Eigen value Problem:

Let $A = (a_{ij})$ be a square matrix of order n. The matrix A may be singular or non-singular. consider the homogeneous system of equations

$$AX = \lambda X \quad \text{or} \quad (A - \lambda I)X = 0 \quad -(1)$$

where λ is a scalar and I is an identity matrix of order n. The homogeneous system of equations (1) always has a trivial solution. we need to find values of λ for which the homogeneous system (1) has non-trivial solution.

→ The values of λ, for which non-trivial solutions of the homogeneous system (1) exist, are called the eigen values or the characteristic values of A and the corresponding non-trivial solution vector X are called the eigen vectors or the characteristic vectors of A.

→ If X is a non-trivial solution of the homogeneous system (1), then αX, where α is any constant, is also a solution of the homogeneous system. Hence, an eigen vector is unique only upto a constant multiple.

The problem of determining the eigen values and the corresponding eigen vectors of a square matrix A is called an <u>eigen value problem</u>.

<u>Eigen values and Eigen vectors</u>:

If the homogeneous system (1) has a non-trivial solution, then the rank of the coefficient matrix $(A-\lambda I)$ is less than n, ie, the coefficient matrix must be singular. Therefore,

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad (2)$$

Expanding the determinant given in eqn (2), we obtain a polynomial of degree n in λ, which is of the form

$$P_n(\lambda) = |A-\lambda I| = (-1)^n[\lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} \cdots + (-1)^n c_n] = 0$$

or $$\lambda^n - c_1\lambda^{n-1} + c_2\lambda^{n-2} - \cdots + (-1)^n c_n = 0 \quad (3)$$

where $c_1, c_2, c_3 \cdots c_n$ can be expressed in term of the elements a_{ij} of the matrix A.

→ This equation (3) is called the characteristic equation of the matrix A.

→ The polynomial equation $P_n(\lambda) = 0$ has n roots which can be real or complex, simple or repeated.

→ The roots $\lambda_1, \lambda_2 \cdots \lambda_n$ of the polynomial equation $P_n(\lambda) = 0$ are called the eigen values.

→ By using the relation between the roots and the coefficients, we can write

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n = c_1 = a_{11} + a_{22} + \cdots + a_{nn}$$

$$\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_4 + \cdots + \lambda_{n-1} \cdot \lambda_n = c_2$$

$$\vdots$$

$$\lambda_1 \cdot \lambda_2 \cdots \lambda_n = c_n$$

If we set $\lambda = 0$ in eqn (2), we get

$$|A| = (-1)^{2n} c_n = c_n = \lambda_1 \lambda_2 \cdots \lambda_n \quad \text{— (4)}$$

$\therefore$ $\boxed{\text{sum of eigen values} = \text{trace}(A)}$

$\boxed{\text{Product of eigen values} = |A|}$

→ The set of eigen values is called the spectrum of A and the largest eigen value in magnitude is called the spectral radius of A and is denoted by $\rho(A)$.

→ If $|A| = 0$, that is the matrix is singular, then from eqn ④, we find that one of the eigen values must be zero. Conversely, if one of the eigen values is zero, then $|A| = 0$.

→ If A is a diagonal or an upper triangular or a lower triangular matrix, then the diagonal elements of the matrix A are the eigen values of A.

→ After determining the eigen values λ_i's, we solve the homogeneous system $(A - \lambda I)X = 0$ for each λ_i, $i = 1, 2, \cdots n$ to obtain the corresponding eigenvectors.

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Properties of eigen values and eigen vector:

Let λ be an eigen value of A and X be its corresponding eigen vector. Then we have the following results.

① αA has eigen value $\alpha\lambda$ and the corresponding eigen vector is X.

$$AX = \lambda X$$
$$\Rightarrow \alpha AX = (\alpha\lambda) X$$

② A^m has eigen value λ^m and the correspoinding eigen vector is X for any positive integer m.

Premultiplying both sides of $AX = \lambda X$ by A, we get

$$A \cdot AX = A\lambda X = \lambda AX = \lambda(\lambda X)$$
$$\Rightarrow A^2 X = \lambda^2 X$$

$\therefore$ A^2 has the eigen value λ^2 and the correspon eigen vector X. Premultiplying successively m times, we get $A^m X = \lambda^m X$

(3) $(A-KI)$ has the eigen value $(\lambda-K)$, for any scalar K and the corresponding eigen vector is X.

$$AX = \lambda X \Rightarrow AX - KIX = \lambda X - KX$$

$$\text{or } (A-KI)X = (\lambda-K)X$$

(4) A^{-1} (if it exists) has the eigen value $\frac{1}{\lambda}$ and the corresponding eigen vector is X.

Premultiplying both sides of $AX = \lambda X$ by A^{-1}, we get

$$A^{-1}AX = \lambda A^{-1}X \Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

(5) $(A-KI)^{-1}$ has the eigen values $\frac{1}{(\lambda-k)}$ and the corresponding eigen vector is X for any scalar k.

(6) A and A^T have the same eigen values, since a determinant can be expanded by rows or columns.

(7) For a real matrix A, if $\alpha+i\beta$ is an eigen value, then its conjugate $\alpha-i\beta$ is also an eigen value (since the char. eqn has real coefficients). When the matrix A is complex, this property does not hold.

Cayley - Hamilton theorem:

"Every square matrix satisfies its own characteristic equation".

If $|A-\lambda I| = (-1)^n[\lambda^n - C_1\lambda^{n-1} + C_2\lambda^{n-2} - \cdots + (-1)^n C_n]$ be the characteristic polynomial of $n \times n$ matrix $A = (a_{ij})$, then the matrix equation

$$X^n - C_1 X^{n-1} + C_2 X^{n-2} - \cdots + (-1)^n C_n I = 0$$

is satisfied by $X = A$ ie

$$A^n - C_1 A^{n-1} + C_2 A^{n-2} - \cdots + (-1)^n C_n I = 0 \quad -(1)$$

Remark (1) we can use eqn (1) to find A^{-1} (if it exists) in terms of the powers of the matrix A.

Premultiplying both sides in eqn (1) by A^{-1} we get

$$A^{n-1} - C_1 A^{n-2} + \cdots + (-1)^{n-1} C_{n-1} \cdot I + (-1)^n C_n \cdot A^{-1} I = A^{-1} 0 = 0$$

or $$A^{-1} = -\frac{(-1)^n}{C_n}\left[A^{n-1} - C_1 A^{n-2} + \cdots + (-1)^{n-1} C_{n-1} I\right)$$

(2) we can use eqn (1) to obtain A^n in terms of lower powers of A as

$$A^n = C_1 A^{n-1} - C_2 A^{n-2} + \cdots + (-1)^{n-1} C_n I.$$

Exp ①: Find the eigen values of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Solution: The characteristic equation of the matrix is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda) - 4 = 0$$

$$\Rightarrow -1 + \lambda^2 - 4 = 0 \Rightarrow \lambda^2 - 5 = 0$$

$$\Rightarrow \lambda = \pm\sqrt{5} \quad \text{ie} \quad \lambda = \sqrt{5} \text{ and } \lambda = -\sqrt{5}$$

Exp ②: Find the eigen values of the matrix

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solution: The characteristic eqn of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-5-\lambda)(-2-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 10 - 4 = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+6) = 0 \Rightarrow \lambda = -1, -6$$

The eigen values of A are -1 and $\underline{\underline{-6}}$.