Introduction to Characteristic Functions (CF)

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Characteristic Functions

Motivation:

There are random variables for which the moment generating function (MGF) does not exist on any real interval with positive length.

Example: Consider a random variable X that follows a Cauchy distribution with the PDF:

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad \text{for all } x \in \mathbb{R}.$$

For this distribution, the MGF does not exist:

$$M_X(s) = \mathbb{E}[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\pi(1+x^2)} dx = \infty, \quad \text{for any } s \neq 0.$$

Solution: If the MGF is not well-defined, we can use the **Characteristic Function** $\phi_X(t)$, which is defined as:

$$\phi_X(t) = \mathbb{E}[e^{itX}]$$

where j is the imaginary unit ($i = \sqrt{-1}$).

Complex-Valued Random Variables and Characteristic Functions

The characteristic function of a random variable X is given by:

$$\phi_X(t) = \mathbb{E}[e^{itX}]$$

where $i = \sqrt{-1}$ and $t \in \mathbb{R}$.
Complex-Valued Random Variables: A complex random variable can be expressed as $X = Y + iZ$, where Y and Z are real-valued random variables. However, for real-valued random variables X:

$$|e^{itX}| = 1.$$

Thus, we have:

$$|\phi_X(t)| = |\mathbb{E}[e^{itX}]| \leq \mathbb{E}[|e^{itX}|] \leq 1.$$

Properties of Characteristic Functions

If X and Y are independent random variables, then:

$$\phi_{X+Y}(t) = \phi(t)\phi_Y(t).$$

Similarly, for $X_1, X_2, \ldots, X_n$ independent random variables:

$$\phi_{X_1+X_2+\cdots+X_n}(t) = \phi_{X_1}(t)\phi_{X_2}(t)\ldots\phi_{X_n}(t).$$

Example: Characteristic Function of Exponential Distribution

Let $X \sim \text{Exponential}(\lambda)$. Find the characteristic function $\phi_X(t)$.
Recall that the PDF of X is given by:

$$f_X(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0,$$

The characteristic function is:

$$\phi_X(t) = \mathbb{E}[e^{itX}] = \int_0^\infty \lambda e^{-\lambda x} e^{itx} dx.$$

Simplifying:

$$\phi_X(t) = \int_0^\infty \lambda e^{-(\lambda - it)x} dx = \left[\frac{\lambda}{\lambda - it}\right] = \frac{\lambda}{\lambda - it}.$$

The characteristic function of an Exponential random variable is:

$$\phi_X(t) = \frac{\lambda}{\lambda - it}, \quad \text{for all } t \in \mathbb{R}.$$

Properties of Characteristic Functions

- The characteristic function always exists, even when the MGF does not.
- It uniquely determines the distribution of a random variable.
- The characteristic function is bounded, i.e., $|\phi_X(t)| \leq 1$ for all $t \in \mathbb{R}$.

Using Characteristic Functions

- The characteristic function provides an alternative method for analyzing distributions when the MGF does not exist.
- Like the MGF, the characteristic function uniquely determines the distribution of a random variable.

Moments from Characteristic Functions:

- The moments of a random variable X can be obtained by taking derivatives of the characteristic function:

$$\mathbb{E}[X^n] = \frac{d^n}{d(it)^n}\phi_X(t)\bigg|_{t=0}, \quad \text{if moments exist.}$$