Assignment

1. Test for Consistency and Solve $2x+3y+7z=5$, $3x+y-3z=12$, $2x+19y-47z=32$.

2. Investigate for what values of λ and μ the Simultaneous equations $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Homogeneous linear equations :-

Consider a system of m homogeneous equations in n unknowns $x_1, x_2 \ldots x_n$ as given below

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \text{---} (1)$$

In matrix notation, the equation (1) Can be written as

$$Ax = 0$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$,

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

It is clear that $x_1 = 0, x_2 = 0, \ldots x_n = 0$ is a Solution of eq (1). This is called trivial solution of $Ax = 0$. The trivial solution is also called zero solution.

Working rule for finding the Solution of the equation $AX=0$

1. If $r = n$ (number of variables) $\Rightarrow$ the system of equations have only trivial solution (i.e., zero solution)

2. If $r < n$ $\Rightarrow$ the system of equations have an infinite number of non-trivial solutions, we shall have $n-r$ linearly independent solutions.

To obtain infinite solutions, set $(n-r)$ variables any arbitrary value and solve for the remaining unknowns.

Problems:-

1. Solve Completely the system of equations
$x+2y+3z=0$; $3x+4y+4z=0$, $7x+10y+12z=0$.

Sol: The system of equations in matrix notation can be written as $AX=0$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the Coefficient matrix A to the Echelon form, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of $A = 3$ and Number of variables $= 3$

$\therefore$ The system of equations has a trivial solution.

$\therefore$ $x=0$, $y=0$, $z=0$ is the only solution.

2. Solve the system of equations $x+3y-2z=0$; $2x-y+4z=0$; $x-11y+14z=0$.

Sol: The given system can be written as $Ax=0$

~~where~~ A

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the coefficient matrix A to the Echelon form

We get:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the Echelon form. Number of non-zero rows is 2 $\therefore$ The rank of matrix is 2.

$\because$ Number of variables is 3, this will have $3-2=1$ non-zero solution.

The given equations are now can be written as

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x+3y-2z=0$$

$$-7y+8z=0$$

Let $z=k$, where k is a constant

$-7y=-8k$

$y=8/7\,k$.

$x+3\,8/7\,k-2k=0$

$x=-\frac{10}{7}k$

$\therefore x=-\frac{10}{7}k$, $y=\frac{8}{7}k$

$z=k$ which is the general solution.

3. Solve the System of equations $x+3y+2z=0$, $2x-y+3z=0$, $3x-5y+4z=0$, $x+17y+4z=0$.

Sol: Given System of equations Can be written as $AX=0$.

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the Coefficient matrix A to the echelon form, We get

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 + 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of $A = 2$

$\therefore$ Number of unknowns $= 3$.

Assigning the arbitrary values to $n-r = 3-2 = 1$ variable

The given equations are now Can be written as

$$\therefore \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x+3y+2z=0$

$-7y-z=0$.

let $z=k$, $y=-k/7$,

$x - 3k/7 + 2k = 0$

$x = \dfrac{-11k}{7}$

$\therefore \quad x = \frac{-11k}{7}, \quad y = \frac{-k}{7}, \quad z = k$.

Giving different value to k, an infinite number of values can be obtained.

4 Solve Completely the system of equations

$x+y-2z+3w=0$; $x-2y+z-w=0$; $4x+y-5z+8w=0$;

$5x-7y+2z-w=0$

Sol: The given System of equation in matrix form is

$$AX = 0$$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the Coefficient matrix A to the Echelon form, we get

$$A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix} \qquad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 4R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in the Echelon form. we have

Rank of $A = 2$ (number of non-zero rows)

Since rank of A $(=2)$ is less than the number of unknowns.

$\therefore$ The given System has infinite number of non-trivial Solutions.

$\therefore$ Number of Independent solutions $= 4-2=2$

$\therefore$ The given system of equations is equivalent to

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x+y-2z+3w=0$$

$$-3y+3z-4w=0$$

Taking $w=k_1$, $z=k_2$

$$\therefore \quad -3y+3k_2-4k_1=0$$

$$-3y = 4k_1-3k_2$$

$$y = \frac{3k_2-4k_1}{3}$$

$$\therefore \quad x+\frac{3k_2-4k_1}{3} - 2k_2 + 3k_1 = 0$$

$$\Rightarrow \quad 3x+3k_2-4k_1-6k_2+9k_1=0$$

$$\Rightarrow \quad 3x-3k_2+5k_1=0$$

$$3x = 3k_2-5k_1$$

$$\Rightarrow \quad x = \frac{3k_2-5k_1}{3}$$

$$\therefore \quad x = k_2-\frac{5}{3}k_1, \quad y = k_2-\frac{4}{3}k_1, \quad z=k_2, \quad w=k_1$$

which is the required solution.

5. S.T the only real number λ for which the system

$x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$; $2x+3y+z=\lambda z$

has non-zero Solution is 6 and solve them, when $\lambda=6$.

Sol:- Given system Can be expressed as $AX=0$ where

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the number of Variables $n=3$

The given system of equations possess a non-zero (non-trivial) solution, if

Rank of A < number of unknowns i.e., Rank of $A<3$.

For this we must $\det A=0$.

$$\therefore \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)^2-6\right]-2\left[3(1-\lambda)-4\right]+3\left[9-2(1-\lambda)\right]=0$$

$$\Rightarrow (1-\lambda)\left[1+\lambda^2-2\lambda-6\right]-2\left[3-3\lambda-4\right]+3\left[9-2+2\lambda\right]=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-2\lambda-5\right]-2\left[-3\lambda-1\right]+3\left[2\lambda+7\right]=0$$

$$\Rightarrow \lambda^2-2\lambda-5-\lambda^3+2\lambda^2+5\lambda+6\lambda+2+6\lambda+21=0$$

$$\Rightarrow \cancel{-\lambda^3+3\lambda^2+3\lambda-24=0} \quad -\lambda^3+3\lambda^2+15\lambda+18=0$$

$$\Rightarrow \cancel{\lambda^3-3\lambda^2-3\lambda+24=0}. \quad \lambda^3-3\lambda^2-15\lambda-18=0.$$

$\lambda=6$ is a root of the equation $(216-108-90-18=0)$

$\therefore (\lambda-6)(\lambda^2+3\lambda+3) = 0$

6	1	-3	-15	-18
	0	6	18	18
	1	3	3	0

$\Rightarrow \lambda = 6$ is the only real value and other values are complex

When $\lambda = 6$, the given system becomes $AX = 0$

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$R_2 \Rightarrow 5R_2 + 3R_1$~~

$R_3 \Rightarrow$

Reducing the coefficient matrix A to the Echelon form,

We get

$$A = \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix} \quad R_2 \to 5R_2 + 3R_1, \quad R_3 \to 5R_3 + 2R_1$$

$$\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$A \sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A = 2$, Number of unknowns $= 3$

$\therefore$ Rank of A < number of unknowns

$\therefore$ The system of equations has infinite number of non-trivial solutions.

The given equations are now can be written as

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x+2y+3z=0$$
$$-19y+19z=0.$$

Assigning the arbitrary values to $n-r = 3-2 = 1$ variable

we get $z=k$, $-19y = -19z$

$y = k.$

$\therefore \; -5x + 2k + 3k = 0$

$\Rightarrow \; x = k$

$\therefore \; x=k, \; y=k, \; z=k.$

Giving different value to k, an ~~if~~ infinite number of values can be obtained.

Assignments:-

6. Solve the system $2x-y+3z=0$, $3x+2y+z=0$ and $x-4y+5z=0$ Ans:- $x=-k$, $y=k$, $z=k$

7. Solve the system of equations $x+y+w=0$, $y+z=0$, $x+y+z+w=0$, $x+y+2z=0$ Ans:- $x=0, y=0, z=0, w=0$

8. Determine whether the following equations will have a non-trivial solution if so solve them.
$4x+2y+z+3w=0$, $6x+3y+4z+7w=0$, $2x+y+w=0$

Ans:- $x=k_1$, $y=-2k_1-k_2$, $z=-k_2$ & $w=k_2$.