary polynomial in λ of degree n where A is a square matrix of order n

(2) In characteristic polynomial of A

(i) the coefficient of λ^n is $(-1)^n$

(ii) the coefficient of λ^{n-1} is trace of A

(iii) the constant term is $|A|$

(3) If A is 3×3 matrix then the characteristic equation can be expressed as

$$|A - \lambda I| = (-1)^3\lambda^3 + (-1)^2 S_1\lambda^2 + (-1)S_2\lambda + |A| = 0$$

Where $S_1 =$ Sum of the diagonal elements of A,

$S_2 =$ Sum of the minors of the diagonal elements of A

Ch. eqn can be written as (3x3 matrix)

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

S_1 = trace of A = sum of diagonal elements of A

S_2 = sum of minors of the diagonal elements of A

PROPERTIES OF THE CHARACTERISTIC ROOTS (EIGEN VALUES):

(1) If A is a square matrix of order n then the degree of the characteristic equation is n and consequently there are exactly n roots (eigenvalues) not necessarily distinct.

(2) Sum of all eigenvalues = The sum of the diagonal elements of A (i.e. trace of A)

(3) Product of all eigenvalues of $A = |A|$ = constant term in the polynomial.

CHARACTERISTIC VECTORS OR EIGEN VECTORS:

let A be a nxn square matrix.

let λ be one of it's Eigen values

Then there exists a non-zero vector X such that

$\boxed{AX = \lambda X} \rightarrow AX - \lambda X = 0$

$[A - \lambda I]X = 0 \rightarrow$ Homogeneous system of eqns

only trivial $\rightarrow$ rank $= n$

non-trivial $\rightarrow$ rank $< n$

$|A - \lambda I| = 0$

λ

This X is the eigen vector corresponding to λ.

WORKING RULE TO FIND THE EIGEN VECTORS OF A:

For the given eigen value λ,

we take $[A - \lambda I]X = 0 \rightarrow$ Homogeneous system

$\downarrow$ row trans.

Echelon form

or use crammer's Rule to find the solution

SOME SOLVED EXAMPLES:

1. Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Soln :- The characteristic Equation is $|A-\lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)\left[(-3-\lambda)(1-\lambda) - 8\right] + 8\left[4(1-\lambda)+6\right]$$
$$-2\left[-16-3(-3-\lambda)\right] = 0$$

$$\boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}$$

OR use the formula instead

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

S_1 = trace of A

$= 8 + (-3) + 1 = 6$

S_2 = sum of minors of diagonal elements of A

= minor of 8 + minor of (-3) + minor of (1)

$$= \begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix}$$

$S_2 = -3 - 8 + 8 + 6 - 24 + 32$

$\therefore S_2 = 11$

$|A| = 6$

$\therefore$ The characteristic equation is

$$\boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}$$

The roots are $\boxed{\lambda = 1, 2, 3}$ → eigen values

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

use crammer's Rule

choose any two rows and write the corresponding equations

$$7x_1 - 8x_2 - 2x_3 = 0$$

$$4x_1 - 4x_2 - 2x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{-x_2}{-6} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2}$$

$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ is eigen vecter correspon to eigen value $\lambda = 1$

12/13/202
1 1:25 PM

For $\lambda = 2$, $[A - \lambda I] X = 0 \Rightarrow [A - 2I] X = 0$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Taking 1st & 2nd row

$$6x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 5x_2 - 2x_3 = 0$$

By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{-x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is the eigen vecter corresponding to $\lambda = 2$

For $\lambda = 3$, $[A - \lambda I] X = 0 \Rightarrow [A - 3I] X = 0$

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Choose 1st & 2nd row

$$5x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 6x_2 - 2x_3 = 0$$

By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$ is an eigen vecter corresponding to $\lambda = 3$

2. Find the Eigen values and Eigen vectors of the matrix. Also verify that the Eigen vectors are linearly independent $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Soln:- The characteristic Equation is $\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0.$$
$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$\therefore$ Eigen values are $\lambda = 1, 2, 3$

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = 0$$

By $\xrightarrow[R_3 - R_1]{R_2 - R_1}$
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 - x_2 + x_3 = 0$$
$$2x_2 - 2x_3 = 0 \Rightarrow x_2 = x_3$$

let $x_3 = t \Rightarrow x_2 = t$

$$x_1 - t + t = 0 \Rightarrow \boxed{x_1 = 0.}$$

$\therefore X_1 = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vecter for $\lambda = 1$.

For $\lambda = 2$, $[A - \lambda I]X = 0 \Rightarrow [A - 2I]X = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \left.\begin{aligned} -x_2 + x_3 &= 0 \\ x_1 - x_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 = x_3$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vecter corresponding to $\lambda = 2$

$\lambda = 3$, $[A - \lambda I]X = 0 \Rightarrow [A - 3I]X = 0$

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\xrightarrow[R_3 + R_1]{R_2 + R_1}$
$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} & & \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad -x_1 - x_2 + x_3 = 0$$

$$-2x_2 = 0 \Rightarrow \boxed{x_2 = 0}$$

$$-x_1 + x_3 = 0 \Rightarrow \boxed{x_1 = x_3}$$

let $x_1 = x_3 = 1$

$\therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigen vecter corresponding to $\lambda = 3$.

Now to check $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

are Linearly Independent.

let $k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$

$$k_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$k_2 + k_3 = 0$$
$$k_1 + k_2 = 0 \quad \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$
$$k_1 + k_2 + k_3 = 0$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_3-R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} k_1 + k_2 = 0 \\ k_2 + k_3 = 0 \end{array}$$

$$\xrightarrow{R_3-R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0 \Rightarrow \quad \begin{aligned} k_2+k_3 &= 0 \\ k_3 &= 0 \end{aligned}$$

$$\Rightarrow k_1 = k_2 = k_3 = 0$$

$\therefore$ The eigen vectors are Linearly Independent.

3. Determine the Eigen values and the associated Eigen vectors for the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Soln:- The characteristic equation is $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

or $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\therefore$ The Eigen values are $\lambda = 5, 1, 1$

For $\lambda = 5$, $[A - \lambda I]X = 0 \Rightarrow [A - 5I]X = 0$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

By crammer's Rule

$-3x_1 + 2x_2 + x_3 = 0$

$x_1 - 2x_2 + x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vecter corresponding to $\lambda = 5$

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Rank $= 1$, $\therefore$ no. of parameters $= 3 - 1 = 2$

$$x_1 + 2x_2 + x_3 = 0$$

let $x_1 = t,\ x_3 = s \Rightarrow x_2 = -\frac{1}{2}(t+s)$

$$\therefore X = \begin{bmatrix} t \\ -\frac{1}{2}(t+s) \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ are eigen vecters corresponding to $\lambda = 1$.

12/15/2021 2:24 PM

4. Find the Eigen values and Eigen vectors for $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

Soln: The characteristic Eqn is $\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

Sum of eigen values $=$ trace $= 5$

$\therefore$ roots are $\lambda = 1, 2, 2$

For $\lambda = 1$, $[A - \lambda I]x = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} x_1 + 2x_2 + 2x_3 = 0 \\ x_1 + 5x_2 + 3x_3 = 0 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ 5 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix}}$$

$$\frac{x_1}{-4} = \frac{-x_2}{1} = \frac{x_3}{3}$$

$\therefore X_1 = \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix}$ is an eigen vector for $\lambda = 1$.

For $\lambda = 2$, $[A - \lambda I]X = 0 \Rightarrow [A - 2I]X = 0$

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}$$

$$\xrightarrow[R_3 + R_1]{R_2 - R_1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ Rank $= 2$

$\therefore$ No. of parameters $= 1$.

$$\begin{bmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ No. of parameters $= 1$.

$\therefore$ No. of L.I. Eigenvectors $= 1$.

$$x_1 + 3x_2 + 3x_3 = 0$$
$$2x_2 + x_3 = 0$$

let $x_2 = t \Rightarrow x_3 = -2t$

$x_1 + 3t - 6t = 0 \Rightarrow x_1 = 3t$

$\therefore X_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ is an eigen vector for $\lambda = 2$

5. Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$\left\{ \begin{array}{l} \text{Upper triangular} \\ \text{Lower triangular} \\ \text{Diagonal.} \end{array} \right.$

Soln: Ch. eqn $\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$$(2-\lambda)^3 = 0$$

$$\Rightarrow \lambda = 2, 2, 2$$

For $\lambda = 2$, $[A - \lambda I]X = 0 \Rightarrow [A - 2I]X = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ rank $= 2$

$\Rightarrow$ no. of parameters $= 3 - 2 = 1$

$\Rightarrow$ 1 Eigen vector.

$\left.\begin{array}{l} x_2 = 0 \\ x_3 = 0 \end{array}\right\}$ let $x_1 = t$

$X = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigen vector for $\lambda = 2$.

$$X = \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is an eigen vector for $\lambda = 2$.

$\longrightarrow$ Unitary matrix $\rightarrow AA^{\theta} = I$

6. Prove that the Eigen values of $\begin{bmatrix} \frac{(1+i)}{2} & \frac{-(1-i)}{2} \\ \frac{(1+i)}{2} & \frac{(1-i)}{2} \end{bmatrix}$ are of unit modulus

Soln:- Ch. eqn is

$$\begin{vmatrix} \frac{(1+i)}{2} - \lambda & \frac{-(1-i)}{2} \\ \frac{(1+i)}{2} & \frac{(1-i)}{2} - \lambda \end{vmatrix} = 0$$

$$\left[\frac{(1+i)}{2} - \lambda\right]\left[\frac{(1-i)}{2} - \lambda\right] - \left[\left(\frac{1+i}{2}\right)\left(\frac{-(1-i)}{2}\right)\right] = 0$$

$$\left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) - \frac{(1+i)}{2}\lambda - \frac{(1-i)}{2}\lambda + \lambda^2 + \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) = 0$$

$$\frac{1}{2} - \lambda + \lambda^2 + \frac{1}{2} = 0$$

$$\lambda^2 - \lambda + 1 = 0$$

$\therefore$ Eigen values are $\lambda = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$$\lambda = \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\lambda_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$|\lambda_1| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} = 1$$

$$|\lambda_2| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{\sqrt{3}}{2}\right)^2} = 1$$

$$|\lambda_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = 1$$

$\therefore$ Eigen values are of unit modulus.

SIMILARITY OF MATRICES

Wednesday, December 15, 2021 2:50 PM

A, B → M (non singular)

$M^{-1}AM = B$ → Similarity

SIMILARITY OF MATRICES

Definition:

(i) If A and B are two square matrices of order n then B is said to be **similar** to A if there exists a non – singular matrix M such that $B = M^{-1}AM$

(ii) A square matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix.

Combining the two definitions we see that A is diagonalizable if there exists a matrix M such that

$M^{-1}AM = D$

where D is a diagonal matrix. In this case M is said to diagonalize A or transform A to diagonal form.

A → M, D → $M^{-1}AM = D$

Theorem 1: If A is similar to B and B is similar to C, then A is similar to C. → transitive.

Theorem 2: If A and B are similar matrices then $|A| = |B|$

Theorem 3: If A is similar to B, then A^2 is similar to B^2

Corollary: If A is diagonalisable then A^2 is diagonalisable.

Theorem 4: If A and B are two similar matrices then they have the same Eigen values

ALGEBRAIC AND GEOMETRIC MULTIPLICITY OF AN EIGEN VALUES (AM, GM)

Definition:

(i) If λ is an eigen value of the matrix A repeated t times then t is called the algebraic multiplicity of λ.

(ii) If s is the number of linearly Independent Eigen vectors corresponding to the eigen value λ then s is called the geometric multiplicity of λ.

EX-1

$\lambda = 1, 2, 3$

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\lambda = 1$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\lambda = 2$

$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\lambda = 3$

E.V.	AM	GM
1	1	1
2	1	1

EX-2

$\lambda = 5, 1, 1$

$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\lambda = 5$

$\left.\begin{matrix} X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \end{matrix}\right\} \lambda = 1$

EV	AM	GM
5	1	1

EX-3

$\lambda = 1, 2, 2$

$X_1 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$ $\lambda = 1$

$X_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ $\lambda = 2$

EV	AM	GM
1	1	1

EX-4

$\lambda = 2, 2, 2$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\lambda = 2$

$\lambda = 2$

AM = 3

1	1	1
2	1	1
3	1	1

✓

EV	AM	GM
5	1	1
1	2	2

✓

EV	AM	GM
1	1	1
2	2	1

X

AM = 3
GM = 1

X

Theorem: The necessary and sufficient condition of a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its Eigen values coincides with the algebraic multiplicity.

diagonalisable

i. e. We can diagonalise a given square matrix if and only if algebraic multiplicity of each of its Eigen values is equal to the geometric multiplicity. for every eigen value AM = GM $\rightarrow$ diagonalisable

If corresponding to any Eigen value, if algebraic multiplicity is **not equal** to geometric multiplicity then the matrix is **not diagonalizable.**

$M^{-1}AM = D$

Corollary: Every matrix whose Eigen values are distinct is similar to a diagonal matrix.

Theorem: A square nonsingular matrix A whose Eigen value are all distinct can be diagonalised by a similarity transformation $D = M^{-1}AM$ where M is the matrix whose columns are the Eigenvectors of A and D is the diagonal matrix whose diagonal elements are the Eigen values of A.

$\lambda = 1, 2, 3$ $\quad X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{array}{cccc} & X_1 & X_2 & X_3 \\ M = & \left[\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right. & \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} & \left.\begin{matrix} 1 \\ 0 \\ 1 \end{matrix}\right] \end{array} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Notes: **1.** If Eigen values of A are not distinct then it may or may not be possible to diagonalise it.

2. A and D are similar matrices and hence, they have the same Eigen values

3. The process of finding the modal matrix M is called diagonalising the matrix A.

12/17/2021 2:14 PM

1. Find the algebraic multiplicity and geometric multiplicity of each Eigen value of the matrix

$$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

Soln: The characteristic Equation is $\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0.$$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0.$$

Eigen values = 2, 2, 3

$\therefore$ For $\lambda = 2$, AM = 2

For $\lambda = 3$, AM = 1

For $\lambda = 2$, $[A - \lambda I]X = 0 \Rightarrow [A - 2I]X = 0$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow[-\frac{1}{5}R_3]{\frac{1}{3}R_2} \begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ Rank = 2

$\therefore$ no. of parameters = 3 − 2 = 1

No. of eigen vectors = 1.

$x_1 + 10x_2 + 5x_3 = 0$

$5x_2 + 2x_3 = 0$

Let $x_3 = 5t \Rightarrow x_2 = -2t$

$x_1 - 20t + 25t = 0 \Rightarrow x_1 = -5t$

$$\therefore X_1 = \begin{bmatrix} -5t \\ -2t \\ 5t \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$$ is an eigen vector for $\lambda = 2$

For $\lambda = 3$, $[A - 3I]X = 0$

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

By crammer's Rule

$x_1 + 3x_2 + 2x_3 = 0 \quad (-\frac{1}{2}R_2)$

$3x_1 + 5x_2 + 4x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix}}$$

$$\frac{x_1}{2} = \frac{-x_2}{-2} = \frac{x_3}{-4}$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is an eigen vecter for $\lambda = 3$

For $\lambda = 2$, AM $= 2$ GM $= 1$.

For $\lambda = 3$, AM $= 1$ GM $= 1$.

Note :- This matrix is not diagonalisable as AM $\neq$ GM for $\lambda = 2$.

2. Show that the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix

Soln :- Ch. eqn $\longrightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda \longrightarrow 0, 3, 15$

$|A| = 0$.

eigen value $= 0$.

If A is a

$\lambda \longrightarrow 0, 3, 15$

Since, all Eigen values are distinct, the matrix A is diagonalisable.

$= 0.$

If A is a singular matrix then atleast one of the eigen values $= 0$

For $\lambda = 0$, $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

For $\lambda = 3$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 15$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

The matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ will be diagonalised to diagonal matrix $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ by transforming matrix $M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ such that $\underline{\underline{M^{-1}AM = D}}$.

3. Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonisable. Find the diagonal form D and the diagonalising matrix M

Soln:- ch- eqn $\longrightarrow \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

Soln: ch. eqn $\longrightarrow \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$\lambda \rightarrow -1, -1, \underline{3}$

For $\lambda = -1$, $[A - \lambda I]X = 0 \Rightarrow [A + I]X = 0$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ Rank $= 1$. $\therefore$ no. of parameters $= 3 - 1 = 2$

$\therefore$ No. of L.I. eigen vectors $= 2$

$$-8x_1 + 4x_2 + 4x_3 = 0$$

let $x_2 = t$, $x_3 = s$

$$8x_1 = 4t + 4s \Rightarrow x_1 = \frac{t}{2} + \frac{s}{2}$$

$$\therefore X = \begin{bmatrix} \frac{t}{2} + \frac{s}{2} \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$= t\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ & $X_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ are eigen vectors for $\lambda = -1$

For $\lambda = 3$, $[A - \lambda I]X = 0 \Rightarrow [A - 3I]X = 0$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is eigen vector for $\lambda = 3$ (check)

$\therefore$ For $\lambda = -1$ AM $= 2$ GM $= 2$

For $\lambda = 3$ AM $= 1$ GM $= 1$

$\therefore$ A is diagonalisable as AM = GM for all the eigen values.

The given matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ will be diagonalised

to the diagonal form $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the

transforming matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ such that $M^{-1}AM = D$.

4. Show that the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is not similar to a diagonal matrix (not diagonalisable).

Soln :- eigen values $\rightarrow 2, 2, 1$

(A is a triangular matrix)

For $\lambda = 2$, $[A - 2I]X = 0$

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_3 - R_2} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ Rank = 2

$\therefore$ no. of parameters = $3-2=1$

no. of L.I. eigen vectors = 1

$$\left.\begin{array}{r} 3x_2+4x_3=0 \\ -x_3=0 \end{array}\right\} \Rightarrow x_2=0, x_3=0$$

$\therefore x_1 = t$

$\therefore X_1 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigen vector for $\lambda=2$

$\therefore$ For $\lambda=2$, AM = 2
GM = 1.

$\therefore$ AM $\neq$ GM for $\lambda=2$

$\Rightarrow$ A is not diagonalisable

Ex:- $A = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} = KI \quad (K \in R)$

Eigen values = $\lambda = 50, 50, 50$ (diagonal matrix)

AM for $\lambda=50$ is 3

For $\lambda=50$, $[A-50I]X=0$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

Rank = 0

no. of parameters = $3-0=3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Rank = 0

no. of parameters = 3 − 0 = 3

Let $x_1 = p$, $x_2 = q$, $x_3 = r$

$$X = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = p\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + q\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

are the eigen vector for $\lambda = 50$.

$\therefore$ GM for $\lambda = 50$ is

$\therefore$ AM = GM for $\lambda = 50$ ⇒ A is diagonalisable.

<u>Note</u>:- X_1, X_2, X_3 are Linearly Independent (check).

12/20/2021 1:15 PM

5. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix}$, prove that both A and B are not diagonable but AB is diagonable

<u>Soln</u>:- $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Eigen values of A are $\lambda = 1, 1$ (upper triangular)

AM of $\lambda = 1$ is 2

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow x_2 = 0$

$$\begin{bmatrix} \cdot & \cdot \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cdot \\ x_2 \end{bmatrix}$$

$\Rightarrow 2x_2 = 0 \Rightarrow x_2 = 0$

Let $x_1 = 1$.

$\therefore X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is eigen vecter corresponding to $\lambda = 1$

$\therefore$ GM of $\lambda = 1$ is 1.

$\therefore$ AM $\neq$ GM for $\lambda = 1$

$\therefore$ matrix A is not diagonalisable

For $B = \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix}$

Eigen values for B, $\lambda = 2, 2$

(lower triangular)

$\therefore$ AM for $\lambda = 2$ is 2

For $\lambda = 2$, $[B - \lambda I]X = 0 \Rightarrow [B - 2I]X = 0$

$$\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \frac{1}{2}x_1 = 0 \Rightarrow x_1 = 0$$

Let $x_2 = 1$

$\therefore X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is eigen vecter for $\lambda = 2$

GM for $\lambda = 2$ is 1

AM $\neq$ GM for $\lambda = 2$

AM $\neq$ GM for $\lambda = 2$

$\therefore$ B is not diagonalisable.

Let $C = AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1/2 & 2 \end{bmatrix}$

Charateristic eqn for C is $\begin{vmatrix} 3-\lambda & 4 \\ 1/2 & 2-\lambda \end{vmatrix} = 0$

$$(3-\lambda)(2-\lambda) - 2 = 0$$
$$\lambda^2 - 5\lambda + 4 = 0$$
$$\lambda = 1, 4$$

C = AB has distinct eigen values.

$\therefore$ C is diagonalisable.

For $\lambda = 1$, $[C - \lambda I]X = 0 \Rightarrow [C - I]X = 0$

$\begin{bmatrix} 2 & 4 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow R_2 - \frac{1}{4}R_1 \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$2x_1 + 4x_2 = 0$$

Let $x_2 = t \Rightarrow x_1 = -2t$

$\therefore X_1 = \begin{bmatrix} -2t \\ t \end{bmatrix} \sim \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is eigen vector for $\lambda = 1$.

For $\lambda = 4$, $[C - \lambda I]X = 0$ $[C - 4I]X = 0$

$X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

6. Find the symmetric matrix $A_{3\times 3}$ having the eigen values $\lambda_1 = 0, \lambda_2 = 3$ and $\lambda_3 = 15$, with the corresponding Eigen vectors $X_1 = [1, 2, 2]', X_2 = [-2, -1, 2]'$ and X_3

Properties of Eigen Values and Eigen Vectors

Monday, December 27, 2021 1:33 PM

Theorem 1: Prove that zero is an eigenvalues of a matrix A if and only if A is singular.

Theorem 2: Prove that the matrix A and its transpose A^T has same characteristic roots.

Theorem 3: Prove that the eigenvalues of a diagonal matrix are precisely the diagonal elements.

Theorem 4: Prove that the eigenvalues of a triangular matrix are precisely the diagonal elements.

Theorem 5: λ is an Eigen value of the matrix A if and only if there exists a non – zero vector X such that ~~$AX = X$~~. $AX = \lambda X$.

Theorem 6: If X is an Eigen vector of a matrix A corresponding to an Eigen value λ then kX (k is a non – zero scalar) is also an Eigen vector of A corresponding to the same Eigen value λ.

Theorem 7: (Uniqueness of Eigen Value):

If X is an Eigen vector of a matrix A then X cannot correspond to more than one Eigen values of A.

Theorem 8: (Linear independence of Eigen Vectors):

Eigen vectors corresponding to distinct Eigen values of a matrix are linearly independent.

Theorem 9: Eigen values of a Hermitian matrix are real. $|A|$ = product of eigenvalues

Corollary 1: The determinant of a Hermitian matrix is real. A^t A^θ $\bar{A}$

Corollary 2: Eigen values of a real symmetric matrix are all real.

Corollary 3: Eigen values of a Skew – Hermitian matrix are either purely imaginary or zero.

Corollary 4: The Eigen values of a real skew – symmetric matrix are purely imaginary or zero.

Theorem 10: The Eigen values of unitary matrix are of unit modulus. (have absolute value one).

Corollary: Eigen values of an orthogonal matrix are of unit modulus.

* **Theorem 11:** The Eigen vectors corresponding to distinct Eigen values of a real symmetric matrix are orthogonal. $X_1 \cdot X_2 = 0$.

Theorem 12: Any two Eigen vectors corresponding to two distinct Eigen values of a unitary matrix are orthogonal.

$x_1 \quad x_2 \quad x_3$

$A \to 1, 2, 3$

$3A \to 3, 6, 9$

$-5A \to -5, -10, -15$

$A^{-1} \to \frac{1}{1}, \frac{1}{2}, \frac{1}{3}$

$A^2 \to 1^2, 2^2, 3^2$

$1, 4, 9$

$A^3 \to 1, 8, 27$

Note:

1. If one Eigen value of a matrix A is $a + ib$ then another eigen value must be $a - ib$.
2. If λ is an Eigen value of A then $\bar{\lambda}$ is an eigen value of A^θ
3. If $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigen values of A then show that $k\lambda_1, k\lambda_2, \dots\dots k\lambda_n$ are the Eigen value kA.
4. If $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigen values of A then show that $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \dots\dots \frac{1}{\lambda_n}$ are the Eigen values of A^{-1}
5. If $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigen values of A then show that $\lambda_1^2, \lambda_2^2, \dots\dots, \lambda_n^2$ are the eigen values of A^2.
6. If λ is an Eigen value of a non – singular matrix A, prove that $\frac{|A|}{\lambda}$ is an eigen value of adj A.
7. If λ is an Eigen value of the matrix A then $\lambda \pm k$ is an eigen value of $A \pm kI$
8. If $f(x)$ is an algebraic polynomial in x and λ is an Eigen value and X is the corresponding Eigen vector of a square matrix A then $f()$ is an eigen value and X is the corresponding eigen vector of $f(A)$.

Solved Examples

1. If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ Where a, b, c are positive integers, then prove that

(i) a + b+ c is an Eigen value of A and **(ii)** ~~if A is non – singular,~~ one of the Eigen values is negative.

The characteristic eqn of A is

$|a-\lambda \quad b \quad c|$

The characteristic eq^n of A is

$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

By $C_1 + C_2 + C_3$

$$\begin{vmatrix} a+b+c-\lambda & b & c \\ a+b+c-\lambda & c-\lambda & a \\ a+b+c-\lambda & a & b-\lambda \end{vmatrix} = 0$$

$$\underline{\underline{(a+b+c-\lambda)}} \begin{vmatrix} 1 & b & c \\ 1 & c-\lambda & a \\ 1 & a & b-\lambda \end{vmatrix} = 0$$

$\therefore$ $a+b+c-\lambda = 0 \Rightarrow \lambda = a+b+c$.

$\therefore$ one of the eigen values is $a+b+c$

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A then

$\underline{\lambda_1 + \lambda_2 + \lambda_3 = \text{trace of } A} = a+b+c$.

but one of the eigen values $= a+b+c$

say $\lambda_1 = a+b+c$

$\Rightarrow \lambda_2 + \lambda_3 = 0$

As A is non singular, λ_2, λ_3 <u>are not</u> zero.

$\Rightarrow$ either of λ_2, or λ_3 must be negative.

2. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of a 3×3 matrix then prove that the Eigen values of $adj\ A$ are $\lambda_1\lambda_2, \lambda_2\lambda_3$ and $\lambda_3\lambda_1$

Soln :- $\lambda_1, \lambda_2, \lambda_3$ are eigen values of A

$\Rightarrow |A| = \lambda_1 \lambda_2 \lambda_3$

$\Rightarrow$ eigen values of adj A $\rightarrow \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$

$\rightarrow \frac{\lambda_1\lambda_2\lambda_3}{\lambda_1}, \frac{\lambda_1\lambda_2\lambda_3}{\lambda_2}, \frac{\lambda_1\lambda_2\lambda_3}{\lambda_3}$

$\rightarrow \lambda_2\lambda_3, \lambda_1\lambda_3, \lambda_1\lambda_2$

$2, 3, 4 \rightarrow A \qquad \rightarrow |A| = 2\times3\times4 = 24$

adj A $\rightarrow \frac{|A|}{2}, \frac{|A|}{3}, \frac{|A|}{4} = 12, 8, 6$

12/29/2021 2:23 PM

3. If A is a real square matrix of order n where n is an **odd** positive integer, then show that A has at least one real Eigen value.

Soln :- Since A is a matrix of odd order, it has characteristic eqn of degree n which is odd.

Case - I : The char. eqn has all real roots

Then we get the req. answer.

Case - II : The char. eqn has complex roots

but the complex roots always occur in pairs

$\therefore$ no. of complex eigen values will always be even.

$\therefore$ atleast one eigen values would be new.

4. The sum of the Eigen values of a 3×3 matrix is 6 and the product of the Eigen values is also 6. If one of the Eigen values is one, find the other two Eigen values.

Soln :- let $\lambda_1, \lambda_2, \lambda_3$ be the e.values

$\lambda_1 + \lambda_2 + \lambda_3 = 6$ also $\lambda_1 = 1$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 6$

$\Rightarrow \left.\begin{array}{l} \lambda_2 + \lambda_3 = 5 \\ \lambda_2 \cdot \lambda_3 = 6 \end{array}\right\} \Rightarrow \lambda_2 = 2, \lambda_3 = 3.$

$\lambda_2 = (5 - \lambda_3)$

$(5 - \lambda_3)\lambda_3 = 6$

$5\lambda_3 - \lambda_3^2 = 6$ $\Rightarrow \lambda_3^2 - 5\lambda_3 + 6 = 0$

$\Rightarrow (\lambda_3 - 2)(\lambda_3 - 3) = 0$

//

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\lambda^3 - 6\lambda^2 + S_2\lambda - 6 = 0$

for $\lambda = 1$, $(1)^3 - 6 + S_2 - 6 = 0 \Rightarrow S_2 = 11$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \lambda = 1, 2, 3.$

5. If $A = \begin{bmatrix} sin\theta & cosec\theta & 1 \\ \sec\theta & cos\theta & 1 \\ tan\theta & cot\theta & 1 \end{bmatrix}$ then prove that there does not exist a real value of θ for which characteristic roots of A are $-1, 1, 3$

$\left\lfloor \tan\theta \quad \cot\theta \quad 1 \right\rfloor$

characteristic roots of A are $-1, 1, 3$

Soln :- Sum of eigen values = trace of A

$$-1+1+3 = \sin\theta + \cos\theta + 1$$

$$3 = \sin\theta + \cos\theta + 1$$

$\Rightarrow \sin\theta + \cos\theta = 2$

$\Rightarrow$ This is not possible for any real value of θ

$\Rightarrow$ $-1, 1, 3$ can not be e.values for any real value of θ

6. Find the characteristic roots of $A^{30} - 9A^{28}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Soln :- ch. eqn of A is $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)^2 - 4 = 0$$

$$1 - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = -1, 3.$$

matrix	λ_1	λ_2
A	-1	3
A^{28}	$(-1)^{28} = 1$	3^{28}
$-9A^{28}$	$9 \times 1 = 9$	$9 \times 3^{28} = 3^{30}$

If λ is eigen value of A then $f(\lambda)$ is eigen value for $f(A)$

$f(A) = A^{30} - 9A^{28}$

$\therefore f(-1)$ & $f(3)$ are the

	$= 9$	$= 3^{30}$
✓ A^{30}	$(-1)^{30} = 1$	3^{30}
$A^{30} - 9A^{28}$	$1 - 9 = -8$	$3^{30} - 3^{30} = 0$

$\therefore f(-1)$ & $f(3)$ are the e-values for $f(A)$

$f(-1) = (-1)^{30} - 9(-1)^{28} = -8$

$f(3) = 3^{30} - 9(3)^{28} = 0$

$\therefore$ The eigen values of $A^{30} - 9A^{28}$ are -8 & 0

7. Find the Eigen values of $A^2 - 2A + I$ if $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

Eigen values of A are $1, 2, 3$ (upper triangular matrix)

$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

$f(A) = A^2 - 2A + I$

Eigen values of $f(A)$ are $f(\lambda_1), f(\lambda_2)$ & $f(\lambda_3)$

$f(\lambda_1) = f(1) = (1)^2 - 2(1) + 1 = 1 - 2 + 1 = 0$

$f(\lambda_2) = f(2) = (2)^2 - 2(2) + 1 = 1$

$f(\lambda_3) = f(3) = (3)^2 - 2(3) + 1 = 4$

$\therefore$ Eigen values of $A^2 - 2A + I$ are $0, 1, 4$

8. Find the Eigen values of $adjA$ if $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$$\begin{array}{c|c|c|c|c} A \longrightarrow & 1 & 2 & 4 & 6 \\ adj A \rightarrow & \frac{|A|}{1} & \frac{|A|}{2} & \frac{|A|}{4} & \frac{|A|}{6} \\ & = 48 & 24 & 12 & 8 \end{array}$$

but $|A|$
$= 1 \times 2 \times 4 \times 6$
$= 48$

9. If A is a square matrix of order 2 with $|A| = 1$ then prove that A and A^{-1} have the same eigen values. Hence verify for $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

Sol^n:- Eigen values of A are α and β

$\therefore$ Eigen values of A^{-1} are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Also $|A| = 1 \Rightarrow \alpha\beta = 1 \Rightarrow \alpha = \frac{1}{\beta}$ & $\beta = \frac{1}{\alpha}$

$\therefore$ Eigen values of A^{-1} are β & α

$\therefore$ A & A^{-1} have same eigen values.

Now $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ $\qquad |A| = -1 + 2 = 1$

Ch. eq^n of A is $\begin{vmatrix} -1-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$(-1-\lambda)(1-\lambda) + 2 = 0$

$-1 + \lambda - \lambda + \lambda^2 + 2 = 0$

$A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ $\qquad \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

Now $A^{-1} = \frac{adj A}{|A|} = adj A = \begin{bmatrix} 1 & -2 \end{bmatrix}$

Now $A^{-1} = \frac{adj A}{|A|} = adj A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

ch. eqn of A^{-1} is $\begin{vmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(-1-\lambda) + 2 = 0$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$\therefore$ verified for the given matrix A.

10. Verify that $X = [2,3,-2,-3]'$ is an eigen vector corresponding to the eigen value $\underline{\lambda = 2}$ of the matrix.

$$A = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$$

Soln :- To check that $\boxed{AX = \lambda X}$ ✓

$$AX = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \\ -6 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix}$$

$$= 2X$$

$\therefore AX = 2X = \lambda X$

$\therefore$ X is the eigen vector for eigen value $\lambda = 2$ for matrix A.

CAYLEY – HAMILTON THEOREM

Monday, December 20, 2021 1:25 PM

STATEMENT: Every square matrix satisfies it's characteristic Equation.

$A \rightarrow$ ch. eq^n $\rightarrow \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0 = 0.$

cayley- Hamilton theorem means that

$$A^n - a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_0 I = 0.$$

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \qquad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

By C-H thm

$$A^3 - 6A^2 + 11A - 6I = 0$$

1. Verify the Cayley-Hamilton theorem for the matrix A and hence, find A^{-1} and A^4 where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Soln: The characteristic equation is $\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

To verify cayley-Hamilton theorem, we have to show that $A^3 - 5A^2 + 9A - I = 0$ ——— (1)

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

Sub in LHS of ①

$A^3 - 5A^2 + 9A - I$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ -10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Hence Cayley-Hamilton Theorem is verified.

$A^3 - 5A^2 + 9A - I = 0$ ——— ①

multiply throughout by A^{-1}

$A^2 - 5A + 9I - A^{-1} = 0$

$$\Rightarrow A^{-1} = A^2 - 5A + 9I = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

multiplying ① by A

$A^4 - 5A^3 + 9A^2 - A = 0$

$$A^4 = 5A^3 - 9A^2 + A = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$$

2. Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

soln. The characteristic eqn of A is $\begin{vmatrix} 2-\lambda & 1 & 1 \end{vmatrix}$

$$\begin{bmatrix} & & \\ 1 & 1 & 2 \end{bmatrix}$$

Soln:- The characteristic eqn of A is $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{——} \ (1)$$

Now dividing $(\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1)$
by $(\lambda^3 - 5\lambda^2 + 7\lambda - 3)$

$$\begin{array}{r|l} & \lambda^5 + \lambda \\ \hline \lambda^3 - 5\lambda^2 + 7\lambda - 3 & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\ & \underset{-}{\lambda^8} \underset{+}{-5\lambda^7} \underset{-}{+7\lambda^6} \underset{+}{-3\lambda^5} \\ \hline & \qquad\qquad \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\ & \qquad\qquad \underset{-}{\lambda^4} \underset{+}{-5\lambda^3} \underset{-}{+7\lambda^2} \underset{+}{-3\lambda} \\ \hline & \qquad\qquad\qquad\qquad \lambda^2 + \lambda + 1 \end{array}$$

$\therefore$ quotient $= \lambda^5 + \lambda$
remainder $= \lambda^2 + \lambda + 1$.

dividend = (divisor) × (quotient) + Remainder
writing this in terms of A.

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= \underbrace{(A^3 - 5A^2 + 7A - 3I)}(A^5 + A) + (A^2 + A + I)$$

$$= 0 + (A^2 + A + I) \qquad (\text{using } 1)$$

$$= A^2 + A + I$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore \text{given expression} = A^2 + A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

3. Apply Cayley-Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that $A^8 = 625I$

Soln :- Characteristic eqn of A $\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(-1-\lambda) - 4 = 0$$

$$\lambda^2 - 5 = 0$$

By cayley-Hamilton theorem

$$A^2 - 5I = 0 \quad \text{—— } 1$$

$$\boxed{A^2 = 5I}$$

$$A^2 \cdot A^2 = 5I \cdot 5I$$

$$A^4 = 25I$$

$$A^4 \cdot A^4 = (25I)(25I)$$

$A^8 = 625 I$

12/27/2021 1:17 PM

4. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$, hence, find A^{50} ✓

Soln :- The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix}$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 1\right] = 0$$

$$\lambda^2 - 1 - \lambda^3 + \lambda = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

$\therefore$ By cayley- Hamilton theorem,

$$A^3 - A^2 - A + I = 0 \quad \text{———} \quad (1)$$

Let $p(n)$: $A^n = A^{n-2} + A^2 - I$, $n \geq 3$

we will prove the result by the method of mathematical Induction.

for $n = 3$,

$p(3)$: $A^3 = A + A^2 - I$ this is true from (1)

Let us assume that $p(n)$ is true for $n = k$

ie $A^k = A^{k-2} + A^2 - I$ ——— (2)

TPT: $p(n)$ is true for $n = k+1$

$A^{k+1} = A \cdot A^k = A \cdot \left(A^{k-2} + A^2 - I\right)$ (using (2)

$= A^{(k+1)-2} + A^3 - A$

$= A^{(k+1)-2} + (A + A^2 - I) - A$ (using 1)

$A^{k+1} = A^{(k+1)-2} + A^2 - I$

$\left\{ p(n) : A^n = A^{n-2} + A^2 - I \right\}$

$\therefore$ $p(n)$ is true for $n = k+1$

$\therefore$ By mathematical induction,

$p(n)$: $A^n = A^{n-2} + A^2 - I$ is true for $n \geq 3$.

To find A^{50}, put $n = 50$ in $p(n)$

$A^{50} = A^{48} + A^2 - I$

$A^{50} = (A^{46} + A^2 - I) + (A^2 - I)$ $(n = 48)$

$A^{50} = A^{46} + 2(A^2 - I)$

$= A^{44} + A^2 - I + 2(A^2 - I)$

$A^{50} = A^{44} + 3(A^2 - I)$

continuing like this 24 steps.

$A^{50} = A^2 + 24(A^2 - I)$

$$A^{50} = 25A^2 - 24I$$

Now $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\therefore A^{50} = 25 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - 24 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

Function of Square matrices

Friday, December 31, 2021 2:10 PM

CALCULATION OF POWERS OF MATRIX (FUNCTIONS OF SQUARE MATRIX):

If A is a non – singular square matrix with distinct Eigen values then we can find any power of A. i.e A^k (k is a positive integer) by the process explained below.

we have $M^{-1}AM = D$

Operating by M on the left and by M^{-1} on the right

$MM^{-1}AMM^{-1} = MDM^{-1}$

$\therefore (MM^{-1})A(MM^{-1}) = MDM^{-1}$

$\therefore A = MDM^{-1}$

$\therefore A^n = (MDM^{-1})(MDM^{-1}) \ldots\ldots\ldots (MDM^{-1})\ (n\ times)$

$\therefore A^n$

$= MD(M^{-1}M)D(M^{-1}M) \ldots\ldots\ldots (M^{-1}M)DM^{-1}$

$= MD \ldots\ldots\ldots DM^{-1}$

$$= MD^nM^{-1} = M\begin{bmatrix} \lambda_1^n & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2^n & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_n^n \end{bmatrix} M^{-1}$$

Note: Above method can be applied for any function of A i.e. $\boxed{f(A) = M\,f(D)\,M^{-1}}$

$M^{-1}AM = D$ (Diagonalisation)

$M = [x_1\ x_2\ x_3]$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$A = MDM^{-1}$

$A^n = MD^nM^{-1}$

$A^n = MD^nM^{-1}$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad D^{10} = \begin{bmatrix} 1^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 3^{10} \end{bmatrix}$$

$f(A) = M\,f(D)\,M^{-1}$

$\cos A = M \cos D\, M^{-1}$

$$\cos D = \begin{bmatrix} \cos 1 & 0 & 0 \\ 0 & \cos 2 & 0 \\ 0 & 0 & \cos 3 \end{bmatrix}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$\cos D = I - \frac{1}{2!}D^2 + \frac{1}{4!}D^4 - \frac{1}{6!}D^6 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2!}\begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} \alpha^4 & 0 \\ 0 & \beta^4 \end{bmatrix} - \frac{1}{6!}\begin{bmatrix} \alpha^6 & 0 \\ 0 & \beta^6 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots & 0 \\ 0 & 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{bmatrix}$$

ANOTHER METHOD: (General method) can be applied to any square matrix.

① If A is 2×2 matrix, we write

$f(A) = \alpha_1 A + \alpha_0 I$ and find α_1 & α_0 using the eigen values of A

② If A is 3×3 matrix, we write

$f(A) = \underline{\alpha_2} A^2 + \underline{\alpha_1} A + \underline{\alpha_0} I$ and find α_2, α_1 & α_0 using the eigen values of A.

$A^{50} \longrightarrow$ divide by the ch. poly

$A^{50} = (\underset{\overset{||}{0}}{\underline{\underline{\text{divisor}}}} \times \text{quotient}) + \underline{\text{Remainder}}$

$A^{50} = \text{Remainder} = \begin{cases} \underline{\alpha_1 A + \alpha_0 I} & (A \text{ is } 2\times2) \\ \underline{\alpha_2 A^2} + \alpha_1 A + \alpha_0 I & (A \text{ is } 3\times3) \end{cases}$

SOME SOLVED EXAMPLES:

1. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find A^{50}

Soln:- Ch. eqn of A is $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

(distinct)

(disti...)

$\therefore$ eigen values of A are $\lambda = 1, 3$.

Find eigen vectors now.

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$$

Let $x_1 = t \Rightarrow x_2 = -t$

$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector for $\lambda = 1$

For $\lambda = 3$, $[A - \lambda I]X = 0$ $\quad [A - 3I]X = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector for $\lambda = 3$.

$\therefore$ Modal matrix $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ & $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Now $f(A) = M f(D) M^{-1}$

$$A^{50} = M D^{50} M^{-1}$$

$$\therefore A^{50} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3^{50} \\ -1 & 3^{50} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$M^{-1} = \dfrac{\operatorname{adj} M}{|M|}$

$|M| = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$

$\operatorname{adj} M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} \cdot & \cdot \\ -1 & 3^{50} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} \cdot & \cdot \\ 1 & 1 \end{bmatrix}$$

$$A^{50} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

Now solving by method - II

let $A^{50} = \alpha_1 A + \alpha_0 I$ — (1) (A is 2×2 matrix)

we assume that this relation is true for λ

$$\lambda^{50} = \alpha_1 \lambda + \alpha_0 \quad \text{— (2)}$$

for $\lambda = 1$, $(1)^{50} = \alpha_1(1) + \alpha_0 \Rightarrow \alpha_1 + \alpha_0 = 1$ — (3)

for $\lambda = 3$, $(3)^{50} = \alpha_1(3) + \alpha_0 \Rightarrow 3\alpha_1 + \alpha_0 = 3^{50}$ — (4)

(4) - (3) $\Rightarrow 2\alpha_1 = 3^{50} - 1 \Rightarrow \alpha_1 = \dfrac{3^{50}-1}{2}$

Sub in (3)

$$\alpha_1 + \alpha_0 = 1 \Rightarrow \alpha_0 = 1 - \alpha_1 = 1 - \frac{3^{50}-1}{2}$$

$$= \frac{3-3^{50}}{2}$$

Sub. α_0 & α_1 in (1)

$$A^{50} = \alpha_1 A + \alpha_0 I = \frac{3^{50}-1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{3-3^{50}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{50}-1+\frac{3-3^{50}}{2} & \frac{3^{50}-1}{2} \\ \frac{3^{50}-1}{2} & 3^{50}-1+\frac{3-3^{50}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{50}+1 & \frac{3^{50}-1}{2} \\ \cdot & \cdot \end{bmatrix} = \frac{1}{\cdot}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ \cdot & \cdot \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} \frac{3^{50}+1}{2} & \frac{3^{50}-1}{2} \\ \frac{3^{50}-1}{2} & \frac{3^{50}+1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

2. If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$, prove that $A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$

Soln:- Ch. eqn of A is $\begin{vmatrix} 2-\lambda & 3 \\ -3 & -4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-4-\lambda) + 9 = 0$

$-8 - 2\lambda + 4\lambda + \lambda^2 + 9 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda = -1, -1$ (repeated).

we use method - 2 here.

let $f(A) = A^{50} = \alpha_1 A + \alpha_0 I$ —— (1)

writing in terms of λ

$f(\lambda) = \lambda^{50} = \alpha_1 \lambda + \alpha_0$ —— (2)

put $\lambda = -1$, $(-1)^{50} = \alpha_1(-1) + \alpha_0$

$-\alpha_1 + \alpha_0 = 1$ —— (3)

differentiate eqn (2) wrt λ.

$50\lambda^{49} = \alpha_1$

put $\lambda = -1$, $50(-1)^{49} = \alpha_1 \Rightarrow \boxed{\alpha_1 = -50}$

Sub in (3), $-\alpha_1 + \alpha_0 = 1 \Rightarrow \alpha_0 = 1 + \alpha_1 = -49$

$\boxed{\alpha_0 = -49}$

Sub α_0, α_1 in (1),

$$A^{50} = \alpha_1 A + \alpha_0 I = -50A - 49I$$

$$= -50\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$$

3. Find e^A and 4^A if $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

Soln :- Ch. eqn of A is $\begin{vmatrix} \frac{3}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2}-\lambda \end{vmatrix} = 0$

$$\left(\frac{3}{2}-\lambda\right)^2 - \frac{1}{4} = 0$$

$$\frac{9}{4} - 3\lambda + \lambda^2 - \frac{1}{4} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 1, 2$$

we will use method - 2.

let $f(A) = e^A = \alpha_1 A + \alpha_0 I$ —— (1)

writting in terms of λ

$$e^\lambda = \alpha_1 \lambda + \alpha_0$$

put $\lambda = 1$, $\quad e = \alpha_1 + \alpha_0$ —— (2)

put $\lambda = 2$, $\quad e^2 = 2\alpha_1 + \alpha_0$ —— (3)

(3) − (2) $\Rightarrow$ $\boxed{\alpha_1 = e^2 - e}$

$e^2 - e + \alpha_0$

$③ - ②$

sub in ② $\Rightarrow$ $e = e^2 - e + \alpha_0$

$\Rightarrow \boxed{\alpha_0 = 2e - e^2}$

Sub in ① $e^A = \alpha_1 A + \alpha_0 I$

$$= (e^2-e)\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} + (2e-e^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3(e^2-e)}{2} + 2e - e^2 & \frac{e^2-e}{2} \\ \frac{e^2-e}{2} & \frac{3}{2}(e^2-e) + 2e - e^2 \end{bmatrix}$$

$$e^A = \begin{bmatrix} \frac{e^2+e}{2} & \frac{e^2-e}{2} \\ \frac{e^2-e}{2} & \frac{e^2+e}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^2+e & e^2-e \\ e^2-e & e^2+e \end{bmatrix}$$

Replacing e by 4

$$4^A = \frac{1}{2}\begin{bmatrix} 4^2+4 & 4^2-4 \\ 4^2-4 & 4^2+4 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

1/3/2022 1:14 PM

4. If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that $3 \tan A = A \tan 3$

Soln :- Ch. eqn of A is $\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 9 = 0$$

Eigen values $\rightarrow$ $\lambda = \pm 3$

Let $f(A) = \tan A = \alpha_1 A + \alpha_0 I$ —— (1)

writing in terms of λ

$$\tan\lambda = \alpha_1\lambda + \alpha_0 \quad \text{—— (2)}$$

put $\lambda = 3$, $\tan 3 = 3\alpha_1 + \alpha_0$ —— (3)

put $\lambda = -3$, $\tan(-3) = -3\alpha_1 + \alpha_0$

$-\tan 3 = -3\alpha_1 + \alpha_0$ —— (4)

Adding (3) & (4) $\Rightarrow$ $2\alpha_0 = 0$ $\Rightarrow$ $\boxed{\alpha_0 = 0}$

sub $\alpha_0 = 0$ in (3) $\Rightarrow$ $\tan 3 = 3\alpha_1$

$\Rightarrow \boxed{\alpha_1 = \frac{1}{3}\tan 3}$

sub α_1 & α_0 in (1)

$$\tan A = \alpha_1 A + \alpha_0 I = \left(\frac{1}{3}\tan 3\right)A + 0I$$

$$\Rightarrow 3\tan A = (\tan 3)A$$

$$\Rightarrow 3\tan A = A\tan 3$$

Hence proved.

5. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{50}

5. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{50}

Soln :- Ch. eqn of A is $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\left[\lambda^2-1\right] = 0$$

$$\lambda^2 - 1 - \lambda^3 + \lambda = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Eigen values are $\lambda = -1, 1, 1$

Let $f(A) = A^{50} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ —— (1)

writing in terms of λ

$$\lambda^{50} = \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 \quad \text{—— (2)}$$

put $\lambda = -1$, $(-1)^{50} = \alpha_2(-1)^2 + \alpha_1(-1) + \alpha_0$

$$1 = \alpha_2 - \alpha_1 + \alpha_0 \quad \text{—— (3)}$$

put $\lambda = 1$, $(1)^{50} = \alpha_2(1)^2 + \alpha_1(1) + \alpha_0$

$$1 = \alpha_2 + \alpha_1 + \alpha_0 \quad \text{—— (4)}$$

diff. (2) wrt λ

$$50\lambda^{49} = 2\alpha_2\lambda + \alpha_1$$

put $\lambda = 1$, $50(1)^{49} = 2\alpha_2(1) + \alpha_1$

$$50 = 2\alpha_2 + \alpha_1 \quad \text{--- (5)}$$

Solving (3), (4) & (5) we get.

$$\alpha_2 = 25, \quad \alpha_1 = 0, \quad \alpha_0 = -24$$

Sub. $\alpha_0, \alpha_1, \alpha_2$ in eq^n (1)

$$A^{50} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I = 25A^2 - 24I$$

$$A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

6. Show that $\cos 0_{3\times3} = I_{3\times3}$

Sol^n :- $O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{vmatrix} 0-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0$

$$\lambda^3 = 0.$$

Eigen values of $O_{3\times3}$ are $\lambda = 0, 0, 0.$

Let $\cos O_{3\times3} = \alpha_2 O^2_{3\times3} + \alpha_1 O_{3\times3} + \alpha_0 O_{3\times3}$ --- (1)

writing in terms of λ

$$\cos \lambda = \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 \quad \text{--- (2)}$$

put $\lambda = 0$, $\cos 0 = \alpha_2 (0)^2 + \alpha_1 (0) + \alpha_0$

$$\boxed{1 = \alpha_0}$$

differentiating (2) wrt λ

differentiating ② wrt λ

$$-\sin\lambda = 2\alpha_2\lambda + \alpha_1 \quad \text{——— ③}$$

put $\lambda = 0$, $\quad -\sin 0 = 2\alpha_2(0) + \alpha_1$

$$\Rightarrow \boxed{\alpha_1 = 0}$$

differentiating ③ wrt λ

$$-\cos\lambda = 2\alpha_2$$

put $\lambda = 0$, $\quad -\cos 0 = 2\alpha_2$

$$\Rightarrow \boxed{\alpha_2 = -\frac{1}{2}}$$

sub $\alpha_0, \alpha_1, \alpha_2$ in eqn ①

$$\cos O_{3\times3} = \alpha_2 O_{3\times3}^2 + \alpha_1 O_{3\times3} + \alpha_0 I$$

$$\cos O_{3\times3} = \alpha_0 I$$

$$\therefore \boxed{\cos O_{3\times3} = I}$$

<u>H.W</u> prove that $\sin O_{3\times3} = O_{3\times3}$.

Minimal Polynomial

Monday, January 3, 2022 1:42 PM

$f(x) = x^{10} - 9x^7 + 6x^3 - x^2 + 1$

$f(A) = A^{10} - 9A^7 + 6A^3 - A^2 + I = 0$

MINIMAL POLYNOMIAL AND MINIMAL EQUATION OF A MATRIX

Let $f(x)$ be a polynomial in x and A be a square matrix of order n.

If $f(A) = 0$ then we say that $f(x)$ **annihilates the matrix** A.

We know that by Caley- Hamilton theorem every matrix satisfies its characteristic equation.

Hence, the characteristic polynomial of the matrix A annihilates A.

$\lambda^3 - 3\lambda^2 + 5\lambda - 6 = 0$

$A^3 - 3A^2 + 5A - 6I = 0$

MONIC POLYNOMIAL:

A polynomial in x, in which the coefficient of the highest power of x is unity is called a **monic polynomial**.

Thus, $x^3 + 2x^2 + 3x - 7$ is a monic polynomial while $2x^3 - 3x^2 + 4x - 9$ is not a **monic polynomial**.

MINIMAL POLYNOMIAL OF A MATRIX:

The monic polynomial of lowest degree that annihilates a matrix A is called **minimal polynomial** of A.

Further, if $f(x)$ is the minimal polynomial of A then the equation $f(x) = 0$ is called the **minimal equation** of the matrix A.

$A \to 3\times3$

deg - 3 $\to$ deg 2 — deg 1

λ^3 - - - - - Characteristic poly

If a matrix is of order n then its characteristic polynomial is of degree n.

We know that the characteristic polynomial of A annihilates A. Hence, the degree of minimal polynomial of A cannot be greater than n.

Note: **(i)** Minimal polynomial of a matrix is unique

(ii) Minimal polynomial of a matrix is a divisor of every polynomial that annihilates this matrix

(iii) Minimal polynomial of a matrix is a divisor of the characteristic polynomial of that matrix

all the roots of minimal poly = some roots of ch. poly.

(iv) Null matrix is only matrix whose minimal polynomial is x

$f(x) = x$
$f(0) = 0$

(v) Unit matrix is the only matrix whose minimal polynomial is $(x - 1)$

Identity

$f(x) = x - I$
$f(I) = I - I = 0.$

DEROGATORY AND NON – DEROGATORY MATRICES:

An n – rowed square matrix is said to be **derogatory or Non – derogatory** according as the degree of its

Identity $f(I) = I$

DEROGATORY AND NON – DEROGATORY MATRICES:

An n – rowed square matrix is said to be **derogatory or Non – derogatory** according as the degree of its minimal equation is less than or equal to n.

A

↓

Minimal poly.

deg = n (ch. poly = min poly) → Non derogatory

deg < n → Derogatory

① If all the eigen values are distinct then characteristic polynomial is equal to the minimal polynomial and the matrix is Non- derogatory

② If there are repeated eigen values, then we need to check whether we can find a lower deg. polynomial which annhilates the matrix and then decide about the minimal poly.

SOME SOLVED EXAMPLES:

1. Show that the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is derogatory

Sol^n :- Ch. eq^n of A is $\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$\therefore$ Eigen values of A are $\lambda = 1, 2, 2$

let us now find the minimal polynomial of A.

we know that each eigen value of A is a root of minimum polynomial of A.

So if $f(x)$ is the minimal polynomial of A then $(x-1)$ and $(x-2)$ are the facters of $f(x)$

let us see whether the polynomial $(x-1)(x-2) = x^2 - 3x + 2$ annhilates A.

ie to check $A^2 - 3A + 2I = 0$

$$A^2 = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}^2 = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix}$$

$$\therefore A^2 - 3A + 2I = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 - 3A + 2I = 0.$$

$\therefore f(x) = x^2 - 3x + 2$ annhilates A.

Thus $f(x)$ is the monic polynomial of lowest degree that annhilates A. Hence $f(x)$ is the minimal polynomial of A.

Since its degree is less than the order of A, A is derogatory.

1/4/2022 10:37 AM

2. Show that the matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ is non- derogatory

Soln :- ch. eqn of A is $\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$\therefore$ Eigen values of A are $\lambda = -2, 1, 3$

All the eigen values are distinct.

$\therefore$ $f(x) = (x+2)(x-1)(x-3)$ is the minimal polynomial

$\therefore$ deg of minimal polynomial is equal to the order of the matrix. Hence the matrix is non - derogatory.

Q.3 :- $A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$$

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

Eigen values $\lambda = 2, \underline{2}, \underline{4}$

$f(x) = (x-2)(x-4) = x^2 - 6x + 8$

check whether $f(x)$ annihilates A.

check $A^2 - 6A + 8I = 0$

$$A^2 - 6A + 8I = \begin{bmatrix} 4 & -18 & 18 \\ 0 & 10 & -6 \\ 0 & -6 & 10 \end{bmatrix} - 6\begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore f(x) = x^2 - 6x + 8$ is the minimal polynomial

deg $f(x)$ < order of A

$\therefore$ A is derogatory.

Ex:- Find the Symmetric matrix $A_{3\times3}$ having the eigen values $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 15$ with the corresponding eigen vectors $X_1 = [1, 2, 2]'$, $X_2 = [-2, -1, 2]'$ and X_3.

Soln:- let $X_3 = [x_1, x_2, x_3]'$ be the third eigen vector corresponding to eigen value $\lambda = 15$.

Since the required matrix A is Symmetric and all eigenvalues are distinct, the three eigenvectors corresponding to 3 eigenvalues are orthogonal.

X_3 is orthogonal to X_1 & X_2

$X_1 \cdot X_3 = 0 \Rightarrow [1\ 2\ 2] \cdot [x_1\ x_2\ x_3] = 0$

$\Rightarrow x_1 + 2x_2 + 2x_3 = 0$

$X_2 \cdot X_3 = 0 \Rightarrow [-2, -1, 2] \cdot [x_1, x_2, x_3] = 0$

$\Rightarrow -2x_1 - x_2 + 2x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{-x_2}{6} = \frac{x_3}{3}$$

$\therefore X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 15$

Since A is symmetric, it is orthogonally similar to a diagonal matrix D.

There exists an orthogonal matrix P such that

$P^{-1}AP = D$ ie $A = PDP^{-1} = PDP^t$

(P is orthogonal $\Rightarrow P^{-1} = P^t$)

Since P is an orthogonal matrix, we divide each vector by its norm $X_1 = [1\ 2\ 2]$

$\|X_1\| = \sqrt{1+4+4} = 3$

$\|X_2\| = \sqrt{4+1+4} = 3$

$$\|x_3\| = \sqrt{4+4+1} = 3$$

$$\therefore P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$A = PDP^t = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

<u>Note</u> : $A = MDM^{-1}$ $\quad M = [x_1 \quad x_2 \quad x_3]$

M^{-1} ?