Zero and Identity transformations :

For vector spaces V and W (over $\mathbb{R}$ or $\mathbb{C}$), we define

$T_0 : V \to W$ by $T_0(x) = 0$ for all $x \in V$.

Then, for any vectors $x, y \in V$ and scalar $\alpha \in \mathbb{R}$ or $\mathbb{C}$, we have

$$T_0(x+y) = 0 = 0+0 = T_0(x) + T_0(y)$$

$$\text{and } T_0(\alpha x) = 0 = \alpha \cdot 0 = \alpha T_0(x)$$

Thus, T_0 is linear. We call T_0 the zero transformation.

Now, we define

$$I_V : V \to V \text{ by } I_V(x) = x, \ \forall x \in V.$$

Then, for any vectors $x, y \in V$ and scalar $\alpha \in \mathbb{R}$ or $\mathbb{C}$ we have

$$I_V(x+y) = x+y = I_V(x) + I_V(y)$$

$$\text{and } I_V(\alpha x) = \alpha x = \alpha I_V(x)$$

Thus, I_V is linear. We call I_V the identity transformation.

Theorem: Let V and W be vector spaces over a R or C. Let $\{v_1, v_2 \dots v_n\}$ be a basis of V a let $w_1, w_2 \dots w_n$ be any vectors in W. Then there exists a unique linear transformation $T: V \to W$ such that

$$T(v_1) = w_1, \quad T(v_2) = w_2 \quad \dots \quad T(v_n) = w_n.$$

Matrices as Linear transformations:

Let A be any real $m \times n$ matrix. A determines a transformation

$$T_A : R^n \to R^m \text{ by}$$

$$T_A(x) = Ax$$

where the vectors in R^n and R^m are written as columns.

→ we show T_A is linear. By matrix multiplication,

$$T_A(x+y) = A(x+y) = Ax + Ay = T_A(x) + T_A(y)$$

and $T_A(\alpha x) = A(\alpha x) = \alpha(Ax) = \alpha T_A(x)$

Thus, the matrix transformation T_A (or A) is linear.

Null space (or kernel) and Range space (or image) of a linear transformation:

Let V and W be vector spaces, and let $T: V \to W$ be linear.

→ We define the null space (kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$

ie
$$\boxed{N(T) = \{x \in V : T(x) = 0\}}$$

→ We define the range (image) $R(T)$ of T to be the subset of W consisting of all images (under T) of vectors in V

ie
$$\boxed{R(T) = \{T(x) : x \in V\}}$$

<u>Exp ①</u>: Let V and W be vector spaces, and let $I: V \to V$ and $T_0 : V \to W$ be the identity and zero transformations resp. Then

$$N(I) = \{0\}, \text{ and } R(I) = V$$

also $N(T_0) = V$, and $R(T_0) = \{0\}$.

Exp ②: Let $T: R^3 \to R^3$ be the projection of a vector v into the xy-plane

ie
$$T(x, y, z) = (x, y, 0)$$

clearly the image of T is the entire xy-plane.

ie the points of the form $(x, y, 0)$.

Moreover, the kernel of T is the z-axis

ie the points of the form $(0, 0, z)$.

Thus

$$\text{Im}(T) \text{ or } R(T) = \{(x, y, z) : z = 0\} = xy\text{-plane}$$

$$\ker(T) \text{ or } N(T) = \{(x, y, z) : x = 0, y = 0\} = z\text{-axis}$$

Exp ③: Let $T: R^3 \to R^3$ be the linear transformation that rotates a vector v about the z-axis through an angle θ ie

$$T(x, y, z) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z)$$

Soln: Observe that the distance of a vector v from the origin O does not change under the rotation, and so only the zero vector 0 is mapping into the zero vector

Thus $N(T) = \ker(T) = \{0\}$

On the other hand, every vector u in R^3 is the image of a vector v in R^3 that can be obtained by rotating u back by an angle θ.

Thus $R(T) = \text{Im}(T) = R^3$, entire space.

Theorem: Let V and W be vector spaces and $T: V \to W$ be linear transformation.

Then $N(T)$ and $R(T)$ are subspaces of V and W, respectively.

Proof: To clarify the notation, we use the symbols 0_V and 0_W to denote the zero vectors of V and W resp.

Since $T(0_V) = 0_W \Rightarrow 0_V \in N(T)$.

Let $x, y \in N(T)$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$).

Then $T(x+y) = T(x) + T(y)$ ($\because$ T is linear)

$T(x+y) = 0_W + 0_W = 0_W$

$\Rightarrow (x+y) \in N(T)$

and $T(\alpha x) = \alpha T(x) = \alpha \cdot 0_W = 0_W$

$\Rightarrow \alpha x \in N(T)$

Hence $x+y \in N(T)$ and $\alpha x \in N(T)$

$\therefore$ $N(T)$ is a subspace of V.

Because $T(0_V) = 0_W$, we have that $0_W \in R(T)$.

Now let $x, y \in R(T)$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$. Then there exist v and w in V such that $T(v) = x$ and $T(w) = y$.

So $T(u+v) = T(u) + T(v) = x + y$ and $T(\alpha u) = \alpha T(u) = \alpha x$

Thus $x+y \in R(T)$ and $\alpha x \in R(T)$

So, $R(T)$ is a subspace of $\underline{\underline{W}}$.

Kernel and Image of Matrix transformations :

Here

Let A be any $m \times n$ matrix over a field R or C viewed as a linear map

$$A: R^n \to R^m.$$

Then

$$\ker(A) = N(A) = \text{null sp}(A)$$

and $\text{Im}(A) = \text{colsp}(A)$

Here colsp (A) denotes the column space of A, and null sp (A) denotes the null space of A.

Exp ①: Consider the matrix mapping

$$A: R^4 \to R^3, \text{ where } A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$$

Find ~~the~~ basis and the dimension of

(a) the image of A.
(b) the kernel of A.

solution: (a) The column space of A is equal Im (A).
Now reduce A^T to echelon form.

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{(1, 1, 3), (0, 1, 2)\}$ is a basis of $\text{Im}(A)$, and $\dim(\text{Im}(A)) = 2$

Here $Ker(A)$ is the solution space of the homogeneous system $AX = 0$; where $X = (x, y, z, t)^T$.

Thus, reduce the matrix A of coefficient to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{matrix} x + 2y + 3z + t = 0 \\ y + 2z - 3t = 0 \end{matrix}$$

The free variables are z and t. Thus, $\dim(\ker A) = 2$

(i) set $z = 1$, $t = 0$ to get the solution $(1, -2, 1, 0)$

(ii) set $z = 0$, $t = 1$ to get the solution $(-7, 3, 0, 1)$!

Thus, $\{(1, -2, 1, 0), (-7, 3, 0, 1)\}$ form a basis for $\ker(A)$

Exp ②: Let $T: R^4 \to R^3$ be the linear transformation defined by

$$T(x, y, z, t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$$

Find a basis and the dim of ⓐ the image of T
ⓑ the kernel of T.

Solⁿ: ⓐ Find the images of the usual basis of R^4

$$F(1,0,0,0) = (1, 1, 1)$$
$$F(0,1,0,0) = (-1, 0, 1)$$
$$F(0,0,1,0) = (1, 2, 3)$$
$$F(0,0,0,1) = (1, -1, -3)$$

$\because$ Image vectors span $Im\ F$.

Hence, form the matrix whose rows are these image vectors, and row reduce to echelon form

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $(1,1,1)$ $(0,1,2)$ form a basis for $Im(T)$.

Hence $\dim(Im(A)) = \underline{2}$

ⓑ Set $T(v) = 0$, where $v = (x, y, z, t)$

ie $T(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$
$= (0,0,0)$.

set corresponding entries equal to each other to form the following homo. system whose solution space is $\ker(T)$.

$$\begin{aligned} x-y+z+t &= 0 \\ x+2z-t &= 0 \\ x+y+3z-3t &= 0 \end{aligned} \sim \begin{aligned} x-y+z+t &= 0 \\ y+z-2t &= 0 \\ 2y+2z-4t &= 0 \end{aligned}$$

or
$$\begin{aligned} x-y+z+t &= 0 \\ y+z-2t &= 0 \end{aligned}$$

The free variables are z and t. Hence, $\dim(\ker T) = 2$

(i) set $z=-1$, $t=0$ to obtain $(2,1,-1,0)$
(ii) set $z=0$, $t=1$ to obtain $(1,2,0,1)$

Thus, $(2,1,-1,0)$ and $(1,2,0,1)$ form a basis of $\ker(T)$.

Dimension Theorem

Let V and W be vector spaces, and let $T: V \to W$ be linear. If V is finite-dimensional, then

$$\boxed{\text{Rank}(T) + \text{Nullity}(T) = \dim(V)}$$

where $\text{Rank}(T) = \dim(\text{Im}(T))$

$\text{Nullity}(T) = \dim(\ker(T))$

Theorem: Let V and W be vector spaces, and let $T: V \to W$ be linear. Then T is one-to-one iff $N(T) = \{0\}$.

→ Theorem: Let V and W be vector spaces of equal (finite) dimension, and let $T: V \to W$ be linear. Then the following are equivalent.

(a) T is one-to-one

(b) T is onto

(c) $\text{Rank}(T) = \dim(V)$

Note: If V is not finite-dimensional and $T: V \to V$ is linear, then it does not follow that one-to-one and onto are equivalent.

→ Theorem: Let V and W be vector spaces, and let $T: V \to W$ be linear. If $\beta = \{v_1, v_2 \dots v_n\}$ is a basis for V, then

$R(T) = \text{span}\{T(v_1), T(v_2) \dots T(v_n)\}$

Exp ①: Define the linear transformation

$$T : P_2(R) \to M_{2\times 2}(R) \text{ by}$$

$$T(f(x)) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Find $R(T)$ and its dim.

Sol^n. $\because$ $\beta = \{1, x, x^2\}$ is a basis for $P_2(R)$ we have

$$R(T) = \text{span}(\{T(1), T(x), T(x^2)\})$$

$$= \text{span}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right)$$

$$= \text{span}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right)$$

Thus we have found a basis for $R(T)$, and so $\dim(R(T)) = 2$

Exp ②: Let $T : P_2(R) \to P_3(R)$ be the linear trans. defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt.$$

Sol^n. $\because$ $R(T) = \text{span}(\{T(1), T(x), T(x^2)\})$

$$= \text{span}\left(\{3x,\ 2+\tfrac{3}{2}x^2,\ 4x+x^3\}\right)$$

Since $\{3x,\ 2+\frac{3}{2}x^2,\ 4x+x^3\}$ is L.I.

$\therefore$ $\text{Rank}(T) = 3$

Since $\dim(P_3(R)) = 4$. From dim theorem,

$$\text{nullity}(T) = 3-3 = 0$$

$\therefore$ $N(T) = \{0\}$ $\Rightarrow$ T is onto-to-one.

Let

Matrix Representation of a linear transformation

Let T be a linear transformation from a vector space V into itself, and suppose $S = \{u_1, u_2, \dots u_n\}$ is a basis of V. Now $T(u_1), T(u_2) \dots T(u_n)$ are vectors in V, and so each is a linear combination of the vectors in the basis S: say

$$T(u_1) = a_{11} u_1 + a_{12} u_2 + \dots + a_{1n} u_n$$
$$T(u_2) = a_{21} u_1 + a_{22} u_2 + \dots + a_{2n} u_n$$
$$\vdots$$
$$T(u_n) = a_{n1} u_1 + a_{n2} u_2 + \dots + a_{nn} u_n$$

<u>Definition</u>: The transpose of the above matrix of coefficients, denoted by $[T]_S$, is called the matrix representation of T relative to the basis S, or simply the matrix of T in the basis S.

→ Using the coordinate (column) vector notation, the matrix representation of T may be written in the form

$$[T]_S = \left[[T(u_1)]_S, [T(u_2)]_S \dots [T(u_n)]_S \right]$$

Exp ①: Let $T: R^2 \to R^2$ be the linear operator de

by $\quad T(x, y) = (2x + 3y,\ 4x - 5y)$.

ⓐ Find the matrix representation of T relative to the basis $S = \{u_1, u_2\} = \{(1, 2), (2, 5)\}$.

ⓑ Find the matrix representation of T relative to the basis $S' = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$.

Solution: ⓐ step①: First find $T(u_1)$, and then write it as a linear combination of the basis vectors u_1 and u_2. we have

$$T(u_1) = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -6 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + 2y &= 8 \\ 2x + 5y &= -6 \end{aligned}$$

Solve the system to obtain: $x = 52,\ y = -22$

Hence $\quad T(u_1) = 52u_1 - 22u_2$

step②: Find $T(u_2)$, and then write it as a linear combination of u_1 and u_2:

$$T(u_2) = T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 19 \\ -17 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + 2y &= 19 \\ 2x + 5y &= -17 \end{aligned}$$

solve the system to obtain: $x = 129,\ y = -55$

Thus $\quad T(u_2) = 129u_1 - 55u_2$

Now write the co-ordinates of $T(u_1)$ and $T(u_2)$ as columns to obtain the matrix

$$[T]_S = \begin{bmatrix} 52 & 129 \\ -22 & -55 \end{bmatrix}$$

(b) Find $T(e_1)$ and write it as a linear combination of the usual basis vectors e_1 and e_2, and then find $T(e_2)$ and write it as a linear combination of e_1 and e_2. We have

$$T(e_1) = T(1,0) = (2,4) = 2e_1 + 4e_2$$
$$T(e_2) = T(0,1) = (3,-5) = 3e_1 - 5e_2$$

so $[T]_{S'} = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$

<u>Note</u>: The coordinates of $T(e_1)$ and $T(e_2)$ form the columns, not the rows, of $[T]_{S'}$.

<u>Exp (2)</u>: Let V be the vector space of functions with basis $S = \{\sin t, \cos t, e^{3t}\}$, and $T: V \to V$ be the differential operator defined by

$$T(f(t)) = f'(t).$$

Find matrix representation of T.

Solution: $T(\sin t) = \cos t = 0(\sin t) + 1(\cos t) + $

$T(\cos t) = -\sin t = -1(\sin t) + 0(\cos t) + 0(e$

$T(e^{3t}) = 3e^{3t} = 0(\sin t) + 0(\cos t) + 3(e^{3t}$

$$\therefore [T]_s = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Matrix Mappings and Their Matrix Representation:

Consider the following matrix A, which may be viewed as a linear transformation on R^2 to R, and basis S of R^2

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix} \text{ and } S = \{u_1, u_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$$

We find the matrix representation of A relative to the basis S.

Step ①: write $A(u_1)$ as a linear combination of u_1 and u_2, we have

$$A(u_1) = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad x + 2y = -1$$
$$2x + 5y = -6$$

$\Rightarrow x = 7, y = -4$ $\quad\therefore A(u_1) = 7u_1 - 4u_2$

②: write $A(u_2)$ as a linear combination of u_1 and u_2, we get

$$A(u_2) = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad x + 2y = -4$$
$$2x + 5y = -7$$

$\Rightarrow \quad x = -6, \ y = 1$ Thus $A(u_2) = -6u_1 + u_2$

writting the co-ordinates of $A(u_1)$ and $A(u_2)$ as columns gives us the following matrix representation of A:

$$[A]_S = \begin{bmatrix} 7 & -6 \\ -4 & 1 \end{bmatrix}$$

<u>Remark</u>: Suppose we want to find the matrix representation of A relative to the usual basis $E = \{e_1, e_2\}$ $= \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ of R^2. we have

$$A(e_1) = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3e_1 + 4e_2$$

$$A(e_2) = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} = -2e_1 - 5e_2$$

Thus $[A]_E = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix} = A$

→ The matrix representation of any $n \times n$ square matrix A over a field R or C relative to the usual basis E of R^n or C^n is the matrix A itself.

ie

$$[A]_E = A$$

Algorithm for finding Matrix Representations:

Algorithm: The input is a linear operator T on a vector space V and a basis $S = \{u_1, u_2 \cdots u_n\}$ of V. The output is the matrix representation $[T]_S$.

Step ①: Find a formula for the co-ordinates of an arbitrary vector v relative to the basis S.

Step ②: Repeat for each basis vector u_k in S:

(i) Find $T(u_k)$

(ii) Write $T(u_k)$ as a linear combination of the basis vectors $u_1, u_2 \cdots u_n$.

Step ③: Form the matrix $[T]_S$ whose columns are the coordinate vector in step 2 (ii).

Exp Let $T: R^2 \to R^2$ be defined by

$$T(x, y) = (2x+3y, 4x-5y)$$

Find the matrix representation $[T]_S$ of T relative to the basis $S = \{u_1, u_2\} = \{(1,-2), (2,-5)\}$

Soln: **Step ①:** First find the coordinates of $(a, b) \in R^2$ relative to the basis S. we have

$$\begin{bmatrix} a \\ b \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -5 \end{bmatrix} \Rightarrow \begin{matrix} x+2y = a \\ -2x-5y = b \end{matrix}$$

Solving for x and y in terms of a and b

$$x = 5a + 2b$$
$$y = -2a - b$$

Thus $\quad (a, b) = (5a + 2b) u_1 + (-2a - b) u_2$

Step ②: Now we find $T(u_1)$ and write it as a linear combination of u_1 and u_2 using the above formula for (a, b), and then we repeat the process for $T(u_2)$. We have

$$T(u_1) = T(1, -2) = (-4, 14) = 8u_1 - 6u_2$$
$$T(u_2) = T(2, -5) = (-11, 33) = 11u_1 - 11u_2$$

Step ③: Finally, we write the co-ordinates of $T(u_1)$ and $T(u_2)$ as columns to obtain the required matrix:

$$[T]_S = \begin{bmatrix} 8 & 11 \\ -6 & -11 \end{bmatrix}$$

Change of Basis :

Let $S = \{u_1, u_2 \dots u_n\}$ be a basis of a vector space V, and let $S' = \{v_1, v_2 \dots v_n\}$ be another basis. (For reference, we will call S the "old" basis and S' the "new" basis). Because S is a basis, each vector in the "new" basis S' can be written uniquely as a linear combination of the vectors in S; say,

$v_1 = a_1$
$v_2 = a_2$

$$v_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$
$$v_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$
$$\vdots$$
$$v_n = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$$

Let P be the transpose of the above matrix of coefficients i.e. let $P = [p_{ij}]$, where $p_{ij} = a_{ji}$. Then P is called the change-of-basis matrix from the old basis S to the new basis S'.

Remark ① The above change-of-basis matrix P may also be viewed as the matrix whose columns are the coordinate column vectors of the new basis vectors v_i relative to the old basis S

$$P = [\,[v_1]_S, [v_2]_S \cdots [v_n]_S\,]$$

② Analogously, there is a change-of-basis matrix Q from the 'new' basis S' to the old basis S. Similarly, Q may be viewed as the matrix whose columns are the coordinate column vectors of the old basis vectors u_i relative to the new basis S'.

$$Q = [\,[u_1]_{S'}, [u_2]_{S'} \cdots [u_n]_{S'}\,]$$

Theorem: Let P and Q be the above change-of-basis matrices. Then $Q = P^{-1}$.

Here ... and ... new ba...

Exp ①: Consider the following two basis of R^2:

$$S = \{u_1, u_2\} = \{(1,2), (3,5)\}$$

and $$S' = \{v_1, v_2\} = \{(1,-1), (1,-2)\}$$

ⓐ Find the change-of-basis matrix P from S to the new basis S'.

ⓑ Find the change-of-basis matrix Q from new basis S' to the 'old' basis S.

Soln: ⓐ Write each of the new basis vectors of S' as a linear combination of the original basis vectors u_1 and u_2 of S. We have

$$(1,-1) = x u_1 + y u_2 = x(1,2) + y(3,5)$$

$$\Rightarrow \begin{array}{l} x + 3y = 1 \\ 2x + 5y = -1 \end{array} \Rightarrow x = -8, \; y = 3$$

Thus $$v_1 = -8u_1 + 3u_2$$

and $$(1,-2) = x u_1 + y u_2 = x(1,2) + y(3,5)$$

$$\Rightarrow \begin{array}{l} x + 3y = 1 \\ 2x + 5y = -2 \end{array} \Rightarrow x = -11, \; y = 4$$

Thus $$v_2 = -11u_1 + 4u_2$$

Hence, $$P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$$

Here we write each of the old basis vectors u_1 and u_2 of S' as a linear combination of the new basis vectors v_1 and v_2 of S'. We have

$$u_1 = (1,2) = x(1,-1) + y(1,-2)$$

$$\Rightarrow \begin{aligned} x + y &= 1 \\ -x - 2y &= 2 \end{aligned} \Rightarrow x = 4, \; y = -3$$

$$\therefore u_1 = 4v_1 - 3v_2$$

and $u_2 = (3,5) = x(1,-1) + y(1,-2)$

$$\Rightarrow \begin{aligned} x + y &= 3 \\ -x - 2y &= 5 \end{aligned} \Rightarrow x = +11, \; y = -8$$

$$\therefore u_2 = 11v_1 - 8v_2$$

and hence $Q = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}$

Exp ②: Consider the following two basis of R^3:

$$E = \{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

and $E' = \{u_1, u_2, u_3\} = \{(1,0,1), (2,1,2), (1,2,2)\}$

(a) Find the change-of-basis matrix P from the basis E to the basis E'

(b) Find the change-of-basis matrix Q from the basis E' to the basis E.

Sol^n: (4) we write

$u_1 = (1,0,1) = 1\cdot e_1 + 0\cdot e_2 + 1\cdot e_3$

$u_2 = (2,1,2) = 2\cdot e_1 + 1\cdot e_2 + 2e_3$

$u_3 = (1,2,2) = 1\cdot e_1 + 2e_2 + 2e_3$

Hence $P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

(5) Here we write

$e_1 = (1,0,0) = xu_1 + yu_2 + zu_3$

$(1,0,0) = (x+2y+z,\ y+2z,\ x+2y+2z)$

$\Rightarrow \quad x+2y+z = 1$
$\quad\quad y+2z = 0$
$\quad\quad x+2y+2z = 0$

$\Rightarrow \quad x = -2,\ y = 2,\ z = -1$

$\therefore\ e_1 = -2u_1 + 2u_2 - u_3$

$e_2 = (0,1,0) = (x+2y+z,\ y+2z,\ x+2y+2z)$

$\Rightarrow \quad x+2y+z = 0$
$\quad\quad y+2z = 1$
$\quad\quad x+2y+2z = 0$

$\Rightarrow\ x = -2,\ y = 1,\ z = 0$

$\Rightarrow \therefore\ e_2 = -2u_1 + u_2$

and $e_3 = (0,0,1) = (x+2y+z,\ y+2z,\ x+2y+2z)$

$$x + 2y + z = 0$$
$$y + 2z = 0 \quad \Rightarrow \quad x = 3, \; y = -2, \; z = 1$$
$$x + 2y + 2z = 1$$

$$\therefore \quad e_3 = 3u_1 - 2u_2 + u_3$$

Hence, $Q = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$

we can verify that $Q = P^{-1}$

Inventible linear transformations :

Let V and W be vector spaces, and let $T: V \to W$ be linear. A function $U: W \to V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$.

→ If T has an inverse, then T is said to be inventible.

→ If T is inventible, then the inverse of T is unique and is denoted by T^{-1}.

The following facts hold for invertible functions T and U.

(1) $(TU)^{-1} = U^{-1}T^{-1}$

(2) $(T^{-1})^{-1} = T$

(3) Let $T: V \to W$ be a linear transformation, where V and W are finite dimensional spaces of equal dimension. Then T is invertible if and only if

$$\text{rank}(T) = \dim(V)$$

Exp Let $T: P_1(R) \to R^2$ be a L.T. defined by

$$T(a+bx) = (a, a+b).$$

Then $T^{-1}: R^2 \to P_1(R)$ is defined by

$$T^{-1}(c,d) = c + (d-c)x$$

Observe that T^{-1} is also linear.

Theorem: Let V and W be vector spaces, and let $T: V \to W$ be linear and invertible. Then $T^{-1}: W \to V$ is linear.

Proof: Let $w_1, w_2 \in W$, and α be any scalar. Since T is invertible, it is one-to-one and onto, so there exist unique vectors v_1 and v_2 in V such

such that

$$T(v_1) = w_1, \quad T(v_2) = w_2$$

$$T^{-1}(w_1 + w_2) = T^{-1}(T(v_1) + T(v_2))$$
$$= T^{-1}(T(v_1 + v_2))$$
$$= v_1 + v_2$$
$$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$$

and $T^{-1}(\alpha w_1) = T^{-1}\{\alpha T(v_1)\}$
$$= T^{-1}\{T(\alpha v_1)\}$$
$$= \alpha v_1$$
$$T^{-1}(\alpha w_1) = \alpha T^{-1}(w_1)$$

Hence T^{-1} is linear transformation.

Definition: A linear transformation $T: V \to W$ from a vector space V to a vector space W is called an isomorphism if it is invertible (or one-to-one and onto). In this case, we say V and W are isomorphism to each other.

Theorem: ① Two vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$

② Any n-dim. vector space V is isomorphic to the n-space R^n.

Properties of Matrix Representations:

Theorem: Let $T: V \to V$ be a linear transform, and let S be a finite basis of V. Then, for any vector v in V,

$$[T]_S [v]_S = [T(v)]_S.$$

Exp Let $T: R^2 \to R^2$ be a linear operator defined by

$$T(x,y) = (2x+3y,\ 4x-5y)$$

and $S = \{u_1, u_2\} = \{(1,-2),\ (2,-5)\}$

Let $v = (5,-7)$ and so $T(v) = (-11,\ 55)$

$\therefore$ $v = a u_1 + b u_2 \Rightarrow (5,-7) = a(1,-2) + b(2,-5)$

$$\Rightarrow \begin{aligned} a+2b &= 5 \\ -2a-5b &= -7 \end{aligned} \Rightarrow a = 11,\ b = -3$$

$$\therefore\ [v]_S = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

and $T(v) = (-11,\ 55) = a(1,-2) + b(2,-5)$

$$\Rightarrow \begin{aligned} a+2b &= -11 \\ -2a-5b &= 55 \end{aligned} \Rightarrow \begin{aligned} a &= 55 \\ b &= -33 \end{aligned}$$

$$\therefore\ [T(v)]_S = \begin{bmatrix} 55 \\ -33 \end{bmatrix}$$

also, we can find $[T]_S = \begin{bmatrix} 8 & 11 \\ -6 & -11 \end{bmatrix}$

$$\therefore\ [T]_S [v]_S = \begin{bmatrix} 8 & 11 \\ -6 & -11 \end{bmatrix} \begin{bmatrix} 11 \\ -3 \end{bmatrix} = \begin{bmatrix} 55 \\ -33 \end{bmatrix} = [T(v)]_S$$

Applications of change-of-Basis Matrix:

Thus.

Theorem: Let P be the change-of-basis matrix from basis S to a basis S' in a vector space V. Then, for any vector $v \in V$, we have

$$P[v]_{S'} = [v]_S$$

and hence

$$P^{-1}[v]_S = [v]_{S'}$$

Proof: Let $\dim V = 3$.

Suppose P is the change-of-basis matrix from the basis $S = \{u_1, u_2, u_3\}$ to the basis $S' = \{v_1, v_2, v_3\}$

$$v_1 = a_1 u_1 + a_2 u_2 + a_3 u_3$$
$$v_2 = b_1 u_1 + b_2 u_2 + b_3 u_3$$
$$v_3 = c_1 u_1 + c_2 u_2 + c_3 u_3$$

and hence
$$P = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Now suppose $v \in V$ and

$$v = k_1 v_1 + k_2 v_2 + k_3 v_3$$

ie
$$v = k_1(a_1 u_1 + a_2 u_2 + a_3 u_3) + k_2(b_1 u_1 + b_2 u_2 + b_3 u_3) + k_3(c_1 u_1 + c_2 u_2 + c_3 u_3)$$
$$= (a_1 k_1 + b_1 k_2 + c_1 k_3) u_1 + (a_2 k_1 + b_2 k_2 + c_2 k_3) u_2 + (a_3 k_1 + b_3 k_2 + c_3 k_3) u_3$$

Thus,

$$[v]_{S'} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \text{ and } [v]_S = \begin{bmatrix} a_1 k_1 + b_1 k_2 + c_1 k_3 \\ a_2 k_1 + b_2 k_2 + c_2 k_3 \\ a_3 k_1 + b_3 k_2 + c_3 k_3 \end{bmatrix}$$

Accordingly,

$$P[v]_{S'} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a_1 k_1 + b_1 k_2 + c_1 k_3 \\ a_2 k_1 + b_2 k_2 + c_2 k_3 \\ a_3 k_1 + b_3 k_2 + c_3 k_3 \end{bmatrix}$$

$$= [v]_S$$

Finally, multiplying the equation

$[v]_S = P[v]_{S'}$ by P^{-1}, we get

$$P^{-1}[v]_S = P^{-1}P[v]_{S'} = I[v]_{S'} = [v]_{S'}$$

<u>Theorem</u>: Let P be the change-of-basis matrix from a basis S to a basis S' in a vector space V. Then, for any linear operator T on V

$$[T]_{S'} = P^{-1}[T]_S P$$

i.e If A and B are the matrix representations of T relative to S and S' resp, then

$$B = P^{-1}AP.$$

Exp Consider the following two bases of R^3:

$E = \{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

and $S = \{u_1, u_2, u_3\} = \{(1,0,1), (2,1,2), (1,2,2)\}$

Soln: The change-of-basis matrix P from E to S and its inverse one

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

(a) write $v = (1, 3, 5)$ as a linear combination of u_1, u_2, u_3 or find $[v]_S$.

$$v = x u_1 + y u_2 + z u_3$$

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + 2y + z &= 1 \\ y + 2z &= 3 \\ x + 2y + 2z &= 5 \end{aligned}$$

$\Rightarrow x = 7, \ y = -5, \ z = 4$

$\therefore v = 7u_1 - 5u_2 + 4u_3$

$$\therefore [v]_S = \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}$$

On the other hand, we know that $[v]_E = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, because is the usual basis, and, Now

$$[v]_S = P^{-1}[v]_E = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}$$

Thus, again $\underline{v = 7u_1 - 5u_2 + 4u_3}$.

(b) Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix}$, which may be viewed as a linear operator on R^3. Find the matrix B that represents A relative to the basis S.

$$\therefore \quad A(u_1) = (-1, 3, 5) = 11u_1 - 5u_2 + 6u_3$$
$$A(u_2) = (1, 2, 9) = 21u_1 - 14u_2 + 8u_3$$
$$A(u_3) = (3, -4, 5) = 17u_1 - 8u_2 + 2u_3$$

and hence $B = \begin{bmatrix} 11 & 21 & 17 \\ -5 & -14 & -8 \\ 6 & 8 & 2 \end{bmatrix}$

On the other hand, because we know P and P^{-1}.

ie $B = P^{-1}AP = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

$$= \underline{\begin{bmatrix} 11 & 21 & 17 \\ -5 & -14 & -8 \\ 6 & 8 & 2 \end{bmatrix}}$$

Vector spaces of linear transformation:

Let V and W be two vector spaces. Let $L(V:W)$ denote the set of all linear transformations from V to W ie

$$L(V:W) = \{T : T \text{ is a linear transformation from } V \text{ into } W\}.$$

For, $S, T \in L(V:W)$ and $\alpha \in R$, define the sum $(S+T)$ and the scalar multiplication (αS) by

$$(S+T)(v) = S(v) + T(v)$$

and $(\alpha S)(v) = \alpha(S(v))$

for any $v \in V$.

Then clearly $(S+T)$ and (αS) belong to $L(V:W)$ so that $L(V:W)$ becomes a vector space.

Theorem: Let V and W be vector spaces with ordered bases β and β', respectively, and let $S, T : V \to W$ be linear. Then we have:

$$[S+T]_{\beta}^{\beta'} = [S]_{\beta}^{\beta'} + [T]_{\beta}^{\beta'}$$

and $[kS]_{\beta}^{\beta'} = k[S]_{\beta}^{\beta'}$.

Note $\dim(L(V:W)) = \dim(V) \cdot \dim(W)$.

Theorem: Let V, W and Z be vector spaces with ordered bases, β, β', β'' resp. Suppose that $S: V \to W$ and $T: W \to Z$ are linear transforms. Then

$$[T \circ S]_{\beta}^{\beta''} = [T]_{\beta'}^{\beta''} [S]_{\beta}^{\beta'}$$

Exp(1): Let β be the standard basis for R^3, and let $S, T: R^3 \to R^3$ be two linear transformations given by

$S(e_1) = (2, 2, 1)$ and $T(e_1) = (1, 0, 1)$
$S(e_2) = (0, 1, 2)$ and $T(e_2) = (0, 1, 1)$
$S(e_3) = (-1, 2, 1)$ and $T(e_3) = (1, 1, 2)$.

compute $[S+T]_\beta$, $[2T-S]_\beta$, and $[T \circ S]_\beta$.

Solution: $\because [S]_\beta = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

and $[T]_\beta = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$\therefore$ (i) $[S+T]_\beta = [S]_\beta + [T]_\beta = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$

(ii) $[2T-S]_\beta = 2[T]_\beta - [S_\beta] = \begin{bmatrix} 0 & 0 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

(iii) $[T \circ S]_\beta = [T]_\beta \cdot [S]_\beta = \begin{bmatrix} 3 & +2 & 0 \\ 3 & 3 & 3 \\ 6 & 5 & 3 \end{bmatrix}$

Exp ②: Let $T: P_2(R) \to P_2(R)$ be the L.T. defined

$$T(f(x)) = (3+x)f'(x) + 2f(x)$$

and $S: P_2(R) \to R^3$ defined by

$$S(a+bx+cx^2) = (a-b, a+b, c)$$

For a basis $\beta = \{1, x, x^2\}$ for $P_2(R)$ and the standard basis $\beta' = \{e_1, e_2, e_3\}$ for R^3, compute

(i) $[S]_\beta^{\beta'}$

(ii) $[T]_\beta$

(iii) $[S \circ T]_\beta^{\beta'}$.

Theorem: Let V and W be vector spaces with ordered basis β and β', resp., and let

$T: V \to W$ be an isomorphism. Then

$$[T^{-1}]_{\beta'}^{\beta} = \left([T]_\beta^{\beta'}\right)^{-1}.$$

Exp Let $T: R^3 \to R^3$ be the L.T. defined by

$$T(x_1, x_2, x_3) = (x_1 + 3x_2 - 2x_3, 2x_1 + 3x_2, x_2 - x_3)$$

Find T^{-1}.

Soln: Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis of R^3.

$T(e_1) = (1, 2, 0)$, $T(e_2) = (3, 3, 1)$, $T(e_3) = (-2, 0, -1)$

ined.

$$[T]_\beta = [T(e_1), T(e_2), T(e_3)] = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 3 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

inverse of matrix $[T]_\beta^{-1} = \begin{bmatrix} 3 & -1 & -6 \\ -2 & 1 & 4 \\ -2 & 1 & 3 \end{bmatrix}$

we know $(T^{-1})_\beta = [T_\beta]^{-1}$

$$T^{-1}(x_1, x_2, x_3) = \begin{bmatrix} 3 & -1 & -6 \\ -2 & 1 & 4 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 - x_2 - 6x_3 \\ -2x_1 + x_2 + 4x_3 \\ -2x_1 + x_2 + 3x_3 \end{bmatrix}$$

Definition: For any square matrices A and B. A is said to be similar to B if there exists a non-singular matrix P such that

$$B = P^{-1} A P$$

Exp Let $\beta = \{v_1, v_2, v_3\}$ be a basis for R^3 consisting of $v_1 = (1, 1, 0)$, $v_2 = (1, 0, 1)$ and $v_3 = (0, 1, 1)$. Let T be the linear transformation on R^3 given by the matrix

$$[T]_\beta = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

Let $\alpha = \{e_1, e_2, e_3\}$ be the standard basis. Find the transition matrix $[Id]_\alpha^\beta$ and $[T]_\alpha$.

Sol^n: Since $v_1 = e_1 + e_2$, $v_2 = e_1 + e_3$, $v_3 = e_2 + e_3$

we have

$$[Id]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and $[Id]_\alpha^\beta = \left([Id]_\beta^\alpha\right)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Therefore,

$$[T]_\alpha = [Id]_\beta^\alpha \, [T]_\beta \, [Id]_\alpha^\beta = \frac{1}{2} \begin{bmatrix} 4 & 2 & 2 \\ 3 & -1 & 1 \\ -1 & 1 & 7 \end{bmatrix}$$

Similarity :

Let $T: V \to W$ be a linear transformation on a vector space V with basis α and α' to another vector space W with basis β and β'. Then

$$[T]_{\alpha'}^{\beta'} = P^{-1} [T]_{\alpha}^{\beta} Q$$

where $Q = [Id_V]_{\alpha'}^{\alpha}$ and $P = [Id_W]_{\beta'}^{\beta}$ are the transition matrices.

Theorem: Let $T: V \to V$ be a linear transformation on a vector space V, and let α and β be ordered bases for V. Let $Q = [Id]_{\beta}^{\alpha}$ be the transition matrix from β to α. Then

(i) Q is invertible and $Q^{-1} = [Id]_{\alpha}^{\beta}$.

(ii) For any $v \in V$

$$[v]_{\alpha} = Q [v]_{\beta}$$

(iii) $$[T]_{\beta} = Q^{-1} [T]_{\alpha} Q.$$

Let $\beta = \{(1,0), (1,-1)\}$

$\beta' = \{(0,1), (1,1)\}$

For identity transformation

$$T(1,0) = (1,0) = -1(0,1) + 1(1,1)$$
$$T(1,-1) = (1,-1) = -2(0,1) + 1(1,1)$$

$$\therefore \ [Id]_{\beta}^{\beta'} = P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

Let $v = (0,3)$ $\quad \therefore [v]_{\beta} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$$\therefore \ [v]_{\beta'} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Definition: The matrix representation $[Id]_{\beta}^{\alpha}$ of the identity transformation $Id: V \to V$ with respect to any two basis α and β is called the transition matrix or the co-ordinate change matrix from β to α

$$[v]_{\alpha} = [Id]_{\beta}^{\alpha} [x]_{\beta}$$

Exp 1 Let $\beta = \{(1,0), (0,1)\}$ and $\beta' = \{(3,1), (-2,1)\}$

Then for identity transformation

$$T(3,1) = (3,1) = 3(1,0) + 1(0,1)$$
$$T(-2,1) = (-2,1) = -2(1,0) + 1(0,1)$$

The transition matrix from β' to β is

$$[Id]_{\beta'}^{\beta} = P = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

Let $v = (4,3)$ then $v = x(3,1) + y(-2,1)$

$$\Rightarrow \begin{array}{l} 3x - 2y = 4 \\ x + y = 3 \end{array}$$

$$\Rightarrow (x, y) = (2,1)$$

$$\therefore [v]_{\beta'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and so $[v]_{\beta} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

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m: Suppose that A represents a linear trans. $T: V \to V$ on a vector space V with respect to an ordered basis $\alpha = \{v_1, v_2 \dots v_n\}$, ie $[T]_\alpha = A$. If $B = Q^{-1}AQ$ for some non-singular matrix Q, then there exists a basis β for V such that $B = [T]_\beta$ and $Q = [Id]_\beta^\alpha$.

__Exp__ Let D be the differential operator on the vector space $P_2(R)$. Two ordered basis $\alpha = \{1, x, x^2\}$ and $\beta = \{1, 2x, 4x^2 - 2\}$ for $P_2(R)$. Find the associated matrix of T w.r.t α and the associated matrix of T w.r.t β. Are they similar ?.

__Soln__:

$$D(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$D(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$D(x^2) = 2x = 0\cdot 1 + 1\cdot x + 0\cdot x^2$$

$$[T]_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D(1) = 0$$
$$D(2x) = 2$$
$$D(4x^2-2) = 8x$$
$$= 0\cdot 1 + 4\cdot 2x + 0\cdot(4x^2-2)$$

$$[T]_\beta = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Now the transition matrix

$$Q = [Id]_\beta^\alpha = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and $Q^{-1} = [Id]_\alpha^\beta = \frac{1}{4}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

we can see that

$$[D]_\beta = [Id]_\alpha^\beta \, [D]_\alpha \, [Id]_\beta^\alpha$$
$$= Q^{-1} [D]_\alpha Q$$

Exp let $M = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

(i) Find the unique L.T. $T: \mathbb{R}^3 \to \mathbb{R}^2$ so that M is the matrix of T w.r.t. the standard basis
(ii) Find $T(x, y, z)$.

Soln.
$$T(1,0,0) = 4(1,0) + 0(0,1)$$
$$T(0,1,0) = 2(1,0) + 1(0,1)$$
$$T(0,0,1) = 1(1,0) + 3(0,1)$$

$$T(x,y,z) = T[(x(1,0,0) + y(0,1,0) + z(0,0,1)]$$
$$= x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$$
$$= (4x, 0) + (2y, y) + (z, 3z)$$
$$= (4x + 2y + z, \; y + 3z)$$

$$\alpha_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$\alpha_2 = \{(1,0), (0,1)\}$$