Exp: Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Also (i) obtain A^{-1} and A^3, (ii) find the eigen values of A, A^2 and verify that eigen values of A^2 are squares of those of A; (iii) find the spectral radius of A.

Solution: The characteristic eqn of A is given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 2\lambda - 3) - 2(\lambda - 3) = 0$$

$$\Rightarrow \quad -\lambda^3 + 3\lambda^2 - \lambda + 3 = 0 \quad \text{—— (1)}$$

Now $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$$

we have

$$-A^3+3A^2-A+3I=-\begin{bmatrix}-1 & 10 & 12\\ 1 & 11 & 10\\ -1 & 16 & 17\end{bmatrix}+3\begin{bmatrix}-1 & 4 & 4\\ 0 & 3 & 4\\ 0 & 6 & 5\end{bmatrix}$$

$$-\begin{bmatrix}1 & 2 & 0\\ -1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}+3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad \text{—— (2)}$$

Hence, A satisfiee the characteristic equation

$$-\lambda^3+3\lambda^2-\lambda+3=0$$

(i) From eqn (2), we get

$$A^{-1}=\frac{1}{3}(A^2-3A+I)$$

$$=\frac{1}{3}\left[\begin{pmatrix}-1 & 4 & 4\\ 0 & 3 & 4\\ 0 & 6 & 5\end{pmatrix}-3\begin{pmatrix}1 & 2 & 0\\ -1 & 1 & 2\\ 1 & 2 & 1\end{pmatrix}+\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]$$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}-3 & -2 & 4\\ 3 & 1 & -2\\ -3 & 0 & 3\end{bmatrix}$$

From eqn (2), we get

$$A^3=3A^2-A+3I=3\begin{bmatrix}-1 & 4 & 4\\ 0 & 3 & 4\\ 0 & 6 & 5\end{bmatrix}-\begin{bmatrix}1 & 2 & 0\\ -1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}+3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$$

(ii) Eigen values of A are the roots of

$$\lambda^3 - 3\lambda^2 + \lambda - 3 = 0$$

or $(\lambda - 3)(\lambda^2 + 1) = 0$

or $\lambda = 3, i, -i$

characteristic eqn of A^2 is given by

$$\begin{vmatrix} -1-\lambda & 4 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 6 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)[(3-\lambda)(5-\lambda) - 24] = 0$$

or $(\lambda+1)(\lambda^2 - 8\lambda - 9) = 0 \Rightarrow (\lambda+1)(\lambda-9)(\lambda+1) = 0$

or $\lambda = 9, -1, -1$

$\therefore$ The eigen values of A^2 are the $(3)^2, (+i)^2, (-i)^2$ eigen values of A.

which are the squares of the eigen values of A.

(iii) The spectral radius of A is given by

$\rho(A) =$ largest eigenvalues in magnitude

$= \max\{|3|, |-i|, |i|\}$

$= \underline{\underline{3}}$

Exp ① Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution: The characteristic equation is $|A - \lambda I| = 0$

ie $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$ or $(5-\lambda)(2-\lambda) - 4 = 0$

or $\lambda^2 - 7\lambda + 6 = 0$ or $(\lambda - 6)(\lambda - 1) = 0$

$\therefore \lambda = 1, 6$

Thus the eigen values are 1 and 6.

(i) when $\lambda = 1$, the corresponding eigen vectors are given by

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

Let $x_2 = k$, then $x_1 = -k$

Hence, eigen vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(ii) when $\lambda = 6$, the corresponding eigen vectors are given by

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + 4x_2 = 0 \Rightarrow x_1 = 4x_2$

Let $x_2 = k$, then $x_1 = 4k$

Hence, eigen vector $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4k \\ k \end{bmatrix}$

$$X_2 = k \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Exp ②: Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: The characteristic equation is $|A - \lambda I| = 0$

ie $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$ ie $\lambda^3 - 7\lambda^2 + 36 = 0$

Since $\lambda = -2$ satisfies it, we can write this eqn as

$(\lambda+2)(\lambda^2 - 9\lambda + 18) = 0$ or $(\lambda+2)(\lambda-3)(\lambda-6) = 0$

Thus the eigen values of A are $\lambda = -2, 3, 6$.

The eigen vectors of the matrix A corresponding to the eigen value λ is given by the non-zero solution of the equation $(A - \lambda I) X = 0$

or $\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ —— (1)

(i) when $\lambda = -2$, the corresponding eigen vector is given by

$$\begin{bmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

applying $R_2 \to R_2 - \frac{1}{3} R_1$ and $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 3 & 1 & 3 \\ 0 & \frac{20}{7} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 3x_1 + x_2 + 3x_3 &= 0 \\ x_2 &= 0 \end{aligned}$$

$$\Rightarrow 3x_1 + 3x_3 = 0$$

$$\Rightarrow x_1 = -x_3$$

Let $x_3 = k$, $\Rightarrow x_1 = -k$, and $x_2 = 0$

Hence, eigen vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(ii) when $\lambda = 3$, the corresponding eigen vector is given by

$$\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & +2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

applying $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

applying $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 2x_2 + x_3 = 0$$
$$x_2 + x_3 = 0$$

Let $x_3 = k$, $\Rightarrow x_2 = -k \quad \Rightarrow x_1 = k$

Hence, eigen vector $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(iii) when $\lambda = 6$, the corr. eigen vector is given by

$$\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ -5 & 1 & 3 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

applying $R_2 \to R_2 + 5R_1$ and $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

applying $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 - x_2 + x_3 = 0$$

$$-4x_2 + 8x_3 = 0$$

$$\Rightarrow x_2 = 2x_3$$

Let $x_3 = k, \Rightarrow x_2 = 2k \Rightarrow x_1 = k$

Hence, the eigen vector $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$