Practice Questions form Module-2

1. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define
   (i) $(x, y) + (x_1, y_1) = (x + y_1, y + x_1)$
   $c(x, y) = (cx, cy)$
   (ii) $(x, y) + (x_1, y_1) = (x + x_1, 0)$
   $c(x, y) = (cx, 0)$
   (iii) $(x, y) + (x_1, y_1) = (0, 0)$
   $c(x, y) = (cx, cy)$
   (iv) $(x, y) + (x_1, y_1) = (xx_1, yy_1)$
   $c(x, y) = (cx, cy)$

   Is V, with these operations, a vector space over the field of real numbers?

2. Rather than use the standard definitions of addition and scalar multiplication in $\mathbb{R}^2$, let these two operations be defined as shown below.
   (a) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c(x, y) = (cx, y)$
   (b) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and $c(x, y) = (cx, cy)$
   (c) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c(x, y) = (\sqrt{c}x, \sqrt{c}y)$

   With each of these new definitions, is $\mathbb{R}^2$ a vector space? Justify your answer.

3. Determine whether the set W is a subspace of $\mathbb{R}^3$ with the standard operations. Justify your answer.
   (a) $W = \{(0, x_2, x_3) :\ x_2$ and x_3 are real numbers$\}$
   (b) $W = \{(x_1, x_2, 4) :\ x_1$ and x_2 are real numbers$\}$
   (c) $W = \{(a, a - 3b, b) :\ a$ and b are real numbers$\}$
   (d) $W = \{(s, t, s + t) :\ s$ and t are real numbers$\}$
   (e) $W = \{(x_1, x_2, x_1x_2) :\ x_1$ and x_2 are real numbers$\}$
   (f) $W = \{(x_1, 1/x_1, x_3) :\ x_1$ and x_3 are real numbers, $x_1 \neq 0\}$.
4. Let V be the set of all matrices of the form $\begin{pmatrix} 1 & p \\ q & 1 \end{pmatrix}$ where $p, q \in R$. Determine whether or not the set V is a vector space under standard addition and scalar multiplication of matrices.
5. Let $\Gamma = R^2$, with addition defined by $(x, y) \oplus (w, z) = (x + w, y + z)$ and scalar multiplication defined by $a \odot (x, y) = (ax + a, ay - 2a)$. Determine whether Γ, a vector space is or not? If yes, find additive identity and inverse.
6. Determine the set of real numbers is a vector space under the addition and scalar multiplication whereas R defined the set of positive real numbers. Define vector addition by $x \oplus y = xy$ and scalar multiplication by $c \odot x = x^c$.
7. Let V=R, Define addition and scaler multiplication by $a \oplus b = 2a + 2b$ and $k.a = ka$, Show that addition is commutative but not associative.
8. Let$M_{n\times n}$ be a set of n-square matrices and let $A \in M_{n\times n}$ be a fixed matrix. Define a set $S = \{B \in M_{n\times n} | \ AB = BA\}$. Is S a subspace? Explain.
9. Let $V = R^2$ and define $\binom{a}{b} + \binom{c}{d} = \binom{a + 2c}{b + 2d}$ and $k\binom{a}{b} = \binom{ka}{kb}$.
   Determine whether V is a vector space. Also find additive identity and additive inverse.

10. Verify that the set $M_{2\times2}$of 2×2 matrices with real number entries is a vector space under the natural entry-by-entry operations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix} \text{ and } k \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}.$$

11. Let $H = \left\{ \begin{bmatrix} p \\ 0 \\ q \end{bmatrix} : p \text{ and } q \text{ are real numbers} \right\}$. Show that H is a subspace of R^3.

12. Let α and β be real numbers. Is the set W of all matrices of the form $\begin{bmatrix} 2\alpha & \beta \\ 3\alpha+\beta & 3\beta \end{bmatrix}$ a subspace of the vector space $M_{2\times2}$? Explain.

13. Show that the subset $\mathrm{W} = \{\mathrm{A} \in \mathrm{M_n(R)} | \mathrm{Trace(A)} = 0\}$ is a subspace of vector space $\mathrm{M_n(R)}$.

14. Let R be the field of real numbers. Which of the following are subspaces of $V_3(R)$.
    (i) $\{(x, 3y, 5z) : x, y, z \in R\}$
    (ii) $\{(x, y, 2z) : x, y, z \in Q\}$

15. Let V be the vector space of all 2 × 2 matrices over the field R. Show that W is not a subspace of V, where W contains all 2 × 2 matrices with zero determinants.

16. Check whether the following vectors are linearly independent or dependent?

$$v_1 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \ v_2 = \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} \text{and } v_1 = \begin{pmatrix} 4 \\ 18 \\ 4 \end{pmatrix}$$

17. Determine whether the set $\{x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\}$ is linearly independent or linearly dependent in $P_3(\mathbb{R})$.

18. Find a basis of $W_1 \cap W_2$, where W_1 and W_2 are the subspaces of R^4 generated by $(1,1,1,1), (1,1,-1,-1), (1,-1,1,-1)$ and $(1,1,0,0), (1,-1,1,0), (2,-1,1,-1)$, respectively.

19. Let W be a subspace of R^5 spanned by the vectors $(1,1,2,1,-2), (2,3,8,1,-1)$ and $(-1,1,6,-3,8)$. Find a basis of W.

20. In $V_3(R)$, where R is the field of real numbers, examine each of the following sets of vectors are linearly dependent:
    (i) $\{(1,2,0), (0,3,1), (-1,0,1)\}$
    (ii) $\{(-1,2,1), (3,0,-1), (-5,4,3)\}$

21. Show that the vectors $\alpha = (1,0,-1)$, $\beta = (1,2,1)$, $\gamma = (0,-3,2)$, form a basis for R^3. Express each of the standard basis vectors as a linear combination of α, β, γ.

22. Find the row rank, column rank, rank, and nullity of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix}$$

23. Find a basis for row space, column space, and null space for the following matrix:

$$M = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & 8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

24. Write the vectors $w_1 = (1,1,1)$ and $w_2 = (1,-2,2)$, if possible, as a linear combination of vectors in the set
$$S = \{(1,2,3),(0,1,2),(-1,0,1)\}.$$
25. Determine whether the set
    (a) $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$ spans $\mathbb{R}^3$.
    (b) $S = \{(1,2,3),(0,1,2),(-1,0,1)\}$ spans $\mathbb{R}^3$.
26. Determine whether the set of vectors is L.I. or L.D.:
    (a) $S = 1 + x - 2x^2, 2 + 5x - x^2, x + x^2$in P_2
    (b) $S = \left\{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\right\}$ in $M_{2\times 2}$.
27. For which value of t is the set linearly independent?
    (a) $S = \{(t,1,1),(1,t,1),(1,1,t)\}$
    (b) $S = \{(t,1,1),(1,0,1),(1,1,3t)\}$
28. Determine whether the following S is a basis for the given vector space:
    (a) $S = \{(2,1,0),(0,-1,1)\}$ for $\mathbb{R}^3$
    (b) $S = \{(1,5,3),(0,1,2),(0,0,6)\}$ for $\mathbb{R}^3$
    (c) $S = \{1 - 2t^2 + t^3, -4 + t^2, 2t + t^3, 5t\}$ for P_3.
    (d) $S = \left\{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\right\}$ for $M_{2\times 2}$.
29. Determine whether $S = \{(4,3,2),(0,3,2),(0,0,2)\}$ is a basis for $\mathbb{R}^3$. If it is, write $u = (8,3,8)$ as a linear combination of the vectors in S.
30. Express the polynomial $p(x) = x^2 + 4x - 3$ in $R_3[x]$ as a linear combination of the polynomials
    $p_1(x) = x^2 - 2x + 5, \quad p_2(x) = 2x^2 - 3x, \quad p_3(x) = x + 1$
31. Find a basis for each of the vector spaces of all 3×3 (i) diagonal matrices (ii) symmetric matrices. What are the dimensions of these vector spaces?
32. Find a basis for the subspace of $\mathbb{R}^3$ spanned by
$$S = \{(-1,2,5),(3,0,3),(5,1,8)\}.$$
33. Check whether $U = \{u_1 = (1,1,1), u_2 = (1,2,3), u_1 = (2,-1,1)\}$ form a basis for R^3 or not.
34. Determine whether the $v = (5,-4,-7)$ in R^3 is a linear combination of the vectors $u_1 = (1,-1,0),\ u_2 = (-2,-1,-1), u_1 = (3,-1,-3)$.
35. Let W be the subspace of R^4 spanned by the vectors $u_1 = (1,-2,5,-3), u_2 = (2,3,1,-4), u_3 = (3,8,-3,-5),$. Find the basis and dimension W.
36. The vectors u_1=$(1,-2)$ & $u_2 = (4,-7)$ from a basis S of R^2. Find the coordinate vector $[v]$ of v relative to S, where :
    (a) $v = (5,3)$ (b) $v = (a,b)$
37. Determine whether $B = \left\{\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}\right\}$ is a basis for $M_{2x2.}$

38. For the given matrix,

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$$

find

(i) the rank and nullity
(ii) the null space and one of its bases
(iii) a basis of row space
(iv) a basis of column space
(v) whether the rows of A are LI
(vi) whether the columns of A are LI
(vii) a subset of the column vectors of A that forms a basis for the column space of A.

39. Determine whether b is in the column space of A. If it is, write b as a linear combination of the column vectors of A.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

40. Let $V = \{(x, y, z, u) \in \mathbb{R}^4 : y + z + u = 0\}$, and $W = \{(x, y, z, u) \in \mathbb{R}^4 : x + y = 0, z = 2u\}$ be two subspaces of $\mathbb{R}^4$. Find bases for $V, W, V + W$ and $V \cap W$.

41. Let V be the subspace of $\mathbb{R}^5$ spanned by $(1, 3, -2, 2, 3), (1, 4, -3, 4, 2), (2, 3, -1, -2, 10)$, and W the subspace spanned by $(1, 3, 0, 2, 1), (1, 5, -6, 6, 3), (2, 5, 3, 2, 1)$. Find a basis for $V + W$ and $V \cap W$.

42. Find a polynomial $p(x) = a + bx + cx^2 + dx^3$ that satisfies $p(0) = 1, p'(0) = 2, p(1) = 4, p'(1) = 4$.