$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in Echelon form. Number of non-zero rows is '2' $\boxed{\therefore \rho(A) = 2.}$

System of linear Simultaneous Equations :

Definition :- An equation of the form

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $x_1, x_2 \ldots x_n$ are unknowns and $a_1, a_2 \ldots a_n, b$ are constants is called a linear equation in n unknowns.

Non - Homogeneous linear equations:

Consider m linear non-homogeneous equations in n unknowns as given below

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots \cdots \cdots \cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The above system of equations in matrix notation can be written as $AX = B$.

where A is the coefficient matrix formed by coefficients of unknowns.

i.e., $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

The matrix $[AB]$ is called the augmented matrix formed by the Coefficient matrix together with column formed by Constants $b_1, b_2 \cdots b_m$.

Condition of Consistency :-

The System of equations $Ax = B$ is Consistent iff the rank of the Coefficient matrix A is equal to the rank of the Augment Matrix $[AB]$

i.e. Consistent $\Rightarrow$ Rank of A = Rank AB

Nature of Solution :- For Non-homogeneous System

The System of equations $Ax = B$ is said to be

(i) Consistent if Rank A = Rank $[AB]$

(ii) Consistent and an unique Solution if

Rank of A = Rank of $[AB]$ = $r = n$.

where r is the rank and n is the number of unknowns.

(iii) Consistent and an infinite number of solutions if Rank A < Rank [AB] $\Rightarrow r < n$

In this case, we have to give arbitrary values to $n-r$ variables and the remaining variables can be expressed in terms of these arbitrary values.

(iv) Inconsistent if Rank of A $\neq$ Rank of [AB]

Problems :-

1. Find whether the following equations are consistent, if so solve them. $x+y+2z=4$; $2x-y+3z=9$; $3x-y-z=2$.

Sol: Given equations are
$$x+y+2z=4$$
$$2x-y+3z=9$$
$$3x-y-z=2$$

The given system of equations in matrix notation can be written as $AX=B$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

The augmented matrix
$$[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{array}\right]$$

Reducing the matrix [AB] to Echelon form by elementary row transformations we get $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$[A\ B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{array}\right] \quad R_3 \to 3R_3 - 4R_2$$

$$[AB] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{array}\right]$$

$\therefore$ Rank of $[AB] = 3$ (number of non-zero rows)

Rank of $A = 3$ (leaving last column in $[AB]$)

$\therefore$ Rank of A = Rank of $[AB] = 3$

Here number of unknowns $= 3$

$\therefore$ The given equations is consistent and have a unique solution.

Now, the given equations can be written as

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x + y + 2z &= 4 \quad \text{--- (1)} \\ -3y - z &= 1 \quad \text{--- (2)} \\ -17z &= -34 \quad \text{--- (3)} \end{aligned}$$

$\Rightarrow z = 2$

Substituting $z = 2$ in eq (2) we get

$-3y - 2 = 1 \Rightarrow y = -1$

Substituting $z = 2$, $y = -1$ in eq (1) we get

$x - 1 + 4 = 4 \Rightarrow x = 1$

$$\boxed{\therefore \quad x = 1, \; y = -1, \; z = 2 \quad \text{is the solution}}$$

2. Prove that the following set of equations are Consistent and solve them

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

Sol: The given system of equations Can be written in the matrix form i.e $AX = B$

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

The Augmented matrix

$$[AB] = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right] \qquad \begin{array}{l} R_2 \to 3R_2 - R_1 \\ R_4 \to 3R_4 - 2R_1 \end{array}$$

$-9-6$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 10 & 3 & -2 \\ 0 & -15 & -7 & 13 \end{array}\right] \qquad \begin{array}{l} R_3 \to 3R_3 - 10R_2 \\ R_4 \to R_4 + 5R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & -17 & 68 \end{array}\right] \qquad R_3/29, \quad R_4/17$$

$$\sim \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{array}\right] \qquad R_4 \to R_4 + R_3$$

$$[AB] \sim \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 0 & 3 & -2 & 11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of $A = 3$; Rank of $[AB] = 3$

and number of unknowns $= 3$

$\therefore$ The given equations is consistent and have a unique solution.

Now the given equations can be written as

$$\begin{bmatrix} 3 & 3 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \\ -4 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 3x + 3y + 2z = 1 \quad \text{--- (1)}$

$3y - 2z = 11 \quad \text{--- (2)}$

$z = -4$

Substituting $z = -4$ in eq (2) we get

$3y + 8 = 11$

$3y = 3 \Rightarrow y = 1$

Substituting $z = -4$, $y = 1$ in eq (1), we get

$3x + 3 - 8 = 1 \Rightarrow x = 2$

$\boxed{\therefore\ x = 2,\ y = 1,\ z = -4 \text{ is the solution}}$

3. Show that the equations $x - 4y + 7z = 14$, $3x + 8y - 2z = 13$, $7x - 8y + 26z = 5$ are not consistent.

Sol: The given system of equations can be written as

$AX = B.$ $$\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}$$

$$[AB] = \left[\begin{array}{ccc|c} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{array}\right] \quad R_4 \to R_4 - R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & -64 \end{array}\right]$$

$\therefore \quad \rho(AB) = 3 \quad \rho(A) = 2$

$\therefore \quad \rho(A) \neq \rho(AB)$

$\therefore$ The given system of equation is inconsistent.

Assignment:-

1. Solve the equations $x+y+z=9$; $2x+5y+7z=52$; $2x+y-z=0$ Ans: $x=1$, $y=3$, $z=5$

2. Solve the equations $x+y+z=6$; $x-y+2z=5$; $2x-2y+3z=7$ Ans: $x=1$, $y=2$, $z=3$.

3. Show that the system of equations $x+2y+z=3$, $2x+3y+2z=5$, $3x-5y+5z=2$, $3x+9y-z=4$ are consistent and solve them. Ans: $x=-1$, $y=1$, $z=2$

4. Show that the equations $x+y+z=4$, $2x+5y-2z=3$, $x+7y-7z=5$ are not consistent.