Orthogonal projection matrices:

Let W be a subspace of the Euclidean m-space R^m. An $m \times m$ matrix P is called the (orthogonal) projection matrix on a subspace W if $Proj_W(x) = Px$ for any vector x in R^m. Equivalently, P is the matrix representation of the orthogonal projection $Proj_W$ of R^m onto W with respect to the standard basis for R^m.

Theorem: For any subspace W of R^m, the projection matrix P on W can be written as

$$P = [Proj_W]_\alpha = A(A^TA)^{-1}A^T$$

for a matrix A whose columns form a basis for W.

Exp Find the projection matrix P on the plane $2x-y-3z=0$ in R^3 and calculate Pb for $b=(1,0,1)$.

Solution: Choose any basis for the plane $2x-y-3z=0$, say

$$v_1 = (0,3,-1) \text{ and } v_2 = (1,2,0)$$

Let $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}$ be the matrix with v_1 and v_2 as as columns.

Then

$$(A^TA)^{-1} = \begin{bmatrix} 10 & 6 \\ 6 & 5 \end{bmatrix}^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}$$

The projection matrix is

$$P = A(A^TA)^{-1}A^T$$

$$= \frac{1}{14}\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}\begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$P = \frac{1}{14}\begin{bmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{bmatrix}$$

and

$$Pb = \frac{1}{14}\begin{bmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \underline{\underline{\frac{1}{14}\begin{bmatrix} 16 \\ -1 \\ 11 \end{bmatrix}}}$$

<u>Remark</u>: In particular, if the columns of A consist of an orthonormal basis $\alpha = \{u_1, u_2 \cdots u_n\}$ for W, then

$$A^TA = \begin{bmatrix} \cdots & u_1^T & \cdots \\ & \vdots & \\ \cdots & u_n^T & \cdots \end{bmatrix}\begin{bmatrix} \vdots & & \vdots \\ u_1 & \cdots & u_n \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = I_n$$

since $u_i^Tu_j = \delta_{ij}$. Hence, the normal equation

$$A^TAx = A^Tb \text{ becomes}$$

$$x = A^T b = \begin{bmatrix} \cdots & u_1^T & \cdots \\ & \vdots & \\ \cdots & u_n^T & \cdots \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \langle u_1, b \rangle \\ \vdots \\ \langle u_n, b \rangle \end{bmatrix},$$

which is just the expression of $\mathrm{Proj}_W(b)$ with respect to the orthonormal basis α for W that are the columns of A.

<u>Theorem</u>: Suppose that the column vectors $\{u_1, u_2 \cdots u_n\}$ of A form an ~~orthogonal~~ orthonormal basis for W in $\mathbb{R}^m$. Then we get

$$P = A(A^TA)^{-1}A^T = AA^T$$

$$P = u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T$$

In particular, if A is an $m \times m$ orthogonal matrix, then, for all $b \in \mathbb{R}^m$, $Ax = b$ has the unique solution

$$x = A^{-1}b = \underline{\underline{A^Tb}}$$

<u>Exp</u>. If $A = [c_1, c_2]$, where $c_1 = (1,0,0)$, $c_2 = (0,1,0)$, then the column vectors of A are orthonormal, $C(A)$ is the xy-plane, and the projection of $b = (x, y, z) \in \mathbb{R}^3$ onto $C(A)$ is $b_c = (x, y, 0)$. In fact,

$$P = AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem: A square matrix P is a projection matrix if and only if it is symmetric and idempotent.

i.e. $P^T = P$ and $P^2 = P$.

Proof: Let P be a projection matrix.

$$\therefore P = A(A^TA)^{-1}A^T$$

for some matrix A whose columns are L.I..

$$\therefore P^T = (A(A^TA)^{-1}A^T)^T$$

$$= A((A^TA)^{-1})^T A^T$$

$$P^T = A(A^TA)^{-1}A^T = P \quad (\because (A^{-1})^T = A^{-1})$$

$\Rightarrow$ P is symmetric matrix

and

$$P^2 = P \cdot P = (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T)$$

$$= A(A^TA)^{-1}A^T = P$$

$\Rightarrow$ P is idempotent.

In

QR factorization :

Let A be an $m \times n$ matrix with L.I. columns. Applying Gram-schmidt orthogonalization process to the columns of A produces an $m \times n$ matrix Q whose columns are orthogonal. So,

$A = QR$, where R is $n \times n$ upper triangular matrix with +ive entries on the diagonal.

Let $A = [c_1, c_2 \cdots c_n]$. Then from Gram-schmidt orthogonalization,

$$q_1 = c_1$$

$$q_2 = c_2 - \frac{\langle c_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1$$

$$\vdots$$

$$q_n = c_n - \frac{\langle c_n\ q_1 \rangle}{\langle q_1\ q_1 \rangle} q_1 - \cdots - \frac{\langle c_n\ q_{n-1} \rangle}{\langle q_{n-1}, q_{n-1} \rangle} q_{n-1}$$

gives an orthogonal basis $\{q_1, q_2 \cdots q_n\}$ for $C(A)$.

By taking normalization of these vectors, we obtain an orthonormal basis $\{u_1, u_2 \cdots u_n\}$ for $C(A)$, where $u_i = \frac{q_i}{\|q_i\|}$

Rewriting these equations gives us

$$c_1 = q_1 \qquad = \|q_1\| \, u_1 = b_{11} u_1$$

$$c_2 = a_{21} q_1 + q_2 = b_{21} u_1 + b_{22} u_2$$

$$\vdots$$

$$c_n = a_{n1} q_1 + \cdots + a_{nn-1} q_{n-1} + q_n$$

$$= b_{n1} u_1 + b_{n2} u_2 + \cdots + b_{nn} u_n$$

where $a_{ij} = \dfrac{\langle c_i, q_j \rangle}{\langle q_j, q_j \rangle}$ for $i > j$, $a_{ii} = 1$

and $b_{ij} = a_{ij} \|q_j\|$ for $i \geq j$. Hence

$$A = [c_1, c_2 \cdots c_n] = [u_1 \cdots u_n] \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ 0 & b_{22} & \cdots & b_{n2} \\ \vdots & & & \\ 0 & 0 & & b_{nn} \end{bmatrix}$$

ie $A = QR$.

The matrix $Q = [u_1, u_2 \cdots u_n]$ is an $m \times n$ matrix with orthonormal columns, called the orthogonal part of A, and

$$R = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ 0 & b_{22} & \cdots & b_{n2} \\ \vdots & & & \\ 0 & 0 & & b_{nn} \end{bmatrix}$$

is an invertible upper triangular matrix, called the upper triangular part of A.

such an $A = QR$ is called the QR factorization of an $m \times n$ matrix A, when rank $(A) = n$.

→ with this decomposition of A, the projection matrix can now be calculated easily as

$$P = A(A^TA)^{-1}A^T$$

$$P = QR(R^TQ^TQR)^{-1}R^TQ^T = QQ^T$$

and $x = (A^TA)^{-1}A^Tb = R^{-1}Q^Tb.$

Exp compute QR factorization for the matrix.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution: Let $A = [c_1 \; c_2 \; c_3]$

First we find the orthogonal part of A ie Q.

$q_1 = c_1 = (1, 0, 1)$

$q_2 = c_2 - \dfrac{\langle c_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 = (2, 1, 0) - \dfrac{2}{2}(1, 0, 1)$

$= (1, 1, -1)$

$q_3 = c_3 - \dfrac{\langle c_3, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 - \dfrac{\langle c_3, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2$

$= (0, 1, 1) - \dfrac{1}{2}(1, 0, 1) - \dfrac{0}{3}(1, 1, -1) = \dfrac{1}{2}(-1, 2, 1)$

and $\|q_1\| = \sqrt{2}$, $\|q_2\| = \sqrt{3}$, $\|q_3\| = \frac{\sqrt{6}}{2}$.

Hence,

$$u_1 = \frac{q_1}{\|q_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$u_2 = \frac{q_2}{\|q_2\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$u_3 = \frac{q_3}{\|q_3\|} = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

and

$$b_{11} = a_{11}\|q_1\| = \|q_1\| = \sqrt{2}$$

$$b_{21} = a_{21}\|q_1\| = \frac{\langle c_2, q_1\rangle}{\langle q_1, q_1\rangle}\|q_1\| = 1\cdot\sqrt{2} = \sqrt{2}$$

$$b_{22} = a_{22}\|q_2\| = 1\cdot\sqrt{3} = \sqrt{3}$$

$$b_{31} = a_{31}\|q_1\| = \frac{\langle c_3, q_1\rangle}{\langle q_1, q_1\rangle}\|q_1\| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$b_{32} = a_{32}\|q_2\| = \frac{\langle c_3 . q_2\rangle}{\langle q_2, q_2\rangle}\|q_2\| = 0.$$

$$b_{33} = a_{33}\|q_3\| = 1\cdot\frac{\sqrt{6}}{2} = \frac{\sqrt{6}}{2}$$

Then

$$c_1 = b_{11}u_1 = \sqrt{2}\,u_1$$

$$c_2 = b_{21}u_1 + b_{22}u_2 = \sqrt{2}\,u_1 + \sqrt{3}\,u_2$$

$$c_3 = b_{31}u_1 + b_{32}u_2 + b_{33}u_3 = \frac{1}{\sqrt{2}}u_1 + 0u_2 + \sqrt{6}\,u_3$$

$$\therefore A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [u_1, u_2, u_3]\begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{bmatrix} = Q \cdot R.$$

and hence the projection matrix

$$P = Q Q^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Exp ②: Find QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence find its projection matrix P.

Solⁿ:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{21}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{2}{\sqrt{21}} \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{7}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{\sqrt{7}}{\sqrt{3}} \end{bmatrix} = QR$$

and $P = Q \cdot Q^T = \begin{bmatrix} \frac{6}{7} & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{7} & \frac{6}{7} & -\frac{1}{7} & \frac{2}{7} \\ -\frac{1}{7} & -\frac{1}{7} & \frac{6}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{2}{7} & \frac{2}{7} & \frac{3}{7} \end{bmatrix}$

Ex. Find QR decomposition of

$$A = [c_1 \; c_2 \; c_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Soln:

$q_1 = c_1 = (1, 1, 0, 0)$

$q_2 = c_2 - \dfrac{\langle c_2\, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0 \right)$

$q_3 = c_3 - \dfrac{\langle c_3\, q_2 \rangle}{\langle c_3\, q_2 \rangle} q_2 - \dfrac{\langle c_3\, q_1 \rangle}{\langle q_1\, q_1 \rangle} q_1 = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1 \right)$

and $\|q_1\| = \sqrt{2}$, $\|q_2\| = \sqrt{\frac{3}{2}}$, $\|q_3\| = \sqrt{\frac{7}{3}}$

$\therefore u_1 = \dfrac{q_1}{\|q_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right)$

$u_2 = \quad = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}, 0 \right)$

$u_3 = \quad = \left(-\frac{2}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{\sqrt{3}}{\sqrt{7}} \right)$

Then

$c_1 = b_{11} u_1$

$c_2 = b_{21} u_1 + b_{22} u_2$

$c_3 = b_{31} u_1 + b_{32} u_2 + b_{33} u_3$

Here $a_{ij} = \dfrac{\langle c_i\, q_j \rangle}{\langle q_j, q_j \rangle}$

$b_{ij} = a_{ij} \|q_j\|$

$\Rightarrow$

$c_1 = \sqrt{2}\, u_1$

$c_2 = \frac{1}{\sqrt{2}} u_1 + \sqrt{\frac{3}{2}}\, u_2$

$c_3 = \frac{1}{\sqrt{2}} u_1 + \frac{1}{\sqrt{6}} u_2 + \sqrt{\frac{7}{3}}\, u_3$

$\therefore b_{11} = \sqrt{2}$

$b_{21} = \frac{1}{\sqrt{2}}$

$b_{22} = \sqrt{\frac{3}{2}}$

$b_{31} = \frac{1}{\sqrt{2}}$

$b_{32} = \frac{1}{\sqrt{6}}$ $\quad b_{33} = \sqrt{\frac{7}{3}}$

$$\therefore A = [u_1, u_2, u_3] \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3}/\sqrt{2} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{\sqrt{7}}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{21}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{2}{\sqrt{21}} \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{7}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{7}{3}} \end{bmatrix} = QR$$

$$\therefore P = QQ^T = \begin{bmatrix} \frac{6}{7} & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{7} & \frac{6}{7} & -\frac{1}{7} & \frac{2}{7} \\ -\frac{1}{7} & -\frac{1}{7} & \frac{6}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{2}{7} & \frac{2}{7} & \frac{3}{7} \end{bmatrix}$$

Singular Value Decomposition

The singular value decomposition of a matrix A can be expressed in terms of the factorization of A into the product of three matrices as

$$A = U D V^T$$

ie $$A_{m\times n} = U_{m\times k} \cdot D_{k\times k} \cdot V^T_{k\times n}$$

the columns of U and V are orthonormal and the matrix D is diagonal with +ive real entries.

Exp ①: Find the singular value decomposition of a matrix

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Solⁿ. First find eigen values of A^TA:

$$\therefore A^TA = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

$$\therefore |A^TA - \lambda I| = \begin{vmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (25-\lambda)^2 - 225 = 0$$

$$\Rightarrow \lambda^2 - 50\lambda + 400 = 0$$

$$\Rightarrow (\lambda - 10)(\lambda - 40) = 0$$

$\lambda = 10, 40$

$\therefore$ The eigen values of A^TA are $10, 40$.

Now eigen vector for $\lambda = 10$ is

$$(A^TA - 10I)X_1 = 0$$

$$\Rightarrow \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 15v_1 - 15v_2 = 0$$

$$\Rightarrow v_1 = v_2$$

So $X_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v$

Normalizing $v \Rightarrow u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

eigen vector for $\lambda = 40$ is

$$(A^TA - 40I)X_2 = 0$$

$$\Rightarrow \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -15w_1 - 15w_2 = 0$$

$$\Rightarrow w_1 = -w_2$$

So $X_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = w$

Normalizing $w \Rightarrow u_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Hence the matrix $V = \begin{array}{c} \begin{matrix} \lambda = 10 & \lambda = 40 \end{matrix} \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{array}$

Now $AA^T = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix}$

Finding eigen values for AA^T

$$|AA^T - \lambda I| = \begin{vmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (16-\lambda)(34-\lambda) - 144 = 0$$

$$\Rightarrow \lambda^2 - 50\lambda + 400 = 0$$

$$\Rightarrow (\lambda-10)(\lambda-40) = 0$$

Hence $\lambda = 10, 40$

The eigen vector ~~of~~ for ~~AA^T~~ $\lambda = 10$ is

$$(AA^T - 10I)x_1' = 0$$

$$\begin{bmatrix} 16 & 12 \\ 12 & 24 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 6u_1' + 12u_2' = 0$$
$$\Rightarrow u_1' = -2u_2'$$

So $u' = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ Normalizing $u_1' = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$

Now eigen vector for $\lambda = 40$ is

$$\begin{bmatrix} -24 & 12 \\ 12 & -6 \end{bmatrix}\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -24w_1' + 12w_2' = 0$$
$$\Rightarrow 2w_1' = w_2'$$

So $w' = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ Normalizing $u_2' = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$

Hence the matrix $U = \begin{array}{cc} \lambda = 10 & \lambda = 40 \end{array}$
$$U = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

eige.

So the singular value decomposition of A is

$$A = U D V^T$$

$$\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{40} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

=

Exp ②: Find the singular value decomposition of a Matrix $A = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$.

Soln: $AA^T = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 50 & 30 \\ 30 & 50 \end{bmatrix}$

Finding the eigen vector of AA^T

$\therefore |AA^T - \lambda I| = \begin{vmatrix} 50-\lambda & 30 \\ 30 & 50-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 100\lambda + 1600 = 0$

$\Rightarrow \lambda = 20, 80$

Now eigen vector for $\lambda = 80$ is

$(AA^T - 80I)v = 0$

$\begin{bmatrix} -30 & 30 \\ 30 & -30 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = v_2$

$\therefore$ eigen vector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Normalizing $= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

eigen vector for $\lambda = 20$ is

$$\begin{bmatrix} 50-20 & 30 \\ 30 & 50-20 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 30w_1 + 30w_2 = 0 \\ w_1 = -w_2 \end{array}$$

so eigen vector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow$ Normalizing $= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

Hence $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Now $A^T A = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix}$

Finding eigen vectors of $A^T A$

$$\Rightarrow |A^T A - \lambda I| = \begin{vmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 100\lambda + 1600 = 0$$

$$\Rightarrow \lambda = 20, 80$$

eigen vector for $\lambda = 80$ is

$$\begin{bmatrix} 26-80 & 18 \\ 18 & 74-80 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3w_1' = v_2$$

$\therefore$ eigen vector $= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Normalizing $= \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$

Now eigen vector for $\lambda = 20$ is

$$\begin{bmatrix} 20-20 & 18 \\ 18 & 74-20 \end{bmatrix} \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow w_1' = -3w_2'$$

$\therefore$ eigen vector $= \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, Normalizing $= \begin{bmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{bmatrix}$

Hence $V = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$

singular values are $\sqrt{80}$, and $\sqrt{20}$

Hence

$$A = UDV^T$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{80} & 0 \\ 0 & \sqrt{20} \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$= \frac{1}{\cancel{\sqrt{20}}} \times \cancel{\sqrt{20}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$