Module 1:- Matrices and Linear System of Equations:

Elementary Row-operations, Elementary Matrices, Echelon form, Rank of a matrix by row-reduction, solutions of system of linear equations by row reduction, Matrix Factorization, LU factorization, LDU factorization.

Definition of Matrix:

An ordered set of 'mn' numbers, real or Complex arranged in a rectangular array of 'm' rows and 'n' Columns written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called an $m \times n$ (read as m by n) matrix.

These mn numbers are also called the elements of the matrix.

Thus we write $A = [a_{ij}]_{m \times n}$ where $1 \leq i \leq m$, $1 \leq j \leq n$

The Symbol a_{ij} denotes the element in the i^{th} row and j^{th} Column.

Types of matrices:

1. **Row matrix:** A matrix which consists of a single row is called a Row matrix or Row vector.

Ex : $\begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}$

2. Column matrix : A matrix which consists of a single column is called a column matrix or column vector.

Ex : $\begin{bmatrix} 1 \\ 3 \\ 2 \\ -8 \end{bmatrix}$

3. Square Matrix : A matrix in which rows and columns are equal is called a square matrix

Ex : $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}_{3\times3}$ $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{4\times4}$

Note :- Trace - The sum of the diagonal elements of a square matrix is called the trace of A.

4. Determinant of matrix : A determinant which has the same elements as the square matrix A is known as determinant of the matrix and is denoted by $|A|$.

Ex :- If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix}$

Note :- If $|A| = 0$, the matrix A is called Singular matrix

$|A| \neq 0$, the matrix A is called Non-Singular matrix

Note 2 :- Difference between matrix and a Determinant

1. In determinant the rows and columns must be equal where as in a matrix, the number of rows and columns may or may not be equal.

2. On interchanging the rows and columns, a different matrix is formed whereas in a determinant, an interchange of rows and columns does not change the value of the determinant.

5. **Diagonal matrix :-** A square matrix in which all the elements except those along the diagonal are zero is called a diagonal matrix.

Ex :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}_{4\times4} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}_{5\times5}$$

6. **Scalar Matrix :-** A diagonal matrix in which all the diagonal elements are equal, is called a scalar matrix

Ex :-

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times3} \qquad \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}_{4\times4}$$

7. **Unit Matrix or Identity Matrix :-** A diagonal matrix in which all the diagonal elements are equal to Unity is called a unit matrix.

A unit matrix of order n and is denoted by I_n.

Ex:- $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8. **Zero Matrix or Null Matrix:-** A matrix in which all the elements are zero, is called a zero or Null matrix and is denoted by O.

Ex:- $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Note:- Null matrix need not be a square matrix where as unit matrix must be a square matrix.

9. **Triangular Matrix:-** A square matrix in which every element either ~~above~~ above or below the principal diagonal is zero, is called a Traingular Matrix.

(i) If every element below the principal diagonal is zero, the matrix is called upper triangular matrix.

Ex:- $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$ $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}_{4\times 4}$

upper

(ii) If every element above the principal diagonal is zero, the matix is called lower triangular matrix.

Ex:
$$\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 2 \end{bmatrix}$$

10. **Symmetric and Skew symmetric Matrices :-**

A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ (or)

Ex:-
$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $a_{ij} = -a_{ji}$

Ex:-
$$\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 8 \\ 3 & -8 & 0 \end{bmatrix}$$

11. **Transpose of a matrix :-** Matrix obtained by interchanging rows and columns is known as transpose of the given matrix. The transpose of the matrix A is denoted by A' or A^T

Ex:- If $A = \begin{bmatrix} 1 & 4 & 5 \\ 6 & 7 & 2 \\ 9 & 3 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 6 & 9 \\ 4 & 7 & 3 \\ 5 & 2 & 6 \end{bmatrix}$$

12. **Idempotent Matrix :-** A matrix A such that $A^2 = A$ is called Idempotent

Ex :- $$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

13. **Involutory Matrix :-** A matrix A such that $A^2 = I$ is called involutory

Ex :- $$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}.$$

Elementary Transformations :-

Following transforms are known as elementary transformations.

1. The interchange of any two rows (or columns)

Notations $R_i \leftrightarrow R_j$ stands for interchange of i^{th} row and j^{th} row

$C_i \leftrightarrow C_j$ stands for interchange of i^{th} column and j^{th} column.

2. The multiplication of elements of any row (or column) by any non-zero number.

Notation $R_i \rightarrow KR_i$ stands for the multiplication of all elements of i^{th} row by K

$C_i \to kC_i$ stands for the multiplications of all elements of j^{th} column by k.

3. The addition to the elements of any other row (or column) the corresponding elements of any other row (or column) multiplied by any number.

Notation $R_i \to R_i + k \cdot R_j$ means add to the elements of i^{th} row, k times the elements of j^{th} row.

$C_i \to C_i + kC_j$ means add to the elements of j^{th} column, k times the elements of j^{th} column.

Rank of a Matrix: A matrix is said to be rank r if

(i) it has atleast one non-zero minor of order r

and

(ii) every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the highest order of any non-vanishing minor of the matrix.

The rank of A is denoted by $\rho(A)$.

Note :-

1. The rank of a null matrix is zero i.e., $\rho(A) = 0$
2. Every matrix will have a rank
3. Rank of a matrix is unique

3. For a non-zero matrix, $\rho(A) \geq 1$

4. The rank of a unit matrix of order n is n.

5. The rank of an $m \times n$ matrix $\leq \min(m, n)$

6. The rank of every non-singular matrix of order n is n the rank of a singular matrix of order n is less than n.

Echelon Form of a matrix :-

A matrix is said to be Echelon form if

(i) all non-zero rows, if any, precede the zero rows.

(ii) The number of zeros preceding the first non-zero element in a row less than the number of such zeros in the succeeding row.

(iii) The first non-zero element in each non zero row is unity.

∴ The rank of a matrix in Echelon form is the number of non-zero rows of the matrix

Note :- The condition (iii) is optional.

Zero row :- If all the elements in a row of a matrix are zeros, then it is called a zero row.