4 Show that the equations $x+y+z=6$, $x+2y+3z=14$, $x+4y+7z=30$ are consistent and solve them

Sol: The system of equations can be written in the matrix form $AX=B$ i.e.,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Consider $[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right]$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right]$ $R_3 \to R_3 - 3R_2$

$[AB] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$

This is in Echelon form.

$\rho(A) = 2$, $\rho(AB) = 2$, the system of equations is consistent.

Number of unknowns $= 3$

Since rank of A is less than the number of unknowns, therefore the system of equations will have infinite number of solutions.

Now the given equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$x+y+z=6, \quad y+2z=8$

Now rank of A = rank of AB = $2 < 3$ (number of unknowns)

Hence the system of equations has infinite number of solutions.

$\therefore$ Number of independent solutions $= n-r = 3-2 = 1$

let $z = k$; $\quad y + 2k = 8$

$\Rightarrow y = 8-2k$.

$\therefore \quad x + 8 - 2k + k = 6$

$x = k-2$

$\therefore \quad x = k-2, \quad y = 8-2k, \quad z = k$ is the solution,

Where k is an arbitrary constant.

5. Discuss for what values of λ, μ the simultaneous equations $x+y+z=6$, $\cancel{x+2y+\lambda z = \mu}$ $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Sol: The systems of equations can be written in matrix form $AX = B$ i.e.,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Consider the Augmented matrix $[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$[AB] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

Case (i): Suppose $\lambda = 3$, $\mu \neq 10$

$\therefore$ Rank of $A = 2$, Rank of $[AB] = 3$

$\therefore$ Rank of $A \neq$ Rank of $[AB]$.

$\therefore$ The System is inconsistent and so the equations have no solution.

Case (ii): Suppose $\lambda \neq 3$, $\mu \neq 10$ $\therefore$ Rank of $A = 3$
Rank of $[AB] = 3$

$\therefore$ Number of unknowns $= 3$

$\therefore$ The system is Consistent and so the equations have an unique Solution.

Case (iii): Suppose $\lambda = 3, \mu = 10$, Rank of $A =$ Rank of AB

i.e. $2 < 3$ (number of unknowns).

$\therefore$ The System is Consistent and so the equations have an Infinite number of Solutions.

6. For what values of 'a' and 'b' for which the equations $x+y+z=3$; $x+2y+2z=6$; $x+ay+3z=b$ have (i) no Solution (ii) a unique solution (iii) Infinite number of solutions.

Sol: Given equations can be expressed in the matrix form $AX=B$ i.e.,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix}$$

Consider the Augmented matrix

$$[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-3 \end{array}\right] \qquad R_3 \to R_3 - 2R_2$$

$$[AB] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-3 & 0 & b-9 \end{array}\right]$$

(i) Suppose $a=3$ and $b \neq 9$

$\therefore$ Rank $A = 2$ and Rank $[AB] = 3$

$\therefore$ The system of equations is not consistent and it has no solution.

(ii) Suppose $a \neq 3$, $b \neq 9$, Rank of A = Rank of $[AB] = 3$

$\therefore$ Number of variables $= 3$

$\therefore$ The system of equations is consistent and it has unique solution.

(iii) Suppose $a=3$, $b=9$, Rank of A = Rank of $[AB] = 2$

Here number of variables $= 3$ $(\because r < n)$ $(2<3)$

The system will have infinite number of solutions with $n-r = 3-2 = 1$ arbitrary value.

Assignment

1. Test for Consistency and Solve $2x+3y+7z=5$, $3x+y-3z=12$, $2x+19y-47z=32$.

2. Investigate for what values of λ and μ the Simultaneous equations $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Homogeneous linear equations :-

Consider a system of m homogeneous equations in n unknowns $x_1, x_2 \ldots x_n$ as given below

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \text{---} (1)$$

In matrix notation, the equation (1) Can be written as

$$AX = 0$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$,

$$O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

It is clear that $x_1 = 0, x_2 = 0, \ldots x_n = 0$ is a Solution of eq (1). This is called trivial solution of $AX = 0$. The trivial solution is also called zero solution.