Vector spaces:

Let V be a non-empty set of certain objects, which may be vectors, matrices, functions or some other objects. Each object is an element of V is called a vector. The elements of V are denoted by a, b, c, u, v, .. etc. Assume that the two algebraic operations

(i) vector addition and
(ii) scalar multiplication

are defined on elements of V.

If the vector addition is defined as the usual addition of vectors, then

$$a+b = (a_1, a_2, a_3, \ldots a_n) + (b_1, b_2, b_3, \ldots b_n)$$
$$= (a_1+b_1, a_2+b_2, a_3+b_3, \ldots a_n+b_n)$$

If the scalar multiplication is defined as the usual scalar multiplication of a vector by scalar α, then

$$\alpha a = \alpha(a_1, a_2, a_3 \ldots a_n)$$
$$= (\alpha a_1, \alpha a_2, \alpha a_3, \ldots \alpha a_n)$$

The set V defines a vector space if for any elements a, b, c in V and any scalars α, β the following properties (axioms) are satisfied.

Properties (axioms) with respect to vector addition

(I) $a+b$ is in V

(II) $a+b = b+a$ (commutative law)

(III) $(a+b)+c = a+(b+c)$ (Associative law)

(IV) $a+0 = 0+a = a$ (existence of a unique zero element in V)

(V) $a+(-a) = 0$ (existence of additive inverse or negative vector in V).

Properties (axioms) with respect to scalar multiplication

(VI) αa is in V

(VII) $(\alpha+\beta)a = \alpha a + \beta a$ (left distributive law)

(VIII) $(\alpha\beta)a = \alpha(\beta a)$

(IX) $\alpha(a+b) = \alpha a + \alpha b$ (right distributive law)

(X) $1 \cdot a = a$.

The properties (i) to (vi) are called the closure properties. When these two properties are satisfied, we say that the vector space is closed under the vector addition and scalar multiplication.

→ The vector addition and scalar multiplication defined above need not always be usual addition and multiplication operators. Thus, the vector space depends not on the set V of vectors, but also on the definition of vector addition and scalar multiplication on V.

Remark ① If even one of the above properties is not satisfied, then V is not a vector space. We usually check the closure properties first before checking the other properties.

② The set of real numbers and complex numbers are called fields of scalars.

③ The vector space $V = \{0\}$ is called a trivial vector space.

→ The following are some examples of vector sp[ace] under the usual operations of vector addition and scalar multiplication.

(I) The set V of real or complex numbers.

(II) The set of real valued continuous functions f on any closed interval [a, b]. The 0 vector defined in property (IV) is the zero function.

(III) The set of polynomials P_n of degree less than or equal to n.

(IV) The set V of n-tuples in R^n or C^n.

(V) The set V of all m × n matrices. The element 0 defined in property (IV) is the null matrix of order m × n.

The following are some examples which are no vector spaces. Assume that usual operations of vector addition and scalar multiplication are being used.

(I) The set V of all polynomials of degree n. Let P_n and Q_n be two polynomials of degree n in V. Then, $\alpha P_n + \beta Q_n$ need not be a polynomial of degree and thus may not be in V.

Exp: If $P_n = x^n + a_1 x^{n-1} + \cdots + a_n$

and $Q_n = -x^n + b_1 x^{n-1} + \cdots + b_n$

then $(P_n + Q_n)$ is a polynomial of degree $(n-1)$.

(2) The set V of all real-valued functions of one variable x, defined and continuous on the closed interval $[a, b]$ such that the value of the function at b is some non-zero constant p.

Exp: Let $f(x)$ and $g(x)$ be two elements in V.

Now, $f(b) = g(b) = p$

$\because$ $f(b) + g(b) = 2p$, $f(x) + g(x)$ is not in V.

Note: If $p = 0$, the V forms a vector space.

Exp (1): Let V be the set of all ordered pairs (x, y), where x, y are real numbers.

Let $a = (x_1, y_1)$ and $b = (x_2, y_2)$ be two elements in V.

Define the addition as

$$a + b = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$

and the scalar multiplication as

$$\alpha a = \alpha(x_1, y_1) = \left(\frac{\alpha x_1}{3}, \frac{\alpha y_1}{3}\right)$$

Show that V is not a vector space. which of the properties are not satisfied ?

Soln: we illustrate the properties that are not satisfied.

(i) $(x_2, y_2) + (x_1, y_1) = (2x_2 - 3x_1, y_2 - y_1) \neq (x_1, y_1) + (x_2, y_2)$

$\therefore$ property (ii) (commutative law) does not hold.

(ii) $[((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)]$

$= (2x_1 - 3x_2, y_1 - y_2) + (x_3, y_3)$

$= (4x_1 - 6x_2 - 3x_3, y_1 - y_2 - y_3)$

and

$[(x_1, y_1) + ((x_2, y_2) + (x_3, y_3))]$

$= (x_1, y_1) + (2x_2 - 3x_3, y_2 - y_3)$

$= (2x_1 - 6x_2 - 9x_3, y_1 - y_2 + y_3)$

$\therefore$ property (iii) (associative law) is not satisfied.

(iii) $1 \cdot (x_1, y_1) = \left(\frac{x_1}{3}, \frac{y_1}{3}\right) \neq (x_1, y_1)$

$\therefore$ property (X) (existence of multiplicative inverse) is not satisfied.

Hence, V is not a vector space.

(2). Let V be the set of all ordered pairs (x, y), where x, y are real numbers. Let $a = (x_1, y_1)$ and $b = (x_2, y_2)$ be two elements in V. Define the addition as

$$a + b = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

and the scalar multiplication as

$$\alpha (x_1, y_1) = (\alpha x_1, \alpha y_1)$$

Show that V is not a vector space. Which of the properties are not satisfied?

Solⁿ: Note that $(1, 1)$ is an element of V. From the given definition of vector addition, we find that

$$(x_1, y_1) + (1, 1) = (x_1, y_1)$$

and this is true only for the element $(1, 1)$. Therefore, the element $(1, 1)$ plays the role of 0 element as defined in property (IV).

Now, there is no element in V for which

$$a + (-a) = 0 = (1, 1), \text{ since}$$

$$(x_1, y_1) + (-x_1, -y_1) = (-x_1^2, -y_1^2) \neq (1, 1)$$

$\therefore$ property (V) is not satisfied.

Now, let $\alpha = 1$, $\beta = 2$ be any two scalars.

we have

$$(\alpha+\beta)(x_1, y_1) = 3(x_1, y_1) = (3x_1, 3y_1)$$

and

$$\alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1)$$
$$= (x_1, y_1) + (2x_1, 2y_1)$$
$$= (2x_1^2, 2y_1^2)$$

$\therefore$ $(\alpha+\beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$ and property (VII) is not satisfied.

Similarly, it can be shown that property (IX) is not satisfied.

Hence, V is not a vector space.

Theorem: Any intersection of subspaces of a vector space V is a subspace of V.

Solution: Let C be a collection of subspaces of V, and let W denote the intersection of the subspaces in C.

(i) Since every subspace contains the zero vector

$$\Rightarrow \quad 0 \in W.$$

(ii) Let $a, b \in W$. Then a and b are contained in each subspace in C. Because each subspace in C is closed under addition ~~and scalar multiplication~~, it follows that $(a+b)$ is contained in each subspace in C.

$$\therefore \quad (a+b) \in W$$

(iii) Let $\alpha \in \mathbb{R}$ or $\mathbb{C}$ and a contained in each subspace in C. Also, each subspace in C is closed under scalar multiplication, it follows that

αa is contained in each subspace in C

$$\therefore \quad \alpha a \in W$$

Hence W is a subspace of V.

Theorem: Union of two subspace need not a sub.

Proof: Let $V = R^2(R)$

and W_1 = Real axis ie x-axis $= \{(x,0) : x \in R\}$

and W_2 = Imaginary axis ie y-axis $= \{(0,y) : y \in R\}$

Here W_1 and W_2 both are subspaces of vector space $V = R^2(R)$.

Now $W_1 \cup W_2 = \{(x,0), (y,0) : x, y \in R\}$ includes all the elements of the from $(x,0)$ and $(y,0)$ But

$$(x,0) + (0,y) = (x,y) \notin W_1 \cup W_2$$

Hence $W_1 \cup W_2$ is not a subspace.

Theorem: If W_1 and W_2 are subspaces then $W_1 \cup W_2$ is a subspace iff $W_1 \subset W_2$ or $W_2 \subset W_1$.

Proof: If $W_1 \subset W_2$ or $W_2 \subset W_1$ then we have.

$$W_1 \cup W_2 = W_2 \text{ or } W_1 \cup W_2 = W_2 \text{ resp.}$$

So $W_1 \cup W_2$ is a subspees as W_1 and W_2 are subspaces.

Linear combination:

Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors of S if there exist a finite number of vectors $u_1, u_2 \cdots u_n$ in S and scalars $a_1, a_2 \cdots a_n$ in F ($\mathbb{R}$ or $\mathbb{C}$) such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n .$$

Spanning set:

Let S be a non-empty subset of a vector space V. The span of S, denoted by span(S), is the set consisting of all linear combinations of the vectors in S.

Exp ①: Let V be the vector space of all 2×2 real matrices. Show that the sets

(i) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

(ii) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$

span V.

Soln: Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element of V

(i) We write

xp(2) Polynomi

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Since every element of V can be written as a linear combination of the elements of S, the set S spans the vector space V.

(ii) We need to determine the scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Equating the corresponding elements, we obtain the system of equations

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = a$$

$$\alpha_2 + \alpha_3 + \alpha_4 = b$$

$$\alpha_3 + \alpha_4 = c$$

$$\alpha_4 = d$$

The solution of this system of equations is

$\alpha_4 = d, \quad \alpha_3 = c-d, \quad \alpha_2 = b-c, \quad \alpha_1 = a-b$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b-c)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (c-d)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\because$ every elements of V can be written as a linear combination of the elements of S, the set S spans the vector space V.

Exp (2): Let V be the vector space of all the polynomials of degree ≤ 3. Determine whether or not the set

$$S = \{ t^3, t^2+t, t^3+t+1 \}$$

spans V?

Sol^n: Let $P(t) = \alpha t^3 + \beta t^2 + \gamma t + \delta$ be an arbitrary element in V. We need to find whether or not there exist scalars a_1, a_2, a_3 such that

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = a_1(t^3) + a_2(t^2+t) + a_3(t^3+t+1)$$

$$= (a_1+a_3)t^3 + a_2 t^2 + (a_2+a_3)t + a_3$$

$$\Rightarrow \quad a_1+a_3 = \alpha, \quad a_2 = \beta, \quad a_2+a_3 = \gamma, \quad \delta = a_3$$

The solution of the first three eq^n is given by

$$a_1 = \alpha+\beta-\gamma, \quad a_2 = \beta, \quad a_3 = \gamma-\beta$$

Substituting in the last eq^n, we obtain

$$\gamma - \beta = \delta$$

which may not be true for all the elements in V.

For example, the polynomial t^3+2t^2+t+3 does not satisfy this condition and therefore, it can not be written as a linear combination of the elements of S.

$\therefore$ S does not span the vector space V.

Linear Independence of Vectors:

Let V be a vector space. A finite set $\{u_1, u_2 \ldots u_n\}$ of the elements of V is said to be linearly dependent if there exist scalars $a_1, a_2, a_3 \ldots a_n$, not all zero, such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0 \quad \text{--- (1)}$$

If eqn (1) is satisfied only for

$$a_1 = a_2 = \cdots = a_n = 0,$$

then the set of vectors is said to be linearly independent.

Theorem: The set of vectors $\{u_1, u_2 \ldots u_n\}$ is linearly dependent iff at least one element of the set is a linear combination of the remaining elements.

Exp (1): Let
$$u_1 = (1, -1, 0)$$
$$u_2 = (0, 1, -1)$$
$$u_3 = (0, 0, 1)$$
be elements of $\mathbb{R}^3$. Show that the set of vectors $\{u_1, u_2, u_3\}$ is linearly independent.

Soln: We consider the vector eqn

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0$$

$$\Rightarrow a_1(1, -1, 0) + a_2(0, 1, -1) + a_3(0, 0, 1) = \bar{0}$$

$$(a_1, -a_1+a_2, -a_2+a_3) = \bar{0} = (0,0,0)$$

Comparing, we obtain $a_1 = 0$, $-a_1+a_2=0$ and $-a_2+a_3$

The solution of these equations is

$$a_1 = a_2 = a_3 = 0$$

$\therefore$ the given set of vectors is linearly independent.

Exp (2): Let $u_1 = (1,-1,0)$, $u_2 = (0,1,-1)$, $u_3 = (0,2,1)$ and $u_4 = (1,0,3)$ be elements of R^3. show that the set of vectors $\{u_1, u_2, u_3, u_4\}$ is L.D.

Solⁿ: The given set of elements will be L.D if there exists scalars a_1, a_2, a_3, a_4 not all zero, such that

$$a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 = \bar{0} \quad \text{--- (1)}$$

$$a_1(1,-1,0) + a_2(0,1,-1) + a_3(0,2,1) + a_4(1,0,3) = (0,0,0)$$

$$\Rightarrow a_1 + a_4 = 0, \quad -a_1+a_2+2a_3 = 0, \quad -a_2+a_3+3a_4=0$$

The solution of this system of eqⁿs is

$$a_1 = -a_4, \quad a_2 = \frac{5}{3}a_4, \quad a_3 = -\frac{4}{3}a_4, \quad a_4 \text{ arbitrary}$$

eqⁿ (1) $\Rightarrow$ $-u_1 + \frac{5}{3}u_2 - \frac{4}{3}u_3 + u_4 = 0$

Hence, there exist scalars not all zero, such that eqⁿ (1) is satisfied. $\therefore$ the set of vectors is L.D.