Applications of Linear Algebra

Dr. Dileep Kumar

Assistant Professor

Department of Mathematics

VIT Bhopal University

Applications of Linear Systems

Network Analysis

The concept of a *network* appears in a variety of applications. Loosely stated, a ***network*** is a set of ***branches*** through which something "flows." For example, the branches might be electrical wires through which electricity flows, pipes through which water or oil flows, traffic lanes through which vehicular traffic flows, or economic linkages through which money flows, to name a few possibilities. In most networks, the branches meet at points, called ***nodes*** or ***junctions***.

A common problem in network analysis is to use known flow rates in certain branches to find the flow rates in all of the branches.

EXAMPLE 1 | Network Analysis Using Linear Systems

Figure 1.10.1 shows a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

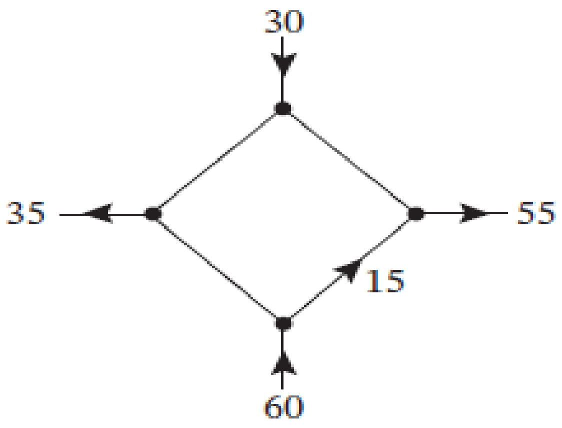


FIGURE 1.10.1

Solution As illustrated in Figure 1.10.2, we have assigned arbitrary directions to the unknown flow rates x_1, x_2, and x_3. We need not be concerned if some of the directions are incorrect, since an incorrect direction will be signaled by a negative value for the flow rate when we solve for the unknowns.

It follows from the conservation of flow at node A that

$$x_1 + x_2 = 30$$

Similarly, at the other nodes we have

$$\begin{aligned} x_2 + x_3 &= 35 \quad (\text{node } B) \\ x_3 + 15 &= 60 \quad (\text{node } C) \\ x_1 + 15 &= 55 \quad (\text{node } D) \end{aligned}$$

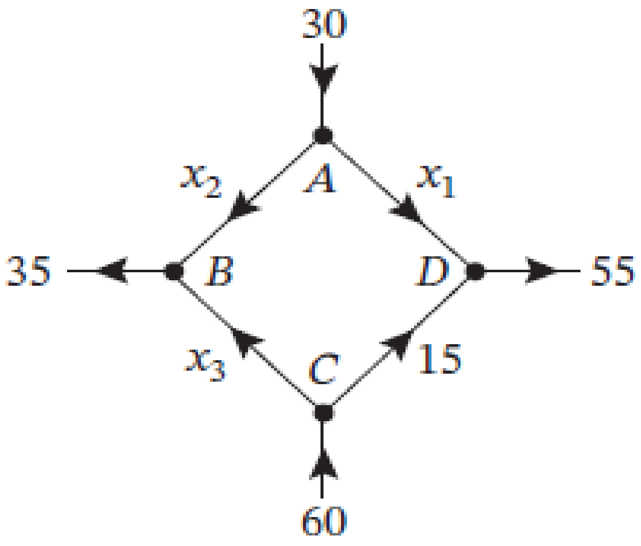


FIGURE 1.10.2

These four conditions produce the linear system

$$\begin{aligned}
x_1 + x_2 \phantom{{}+x_3} &= 30 \\
x_2 + x_3 &= 35 \\
x_3 &= 45 \\
x_1 \phantom{{}+x_2+x_3} &= 40
\end{aligned}$$

which we can now try to solve for the unknown flow rates. In this particular case the system is sufficiently simple that it can be solved by inspection (work from the bottom up). We leave it for you to confirm that the solution is

$$x_1 = 40, \quad x_2 = -10, \quad x_3 = 45$$

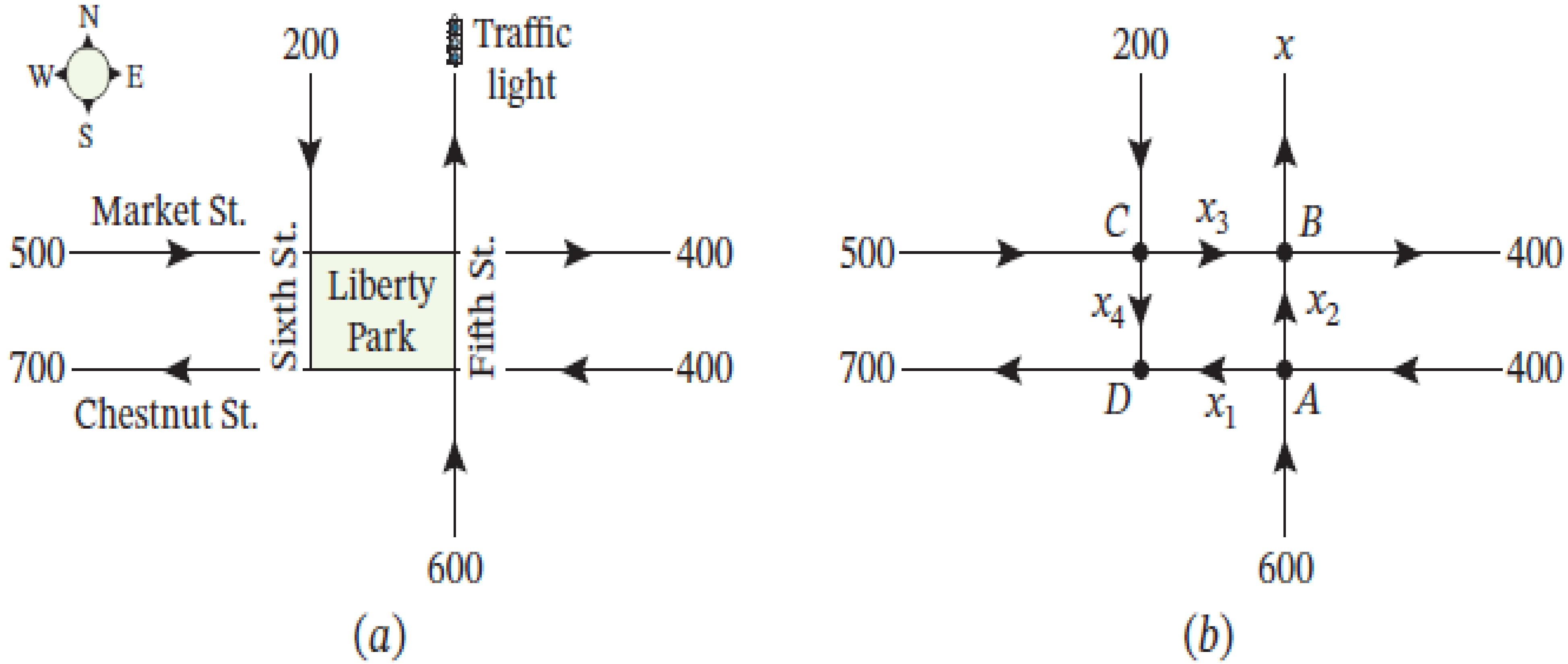
The fact that x_2 is negative tells us that the direction assigned to that flow in Figure 1.10.2 is incorrect; that is, the flow in that branch is *into* node A.

EXAMPLE 2 | Design of Traffic Patterns

The network in Figure 1.10.3*a* shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

(*a*) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(*b*) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?



N
W
E
S
200
Traffic light
Market St.
500
400
Sixth St.
Liberty Park
Fifth St.
700
400
Chestnut St.
600
(a)
200
x
C
x_3
B
500
400
x_4
x_2
700
400
D
x_1
A
600
(b)

FIGURE 1.10.3

Solution (*a*) If, as indicated in Figure 1.10.3*b*, we let x denote the number of vehicles per hour that the traffic light must let through, then the total number of vehicles per hour that flow in and out of the complex will be

Flowing in: $500 + 400 + 600 + 200 = 1700$

Flowing out: $x + 700 + 400$

Equating the flows in and out shows that the traffic light should let $x = 600$ vehicles per hour pass through.

Solution (*b*) To avoid traffic congestion, the flow in must equal the flow out at each intersection. For this to happen, the following conditions must be satisfied:

Intersection	Flow In		Flow Out
A	$400 + 600$	$=$	$x_1 + x_2$
B	$x_2 + x_3$	$=$	$400 + x$
C	$500 + 200$	$=$	$x_3 + x_4$
D	$x_1 + x_4$	$=$	700

Thus, with $x = 600$, as computed in part (a), we obtain the following linear system:

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 1000 \\ x_2 + x_3 \qquad &= 1000 \\ x_3 + x_4 &= \ \ 700 \\ x_1 \qquad\qquad + x_4 &= \ \ 700 \end{aligned}$$

We leave it for you to show that the system has infinitely many solutions and that these are given by the parametric equations

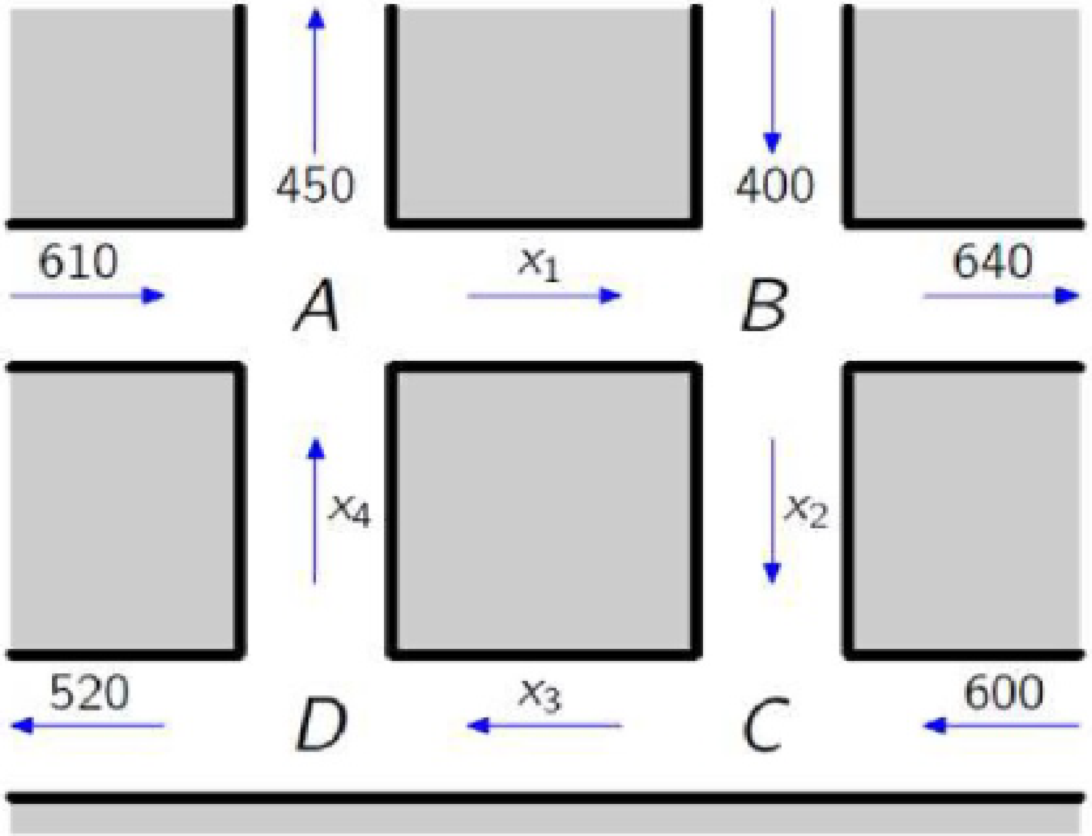
$$x_1 = 700 - t, \quad x_2 = 300 + t, \quad x_3 = 700 - t, \quad x_4 = t \tag{1}$$

However, the parameter t is not completely arbitrary here, since there are physical constraints to be considered. For example, the average flow rates must be nonnegative since we have assumed the streets to be one-way, and a negative flow rate would indicate a flow in the wrong direction. This being the case, we see from (1) that t can be any real number that satisfies $0 \leq t \leq 700$, which implies that the average flow rates along the streets will fall in the ranges

$$0 \leq x_1 \leq 700, \quad 300 \leq x_2 \leq 1000, \quad 0 \leq x_3 \leq 700, \quad 0 \leq x_4 \leq 700$$

Traffic flow (sum of in flow=sum of out flow)

Example In the downtown section of Vijayawada two sets of one-way streets intersect. The average hourly volume of traffic entering and leaving this section during rush hour is given in diagram. Determine the amount of traffic between each of the four intersection.



Answer

$$x_1 - x_4 = 160$$

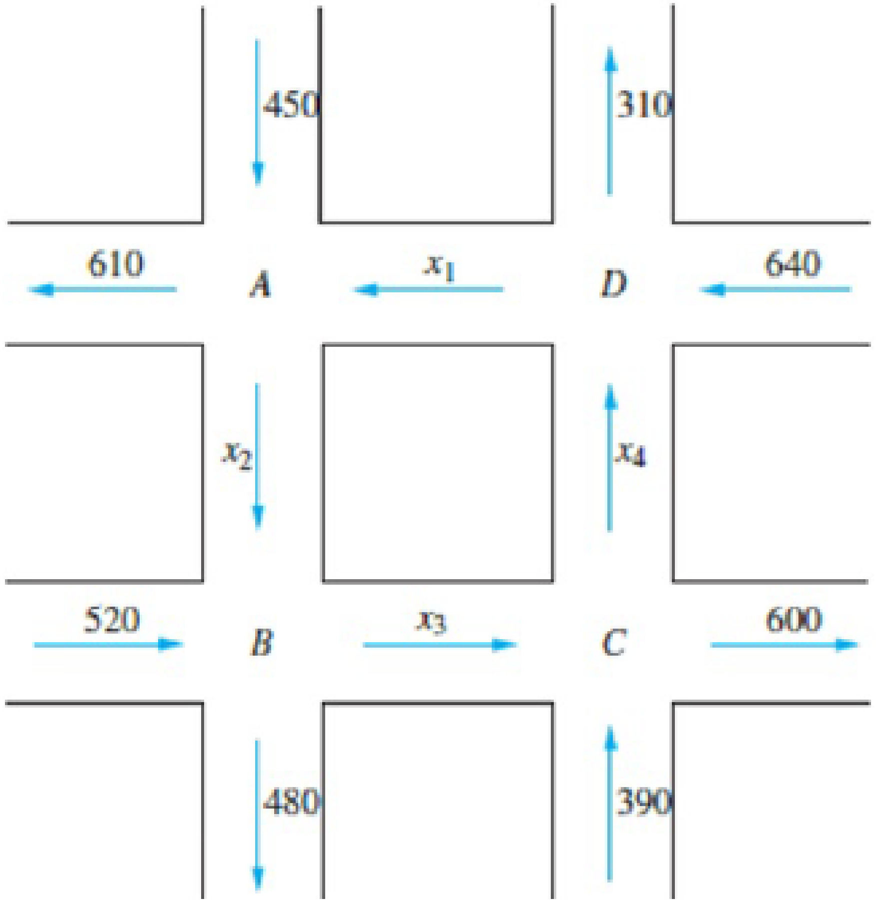
$$x_1 - x_2 = 240$$

$$-x_2 + x_3 = 600$$

$$x_3 - x_4 = 520$$

$x_1 = 460, x_2 = 220, x_3 = 820$
and $x_4 = 300$

Example In the downtown section of Bhubaneswar two sets of one-way streets intersect. The average hourly volume of traffic entering and leaving this section during rush hour is given in diagram. Determine the amount of traffic between each of the four intersection.



Answer

$$x_1 - x_2 = 160$$

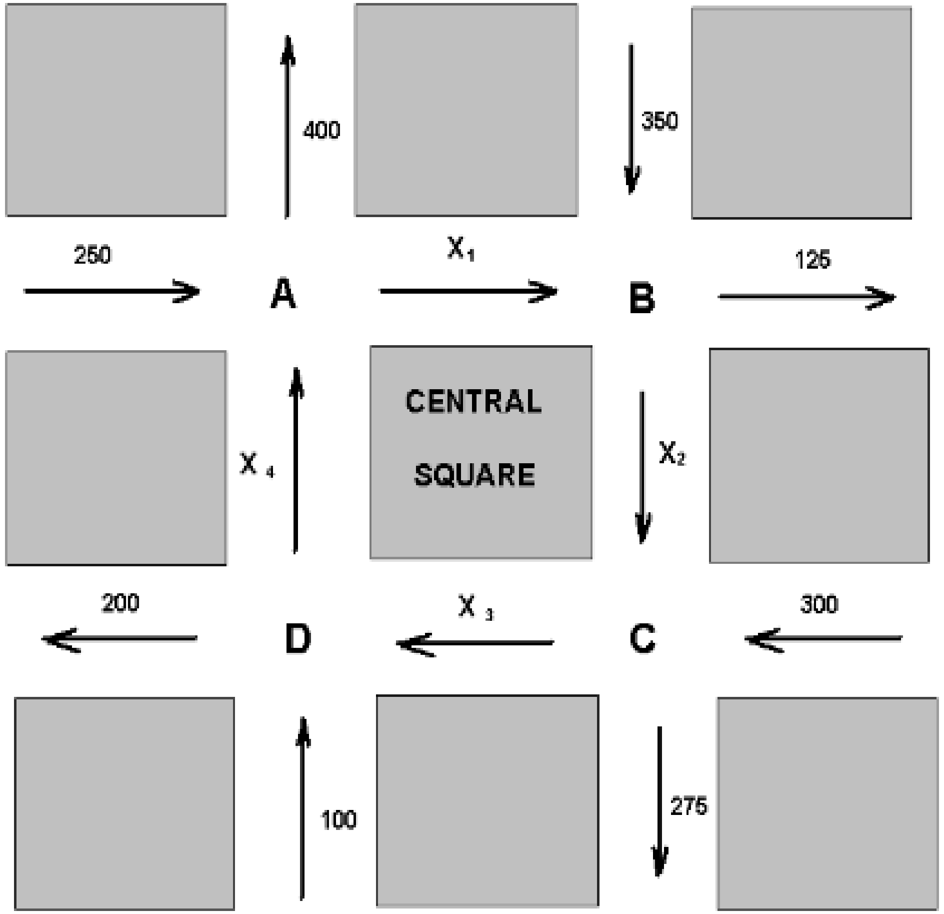
$$-x_2 + x_3 = 40$$

$$x_3 - x_4 = 210$$

$$x_1 - x_4 = 330$$

A

- Consider the following traffic diagram showing four one-way streets, with the average amount of rush-hour traffic entering and leaving each intersection Given that the number of cars entering each intersection equals amount of cars leaving that intersection, set up a system of four equations on $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$ and $\mathbf{x_4}$. For example, for intersection A we have $250+x_4=400+x_1$



At A:

$250 + x_4 = 400 + x_1$

Thus $x_1 - x_4 = -150$ ------(1)

At B: $x_1 - x_2 = -225$ ------(2)

At C: $x_2 - x_3 = -25$ ------(3)

At D: $x_3 - x_4 = 100$ ------(4)

Solution: Home Work

(Hint: infinitely many solutions)

Example Ace Novelty wishes to produce three types of souvenirs types $A, B,$ and C. To manufacture a type$-A$ souvenir requires 2 minutes on machine I, 1 minute on machine II, and 2 minute machine III. A type$-B$ souvenir requires 1 minutes on machine I, 3 minute on machine II, and 1 minute machine III. type$-C$ souvenir requires 1 minutes on machine I and 2 minutes each on machines II, III. There ara 3 hours available on machine I, 5 hours available on machine II, and 4 hours available on machine III for processing the order. How many souvenirs of each type should Ace Novelty make in order to use all of the available time?

Answer

$$2x_1 + x_2 + x_3 = 180$$

$$x_1 + 3x_2 + 2x_3 = 300$$

$$2x_1 + x_2 + 2x_3 = 240$$

$x_1 = 36, x_2 = 48$ and $x_3 = 60$.

Balancing Chemical Equations

Balancing a chemical equation

When hydrogen (H) burns in oxygen (O) it produces water.

$$H_2 + O_2 \to H_2O$$

Number of atoms of H and O are same on both sides of this reaction.

$$2H_2 + O_2 \to 2H_2O$$

EXAMPLE 5 | Balancing Chemical Equations Using Linear Systems

Balance the chemical equation

$$\begin{array}{ccccccc} \text{HCl} & + & \text{Na}_3\text{PO}_4 & \longrightarrow & \text{H}_3\text{PO}_4 & + & \text{NaCl} \\ [\text{hydrochloric acid}] & + & [\text{sodium phosphate}] & \longrightarrow & [\text{phosphoric acid}] & + & [\text{sodium chloride}] \end{array}$$

Solution Let x_1, x_2, x_3, and x_4 be positive integers that balance the equation

$$x_1\,(\text{HCl}) + x_2\,(\text{Na}_3\text{PO}_4) \longrightarrow x_3\,(\text{H}_3\text{PO}_4) + x_4\,(\text{NaCl}) \tag{7}$$

Equating the number of atoms of each type on the two sides yields

$$\begin{aligned}
1x_1 &= 3x_3 \quad &&\textbf{Hydrogen (H)}\\
1x_1 &= 1x_4 \quad &&\textbf{Chlorine (Cl)}\\
3x_2 &= 1x_4 \quad &&\textbf{Sodium (Na)}\\
1x_2 &= 1x_3 \quad &&\textbf{Phosphorus (P)}\\
4x_2 &= 4x_3 \quad &&\textbf{Oxygen (O)}
\end{aligned}$$

from which we obtain the homogeneous linear system

$$\begin{array}{rcrcrcrcl}
x_1 & & & - & 3x_3 & & & = & 0\\
x_1 & & & & & - & x_4 & = & 0\\
 & & 3x_2 & & & - & x_4 & = & 0\\
 & & x_2 & - & x_3 & & & = & 0\\
 & & 4x_2 & - & 4x_3 & & & = & 0
\end{array}$$

We leave it for you to show that the reduced row echelon form of the augmented matrix for this system is

$$\begin{bmatrix}
1 & 0 & 0 & -1 & 0\\
0 & 1 & 0 & -\frac{1}{3} & 0\\
0 & 0 & 1 & -\frac{1}{3} & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

from which we conclude that the general solution of the system is

$$x_1 = t, \quad x_2 = t/3, \quad x_3 = t/3, \quad x_4 = t$$

where t is arbitrary. To obtain the smallest positive integers that balance the equation, we let $t = 3$, in which case we obtain $x_1 = 3, x_2 = 1, x_3 = 1$, and $x_4 = 3$. Substituting these values in (7) produces the balanced equation

$$3HCl + Na_3PO_4 \longrightarrow H_3PO_4 + 3NaCl$$

Balance the following chemical equation. When solutions of sodium phosphate and barium nitrate are mixed, the result is barium phosphate and sodium nitrate. The unbalanced equation is

$$Na_3PO_4 + Ba(NO_3)_2 \rightarrow Ba_3(PO_4)_2 + NaNO_3$$

Example Balance the chemical equation and derive the balanced equation $HCl + Na_3PO_4 \rightarrow H_3PO_4 + NaCl.$

Answer

$$x_1HCl + x_2Na_3PO_4 \rightarrow x_3H_3PO_4 + x_4NaCl$$

$$x_1 - 3x_3 = 0$$

$$x_1 - x_4 = 0$$

$$3x_2 - x_4 = 0$$

$$x_2 - x_3 = 0$$

Example When propane burns in oxygen, it produces carbon dioxide and water, balance the chemical equation and derive the balanced equation $C_2H_8 + O_2 \to CO_2 + H_2O$.

$Answer\ x_1C_2H_8 + x_2O_2 \to x_3CO_2 + x_4H_2O.$

$$2x_1 - x_3 = 0$$

$$4x_1 - x_4 = 0$$

$$2x_2 - 2x_3 - x_4 = 0$$

$x_1 = 2, x_2 = 4, x_3 = 2, x_4 = 4.$

Electrical Circuits

The technical term for electrical pressure is ***electrical potential***; it is commonly measured in ***volts*** (V). The degree to which a resistor reduces the electrical potential is called its ***resistance*** and is commonly measured in ***ohms*** (Ω). The rate of flow of electrons in a wire is called ***current*** and is commonly measured in ***amperes*** (also called ***amps***) (A).

Ohm's Law If a current of I amperes passes through a resistor with a resistance of R ohms, then there is a resulting drop of E volts in electrical potential that is the product of the current and resistance; that is,

$$E = IR$$

$$V = RI$$

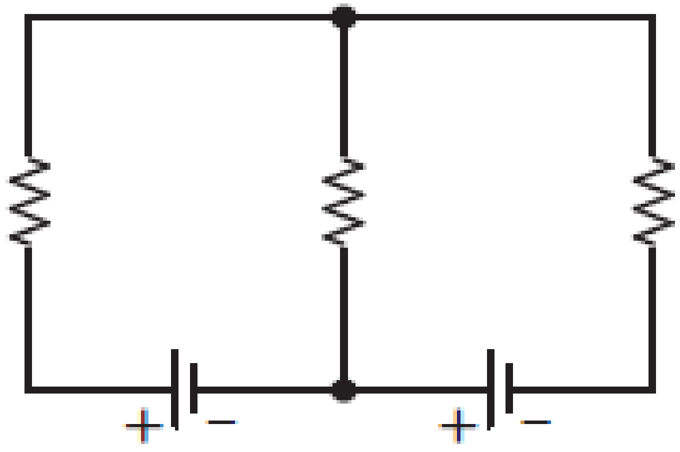
$$\text{Volts} = \text{Resistance} \times \text{Current Flow } I$$

$$RI = V$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Resistance Current Flow Volts

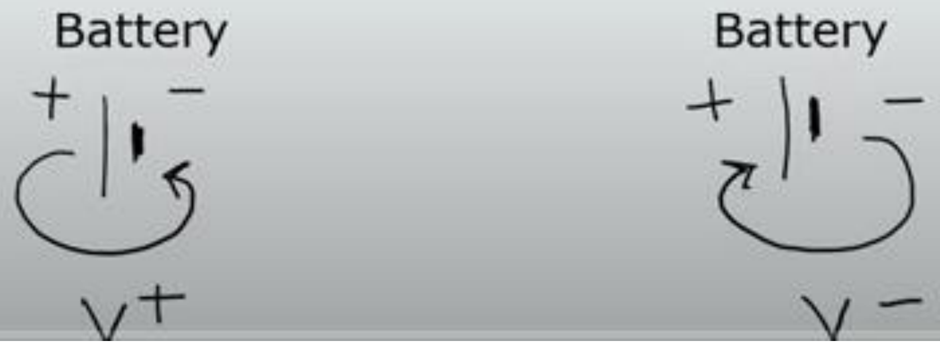
A point at which three or more wires in a network are joined is called a ***node*** (or ***junction point***).



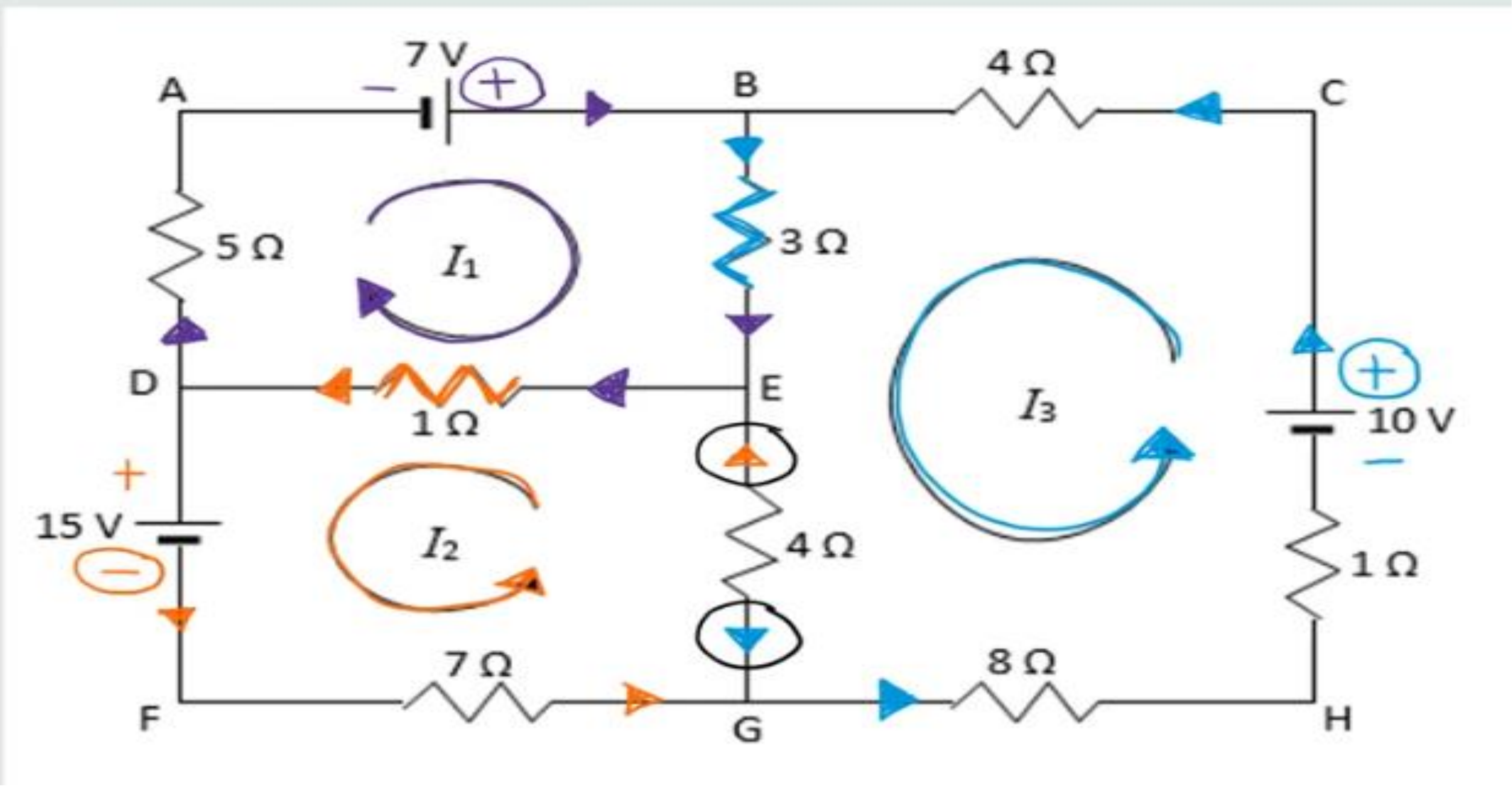
Kirchhoff's Current Law The sum of the currents flowing into any node is equal to the sum of the currents flowing out.

Kirchhoff's Voltage Law In one traversal of any closed loop, the sum of the voltage rises equals the sum of the voltage drops.

- If the loop current comes from the positive end, then the voltage will be positive.
- If the loop current comes from the negative end, then the voltage will be negative.



Question: For the figure below write a matrix equations and determine the loop currents.



A
B
C
D
E
F
G
H
7 V
4 Ω
5 Ω
3 Ω
1 Ω
15 V
4 Ω
10 V
1 Ω
7 Ω
8 Ω
I_1
I_2
I_3

Loop 1: $(3+1+5)I_1 + 1I_2 + 3I_3 = 7$

Loop 2: $1I_1 + (7+4+1)I_2 - 4I_3 = -15$

Loop 3: $3I_1 - 4I_2 + (4+3+4+8+1)I_3 = 10$

$$\Rightarrow \begin{bmatrix} 9 & 1 & 3 \\ 1 & 12 & -4 \\ 3 & -4 & 20 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -15 \\ 10 \end{bmatrix}$$

Solve for the Loop Currents:

$$\begin{bmatrix} 9 & 1 & 3 & 7 \\ 1 & 12 & -4 & -15 \\ 3 & -4 & 20 & 10 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} I_1 & I_2 & I_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0.88 \\ 0 & 1 & 0 & -1.29 \\ 0 & 0 & 1 & 0.11 \end{bmatrix} \end{array}$$

$$\boxed{\begin{aligned} I_1 &= 0.88 \text{ amps} \\ I_2 &= -1.29 \text{ amps} \\ I_3 &= 0.11 \text{ amps} \end{aligned}}$$

A Circuit with Three Closed Loops: Determine the currents I_1, I_2, and I_3 in the circuit shown in Figure 4

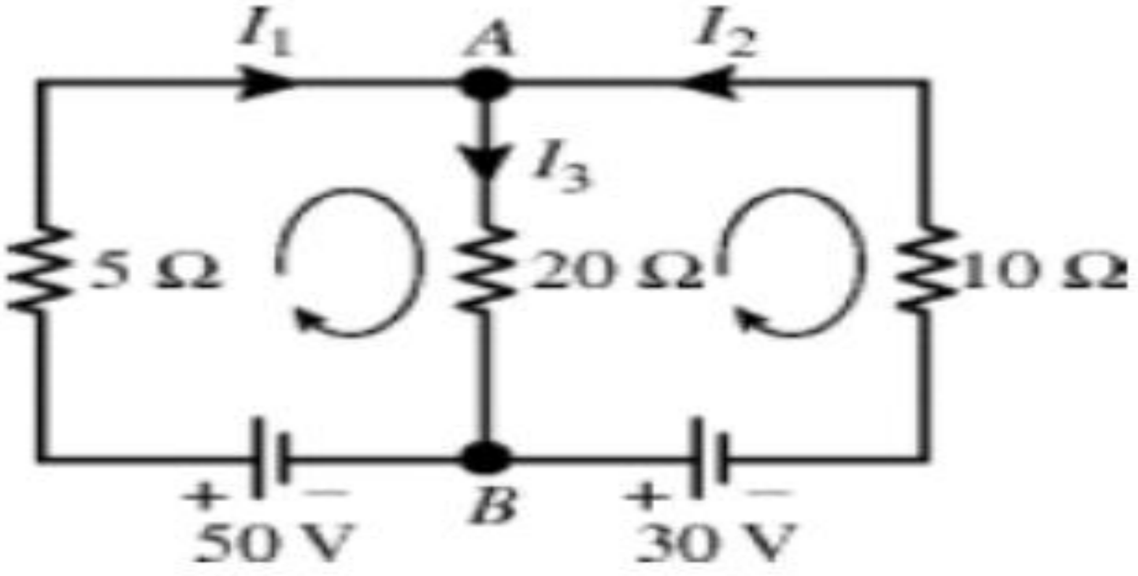


Figure: (iv)

Solution: Using the assigned directions for the currents, Kirchhoff s current law provides one equation for each node:

Node	Current In		Current Out
A	$I_1 + I_2$	$=$	I_3
B	I_3	$=$	$I_1 + I_2$

However, these equations are really the same, since both can be expressed as

$$I_1 + I_2 - I_3 = 0$$

To find unique values for the currents we will need two more equations, which we will obtain from Kirchhoff's voltage law. Thus from the network diagram

	Voltage Rises	Voltage Drops
Left Inside Loop	50	$5I_1 + 20I_3$
Right Inside Loop	$30 + 10I_2 + 20I_3$	0
Outside Loop	$30 + 50 + 10I_2$	$5I_1$

These conditions can be rewritten as

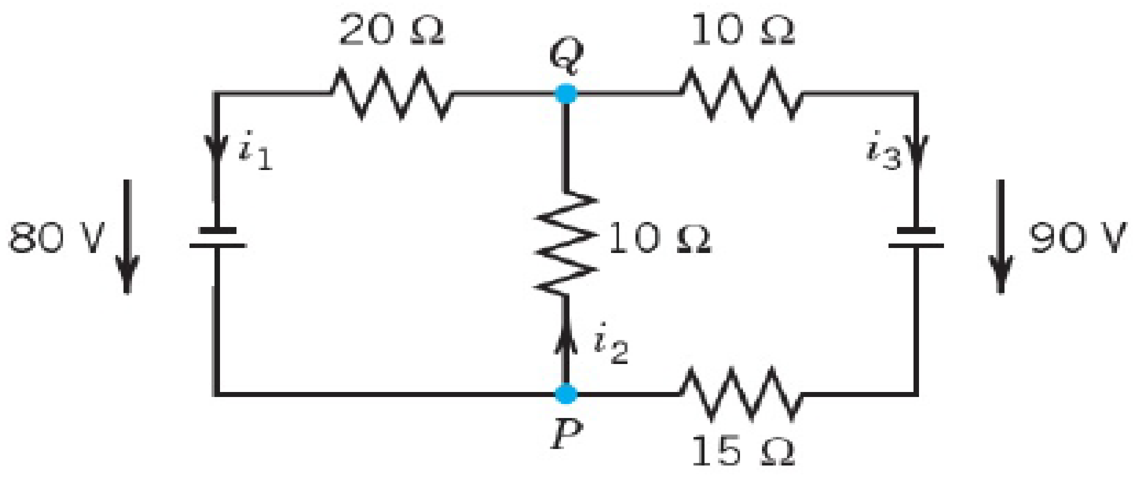
$$5I_1 + 20I_3 = 50$$
$$10I_2 + 20I_3 = -30$$
$$5I_1 - 10I_2 = 80$$

After combining all the equations we obtain

$$I_1 + i_2 - I_3 = 0$$
$$5I_1 + 20I_3 = 50$$
$$10I_2 + 20I_3 = -30$$

Thus after solving the system of equations we get $I_1 = 6A$, $I_2 = -5A$ and $I_3 = 1A$.

Example Find the current of the following electrical network.



Answer

$$i_1 - i_2 + i_3 = 0$$

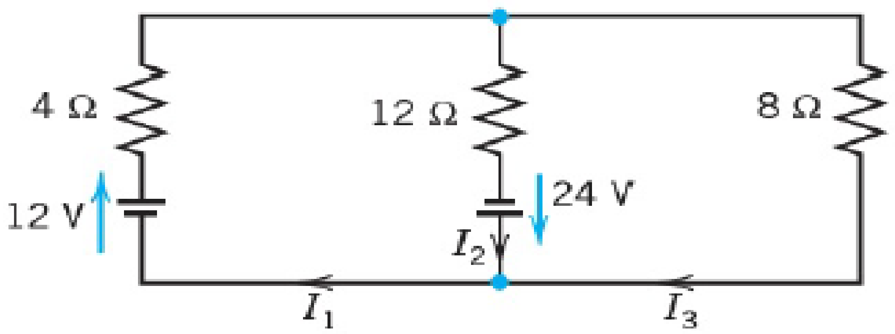
$$-i_1 + i_2 - i_3 = 0$$

$$10i_2 + 25i_3 = 90$$

$$20i_1 + 10i_2 = 80$$

$i_3 = 2, i_2 = 4, i_1 = 2$

Example Find the current of the following electrical network.



$-i_1 + i_2 + i_3 = 0,\ 4i_1 + 12i_2 = 36$

$12i_2 - 8i_3 = 24$

Polynomial Interpolation

Now let us consider the more general problem of finding a polynomial whose graph passes through n points with distinct x-coordinates

$$(x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3), \ldots, \quad (x_n, y_n) \tag{10}$$

Since there are n conditions to be satisfied, intuition suggests that we should begin by looking for a polynomial of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} \tag{11}$$

Theorem 1.10.1

Polynomial Interpolation

Given any n points in the xy-plane that have distinct x-coordinates, there is a unique polynomial of degree $n - 1$ or less whose graph passes through those points.

Let us now consider how we might go about finding the interpolating polynomial (11) whose graph passes through the points in (10). Since the graph of this polynomial is the graph of the equation

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \tag{12}$$

it follows that the coordinates of the points must satisfy

$$\begin{array}{ccccccccc} a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + \cdots + & a_{n-1} x_1^{n-1} & = & y_1 \\ a_0 & + & a_1 x_2 & + & a_2 x_2^2 & + \cdots + & a_{n-1} x_2^{n-1} & = & y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_0 & + & a_1 x_n & + & a_2 x_n^2 & + \cdots + & a_{n-1} x_n^{n-1} & = & y_n \end{array} \tag{13}$$

In these equations the values of x's and y's are assumed to be known, so we can view this as a linear system in the unknowns $a_0, a_1, \ldots, a_{n-1}$. From this point of view the augmented matrix for the system is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{bmatrix} \tag{14}$$

Hence, the interpolating polynomial can be found by reducing this matrix to reduced row echelon form

Question.

Find a cubic polynomial whose graph passes through the points

$$(1, 3), \quad (2, -2), \quad (3, -5), \quad (4, 0)$$

Solution Since there are four points, we will use an interpolating polynomial of degree $n = 3$. Denote this polynomial by

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

and denote the x- and y-coordinates of the given points by

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 4 \quad \text{and} \quad y_1 = 3, \quad y_2 = -2, \quad y_3 = -5, \quad y_4 = 0$$

Thus, it follows from (14) that the augmented matrix for the linear system in the unknowns a_0, a_1, a_2, and a_3 is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 \\ 1 & x_2 & x_2^2 & x_2^3 & y_2 \\ 1 & x_3 & x_3^2 & x_3^3 & y_3 \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

We leave it for you to confirm that the reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

from which it follows that $a_0 = 4, a_1 = 3, a_2 = -5, a_3 = 1$. Thus, the interpolating polynomial is

$$p(x) = 4 + 3x - 5x^2 + x^3$$

The graph of this polynomial and the given points are shown in Figure 1.10.12.

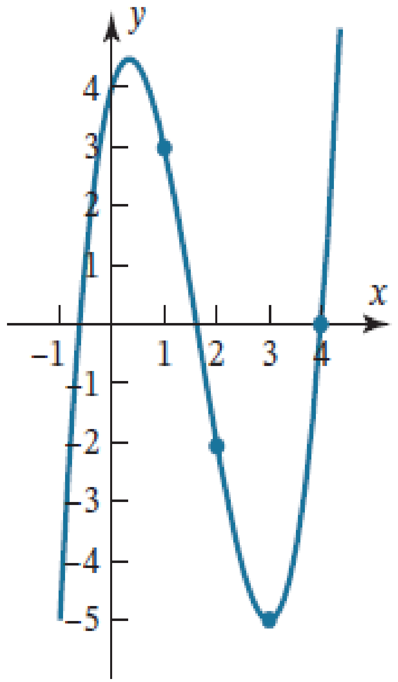


FIGURE 1.10.12

EXERCISES:

Find the quadratic polynomial whose graph passes through the points $(0,0)$, $(-1,1)$, and $(1,1)$.

Find the cubic polynomial whose graph passes through the points $(-1,-1)$, $(0,1)$, $(1,3)$, $(4,-1)$.

Leontief Input-Output Models

Leontief Model of an Open Economy

Let us consider a simple open economy with one open sector and three product-producing sectors: manufacturing, agriculture, and utilities. Assume that inputs and outputs are measured in dollars and that the inputs required by the productive sectors to produce one dollar's worth of output are in accordance with Table 1.

TABLE 1

		Input Required per Dollar Output		
		Manufacturing	**Agriculture**	**Utilities**
Provider	**Manufacturing**	\$ 0.50	\$ 0.10	\$ 0.10
	Agriculture	\$ 0.20	\$ 0.50	\$ 0.30
	Utilities	\$ 0.10	\$ 0.30	\$ 0.40

$$C = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.4 \end{bmatrix} \tag{1}$$

This is called the ***consumption matrix*** (or sometimes the ***technology matrix***) for the economy. The column vectors

$$\mathbf{c}_1 = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.3 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \end{bmatrix}$$

C is the list of inputs required by the manufacturing, agricultural, and utilities sectors, respectively.

$$\underbrace{x_1\begin{bmatrix}0.5\\0.2\\0.1\end{bmatrix}}_{\boxed{\text{Fractions consumed by manufacturing}}} + \underbrace{x_2\begin{bmatrix}0.1\\0.5\\0.3\end{bmatrix}}_{\boxed{\text{Fractions consumed by agriculture}}} + \underbrace{x_3\begin{bmatrix}0.1\\0.3\\0.4\end{bmatrix}}_{\boxed{\text{Fractions consumed by utilities}}} = \begin{bmatrix}0.5 & 0.1 & 0.1\\0.2 & 0.5 & 0.3\\0.1 & 0.3 & 0.4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = C\mathbf{x}$$

The vector $C\mathbf{x}$ is called the ***intermediate demand vector*** for the economy. Once the intermediate demand is met, the portion of the production that is left to satisfy the outside demand is $\mathbf{x} - C\mathbf{x}$. Thus, if the outside demand vector is $\mathbf{d}$, then $\mathbf{x}$ must satisfy the equation

$$\underbrace{\mathbf{x}}_{\boxed{\text{Amount produced}}} - \underbrace{C\mathbf{x}}_{\boxed{\text{Intermediate demand}}} = \underbrace{\mathbf{d}}_{\boxed{\text{Outside demand}}}$$

x_1 dollars of manufactured goods

x_2 dollars of agricultural products

x_3 dollars of utilities

$$\boxed{(I - C)\mathbf{x} = \mathbf{d}} \tag{2}$$

The matrix $I - C$ is called the ***Leontief matrix*** and (2) is called the ***Leontief equation***.

EXAMPLE 1 | Satisfying Outside Demand

Consider the economy described in Table 1. Suppose that the open sector has a demand for \$7900 worth of manufacturing products, \$3950 worth of agricultural products, and \$1975 worth of utilities.

(*a*) Can the economy meet this demand?

(*b*) If so, find a production vector $\mathbf{x}$ that will meet it exactly.

Solution The consumption matrix, production vector, and outside demand vector are

$$C = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix} \tag{3}$$

To meet the outside demand, the vector $\mathbf{x}$ must satisfy the Leontief equation (2), so the problem reduces to solving the linear system

$$\underbrace{\begin{bmatrix} 0.5 & -0.1 & -0.1 \\ -0.2 & 0.5 & -0.3 \\ -0.1 & -0.3 & 0.6 \end{bmatrix}}_{I-C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix}}_{\mathbf{d}} \tag{4}$$

(if consistent). We leave it for you to show that the reduced row echelon form of the augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 27{,}500 \\ 0 & 1 & 0 & 33{,}750 \\ 0 & 0 & 1 & 24{,}750 \end{array}\right]$$

This tells us that (4) is consistent, and the economy can satisfy the demand of the open sector exactly by producing \$27,500 worth of manufacturing output, \$33,750 worth of agricultural output, and \$24,750 worth of utilities output.

In this case, the consumption matrix, production vector, and outside demand vector have the form

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

where all entries are nonnegative and

c_{ij} = the monetary value of the output of the ith sector that is needed by the jth sector to produce one unit of output

x_i = the monetary value of the output of the ith sector

d_i = the monetary value of the output of the ith sector that is required to meet the demand of the open sector

As discussed in our example above, a production vector $\mathbf{x}$ that meets the demand $\mathbf{d}$ of the outside sector must satisfy the Leontief equation

$$(I - C)\mathbf{x} = \mathbf{d}$$

If the matrix $I - C$ is invertible, then this equation has the unique solution

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} \tag{5}$$

Theorem 1.11.1

If C is the consumption matrix for an open economy, and if all of the column sums are less than 1, then the matrix $I - C$ is invertible, the entries of $(I - C)^{-1}$ are nonnegative, and the economy is productive.

EXAMPLE 2 | An Open Economy Whose Sectors Are All Profitable

The column sums of the consumption matrix C in (1) are less than 1, so $(I - C)^{-1}$ exists and has nonnegative entries. Use a calculating utility to confirm this, and use this inverse to solve Equation (4) in Example 1.

Solution We leave it for you to show that

$$(I - C)^{-1} \approx \begin{bmatrix} 2.65823 & 1.13924 & 1.01266 \\ 1.89873 & 3.67089 & 2.15190 \\ 1.39241 & 2.02532 & 2.91139 \end{bmatrix}$$

This matrix has nonnegative entries, and

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} \approx \begin{bmatrix} 2.65823 & 1.13924 & 1.01266 \\ 1.89873 & 3.67089 & 2.15190 \\ 1.39241 & 2.02532 & 2.91139 \end{bmatrix} \begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix} \approx \begin{bmatrix} 27{,}500 \\ 33{,}750 \\ 24{,}750 \end{bmatrix}$$

which is consistent with the solution in Example 1.

Example:

Consider the open economy described by the accompanying table, where the input is in dollars needed for $1.00 of output.

a. Find the consumption matrix for the economy.

b. Suppose that the open sector has a demand for $1930 worth of housing, $3860 worth of food, and $5790 worth of utilities. Use row reduction to find a production vector that will meet this demand exactly.

Input Required per Dollar Output

Provider		Housing	Food	Utilities
	Housing	$ 0.10	$ 0.60	$ 0.40
	Food	$ 0.30	$ 0.20	$ 0.30
	Utilities	$ 0.40	$ 0.10	$ 0.20

Applications: Finding the points of intersection

- Let ℓ_1 be the line through the point $P = (4, -7, 2)$ with direction vector $\mathbf{u} = [3, 2, 2]$, and let ℓ_2 be the line through the point $Q = (1, 0, -3)$ with direction vector $\mathbf{v} = [8, -3, -2]$.
  (a) Write down parametric equations for ℓ_1 and ℓ_2 in terms of parameters $s \in \mathbb{R}$ and $t \in \mathbb{R}$, respectively.
  (b) State a system of linear equations involving s and t whose solution (if it exists) would give a point of intersection of ℓ_1 and ℓ_2.
  (c) By first reducing the associated augmented matrix to row echelon form, determine whether or not ℓ_1 and ℓ_2 intersect.

(a) l_1:

$$P + su = (4, -7, 2) + s <3, 2, 2>$$
$$= < 4 + 3s, -7 + 2s, 2 + 2s>$$

l_2:

$$Q + tv = (1, 0, -3) + t <8, -3, -2>$$
$$= < 1 + 8t, 0 - 3t, -3 - 2t>$$

(b) $< 4 + 3s, -7 + 2s, 2 + 2s> \; = < 1 + 8t, -3t, -3 - 2t>$

Thus $4 + 3s = 1 + 8t \quad \Rightarrow 3s - 8t = -3$ ------(1)

$-7 + 2s = -3t \quad \Rightarrow 2s + 3t = 7$ ------(2)

$2 + 2s = -3 - 2t \Rightarrow 2s + 2t = -5$ ------(2)

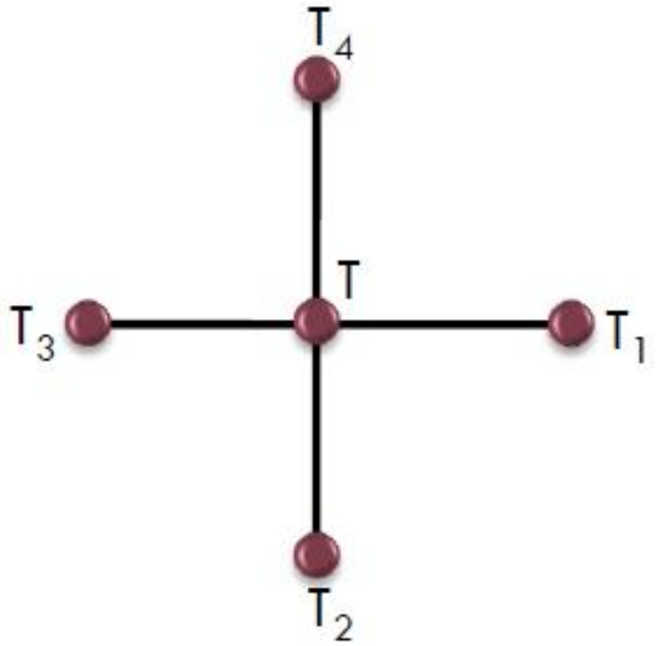
(c) $3s - 8t = -3,\ 2s + 3t = 7,\ 2s + 2t = -5$

$$\begin{bmatrix} 3 & -8 & -3 \\ 2 & 3 & 7 \\ 2 & 2 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -8 & -3 \\ 0 & 25/3 & 9 \\ 0 & 22/3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -8 & -3 \\ 0 & 25/3 & 9 \\ 0 & 0 & -273/25 \end{bmatrix}$$

Thus the system is inconsistent

Applications: Temperature Distribution

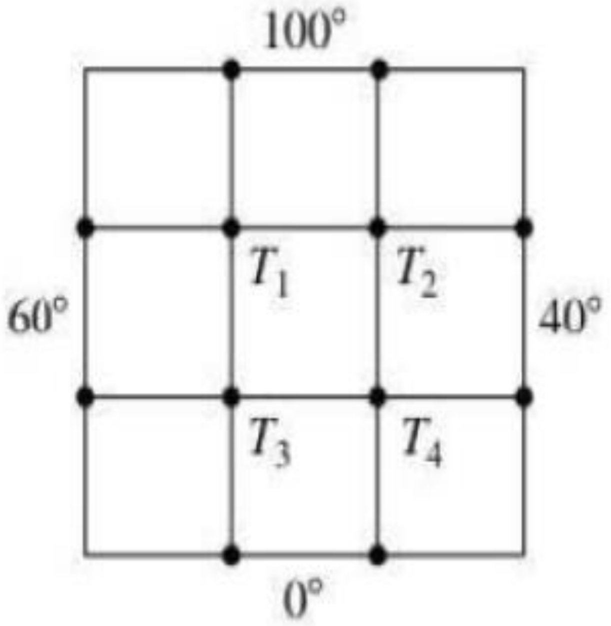
A simple model for estimating the temperature distribution on a square plate gives rise to a linear system of equations. To construct the appropriate linear system, we use the following information: The square plate is perfectly insulated on its top and bottom so that the only heat flow is through the plate itself. The four edges are held at various temperatures. To estimate the temperature at an interior point on the plate, we use the rule that it is the average of the temperatures at its four compass-point neighbors, to the west, north, east, and south.



$$T = (T_1 + T_2 + T_3 + T_4)/\ 4$$

Example

- Estimate the temperatures T_i, $i = 1, 2, 3, 4$, at the four equispaced interior points on the plate shown as



$$
\begin{aligned}
T_1 &= \frac{60 + 100 + T_2 + T_3}{4} && \text{or} && 4T_1 - T_2 - T_3 = 160 \\
T_2 &= \frac{T_1 + 100 + 40 + T_4}{4} && \text{or} && -T_1 + 4T_2 - T_4 = 140 \\
T_3 &= \frac{60 + T_1 + T_4 + 0}{4} && \text{or} && -T_1 + 4T_3 - T_4 = 60 \\
T_4 &= \frac{T_3 + T_2 + 40 + 0}{4} && \text{or} && -T_2 - T_3 + 4T_4 = 40.
\end{aligned}
$$

The augmented matrix for this linear system is

$$
\left[A \mid \mathbf{b}\right] = \left[\begin{array}{rrrr|r} 4 & -1 & -1 & 0 & 160 \\ -1 & 4 & 0 & -1 & 140 \\ -1 & 0 & 4 & -1 & 60 \\ 0 & -1 & -1 & 4 & 40 \end{array}\right]
$$

we obtain the unique solution (verify)

$$
T_1 = 65^\circ, \quad T_2 = 60^\circ, \quad T_3 = 40^\circ, \quad \text{and} \quad T_4 = 35^\circ.
$$