Theorem: If V is an inner product space, then

(i) $d(x,y) \geq 0$

(ii) $d(x,y) = 0$ iff $x = y$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,y) \leq d(x,z) + d(z,y)$

Defn: Two vectors u and v in an inner product space are said to be orthogonal if

$$\langle u, v \rangle = 0$$

Pythagorean theorem: Let V be an inner product space, and let u, v be any two vectors in V with the angle θ. Then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\| \, \|y\| \cos\theta.$$

Theorem: If $\{v_1, v_2 \ldots v_n\}$ non-zero vectors in an inner product space V are mutually orthogonal (ie each vector is orthogonal to every other vector) then they are linearly independent.

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Orthogonal basis and linear combinations, Fourier the coefficients of;

Let S consist of the following three vectors in R^3

$u_1 = (1, 2, 1)$, $u_2 = (2, 1, -4)$, $u_3 = (3, -2, 1)$

$\because$ $\langle u_1, u_2 \rangle = 0$

$\langle u_2, u_3 \rangle = 0$

$\langle u_1, u_3 \rangle = 0$

$\therefore$ the vectors are orthogonal

$\therefore$ they are linearly independent.

Thus, S is an orthogonal basis of R^3.

Suppose we want to write $v = (7, 1, 9)$ as a linear combination of u_1, u_2, u_3.

$$\therefore \quad v = x u_1 + y u_2 + z u_3 \quad \text{--- } (*)$$

or $(7, 1, 9) = x(1, 2, 1) + y(2, 1, -4) + z(3, -2, 1)$

$$\Rightarrow \quad \begin{aligned} x + 2y + 3z &= 7 \\ 2x + y - 2z &= 1 \\ x - 4y + z &= 9 \end{aligned}$$

$$\Rightarrow \quad x_1 = 3, \quad x_2 = -1, \quad x_3 = 7$$

Thus $v = 3u_1 - u_2 + 2u_3$

Method ②: Here we use that the basis vectors are orthogonal. If we take the inner product of each side of ⊛ w.r.t. u_i, we get

$$\langle v, u_i \rangle = \langle x u_1 + y u_2 + z u_3, u_i \rangle$$

or $$\langle v, u_i \rangle = x_i \langle u_i, u_i \rangle$$

or $$x_i = \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle}$$

Here two term drop out, because u_1, u_2, u_3 are orthogonal. Accordingly,

$$x_1 = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{7+2+9}{1+4+1} = \frac{18}{6} = 3$$

$$x_2 = \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{14+1-36}{4+1+16} = \frac{-21}{21} = -1$$

$$x_3 = \frac{\langle v, u_3 \rangle}{\langle u_3, u_3 \rangle} = \frac{21-2+9}{9+4+1} = \frac{28}{14} = 2$$

Thus, we get $v = 3u_1 - u_2 + 2u_3$

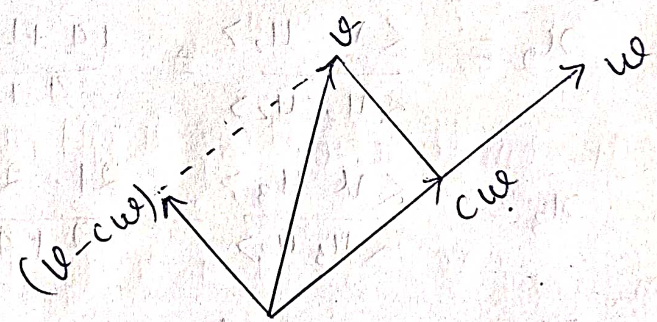
Theorem: Let $\{u_1, u_2, \ldots, u_n\}$ be an orthogonal basis of V. Then, for any $v \in V$,

$$v = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 + \cdots + \frac{\langle v, u_n \rangle}{\langle u_n, u_n \rangle} u_n$$

Note: The scalar $k_i = \dfrac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle}$ is called the Fourier coefficient of v with respect to u_i, because it is analogous to a coefficient in the Fourier series of a function.

Projections:

Let V be an inner product space. Suppose w is a given non-zero vector in V, and suppose v is another vector. We seek the "projection of v along w" will be multiple cw of w such that $v' = (v - cw)$ is orthogonal to w.



Thus

$\langle v - cw, w \rangle = 0$

or

$\langle v, w \rangle - c \langle w, w \rangle = 0$

or $c = \dfrac{\langle v, w \rangle}{\langle w, w \rangle}$

Accordingly, the projection of v along w is denoted by $\text{proj}(v, w)$ and defined by

$$\boxed{\text{proj}(v, w) = cw = \frac{\langle v, w \rangle}{\langle w, w \rangle} w}$$

where c - Fourier coefficient w.r.t $\underline{\underline{w}}$.

Theorem: Suppose $(w_1, w_2 \dots w_r)$ form an orthogonal set of non-zero vectors in V. Let v be any vector in V. Define

$$v' = v - (c_1 w_1 + c_2 w_2 + \dots + c_r w_r)$$

where $c_1 = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle}$, $c_2 = \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} \dots c_r = \frac{\langle v, w_r \rangle}{\langle w_r, w_r \rangle}$

Then v' is orthogonal to $w_1, w_2 \dots w_r$.

Defn: The projection of a vector $v \in V$ along a subspace W of V is defined as

$$\text{proj}(v, W) = c_1 w_1 + c_2 w_2 + \dots + c_r w_r$$

where c_i are the components of v along w_i and

$W = \text{span}\{w_1, w_2 \dots w_r\}$, where w_i form an orthogonal set.

Exp Find the Fourier coefficient c and the projection of $v = (1, -2, 3, 4)$ along $w = (1, 2, 1, 2)$ in R^4.

Soln.

$$\therefore c = \frac{\langle v, w \rangle}{\langle w, w \rangle} = \frac{1 - 4 + 3 + 8}{1 + 4 + 1 + 4} = \frac{-8}{10} = -\frac{4}{5}$$

and $\text{proj}(v, w) = cw = -\frac{4}{5}(1, 2, 1, 2)$

$$= \left(-\frac{4}{5}, -\frac{8}{5}, -\frac{4}{5}, -\frac{8}{5}\right)$$

Gram - Schmidt orthogonalization process

Suppose $\{v_1, v_2 \dots v_n\}$ is a basis of an inner product space V. one can use this basis to construct an orthogonal basis $\{w_1, w_2, \dots w_n\}$ of V as follows: Set

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2\, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3\, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3\, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$\vdots$

$$w_n = v_n - \frac{\langle v_n\, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \dots - \frac{\langle v_n\, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}$$

In other words, for $k = 2, 3, 4 \dots n$, we define.

$$w_k = v_k - c_{k1} w_1 - c_{k2} w_2 - \dots - c_{k,k-1} w_{k-1}$$

where $c_{ki} = \frac{\langle v_k\, w_i\rangle}{\langle w_i, w_i\rangle}$ is component of v_k along w_i.

Exp Let V be the vector space of polynomials $f(t)$ with inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$$

Apply the Gram-Schmidt orthogonalization process to ~~fin~~ $\{1, t, t^2, t^3\}$ to find an orthogonal basis $\{f_0, f_1, f_2, f_3\}$ with integer coefficients for $P_3(t)$.

Soln: (i) $f_0 = 1$

(ii) $$f_1 = t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle}\cdot 1 = t - \frac{\int_{-1}^{1} t\, dt}{\int_{-1}^{1} 1\, dt}\cdot 1$$

$$f_1 = t - 0 = t$$

(iii) $$f_2 = t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}\cdot 1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle}\cdot t$$

Now $$\langle t^2, 1\rangle = \int_{-1}^{1} t^2\, dt = \frac{2}{3}$$

$$\langle 1, 1\rangle = \int_{-1}^{1} 1\, dt = 2$$

$$\langle t^2, t\rangle = \int_{-1}^{1} t^3\, dt = 0$$

$$\langle t, t\rangle = \int_{-1}^{1} t^2\, dt = \frac{2}{3}$$

$$\therefore\ f_2 = t^2 - \frac{\frac{2}{3}}{2}\cdot 1 - 0 = \left(t^2 - \frac{1}{3}\right)$$

or $$f_2 = (3t^2 - 1)$$

(iv) $f_4 = t^3 - \frac{\langle t^3, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle t^3, t\rangle}{\langle t, t\rangle} t - \frac{\langle t^3, t^2\rangle}{\langle t^2, t^2\rangle} t^2$

Now $\langle t^3, 1\rangle = \int_{-1}^{1} t^3 dt = 0$

$\langle t^3, t\rangle = \int_{-1}^{1} t^4 dt = \frac{2}{5}$

$\langle t^3, t^2\rangle = \int_{-1}^{1} t^5 dt = 0$

$\langle t^2, t^2\rangle = \int_{-1}^{1} t^4 dt = \frac{2}{5}$

$\therefore f_4 = t^3 - 0 - \frac{\frac{2}{5}}{\frac{2}{3}} t - 0 = t^3 - \frac{3}{5} t$

or $f_4 = (5t^3 - 3t)$

Thus, $\{f_0, f_1, f_2, f_3\} = \{1, t, 3t^2 - 1, 5t^3 - 3t\}$ is the required orthogonal basis.

Orthogonal Projection:

Let U and W be subspaces of a vector space V. A linear transformation $T: V \to V$ is called the projection of V onto the subspace U along W if

$$V = U \oplus W$$

and $\quad T(x) = u \quad$ for $\quad x = u + w \in U \oplus W$

Theorem: Let W be a subspace of V. Then V is the direct sum of W and $W^{\perp}$

ie $\quad V = W \oplus W^{\perp}$

Hence, $v \in V$ may be expressed uniquely in the form

$$v = w + w' \text{, where } w \in W \text{ and } w' \in W^{\perp}$$

→ We define w to be the projection of v along W, and denote it by $\text{proj}(v, W)$. In particular, if $W = \text{span}(w_1, w_2 \dots w_n)$, where the w_i form an orthogonal set, then

$$\text{proj}(v, W) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \dots + \dots + \frac{\langle v, w_n \rangle}{\langle w_n, w_n \rangle} w_n$$

Exp suppose $v = (1, 3, 5, 7)$. Find the projection of v onto W, where W is the subspace of R^4 spanned by

(a) $u_1 = (1, 1, 1, 1)$ and $u_2 = (1, -3, 4, -2)$

(b) $v_1 = (1, 1, 1, 1)$ and $v_2 = (1, 2, 3, 2)$

solution: (a) Because u_1 and u_2 are orthogonal $(\langle u, v \rangle = 0)$

$$\therefore \operatorname{proj}(v, W) = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$$= \frac{1+3+5+7}{1+1+1+1}(1, 1, 1, 1) + \frac{1-9+20-14}{1+9+16+4}(1, -3, 4, -2)$$

$$= 4(1, 1, 1, 1) - \frac{1}{15}(1, -3, 4, -2)$$

$$= \left(\frac{59}{15}, \frac{63}{5}, \frac{56}{15}, \frac{62}{15}\right)$$

(b) Because v_1 and v_2 are not orthogonal, first apply the Gram-schmidt algorithm to find an orthogonal basis for W.

$$w_1 = v_1 = (1, 1, 1, 1)$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (1, 2, 3, 2) - \frac{8}{4}(1, 1, 1, 1) = (-1, 0, 1, 0)$$

$$\therefore \operatorname{proj}(v, W) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= \frac{1+3+5+7}{1+1+1+1}(1, 1, 1, 1) + \frac{-1+0+5+0}{1+0+1+0}(-1, 0, 1, 0)$$

$$= (7, 4, 1, 9)$$

Theorem: A linear transformation $T: V \to V$ is a projection onto a subspace U if and only if

$$(T \circ T) = T^2 = T$$

Exp $T: R^3 \to R^3$ defined by $T(x, y, z) = (x, y, 0)$.

Theorem: Let U be a subspace of an inner product space V. Then

(i) $\dim U + \dim U^{\perp} = \dim V$

(ii) $(U^{\perp})^{\perp} = U$

(iii) $V = U + U^{\perp}$

This is called the orthogonal decomposition of V by U.

Theorem: Let U & W be subspaces of an inner product space V. Then

(i) $(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$

(ii) $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$

Definition: Let V and W be two inner product spaces. A linear transformation $T: V \to W$ is called an isometry, or an orthogonal transformation, if it preserve the lengths of vectors,

i.e. for every vector $x \in V$

$$\| T(x) \| = \| x \|$$

Theorem: Let $T: V \to W$ be a linear transformation of an inner product space V to W. Then T is an isometry if and only if T preserves inner products, i.e.

$$\langle T(x), T(y) \rangle = \langle x, y \rangle$$

for any vectors, $x, y \in V$.

Theorem: Let A be an $n \times n$ matrix. Then, A is an orthogonal matrix iff $A: \mathbb{R}^n \to \mathbb{R}^n$, as a linear transformation, preserves the dot product.

i.e. for any vectors $x, y \in \mathbb{R}^n$

$$Ax \cdot Ay = x \cdot y.$$

Theorem: A linear transformation $T: V \to W$ is an isometry if and only if

$$d(T(x), T(y)) = d(x, y)$$

for any $x, y \in V$.

Relations of fundamental subspaces:

Theorem: For any $m \times n$ matrix A, the null space $N(A)$ and the row space $R(A)$ are orthogonal in R^n. Similarly, the null space $N(A^T)$ of A^T and the column space $C(A) = R(A^T)$ are orthogonal in R^m.

Theorem (The second fundamental theorem)

For any $m \times n$ matrix A,

(i) $N(A) \oplus R(A) = R^n$
(ii) $N(A^T) \oplus C(A) = R^m$

Theorem: ① $N(A) = R(A)^{\perp}$ and $R(A) = N(A)^{\perp}$

② $N(A^T) = C(A)^{\perp}$ and $C(A) = N(A^T)^{\perp}$.

Least square solutions:

Theorem: Let A be an $m \times n$ matrix, and let $b \in R^m$ be any vector. Then a vector $x_0 \in R^n$ is a least square solution of $Ax = b$ if and only if x_0 is a solution of the normal equation

$$A^T A x = A^T b.$$

Exp Find all the least square solutions to $Ax=$ and then determine the orthogonal projection b_c of b into the column space $C(A)$ of A, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution:

$$A^T A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix}$$

and

$$A^T b = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

From the normal equation, a least square solution of $Ax=b$ is a solution of

$$A^T A x = A^T b$$

i.e

$$\begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

Hence $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -8 \\ -5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ for any t.

Now

$$b_c = Ax = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

=

<u>Theorem</u>: For any $m \times n$ matrix A, $A^T A$ is a symmetric $n \times n$ square matrix and

$$\text{rank}(A^T A) = \text{rank}(A).$$

<u>Theorem</u>: If the columns of A are L.I, then

(i) $A^T A$ is invertible so that $(A^T A)^{-1} A^T$ is a left inverse of A,

(ii) the vector $x = (A^T A)^{-1} A^T b$ is the unique least square solution of a system $Ax = b$, and

(iii) $Ax = A(A^T A)^{-1} A^T b$ is the projection b_c of b into the column space $C(A)$.

=

Exp Consider the following system of linear eq^ns:

$$Ax = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix} = b.$$

Sol^n: Clearly, the two columns of A are L.I. and C(A) is the xy-plane. Thus $b \notin C(A)$.

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 7 & 29 \end{bmatrix}$$

which is invertible.

$$\therefore (A^T A)^{-1} = \frac{1}{9} \begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix}$$

Hence

$$x = (A^T A)^{-1} A^T b = \frac{1}{9} \begin{bmatrix} 29 & -7 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 23 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14/3 \\ -1/3 \end{bmatrix}$$

is ~~a~~ least square solution, and the orthogonal projection of b is C(A) is

$$b_c = Ax = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 14/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$