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# SCHOOL OF ADVANCED SCIENCES

**Fall Semester 2024-2025**

**Problem Set 3**

**Programme Name & Branch : M.Sc. & Data Science**

**Course Name & code : Probability and Distribution Models & PMDS502L**

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(1) Given $P(A) = \frac{3}{4}$ and $P(B) = \frac{3}{8}$ show that $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$.

(2) Given $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(A \cap B) = \frac{1}{6}$, find the probabilities:

  (i) $P(A^c \cup B)$

  (ii) $P(A^c \cap B^c)$

(3) A fruit basket contains 25 apples and oranges, of which 20 are apples. If two fruits are randomly picked in sequence, what is the probability that both the fruits are apples?

(4) A die is loaded (not all outcomes are equally likely) such that the probability that the number $i$ shows up is $i \times K$, where $i = 1, 2, \ldots, 6$, where $K$ is a constant. Find

  (i) the value of $K$.

  (ii) the probability that a number greater than 3 shows up.

(5) A population comprises of 40% female and 60% male.Suppose that 15% of female and 30% male in the the population smoke. A person is selected at random from the population.

  (a) Find the probability that he/she is a smoker.

  (b) Given that the selected person is smoker, find the probability that he is a male.

(6) At the college entrance examination each candidate is admitted or rejected according to whether he has passed or failed the test. Of the candidates who are really capable, 80% pass the test; and of the incapable, 25% pass the test. Given that 40% of the candidates are really capable, find the proportion of capable college students.

(7) Let $X$ and $Y$ be two random variables with the joint density function given by

$$f(x, y) = \begin{cases} x^2 + xy, & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional distribution $f_{X|Y}(x \mid y)$ and $f_{Y|X}(y \mid x)$.

(8) Let $X$ and $Y$ be jointly continuous random variables with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

  (a) Are $X$ and $Y$ independent?

  (b) Find $f(x|y)$.

  (c) Find $E[X]$, $E[Y]$, $Var(X)$ & $Var(Y)$.

(9) Consider two random variables $X$ and $Y$ with joint PMF given in Table 1.

  (a) Find $P(X = 0, Y \leq 1)$.

  (b) Are $X$ and $Y$ independent?

  (c) Find $E(X), E(Y), E(XY)$.

Table 1: Joint PMF of $X$ and $Y$

| | $Y = 0$ | $Y = 1$ | $Y = 2$ |
|---|---|---|---|
| $X = 0$ | $\frac{1}{6}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |
| $X = 1$ | $\frac{1}{8}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

(10) The probability function of a random variable $X$ is given by

$$f(x) = \begin{cases} \frac{x^2}{81}, & -3 < x < 6, \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability density for the random variable $U = \frac{1}{3}(12 - X)$ and $E(\frac{1}{X})$.

(11) Given the joint density of $X$ and $Y$:

$$f(x,y) = \begin{cases} \frac{1}{2} y e^{-xy}, & 0 < x < \infty,\ 0 < y < 2, \\ 0, & \text{otherwise} \end{cases}$$

find $E(X \mid Y = y)$.

(12) The marks scored by students of a college in a test are realizations of a random variable with a mean of 120 and a standard deviation of 5. According to the declared grading scheme, students securing marks between 112 and 128 will be awarded a B grade. Using Chebyshev's inequality, find a lower bound on the proportion of students likely to receive a B grade.

(13) Prove that $P(|X - 2| \leq 2) > \frac{1}{2}$, where $X$ is a a geometric random variable with the probability mass function given by:

$$P(X = x) = \begin{cases} \frac{1}{2^x}, & x = 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

(14) Compute the conditional expectation of $X$ given that $Y = y,\ 0 < y < 1$, where the joint density of $X$ and $Y$ is given by

$$f(x,y) = \begin{cases} 6xy(2 - x - y), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(15) Suppose that $N$ is a counting variable with $P(N = k) = pq^{k-1}$ for $k = 1, 2, \dots$, where $0 < p < 1$ and $q = 1 - p$. Find the probability generating function for $N$.

(16) Let $X$ be a random variable whose probability density function is given by

$$f_X(x) = \begin{cases} e^{-2x} + \frac{e^{-x}}{2}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Write down the moment generating function for $X$. Use this moment generating function to compute the first and second moments of $X$. Determine whether the following statement is true or false: If $X \sim \text{Exp}(\lambda_X)$ and $Y \sim \text{Exp}(\lambda_Y)$, where $X$ and $Y$ are two random variables, then $X + Y \sim \text{Exp}(\lambda_X + \lambda_Y)$. Provide justification for your answer.

(17) A statistician finds that the weekly sales of a local bookstore, represented by the random variable $Y$, have a moment generating function given by $m_Y(t) = \frac{1}{(1-1500t)^5}, t > 0$. Determine the standard deviation of the weekly sales for the bookstore.

(18) One per thousand of a population is subject to certain kinds of accident each year. Given an insurance company has insured 5000 persons from the population, find the probability that at most 2 persons will incur this accident.

(19) A certain airline company, having observed that 5% of the persons making reservations on a flight do not show up for the flight, sells 100 seats on a plane that has 95 seats. What is the probability that there will be a seat available for every person who shows up for the flight?

(20) Workers in a factory incur accidents at the rate of two accidents per week. Calculate the probability that there will be at most two accidents $(i)$ during one week, $(ii)$ during two weeks, $(iii)$ in each of two weeks.

(21) Products produced by a machine has a 3% defective rate. What is the probability that the first defective occurs in the fifth item inspected?

(22) In a factory, a machine produces a batch of 10 products. A random sample of 200 batches yields the following defect counts:

- 0 defects: 90 batches
- 1 defect: 60 batches
- 2 defects: 30 batches
- 3 defects: 15 batches
- 4 or more defects: 5 batches

Fit a binomial distribution and estimate the expected frequencies of the defect counts.

(23) A hospital records the number of emergency admissions per day over the past 60 days:

- 0 admissions: 12 days
- 1 admission: 18 days
- 2 admissions: 15 days
- 3 admissions: 10 days
- 4 or more admissions: 5 days

Fit a Poisson distribution and calculate the expected frequencies for each level of admissions.

(24) The height of the men is normally distributed with mean $\mu = 167\ cm$ and standard deviation $\sigma = 3\ cm$. What is the percentage of population of men that have height

(a) greater than 167 $cm$.

(b) greater than 170 $cm$.

(c) between 161 $cm$ and 173 $cm$.

(25) The time to failure of a particular brand of light bulb is normally distributed with mean $\mu = 400$ hours and standard deviation $\sigma = 20$ hours.

(a) What percentage of the bulbs will last longer than 438 hours?

(b) What percentage of the bulbs will fail before 360 hours?

(c) What percentage of the bulbs will last between 380 and 420 hours?

(26) A monitor issues a warning signal when an action is needed as part of a production process. The interval, $X$ hours, between successive signals follows an exponential distribution with parameter $\lambda = 0.08$.

(i) Find the probability that the interval between the next two signals is:

i. Between 10 and 20 hours;

ii. Less than two hours;

iii. Longer than 50 hours.

(ii) State the mean and standard deviation of the intervals between successive signals.

(iii) Following a warning signal, what is the longest time the production process could be left unsupervised whilst ensuring the probability of missing the next signal is less than 0.01?

(27) The lifetime of a light bulb is $X$ hours, where $X$ can be modelled by an exponential distribution with parameter $\lambda = 0.0125$.

a) Find the mean and variance of the lifetime of a light bulb.

b) Find the probability that the lifetime of a bulb is:

(i) less than 100 hours;

(ii) between 50 hours and 150 hours.

(28) Suppose that on average 1 customer per minute arrives at a shop. What is the probability that the shopkeeper will wait more than 5 minutes before:

(i) both of the first two customers arrive, and

(ii) the first customer arrives?

Assume that the waiting times follow a gamma distribution.

(29) Telephone calls arrive at a switchboard at an average rate of 2 per minute. Let $X$ denote the waiting time in minutes until the 4th call arrives, and assume that the waiting time follows a Gamma distribution.

Write the probability density function (PDF) of $X$, and also find its mean and variance.

(30) Determine the constant $k$ such that the function

$$f(x) = \begin{cases} kx(1-x), & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is the probability density function (p.d.f.) of the Beta distribution of the second kind. Also, find its mean and variance.