## Gauss - Jordan elimination method:

Consider the system of linear eqns as:

$$AX = B \quad \text{--- (1)}$$

In Gauss - Jordan elimination method, apply elementary row operations on A and B such that A reduces to the normal form $I_r$. Then the solution is obtained.

**Exp (1):** Solve the system by (i) Gauss-elimination method (ii) Gauss - Jordan method

$$2x_1 + 5x_2 + 2x_3 - 3x_4 = 3$$
$$3x_1 + 6x_2 + 5x_3 + 2x_4 = 2$$
$$4x_1 + 5x_2 + 14x_3 + 14x_4 = 11$$
$$5x_1 + 10x_2 + 8x_3 + 4x_4 = 4$$

**Solution:** Consider the augmented matrix $C = [A : B]$

$$C = [A:B] = \left[\begin{array}{cccc:c} 2 & 5 & 2 & -3 & 3 \\ 3 & 6 & 5 & 2 & 2 \\ 4 & 5 & 14 & 14 & 11 \\ 5 & 10 & 8 & 4 & 4 \end{array}\right]$$

applying $R_4 \to R_4 - R_3$, $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$

$$\sim \left[\begin{array}{cccc:c} 2 & 5 & 2 & -3 & 3 \\ 1 & 1 & 3 & 5 & -1 \\ 1 & -1 & 9 & 12 & 9 \\ 1 & 5 & -6 & -10 & -7 \end{array}\right]$$

$R_4 \longleftrightarrow R_1$ and $R_2 \longleftrightarrow R_3$

$$\sim \left[\begin{array}{cccc:c} 1 & 5 & -6 & -10 & -7 \\ 1 & -1 & 9 & 12 & 9 \\ 1 & 1 & 3 & 5 & -1 \\ 2 & 5 & 2 & -3 & 3 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{cccc:c} 1 & 5 & -6 & -10 & -7 \\ 0 & -6 & 15 & 22 & 16 \\ 0 & -4 & 9 & 15 & 6 \\ 0 & -5 & 14 & 17 & 17 \end{array}\right]$$

$R_2 \to R_2 - R_4$ and $R_2 \to -R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & 5 & -6 & -10 & -7 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & -4 & 9 & 15 & 6 \\ 0 & -5 & 14 & 17 & 17 \end{array}\right]$$

$R_3 \to R_3 + 4R_2$, $R_4 \to R_4 + 5R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & 5 & -6 & -10 & -7 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & -5 & 5 & -10 \\ 0 & 0 & 9 & -8 & 22 \end{array}\right]$$

applying $R_3 \to -\frac{1}{5} R_3$ and $R_4 \to R_4 - 9R_3$

$$\sim \left[\begin{array}{cccc:c} 1 & 5 & -6 & -10 & -7 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

By back substitution:

$$x_4 = 4$$

$$x_3 - x_4 = 2 \Rightarrow x_3 = 6$$

$$x_2 - x_3 - 5x_4 = 1 \Rightarrow x_2 = 27$$

$$x_1 + 5x_2 - 6x_3 - 10x_4 = -7 \Rightarrow x_1 = -66$$

Gauss-Jordan method:

$$\left[\begin{array}{cccc:c} 1 & 5 & -6 & -10 & -7 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

applying $R_3 \to R_3 + R_4$, $R_2 \to R_2 + 5R_4$, $R_1 \to R_1 + 10R_4$

$$\sim \left[\begin{array}{cccc:c} 1 & 5 & -6 & 0 & 33 \\ 0 & 1 & -1 & 0 & 21 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

applying $R_2 \to R_2 + R_3$, $R_1 \to R_1 + 6R_3$

$$\sim \left[\begin{array}{cccc:c} 1 & 5 & 0 & 0 & 69 \\ 0 & 1 & 0 & 0 & 27 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

applying $R_1 \to R_1 - 5R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & -66 \\ 0 & 1 & 0 & 0 & 27 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$\therefore \; x_1 = -66, \; x_2 = 27, \; x_3 = 6, \; x_4 = 4$

compute the inverse of a matrix from elementary

(Gauss - Jordan method)

If $A$ is reduced to $I$ by elementary operation then

$$PA = I \quad \text{where} \quad P = P_n P_{n-1} \cdots P_2 \cdot P_1$$

$$\Rightarrow P = A^{-1} \qquad = \text{elementary matrix.}$$

Working rule: write $A = IA$

Perform elementary row operation on $A$ of the left side and on $I$ of the right hand side so that $A$ is reduced to $I$ and $I$ of right hand side is reduced to $P$ getting

$$I = PA$$

Then $P$ is the inverse of $A$.

Exp (1): Find the inverse of the given matrix by Gauss - Jordan method.

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Sol^n: write $A = IA$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

applying $R_1 \to \frac{1}{3} R_1$

$$\sim \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

applying $R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

applying $R_2 \to -R_2$

$$\sim \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

applying $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 1 \end{bmatrix}$$

applying $R_3 \to -3R_3$

$$\begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A$$

applying $R_1 \to R_1 - \frac{4}{3} R_3$ & $R_2 \to R_2 + \frac{4}{3} R_3$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

applying $R_1 \to R_1 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

Hence $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

Exp (2): Find the inverse, by Gauss-Jordan method, of the matrix:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Soln: write $A = IA$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$R_3 \to \frac{1}{2} R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

$R_1 \to R_1 - 3R_3, \quad R_2 \to R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2} & +\frac{11}{2} & -\frac{3}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

$$\Rightarrow \quad I = A^{-1} I$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

Exercise (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$ Ans: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

(2) $A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 0 & 1 & 5 \end{bmatrix}$ Ans: $\begin{bmatrix} 0 & 1 & 4 \\ 0 & 5 & -19 \\ 0 & 0 & 22 \end{bmatrix}$

# LDU Factorization:

In general, if an $n \times n$ matrix $A$ can be reduced to an upper triangular form $U$ without any interchanges of rows, then $A$ can be factored into a product $LU$, where $L$ is lower triangular with 1's on the diagonal. The $(i, j)$ entry of $L$ below the diagonal will be the multiple of the $i$th row that was subtracted from the $j$th row during the elementary process.

**Exp (1):**

$$A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 5 \end{bmatrix}$$

**Sol$^n$:** write $A = IA$

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

apply: $R_2 \to R_2 - \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 4 & 2 & -2 \\ 0 & 9 & 3 \\ -2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Here $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

apply: $R_3 \to R_3 + \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 4 & 2 & -2 \\ 0 & 9 & 3 \\ 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} A$$

Here $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

apply: $R_3 \to R_3 - \frac{1}{3} R_2$

$$\sim \begin{bmatrix} 4 & 2 & -2 \\ 0 & 9 & 3 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} A$$

Here $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}$

Thus $U = E_3 \cdot E_2 \cdot E_1 \cdot A$

$$\begin{bmatrix} 4 & 2 & -2 \\ 0 & 9 & 3 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 5 \end{bmatrix}$$

Since the elementary matrices are non-singular, it follows that

$$A = \left(E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}\right) U$$

When the inverse elementary matrices are multiplied in this order, the result is a lower triangular matrix $L$ with 1's on the diagonal. The entries below the diagonal of $L$ will just be the multiples that were subtracted during the elimination process.

$$\therefore \quad E_1^{-1} E_2^{-2} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} = L.$$

$$\therefore \quad A = LU.$$

It is possible to go one step further and factor $U$ into a product $DU_1$, where $D$ is diagonal and $U_1$ is upper triangular with 1's on the diagonal:

$$DU_1 = \begin{bmatrix} u_{11} & \cdots & \cdots & 0 \\ 0 & u_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \frac{u_{13}}{u_{11}} & \cdots & \frac{u_{1n}}{u_{11}} \\ 0 & 1 & \frac{u_{23}}{u_{22}} & \cdots & \frac{u_{2n}}{u_{22}} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

It follows, then, that

$$A = LDU_1$$

The matrices $L$ and $U_1$ are referred to as unit triangular matrices since they are triangular and their diagonal entries are all equal to 1.

→ The representation of a square matrix $A$ as a product of the form $LDU$, where $L$ is a unit lower triangular matrix, $D$ is diagonal, and $U$ is a unit upper triangular matrix, is refferred to as an $LDU$ factorization of $A$.

→ In general if $A$ has an $LDU$ factorization, then it is unique.

$$\therefore A = LU$$

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & -2 \\ 0 & 9 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore \quad U = DU'$

$$\begin{bmatrix} 4 & 2 & -2 \\ 0 & 9 & 3 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$A = LDU'$$

$$\Rightarrow \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

**Exp (2):** Find LDU factorization of the given matrix

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}$$

**Soln:** write $A = IA$

$$\begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

apply: $R_2 \to R_2 - \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 4 & 2 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} A$$

Here $E_1 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$

ie $U = E_1 A$

$\Rightarrow A = E^{-1} U$

$\Rightarrow A = LU$ , where $L = E^{-1}$

ie $\begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 9 \end{bmatrix}$

$\because U = DU' = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

$\therefore \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

Exp (3): Find LDU factorization of

$$A = \begin{bmatrix} 9 & 3 & -6 \\ 3 & 4 & 1 \\ -6 & 1 & 9 \end{bmatrix}$$

Sol$^n$: write $A = IA$

$$\begin{bmatrix} 9 & 3 & -6 \\ 3 & 4 & 1 \\ -6 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

apply: $R_2 \to R_2 - \frac{1}{3} R_1$

$$\sim \begin{bmatrix} 9 & 3 & -6 \\ 0 & 3 & 3 \\ -6 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Here $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

apply: $R_3 \to R_3 + \frac{2}{3} R_1$

$$\sim \begin{bmatrix} 9 & 3 & -6 \\ 0 & 3 & 3 \\ 0 & 3 & +5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +\frac{2}{3} & 0 & 1 \end{bmatrix} A$$

Here $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{bmatrix}$

apply: $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 9 & 3 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & +2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A$$

Here $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore \quad U = E_3 \cdot E_2 \cdot E_1 \, A$

$$\begin{bmatrix} 9 & 3 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & +2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +\frac{2}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & - \\ 3 & 4 & 1 \\ -6 & 1 & 9 \end{bmatrix}$$

$\Rightarrow \quad A = (E_1^{-1} E_2^{-1} E_3^{-1}) \, U = LU$

$$\therefore \; L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A = LU$$

$$\Rightarrow \begin{bmatrix} 9 & 3 & -6 \\ 3 & 4 & 1 \\ -6 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & +2 \end{bmatrix}$$

Now, write

$$U = DU'$$

$$\begin{bmatrix} 9 & 3 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & +2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & +2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\quad A = LDU'$

$$\Rightarrow \begin{bmatrix} 9 & 3 & -6 \\ 3 & 4 & 1 \\ -6 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

[illegible] 3, 7, 16, 23, 53, ~~[illegible]~~, 76,