Vellore Institute of Technology

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SCHOOL OF ADVANCED SCIENCES
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PMDS605L - DATA STRUCTURES AND ALGORITHMS

Tutorial- 1
(To access the students' understanding on asymptotic bounds)

Following table might help you to understand the asymptotic notations:

Definition	$\boxed{?}$ $c > 0$	$\boxed{?}$ $n_0 \geq 1$	$f(n)$ $\boxed{?}$ $c \cdot g(n)$
$O()$	$\exists$	$\exists$	$\leq$
$o()$	$\forall$	$\exists$	$<$
$\Omega()$	$\exists$	$\exists$	$\geq$
$\omega()$	$\forall$	$\exists$	$>$

$$f = O(g) \text{ and } f = \Omega(g) \Leftrightarrow f = \Theta(g) \qquad f = o(g) \Rightarrow f = O(g)$$
$$f = O(g) \Leftrightarrow g = \Omega(f) \qquad f = \omega(g) \Rightarrow f = \Omega(g)$$
$$f = o(g) \Leftrightarrow g = \omega(f) \qquad f \sim g \Rightarrow f = \Theta(g)$$

The definitions of the various asymptotic notations are closely related to the definition of a limit. As a result, $\lim_{n\to\infty} f(n)/g(n)$ reveals a lot about the asymptotic relationship between f and g, provided the limit exists. The table below translates facts about the limit of f/g into facts about the asymptotic relationship between f and g.

$$\lim_{n\to\infty} f(n)/g(n) \neq 0, \infty \Rightarrow f = \Theta(g) \qquad \lim_{n\to\infty} f(n)/g(n) = 1 \Rightarrow f \sim g$$
$$\lim_{n\to\infty} f(n)/g(n) \neq \infty \Rightarrow f = O(g) \qquad \lim_{n\to\infty} f(n)/g(n) = 0 \Rightarrow f = o(g)$$
$$\lim_{n\to\infty} f(n)/g(n) \neq 0 \Rightarrow f = \Omega(g) \qquad \lim_{n\to\infty} f(n)/g(n) = \infty \Rightarrow f = \omega(g)$$

Therefore, skill with limits can be helpful in working out asymptotic relationships. In particular, recall L'Hospital's Rule:

$$\text{If } \lim_{n\to\infty} f(n) = \infty \text{ and } \lim_{n\to\infty} g(n) = \infty, \text{ then } \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

1. Prove the following:
   (i) $3n + 2 \in O(n)$
   (ii) $3n + 3 \in O(n)$
   (iii) $100n + 6 \in O(n)$
   (iv) $10n^2 + 4n + 2 \in O(n^2)$
   (v) $100n^2 + 100n - 6 \in O(n^2)$
   (vi) $6 \cdot 2^n + n^2 \in O(2^n)$

2. Verify that $3n^2 + 4n - 2$ is $O(n^2)$.

3. Show that $n^3 \neq O(n^2)$.

4. Prove that $4n^2 + 7n + 12 \in O(n^2)$.

5. Prove that $\log n + \log(\log n) = O(\log n)$.

6. Prove that $3n^2 + 7n - 5 = \Theta(n^2)$.

7. Prove that $2^{n+1} \in O(2^n)$.

8. Verify whether $2^{2n} \in O(2^n)$.

9. Let $f(n) = 7n + 8$ and $g(n) = n$. Is $f(n) \in o(g(n))$?

10. Let $f(n) = 7n + 8$ and $g(n) = n$. Is $f(n) \in O(g(n))$?

11. Prove that $n^2 = O(2^n)$.

12. Prove that $n^3 + 2n^2 + 3 = O(n^3)$.

13. Consider $f(n) = n + \log n$ and $g(n) = \sqrt{n}$. Is $f(n) = O(g(n)$ or $g(n) = O(f(n))$ or both?

14. If $f(n) = n + \log n + \sqrt{n}$, find a simple function g such that $f(n) = \Theta(g(n))$.

15. Is $7n + 8 \in o(n^2)$?

16. Express $\frac{n^3}{1000} - 100n^2 - 100n + 3$ in terms of Θ notation.

Asymptotic notation consists of six funny symbols used to describe the relative growth rates of functions. These six symbols are defined in the table below.

$f = \Theta(g)$	f grows at the same rate as g	There exists an n_0 and constants $c_1, c_2 > 0$ such that for all $n > n_0$, $c_1 g(n) \leq \lvert f(n)\rvert \leq c_2 g(n)$.
$f = O(g)$	f grows no faster than g	There exists an n_0 and a constant $c > 0$ such that for all $n > n_0$, $\lvert f(n)\rvert \leq c g(n)$.
$f = \Omega(g)$	f grows at least as fast as g	There exists an n_0 and a constant $c > 0$ such that for all $n > n_0$, $c g(n) \leq \lvert f(n)\rvert$.
$f = o(g)$	f grows slower than g	For all $c > 0$, there exists an n_0 such that for all $n > n_0$, $\lvert f(n)\rvert \leq c g(n)$.
$f = \omega(g)$	f grows faster than g	For all $c > 0$, there exists an n_0 such that for all $n > n_0$, $c g(n) \leq \lvert f(n)\rvert$.
$f \sim g$	f/g approaches 1	$\lim_{n \to \infty} f(n)/g(n) = 1$