ALAGAPPA UNIVERSITY

[Accredited with 'A+' Grade by NAAC (CGPA:3.64) in the Third Cycle
and Graded as Category–I University by MHRD-UGC]

(A State University Established by the Government of Tamil Nadu)

KARAIKUDI – 630 003

Directorate of Distance Education

M.Com. [Finance & Control]

II - Semester

335 21

OPERATIONS RESEARCH

Authors

CR Kothari, *Ex-Associate Professor, Depatment of Economic Administration and Fin. Management, University of Rajasthan*
Units: (1.0-1.1, 1.3-1.11, 2.0-2.2.3)

S Kalavathy, *Dean, Department of Science and Humanities, K Ramakrishnan College of Technology, Samayapuram, Trichy*
Units: (1.2, 2.2.4-2.7, 3-14)

VIKAS®

Vikas® is the registered trademark of Vikas® Publishing House Pvt. Ltd.

VIKAS® PUBLISHING HOUSE PVT. LTD.
E-28, Sector-8, Noida - 201301 (UP)
Phone: 0120-4078900 • Fax: 0120-4078999
Regd. Office: 7361, Ravindra Mansion, Ram Nagar, New Delhi 110 055
• Website: www.vikaspublishing.com • Email: helpline@vikaspublishing.com

Work Order No. AU/DDE/DE1-291/Preparation and Printing of Course Materials/2018 Dated 19.11.2018 Copies - 500

SYLLABI-BOOK MAPPING TABLE

Operations Research

Syllabi	Mapping in Book
BLOCK - I: OPERATIONS RESEARCH AND LINEAR PROGRAMMING PROBLEM CONCEPT	
Unit 1: Operations Research: Historical Background, Scope of Operations Research, Features of Operations Research, Phases of Operations Research, Types of Operations Research Models, Operations Research Techniques and Tools, Structure of the Mathematical Model, Limitations of Operations Research.	**Unit 1:** Operations Research **(Pages 1-32)**
Unit 2: Linear Programming: Linear Programming Problem, Requirements of LPP, Mathematical Formulation of LPP, Case Studies of LPP, Graphical Methods to Solve Linear Programming Problems, Applications, Advantages, Limitations.	**Unit 2:** Linear Programming **(Pages 33-48)**
Unit 3: Graphical Analysis of Linear Programming Problems: Graphical Analysis, Some Basic Definitions, Graphical Methods to Solve LPP, Some Exceptional Cases, Important Geometric Properties of LPP.	**Unit 3:** Graphical Analysis of LPP **(Pages 49-62)**
Unit 4: Simplex Method: Standard Form of LPP, Fundamental Theorem of LPP, Solution of LPP - Simplex Method, The Simplex Algorithm, Penalty Cost Method or Big M-method, Two Phase Method, Solved Problems on Minimisation.	**Unit 4:** Simplex Method **(Pages 63-92)**
Unit 5: Duality in Linear Programming Problem: Importance of Duality Concepts, Formulation of Dual Problem, Economic Interpretation of Duality, Sensitivity Analysis.	**Unit 5:** Duality in Linear Programming Problem **(Pages 93-106)**
Unit 6: Transportation Problem: Formulation of Transportation Problem (TP), Transportation Algorithm (MODI Method), the Initial Basic Feasible Solution, Moving Towards Optimality.	**Unit 6:** Transportation Problem **(Pages 107-125)**
Unit 7: Assignment Problem: Mathematical Formulation of the Problem, Hungarian Method Algorithm, Routing Problem, Travelling Salesman Problem.	**Unit 7:** Assignment Problem **(Pages 126-150)**
BLOCK - II: INTEGER PROGRAMMING AND QUEUING CONCEPTS	
Unit 8: Integer Programming Problem: Types of Integer Programming Problems, Gomory's All-IPP Method, All IPP Algorithm, Branch and Bound Technique.	**Unit 8:** Integer Programming Problem **(Pages 151-181)**
Unit 9: Infinite Queuing Models: Queuing Theory, Operating Characteristics of a Queuing System, Constituents of a Queuing System, Service Facility, Queue Discipline.	**Unit 9:** Infinite Queuing Models **(Pages 182-189)**
Unit 10: Mathematical Analysis of Queuing Theory: Mathematical Analysis of Queuing Process, Properties of Queuing System, Notations, Service System, Single Channel Models, Multiple Service Channels, Erlang Family of Distribution of Service Times, Applications of Queuing Theory, Limitations of Queuing Theory.	**Unit 10:** Mathematical Analysis of Queuing Theory **(Pages 190-210)**

CONTENTS

INTRODUCTION

NOTES

Operational Research, or simply OR, originated in the context of military operations, but today it is widely accepted as a powerful tool for planning and decision-making, especially in business and industry. The OR approach has provided a new tool for managing conventional management problems. In fact, operational research techniques do constitute a scientific methodology of analysing the problems of the business world. They provide an improved basis for taking management decisions. The practice of OR helps in tackling intricate and complex problems, such as that of resource allocation, product mix, inventory management, sequencing and scheduling, replacement and a host of similar problems of modern business and industry. With IT facilities becoming widely available, the significance and scope of OR has grown, and is still growing. Hence, OR is now an integral part of courses of computer science, economics, business management, public administration and several other disciplines.

Operations research (OR) is an interdisciplinary branch of applied mathematics and formal science that uses mathematical methods, such as mathematical modelling, statistics and algorithms to arrive at optimal or near optimal solutions to complex problems. Basically, it is concerned with optimizing the maxima (profit, assembly line performance, bandwidth, etc.) or minima (loss, risk, etc.) of some objective function. It also helps management achieve its goals using scientific methods. The field of operations research is closely related to Industrial Engineering and Industrial Engineers consider operations research techniques as their major toolset. Some of the primary tools used by operations researchers are statistics, optimization, probability theory, queuing theory, game theory, graph theory, decision analysis and simulation. Because of the computational nature of these fields, OR is linked to computer science and OR professionals use specific custom-written software for computation of data and decision-making.

The uniqueness of OR prompted industries to use its formal tools, such as operations analysis, system analysis, management science, decision science, etc. Commercial industries such as airlines, automobiles, communications, electronics, transportation, chemicals and mining use OR techniques to optimally utilize their limited resources and thereby maximize profits. Hence, OR is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system, incorporating measurement of factors, such as chance and risk, with which to predict and compare the outcomes of alternative decision strategies and controls.

NOTES

This book, *Operations Research,* has been carefully and painstakingly planned to prepare the students of distance learning programmes to become successful managers and practitioners. The material has been presented in the self-instructional mode or the SIM format. In this approach, we have begun with an Introduction of the topic of the unit; then, we have outlined the Objectives; which is followed by the details of the contents in a simple and easy-to-learn format. At the end of each unit, we have highlighted a Summary and 'Key Words' for quick recollection. Finally, we have carefully posed Self Assessment Questions and Exercises along with 'Check Your Progress' questions to increase your comprehension of the subject.

BLOCK - I
OPERATIONS RESEARCH AND LINEAR PROGRAMMING PROBLEM CONCEPT

NOTES

UNIT 1 OPERATIONS RESEARCH

Structure

1.0 INTRODUCTION

Modern technological progress is accompanied by growth in scientific techniques. While existing methods have been improved to meet the challenges arising from the development of commerce and industry, a large number of new techniques or so-called sophisticated tools of analysis have been and are even now being devised to enlarge the application of scientific knowledge to an unlimited extent. Such techniques have brought about a virtual revolution and can be reckoned as the controlling forces of different walks of life. Operations Research, popularly known as OR, is a recent addition to a long list of scientific tools that provide a new outlook to many conventional management problems. OR adds greater sophistication to solving management problems. It seeks to determine the best (optimum) course of action of a decision problem in the face of limited resources. Therefore, OR has become a versatile tool in the field of management and its potential for future use is indeed substantial.

With the advent of automation, the centralization of organizational management has become disintegrated. In a single unit of industry, different

NOTES

departments of production, sales and inventory are governed by specialized agencies. Their targets and goals often diverge, even as they serve the common goals of the organization. The policy decisions required to coordinate these conflicting directives may be taken quite effectively if an optimal solution can be determined from all the available alternative compromise formulae. OR provides an effective scientific technique to solve such decision-making problems of modern business and industry.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss the nature and scope of operations research
- Analyse the significance of operations research
- Explain the scientific methods used in operations research
- Interpret problem formulation
- Describe the models in operations research
- Mention the techniques of operations research
- Assess the application areas of operations research

1.2 HISTORICAL BACKGROUND

The term, 'Operations Research' was first coined by well-known authors J. F. McCloskey and F.N. Trefethen in 1940. This new science came into existence in a military context. During World War II, military management called on scientists from various disciplines and organized them into teams to assist in solving strategic and tactical problems, relating to air and land defence of the country. Their mission was to formulate specific proposals and plans for aiding the military commands to arrive at decisions on optimal utilization of scarce military resources and efforts and also to implement the decisions effectively. This new approach to the systematic and scientific study of the operations of the system was called Operations Research (OR). Hence, OR can be referred to as 'an art of winning the war without actually fighting it'.

1.2.1 Nature and Scope of OR

Operations Research (OR) has been defined so far in various ways and it is perhaps still too early to be defined in some authoritative way. It is not possible to give uniformly-acceptable definitions of OR. The following definitions are proposed by various specialists in the field of OR. These have been changed according to the development of the subject.

> OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. *P. M. Morse, GE and Kimball*

NOTES

> OR is the scientific method of providing the executive with an analytical and objective basis for decisions.
>
> *P. M. S. Blackett*

> OR is the art of giving bad answers to problems to which otherwise worse answers are given.
>
> *T. L. Saaty*

> OR is a systematic method oriented study of the basic structures, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision-making.
>
> *E. L. Arnoff and M. J. Netzorg*

> OR is a scientific approach to problem solving for executive management.
>
> *H. M. Wagner*

> OR is an aid for the executive in making his decisions by providing him with the quantitative information based on the scientific method of analysis.
>
> *C. Kittee*

> OR is the scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.
>
> *H. A. Taha*

The various definitions given here, bring out the following essential characteristics of operations research:

- Systems orientation
- Use of interdisciplinary terms
- Application of scientific methods
- Uncovering new problems
- Quantitative solutions
- Human factors

Scope of Operations Research

There is great scope for economists, statisticians, administrators and technicians working as a team to solve problems of defence by using the OR approach. Besides this, OR is useful in various other important fields, such as the following:

- Agriculture
- Finance
- Industry
- Marketing
- Personnel management

- Production management
- Research and development

1.2.2 Phases of Operations Research

NOTES

The procedure to be followed in the study of OR generally involves the following major phases:

- Formulating the problem
- Constructing a mathematical model
- Deriving a solution from the model
- Testing the model and its solution (updating the model)
- Controlling the solution
- Implementation

1.2.3 Features of Operations Research

OR has gained increasing importance since World War II in the technology of business and industry administration. It greatly helps in tackling the intricate and complex problems of modern business and industry. OR techniques are, in fact, examples of the use of scientific method of management. The features of OR can be well understood under the following heads:

- **OR provides a tool for scientific analysis:** OR provides the executives with a more precise description of the cause-and-effect relationship and risks underlying the business operations in measurable terms. This eliminates the conventional intuitive and subjective basis on which managements used to formulate their decisions decades ago. In fact, OR replaces the intuitive and subjective approach of decision-making by an analytical and objective approach. The use of OR has transformed the conventional techniques of operational and investment problems in business and industry. As such, OR encourages and enforces disciplined thinking about organizational problems.
- **OR provides solution for various business problems:** The OR techniques are being used in the field of production, procurement, marketing, finance and other allied fields. Problems like the following, and similar other problems, can be solved with the help of OR techniques: How best can the managers and executives allocate the available resources to various products so that in a given time, the profits are maximum or the cost is minimum?

  Is it possible for an industrial enterprise to arrange the time and quantity of orders of its stocks such that the overall profit with given resources is maximum?

  How far is it within the competence of a business manager to determine the number of men and machines to be employed and used in such a manner

NOTES

that neither remains idle and at the same time, the customer or the public has not to wait unduly long for service?. Similarly, we might have a complex of industries—steel, machine tools and others—all employed in the production of one item, say steel. At any particular time, we have a number of choices of allocating resources, such as money, steel and tools for producing autos, building steel factories or tool factories. What should be the policy which optimizes the total number of autos produced over a given period? OR techniques are capable of providing an answer in such a situation.

Planning decisions in business and industry are largely governed by the picture of anticipated demands. The potential long-range profits of the business may vary in accordance with different possible demand patterns. The OR techniques serve to develop a scientific basis for coping with the uncertainties of future demands. Thus, in dealing with the problem of uncertainty over future sales and demands, OR can be used to generate 'a least risk' plan.

At times, there may be a problem in finding an acceptable definition of long-range company objectives. Management may be confronted with different viewpoints—some may stress the desirability of maximizing net profit whereas others may focus attention primarily on the minimization of costs. OR techniques (especially that of mathematical programming, such as linear programming) can help resolve such dilemmas by permitting systematic evaluation of the best strategies for attaining different objectives. These techniques can also be used for estimating the worth of technical innovations as also of potential profits associated with the possible changes in rules and policies.

How much changes can be there in the data on which a planning formulation is based without undermining the soundness of the plan itself? How accurately must managements know the cost coefficients, production performance figures and other factors before it can make planning decisions with confidence? Many of the basic data required for the development of long-range plans are uncertain. Such uncertainties though cannot be avoided, but through various OR techniques, the management can know how critical such uncertainties are and this in itself is a great help to business planners.

- **OR enables proper deployment of resources:** OR renders valuable help in proper deployment of resources. For instance, Programme Evaluation and Review Technique (PERT) enables us to determine the earliest and the latest times for each of the events and activities and, thereby, helps in the identification of the critical path. All these help in the deployment of resources from one activity to another to enable the project completion on time. Thus, this technique provides for determining the probability of completing an event or project itself by a specified date.

NOTES

- **OR helps in minimizing waiting and servicing costs:** The waiting line or queuing theory helps the management in minimizing the total waiting and servicing costs. This technique also analyses the feasibility of adding facilities and, thereby, helps the business people in taking a correct and profitable decision.
- **OR enables the management to decide when to buy and how much to buy:** The main objective of inventory planning is to achieve balance between the cost of holding stocks and the benefits from stock holding. Hence, the technique of inventory planning enables the management to decide when to buy and how much to buy.
- **OR assists in choosing an optimum strategy:** Game theory is especially used to determine the optimum strategy in a competitive situation and enables the businessmen to maximize profits or minimize losses by adopting the optimum strategy.
- **OR renders great help in optimum resource allocation:** The linear programming technique is used to allocate scarce resources in an optimum manner in problems of scheduling, product mix, and so on. This technique is popularly used by modern managements in resource allocation and in affecting optimal assignments.
- **OR facilitates the process of decision-making:** Decision theory enables the businessmen to select the best course of action when information is given in the probabilistic form. Through decision tree (a network showing the logical relationship between the different parts of a complex decision and the alternative courses of action in any phase of a decision situation) technique, executive's judgement can systematically be brought into the analysis of the problems. Simulation is another important technique used to imitate an operation or process prior to actual performance. The significance of simulation lies in the fact that it enables in finding out the effect of alternative courses of action in a situation involving uncertainty where mathematical formulation is not possible. Even complex groups of variables can be handled through this technique.
- **Through OR management, you can know the reactions of the integrated business systems:** The Integrated Production Models technique is used to minimize cost with respect to workforce, production and inventory. This technique is quite complex and is usually used by companies having detailed information concerning their sales and costs statistics over a long period. Besides, various other OR techniques also help management people in taking decisions concerning various problems of business and industry. The techniques are designed to investigate how the integrated business system would react to variations in its component elements and/or external factors.

Check Your Progress

1. State the essential characteristics of operations research.
2. How is operations research used as a tool for scientific analysis?

NOTES

1.3 SCIENTIFIC METHODS IN OPERATIONS RESEARCH

The methodology of operations research generally involves the following steps:

- **Formulating the problem:** The first step in an OR study is to formulate the problem in an appropriate form. Formulating a problem consists of identifying, defining and specifying the measures of the components of a decision model. This means that all quantifiable factors which are pertinent to the functioning of the system under consideration are defined in mathematical language: variables (factors which are not controllable) and parameters or coefficient, along with the constraints on the variables and the determination of suitable measures of effectiveness.
- **Constructing the model:** The second step comprises constructing the model, by which we mean that appropriate mathematical expressions are formulated that describe interrelations of all variables and parameters. In addition, one or more equations or inequalities are required to express the fact that some or all of the controlled variables can only be manipulated within limits. Such equations or inequalities are termed as constraints or the restrictions. The model must also include an objective function, which defines the measure of effectiveness of the system. The objective function and the constraints, together constitute a model of the problem that we want to solve. This model describes the technology and the economics of the system under consideration through a set of simultaneous equations and inequalities.
- **Deriving the solution:** Once the model is constructed, the next step in an OR study is that of obtaining the solution to the model, that is, finding the optimal values of the controlled variables—values that produce the best performance of the system for specified values of the uncontrolled variables. In other words, an optimum solution is determined on the basis of the various equations of the model satisfying the given constraints and interrelations of the system, and at the same time maximizing profit or minimizing cost or coming as close as possible to some other goal or criterion. How the solution can be derived depends on the nature of the model. In general, there are three methods available for the purpose, namely the analytical methods, the numerical methods and the simulation methods. Analytical methods involve expressions of the model by mathematical computations and the kind of mathematics required depends

NOTES

upon the nature of the model under consideration. This sort of mathematical analysis can be conducted only in some cases without any knowledge of the values of the variables, but in others the values of the variables must be known concretely or numerically. In latter cases, we use the numerical methods, which are concerned with iterative procedures through the use of numerical computations at each step. The algorithm (or the set of computational rules) is started with a trial or initial solution and continued with a set of rules for improving it towards optimality. The initial solution is then replaced by the improved one and the process is repeated until no further improvement is possible. However, in those cases where the analytical as well as the numerical methods cannot be used for deriving the solution, then simulation methods are used, that is, experiments are conducted on the model in which values of the uncontrolled variables are selected with the relative frequencies dictated by their probability distributions. The simulation methods involve the use of probability and sampling concepts, and are generally used with the help of computers. Whichever method is used, our objective is to find an optimal or near optimal solution, that is, a solution which optimizes the measure of effectiveness in a model.

- **Testing the validity:** The solution values of the model, obtained as stated in the previous step, are then tested against actual observations. In other words, effort is made to test the validity of the model used. A model is supposed to be valid if it can give a reliable prediction of the performance of the system represented through the model. If necessary, the model may be modified in the light of actual observations and the whole process is repeated till a satisfactory model is attained. The operational researcher quite often realizes that his model must be a good representation of the system and must correspond to reality, which in turn requires this step of testing the validity of the model in an OR study. In effect, performance of the model must be compared with the policy or procedure that it is meant to replace.
- **Controlling the solution:** This step of an OR study establishes control over the solution by proper feedback of information on variables, which might have deviated significantly. As such, the significant changes in the system and its environment must be detected and the solution must accordingly be adjusted. This is particularly true when solutions are rules for repetitive decisions or decisions that extend over time.
- **Implementing the results:** Implementing the results constitutes the last step of an OR study. Since the objective of OR is not merely to produce reports but to improve the performance of systems, the results of the research must be implemented, if they are accepted, by the decision-makers. It is through this step that the ultimate test and evaluation of the research is

made and it is in this phase of the study where the researcher has the greatest opportunity for learning.

Thus, the procedure for an OR study generally involves some major steps, namely formulating the problem, constructing the mathematical model to represent the system under study, deriving a solution from the model, testing the model and the solution so derived, establishing controls over the solution, and lastly, putting the solution to work-implementation. (Although the said phases and the steps are usually initiated in the order listed in an OR study, it should always be kept in mind that they are likely to overlap in time and interact, that is, each phase usually continues until the study is completed.)

The OR approach can as well be illustrated by the following flow chart shown is Figure 1.1.

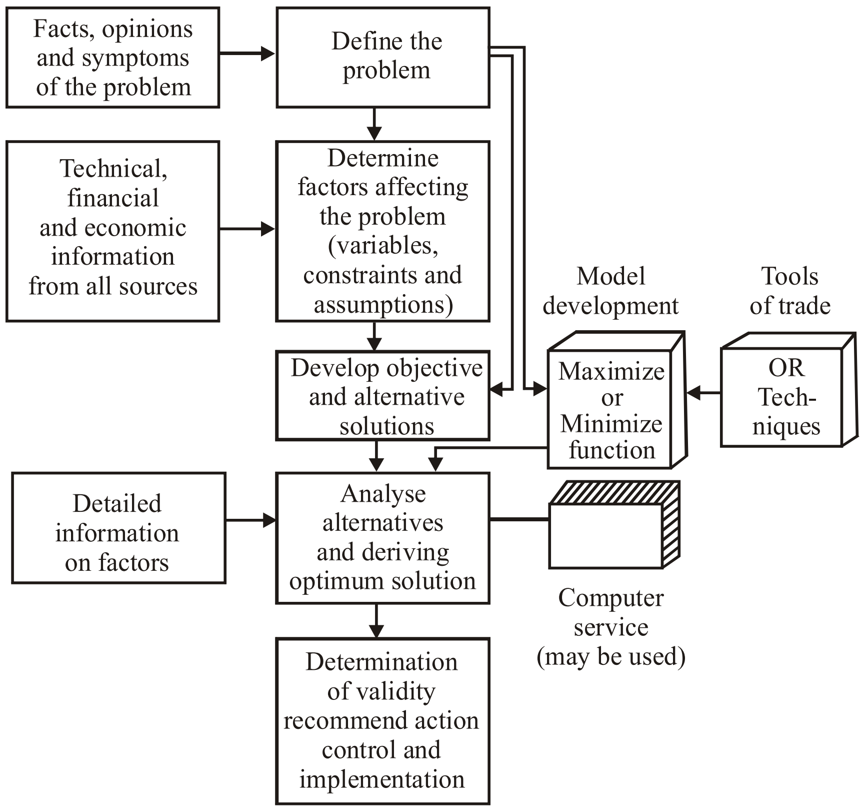


Fig. 1.1 *Flow Chart Showing OR Approach*

NOTES

Check Your Progress

3. State the first step in the scientific method of operations research.
4. How are the solution values of a decision model (obtained) validated?

1.4 PROBLEM FORMULATION

NOTES

To formulate a problem correctly, one must know what a problem is and must understand the nature of the problem. A problem is said to exist when we have the following conditions satisfied:

- There must be an individual or a group, which has some difficulty or problem.
- There must be some objectives to be attained. If one wants nothing, one cannot have a problem.
- There must be alternative means (or the courses of action) for attaining the objective(s) one wishes to attain. This, in other words, means that there must be atleast two means available. If there is no choice of means, one cannot have a problem.
- There must remain doubt with regard to selection of alternatives, that is, the relative efficiency of the alternative means must differ.
- There must be some environment(s) to which the difficulty pertains.

The nature of the problem must be understood in context of the above points before its formulation takes place. It is only after this that the problem must be stated along with the objective(s) and the bounds within which it is to be studied. In fact, this involves the task of laying down boundaries within which we shall study the problem with a predetermined objective in view. In brief, the task of formulating the problem consists in making various components of the problem explicit.

Formulating the Objectives

Generally, the objectives of an organization may either be retentive or acquisitive. Retentive objectives are concerned with preserving either resources of value, such as money, time, energy, or states such as comfort, safety, stability, and the like. In other words, retentive objectives are concerned with what is consumed by courses of action and as such can easily be translated as minimizing objectives. On the other hand, we may have acquisitive objectives which are concerned with acquiring resources or attaining states and can easily be translated as maximizing objectives. Essentially then, we may have either minimizing objectives or maximizing objectives. Accordingly, profits, sales, revenues, output and the like are always to be maximized, and cost, time and the like are to be minimized.

Formulation of the problem and the objective often requires a detailed knowledge of how the concerning system operates, which can be learnt through performing a system analysis. This is a process analogous to the physical examination that a doctor performs on his patient after initial discussion of symptoms. Such an analysis can provide the background information required in formulating the problem and the model required to solve it.

Measure of Performance

After understanding the nature of the problem and after identifying the courses of action, the uncontrolled variables and the objectives, it is necessary to construct a measure of performance that can be used to determine which alternative is best and what function of this measure (usually described as the objective function) should be used as the criterion of best solution of the concerning problem. Selecting a criterion of the best solution often requires a good knowledge of decision theory. The type of appropriate criterion depends on the state of knowledge of outcomes that we assume to be available. Usually, we may have one of the three situations with regard to the state of outcomes, namely, deterministic situation or state of certainty, stochastic situation or risk situation, and finally the situation of uncertainty. They are discussed as follows:

- **Deterministic situations:** These are situations in which each course of action results in only one outcome. In such situations, the probability of each outcome is known or believed to be either equal to unity or zero. Due to this, the criterion or the measure of performance used in deterministic situations is that of the maximization of utility.
- **Stochastic situations:** These are situations in which each course of action can result in alternative outcomes; the probabilities of which are known or can be estimated. The criterion that is generally accepted as the most appropriate in such situations is that of maximizing the expected utility and is often termed as either the Expected Monetary Value (EMV) criterion or the Expected Opportunity Loss (EOL) criterion. To select the best alternative, we select that alternative of which EMV is highest possible (or EOL is the lowest possible).
- **Uncertainty situations:** These are situations in which each course of action can result in alternative outcomes, the probabilities of which cannot be assigned, as the decision-maker does not know which outcomes can or will occur. This means that the decision-maker has to act with imperfect information in such a situation. Consequently, there is no single best criterion for selecting a strategy to deal with such a situation. Accordingly, various criteria have been developed for selecting a course of action in uncertainty situations. The following criteria deserves mention in this context:
    (i) Maximin decision rule
    (ii) Maximax decision rule
    (iii) Minimax decision rule (or regret rule)
    (iv) Hurwicz decision rule
    (v) Laplace decision rule

Nothing can be said as to which one of these criteria is the best criterion in any given situation involving uncertainty. The choice for the selection must be left to the decision-maker who must resort to his own judgement, skill and experience.

NOTES

Thus, problem formulation, which is considered as the first phase of an OR study, requires a statement of the problem's components that include the controllable or the decision variables, the uncontrollable parameters or the environment, the constraints on the variables and also the objective along with an appropriate measure of performance.

Check Your Progress

5. What are stochastic situations?
6. What are retentive objectives?

1.5 TYPES OF MODELS IN OPERATIONS RESEARCH

After problem formulation, modelling plays an important part in OR. In fact, it is considered as the essence of operations research approach. Modelling permits the operational researcher to test rigorously the implications of new plans, new schemes and new ideas. The term 'modelling' generally refers to the idea of constructing a model of the given problem and then deriving the solution from it for the problem under consideration. Modelling is a device used to arrive at a well-structured view of reality. By analysing or experimenting on models, we can usually determine how changes in the relevant system will affect its performance.

What is a model?

A **model** is a physical or symbolic representation of the relevant aspects of the reality or system with which one is concerned. In other words, a model is a means of portraying the system or reality of concern to the decision-maker. As such, the concept of a model generally implies a series of connected and identifiable relationships that essentially demonstrate the proposition: if this action, then this result. Thus, 'the model, being an abstraction of the assumed real system, identifies the pertinent relationships of the system in the form of an objective and a set of constraints.'

Types of models

Models can be of several types, but models commonly referred are as follows:

- **Iconic models:** Iconic model is a pictorial or visual representation of certain aspects of a system. In iconic models, the relevant properties of the real thing are represented by the properties themselves, usually with a change of scale. Such models generally look like what they represent, but differ in size. An iconic model is said to be 'scaled down' or 'scaled up' as the dimensions of the model are smaller or bigger than those of the real thing. For instance, iconic models of the sun housed in planetoriums, the model of a huge building or of an airplane or of an automobile for exhibition purposes are scaled down; whereas models of the atom or of a cell used for, say teaching purposes are scaled up. Similarly, maps, photographs, drawings also constitute examples of iconic models. Iconic models, such as models of aircraft, machine, engine or of any tangible thing may also be termed as physical models. Iconic models are generally specific, concrete and easy to observe, but are difficult to manipulate for experimental purposes and are not very useful for the purpose of prediction. Moreover, such models represent a static event only.

- **Analogue models:** Analogue models use one set of properties to represent another set of properties. They are more abstract than iconic models. However, such models are easier to manipulate and can represent dynamic situations. Graphs representing time series, stock market changes or other phenomenons, flow charts, demand curves and frequency graphs are examples of analogue models for they use geometrical magnitudes and location to represent a wide variety of variables and the relationships between them. Similarly, a hydraulic system can be used as an analogue model of electrical, traffic and economic systems. Maps in different colours, each colour in a map representing a particular characteristic, such as blue colour representing water, brown representing land, and so on, are also termed as analogue models. Such models are generally more useful than iconic models because of their relatively greater capacity to represent the characteristics of the real system.
- **Symbolic or mathematical models:** In symbolic models, the components of what is represented and their interrelationships are given by symbols. These models use letters, numbers and other types of symbols to represent variables and the relationships between them. Such models generally assume the form of equations or inequalities depicting the relationships amongst the variables of the system which they represent. For instance, a mathematical equation representing a relationship between constants and variables may be considered a symbolic model. Thus, a symbolic model means a mathematical description of an activity, which expresses the relationships among the various elements with sufficient accuracy that it can be used to predict the actual outcome under any expected set of circumstances. Symbolic models are most general and precise, and also are most abstract in nature. They are usually the easiest to manipulate experimentally.

  A great advantage of such models is that they lend themselves easily for manipulation on computers. In most OR applications, it is assumed that the objective and constraints can be expressed mathematically as functions of decision variables. In such a case, we say that we are dealing with a symbolic or a mathematical model. Important symbolic or mathematical models often used in operations research are as follows:

  (*i*) *Allocation models:* They are used in finding a solution of optimizing a given objective such as profit maximization under certain restrictions.

NOTES

(*ii*) *Scheduling models:* They are used in determining an optimum sequence of performing certain operations of jobs with a view to minimize overall time and/or cost. PERT and other network techniques fall under this category.

(*iii*) *Queuing models:* They represent the random arrival of customers at a service station where the facility is limited and the object of such models happens to be to minimize the costs of both servicing and waiting.

(*iv*) *Inventory models:* These models relate mainly to optimizing inventory levels. The main emphasis in such models is on minimizing the total costs associated with inventories.

(*v*) *Replacement models:* These models are concerned with replacement strategies.

(*vi*) *Competitive models:* These models are concerned with the determination of optimum strategies under competitive conditions. Game theory models are examples of competitive theory.

These wide variety of models used in OR give rise to a corresponding number of solution techniques, such as linear, integer, dynamic and non-linear programming that represent algorithms for solving the models.

- **Simulation models:** At times, however, it may not be possible to depict a real system in mathematical formulation, that is, it may just happen that a mathematical model of a given problem cannot be constructed, and as such, we may adopt simulation models.

Simulation models differ from mathematical models in that the relationships between input and output are not explicitly stated. A simultation model breaks down the modelled system into basic modules that are then linked to one another by a well-defined logical relationship.

1.5.1 Structure of the Mathematical Model

- **Functional, operational and accounting models:** A functional model is a model without values specified for the constants and as such the variables can only be identified in abstract mathematical notation. A functional model specifies the form of the relationship, but not the exact value of the relationship. However, when the values which must be associated with some measurable quantity in the environment being represented have been identified, we have an operating model. For instance, $Y_i = a_i + b_i X_i$ is an example of a functional model, but when we identify the values of the two constants, such as $a_1 = 4$ and $b_1 = 3$ or $a_2 = 5$ and $b_2 = 4$, then we can write the said model either as $Y_1 = 4 + 3X_1$ or $Y_2 = 5 + 4X_2$, and in this form, it is known as an operating model. Certain equations are essentially tautologies because they merely represent addition of values to give a total value; thus, in a sense,

NOTES

they are related to accounts. Suppose, we are given Y_1 and Y_2. The total production, Y_i is equal to $Y_1 + Y_2$. This is nothing but an accounting statement. Does it constitute a model? In the business world, we assume it to be a model (and consider it an accounting model) because it does appear to represent symbolically relevant aspects of the reality with which we are often concerned.

- **Prediction, improvement and optimization models:** Mostly this classification is stated in the context of managerial analysis. Optimization models are the result of quantitative analysis and allow the measurable criterion to assume the highest or the lowest possible value, given the model structure and constraints. Improvement models are the result of system analysis and consequently the highest value of the criterion is chosen from among the alternatives investigated. Either of these models depends upon inputs which predict outcomes of relationship; thus, they depend upon a prediction model.
- **Static, dynamic, linear and non-linear models:** According to Joseph Wright Forrester (a pioneer in operations research), all models possess differing degrees of various characteristics, namely staticness, linearity, stability and openness. In view of this, models can as well be classified as static model or dynamic model, linear model or non-linear model. A static model is one in which time plays no role, that is, the relationships do not vary with time. As against this, a dynamic model contains time-varying interactions. A linear model is one in which all of the equations or inequalities are linear, but a model which contains one or more non-linear equations is known as a non-linear model. In case of non-linear models, the degree of non-linearity may vary greatly from model to model.
- **Stable and unstable models:** In the words of Forrester, 'A stable system is one that tends to return to its initial condition after being disturbed. It may overshoot and oscillate like a single pendulum that is set in motion, but the disturbances decline and die out. In an unstable system that starts at rest, an initial disturbance is amplified, leading to growth or oscillations whose amplitude increases.' Another sub-element of stability is concerned with the presence of steady state or transient models. Using the words of Forrester again, we can say that a steady state pattern is one that is repetitive with time in which the behaviour in one period is of the same nature as any other period. On the other hand, the transient behaviour refers to those changes where the character of the system changes with time.
- **Open and close models:** On the basis of the characteristic of openness, a model can be classified either as an open model or as a closed model. A closed model is one that internally generates the values of variables through time by the interaction of variables one on another, whereas the open model is one in which some variables are supplied from the external environment.

NOTES

Properties of a good model

We can list down the properties of a good model as follows:

- A good model must represent the reality or the system with which it is concerned as accurately as possible and must not compromise with any of the critical elements of the system. However, this does not mean that a model should be as close as possible to reality in every respect, for then, it would require an infinite length of time to construct such a model, which at times may even be beyond human comprehension. Thus, a good model must be economical of time and also be readily workable. Usable approximate models are much better than more exact models that cannot be used.
- Since a model happens to be a simplified representation of an empirical situation, it must possess the characteristic of predicting outcomes reasonably well and must be consistent with effective action.
- A good model must be simple enough so as to be readily understood and, if necessary, must be capable of being modified quickly and effectively.
- A good model must identify the factors that can influence the course of a particular decision and must also determine how these factors interact to produce departures from the expected or desired outcome.
- A model must make certain assumptions about the structure of the problem under study for no system of equations can produce meaningful results if it consists entirely of unknown, but the number of assumptions should be as small as possible in a good model.
- A good model is one that is capable of answering in case when small perturbations take place in the data. This sensitivity of the model, that is, its capacity to answer in context of small changes constitutes an important virtue of a good model.

Advantages of modelling

There are several advantages that result from modelling. Important ones are as follows:

- The main advantage of modelling is that the model formulated under it can be manipulated in a variety of ways until the best solution of the problem under consideration is found. The actual system or the organization can then be operated in an optimum fashion with a minimum of costly experiments and adjustments. For instance, models of aircraft are always tested in wind tunnels before production is started.
- Models can be used on a 'what if' basis to explore the possible future consequences of decisions that might be made today. This enables the creative manager to test the implications of new plans, new schemes and

new ideas that are most critical to the success or failure of a particular project. Modelling, thus, helps in finding avenues for research and improvements.

NOTES

- Modelling provides an easy and systematic approach for studying the problem concerned. Through this technique, the problem under study becomes controllable. Had the models been as complex and difficult to control as reality, then there would have been no advantage in their use. In fact, modelling strips a natural phenomenon of its bewildering complexity and duplicates the essential behaviour of the natural phenomenon with a few variables, simply related.
- Modelling provides an opportunity to explore a broad range of solutions and to estimate the sensitivity of the solution to changes in individual elements or in combination of these elements. This, in turn, will enable the decision-maker to move away safely from the precise optimum point by a given margin in order to accommodate other factors. Thus, modelling indicates the limitations and scope of an activity.
- Model building and analysis does not replace intuition and judgement; it rather supports them with tools for handling complexity and uncertainty with which the human intuition cannot cope unaided. In a sense, the use of a model frees the intuition and permits it to concentrate on those problems to which it is particularly suited.
- Modelling enables quick and economical experimentation for finding an optimum solution for a given problem. The results of such experiments can be evaluated to determine which strategy is most consistent with a desired objective. Besides, the validity of the model (that is, whether the model results in a reliable prediction of the system's performance) can be checked, and in case of even little doubt, the model can accordingly be modified quickly and effectively.

Deriving Solutions from Models

(a) Abstraction

Abstraction happens to be the first step in modelling and consists in selecting the critical factors or variables from the empirical situation. There are usually an uncountable number of 'facts' in any empirical situation and the decision-maker must intelligently abstract those factors which he considers to be most relevant to the problem he faces. Relevance has two charactreristics: (i) There may be variables which affect the environment, but whose impact is negligible and as such they are not considered relevant. (ii) There are some variables which do not affect the environment and thus, need not be even considered tentatively as a part of the model. Essentially, the model must deal with the relevant aspects of the reality with which we are concerned. For instance, one can classify automobiles in several ways: on the basis of car's origin one can differentiate between domestic and

NOTES

foreign cars; they can as well be classified on the basis of age, cost or manufacturer. One can as well classify them on the basis of body style, colour, size of engine or on the basis of the other factor. The point is that all these factors are valid in a particular situation. However, if someone is interested in conducting the study of the effect of the weather on automobiles, the paint colour would be very useful in such a study. If one is, however, concerned with automobile accidents, then the classification on the basis of colour would be useless and the classification on the basis of engine type and size would be considered as appropriate and valuable.

(b) Model building

The next step in modelling is that of model building. The relevant factors or variables selected as stated in (*a*) are put in some logical manner, so that they form a model of the given problem. In model building, it is generally considered desirable to simplify reality but only to the point where there is no significant loss of accuracy. In fact, the decision-maker tries his level best to construct the simplest possible model that predicts outcomes reasonably well and is amenable to quick modification in case there is need for the same models to be represented in a variety of ways. However, in the relatively recent past, emphasis is laid on the development of symbolic models particularly with regard to business problems. As such, one often talks about symbolic models in the context of model building. The question then is: 'How to construct a mathematical model for a given problem?' The answer depends on that the variables and parameters which are relevant and are also capable of quantitative measurement of some form which must be put in logical manner indicating the necessary relationships along with constraints and the objective function. Such an arrangement is often described as a model of the given problem. It is not possible to prepare a manual of constructions for model building as much depends upon the imagination, insight and experience of the decision-maker himself. However, past experience of model building proves a good guide in developing an appropriate model. Accordingly, we give the following types of illustrative symbolic models:

- **Case 1 (Total Expense model):** Symbolic model can just be presented in the equation form. A very simple model of total expense can be as follows:

  $$T = a + bX$$

  where, T = total costs; a = fixed costs; b = variable costs; X = number of units.

  This model can be used as a predictor of the total costs associated with a given level of output, of course, with certain assumptions. This model implies that total costs are dependent upon the number of units produced. Thus, T is the dependent variable and X is the independent variable in the given context. On the other hand, a and b are constants (or the parameters) of the given model. The model, thus, represents the relationship between the

NOTES

total cost on one hand and number of units produced along with given constants on the other. If the object happens to have the total costs at its minimum level, then we have to find that value of X corresponding to which T happens to be minimum possible.

- **Case 2 (Milkman's model):** Consider the situation of a milkman who must decide how many cases of milk to order to maximize his expected profits. He buys a certain number of milk cases each day and sells some or all of them. He makes a profit on each one he sells. However, the unsold case brings him a loss. The demand for milk cases in the market varies from day to day, but the probability that any specified number of milk cases will be sold on a particular day can be determined on the basis of past data. We can develop the model for this type of problem by identifying the relevant variables and assigning symbols to them as follows:

  n = The number of milk cases ordered per day

  p = The profit made on each milk case sold

  l = The loss incurred on each milk case unsold

  d = The demand, that is, the number of milk cases that could be sold per day if $n \geq d$

  pr (d) = The probability that the demand will equal supply on a randomly selected day

  P = Net profit per day (negative P represents a loss)

  If the demand on a particular day exceeds the number ordered, that is, If $d > n$, his profit would be:

  $P(d > n) - n.p$

  On the other hand, if demand does not exceed the number ordered, his profit would be as follows:

  $$P(n \geq d) = d.p - (n - d).1$$

  Then the exacted net profit per day ($\overline{P}$) can be expressed as follows:

  $$\overline{P} = \sum_{d=1}^{n} P(d)\left[dp - (n-d)l\right] + \sum_{d=n+1}^{\infty} P(d)n.p$$

  This is a decision model under risk situation, wherein $\overline{P}$ is the measure of performance, n is the controlled variable, d is an uncontrolled variable, and p and l are uncontrolled constants. To solve the problem represented through this model, we should find the value of n that maximizes $\overline{P}$.

- **Case 3 (Linear Programming model):** A simple linear programming model can be stated as follows:

  Maximize: $Z = \sum_{j=1}^{n} c_j X_j \quad (j = 1,2,\ldots\ldots n)$ (Objective Function)

NOTES

Subject to the constraints:

$$\sum_{j=1}^{n} aij\ Xj < bi \quad (i = 1,2,\ldots\ldots\ m) \qquad \text{(Constraints concerning resources)}$$

and $X_j \geq 0$ (Non-negativity condition)

The linear programming model pertains to maximization case. It assumes prior knowledge of all the coefficients, that is, it operates only under conditions of certainty. It has three components, namely the objective function, constraints and the non-linearity condition. The various assumptions are the linearity of the objective function and that of the constraints, additivity of resources and divisibility of decision variables. Non-negativity condition implies that no X can be negative. The model requires the determination of n production activities pursued at level X subject to a limited amount of m resources being available. Each unit of the jth activity yields a return of c, and uses an amount aij of the ith resource. Z denotes the optimal value of the objective function for a given system.

- **Case 4 (Simple inventory model concerning economic lot size):** After the model has been built, certain conclusions may be derived about its behaviour by means of logical analysis. Some mathematical calculations are often required for obtaining the solution to the model. The solution to the model means those values of the controllable variables that optimize the given objective. If the logic adopted in deriving the conclusions from the model is correct, then the solution to the model will also serve as the effective solution for the empirical problem represented by the said model.

(c) The question of error

It should always be remembered that the solution derived from the model always contains some degree of error because of the abstraction process under which the decision-maker may select the wrong variables, or not enough variables for constructing his model. If the error involved is not big enough to disturb the validity of the concerned model, then we generally do not mind the error and accordingly accept the solution provided by the model. However, when the error becomes significant, then we have to rethink about the model. The question of when the error becomes so large that the conclusions must be modified before it can be adopted as a solution, is one of judgement of the decision-maker depending upon his experience and knowledge of the given situation.

(d) Updating the model

A good model is one which gives valid results and permits reliable prediction of the system's performance. As such the model once built up to represent a system must be continuously updated to take account of the past, present and future specifics of the problem. This, in other words, means that the model should be modified as soon as one or more of the controlled variables change significantly increasing the size of the error involved.

NOTES

A word of caution in modelling

The usefulness of modelling has already been stated, but one must note that the same follows only when the model happens to be a reasonable approximation of reality. If the model is a good representation of the problem, the solution of the model will be a good solution of the problem. However, the solution of a poor model, even though it may be an exact one, will not be a good solution of the problem. Hence, the construction of the model and the translation of the results to the actual problem must be done with utmost care. The key to successful modelling is to represent the decision situation as accurately as possible and in particular, not to compromise the critical elements of the problem. For this purpose, the problem must be well defined before constructing the model along with a clear and precise specification of all relevant factors and the interrelationships among them.

It should also be noted that though many problems can be easily represented in the form of models, but there may be problems in respect of which we may not construct reasonably good models. Accordingly, we must construct an appropriate model of a problem if we can, otherwise we must use an intuitive or heuristic approach to solve the given problem. 'Finally, if a problem is so complex that it cannot be modelled adequately for solution by one of the available methods or it cannot be solved adequately with a heuristic method, then the problem solver usually resorts to simulation. Of course, simulation is not the answer to all problems; nevertheless, it does have a great deal of merit in studying large, complex systems where the components are highly interrelated.'

Check Your Progress

7. Mention the properties of a good model.
8. What is quadratic programming?

1.6 APPLICATION AREAS OF OPERATIONS RESEARCH IN MANAGEMENT

Operation research techniques, having their origin with war operations analysis during the World War II, have since then assisted defence management to a larger extent. Defence management being vitally concerned with problems of logistics, such as provisioning, distributing, search and forecasting, OR has given valuable help in decision formation. Although OR originated in context of military operations, its impact nowadays can be observed in many other areas. It has successfully entered into different research areas relating to modern business and industry. In fact, it is being reckoned as an effective scientific technique to solve several managerial decision-making problems, such as inventory management problems, resource allocation problems, problems relating to replacement of both men as well as machines, sequencing and scheduling problems, queuing problems, competitive problems, search problems, and the others.

NOTES

Big and Small Organizations

The OR approach, though being practised by big organized business and industrial units, is equally applicable to small organizations. For instance, whenever a departmental store faces the problem, such as employing additional sales girls and purchasing an additional van, one can apply some techniques of the operations research to minimize cost and maximize benefit for such decision. As such, it is often said that wherever there is a problem, there is scope for applying operations research techniques.

Military, business and research

In recent years of organized development, operations research has successfully entered, apart from its military and business applications, into several other areas of research. For instance, the basic problem in most of the developing countries in Asia and Africa is to remove poverty as quickly as possible. There is a considerable scope to solve this problem by applying OR techniques. Similarly, with the explosion of population and consequent shortage of food, every country is facing the problem of optimum allocation of land for various crops in accordance with the climatic conditions and available facilities. The problem of optimal distribution of water from a source, like a canal for irrigation purposes, is faced by each developing country. Hence, good amount of scientific work related to the OR can be done in this direction. Thus, there is considerable scope of applying OR techniques for solving problems relating to national planning. National planning can be largely improved if optimum coordination can be arrived at through central planning agency by resorting to operations research models and techniques. In India, there is a promising field for application of operations research techniques in this direction.

Finance

In the field of finance, the progress of operations research has been rather slow. Financial institutions have many situations similar to those prevailing in other commercial organizations. Investment policy must maximize the return on it, keeping the factor of risk below a specified level. Long-range corporate objectives of an institution are always studied along with functional objectives of individual departments, consistency between the two being essential for corporate planning. In particular, banking institutions require precise and accurate forecasting on cash management and capital budgeting. Linear programming models for many problems and quadratic programming formulation for some complex problems may prove the usefulness of operations research techniques to credit institutions.

Hospital management, health planning and transportation system

The application of operations research techniques can as well be noticed in the context of hospital management, health planning programmes, transportation system and in several other sectors. For instance, hospital management quite often faces the problem of allotting its limited resources, of its multiphased activities. Optimum

NOTES

allocation of resources, so as to ensure a certain desirable level of service to patients, can be reached through OR. Similarly, for operating a transportation service profitably, an optimum schedule for vehicles and crew members becomes necessary. Punctuality, waiting time, total travel time, speed of the vehicles and agreement with trade unions of crew members are some of the variables to be taken into account while preparing optimum schedules, which can be worked out utilizing OR techniques.

As stated earlier, operations research is generally concerned with problems that are tactical rather than strategic in nature, that is, its use to long-range organizational planning problems is very much limited. However, one can hope that with further developments taking place in the field, operations research will be able to deal with organizations in their entirety, rather than with slices through them.

This description brings home the point that during the last four decades, the scope of OR has extensively been widened and hopefully shall attain new strides by the end of the present century. The art of systems analysis, so well developed in the military context, will spread to other contexts—notably such civil government branches as criminal justice, urban problems, housing, health, care, education and social services. There is growing awareness amongst the people that unless they make themselves familiar with OR techniques, they would not be able to understand and appreciate the problems of modern business units. With computer facilities becoming widespread the significance and scope of OR is likely to grow in the coming years. In fact, the future holds great promise for the growth of OR in power, scope and practical importance and utility. OR will continue its currently vigorous efforts to reach out to new arenas of exploration and application.

OR and Modern Business Management

From what has just been stated, we can say that OR renders valuable service in the field of business management. It ensures improvement in the quality of managerial decisions in all functional areas of management. The role of OR in business management can be mentioned as follows:

- **Helps the directing authority:** OR techniques help the directing authority in optimum allocation of various limited resources, namely men, machines, money, material, time, and so on, to different competing opportunities on an objective basis for achieving effectively the goal of a business unit. They help the chief of executive in broadening management vision and perspectives in the choice of alternative strategies to the decision problems, such as forecasting manpower, production capacities, capital requirements and plans for their acquisition.
- **Useful to the production management:** OR is useful to the production management in (i) selecting the building site for a plant, scheduling and controlling its development and designing its layout; (ii) locating within the

NOTES

plant and controlling the movements of required production materials and finished goods inventories; (iii) scheduling and sequencing production by adequate preventive maintenance with optimum number of operatives by proper allocation of machines; and (iv) calculating the optimum product-mix.

- **Useful in personnel management:** OR is useful to the personnel management to find out (i) optimum manpower planning, (ii) the number of persons to be maintained on the permanent or full- time roll, (iii) the number of persons to be kept in a work pool intended for meeting the absenteeism, (iv) the optimum manner of sequencing and routing of personnel to a variety of jobs, and (v) in studying personnel recruiting procedures, accident rates and labour turnover.
- **Help in marketing management:** OR techniques equally help the marketing management to determine (i) where distribution points and warehousing should be located, their size, quantity to be stocked and the choice of customers; (ii) the optimum allocation of sales budget to direct selling and promotional expenses; (iii) the choice of different media of advertising and bidding strategies; and (iv) the consumer preferences relating to size, colour, packaging, and so forth, for various products as well as to outbid and outwit competitors.
- **Useful in financial management:** OR is also very useful to the financial management in (i) finding long-range capital requirements as well as how to generate these requirements, (ii) determining optimum replacement policies, (iii) working out a plan for the firm, (iv) developing capital investments plans, and (v) estimating credit and investment risks.

In addition to all this, OR provides the business executives an understanding of the business operations which gives them new insights and capability to determine better solutions for several decision-making problems with great speed, competence and confidence. When applied on the level of management where policies are formulated, OR assists the executives in an advisory capacity. However, on the operational level where production, personnel, purchasing, inventory and administrative decisions are made, it provides management with a means for handling and processing information. Thus, in brief, OR can be considered as scientific method of providing executive departments with a quantitative basis for taking decisions regarding operations under their control.

1.6.1 Limitations of OR

OR though is a great aid to management, as outlined, but still it cannot be a substitute for decision-making. The choice of a criterion as to what is actually best for a business enterprise is still that of an executive who has to fall back upon his experience and judgement. This is so because of the several limitations of OR. Important limitations are given as follows:

NOTES

- **The inherent limitations concerning mathematical expressions:** OR involves the use of mathematical models, equations and similar other mathematical expressions. Assumptions are always incorporated in the derivation of an equation or model and such an equation or model may be correctly used for the solution of the business problems. This happens when the underlying assumptions and variables in the model are present in the concerning problem. If this caution is not given due care, then there always remains the possibility of wrong application of OR techniques. Quite often the operations researchers have been accused of having many solutions without being able to find problems that fit.
- **High costs are involved in the use of OR techniques:** OR techniques usually prove very expensive. Services of specialized persons are invariably called for (and along with it the use of computer) while using OR techniques. As such only big concerns can think of using such techniques. Even in big business organizations we can expect that OR techniques will continue to be of limited use simply because they are not in many cases worth their cost. As opposed to this, a typical manager, exercising intuition and judgement, may be able to make a decision very inexpensively. Thus, the use of OR is a costlier affair and this constitutes an important limitation of operations research.
- **OR does not take into consideration the intangible factors, that is, non-measurable human factors:** OR makes no allowance for intangible factors, such as skill, attitude, vigour of the management people in taking decisions, but in many instances success or failure hinges upon the consideration of such non-measurable intangible factors. There cannot be any magic formula for getting an answer to management problems; much depends upon proper managerial attitudes and policies.
- **OR is only a tool of analysis and not the complete decision-making process:** It should always be kept in mind that OR alone cannot make the final decision. It is just a tool and simply suggests best alternatives, but in the final analysis many business decisions will involve human element. Thus, OR is at best a supplement rather than a substitute for management; subjective judgement is likely to remain a principal approach to decision-making.
- **Other limitations:** Among other limitations of OR, the following deserve to be mentioned:
  (i) ***Bias:*** The operational researchers must be unbiased. An attempt to shoehorn results into a confirmation of management's prior preferences can greatly increase the likelihood of failure.
  (ii) ***Inadequate objective functions:*** The use of a single objective function is often an insufficient basis for decisions. Laws, regulations, public relations, market strategies, and so on, may all serve to overrule a choice arrived at in this way.

NOTES

(iii) ***Internal resistance:*** The implementation of an optimal decision may also confront internal obstacles, such as trade unions or individual managers with strong preferences for other ways of doing the job.

(iv) ***Competence:*** Competent OR analysis calls for the careful specification of alternatives, a full comprehension of the underlying mathematical relationships and a huge mass of data. Formulation of an industrial problem to an OR set programme is quite often a difficult task.

(v) ***Reliability of the prepared solution:*** At times, a non-linear relationship is changed to linear for fitting the problem to linear programming pattern. This may disturb the solution.

1.6.2 Techniques Tools of Operations Research

Mathematical models have been constructed for the categorized OR problems and methods for solving the models available in many cases. Such methods are usually termed as OR techniques. Some of the important OR techniques often used by decision-makers in modern times in business and industry are as follows:

- **Linear programming:** This technique is used in finding a solution for optimizing a given objective, such as profit maximization or cost minimization under certain constraints. This technique is primarily concerned with the optimal allocation of limited resources for optimizing a given function. The name linear programming is because of the fact that the model in such cases consists of linear equations indicating linear relationship between the different variables of the system. Linear programming technique solves product-mix and distribution problems of business and industry. It is a technique used to allocate scarce resources in an optimum manner in problems of scheduling, product-mix, and so on. Key factors under this technique include an objective function, choice among several alternatives, limits or constraints stated in symbols and variables assumed to be linear.
- **Waiting line or queuing theory:** This theory deals with mathematical study of queues. The queues are formed whenever the current demand for service exceeds the current capacity to provide that service. Waiting line technique concerns itself with the random arrival of customers at a service station where the facility is limited. Providing too much of capacity will mean idle time for servers and will lead to waste of money. On the other hand, if the queue becomes long, there will be a cost due to waiting of units in the queue. Waiting line theory, therefore, aims at minimizing the costs of both servicing and waiting. In other words, this technique is used to analyse the feasibility of adding facilities and to assess the amount and cost of waiting time. With its help, we can find the optimal capacity to be installed, which will lead to a sort of an economic balance between cost of service and cost of waiting.

- **Inventory control/planning:** It aims at optimizing inventory levels. Inventory may be defined as a useful idle resource which has economic value, for example, raw materials, spare parts, finished products, and so forth. Inventory planning, in fact, answers the two questions, namely: how much to buy and when to buy? Under this technique, the main emphasis is on minimizing costs associated with holding of inventories, procurement of inventories and the shortage of inventories.
- **Game theory:** This theory is used to determine the optimum strategy in a competitive situation. The simplest possible competitive situation is that of two persons playing zero-sum game, that is, a situation in which two persons are involved and one person wins exactly what the other loses. More complex competitive situations of real life can as well be imagined where game theory can be used to determine the optimum strategy.
- **Decision theory:** This theory concerns with making sound decisions under conditions of certainty, risk and uncertainty. As a matter of fact, there are three different kinds of under which decisions are made, namely deterministic, stochastic and uncertainty. The decision theory explains how to select a suitable strategy to achieve some object or goal under each of these three states.
- **Network analysis:** It involes the determination of an optimum sequence of performing certain operations concerning some jobs in order to minimize overall time and/or cost. Programme Evaluation and Review Technique (PERT), Critical Path Method (CPM) and other network techniques, such as Gantt Chart comes under Network Analysis. Key concepts under this technique are network of events and activities, resource allocation, time and cost considerations, and network paths and critical paths.
- **Simulation:** It is a technique of testing a model that resembles a real-life situation. This technique is used to imitate an operation prior to actual performance. Two methods of simulation are there—Monte Carlo method of simulation and System simulation method. The former one, using random numbers, is used to solve problems which involve conditions of uncertainty and the mathematical formulation is impossible. However, in case of System simulation, there is a reproduction of the operating environment and the system allows for analysing the response from the environment to alternative management actions. This method draws samples from a real population instead of drawing samples from a table of random numbers.
- **Integrated production models:** This technique aims at minimizing cost with respect to workforce, production and inventory. This technique is a highly complex one and is used only by big business and industrial units. This technique can be used only when sales and costs statistics for a considerable long period are available.

NOTES

- **Some other OR techniques:** In addition, there are several other techniques, such as non-linear programming, dynamic programming, search theory, the theory of replacement, and so on. A brief mention of some of these is as follows:
  - (i) ***Non-linear programming:*** It is that form of programming in which some or all of the variables are curvilinear. In other words, this means that either the objective function or constraints or both are not in linear form. In most of the practical situations, we encounter non-linear programming problems, but for computation purpose, we approximate them as linear programming problems. Even then, there may remain some non-linear programming problems which may not be fully solved by presently known methods.
  - (ii) ***Dynamic programming:*** It refers to a systematic search for optimal solutions to problems that involve many highly complex interrelations that are, moreover, sensitive to multistage effects such as successive time phases.
  - (iii) ***Heuristic programming:*** It is also known as discovery method and refers to step-by-step search towards an optimum when a problem cannot be expressed in mathematical programming form. The search procedure examines successively a series of combinations that lead to stepwise improvements in the solution and the search stops when a near optimum has been found.
  - (iv) ***Integer programming:*** It is a special form of linear programming in which the solution is required in terms of integral numbers (that is, whole numbers) only.
  - (v) ***Algorithmic programming:*** It is just the opposite of heuristic programming. It may also be termed as near mathematical programming. This programming refers to a thorough and exhaustive mathematical approach to investigate all aspects of the given variables in order to obtain optimal solution.
  - (vi) ***Quadratic programming:*** It refers to a modification of linear programming in which the objective equations appear in quadratic form, that is, they contain squared terms.
  - (vii) ***Parametric programming:*** It is the name given to linear programming when the latter is modified for the purpose of inclusion of several objective equations with varying degrees of priority. The sensitivity of the solution to these variations is then studied.
  - (viii) ***Probabilistic programming:*** It also known as stochastic programming and refers to linear programming that include an evaluation of relative risks and uncertainties in various alternatives of choice for management decisions.

NOTES

(ix) ***Search theory:*** It concerns itself with search problems. A search problem is characterized by the need for designing a procedure to collect information on the basis of which one or more decisions are made. This theory is useful in places in which some events are known to occur, but the exact location is not known. The first search model was developed during World War II to solve decision problems connected with air patrols and their search for submarines. Advertising agencies' search for customers, personnel departments' search for good executives, and so on, are some of the examples of search theory's application in business.

(x) ***The theory of replacement:*** It is concerned with the prediction of replacement costs and determination of the most economic replacement policy. There are two types of replacement models—one type of model deals in replacing equipment that deteriorate with time and the other type of model helps in establishing replacement policy for those equipment which fail completely and instantaneously.

All these techniques are not simple, but involve higher mathematics. The tendency today is to combine several of these techniques and put them together into more sophisticated and advanced programming models.

Check Your Progress

9. Mention the limitations of operations research.
10. How is operations research applied in the field of hospital management?

1.7 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The essential characteristics of operations research are as follows:
   - Systems orientation
   - Use of interdisciplinary terms
   - Application of scientific methods
   - Uncovering new problems
   - Quantitative solutions
   - Human factors
2. Operations research provides the executives with a more precise description of the cause-and-effect relationship and risks underling the business operations in measurable terms. This eliminates the conventional intuitive and subjective basis on which managements used to formulate their decisions decades ago.

NOTES

3. The first step in the scientific method of operations research is to formulate the problem in an appropriate form. Formulating a problem consists of identifying, defining and specifying the measures of the components of a decision model.
4. The solution values of a decision model (obtained) are tested against actual observations. In other words, effort is made to test the validity of the model used. A model is supposed to be valid if it can give a reliable prediction of the performance of the system represented through the model.
5. Stochastic situations are those in which each course of action can result in alternative outcomes; the probabilities of which are known or can be estimated.
6. Retentive objectives are concerned with preserving either resources of value, such as money, time, energy or states, such as comfort, safety, stability and others.
7. The properties of a good model are as follows:
    - A good model must represent the reality or the system with which it is concerned as accurately as possible and must not compromise with any of the critical elements of the system.
    - A good model must be simple enough so as to be readily understood and, if necessary, must be capable of being modified quickly and effectively.
    - A good model is one that is capable of answering in case when small perturbations take place in the data.
8. Quadratic programming refers to a modification of linear programming in which the objective equations appear in quadratic forms, that is, they contain squared terms.
9. The important limitations of operations research are as follows:
    - The inherent limitations concerning mathematical expressions.
    - High costs are involved in the use of operations research techniques.
    - Operations research does not taken into consideration the intangible factors.
    - Operations research is only a tool of analysis and not the complete decision-making process.
10. Operations research has been applied in the field of hospital management. For example, hospital management quite often faces the problem of allotting its limited resources, of its multiphased activities. Optimum allocation of resources, so as to ensure a certain desirable level of service to patients, can be reached through operations research.

1.8 SUMMARY

NOTES

- There have been various definitions for operations research, such as applied decision-making, quantitative common sense and making of economic decisions.
- An OR study generally involves three phases, namely the judgment phase, the research phase and the action phase. Of these three, the research phase is the longest and the largest, but the remaining two phases are very important since they provide, respectively, the basis for and implementation of the research phase.
- Mathematical models have been constructed for the categorized OR problems and methods for solving the models available in many cases. Such methods are usually termed as OR techniques.
- OR has gained increasing importance since World War II in the technology of business and industry administration.
- OR, though a great aid to management, cannot be a substitute for decision-making. The choice of a criterion as to what is actually best for a business enterprise is still that of an executive who has to fall back upon his experience and judgement.
- OR techniques, having their origin with war operations analysis during World War II, have since then assisted defence management to a larger extent.
- To formulate a problem correctly, one must know what a problem is and must understand the nature of the problem.
- Generally, the objectives of an organization may either be retentive or acquisitive. Retentive objectives are concerned with preserving either resources of value, such as money, time and energy, or states, such as comfort, safety, stability and others.
- Modelling plays an important part in operational research. In fact, it is considered as the essence of operations research approach.

1.9 KEY WORDS

- **Gantt chart:** It is a type of bar chart, adapted by Karol Adamiecki in 1896 and independently by Henry Gantt in the 1910s, which illustrates a project schedule.
- **Simulation:** It is a technique of testing a model that resembles a real-life situation.
- **Linear programing:** This technique is used in finding a solution for optimizing a given objective, such as profit maximization or cost minimization under certain constraints.

NOTES

- **Model:** It is a physical or symbolic representation of the relevant aspects of the reality or system with which one is concerned.
- **Programme evaluation and review technique (PERT):** It is a project management tool used to schedule, organize and coordinate tasks within a project.

1.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Write a short note on the important operations research techniques used in modern business and industrial units.
2. What is the role of operations research techniques in business and industry?
3. Write a brief note on the various phases in solving an operations research problem.
4. Write a short note on problem formulation in operational research.
5. What are the properties of a good model?
6. Outline the important types of operations research models.
7. What are the important aspects pertaining to modelling?

Long-Answer Questions

1. Define operations research. Explain some of its features.
2. Discuss the limitations of operations research techniques.
3. Is it possible to make all business decisions with the assistance of operations research? Give detailed reasons to support your answer.
4. Enumerate the advantages of modelling in OR.
5. 'Modelling enables quick and economical experimentation for finding an optimum solution to a given problem.' Do you agree? Give reasons for your answer.

1.11 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 2 LINEAR PROGRAMMING

NOTES

Structure

2.0 INTRODUCTION

In a mathematical model, linear programming is one method to achieve the best outcome whose requirements are represented by linear relationships. Basically, linear programming is a technique for the optimization of a linear objective function. In this unit, you will study about formulation and meaning of linear programming, the fields where linear programming can be used. You will also learn about the components and presentation of a Linear Programming Problem (LPP).

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the meaning and formulation of linear programming
- Discuss the components of linear programming problem
- Describe the general form of the linear programming model

2.2 FORMULATION OF LINEAR PROGRAMMING

Decision-making has always been very important in the business and industrial world, particularly with regard to the problems concerning production of commodities.

The main questions that come up before a production manager are as follows:

(i) Which commodity(s) should be produced?

(ii) In what quantities and by which process or processes should they be produced?

NOTES

Alfred Marshall, an influential British economist, in this connection points out that the businessman always studies his production function and input prices, and substitutes one input for another till his costs become the minimum possible. This sort of substitution, in the opinion of Marshall, is done by the businessman's trained instinct rather than with formal calculations. However, now there does exist a method of formal calculations often termed as Linear Programming. This method was first formulated by a Russian mathematician L.V. Kantorovich, but it was developed later in 1947 by an American mathematical scientist George B. Dantzig 'for the purpose of scheduling the complicated procurement activities of the United States Air Force'. Today, this method is being used in solving a wide range of practical business problems. The advent of electronic computers has further increased its applications to solve many other problems in industry. It is being considered as one of the most versatile management techniques.

2.2.1 Meaning of Linear Programming

Linear Programming (LP) is a major innovation since World War II in the field of business decision-making, particularly under conditions of certainty. The word 'Linear' means that the relationships are represented by straight lines, that is, the relationships are of the form $y = a + bx$, and the word 'Programming' means taking decisions systematically. Thus, LP is a decision-making technique under given constraints on the assumption that the relationships amongst the variables representing different phenomena happen to be linear. In fact, Dantzig originally called it 'programming of interdependent activities in a linear structure', but later on shortened it to 'Linear Programming'.

LP is generally used in solving maximization (sales or profit maximization) or minimization (cost minimization) problems subject to certain assumptions. Putting in a formal way, 'Linear Programming is the maximization (or minimization) of a linear function of variables subject to a constraint of linear inequalities.' Hence, LP is a mathematical technique designed to assist the organization in optimally allocating its available resources under conditions of certainty in problems of scheduling, product mix, and so on.

2.2.2 Fields where Linear Programming can be Used

The problem for which LP provides a solution may be stated as, maximize or minimize for some dependent variable which is a function of several independent variables when the independent variables are subject to various restrictions. The dependent variable is usually some economic objective such as profits, production, costs, workweeks, tonnage to be shipped, and so on. More profits are generally preferred to less profits and lower costs are preferred to higher costs. Hence, it is appropriate to represent either maximization or minimization of the dependent variable as one of the firm's objective. LP is usually concerned with such objectives under given constraints with linearity assumptions. In fact, it is powerful to take in its stride a wide range of business applications. The applications of LP are numerous

NOTES

and are increasing everyday. LP is extensively used in solving resource allocation problems. Production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning, and so on, are some of its most fertile application areas. More specifically, LP has been successfully applied in the following fields:

- **Agriculture:** LP can be applied in farm management problems as it relates to the allocation of resources, such as acreage, labour, water supply or working capital in such a way that it maximizes net revenue.
- **Contract awards:** Evaluation of tenders by recourse to LP guarantees that the awards are made in the cheapest way.
- **Industries:** Applications of LP in business and industry are of the most diverse type. Transportation problems concerning cost minimization can be solved by this technique. The technique can also be adopted in solving problems of production (product mix) and inventory control.

Thus, LP is the most widely used technique of decision-making in business and industry in modern times in various fields.

2.2.3 Components of a Linear Programming Problem

There are certain basic concepts and notations to be first understood for easy adoption of the LP technique. A brief mention of such concepts is as follows:

- **Linearity:** The term 'linearity' implies straight line or proportional relationships among the relevant variables. Linearity in economic theory is known as constant returns, which means that if the amount of the input doubles, the corresponding outputs and profits are also doubled. Linearity assumption, thus, implies that two machines and two workers can produce twice as much as one machine and one worker; four machines and four workers twice as much as two machines and two workers, and so on.
- **Process and its level:** Process means the combination of particular inputs to produce a particular output. In a process, factors of production are used in fixed ratios, of course, depending upon technology and as such no substitution is possible with a process. There may be many processes open to a firm for producing a commodity and one process can be substituted for another. There is, thus, no interference of one process with another when two or more processes are used simultaneously. If a product can be produced in two different ways, then there are two different processes (or activities or decision variables) for the purpose of a linear programme.
- **Criterion function:** This is also known as the objective function which states whether the determinants of the quantity should be maximized or minimized. For example, revenue or profit is such a function when it is to be maximized or cost is such a function when the problem is to minimize it. An objective function should include all the possible activities with the revenue (profit) or cost coefficients per unit of production or acquisition. The goal

NOTES

may be either to maximize this function or to minimize this function. In symbolic form, let ZX denote the value of the objective function at the X level of the activities included in it. This is the total sum of individual activities produced at a specified level. The activities are denoted *as* $j = 1, 2, ..., n$. The revenue or cost coefficient of the jth activity is represented by C_j. Thus, $2X_1$, implies that X units of activity $j = 1$ yields a profit (or loss) of $C_1 = 2$.

- **Constraints or inequalities:** These are the limitations under which one has to plan and decide, that is, restrictions imposed upon decision variables. For example, a certain machine requires one worker to operate; another machine requires at least four workers (that is, > 4); there are at most 20 machine hours (that is, < 20) available; the weight of the product should be, say 10 lbs., and so on, are all examples of constraints or what are known as inequalities. Inequalities like $X > C$ (reads X is greater than C) or $X < C$ (reads X is less than C) are termed as strict inequalities. The constraints may be in the form of weak inequalities like $X \leq C$ (reads X is less than or equal to C) or $X \geq C$ (reads C is greater than or equal to C). Constraints may be in the form of strict equalities like $X = C$ (reads X is equal to C).

  Let b_i denote the quantity b of resource i available for use in various production processes. The coefficient attached to resource i is the quantity of resource i required for the production of one unit of product j.
- **Feasible solutions:** These are all those possible solutions which can be worked upon under given constraints. The region comprising all feasible solutions is referred to as Feasible Region.
- **Optimum solution:** Optimum solution is the best of all the feasible solutions.

General form of the linear programming model

Linear programming problem mathematically can be stated as follows:

Choose the quantities,

$$X_j \geq 0 \quad (j = 1, ..., n) \qquad ...(2.1)$$

This is also known as the non-negativity condition and in simple terms means that no X can be negative.

To maximize,

$$Z = \sum_{j=1}^{n} C_j X_j \qquad ...(2.2)$$

Subject to the constraints,

$$\sum_{j=1}^{n} a_{ij} X_j \leq b_i \qquad (i = 1,...,m) \qquad ...(2.3)$$

NOTES

This is the usual structure of a linear programming model in the simplest possible form. This model can be interpreted as a profit maximization situation, where n production activities are pursued at level X_j which have to be decided upon, subject to a limited amount of m resources being available. Each unit of the jth activity yields a return C and uses an amount a_{ij} of the ith resource. Z denotes the optimal value of the objective function for a given system.

Assumptions or conditions to be fulfilled

LP model is based on the assumptions of proportionality, certainty, additivity, continuity and finite choices.

Proportionality is assumed in the objective function and the constraint inequalities. In economic terminology, this means that there are constant returns to scale, that is, if one unit of a product contributes ₹ 5 toward profit, then 2 units will contribute ₹ 10, 4 units ₹ 20, and so on.

Certainty assumption means the prior knowledge of all the coefficients in the objective function, the coefficients of the constraints and the resource values. LP model operates only under conditions of certainty.

Additivity assumption means that the total of all the activities is given by the sum of each activity conducted separately. For example, the total profit in the objective function is equal to the sum of the profit contributed by each of the products separately.

Continuity assumption means that the decision variables are contiunous. Accordingly, the combinations of output with fractional values, in case of product mix problems, are possible and obtained frequently.

Finite choices assumption implies that finite number of choices are available to a decision-maker and the decision variables do not assume negative values.

2.2.4 Mathematical Formulation of a Linear Programming Problem

The procedure for mathematical formulation of an LPP consists of the following steps:

Step 1: The decision variables of the problem are noted.

Step 2: The objective function to be optimized (maximized or minimized) as a linear function of the decision variables is formulated.

Step 3: The other conditions of the problem, such as resource limitation, market constraints, interrelations between variables, and so on, are formulated as linear inequations or equations in terms of the decision variables.

Step 4: The non-negativity constraint from the considerations is added so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative constraint together form a linear programming problem.

General formulation of a linear programming problem

NOTES

The general formulation of the LPP can be stated as follows:

In order to find the values of n decision variables $x_1 x_2 ... x_n$ to maximize or minimize the objective function,

$$Z = C_1x_1 + C_2x_2 + \ldots + C_nx_n \qquad \text{... (2.4)}$$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\,(\leq, =, \geq)b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n\,(\leq, =, \geq)b_2 \\ : \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\,(\leq, =, \geq)b_i \\ : \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\,(\leq=\geq)b_m \end{array}\right\} \qquad \text{... (2.5)}$$

Here, the constraints can be inequality $\leq$ or $\geq$ or even in the form of an equation (=) and finally satisfy the non-negative restrictions:

$$x_1 \geq 0,\, x_2 \geq 0 \ldots x_n \geq 0 \qquad \text{... (2.6)}$$

Matrix form of a linear programming problem

The LPP can be expressed in the matrix form as follows:

Maximize or minimize $Z = Cx \rightarrow$ Objective function

Subject to $Ax\ (\leq, =, \geq)\ b \rightarrow$ Constant equation

$b > 0, x \geq 0 \rightarrow$ Non-negativity restrictions

Where, $x = (x_1, x_2 \cdots x_n)$

$C = (C_1, C_2 \cdots C_n)$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_m \end{pmatrix} \quad A = \begin{pmatrix} a_{11}a_{12}\cdots a_{1n} \\ a_{21}a_{22}\cdots a_{2n} \\ : \\ a_{m1}a_{m2}\cdots a_{mn} \end{pmatrix}$$

2.2.5 Case Studies

Example 2.1: *A* manufacturer produces two types of models, M_1 and M_2. Each model of the type M_1 requires 4 hours of grinding and 2 hours of polishing, whereas each model of the type M_2 requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. The profit on M_1 model is ₹ 3.00 and on model M_2 is ₹ 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?

Solution:

Decision variables: Let X_1 and X_2 be the number of units of M_1 and M_2.

Objective function: Since the profit on both the models are given, we have to maximize the profit, namely:

Max $Z = 3X_1 + 4X_2$

Constraints: There are two constraints: one for grinding and the other for polishing.

The number of hours available on each grinder for one week is 40 hours. There are 2 grinders. Hence, the manufacturer does not have more than $2 \times 40 = 80$ hours for grinding. M_1 requires 4 hours of grinding and M_2 requires 2 hours of grinding.

The grinding constraint is given by,

$4X_1 + 2X_2 \leq 80$

Since there are 3 polishers, the available time for polishing in a week is given by $3 \times 60 = 180$. M_1 requires 2 hours of polishing and M_2 requires 5 hours of polishing. Hence, we have $2X_1 + 5X_2 \leq 180$

Thus, we have,

Max $Z = 3X_1 + 4X_2$

Subject to $4X_1 + 2X_2 \leq 80$

$2X_1 + 5X_2 \leq 180$

$X_1, X_2 \geq 0$

Example 2.2: A firm produces two different types of products, Product M and Product N. The firm uses the same machinery for manufacturing both the products. One unit of Product M requires 10 minutes while one unit of Product N requires 2 minutes. The maximum hours the machine can function optimally for a week is 35 hours. The raw material requirement for Product M is 1 kg, while that of Product N is 0.5 kg. Also, the market constraint on product M is 600 kg, while that of Product N is 800 units per week. The cost of manufacturing Product M is ₹ 5 per unit and it is sold at ₹ 10, while the cost of Product N is ₹ 6 per unit and sold at ₹ 8 per unit. Calculate the total number of units of Product M and Product N that should be produced per week, so as to derive maximum profit.

NOTES

Solution:

Decision variables: Let X_1 and X_2 be the number of products of A and B.

Objective function: Cost of product A per unit is ₹ 5 and is sold at ₹ 10 per unit.

Profit on one unit of product $A = 10 - 5 = 5$.

X_1 units of product A contribute a profit of ₹ $5X_1$ from one unit of product.

Similarly, profit on one unit of $B = 8 - 6 = 2$.

X_2 units of product B contribute a profit of ₹ $2X_2$.

The objective function is given by,

Max $Z = 5X_1 + 2X_2$

Constraints: Time requirement constraint is given by,

$$10X_1 + 2X_2 \leq (35 \times 60)$$
$$10X_1 + 2X_2 \leq 2100$$

Raw material constraint is given by,

$$X_1 + 0.5X_2 \leq 600$$

Market demand on product B is 800 units every week.

$X_2 \geq 800$

The complete LPP is,

Max $Z = 5X_1 + 2X_2$

Subject to, $10X_1 + 2X_2 \leq 2100$

$X_1 + 0.5X_2 \leq 600$

$X_2 \geq 800$

$X_1, X_2 \geq 0$

Example 2.3: A person requires 10, 12 and 12 units of chemicals *A, B* and *C*, respectively, for his garden. A liquid product contains 5, 2 and 1 units of *A, B* and *C* respectively per jar. A dry product contains 1, 2 and 4 units of *A, B, C* per carton. If the liquid product sells for ₹ 3 per jar and the dry product sells for ₹ 2 per carton, what should be the number of jars that needs to be purchased, in order to bring down the cost and meet the requirements?

Solution:

Decision variables: Let X_1 and X_2 be the number of units of liquid and dry products.

Objective function: Since the cost for the products is given, we have to minimize the cost.

Min $Z = 3X_1 + 2X_2$

Constraints: As there are three chemicals and their requirements are given, we have three constraints for these three chemicals.

NOTES

$$5X_1 + X_2 \geq 10$$
$$2X_1 + 2X_2 \geq 12$$
$$X_1 + 4X_2 \geq 12$$

Hence, the complete LPP is,

Min $Z = 3X_1 + 2X_2$

Subject to,

$$5X_1 + X_2 \geq 10$$
$$2X_1 + 2X_2 \geq 12$$
$$X_1 + 4X_2 \geq 12$$
$$X_1, X_2 \geq 0$$

Example 2.4: A paper mill produces two grades of paper, *X* and *Y*. Because of raw material restrictions, it cannot produce more than 400 tonnes of grade *X* and 300 tonnes of grade *Y* in a week. There are 160 production hours in a week. It requires 0.2 and 0.4 hours to produce a tonne of products *X* and *Y* respectively with corresponding profits of ₹ 200 and ₹ 500 per tonne. Formulate this as an LPP to maximize profit and find the optimum product mix.

Solution:

Decision variables: Let X_1 and X_2 be the number of units of the two grades of paper, *X* and *Y*.

Objective function: Since the profit for the two grades of paper *X* and *Y* are given, the objective function is to maximize the profit.

Max $Z = 200X_1 + 500X_2$

Constraints: There are two constraints, one with reference to raw material, and the other with reference to production hours.

Max $Z = 200X_1 + 500X_2$

Subject to,

$$X_1 \leq 400$$
$$X_2 \leq 300$$
$$0.2X_1 + 0.4X_2 \leq 160$$

Non-negative restriction $X_1, X_2 \geq 0$

NOTES

Example 2.5: *A* company manufactures two products, *A* and *B*. Each unit of *B* takes twice as long to produce as one unit of *A* and if the company were to produce only *A*, it would have time to produce 2000 units per day. The availability of the raw material is enough to produce 1500 units per day of both *A* and *B* together. Product *B* requiring a special ingredient, only 600 units of it can be made per day. If *A* fetches a profit of ₹ 2 per unit and *B* a profit of ₹ 4 per unit, find the optimum product mix by the graphical method.

Solution: Let X_1 and X_2 be the number of units of products *A* and *B*, respectively.

The profit after selling these two products is given by the objective function,

Max $Z = 2X_1 + 4X_2$

Since the company can produce at the most 2000 units of the product in a day and Product *B* requires twice as much time as that of Product *A*, production restriction is given by,

$$X_1 + 2X_2 \leq 2000$$

Since the raw material is sufficient to produce 1500 units per day of both *A* and *B*, we have $X_1 + X_2 \leq 1500$.

There are special ingredients for Product *B*; hence, we have $X_2 \leq 600$.

Also, since the company cannot produce negative quantities $X_1 \geq 0$ and $X_2 \geq 0$.

Hence, the problem can be finally put in the form:

Find X_1 and X_2 such that the profits, $Z = 2X_1 + 4X_2$ is maximum.

Subject to,
$$X_1 + 2X_2 \leq 2000$$
$$X_1 + X_2 \leq 1500$$
$$X_2 \leq 600$$
$$X_1, X_2 \geq 0$$

Example 2.6: *A* firm manufacturers 3 products *A*, *B* and *C*. The profits are ₹ 3, ₹ 2 and ₹ 4, respectively. The firm has 2 machines and the following is the required processing time in minutes for each machine on each product.

		Product		
		A	*B*	*C*
Machines	*C*	4	3	5
	D	3	2	4

Machines *C* and *D* have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 units of *A*, 200 units of *B* and 50 units of *C*, but not more than 150 units of *A*. Formulate an LP problem to maximize the profit.

NOTES

Solution: Let X_1, X_2, X_3 be the number of units of the product *A, B, C,* respectively.

Since the profits are ₹ 3, ₹ 2 and ₹ 4, respectively, the total profit gained by the firm after selling these three products is given by,

$$Z = 3X_1 + 2X_2 + 4X_3$$

The total number of minutes required in producing these three products at Machine *C* is given by $4X_1 + 3X_2 + 5X_3$ and at Machine *D* is given by $3X_1 + 2X_2 + 4X_3$.

The restrictions on machines *C* and *D* are given by 2000 minutes and 2500 minutes.

$$4X_1 + 3X_2 + 5X_3 \le 2000$$
$$3X_1 + 2X_2 + 4X_3 \le 2500$$

Also, since the firm manufactures 100 units of *A,* 200 units of *B* and 50 units of *C*, but not more than 150 units of *A*, the further restriction becomes,

$$100 \le X_1 \le 150$$
$$200 \le X_2 \ge 0$$
$$50 \le X_3 \ge 0$$

Hence, the allocation problem of the firm can be finally put in the form:

Find the value of X_1, X_2, X_3 so as to maximize,

$Z = 3X_1 + 2X_2 + 4X_3$

Subject to the constraints,

$$4X_1 + 3X_2 + 5X_3 \le 2000$$
$$3X_1 + 2X_2 + 4X_3 \le 2500$$

$$100 \le X_1 \le 150,\ 200 \le X_2 \ge 0, 50 \le X_3 \ge 0$$

Example 2.7: A peasant has 100 acres of farm. He can sell all potatoes, cabbage or brinjals and can increase the cost to get ₹ 1.00 per kg for potatoes, ₹ 0.75 head for cabbage and ₹ 2.00 per kg for brinjals. The average yield per acre is 2000 kg of potatoes, 3000 heads of cabbage and 1000 kg of brinjals. Fertilizers can be bought at ₹ 0.50 per kg and the amount needed per acre is 100 kg each for potatoes and cabbage and 50 kg for brinjals. The manpower required for sowing, cultivating and harvesting per acre is 5 man-days for potatoes and brinjals and 6 man-days for cabbage. A total of 400 man-days of labour is available at ₹ 20 per man-day. Solve this example as a linear programming model to increase the peasant's profit.

Solution: Let X_1, X_2, X_3 be the area of his farm to grow potatoes, cabbage and brinjals respectively. The peasant produces $2000X_1$ kg of potatoes, $3000X_2$ heads of cabbage and $1000X_3$ kg of brinjals.

NOTES

The total sales of the peasant will be,

$= ₹\ (2000X_1 + 0.75 \times 3000X_2 + 2 \times 1000X_3)$

Fertilizer expenditure will be,

$= ₹\ 20\ (5X_1 + 6X_2 + 5X_3)$

Peasant's profit will be,

Z = Sale (in ₹) – Total expenditure (in ₹)

$= (2000X_1 + 0.75 \times 3000X_2 + 2 \times 1000X_3) - 0.5 \times [100(X_1 + X_2) + 50X_2]$

$-20 \times (5X_1 + 6X_2 + 5X_3)$

$Z = 1850X_1 + 2080X_2 + 1875X_3$

Since the total area of the farm is restricted to 100 acres,

$X_1 + X_2 + X_3 \leq 100$

Also, the total man-days manpower is restricted to 400 man-days.

$5X_1 + 6X_2 + 5X_3 \leq 400$

Hence, the peasant's allocation problem can be finally put in the form:

Find the value of X_1, X_2 and X_3 so as to maximize,

$Z = 1850X_1 + 2080X_2 + 1875X_3$

Subject to,

$$X_1 + X_2 + X_3 \leq 100$$
$$5X_1 + 6X_2 + 5X_3 \leq 400$$
$$X_1, X_2, X_3 \geq 0$$

Example 2.8: ABC Company produces two products: juicers and washing machines. Production happens in two different departments, I and II. Juicers are made in Department I and washing machines in Department II. These two items are sold weekly. The weekly production should not cross 25 juicers and 35 washing machines. The organization always employs a total of 60 employees in the two departments. A juicer requires two man-weeks' labour, while a washing machine needs one man-week's labour. A juicer makes a profit of ₹ 60 and a washing machine contributes a profit of ₹ 40. How many units of juicers and washing machines should the organization make to achieve the maximum profit? Formulate this as an LPP.

Solution: Let X_1 and X_2 be the number of units of juicers and washing machines to be produced.

Each juicer and washing machine contributes a profit of ₹ 60 and ₹ 40. Hence, the objective function is to maximize $Z = 60X_1 + 40X_2$.

There are two constraints which are imposed: weekly production and labour.

Since the weekly production cannot exceed 25 juicers and 35 washing machines,

$X_1 \leq 25$

$X_2 \leq 35$

A juicer needs two man-weeks of hard work and a washing machine needs one man-week of hard work and the total number of workers is 60.

$2X_1 + X_2 \leq 60$

Non-negativity restrictions: Since the number of juicers and washing machines produced cannot be negative, we have $X_1 \geq 0$ and $X_2 \geq 0$.

Hence, the production of juicers and washing machines problem can be finally put in the form of an LP model given as follows:

Find the value of X_1 and X_2 so as to maximize,

$Z = 60X_1 + 40X_2$

Subject to,

$$\begin{aligned} X_1 &\leq 25 \\ X_2 &\leq 35 \\ 2X_1 + X_2 &\leq 60 \\ \text{and } X_1, X_2 &\geq 0 \end{aligned}$$

It is take noted that graphical methods to solve linear programming problem have been discussed in detail in Unit 3.

2.2.6 LPP: Advantages/Applications and Limitations

Advantages/Applications

Linear programming helps to reduce the cost of production for maximum output. In brief, linear programming models assist a decision-maker can most efficiently and effectively employ his production factor and limited resources to get maximum profit at minimum cost.

Limitations

(i) Linear programming can be applied only when the objective function and all the restraints can be expressed in terms of linear equations.

(ii) Linear programming techniques provide solutions only when all the elements related to a problem can be quantified.

(iii) Linear programming technique may give fractional valued answer which is not desirable in some problems.

NOTES

Check Your Progress

1. When and by whom linear programming method was formulated and developed?
2. List the various fields where Linear Programming has been frequently applied.
3. What do you mean by the term 'linearity'?
4. What does the assumption 'finite choices' in an LP problem imply?
5. Mention the assumptions on which LP model is based.

2.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The linear programming method was first formulated by a Russian mathematician L.V. Kantorovich, but it was later developed by an American mathematical scientist George B. Dantzig in 1947 'for the purpose of scheduling the complicated procurement activities of the United States Air Force'.
2. Linear programming has been successfully applied in the following fields:
   - **Agriculture:** LP can be applied in farm management problems as it relates to the allocation of resources, such as acreage, labour, water supply or working capital in such a way that it maximizes net revenue.
   - **Contract awards:** Evaluation of tenders by recourse to LP guarantees that the awards are made in the cheapest way.
   - **Industries:** Applications of LP in business and industry are of the most diverse type. Transportation problems concerning cost minimization can be solved by this technique. The technique can also be adopted in solving problems of production (product mix) and inventory control.
3. The term 'linearity' implies straight line or proportional relationships among the relevant variables. In economic theory, linearity is known as constant returns, which means that if the amount of the input doubles, the corresponding outputs and profits are also doubled.
4. Finite choices assumption implies that finite number of choices are available to a decision-maker and the decision variables do not assume negative values.
5. The LP model is based on a few assumptions or conditions such as proportionality, certainty, additivity, continuity, and finite choices.

2.4 SUMMARY

NOTES

- Linear programming (LP) is a decision-making technique under given constraints on the assumption that the relationships among the variables representing different phenomena happen to be linear.
- LP is the most widely used technique of decision-making in business and industry. Thus, it finds application in production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning, and many other fields.
- LP is extensively used in solving resource allocation problems. Production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning, and so on, are some of its most fertile application areas.
- The term 'linearity' implies straight line or proportional relationships among the relevant variables. Linearity in economic theory is known as constant returns, which means that if the amount of the input doubles, the corresponding outputs and profits are also doubled.
- Criterion function, which is also known as objective function, is a component of LP problems. It states whether the determinants of quantity should be maximized or minimized.
- Constraints are the limitations under which one has to plan and decide, that is, restrictions imposed upon decision variables.
- Feasible solutions are all those possible solutions which can be worked upon under given constraints.
- LP model is based on the assumptions of proportionality, certainty, additivity, continuity and finite choices.

2.5 KEY WORDS

- **Linear Programming:** It refers to a mathematical technique that is used to determine the best possible outcome or solution from a given set of parameters or list of requirements, which are represented in the form of linear relationships.
- **Criterion function:** It refers to an objective function which states the determinants of the quantity to be either maximized or minimized.
- **Linearity:** It refers to the property of a mathematical relationship or function which means that it can be graphically represented as a straight line.

2.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short Answer Questions

1. What do you mean by linear programming?
2. What is meant by proportionality in linear programming?
3. Write a brief note on certainty in linear programming.
4. What are the basic constituents of an LP model?

Long Answer Questions

1. Analyse the fields where linear programming can be used.
2. Discuss the components of a linear programming problem.
3. Describe the steps used in formulation of a Linear Programming problem (LPP).

2.7 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 3 GRAPHICAL ANALYSIS OF LPP

NOTES

Structure

3.0 INTRODUCTION

The graphical method for linear programming is used to solve problems by finding the highest or lowest point of intersection between the objective function line and the feasible region on a graph. It is regarded as the best alternative for the representation and solving linear programming models that have two decision variables.

In this unit, you will study about the graphical analysis of LPP and will also learn the procedure to solve LPP by graphical method. In addition to this, you will also analyse the general formulation of LPP evaluate the matrix form of LPP.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of graphical analysis of LPP
- Examine the procedure for solving LPP by graphical method
- State the general formulation of LPP
- Assess the matrix form of LPP

NOTES

3.2 PROCEDURE FOR SOLVING LPP BY GRAPHICAL METHOD: SOME EXCEPTIONAL CASES

The steps involved in graphical method are as follows:

Step 1 Consider each inequality constraint as an equation.

Step 2 Plot each equation on the graph, as each will geometrically represent a straight line.

Step 3 Mark the region. If the inequality constraint corresponding to that line is ≤, then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint ≥ sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the '*feasible region*'.

Step 4 Assign an arbitrary value, say zero, to the objective function.

Step 5 Draw the straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

Step 6 Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin, passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin, passing through at least one corner of the feasible region.

Step 7 Find the co-ordinates of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Note: As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one that gives the optimal solution. That is, in the case of maximization problem, the optimal point corresponds to the corner point at which the objective function has a maximum value, and in the case of minimization, the optional solution is the corner point which gives the objective function the minimum value for the objective function.

Example 3.1: Solve the following LPP by graphical method.

Minimize $Z = 20X_1 + 10X_2$

Subject to,

$$X_1 + 2X_2 \leq 40$$
$$3X_1 + X_2 \geq 30$$
$$4X_1 + 3X_2 \geq 60$$
$$X_1, X_2 \geq 0.$$

Solution: Replace all the inequalities of the constraints by equation

$$X_1 + 2X_2 = 40 \qquad \text{If } X_1 = 0 \Rightarrow X_2 = 20$$

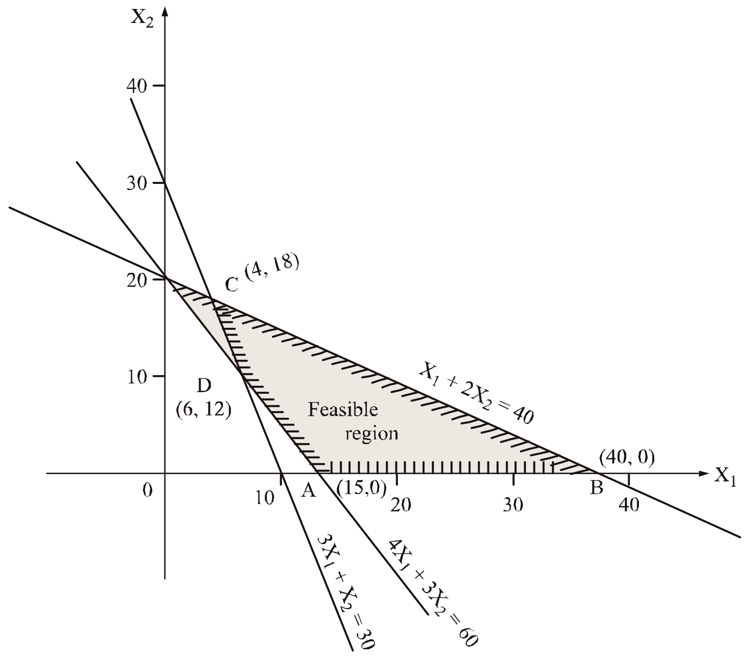
$$\text{If } X_2 = 0 \Rightarrow X_1 = 40$$

$\therefore$ $X_1 + 2X_2 = 40$ passes through (0, 20) (40, 0)

$3X_1 + X_2 = 30$ passes through (0, 30) (10, 0)

$4X_1 + 3X_2 = 60$ passes through (0, 20) (15, 0)

Plot each equation on the graph.



The feasible region is *ABCD*.

C and *D* are points of intersection of lines.

C intersect $X_1 + 2X_2 = 40, \quad 3X_1 + X_2 = 30$

and, *D* intersect $4X_1 + 3X_2 = 60, \quad X_1 + X_2 = 30$

$$C = (4, 18)$$

$$D = (6, 12)$$

Corner points	*Value of* $Z = 20X_1 + 10X_2$
A (15, 0)	300
B (40, 0)	800
C (4, 18)	260
D (6, 12)	240 (Minimum value)

$\therefore$ The minimum value of *Z* occurs at *D* (6, 12). Hence, the optimal solution is $X_1 = 6, X_2 = 12$.

NOTES

Example 3.2: Find the maximum value of $Z = 5X_1 + 7X_2$

Subject to the constraints,

$$x_1 + x_2 \le 4$$
$$3x_1 + 8x_2 \le 24$$
$$10x_1 + 7x_2 \le 35$$
$$x_1, x_2 \ge 0$$

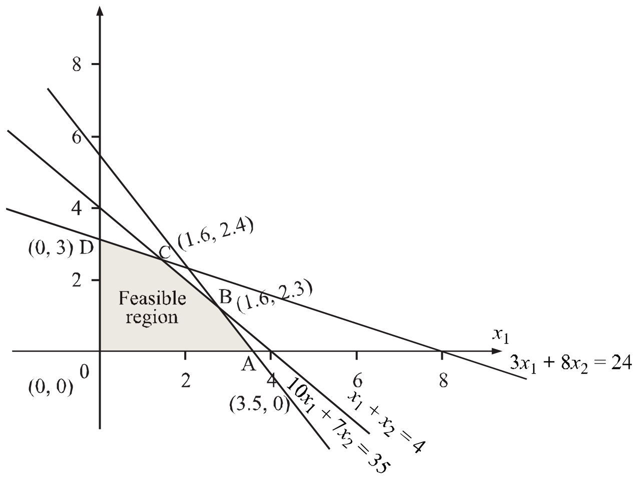
Solution: Replace all the inequalities of the constraints by forming equations

$x_1 + x_2 = 4$ passes through (0, 4) (4, 0)

$3x_1 + 8x_2 = 24$ passes through (0, 3) (8, 0)

$10x_1 + 7x_2 = 35$ passes through (0, 5) (3.5, 0)

Plot these lines on the graph and mark the region below the line as the inequality of the constraint as ≤ which is also lying in the first quadrant.



The feasible region is *OABCD*.

B and *C* are points of intersection of lines

B intersect $x_1 + x_2 = 4$, $10x_1 + 7x_2 = 35$

and *C* intersect $3x_1 + 8x_2 = 24$, $x_1 + x_2 = 4$.

On solving we get, $B = (1.6, 2.3)$

$C = (1.6, 2.4)$

Corner points	*Value of* $Z = 5x_1 + 7x_2$
O (0, 0)	0
A (3.5, 0)	17.5
B (1.6, 2.3)	25.1
C (1.6, 2.4)	24.8 (Maximum value)
D (0, 3)	21

$\therefore$ The maximum value of Z occurs at C (1.6, 2.4) and the optimal solution is x_1 = 1.6, x_2 = 2.4.

Example 3.3: A company produces 2 types of hats. Every hat A requires twice as much labour time as the second hat B. If the company produces only hat B then it can produce a total of 500 hats a day. The market limits daily sales of hat A and B to 150 and 250 respectively. The profits on hat A and B are Rs. 8 and Rs. 5 respectively. Solve graphically to get the optimal solution.

Solution: Let X_1 and X_2 be the number of units of type A and type B hats respectively.

Max $Z = 8X_1 + 5X_2$

Subject to,
$$2X_1 + 2X_2 \leq 500$$
$$X_1 \leq 150$$
$$X_2 \leq 250$$
$$X_1, X_2 \geq 0.$$

First rewrite the inequality of the constraint into an equation and plot the lines on the graph.

$2X_1 + X_2 = 500$ passes through (0, 500) (250, 0)

$X_1 = 150$ passes through (150, 0)

$X_2 = 250$ passes through (0, 250)

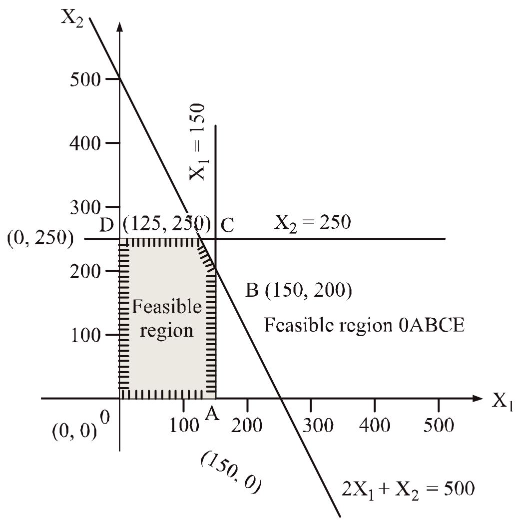
We mark the region below the lines lying in the first quadrant as the inequality of the constraints are ≤. The feasible region is $OABCD$. B and C are points of intersection of lines

$$2X_1 + X_2 = 500,$$

$X_1 = 150$ (B intersect) and $X_2 = 250$ (C intersect)

On solving, we get $B = (150, 200)$

$C = (125, 250)$



NOTES

Corner points	*Value of* $Z = 8X_1 + 5X_2$
O (0, 0)	0
A (150, 0)	1, 200
B (150, 200)	2, 200
C (125, 250)	2, 250 (Maximum Z = 2,250)
D (0, 250)	1, 250

The maximum value of Z is attained at C (125, 250)

$\therefore$ The optimal solution is $X_1 = 125, X_2 = 250$.

i.e., The company should produce 125 hats of type A and 250 hats of type B in order to get the maximum profit of Rs. 2, 250.

Example 3.4: By graphical method solve the following LPP.

Max $Z = 3X_1 + 4X_2$

Subject to,
$$5X_1 + 4X_2 \leq 200$$
$$3X_1 + 5X_2 \leq 150$$
$$5X_1 + 4X_2 \geq 100$$
$$8X_1 + 4X_2 \geq 80$$

and $$X_1, X_2 \geq 0.$$

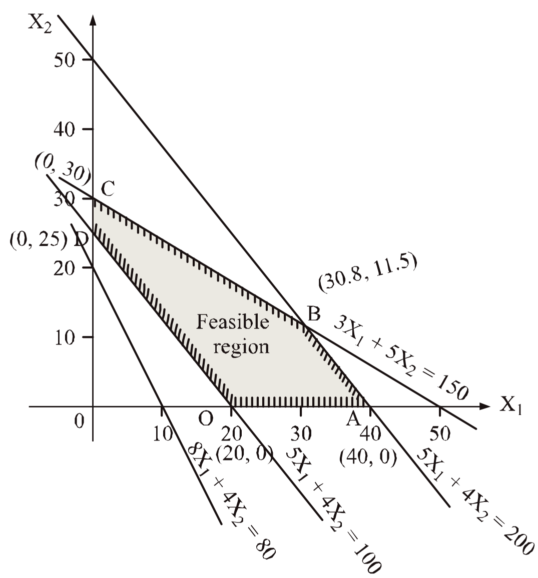
Solution:

Replacing the inequality by equality

$5x_1 + 4x_2 = 200$ passes through (0, 50), (40, 0)

$3x_1 + 4x_2 = 100$ passes through (0, 30), (50, 0)

$8x_1 + 4x_2 = 80$ passes through (0, 20), (10, 0)



NOTES

Feasible region is given by *OABCD*.

Corner points	*Value of* $Z = 3X_1 + 4X_2$
O (20, 0)	60
A (40, 0)	120
B (30.8, 11.5)	138.4 (Maximum value)
C (0, 30)	120
D (0, 25)	100

$\therefore$ The maximum value of Z is attained at B (30.8, 11.5)

$\therefore$ The optimal solution is $X_1 = 30.8, X_2 = 11.5$.

Example 3.5: Use graphical method to solve the LPP.

Maximize $Z = 6X_1 + 4X_2$

Subject to,

$$-2X_1 + X_2 \leq 2$$
$$X_1 - X_2 \leq 2$$
$$3X_1 + 2X_2 \leq 9$$
$$X_1, X_2 \geq 0.$$

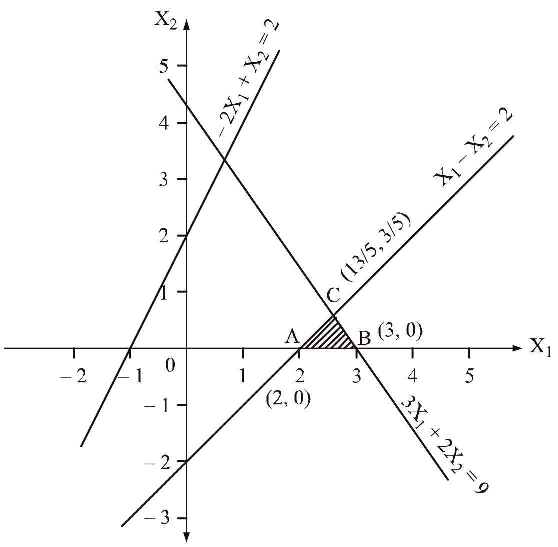
Solution:

Replacing the inequality by equality

$-2x_1 + x_2 = 2$ passes through (0, 2), (–1, 0)

$x_1 - x_2 = 2$ passes through (0, –2), (2, 0)

$3x_1 + 2x_2 = 9$ passes through (0, 4.5), (3, 0)



NOTES

Feasible region is given by *ABC*.

Corner points	*Value of* $Z = 6X_1 + 4X_2$
0 (0, 0)	0
A (2, 0)	12
B (13/5, 3/5)	$\frac{78+12}{5} = \frac{90}{5} = 18$ (Maximum value)

$C\left(\frac{5}{7}, \frac{24}{7}\right) = \frac{126}{7} = 18$ (Maximum value)

D (0, 2) = 8

The maximum value of *Z* is attained at *C* (13/5, 3/5) or at *D* (5/7, 24/7)

$\therefore$ The optimal solution is $X_1 = 13/5, X_2 = 3/5$, or $x_1 = 5/7, x_2 = 24/7$.

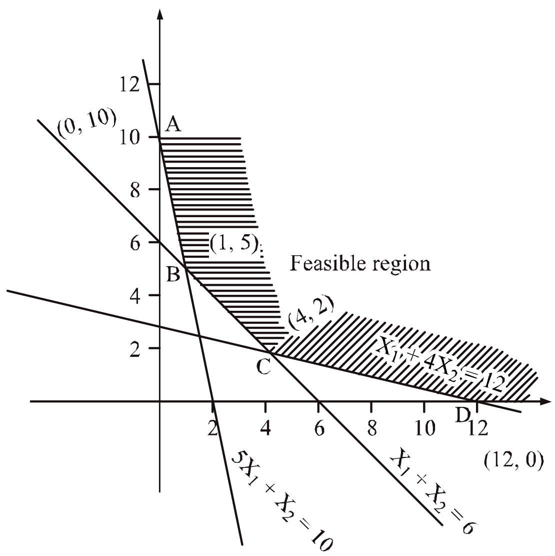
Example 3.6: Use graphical method to solve the LPP.

Maximize $Z = 3X_1 + 2X_2$

Subject to,

$$5X_1 + X_2 \geq 10$$
$$X_1 + X_2 \geq 6$$
$$X_1 + 4X_2 \geq 12$$
$$X_1, X_2 \geq 0.$$

Solution:



Corner points	Value of $Z = 3X_1 + 2X_2$
A (0, 10)	20
B (1, 5)	13 (Minimum value)
C (4, 2)	16
D (12, 0)	36

Since the minimum value is attained at B (1, 5) the optimum solution is $X_1 = 1, X_2 = 5$.

Note: In the above problem if the objective function is maximization, then the solution is unbounded, as maximum value of Z occurs at infinity.

3.2.1 Important Geometric Properties of LPP

There are some linear programming problems which may have,

(*i*) a unique optimal solution

(*ii*) an infinite number of optimal solutions

(*iii*) an unbounded solution

(*iv*) no solution.

The following examples will illustrate these cases.

Example 3.7: Solve the LPP by graphical method.

$$\text{Maximize} \quad Z = 100X_1 + 40X_2$$

$$\text{Subject to,} \quad 5X_1 + 2X_2 \leq 1{,}000$$

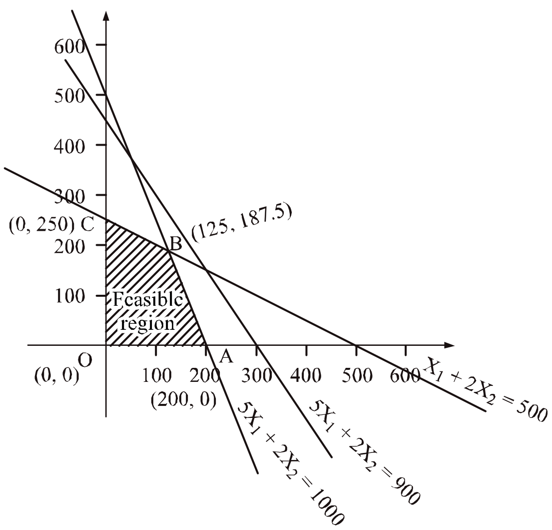
$$3X_1 + 2X_2 \leq 900$$

$$X_1 + 2X_2 \leq 500$$

$$\text{and,} \quad X_1, X_2 \geq 0$$

Solution:

The solution space is given by the feasible region *OABC*.



Corner points	*Value of* $Z = 100X_1 + 40X_2$
O (0, 0)	0
A (200, 0)	20,000 (Max value of *Z*)
B (125, 187.5)	20,000
C (0, 250)	10,000

NOTES

$\therefore$ The maximum value of Z occurs at two vertices A and B.

Since there are infinite number of points on the line joining A and B it gives the same maximum value of Z.

Thus, there are infinite number of optimal solutions for the LPP.

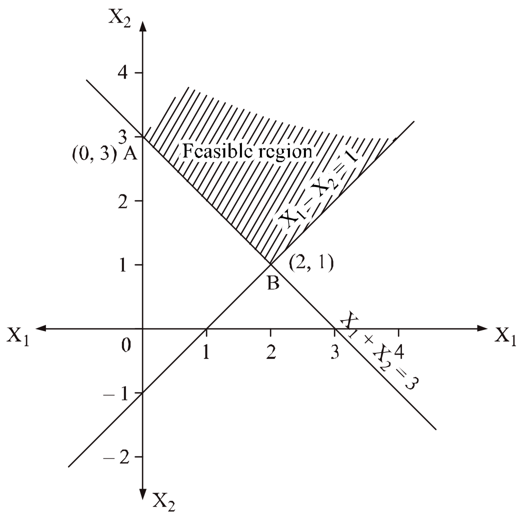
Example 3.8: Solve the following LPP.

Max $\quad Z = 3X_1 + 2X_2$

Subject to, $\quad X_1 + X_2 \geq 1$

$X_1 + X_2 \geq 3$

$X_1, X_2 \geq 0$



Solution: The solution space is unbounded. The value of the objective function at the vertices A and B are $Z(A) = 6$, $Z(B) = 6$. But there exist points in the convex region for which the value of the objective function is more than 8.

In fact, the maximum value of Z occurs at infinity. Hence, the problem has an *unbounded solution*.

No feasible solution

When there is no feasible region formed by the constraints in conjunction with non-negativity conditions, then no solution to the LPP exists.

Example 3.9: Solve the following LPP.

Max $\quad Z = X_1 + X_2$

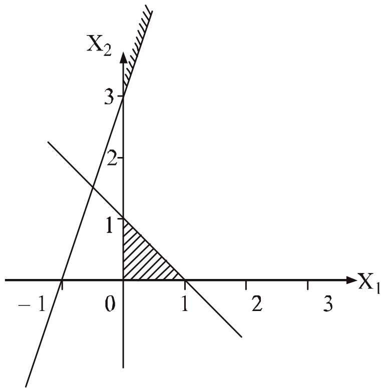
Subject to the constraints, $\quad X_1 + X_2 \leq 1$

$-3X_1 + X_2 \geq 3$

$X_1, X_2 \geq 0$

Solution: There being no point (X_1, X_2) common to both the shaded regions, we cannot find a feasible region for this problem. So the problem cannot be solved. Hence, the problem has no solution.

NOTES



General formulation of LPP

The general formulation of the LPP can be stated as follows:

Maximize or Minimize

$$Z = C_1X_1 + C_2X_2 + ... + C_nX_n \quad ...(1)$$

Subject to m constraints

$$\left.\begin{array}{ll} a_{11}X_1 + a_{12}X_2 + ... + a_{1j}X_j + ... + a_{1n}X_n & (\le \; = \; \ge)\, b_1 \\ a_{21}X_1 + a_{22}X_2 + ... + a_{2j}X_j + ... + a_{2n}X_n & (\le \; = \; \ge)\, b_2 \\ \vdots & \\ a_{i1}X_1 + a_{i2}X_2 + ... + a_{1j}X_j + ... + a_{in}X_n & (\le \; = \; \ge)\, b_i \\ \vdots & \\ a_{m1}X_1 + a_{m2}X_2 + ... + a_{mj}X_j + ... + a_{mn}X_n & (\le \; = \; \ge)\, b_m \end{array}\right\} \quad ...(2)$$

In order to find the values of n decision variables $X_1\, X_2 \dots X_n$ to maximize or minimize the objective function and the non-negativity restrictions

$$X_1 \ge 0,\ X_2 \ge 0 ...\ X_n \ge 0 \quad \dots(3)$$

Matrix form of LPP

The linear programming problem can be expressed in the matrix form as follows:

Maximize or Minimize $Z = CX$

Subject to $AX \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b$

$X \ge 0.$

where, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},\ C = (C_1\ C_2\ ...\ C_n)$

and, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$

NOTES

3.3 CANONICAL OR STANDARD FORMS OF LPP

The general LPP can be classified as canonical or standard forms.

In *standard form*, irrespective of the objective function, namely, maximize or minimize, all the constraints are expressed as equations. Moreover RHS of each constraint and all variables are non-negative.

Characteristics of the Standard Form

(*i*) The objective function is of maximization type.

(*ii*) All constraints are expressed as equations.

(*iii*) Right hand side of each constraint is non-negative.

(*iv*) All variables are non-negative.

In *canonical form,* if the objective function is of maximization, all the constraints other than non-negative conditions are '≤' type. If the objective function is of minimization, all the constraints other than non-negative condition are '≥' type.

Characteristics of the Canonical Form

(*i*) The objective function is of maximization type.

(*ii*) All constraints are of (≤) type.

(*iii*) All variables X_i are non-negative.

Note:

(*i*) Minimization of a function Z is equivalent to maximization of the negative expression of this function, i.e., Min Z = – Max (– Z)

(*ii*) An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by (–1).

(*iii*) Suppose we have the constraint equation,

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n = b_1$$

This equation can be replaced by two weak inequalities in opposite directions,

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n \leq b_1$$

and,

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n \geq b_1$$

(*iv*) If a variable is unrestricted in sign, then it can be expressed as a difference of two non-negative variables, i.e., if X_1 is unrestricted in sign, then $X_1 = X_1' - X_1''$, where X_1', X_1'' are ≥ 0.

(*v*) In standard form, all the constraints are expressed in equation, which is possible by introducing some additional variables called 'slack variables' and 'surplus variables' so that a system of simultaneous linear equations is obtained. The necessary transformation will be made to ensure that $b_i \geq 0$.

NOTES

Check Your Progress

1. Define feasible region and feasible solution.
2. What do you mean by optimal solution?
3. What is the difference between feasible solution and basic feasible solution?
4. What are the two forms of LPP?
5. What do you mean by standard form of LPP?

3.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A region in which all the constraints are satisfied simultaneously is called a feasible region. While any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its feasible solution.
2. Any feasible solution which optimizes (minimizes or maximizes) the objective function is called its optimal solution.
3. The solution of *m* basic variables when each of the (n-m) non-basic variable is set to zero is called basic solution. A basic solution in which all the basic variables are P 0 is called a basic feasible solution.
4. The two forms of LPP are (i) Standard form and (ii) Canonical form.
5. In standard form, irrespective of the objective function namely maximize or minimize, all the constraints are expressed as equations, also right hand side constants are non-negative, i.e., all the variables are non-negative.

3.5 SUMMARY

- Simple linear programming problems with two decision variables can be easily solved by graphical method.
- As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one that gives the optimal solution.
- In the case of maximization problem, the optimal point corresponds to the corner point at which the objective function has a maximum value, and in the case of minimization, the optional solution is the corner point which gives the objective function the minimum value for the objective function.
- Any feasible solution, which optimizes (minimizes or maximizes) the objective function of the LPP is called its optimum solution.
- Basic feasible solutions are of two types: non-degenerate and degenerate.

- If the value of the objective function can be increased or decreased indefinitely, such solutions are called unbounded solutions.

NOTES

3.6 KEY WORDS

- **Linear Programming Problem:** It consists of a linear function to be maximized or minimized subject to certain constraints in the form of linear equations or inequalities.
- **Optimal Solution:** It refers to the feasible solution with the largest objective function value (for a maximization problem).
- **Unbounded Solution:** It refers to a situation where objective function is infinite. A LPP is said to have unbounded solution if its solution can be made infinitely large without violating any of its constraints in the problem.

3.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What do you mean by a general LPP?
2. What do you mean by canonical form of a LPP?
3. Define unbound solution.
4. Mention the characteristics of the standard form.

Long Answer Questions

1. Discuss the procedure for solving LPP by graphical method.
2. Give a matrix form of representing a general LPP.
3. State the characteristics of canonical form and write the canonical form of LPP in matrix form.

3.8 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 4 SIMPLEX METHOD

NOTES

Structure

4.0 INTRODUCTION

Simplex method is used to find the optimal solution to multivariable problems. The method is basically an algorithm which is used by an examinee to examine corner points in a methodical fashion until he/she arrive at the best solution. In this method, slack variables are added to change the constraints into equations and write all variables to the left of the equal sign and constants to the right.

In this unit, you will study in detail about the simplex method in LPP and you will also get to know the steps involved in the simplex algorithm. In addition to this, you will also learn about Charne's BIG M method and get to know the way to solve an LPP using the same. The two-phase simplex method used to solve a LPP is also explained in the unit.

It is to be noted that the standard from of LPP has been discussed in detail in Unit 3 and the fundamental theorem of LPP has been discussed in Unit 2.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss the simplex method in LPP
- Comprehend the steps of simplex algorithm
- Find the solution of any LPP by using simplex algorithm
- Understand solving an LPP using the Charne's BIG M method
- Know the two-phase simplex method to solve a given LPP

4.2 SOLUTION OF LPP – SIMPLEX METHOD

NOTES

Simplex method is an iterative procedure for solving LPP in a finite number of steps. It provides an algorithm, which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more, as the case may be, than at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solution.

Definition

(*i*) Let X_B be a basic feasible solution to the LPP.

Max $Z = CX$

Subject to, $AX = b$

and, $X \geq 0$. Such that it satisfies $X_B = B^{-1}b$

where B is the basis matrix formed by the column of basic variables.

The vector $C_{Bj} = (C_{B1}, C_{B2} \cdots C_{Bm})$ where C_{Bj} are components of C associated with the basic variables, called the *cost vector* associated with the basic feasible solution X_B.

(*ii*) Let X_B be a basic feasible solution to the LPP.

Max $Z = CX$ where,

$AX = b$ and $X \geq 0$.

Let C_B be the cost vector corresponding to X_B. For each column vector a_j in A_1, which is not a column vector of B, let

$$a_j = \sum_{i=1}^{m} a_{ij}\, b_i$$

Then the number $Z_j = \sum_{i=1}^{m} C_{Bi}\, a_{ij}$

is called the *evaluation* corresponding to a_j and the number $(Z_j - C_j)$ is called the *net evaluation* corresponding to j.

Simplex algorithm

The simplex algorithm is the classical method to solve the optimization problem of linear programming.

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows:

Step 1 Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximization by

$$\text{Min } Z = -\text{Max } (-Z)$$

NOTES

Step 2 Check whether all b_i ($i = 1, 2 \cdots m$) are positive. If any b_i is negative then multiply the inequation of the constraint by -1 so as to get all b_i to be positive.

Step 3 Express the problem in standard form by introducing slack/surplus variables to convert the inequality constraints into equations.

Step 4 Obtain an initial basic feasible solution to the problem in the form $X_B = B^{-1}b$ and put it in the first column of the simplex table. Form the initial simplex table as given below:

		C_j = C_1		C_2	C_3	..	..	0	0	 0
C_B	S_B	X_B	X_1	X_2	X_3	X_4	..... X_n	S_1	S_2	..... S_m
C_{B1}	S_1	b_1	a_{11}	a_{12}	a_{13}	a_{14}	..... a_{1n}	1	0	 0
C_{B2}	S_2	b_2	a_{21}	a_{22}	a_{23}	a_{24}	..... a_{2n}	1	0	 0

Step 5 Compute the net evaluations $Z_j - C_j$ by using the relation $Z_j - C_j = C_B (a_j - c_j)$

Examine the sign of $Z_j - C_j$

(*i*) If all $Z_j - C_j \geq 0$, then the initial basic feasible solution X_B is an optimum basic feasible solution.

(*ii*) If at least one $Z_j - C_j < 0$, then proceed to next step as the solution is not optimal.

Step 6 (To find the entering variable, i.e., key column)

If there are more than one negative $Z_j - C_j$, choose the most negative of them. Let it be $Z_r - C_r$ for some $j = r$. This gives the entering variable X_r and is indicated by an arrow at the bottom of the r^{th} column. If there are more than one variables having the same most negative $Z_j - C_j$ then, any one of them can be selected arbitrarily as the entering variable.

(*i*) If all $a_{ir} \leq 0$ ($i = 1, 2 \ldots m$) then there is an unbounded solution to the given problem.

(*ii*) If at least one $a_{ir} > 0$ ($i = 1, 2 \ldots m$) then the corresponding vector X_r enters the basis.

Step 7 (To find the leaving variable or key row)

Compute the ratio (X_{Bi}/a_{ir}, $a_{ir} > 0$)

If the minimum of these ratios be X_{Bi}/a_{kr}, then choose the variable X_k to leave the basis called the *key row* and the element at the intersection of key row and key column is called the *key element*.

Step 8 Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under C_B column. Convert the leading element to unity by dividing the key equation by the key element and all other elements in its column to zero by using Gauss Elimination on the formula

$$\text{New element} = \text{old element} - \left[\frac{\text{Product of elements in key row and column}}{\text{key element}}\right]$$

NOTES

Step 9 Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of unbounded solution.

Example 4.1 Use simplex method to solve the LPP.

Max $Z = 3X_1 + 2X_2$

Subject to,
$$X_1 + X_2 \leq 4$$
$$X_1 - X_2 \leq 2$$
$$X_1, X_2 \geq 0$$

Solution By introducing the slack variables S_1, S_2, convert the problem in standard form.

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2$

Subject to,
$$X_1 + X_2 + S_1 = 4$$
$$X_1 - X_2 + S_2 = 2$$
$$X_1, X_2, S_1, S_2 \geq 0$$

Writing is Martix form $AX = 6$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

An initial basic feasible solution is given by

$$X_B = B^{-1}b,$$

where, $B = I_2, X_B = (S_1\ S_2).$

i.e., $(S_1\ S_2) = I_2\ (4, 2) = (4, 2)$

Initial simplex table

$$Z_j = C_B\ a_j$$

$$Z_1 - c_1 = C_B\ a_1 - c_1 \quad = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 \ = -3$$

$$Z_2 - c_2 = C_B\ a_2 - c_2 \quad = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 - 1) - 2 \ = -2$$

$$Z_3 - c_3 = C_B\ a_3 - c_3 \quad = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1\ 0) - 0 \ = -0$$

$$Z_4 - c_4 = C_B\ a_4 - c_4 \quad = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (0\ 1) - 0 \ = -0.$$

NOTES

		C_j	3	2	0	0	
C_B	Basis	X_B	X_1	X_2	S_1	S_2	$Min \frac{X_B}{X_1}$
0	S_1	4	1	1	1	0	4/1 = 4
←0	S_2	2	1	–1	0	1	2/1 = 2
	Z_j	0	0	0	0	0	
	$Z_j - C_j$		–3↑	–2	0	0	

Since there are some $Z_j - C_j < 0$, the current basic feasible solution is not optimum.

Since $Z_1 - C_1 = -3$ is the most negative, the corresponding non-basic variable X_1 enters the basis.

The column corresponding to this X_1 is called the *key column*.

$$\text{To find the ratio} = \text{Min}\left\{\frac{X_{Bi}}{X_{ir}}, X_{ir} > 0\right\}$$

$$= \text{Min}\left\{\frac{4}{1}, \frac{2}{1}\right\} = 2, \text{ which corresponds to } S_2.$$

∴ The leaving variable is the basic variable S_2. This row is called the *key row*. Convert the leading element X_{21} to units and all other elements in its column *n* i.e. (X_1) to zero by using the formula or Gauss Elimination:

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{key element}}\right]$$

To apply this formula, first we find the ratio, namely

$$\frac{\text{the element to be zero}}{\text{key element}} = \frac{1}{1} = 1$$

Apply this ratio for the number of elements that are converted in the key row. Multiply this ratio by key row elements as shown below.

$$1 \times 2$$
$$1 \times 1$$
$$1 \times -1$$
$$1 \times 0$$
$$1 \times 1$$

Now subtract this element from the old element. The element to be converted into zero is called the *old element row*. Finally we have,

$$4 - 1 \times 2 = 2$$
$$1 - 1 \times 1 = 0$$
$$1 - 1 \times - 1 = 2$$
$$1 - 1 \times 0 = 1$$
$$0 - 1 \times 1 = -1$$

NOTES

$\therefore$ The improved basic feasible solution is given in the following simplex table

First iteration

		C_j	3	2	0	0	
C_B	**Basis**	X_B	X_1	X_2	S_1	S_2	$Min \frac{X_B}{X_2}$
←0	S_1	2	0	2	1	–1	2/1 = 1
0	X_1	2	1	–1	0	1	—
	Z_j	6	3	–3	0	0	
	$Z_j - C_j$		0	–5↑	0	0	

Since $Z_2 - C_2$ is most negative, X_2 enters the basis.

To find $\text{Min}\left(\frac{X_B}{X_{i2}}, X_{i2} > 0\right)$

$$\text{Min}\left(\frac{2}{2}, \frac{2}{-1}\right) = 1. \quad (\because \text{ negative or zero value are not considered})$$

This gives the outgoing variables. Convert the leading element into one. This is done by dividing all the elements in the key row by 2. The remaining elements should be made zero using the formula as shown below.

$-\frac{1}{2}$ is the common ratio. Put this ratio 5 times and multiply each ratio by key row elements.

$$-\frac{1}{2} \times 2$$

$$-\frac{1}{2} \times 0$$

$$-\frac{1}{2} \times 2$$

$$-\frac{1}{2} \times 2$$

$$-\frac{1}{2} \times -1$$

Subtract this from the old element. All the row elements that are converted into zero, are called the *old element*.

$$2 - \left(-\frac{1}{2} \times 2\right) = 3$$

$$1 - \left(-\frac{1}{2} \times 0\right) = 1$$

$$-1 - \left(-\frac{1}{2} \times 2\right) = 0$$

$$0-\left(-\frac{1}{2}\times 1\right) = \frac{1}{2}$$

$$1-\left(-\frac{1}{2}\times -1\right) = \frac{1}{2}$$

NOTES

Second iteration

		C_j	3	2	0	0
C_B	*Basis*	X_B	X_1	X_2	S_1	S_2
2	X_2	1	0	1	1/2	–1/2
3	X_1	3	1	0	1/2	1/2
	Z_j	11	3	2	5/2	1/2
	$Z_j - C_j$		0	0	5/2	1/2

Since all $Z_j - C_j \geq 0$, the solution is optimum. The optimal solution is Max $Z =$ 11, $X_1 = 3$, and $X_2 = 1$.

Example 4.2 Solve the LPP

Max $Z = 3X_1 + 2X_2$

Subject to,
$$4X_1 + 3X_2 \leq 12$$
$$4X_1 + X_2 \leq 8$$
$$4X_1 - X_2 \leq 8$$
$$X_1, X_2 \geq 0$$

Solution Convert the inequality of the constraint into an equation by adding slack variables S_1, S_2, S_3,.

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to,
$$4X_1 + 3X_2 + S_1 = 12$$
$$4X_1 + X_2 + S_2 = 8$$
$$4X_1 - X_2 + S_3 = 8$$
$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 & S_3 \\ 4 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 8 \end{bmatrix}$$

NOTES

Initial table

		C_j	3	2	0	0	0	
C_B	Basis	X_B	X_1	X_2	S_1	S_2	S_3	$Min \frac{X_B}{X_1}$
0	S_1	12	4	3	1	0	0	12/4 = 3
0	S_2	8	4	1	0	1	0	8/4 = 2
←0	S_3	8	4	–1	0	0	1	8/4 = 2
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		–3↑	–2	0	0	0	

$\because$ $Z_1 - C_1$ is most negative, X_1 enters the basis. And the $\min\left(\frac{x_B}{x_{il}}, x_{il} > 0\right) =$ $\min(3, 2, 2) = 2$ gives S_3 as the leaving variable.

Convert the leading element into 1, by dividing key row element by 4 and the remaining elements into 0.

First iteration

		C_j	3	2	0	0	0	
C_B	Basis	X_B	X_1	X_2	S_1	S_2	S_3	$Min \frac{X_B}{X_2}$
0	S_1	4	0	4	1	0	–1	4/4 = 1
←0	S_2	0	0	2	0	1	–1	0/2 = 1
3	X_1	2	1	–1/4	0	0	¼	—
	Z_j	6	3	–3/4	0	0	¾	
	$Z_j - C_j$		0	–11/4↑	0	0	¾	

$$8 - \frac{4}{4} \times 8 = 0 \qquad 12 - \frac{4}{4} \times 8 = 4$$

$$4 - \frac{4}{4} \times 4 = 0 \qquad 4 - \frac{4}{4} \times 4 = 0$$

$$1 - \frac{4}{4} \times -1 = 2 \qquad 3 - \frac{4}{4} \times -1 = 4$$

$$0 - \frac{4}{4} \times 0 = 0 \qquad 1 - \frac{4}{4} \times 0 = 1$$

$$1 - \frac{4}{4} \times 0 = 1 \qquad 0 - \frac{4}{4} \times 0 = 0$$

$$0 - \frac{11}{4} \times 1 = -1 \qquad 0 - \frac{4}{4} \times 1 = -1$$

Since $Z_2 - C_2 = -\frac{11}{4}$ is the most negative, x_2 enters the basis.

To find the outgoing variable, find $\text{Min}\left(\frac{x_B}{x_{i2}}, x_{i2} > 0\right)$

NOTES

$$\text{Min}\left(\frac{4}{4}, \frac{0}{2}, -\right) = 0$$

First iteration

Therefore, S_2 leaves the basis. Convert the leading element into 1 by dividing the key row elements by 2 and make the remaining elements in that column as zero using the formula.

$$\text{New element} = \text{old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{key element}}\right]$$

		C_j	3	2	0	0	0	
C_B	*Basis*	X_B	X_1	X_2	S_1	S_2	S_3	*Min* $\frac{X_B}{X_3}$
←0	S_1	4	0	0	1	–2	1	4/1 = 1
2	X_2	0	0	1	0	1/2	–1/2	—
3	X_1	2	1	0	0	1/8	1/8	2/1/8 = 16
	Z_j	6	3	2	0	11/8	–5/8	
	$Z_j - C_j$		0	0	0	11/8	–5/8↑	

Second iteration

Since $Z_5 - C_5 = -5/8$ is most negative, S_3 enters the basis and

$$\text{Min}\left(\frac{X_B}{S_{13}}, S_{i3}\right) = \text{Min}\left(\frac{4}{1}, -1\frac{2}{1/18}\right) = 4.$$

Therefore, S_1 leaves the basis. Convert the leading element into one and remaining elements as zero.

Third iteration

		C_j	3	2	0	0	0
C_B	*Basis*	X_B	X_1	X_2	S_1	S_2	S_3
0	S_3	4	0	0	1	–2	1
2	X_2	2	0	1	1/2	–1/2	0
3	X_1	3/2	1	0	–1/8	3/8	0
	Z_j	17/2	3	2	5/8	1/8	0
	$Z_j - C_j$		0	0	5/8	1/8	0

Since all $Z_j - C_j \geq 0$, the solution is optimum and it is given by $X_1 = 3/2$, $X_2 = 2$ and Max $Z = 17/2$.

NOTES

Example 4.3 Using simplex method solve the LPP.

$$\text{Max } Z = X_1 + X_2 + 3X_3$$

$$\text{Subject to,} \quad 3X_1 + 2X_2 + X_3 \le 3$$
$$2X_1 + X_2 + 2X_3 \le 2$$
$$X_1, X_2, X_3 \ge 0$$

Solution Rewrite the inequality of the constraints into an equation by adding slack variables.

$$\text{Max } Z = X_1 + X_2 + 3X_3 + 0S_1 + 0S_2$$

$$\text{Subject to,} \quad 3X_1 + 2X_2 + X_3 + S_1 \le 3$$
$$2X_1 + X_2 + 2X_3 + S_2 \le 2$$

Initial basic feasible solution is,

$$X_1 = X_2 = X_3 = 0$$
$$S_1 = 3,\ S_2 = 2 \text{ and } Z = 0$$

$$\begin{bmatrix} X_1 & X_2 & X_3 & S_1 & S_2 \\ 3 & 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & 0 \end{bmatrix}$$

		C_j	1	1	3	0	0	
C_B	Basis	X_B	X_1	X_2	X_3	S_1	S_2	$Min \frac{X_B}{X_3}$
0	S_1	3	3	2	1	1	0	3/1 = 3
←0	S_2	2	2	1	2	0	1	2/2 = 1
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		–1	–1	–3↑	0	0	

Since $Z_3 - C_3 = -3$ is the most negative, the variable X_3 enters the basis. The column corresponding to X_3 is called the *key column*.

$$\left(\text{To determine the key row or leaving variable, find Min}\left(\frac{x_B}{x_{i3}}, x_{i3} > 0\right) \text{Min}\left(\frac{3}{1} = 3, \frac{2}{2} = 1\right) = 1\right)$$

Therefore, the leaving variable is the basic variable S_2, the row is called the *key row* and the intersection element 2 is called the *key element*.

Convert this element into one by dividing each element in the key row by 2 and the remaining elements in that key column as zero using the formula

$$\text{New element} = \text{old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{key element}}\right]$$

First iteration

		C_j	1	1	3	0	0
C_B	Basis	X_B	X_1	X_2	X_3	S_1	S_2
0	S_1	2	2	3/2	0	1	–1/2
3	X_3	1	1	1/2	1	0	1/2
	Z_j	3	3	3/2	3	0	3/2
	$Z_j - C_j$		2	1/2	0	0	3/2

NOTES

Since all $Z_j - C_j \geq 0$, the solution is optimum and it is given by $X_1 = 0$, $X_2 = 0$, $X_3 = 1$, Max $Z = 3$.

4.2.1 Solved Problems on Minimisation

Example 4.4 Use simplex method to solve the LPP.

Min $Z = X_2 - 3X_3 + 2X_5$

Subject to,
$$3X_2 - X_3 + 2X_5 \leq 7$$
$$-2X_2 + 4X_3 \leq 12$$
$$-4X_2 + 3X_3 + 8X_5 \leq 10$$
$$X_2, X_3, X_5 \geq 0$$

Solution Since the given objective function is of **minimization** we shall convert it into maximization using Min Z = –Max(–Z) = –Max Z^*

Max $Z^* = -X_2 + 3X_3 - 2X_5$

Subject to,
$$3X_2 - X_3 + 2X_5 \leq 7$$
$$-2X_2 + 4X_3 \leq 12$$
$$-4X_2 + 3X_3 + 8X_5 \leq 10$$

We rewrite the inequality of the constraints into an equation by adding slack variables S_1, S_2, S_3 and the standard form of LPP becomes.

Max $Z = -X_2 + 3X_3 - 2X_5 + 0S_1 + 0S_2 + 0S_3$

Subject to
$$3X_2 - X_3 + 2X_5 + S_1 = 7$$
$$-2X_2 + 4X_3 + S_2 = 12$$
$$-4X_2 + 3X_3 + 8X_5 + S_3 = 10$$
$$X_2, X_3, X_5, S_1, S_2, S_3 \geq 0$$

$\therefore$ The initial basic feasible solution is given by $S_1 = 7$, $S_2 = 12$, $S_3 = 10$. ($X_2 = X_3 = X_5 = 0$)

NOTES

Initial table

		C_j	-1	3	-2	0	0	0	
C_B	*Basis*	X_B	X_2	X_3	X_5	S_1	S_2	S_3	$Min \frac{X_B}{X_3}$
0	S_1	7	3	-1	2	1	0	0	—
←0	S_2	12	-2	4	0	0	1	0	12/4 = 3
0	S_3	10	-4	3	8	0	0	1	10/3 = 3.33
	Z_j	0	0	0	0	0	0		
	$Z_j - C_j$		1	-3	2	0	0	0	

Since $Z_2 - C_2 = -3 < 0$, the solution is not optimum.

The incoming variable is X_3 (key column) and the outgoing variable (key row) is given by,

$$\text{Min}\left(\frac{X_B}{X_{i3}} X_{i3} > 0\right) = \text{Min}\left(-, \frac{12}{4}, \frac{10}{3}\right) = 3.$$

Hence, S_2 leaves the basis.

First iteration

		C_j	-1	3	-2	0	0	0	
C_B	*B*	X_B	X_2	X_3	X_5	S_1	S_2	S_3	$Min \frac{X_B}{X_2}$
←0	S_1	10	5/2	0	2	1	1/4	0	10/5/2 = 4
3	X_3	3	$-1/2$	1	0	0	1/4	0	—
0	S_3	1	5/2	0	8	0	$-3/4$	1	—
	Z_j	9	$-3/2$	3	0	0	3/4	0	
	$Z_j - C_j$		$-1/2$↑	0	2	0	3/4	0	

Since $Z_1 - C_1 < 0$, the solution is not optimum. Improve the solution by allowing the variable X_2 to enter into the basis and the variable S_1 to leave the basis.

Second iteration

		C_j	-1	3	-2	0	0	0
C_B	*B*	X_B	X_2	X_3	X_5	S_1	S_2	S_3
-1	X_2	4	1	0	4/5	2/5	1/10	0
3	X_3	5	0	1	2/5	1/5	3/10	0
0	S_3	11	0	0	10	1	$-½$	1
	Zj	11	-1	3	2/5	1/5	8/10	0
	$Z_j - C_j$		0	0	12/5	1/5	8/10	0

Since all $Z_j - C_j \geq 0$, the solution is optimum.

$\therefore$ The optimal solution is given by Max $z^x = 11$

$$X_2 = 4, X_3 = 5, X_5 = 0$$

$\therefore$ $\quad$ Min $Z = -$Max $(-Z) = -11$

$\therefore$ $\quad$ Min $Z = -11, X_2 = 4, X_3 = 5, X_5 = 0.$

NOTES

Example 4.5 Solve the following LPP using simplex method.

Max $Z = 15X_1 + 6X_2 + 9X_3 + 2X_4$

Subject to, $\quad 2X_1 + X_2 + 5X_3 + 6X_4 \leq 20$

$$3X_1 + X_2 + 3X_3 + 25X_4 \leq 24$$

$$7X_1 + X_4 \leq 70$$

$$X_1, X_2, X_3, X_4 \geq 0$$

Solution Rewriting the inequality of the constraint into an equation by adding slack variables S_1, S_2 and S_3 the standard form of LPP becomes.

Max $Z = 15X_1 + 6X_2 + 9X_3 + 2X_4 + 0S_1 + 0S_2 + 0S_3$

Subject to,$2X_1 + X_2 + 5X_3 + 6X_4 + S_1 = 20$

$$3X_1 + X_2 + 3X_3 + 25X_4 + S_2 = 24$$

$$7X_1 + X_4 + S_3 = 70$$

$$X_1, X_2, X_3, X_4, S_1, S_2, S_3 \geq 0$$

The initial basic **feasible** solution is $S_1 = 20, S_2 = 24, S_3 = 70$ ($X_1 = X_2 = X_3 = X_4 = 0$ non-basic)

The initial simplex table is given by

		C_j	15	6	9	2	0	0	0	
C_B	Basis	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3	$Min \frac{X_B}{X_1}$
0	S_1	20	2	1	5	6	1	0	0	20/10 = 10
←0	S_2	24	3	1	3	25	0	1	0	24/3 = 8
0	S_3	70	7	0	0	1	0	0	1	70/7 = 10
	Z_j	0	0	0	0	0	0	0	0	
	$Z_j - C_j$		–15↑	–6	–9	–2	0	0	0	

$\therefore$ As some of $Z_j - C_j \leq 0$ the current basic **feasible** solution is not optimum. $Z_1 - C_1 = -15$ is the most negative value and hence X_1 enters the basis and the variable S_2 leaves the basis.

NOTES

First iteration

		C_j	15	6	9	2	0	0	0	
C_B	Basis	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3	Min $\frac{X_B}{X_2}$
$\leftarrow 0$	S_1	4	0	1/3	3	–32/3	1	–2/3	0	4/1/3 = 12
15	X_1	8	1	1/3	1	25/3	0	1/3	0	8/1/3 = 24
0	S_3	14	0	–7/3	–7	–172/3	0	–7/3	1	—
	Z_j	120	15	5	15	125	0	5	0	
	$Z_j - C_j$		0	$-1\uparrow$	6	123	0	5	0	

Since $Z_2 - C_2 = -1 < 0$ the solution is not optimal therefore, X_2 enters the basis and the basic variable S_1 leaves the basis.

Second iteration

		C_j	–5	6	9	–2	0	0	0
C_B	B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3
6	X_2	12	0	1	9	–32	3	–2	0
15	X_1	4	1	0	–2	57/3	–1	1	0
0	S_3	42	0	0	14	–132	7	–7	1
	Z_j	132	15	6	24	93	3	3	0
	$Z_j - C_j$		0	0	15	91	3	3	0

Since all $Z_j - C_j \geq 0$, the solution is optimal and is given by,

Max Z =132, $X_1 = 4$, $X_2 = 12$, $X_3 = 0$, $X_4 = 0$.

Example 4.6

$$\text{Max } Z = 3X_1 + 2X_2 + 5X_3$$

$$\text{Subject to} \quad \begin{aligned} X_1 + 2X_2 + X_3 &\leq 430 \\ 3X_1 + 2X_3 &\leq 260 \\ X_1 + 4X_2 &\leq 420 \\ X_1, X_2, X_3 &\geq 0 \end{aligned}$$

Solution Rewrite the constraint into equation by adding slack variables S_1, S_2, S_3. The standard form of LPP becomes

$$\text{Max } Z = 3X_1 + 2X_2 + 5X_3 + 0S_1 + 0S_2 + 0S_3$$

$$\text{Subject to,} \quad \begin{aligned} X_1 + 2X_2 + X_3 + S_1 &= 430 \\ 3X_1 + 2X_3 + S_2 &= 460 \end{aligned}$$

$$X_1 + 4X_2 + S_3 = 420$$

$$X_1, X_2, X_3, S_1, S_2, S_3 \geq 0.$$

The initial basic feasible solution is,

$S_1 = 430,\ S_2 = 460,\ S_3 = 420\ (X_1 = X_2 = X_3 = 0)$

NOTES

Initial table

		C_j	3	2	5	0	0	0	
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	$Min \frac{X_B}{X_3}$
0	S_1	430	1	2	1	1	0	0	430/1 = 430
←0	S_2	460	3	0	2	0	1	0	160/2 = 230
0	S_3	420	1	4	0	0	0	1	
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$		−3	−2	−5↑	0	0	0	

Since some of $Z_j - C_j \leq 0$, the current basic feasible solution is not optimum. Since $Z_3 - C_3 = -5$ is the most negative, the variable X_3 enters the basis. To find the variable leaving the basis find,

$$\text{Min}\left(\frac{X_B}{X_{i3}}, X_{i3} > 0\right) = \text{Min}\left(\frac{430}{1}, \frac{460}{2}, -\right) = 230.$$

$\therefore$ the variable S_2 leaves the basis.

First iteration

		C_j	3	2	5	0	0	0	
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	$Min \frac{X_B}{X_3}$
←0	S_1	200	−1/2	2	0	1	1/2	0	200/2 = 100
5	X_3	230	3/2	0	1	0	1/2	0	—
0	S_3	420	1	4	0	0	0	1	420/4 = 105
	Z_j	1150	15/2	0	5	0	5/2	0	
	$Z_j - C_j$		9/2	−2↑	0	0	5/2	0	

Since $Z_2 - C_2 = -2$ is negative, the current basic **feasible** solution is not optimum. Therefore, the variable X_2 enters the basis and the variable S_1 leaves the basis.

Second iteration

		C_j	3	2	5	0	0	0
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
2	X_2	100	– 1/4	1	0	1/2	– 1/4	0
5	X_3	230	3/2	0	1	0	1/2	0
0	S_3	20	2	0	0	–2	1	1
	Z_j	1350	7	2	5	1	+2	0
	$Z_j - C_j$		4	0	0	1	2	0

NOTES

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $X_1 = 0$, $X_2 = 100$, $X_3 = 230$ and Max $Z = 1350$.

4.3 THE SIMPLEX ALGORITHM

In this section, we introduce a new type of variable called the artificial variable. These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. To solve such LPP there are two methods:

(i) The Charne's Big M method

(ii) The Two-phase Simplex method

4.3.1 Penalty Cost Method or Big M-Method

The following steps are involved in solving an LPP using the Big M method.

Step 1 Express the problem in the standard form.

Step 2 Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type $\geq$ or =. However, addition of these artificial variables causes violation of the corresponding constraints. Therefore, we would like to get rid of these variables and not allow them to appear in the final solution. This is achieved by assigning a very large penalty ($-M$ for maximization and M for minimization) in the objective function.

Step 3 Solve the modified LPP by simplex method, until any one of the three cases may arise.

1. If no artificial variable appears in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.
2. If at least one artificial variable in the basis at zero level and the optimality condition is satisfied, then the current solution is an optimal basic feasible solution (though degenerated).
3. If at least one artificial variable appears in the basis at positive level and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the

objective function, since it contains a very large penalty M and is called *pseudo optimal solution*.

Note: While applying simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Example 4.6 Use penalty method to

Maximize $Z = 3x_1 + 2x_2$

Subject to the constraints,

$$2x_1 + x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Solution By introducing slack variable $S_1 \geq 0$, surplus variable $S_2 \geq 0$ and artificial variable $A_1 \geq 0$, the given LPP can be reformulated as:

Maximize $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$

Subject to, $2x_1 + x_2 + S_1 = 2$

$$3x_1 + 4x_2 - S_2 + A_1 = 12$$

The starting feasible solution is $S_1 = 2$, $A_1 = 12$.

Initial table

		C_j	3	2	0	0	$-M$	
C_B	B	x_B	x_1	x_2	S_1	S_2	A_1	$\frac{x_B}{x_2}$
$\leftarrow 0$	S_1	2	2	1	1	0	0	2/1 = 2
$-M$	A_1	12	3	4	0	–1	1	12/4 = 3
	Z_j	$-12M$	$-3M$	$-4M$	0	M	$-M$	
	$Z_j - C_j$	–	$-3M - 3$	$-4M - 2$ $\uparrow$	0	M	0	

Since some of the $Z_j - C_j \leq 0$, the current feasible solution is not optimum. Choose the most negative $Z_j - C_j = -4M - 2$.

$\therefore$ x_2 variable enters the basis, and the basic variable S_1 leaves the basis.

First iteration

		C_j	3	2	0	0	$-M$
C_B	B	x_B	x_1	x_2	S_1	S_2	A_1
2	x_2	2	2	1	1	0	0
$-M$	A_1	4	–5	0	–4	–1	1
	Z_j	$4 - 4M$	$4 + 5M$	2	$2 + 4M$	M	$-M$
	$Z_j - C_j$		$5M + 1$	0	$4M + 2$	M	0

Since all $Z_j - C_j \geq 0$ and an artificial variable appears in the basis at positive level, the given LPP does not possess any feasible solution. But the LPP possesses a *pseudo optimal solution*.

NOTES

Example 4.7 Solve the LPP.

Minimize $Z = 4x_1 + x_2$

Subject to,
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution Since the objective function is minimization, we covert it into maximization using,

Min $Z = -$ Max $(-z) = -$Max z^* $\qquad (\because \quad Z^* = -Z)$

Maximize $z^* = -4x_1 - x_2$

Subject to,
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Convert the given LPP into standard form by adding artificial variables A_1, A_2, surplus variable S_1 and slack variable S_2 to get the initial basic feasible solution.

Minimize $Z^* = -4x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to,
$$3x_1 + x_2 + A_1 = 3$$
$$4x_1 + 3x_2 - S_1 + A_2 = 6$$
$$x_1 + 2x_2 + S_2 = 4$$
$$x_1x_2, S_1S_2, A_1A_2 \leq 0$$

The starting feasible solution is $A_1 = 3, A_2 = 6, S_2 = 4$.

Initial solution

		C_j	-4	-1	$-M$	0	$-M$	0	
C_B	B	x_B	x_1	x_2	A_1	S_1	A_2	S_2	$Min \frac{x_B}{x_1}$
$-M$	A_1	3	3	1	1	0	0	0	3/3 = 1
$-M$	A_2	6	4	3	0	-1	1	0	6/4 = 3/2
$\leftarrow 0$	S_2	4	1	2	0	0	0	1	4/1 = 4
	Z_j	$-9M$	$-7M$	$-4M$	$-M$	$-M$	M	$-M$	
	$Z_j - C_j$		$-7M + 4\uparrow$	$-4M + 1$	0	M	0	0	

Since some of the $Z_j - C_j \leq 0$, the current feasible solution is not optimum. As $Z_1 - C_1$ is most negative, x_1 enters the basis and the basic variable A_1 leaves the basis.

NOTES

First iteration

		C_j	−4	−1	−M	0	−M	0	
C_B	B	x_B	x_1	x_2	A_1	S_1	A_2	S_2	$Min \frac{x_B}{x_1}$
$-M$	A_1	3/2	5/2	0	1	0	0	−1/2	3/5
$\leftarrow -M$	A_2	3/2	5/2	0	0	−1	1	−3/2	3/5
−1	x_2	3/2	1/2	1	0	0	0	1/2	3
	Z_j	$-3M - 3/2$	$-5M - 1/2$	−1	$-M$	$+M$	$-M$	$2M - 1/2$	
	$Z_j - C_j$		$-5M + 7/2\uparrow$	0	0	M	0	$2M - 1/2$	

Since $Z_1 - C_1$ is negative, the current feasible solution is not optimum. Therefore, x_1 variable enters the basis and the artificial variable A_2 leaves the basis.

Second iteration

		C_j	−4	−1	−M	0	0	
C_B	B	x_B	x_1	x_2	A_1	S_1	S_2	$Min \frac{x_B}{x_2}$
$\leftarrow -M$	A_1	0	0	0	1	1	1	0
−4	x_1	$\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{3}{5}$	—
−1	x_2	$\frac{6}{5}$	0	1	0	$-\frac{1}{5}$	$\frac{4}{5}$	—
	Z_j	$-\frac{18}{5}$	−4	−1	$-M$	$-M+\frac{9}{5}$	$-M+\frac{8}{5}$	
	$Z_j - C_j$		0	0	0	$-M+\frac{9}{5}$ $\uparrow$	$-M+\frac{8}{5}$	

Since $Z_4 - C_4$ is most negative, S_1 enters the basis and the artificial variable A_1 leaves the basis.

Third iteration

		C_j	−4	−1	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	$Min \frac{x_B}{x_2}$
$\leftarrow 0$	S_1	0	0	0	1	1	0
−4	x_1	$\frac{3}{5}$	1	0	0	$-\frac{1}{5}$	—
−1	x_2	6/5	0	1	0	1	6/5
	Z_j	−18/5	−4	−1	0	−1/5	
	$Z_j - C_j$		0	0	0	$-1/5\uparrow$	

Since $Z_4 - C_4$ is most negative, S_2 enters the basis and S_1 leaves the basis.

NOTES

Fourth iteration

		C_j	−4	−1	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2
0	S_2	0	0	0	1	1
−4	x_1	3/5	1	0	1/5	0
−1	x_2	6/5	0	1	−1	0
	Z_j	−18/5	−4	−1	1/5	0
	$Z_j - C_j$		0	0	1/5	0

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = 3/5$, $x_2 = 6/5$, and Max $Z = -\,18/5$

$\therefore$ Min $Z = -$ Max $(-Z) = 18/5$.

Example 4.8 Solve by Big M method.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to, $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

Solution Since the constraints are equations, introduce artificial variables A_1, $A_2 \geq 0$. The reformulated problem is given as follows.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

Subject to, $x_1 + 2x_2 + 3x_3 + A_1 = 15$

$2x_1 + x_2 + 5x_3 + A_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

Initial solution is given by $A_1 = 15$, $A_2 = 20$ and $x_4 = 10$.

		C_j	1	2	3	−1	$-M$	$-M$	
C_B	B	x_B	x_1	x_2	x_3	x_4	A_1	A_2	Min $\frac{x_B}{x_3}$
$-M$	A_1	15	1	2	3	0	1	0	15/3 = 5
← $-M$	A_2	20	2	1	5	0	0	1	20/5 = 4
−1	x_4	10	1	2	1	1	0	0	10/1 = 10
	Z_j	$-35M - 10$	$-3M - 1$	$-3M - 2$	$-8M - 1$	−1	$-M$	$-M$	
	$Z_j - C_j$		$-3M - 2$	$-3M - 4$	$-8M - 4$ ↑	0	0	0	

Since $Z_3 - C_3$ is most negative, x_3 enters the basis and the basic variable A_2 leaves the basis.

First iteration

NOTES

		C_j	1	2	3	−1	−M	
C_B	B	x_B	x_1	x_2	x_3	x_4	A_1	$Min\frac{x_B}{x_2}$
$\leftarrow -M$	A_1	3	−1/5	7/5	0	0	1	$\frac{3}{7/5}=\frac{15}{7}$
3	x_3	4	2/5	1/5	1	0	0	$\frac{4}{1/5}=\frac{20}{1}$
−1	x_4	6	3/5	9/5	0	1	0	$\frac{6}{9/5}=\frac{30}{9}$
	Z_j	$-3M+6$	$\frac{1}{5}M+\frac{3}{5}$	$-\frac{7}{5}M-\frac{6}{5}$	3	−1	−M	
	Z_j-C_j		$\frac{1}{5}M-\frac{2}{5}$	$-\frac{7}{5}M-\frac{16}{5}$ ↑	0	0	0	

Since $Z_2 - C_2$ is most negative, x_2 enters the basis and the basic variable A_1 leaves the basis.

Second iteration

		C_j	1	2	3	−1	
C_B	B	x_B	x_1	x_2	x_3	x_4	$Min\frac{x_B}{x_1}$
2	x_2	15/7	$-\frac{1}{7}$	1	0	0	—
3	x_3	25/7	3/7	0	1	0	25/3
$\leftarrow -1$	x_4	15/7	$\frac{6}{7}$	0	0	1	15/6
	Z_j	$\frac{90}{7}$	$\frac{1}{7}$	2	3	−1	
	Z_j-C_j		−6/7 ↑	0	0	0	

Since $Z_1 - C_1 = -6/7$ is negative, the current feasible solution is not optimum. Therefore, x_1 enters the basis and the basic variable x_4 leaves the basis.

Third iteration

		C_j	1	2	3	−1
C_B	B	x_B	x_1	x_2	x_3	x_4
2	x_2	15/6	0	1	0	1/6
3	x_3	15/6	0	0	1	3/6
3	x_1	15/6	1	0	0	7/6
	Z_j	15	1	2	3	3
	Z_j-C_j		0	0	0	4

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = x_2 = x_3 = 15/6 = 5/2$, and Max $Z = 15$.

Example 4.9 Use penalty method to solve the following LPP:

NOTES

Minimize $Z = 5x + 3y$

Subject to,

$$2x + 4y \leq 12$$
$$2x + 2y = 10$$
$$5x + 2y \geq 10$$
$$x, y \geq 0$$

Solution First we convert the objective function minimization to maximization using

Min $Z = -\text{Max}(-Z)$.

Rewrite the given LPP into standard form by adding slack variable $S_1 \geq 0$, surplus variable $S_2 \geq 0$ and the artificial variables $A_1, A_2 \geq 0$.

Maximize $Z = -5x - 3y + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to,

$$2x + 4y + S_1 = 12$$
$$2x + 2y + A_1 = 10$$
$$5x + 2y - S_2 + A_2 = 10$$
$$x, y, S_1, S_2, A_1, A_2 \geq 0$$

Initial feasible solution is given by $S_1 = 12$, $A_1 = 10$ and $A_2 = 10$

Since all $Z_j - C_j \geq 0$ (see Table 4.1) and no artificial variable is in the basis, the solution is optimum and is given by:

$x = 4$, $y = 1$, Max $Z = -23$

Min $Z = -\text{Max}(-Z) = 23$

Table 4.1

		C_j	-5	-3	0	$-M$	0	$-M$	
C_B	B	x_B	x	y	S_1	A_1	S_2	A_2	$Min \frac{x_B}{x}$
0	S_1	12	2	4	1	0	0	0	12/2 = 6
$-M$	A_1	10	2	2	0	1	0	0	10/2 = 5
$\leftarrow -M$	A_2	10	5	2	0	0	-1	1	10/5 = 2
	Z_j	$20M$	$-7M$	$-4M$	0	$-M$	M	$-M$	
	$Z_j - C_j$		$-7M + 5\uparrow$	$-4M + 3$	0	0	M	0	$Min \frac{x_B}{y}$
$\leftarrow 0$	S_1	8	0	16/5	1	0	2/5		$\frac{8 \times 5}{16} = 40/16$
$-M$	A_1	6	0	6/5	0	1	2/5	—	30/6 = 5
-5	x	2	1	2/5	0	0	$-1/5$	—	5

NOTES

		C_j	-5	-3	0	$-M$	0	$-M$	
C_B	B	x_B	x	y	S_1	A_1	S_2	A_2	$Min \frac{x_B}{x}$
	Z_j	$-6M-10$	-5	$-6/5\,M-2$	0	$-M-$	$2/5\,M+1$	—	
	Z_j-C_j		0	$-6/5\,M+1\uparrow$	0	$0-$	$\frac{2}{5}M+1$		$Min \frac{x_B}{S_2}$
-3	y	$5/2$	0	1	$5/16$	0	$1/8$	—	20
$\leftarrow -M$	A_1	3	0	0	$-3/8$	1	$-1/4$	—	12
-5	x	1	1	0	$-1/8$	0	$-1/4$	—	—
	Z_j	$-3M-25/2$	-5	-3	$-3/8M$ $-5/16$	$-M$ 0	$+7/8$ $-1/4M$	—	
	Z_j-C_j		0	0	$3/8M$ $-5/16$	0	$7/8$ $-M/4\uparrow$	—	
-3	y	1	0	1	$1/2$	—	0	—	
0	S_2	12	0	0	$-3/2$	—	1	—	
-5	x	4	1	0	$-1/2$	—	0	—	
	Z_j	-23	-5	-3	1	—	0	—	
	Z_j-C_j		0	0	1	—	0	—	

4.3.2 The Two-Phase Simplex Method

The two-phase simplex method is another method to solve a given LPP involving some artificial variables. The solution is obtained in two phases.

Phase I

In this phase, we construct an auxiliary LPP leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 Assign a cost –1 to each artificial variable and a cost 0 to all other variables and get a new objective function $Z^* = -A_1 - A_2 - A_3 ...$ where A_i are artificial variables.

Step 2 Write down the auxiliary LPP in which the new objective function is to be maximized, subject to the given set of constraints.

Step 3 Solve the auxiliary LPP by simplex method until either of the following three cases arise:

(*i*) Max $Z^* < 0$ and at least one artificial variable appears in the optimum basis at positive level.

(*ii*) Max $Z^* = 0$ and at least one artificial variable appears in the optimum basis at zero level.

(*iii*) Max $Z^* = 0$ and no artificial variable appears in the optimum basis.

In case (*i*), given LPP does not possess any feasible solution, whereas in cases (*ii*) and (*iii*) we go to phase II.

NOTES

Phase II

Use the optimum basic feasible solution of phase I as a starting solution for the original LPP. Assign the actual costs to the variable in the objective function and a zero cost to every artificial variable in the basis at zero level. Delete the artificial variable column that is eliminated from the basis in phase I from the table. Apply simplex method to the modified simplex table obtained at the end of phase I till an optimum basic feasible solution is obtained or till there is an indication of unbounded solution.

Example 4.10 Use two-phase simplex method to solve,

Maximize $Z = 5x_1 + 3x_2$

Subject to, $2x_1 + x_2 \leq 1$

$x_1 + 4x_2 \geq 6$

$x_1, x_2 \geq 0$

Solution We convert the given problem into a standard form by adding slack, surplus and artificial variables. We form the auxiliary LPP by assigning the cost –1 to the artificial variable and 0 to all the other variables.

Phase I

Maximize $Z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 - 1.\ A_1$

Subject to, $2x_1 + x_2 + S_1 = 1$

$x_1 + 4x_2 - S_2 + A_1 = 6$

Initial basic feasible solution is given by $S_1 = 1, A_1 = 6$.

		C_j	0	0	0	0	–1	
C_B	B	x_B	x_1	x_2	S_1	S_2	A_1	$Min\ \frac{x_B}{x^2}$
← 0	S_1	1	2	1	1	0	0	1/1 = 1
–1	A_1	6	1	4	0	–1	1	6/4 = 1.5
	Z_j	– 6	–1	– 4	0	1	–1	
	$Z_j - C_j$		–1	– 4↑	0	1	0	
0	x_2	1	2	1	1	0	0	
–1	A_1	2	–7	0	– 4	–1	1	
	Z_j	–2	7	0	4	1	–1	
	$Z_j - C_j$		7	0	4	1	0	

Since all $Z_j - C_j \geq 0$, an optimum feasible solution to the auxiliary LPP is obtained. But as Max $Z^* < 0$, and an artificial variable A_1 is in the basis at a positive level, the original LPP does not possess any feasible solution.

NOTES

Example 4.11 Use two-phase simplex method to solve,

Minimize $Z = x_1 - 2x_2 - 3x_3$

Subject to,$- 2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

$x_1, x_2, x_3 \geq 0$

Solution Convert the objective function into maximization using,

Min Z = – Max (– Z)

Max $Z = - x_1 + 2x_2 + 3x_3$

Introducing the artificial variables $A_1, A_2 \geq 0$, the constraints of the given problem become,

$- 2x_1 + x_2 + 3x_3 + A_1 = 2$

$2x_1 + 3x_2 + 4x_3 + A_2 = 1$

$x_1, x_2, x_3, A_1, A_2 \geq 0$

Phase I

Assigning a cost –1 to the artificial variables A_1 and A_2, and cost 0 to other variables, the objective function of the auxiliary LPP is,

Maximize $Z^* = 0x_1 + 0x_2 + 0x_3 - 1 \,.\, A_1 - 1 \,.\, A_2$, subject to the given constraints.

		C_j	0	0	0	–1	–1	
C_B	B	x_B	x_1	x_2	x_3	A_1	A_2	$Min \frac{x_B}{x_3}$
–1	A_1	2	–2	1	3	1	0	2/3
← –1	A_2	1	2	3	4	0	1	1/4
	Z_j	–3	0	–4	–7	–1	–1	
	$Z_j - C_j$		0	–4	–7↑	0	0	
–1	A_1	5/4	–7/2	–5/4	0	1	–3/4	
0	x_3	1/4	1/2	3/4	1	0	1/4	
	Z_j	–5/4	7/2	5/4	0	–1	3/4	
	$Z_j - C_j$		7/2	5/4	0	0	7/4	

Since all $Z_j - C_j \geq 0$, an optimum basic feasible solution to the auxiliary LPP has been attained. But since max Z^* is negative and the artificial variable A_1 appears in the basis at positive level, the original problem does not possess any feasible solution.

NOTES

Example 4.12

Maximize $Z = 5x_1 - 4x_2 + 3x_3$

Subject to, $2x_1 + x_2 - 6x_3 = 20$

$6x_1 + 5x_2 + 10x_3 \le 76$

$8x_1 - 3x_2 + 6x_3 \le 50$

$x_1, x_2, x_3 \ge 0$

Solution Introducing slack variables $S_1, S_2 \ge 0$ and an artificial variable $A_1 \ge 0$ in the constraints of the given LPP, the problem is reformulated in the standard form.

Initial basic feasible solution is given by $A_1 = 20$, $S_1 = 76$ and $S_2 = 50$.

Phase I

Assigning a cost –1 to the artificial variable A_1 and cost 0 to other variables, the objective function of the auxiliary LPP is,

Maximize $Z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1 \cdot A_1$

Subject to, $2x_1 + x_2 - 6x_3 + A_1 = 20$

$6x_1 + 5x_2 + 10x_3 + S_1 = 76$

$8x_1 - 3x_2 + 6x_3 + S_2 = 50$

$x_1, x_2, x_3, S_1, S_2, A_1 \ge 0$

		C_j	0	0	0	–1	0	0	
C_B	B	x_B	x_1	x_2	x_3	A_1	S_1	S_2	$Min \frac{x_B}{x_1}$
–1	A_1	20	2	1	–6	1	0	0	20/2 = 10
0	S_1	76	6	5	10	0	1	0	76/6 = 12.66
←0	S_2	50	8	–3	6	0	0	1	50/8 = 6.25
	Z_j	–20	–2	–1	6	–1	0	0	
	$Z_j - C_j$		–2↑	–1	6	0	0	0	

		C_j	0	0	0	–1	0	0	
C_B	B	x_B	x_1	x_2	x_3	A_1	S_1	S_2	$Min \frac{x_B}{x_2}$
← –1	A_1	15/2	0	7/4	–15/2	1	0	–1/4	30/7
0	S_1	77/2	0	29/4	11/2	0	1	–3/4	154/29
0	x_1	25/4	1	–3/8	3/4	0	0	1/8	—
	Z_j	–15/2	0	–7/4	15/2	–1	0	1/4	
	$Z_j - C_j$		0	–7/4↑	15/2	0	0	1/4	
0	x_2	30/7	0	1	–30/7	4/7	0	–1/7	
0	S_1	52/7	0	1	256/7	–29/7	1	2/7	
0	x_1	55/7	1	0	–6/7	3/4	0	1/14	
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$		0	0	0	0	1	0	

Since all $Z_j = C_j \geq 0$, an optimum solution to the auxiliary LPP has been obtained. Also Max $Z^* = 0$ with no artificial variables in the basis. We go to phase II.

NOTES

Phase II

Consider the final simplex table of phase I. Consider the actual cost associated with the original variables. Delete the artificial variable A_1 column from the table as it is eliminated in phase I.

		C_j	5	–4	3	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	S_2
–4	x_2	30/7	0	1	–30/7	0	–1/7
0	S_1	52/7	0	0	256/7	1	2/7
5	x_1	55/7	1	0	–6/7	0	1/14
	Z_j	155/7	5	–4	90/7	0	13/4
	$Z_j - C_j$	0	0	0	69/7	0	13/14

Since all $Z_j - C_j \geq 0$, an optimum basic feasible solution has been reached. Hence, an optimum feasible solution to the given LPP is $x_1 = 55/7$, $x_2 = 30/7$, $x_3 = 0$ and Max $Z = 155/7$.

Example 4.13 Solve by two-phase simplex method

Maximize $Z = -4x_1 - 3x_2 - 9x_3$

Subject to
$$2x_1 + 4x_2 + 6x_3 \geq 15$$
$$6x_1 + x_2 + 6x_3 \geq 12$$
$$x_1, x_2, x_3 \geq 0$$

Solution Convert the given LPP into standard form by introducing surplus variables S_1, S_2 and artificial variables A_1, A_2.The initial solution is given by $A_1 = 15$, $A_2 = 12$.

Phase I

Construct an auxiliary LPP by assigning a cost 0 to all the variables and –1 to each artificial variable subject to the given set of constraints, and it is given by

Max $Z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1A_1 - 1A_2$

Subject to, $2x_1 + 4x_2 + 6x_3 + S_1 + A_1 = 15$
$$6x_1 + x_2 + 6x_3 - S_2 + A_2 = 12$$

NOTES

C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	$Min \frac{x_B}{x_3}$
	C_j		0	0	0	0	0	–1	–1	
–1	A_1	15	2	4	6	–1	0	1	0	15/6
← –1	A_2	12	6	1	6	0	[1	0	1	12/6
	Z_j	–27	–8	–5	–12	1	1	–1	–1	
	$Z_j - C_j$		–8	–5	–12↑	1	1	0	0	
										$Min \frac{x_B}{x_2}$
–1	x_1	3	–4	3	0	–1	1	1	[1	3/3 = 1
0	x_3	2	1	1/6	1	0	–1/6	0	1/6	$\frac{2}{1/6} = 12$
	Z_j	–3	4	–3	0	1	–1	–1	1	
	$Z_j - C_j$			–3↑	0	1	–1	0	2	
0	x_2	1	–4/3	1	0	–1/3	1/3	1/3	–1/3	
0	x_3	11/6	22/18	0	1	1/18	–4/18	–1/18	4/18	
	Z_j	0	0	0	0	0	0	0		
	$Z_j - C_j$		0	0	0	1	1	1		

Since all $Z_j - C_j \geq 0$, the current basic feasible solution is optimal. Since Max $Z^* = 0$ and no artificial variable appears in the basis, we go to phase II.

Phase II

Consider the final simplex table of phase I; also consider the actual cost associated with the original variables. Delete the artificial variables A_1, A_2 column from the table as these variables are eliminated from the basis in phase I.

C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	$Min \frac{x_B}{x_1}$
	C_j		–4	–3	–9	0	0	
–3	x_2	$1 \frac{-4}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	—	
← –9	x_3	$\frac{11}{6}$	$\frac{22}{18}$	0	1	$\frac{1}{18}$	$\frac{-4}{18}$	$\frac{3}{2}$
	Z_j	$\frac{-39}{2}$	–1	–3	–9	$\frac{1}{2}$	1	
	$Z_j - C_j$	–3↑	0	0	$\frac{1}{2}$	1		
–3	x_2	30	1	$\frac{12}{11}$	$\frac{-3}{11}$	$\frac{-1}{11}$		
–4	x_1	$\frac{3}{2}$ 1	0	$\frac{18}{22}$	$\frac{1}{22}$	$\frac{4}{22}$		
	Z_j	–15	–4	–3	$\frac{-72}{11}$	$\frac{7}{11}$	1	
	$Z_j - C_j$	0	0	$\frac{27}{11}$	$\frac{7}{11}$	1		

Since all $Z_j - C_j \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is given by Max $Z = -15$, $x_1 = 3/2$, $x_2 = 3$, $x_3 = 0$.

Check Your Progress

1. What is simplex method?
2. Define simplex algorithm.
3. What is a two-phase simplex method?
4. What are the methods used to solve an LPP involving artificial variables?

NOTES

4.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Simplex method is an iterative procedure for solving LPP in a finite number of steps. It provides an algorithm, which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more, as the case may be, than at the previous vertex.
2. The simplex algorithm is the classical method to solve the optimization problem of linear programming.
3. The two-phase simplex method is another method to solve a given LPP involving some artificial variables. The solution is obtained in two phases.
4. The two methods used to solve an LPP are:
   (i) Big M method
   (ii) Two-phase simplex method

4.5 SUMMARY

- Simplex method is an iterative procedure for solving LPP in a finite number of steps. It provides an algorithm, which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more, as the case may be, than at the previous vertex.
- The simplex algorithm is the classical method to solve the optimization problem of linear programming.
- There are two methods used to solve an LPP involving artificial variables, namely:
  (i) Big M method
  (ii) Two-phase simplex method
- The two-phase simplex method is another method to solve a given LPP involving some artificial variables.

NOTES

4.6 KEY WORDS

- **Simplex Method:** It refers to an iterative procedure for solving LPP in a finite number of steps.
- **Charne's Big M Method:** It refers to surplus and artificial variables to convert all inequalities into that form. The Big 'M' refers to a large number associated with the artificial variables.
- **Pseudo Optimal Solution:** If any artificial variable is in the basis with nonzero value at the optimal solution of the augmented problem, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function, since it contains a very large penalty M and is called pseudo optimal solution.

4.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is pseudo optimal solution?
2. What are the phases involved in two-phase simplex method?

Long Answer Questions

1. Discuss the steps for the computation of an optimum solution using simplex algorithm.
2. What are steps involved in solving an LPP using the Big M method? Discuss.

4.8 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 5 DUALITY IN LINEAR PROGRAMMING PROBLEM

NOTES

Structure

5.0 INTRODUCTION

The concept of duality in linear programming states that every LPP (called the primal) is associated with another LPP (called its dual). Either of the problem can be considered as primal with the other as dual.

In this unit, you will study about the concept of duality, important results in duality and sensitivity analysis.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of duality
- Discuss the important results in duality
- Examine sensitivity analysis as the optimal solution of LP

5.2 CONCEPT OF DUALITY AND SENSITIVITY ANALYSIS

5.2.1 Importance of Duality Concept

Two reasons are attributed to the importance of the concept of duality:

1. If a large number of constraints and a lesser number of variables constitute the primal, then the procedure for computation can be minimized by converting the primal into its dual and then finding its solution.

2. While taking future decisions regarding the activities being programmed, the interpretation of the dual variables from the cost or economic point of view proves extremely useful.

NOTES

5.2.2 Formulation of a Dual Problem

In the formulation of a dual problem, it should be first converted into its canonical form. Formulation of a dual problem involves the following modifications:

(i) The objective function of minimization in the primal should be converted into that of maximization in the dual and vice versa.

(ii) The number of constraints in the dual will be equal to the number of variables in its primal and vice versa.

(iii) The right hand side constraints in the dual will be constituted by the cost coefficients C1, C2 ... Cn in the objective function of the primal and vice versa.

(iv) While forming the constraints for the dual, the transpose of the body matrix of the primal problem should be considered.

(v) The variables in both problems should be positive, that is, there should be no negative values.

(vi) If the variable in the primal is unrestricted in sign, then the corresponding constraint in the dual will be an equation and vice versa.

5.2.3 Economic Interpretation of Duality

Let the primal problem be:

$$\text{Max } Z = C_1x_1 + C_2x_2 + \ldots + C_nx_n$$

$$\begin{aligned}\text{Subject to: } \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1\\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2\\ & \vdots\\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m\\ & x_1, x_2 \cdots x_n \ge 0\end{aligned}$$

Dual: The dual problem is defined as:

$$\text{Min } Z^1 = b_1w_1 + b_2w_2 + \ldots + b_mw_m$$

$$\begin{aligned}\text{Subject to: } \quad & a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m \ge C_1\\ & a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \ge C_2\\ & \vdots\\ & a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_n \ge C_n\\ & w_1, w_2 \cdots w_m \ge 0\end{aligned}$$

Here, $w_1, w_2, w_3 \ldots w_m$ are called dual variables.

NOTES

Example 5.1: Write the dual of the following primal LP problem.

Max $Z = x_1 + 2x_2 + x_3$

Subject to:
$$2x_1 + x_2 - x_3 \leq 2$$
$$-2x_1 + x_2 - 5x_3 \geq -6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Since the problem is not in the canonical form, we interchange the inequality of the second constraint.

Max $Z = x_1 + 2x_2 + x_3$

Subject to:
$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$\text{and } x_1, x_2, x_3 \geq 0$$

Dual: Let w_1, w_2, w_3 be the dual variables.

Min $Z^1 = 2w_1 + 6w_2 + 6w_3$

Subject to:
$$2w_1 + 2w_2 + 4w_3 \geq 1$$
$$+w_1 - w_2 + w_3 \geq 2$$
$$-w_1 + 5w_2 + w_3 \geq 1$$
$$w_1, w_2, w_3 \geq 0$$

Example 5.2: Find the dual of the following LPP.

Max $Z = 3x_1 - x_2 + x_3$

Subject to:
$$4x_1 - x_2 \leq 8$$
$$8x_1 + x_2 + 3x_3 \geq 12$$
$$5x_1 - 6x_3 \leq 13$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Since the problem is not in the canonical form, we interchange the inequality of the second constraint.

Max $Z = 3x_1 - x_2 + x_3$

Subject to:
$$4x_1 - x_2 + 0x_3 \leq 8$$
$$-8x_1 - x_2 - 3x_3 \leq -12$$
$$5x_1 + 0x_2 - 6x_3 \leq 13$$
$$x_1, x_2, x_3 \geq 0$$

NOTES

Max Z = Cx

Subject to: $Ax \leq B$

$x \geq 0$

$$C = (3 - 11) \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} b = \begin{pmatrix} 8 \\ -12 \\ 13 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -8 & -1 & -3 \\ 5 & 0 & -6 \end{pmatrix}$$

Dual: Let w_1, w_2, w_3 be the dual variables. The dual problem is

Min $Z^1 = b^T W$

Subject to: $A^T W \geq C^T$ and $W \geq 0$

that is, Min $Z^1 = (8 \quad -12 \quad 13) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

Subject to: $\begin{pmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & 6 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \geq \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

Min $Z_1 = 8w_1 - 12w_2 + 13w_3$

Subject to: $4w_1 - 8w_2 + 5w_3 \geq 3$

$-w_1 - w_2 + 0w_3 \geq -1$

$0w_1 - 3w_2 + 6w_3 \geq 1$

$w_1, w_2, w_3 \geq 0$

Example 5.3: Write the dual of the following LPP.

Min $Z = 2x_2 + 5x_3$

Subject to: $x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0$

NOTES

Solution: Since the given primal problem is not in the canonical form, we interchange the inequality of the constraint. Also, the third constraint is an equation. This equation can be converted into two inequations.

Min $Z = 0x_1 + 2x_2 + 5x_3$

Subject to:
$$x_1 + x_2 + 0x_3 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq 6$$
$$x_1 - x_2 + 3x_3 \leq 4$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

Again on rearranging the constraint, we have

Min $Z = 0x_1 + 2x_2 + 5x_3$

Subject to:
$$x_1 + x_2 + 0x_3 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1, x_2, x_3 \geq 0$$

Dual: Since there are four constraints in the primal, we have four dual variables, namely w_1, w_2, w_3', w_3''.

Max $Z' = 2w_1 - 6w_2 + 4w_3' - 4w_3''$

Subject to:
$$w_1 - 2w_2 + w_3' - w_3'' \leq 0$$
$$w_1 - w_2 - w_3' + w_3'' \leq 2$$
$$0w_1 - 6w_2 + 3w_3' - 3w_3'' \leq 5$$
$$w_1, w_2, w_3', w_3'' \geq 0$$

Let $w_3 = w_3' - w_3''$

Max $Z' = 2w_1 - 6w_2 + 4(w_3' - w_3'')$

Subject to:
$$w_1 - 2w_2 + (w_3' - w_3'') \leq 0$$
$$w_1 - w_2 - (w_3' - w_3'') \leq 2$$

Finally, we have, $0w_1 - 6w_2 + 3(w_3' - w_3'') \le 5$

Max $Z' = 2w_1 - 6w_2 + 4w_3$

NOTES

Subject to: $w_1 - 2w_2 + w_3 \le 0$

$$w_1 - w_2 - w_3 \le 2$$

$$0w_1 - 6w_2 + 3w_3 \le 5$$

$w_1, w_2 \ge 0$, w_3 is unrestricted.

Example 5.4: Give the dual of the following problem.

Max $Z = x + 2y$

Subject to: $2x + 3y \ge 4$

$$3x + 4y = 5$$

$x \ge 0$ and y unrestricted.

Solution: Since the variable y is unrestricted, it can be expressed as $y = y' - y''$, $y', y'' \ge 0$. On reformulating the given problem, we have

Max $Z = x + 2(y' - y'')$

Subject to: $-2x - 3\,(y' - y'') \le -4$

$$3x + 4\,(y' - y'') \le 5$$

$$3x + 4\,(y' - y'') \ge 5$$

$$x, y', y'' \ge 0$$

Since the problem is not in the canonical form we rearrange the constraints.

Max $Z = x + 2y' - 2y''$

Subject to: $-2x - 3y' + 3y'' \le -4$

$$3x + 4y' - 4y'' \le 5$$

$$-3x - 4y' + 4y'' \le -5.$$

Dual: Since there are three variables and three constraints, in dual, we have three variables, namely w_1, w_2', w_2''.

Min $Z' = -4w_1 + 5w_2' - 5w_2''$

Subject to: $-2w_1 - 3w_2' - 3w_2'' \ge 1$

$$-3w_1 + 4w_2' - 4w_2'' \ge 2$$

$$3w_1 - 4w_2' + 4w_2'' \ge -2$$

$$w_1, w_2'', w_2'' \ge 0$$

NOTES

Let $w_2 = w_2' - w_2''$, so that the dual variable w_2 is unrestricted in sign. Finally, the dual is:

$$\text{Min } Z^1 = -4w_1 + 5\,(w_2' - 5w_2'')$$

$$\text{Subject to: } \begin{aligned} -2w_1 + 3\,(w_2' - w_2'') &\geq 1 \\ -3w_1 + 4\,(w_2' - w'') &\geq 2 \\ 3w_1 + 4\,(w_2' + w'') &\geq -2 \end{aligned}$$

that is, Min $Z^1 = -4w_1 + 5w_2$

$$\text{Subject to: } \begin{aligned} -2w_1 + 3w_2 &\geq 1 \\ -3w_1 + 4w_2 &\geq 2 \\ 3w_1 - 4w_2 &\geq -2 \end{aligned}$$

$w_1 \geq 0$ and w_2 is unrestricted.

that is, Min $Z^1 = -4w_1 + 5w_2$

$$\text{Subject to: } \begin{aligned} -2w_1 + 3w_2 &\geq 1 \\ -3w_1 + 4w_2 &\geq 2 \\ -3w_1 + 4w_2 &\leq 2 \end{aligned}$$

that is, Min $Z^1 = -4w_1 + 5w_2$

Subject to $-2w_1 + 3w_2 \geq 1$

$-3w_1 + 4w_2 = 2$, $w_1 \geq 0$ and w_2 is unrestricted.

Example 5.5: Write the dual of the following primal LPP.

$$\text{Min } Z = 4x_1 + 5x_2 - 3x_3$$

$$\text{Subject to: } \begin{aligned} x_1 + x_2 + x_3 &= 22 \\ 3x_1 + 5x_2 - 2x_3 &\leq 65 \\ x_1 + 7x_2 + 4x_3 &\geq 120 \end{aligned}$$

$x_1 + x_2 \geq 0$ and x_3 is unrestricted.

Solution: Since the variable x_3 is unrestricted, $x_3 = x_3' - x_3''$. Also, bring the problem into the canonical form by rearranging the constraints.

$$\text{Min } Z = 4x_1 + 5x_2 - 3\,(x_3' - x_3'')$$

$$\text{Subject to: } \begin{aligned} x_1 + x_2 + (x_3' - x_3'') &\leq 22 \\ x_1 + x_2 + x_3' - x_3'' &\geq 22 \\ -3x_1 - 5x_2 + 2\,(x_3' - x_3'') &\geq -65 \\ x_1 + 7x_2 + 4\,(x_3' - x_3'') &\geq 120 \\ x_1, x_2, x_3' - x_3'' &\geq 0 \end{aligned}$$

NOTES

Min $Z = 4x_1 + 5x_2 - 3x_3' + 3x_3''$

Subject to: $x_1 + x_2 + x_3' - x_3'' \geq 22$

$$-x_1 - x_2 - x_3' + x_3'' \geq -22$$

$$-3x_1 - 5x_2 + 2x_3' - 2x_3'' \geq -65$$

$$x_1 + 7x_2 + 4x_3' - 4x_3'' \geq 120$$

$$x_1, x_2, x_3' - x_3'' \geq 0$$

Dual: Since there are four constraints in the primal problem, in dual, there are four variables, namely w_1', w_2'', w_2, w_3 so that the dual is given by:

Max $Z' = 22(w_1' - w_1'') - 65w_2 + 120w_3$

Subject to: $w_1' - w_1'' - 3w_2 + w_3 \leq 4$

$$w_1' - w_1'' - 5w_2 + 7w_3 \leq 5$$

$$w_1' - w_1'' + 2w_2 + 4w_3 \leq -3$$

$$-w_1' + w_1'' - 2w_2 - 4w_3 \leq 3$$

$$-w_1' + w_1'' - 2w_2 - 4w_3 \leq 3$$

$$w_1', w_1'', w_2, w_3 \geq 0$$

Let $w_1 = w_1' - w_{11}''$, i.e., the variable w_1 is unrestricted.

$$\text{Max } Z' = 22(w_1' - w_1'') - 65w_2 + 120w_3$$

Subject to: $w_1' - w_1'' - 3w_2 + w_3 \leq 4$

$$w_1' - w_1'' - 5w_2 + 7w_3 \leq 5$$

$$-(w_1' - w_1'') - 2w_2 - 4w_3 \leq 3$$

$$-(w_1' - w_1'') - 2w_2 - 4w_3 \leq 3$$

that is, Max $Z' = 22w_1 - 65w_2 + 120w_3$

Subject to: $w_1 - 3w_2 + w_3 \leq 4$

$$w_1 - 5w_2 + 7w_3 \leq 5$$

$$-w_1 - 2w_2 - 4w_3 \geq 3$$

$$-w_1 - 2w_2 - 4w_3 \leq 3$$

Finally, we have:

Min $Z' = 22w_1 - 65w_2 + 120w_3$

Subject to: $w_1 - 3w_2 + 4w_3 \leq 4$

$$w_1 - 5w_2 + 7w_2 \leq 5$$

$$-w_1 - 2w_2 - 4w_3 = 3$$

$w_2, w_3 \geq 0$ and w_1 is unrestricted.

NOTES

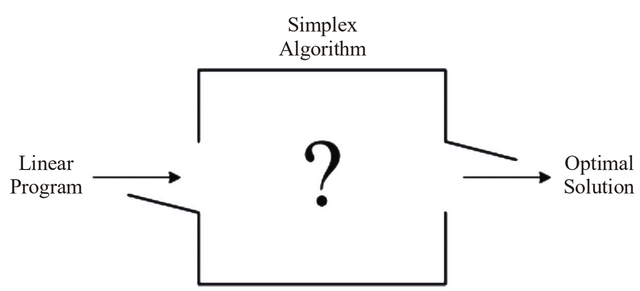
Important Results in Duality

Some of the important results in duality are as follows:

- A primal will always constitute the dual of a dual.
- In a dual, if one is a minimization problem, then the other will be a maximization problem.
- Both the LPP and its dual should have feasible solution, so as to arrive at an optimal solution.
- The fundamental duality theorem states that in the LPP if the primal problem has a finite optimal solution, then its dual also has a finite optimal solution. Moreover, both the problems have the same optimal values of their objective function, that is, Max Z = Min Z. The solution of the other problem can be read from the Zj – Cj row below the columns of slack and surplus variables.
- The existence theorem states that there will be no feasible solution for a problem if either problem in the LPP has an unbounded solution.
- The complementary slackness theorem states that:
  (i) If a primal variable is positive, then its dual constraint will be an equation at the optimum and vice versa.
  (ii) If a primal constraint is an inequality, then the corresponding dual variable is zero at the optimum and vice versa.

5.2.4 Sensitivity Analysis

The optimal solution of a linear programming problem is formulated using various methods. You have learned the use and importance of dual variables to solve the LPP. Here, you will learn how sensitivity analysis helps to solve repeatedly the real problem in a little different form. Generally, these scenarios crop up as an end result of parameter changes due to the involvement of new advanced technologies and the accessibility of well-organized latest information for key (input) parameters or the 'what-if' questions. Thus, **sensitivity analysis** helps to produce optimal solution of simple perturbations for the key parameters. For optimal solutions, consider the simplex algorithm as a 'black box' which accepts the input key parameters to solve LPP as shown below:



NOTES

Example 5.6: Illustrate sensitivity analysis using simplex method to solve the following LPP:

Maximize $Z = 20x_1 + 10x_2$

Subject to, $x_1 + x_2 \leq 3$

$3x_1 + x_2 \leq 7$

And $x_1, x_2 \geq 0$

Solution: Sensitivity analysis is done after making the initial and final tableau using the simplex method. Add slack variables to convert it into equation form.

Maximize $Z = 20x_1 + 10x_2 + 0S_1 + 0S_2$

Subject to: $x_1 + x_2 + S_1 + 0S_2 + = 3$

$3x_1 + x_2 + 0S_1 + S_2 = 7$

Where $x_1, x_2 \geq 0$

To find basic feasible solution, we put $x_1 = 0$ and $x_2 = 0$. This gives $Z = 0$, $S_1 = 3$ and $S_2 = 7$. The initial table will be as follows:

Initial table

		C_j	20	10	0	0	
C_B	B	X_B	X_1	X_2	S_1	S_2	Min $\frac{X_B}{X_i}$
0	S_1	3	1	1	1	0	3/1 = 3
←0	S_2	7	3	1	0	1	7/3 = 2.33
	Z_j	0	0	0	0	0	
	$Z_j - C_j$		–20↑	–10	0	0	

Find $\frac{X_B}{X_i}$ for each row and also find minimum for the second row. Here, $Z_j - C_j$ is maximum negative (–20). Hence, x_1 enters the basis and S_2 leaves the basis. It is shown with the help of arrows.

Key element is 3, key row is second row and key column is x_1. Now convert the key element into entering key by dividing each element of the key row by key element using the following formula:

New element = Old element

$$-\left[\frac{\text{Product of elements in the key row and key column}}{\text{Key element}}\right]$$

The following is the first iteration table:

		C_j	20	10	0	0	
C_B	B	X_B	X_1	X_2	S_1	S_2	
←0	S_1	2/3	0	2/3	1	–1/3	$\frac{2}{3}/\frac{2}{3}=1$
20	X_1	7/3	1	1/3	0	1	$\frac{7}{3}/\frac{1}{3}=7$
	Z_j	140/3	20	20/3	0	20	
	$Z_j - C_j$	–	0	–10/3↑	0	20	

Since $Z_j - C_j$ has one value less than zero, that is, negative value hence, this is not yet the optima solution. Value –10/3 is negative; hence, x_2 enters the basis and S_1 leaves the basis. Key row is upper row.

		C_j	20	10	0	0
C_B	B	X_B	X_1	X_2	S_1	S_2
10	X_2	1	0	1	3/2	–1/2
20	X_1	4/3	1	0	0	4/3
	Z_j	110/3	20	10	0	25
	$Z_j - C_j$		0	0	0	25

$Z_j - C_j \geq 0$ for all; hence, optimal solution is reached, where $x_1 = \frac{4}{3}, x_2 = 1,$

$$Z = \frac{80}{3} + 10 = \frac{110}{3}.$$

Check Your Progress

1. What are the modifications involved in the formulation of a dual problem?
2. State the important results in duality.
3. How does a sensitivity analysis help in producing optimal solution?

5.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

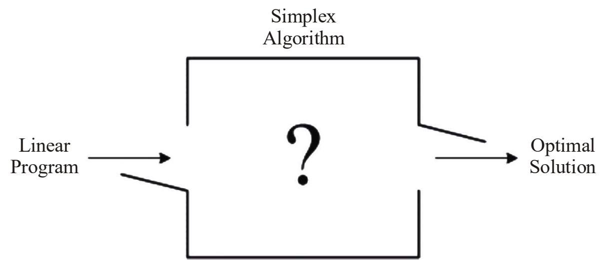
1. The modifications involved in the formulation of a dual problem are as follows:
   - The objective function of minimization in the primal should be converted into that of maximization in the dual and vice versa.

NOTES

NOTES

- The number of constraints in the dual will be equal to the number of variables in its primal and vice versa.
- If the variable in the primal is unrestricted in sign, then the corresponding constraint in the dual will be an equation and vice versa.

2. The important results in duality are as follows:
   - A primal will always constitute the dual of a dual.
   - In a dual, if one is a minimization problem, then the other will be a maximization problem.
   - Both the LPP and its dual should have feasible solution, so as to arrive at an optimal solution.
3. The sensitivity analysis helps to produce optimal solution of simple perturbations for the key parameters. For optimal solutions, consider the simplex algorithm as a 'black box' which accepts the input key parameters to solve LPP as shown below:



5.4 SUMMARY

- Every LPP (called the primal) is associated with another LPP (called its dual). Either of the problem can be considered as primal with the other as dual.
- In the formulation of a dual problem, it should be first converted into its canonical form.
- The number of constraints in the dual will be equal to the number of variables in its primal and vice versa.
- In a dual, if one is a minimization problem, then the other will be a maximization problem.
- The fundamental duality theorem states that in the LPP if the primal problem has a finite optimal solution, then its dual also has a finite optimal solution.

Moreover, both the problems have the same optimal values of their objective function, that is, Max Z = Min Z. The solution of the other problem can be read from the $Z_j - C_j$ row below the columns of slack and surplus variables.

- The complementary slackness theorem states that:
  - (i) If a primal variable is positive, then its dual constraint will be an equation at the optimum and vice versa.
  - (ii) If a primal constraint is an inequality, then the corresponding dual variable is zero at the optimum and vice versa.
- The sensitivity analysis helps to produce optimal solution of simple perturbations for the key parameters.

5.5 KEY WORDS

- **Duality:** It refers to the principle that optimization problems may be viewed from either of two perspectives, the primal problem or the dual problem.
- **Dual problem:** It refers to an LP defined directly and systematically from the primal LP model.
- **Sensitivity Analysis:** It refers to the study of how the uncertainty in the output of a mathematical model or system can be divided and allocated to different sources of uncertainty in its inputs.

5.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a brief note on the concept of duality.
2. Illustrate sensitivity analysis using simplex method to solve a hypothetical LPP.
3. What do you mean by complementary slackness theorem?

Long Answer Questions

1. What modifications are involved in the formulation of a dual problem? Discuss.
2. What do you understand by primal problem? Discuss the concept of primal problem with the help of suitable example.

5.7 FURTHER READINGS

NOTES

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 6 TRANSPORTATION PROBLEM

Structure

6.0 INTRODUCTION

The transportation problem is a subclass of a linear programming problem (LPP). Transportation problems deal with the objective of transporting various quantities of a single homogeneous commodity initially stored at various origins, to different destinations, in a way that keeps transportation cost at a minimum.

You will learn about the applications of the transportation problem and the solutions and rules to solve such problems. The solution of any transportation problem is obtained in two stages, namely initial solution and optimal solution. There are three methods of obtaining an initial solution. These are: North West Corner Rule, Least Cost Method and Vogel's Approximation Method (VAM). VAM is preferred since the solution obtained this way is very close to the optimal solution. The optimal solution of any transportation problem is a feasible solution that minimizes the total cost. An optimal solution is the second stage of a solution obtained by improving the initial solution. Modified Distribution (MODI) method is used to obtain optimal solutions and optimality tests.

6.1 OBJECTIVES

After going through this unit, you will be able to:

- Identify a transportation problem
- Discuss the various transportation models
- Explain the optimum solution using MODI method
- Explain the various special cases of a transportation problem

6.2 TRANSPORTATION PROBLEM

NOTES

The **transportation problem** (TP) is one of the subclasses of LPP (Linear Programming Problem) in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum. To achieve this objective, we must know the amount and location of available supplies and the quantities demanded. In addition, we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

6.2.1 Formulation of Transportation Problem (TP)

Consider a transportation problem with m origins (rows) and n destinations (columns). Let C_{ij} be the cost of transporting one unit of the product from the ith origin to jth destination, a_i the quantity of commodity available at origin i, and b_j the quantity of commodity needed at destination j. X_{ij} is the quantity transported from ith origin to jth destination. This transportation problem can be stated in the following tabular form.

Destinations

Origins	1	2	3	…	n	Capacity
1	C_{11} X_{11}	C_{12} X_{12}	C_{13} X_{13}	… …	C_{1n} X_{1n}	a_1
2	C_{21} X_{21}	C_{22} X_{22}	C_{23} X_{23}	… …	C_{2n} X_{2n}	a_2
3	C_{31} X_{31}	C_{32} X_{32}	C_{33}	… …	C_{3n} X_{3n}	a_3
m	C_{m1} X_{m1}	C_{m2} X_{m2}	C_{m3} X_{m3}	… …	C_{mn} X_{mn}	a_m
Demand	b_1	b_2	b_3	…	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The linear programming model representing the transportation problem is given by,

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^{n} X_{ij} = a_i \qquad i = 1, 2, ..., n$$

(Row Sum)

$$\sum_{i=1}^{m} X_{ij} = b_j \qquad j = 1, 2, ..., n$$

(Column Sum)

$$X_{ij} \geq 0 \quad \text{For all } i \text{ and } j$$

The given transportation problem is said to be balanced if,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

that is, the total supply is equal to the total demand.

6.2.2 Initial Basic Feasible Solution

Feasible solution: Any set of non-negative allocations (X_{ij}>0) which satisfies the row and column sum (rim requirement) is called a feasible solution.

Basic feasible solution: A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to $m + n - 1$, where m is the number of rows and n the number of columns in a transportation table.

Non-degenerate basic feasible solution: Any feasible solution to a transportation problem containing m origins and n destinations is said to be non-degenerate, if it contains $m + n - 1$ occupied cells and each allocation is in independent positions.

The allocations are said to be in independent positions if it is impossible to form a closed path. Closed path means by allowing horizontal and vertical lines and when all the corner cells are occupied.

The allocations in the following tables are not in independent positions.

	*	*
	*	*

*		*
*		*

	*	*	
	*		
	*	*	

The allocations in the following tables are in independent positions.

	*	
*	*	*
*		

*	*		
	*		*
		*	*

Degenerate basic feasible solution: If a basic feasible solution contains less than $m + n - 1$ non-negative allocations, it is said to be degenerate.

6.2.3 Moving Towards Optimality

An **optimal solution** is a feasible solution (not necessarily basic) which minimizes the total cost.

NOTES

The solution of a transportation problem (TP) can be obtained in two stages, namely initial solution and optimum solution.

Initial solution can be obtained by using any one of the three methods, namely:

- North West Corner Rule (NWCR)
- Least Cost Method or Matrix Minima Method
- Vogel's Approximation Method (VAM)

VAM is preferred over the other two methods, since the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

The cells in the transportation table can be classified into occupied cells and unoccupied cells. The allocated cells in the transportation table are called *occupied cells* and the *empty cells* in the transportation table are called *unoccupied cells*.

The improved solution of the initial basic feasible solution is called optimal solution which is the second stage of solution, that can be obtained by MODI.

North West Corner Rule

The following steps explain the North West Corner Rule:

Step 1: Starting with the cell at the upper left corner (North West) of the transportation matrix, we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied, that is, $X_{11} = \text{Min}\ (a_1,b_1)$.

Step 2: If $b_1>a_1$, we move down vertically to the second row and make the second allocation of magnitude $X_{22} = \text{Min}(a_2\ b_1 – X_{11})$ in the cell (2, 1).

If $b_1<a_1$, move right horizontally to the second column and make the second allocation of magnitude $X_{12} = \text{Min}\ (a_1, X_{11} – b_1)$ in the cell (1, 2).

If $b_1 = a_1$, there is a tie for the second allocation. We make the second allocations of magnitude,

$$X_{12} = \text{Min}\ (a_1 - a_1, b_1) = 0 \text{ in the cell } (1,2)$$
$$\text{or, } X_{21} = \text{Min}\ (a_2, b_1 - b_1) = 0 \text{ in the cell } (2,1)$$

Step 3: Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

Example 6.1: Obtain the initial basic feasible solution of a transportation problem whose cost and rim requirement table is as follows:

NOTES

Origin\Destination	D_1	D_2	D_3	*Supply*
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	34

Solution: Since $\Sigma a_i = 34 = \Sigma b_j$, there exists a feasible solution to the transportation problem. We obtain the initial feasible solution as follows:

The first allocation is made in the cell (1, 1), the magnitude being

X_{11} = Min (5, 7) = 5.

The second allocation is made in the cell (2, 1) and the magnitude of the allocation is given by X_{21} = Min (8, 7 – 5) = 2.

	D_1	D_2	D_3	*Supply*
O_1	2 5	7	4	~~5~~ 0
O_2	3 2	3 6	1	~~8~~ ~~6~~ 0
O_3	5	4 3	7 4	~~7~~ ~~4~~ 0
O_4	1	6	2 14	~~14~~ 0
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	34

The third allocation is made in the cell (2, 2) the magnitude X_{22} = Min (8 – 2, 9) = 6.

The magnitude of the fourth allocation is made in the cell (3, 2) given by X_{32} = Min (7, 9 – 6) = 3.

The fifth allocation is made in the cell (3, 3) with magnitude X_{33} = Min (7 – 3, 14) = 4.

The final allocation is made in the cell (4, 3) with magnitude X_{43} = Min (14, 18 – 4) = 14.

Hence, we get the initial basic feasible solution to the given TP and is given by,

$X_{11} = 5$; $X_{21} = 2$; $X_{22} = 6$; $X_{32} = 3$; $X_{33} = 4$; $X_{43} = 14$

Total Cost = 2 × 5 + 3 × 2 + 3 × 6 + 3 × 4 + 4 × 7 + 2 × 14

=10 + 6 + 18 + 12 + 28 + 28 = ₹ 102

Example 6.2: Determine an initial basic feasible solution to the following transportation problem using North West Corner Rule.

NOTES

	D_1	D_2	D_3	D_4	*Supply*
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Required	6	10	15	4	35

Solution: The problem is a balanced TP, as the total supply is equal to the total demand. Using the steps, we find the initial basic feasible solution as given in the following table.

	D_1	D_2	D_3	D_4	*Supply*
O_1	$\boxed{6}$ 6	$\boxed{4}$ 8	$\boxed{1}$	$\boxed{5}$	~~14~~ ~~8~~ 0
O_2	$\boxed{8}$	$\boxed{9}$ 2	$\boxed{2}$ 14	$\boxed{7}$	~~16~~ ~~14~~ 0
O_3	$\boxed{4}$	$\boxed{3}$	$\boxed{6}$ 1	$\boxed{2}$ 4	~~5~~ ~~4~~
Demand	~~6~~	~~10~~ ~~2~~ 0	~~15~~ ~~1~~ 0	~~4~~	35

The solution is given by,

$X_{11} = 6; X_{12} = 8; X_{22} = 2; X_{23} = 14; X_{33} = 1; X_{34} = 4$

Total Cost $= 6 \times 6 + 4 \times 8 + 2 \times 9 + 2 \times 14 + 6 \times 1 + 2 \times 4$

$= 36 + 32 + 18 + 28 + 6 + 8 =$ ₹ 128

Least Cost or Matrix Minima Method

The following methods discuss the Least Cost or Matrix Minima Method:

Step 1 : Determine the smallest cost in the cost matrix of the transportation table. Let it be C_{ij}. Allocate $X_{ij} = \text{Min } (a_i, b_j)$ in the cell (i, j).

Step 2: If $X_{ij} = a_i$ cross off the ith row of the transportation table and decrease b_j by a_i. Then go to Step 3.

If $X_{ij} = b_j$ cross off the jth column of the transportation table and decrease a_i by b_j. Go to Step 3.

If $X_{ij} = a_i = b_j$ cross off either the ith row or the jth column, but not both.

Step 3: Repeat steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Example 6.3: Obtain an initial feasible solution to the following TP using Matrix Minima Method.

	D_1	D_2	D_3	D_4	*Supply*
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24

Solution: Since $\Sigma a_i = \Sigma b_j = 24$, there exists a feasible solution to the TP using the steps in the least cost method, the first allocation is made in the cell (3, 1) the magnitude being $X_{31} = 4$. This satisfies the demand at the destination D_1 and we delete this column from the table as it is exhausted.

NOTES

	D_1	D_2	D_3	D_4	Supply
O_1	1	2 (6)	3	4	~~6~~ 0
O_2	4	3	2 (2)	0 (6)	~~8~~ ~~2~~
O_3	0 (4)	2	2 (6)	1	~~10~~ ~~6~~
Demand	~~4~~ 0	~~6~~ 0	~~8~~ ~~2~~ 0	~~6~~ 0	24

The second allocation is made in the cell (2, 4) with magnitude X_{24} = Min (6, 8) = 6. Since it satisfies the demand at the destination D_4, it is deleted from the table. From the reduced table, the third allocation is made in the cell (3, 3) with magnitude X_{33} = Min (8, 6) = 6. The next allocation is made in the cell (2, 3) with magnitude X_{23} of Min (2, 2) = 2. Finally, the allocation is made in the cell (1, 2) with magnitude X_{12} = Min (6, 6) = 6. Now, all the requirements have been satisfied and hence, the initial feasible solution is obtained.

The solution is given by,

$X_{12} = 6;\ X_{23} = 2;\ X_{24} = 6;\ X_{31} = 4;\ X_{33} = 6$

Since the total number of occupied cells = 5 < $m + n - 1$

We get a degenerate solution.

Total cost = $6 \times 2 + 2 \times 2 + 6 \times 0 + 4 \times 0 + 6 \times 2$

= 12 + 4 + 12 = ₹ 28

Example 6.4: Determine an initial basic feasible solution for the following TP, using the Least Cost Method.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	35

Solution: Since $\Sigma a_i = \Sigma b_j$, there exists a basic feasible solution. Using the steps in least cost method, we make the first allocation to the cell (1, 3) with magnitude X_{13} = Min (14, 15) = 14 (as it is the cell having the least cost).

This allocation exhausts the first row supply. Hence, the first row is deleted. From the reduced table, the next allocation is made in the next least cost cell (2, 3) which is chosen arbitrarily with magnitude X_{23} = Min (1, 16) = 1. This exhausts the third column destination.

NOTES

From the reduced table, the next least cost cell is (3, 4) for which allocation is made with magnitude Min (4, 5) = 4. This exhausts the destination D_4 requirement. Delete this fourth column from the table. The next allocation is made in the cell (3, 2) with magnitude X_{32} = Min (1, 10) = 1 which exhausts the third origin capacity. Hence, the third row is exhausted. From the reduced table, the next allocation is given to the cell (2,1) with magnitude X_{21} = Min (6, 15) = 6. This exhausts the first column requirement. Hence, it is deleted from the table.

Finally, the allocation is made to the cell (2, 2) with magnitude X_{22} = Min (9, 9) = 9 which satisfies the rim requirement. These allocations are shown in the following transportation table:

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1 14	5	~~14~~
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	~~15~~ 1	4	

(I Allocation)

	D_1	D_2	D_3	D_4	Supply
O_2	8	9	2 1	7	~~16~~ 15
O_3	4	3	6	2	5
Demand	6	10	~~1~~ 0	4	

(II Allocation)

	D_1	D_2	D_4	Supply
O_2	8	9	7	15
O_3	4	3	2 4	~~5~~ 1
Demand	6	10	~~4~~ 0	

(III Allocation)

	D_1	D_2	Supply
O_2	8	9	15
O_3	4	3 1	~~1~~ 0
Demand	6	~~10~~ 9	

(IV Allocation)

	D_1	D_2	Supply
O_2	8 6	9 9	15
Demand	6	9	

(V, VI Allocation)

The following table gives the initial basic feasible solution.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1 14	5	14
O_2	8 6	9 9	2 1	7	16
O_3	4	3 1	6	2 4	5
Demand	6	10	15	4	

The solution is given by,

X_{13}=14; X_{21}=6; X_{22}= 9; X_{23}= 1; X_{32}= 1; X_{34}= 4

Transportation cost,

$= 14 \times 1 + 6 \times 8 + 9 \times 9 + 1 \times 2 + 3 \times 1 + 4 \times 2$

$= 14 + 48 + 81 + 2 + 3 + 8 =$ ₹ 156

Vogel's Approximation Method (VAM)

The steps involved in Vogel's Approximation Method (VAM) for finding the initial solution are as follows:

Step 1: Find the penalty cost, namely the difference between the smallest and the next smallest costs in each row and column.

Step 2: Among the penalties as found in Step (1), choose the maximum penalty. If this maximum penalty is more than one (that is, if there is a tie), choose any one arbitrarily.

Step 3: In the selected row or column as by Step (2), find out the cell having the least cost. Allocate to this cell as much as possible depending on the capacity and requirements.

Step 4: Delete the row or column which is fully exhausted. Again, compute the column and row penalties for the reduced transportation table and then go to Step (2). Repeat the procedure until all the rim requirements are satisfied.

Note: If the column is exhausted, then there is a change in row penalty and vice versa.

Example 6.5: Find the initial basic feasible solution for the following transportation problem using VAM.

		Destination				
		D_1	D_2	D_3	D_4	***Supply***
Origin	O_1	11	13	17	14	250
	O_2	16	18	14	10	300
	O_3	21	24	13	10	400
	Demand	200	225	275	250	950

Solution: Since $\Sigma a_i = \Sigma b_j = 950$, the problem is balanced and there exists a feasible solution to the problem.

First, we find the row and column penalty P_{I} as the difference between the least and the next least cost. The maximum penalty is 5. Choose the first column arbitrarily. In this column, choose the cell having the least cost name (1, 1). Allocate to this cell with minimum magnitude (that is, Min (250, 200) = 200.) This exhausts the first column. Delete this column. Since a column is deleted, there is a change in row penalty P_{II} while column penalty P_{II} remains the same. Continuing in this manner, we get the remaining allocations as given in the following table:

NOTES

I Allocation

	D_1	D_2	D_3	D_4	Supply	P_I
O_1	11 200	13	17	14	50 ~~250~~	2
O_2	16	18	14	10	300	4
O_3	21	24	13	10	400	3
Demand	~~200~~ 0	225	275	250		
P_I	5↑	5	1	0		

II Allocation

	D_2	D_3	D_4	Supply	P_{II}
O_1	13 50	17	14	~~50~~	1
O_2	18	14	10	300	4
O_3	24	13	10	400	3
Demand	~~225~~ 175	275	250		
P_{II}	5↑	1	0		

III Allocation

	D_2	D_3	D_4	Supply	P_{III}
O_2	18 175	14	10	~~300~~ 125	4
O_3	24	13	10	400	3
Demand	~~175~~ 0	275	250		
P_{III}	6↑	1	0		

IV Allocation

	D_3	D_4	Supply	P_{IV}
O_2	14	10 125	~~125~~ 0	4 ←
O_3	13	10	400	3
Demand	275	~~250~~ 125		
P_{IV}	1	0		

V Allocation

	D_3	D_4	Supply	P_V
O_3	13 275	10	~~400~~ 125	3
Demand	~~275~~ 0	125		
P_V	13↑	10		

VI Allocation

	D_4	Supply	P_{VI}
O_3	10 125	~~125~~ 0	10 ←
Demand	~~125~~ 0		
P_{VI}	10		

Finally, we arrive at the initial basic feasible solution which is shown in the following table:

	D_1	D_2	D_3	D_4	Supply
O_1	11 200	13 50	17	14	250
O_2	16	18 175	14	10 125	300
O_3	21	24	13 275	10 125	400
Demand	200	225	275	250	

There are six positive independent allocations which equals to $m + n - 1 = 3 + 4 - 1$. This ensures that the solution is a non-degenerate basic feasible solution.

Transportation cost

$= 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125$

$=$ ₹ 12,075

Example 6.6: Find the initial solution to the following TP using VAM.

		Destination				
		D_1	D_2	D_3	D_4	***Supply***
Factory	F_1	3	3	4	1	100
	F_2	4	2	4	2	125
	F_3	1	5	3	2	75
	Demand	120	80	75	25	300

Solution: Since $\Sigma a_i = \Sigma b_j$, the problem is a balance TP. Hence, there exists a feasible solution.

	D_1	D_2	D_3	D_4	*Supply*	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
F_1	3 (45)	3	4 (30)	1 (25)	100	2	2 ←	0	1 ←	4	4
F_2	4	2 (80)	4 (45)	2	125	0	0	2 ←	0	4 ←	
F_3	1 (75)	5	3	2	75	1		←			
Demand	120	80	75	25							
P_I	2↑	1	1	1							
P_{II}	1	1	0	1							
P_{III}	1	1	0								
P_{IV}	1		0								
P_V			0								
P_{VI}			4↑								

Finally, we have the initial basic feasible solution as given in the following table.

	D_1	D_2	D_3	D_4	***Supply***
F_1	3 (45)	3	4 (30)	1 (25)	100
F_2	4	2 (80)	4 (45)	2	125
F_3	1 (75)	5	3	2	75
Demand	120	80	75	25	

There are six independent non-negative allocations equal to $m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non-degenerate basic feasible.

Transportation cost

= 3 × 45 + 4 × 30 + 1× 25 + 2 × 80 + 4 × 45 + 1 + 75

= 135 + 120 + 25 + 160 + 180 + 75

= ₹ 695

NOTES

6.2.4 Transportation Algorithm (MODI) Method

In this section, we will discuss the Optimum Solution Using MODI Method.

Optimality Test

Once the initial basic feasible solution has been computed, the next step in the problem is to determine whether the solution obtained is optimum or not.

Optimality test can be conducted to any initial basic feasible solution of a TP, provided such allocations has exactly $m + n - 1$ non-negative allocations, where m is the number of origins and n is the number of destinations. Also, these allocations must be in independent positions.

To perform this optimality test, we shall discuss the modified distribution method (MODI). The various steps involved in the MODI method for performing optimality test are as follows.

MODI Method

Step 1: Find the initial basic feasible solution of a TP by using any one of the three methods.

Step 2: Find out a set of numbers u_i and v_j for each row and column satisfying $u_i + v_j = C_{ij}$ for each occupied cell. To start with, we assign a number '0' to any row or column having the maximum number of allocations. If this maximum number of allocations is more than 1, choose any one arbitrarily.

Step 3: For each empty (unoccupied) cell, we find the sum u_i and v_j written in the bottom left corner of that cell.

Step 4: Find out for each empty cell the net evaluation value $\Delta_{ij} = C_{ij} - (u_i + v_j)$ and which is written at the bottom right corner of that cell. This step gives the optimality conclusion as follows:

- If all $\Delta_{ij} > 0$ (i.e., all the net evaluation values), the solution is optimum and a unique solution exists.
- If $\Delta_{ij} \geq 0$, then the solution is optimum, but an alternate solution exists.
- If at least one $\Delta_{ij} < 0$, the solution is not optimum. In this case, we go to the next step to improve the total transportation cost.

Step 5: Select the empty cell having the most negative value of Δ_{ij}. From this cell, we draw a closed path by drawing horizontal and vertical lines with the corner

NOTES

cells occupied. Assign signs '+' and '–' alternately and find the minimum allocation from the cell having the negative sign. This allocation should be added to the allocation having the positive sign and subtracted from the allocation having the negative sign.

Step 6: The previous step yields a better solution by making one (or more) occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocations, repeat from Step (2) till an optimum basic feasible solution is obtained.

Example 6.7: Solve the following transportation problem.

		Destination				
		P	***Q***	***R***	***S***	***Supply***
Source	*A*	21	16	25	13	11
	B	17	18	14	23	13
	C	32	17	18	41	19
	Demand	6	10	12	15	43

Origin\Dest	*P*	*Q*	*R*	*S*	*Supply*	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
A	21	16	25	13 11	11	3	–	–	–	–	–
B	17 6	18	14 3	23 4	13	4	4	4	4 ←	–	–
C	32	17 10	18 9	48	19	1	1	1	1	1	17
Demand	6	10	12	15	43						
P_I	4	1	4	10↑							
P_{II}	15	1	4	18↑							
P_{III}	15↑	1	4	–							
P_{IV}	–	1	4	–							
P_V	–	17	18↑	–							
P_{VI}	–	17↑	–	–							

Solution: We first find the initial basic feasible solution by using VAM. Since $\Sigma a_i = \Sigma b_j$, the given TP is a balanced one. Therefore, there exists a feasible solution.

Finally, we have the initial basic feasible solution as given in the following table.

		Destination			
		P	***Q***	***R***	***S***
Source	*A*	21	16	23	13 11
	B	17 6	18	14 3	23 4
	C	32	17 10	18 9	41

NOTES

From this table, we see that the number of non-negative independent allocations is $6 = m+ n - 1 = 3 + 4 - 1$.

Hence, the solution is non-degenerate basic feasible.

Therefore, the initial transportation cost

$= 11 \times 13 + 3 \times 14 + 4 \times 23 + 6 \times 17 + 17 \times 10 + 18 \times 9 =$ ₹ 711

To find the optimal solution: We apply the MODI method in order to determine the optimum solution. We determine a set of numbers u_i and v_j for each row and column, with $u_i+v_j= C_{ij}$ for each occupied cell. To start with, we give $u_2 = 0$ as the second row has the maximum number of allocation.

Now, we find the sum u_i and v_j for each empty cell and enter at the bottom left corner of that cell.

$C_{21} = u_2 + v_1 = 17 = 0 + v_1 = 17 \Rightarrow v_1 = 17$

$C_{23} = u_2 + v_3 = 14 = 0 + v_3 = 14 \Rightarrow v_3 = 14$

$C_{24} = u_2 + v_4 = 23 = 0 + v_4 = 23 \Rightarrow v_4 = 23$

$C_{14} = u_1 + v_4 = 13 = u_1 + 23 = 13 \Rightarrow u_1 = 10$

$C_{33} = u_3 + v_3 = 18 = u_3 + 14 = 18 \Rightarrow u_3 = 4$

Next, we find the net evaluations $\Delta_{ji} = C_{ij} - (u_i + v_j)$ for each unoccupied cell and enter at the bottom right corner of that cell.

Initial table

	P	Q	R	S	U_i
A	$\boxed{21}$ 7 14	$\boxed{16}$ 3 13	$\boxed{23}$ 4 21	$\boxed{13}$ 11	$U_i = 10$
B	$\boxed{17}$ 6	$\boxed{18}$ 13 5	$\boxed{14}$ 3	$\boxed{23}$ 4	$U_2=0$
C	$\boxed{32}$ 21 9	$\boxed{17}$ 10	$\boxed{18}$ 9	$\boxed{41}$ 25 16	$U_3=4$
V_j	$V_1=17$	$V_2=13$	$V_3=14$	$V_4=23$	

Since all $\Delta_{ij} > 0$, the solution is optimal and unique. The optimum solution is given by,

$X_{14} = 11; X_{21} = 6; X_{23} = 3; X_{24} = 4; X_{32} = 10; X_{33} = 9$

The minimum transportation cost

$= 11 \times 13 + 17 \times 6 + 3 \times 14 + 4 \times 23 + 10 \times 17 + 9 \times 18 =$ ₹ 711

Example 6.8: Solve the following transportation problem starting with the initial solution obtained by VAM.

NOTES

	D_1	D_2	D_3	D_4	Supply
O_1	2 ③	2	2	1	3
O_2	10	8	5 ③	4 ④	7
O_3	7 ①	6 ③	6 ①	8	5
Demand	4	3	4	4	15

Solution: Since $\Sigma a_i = \Sigma b_j$, the problem is a balanced TP. Therefore, there exists a feasible solution.

	D_1	D_2	D_3	D_4	Supply	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
O_1	2 ③	2	2	1	3	1	–	–	–	–	–
O_2	10	8	5 ③	4 ④	7	1	1	3 ←	–	–	–
O_3	7 ①	6 ③	6 ①	8	5	0	0	0	0	0	6 ←
Demand	4	3	4	4	15						
P_I	5↑	4	4	3							
P_{II}	3	2	1	4↑							
P_{III}	3	2	1	–							
P_{IV}	7↑	6	6	–							
P_V	–	6↑	6	–							
P_{VI}	–	–	6	–							

Finally, the initial basic feasible solution is given as follows:

	D_1	D_2	D_3	D_4	Supply
O_1	2 ③	2	2	1	3
O_2	10	8	5 ③	4 ④	7
O_3	7 ①	6 ③	6 ①	8	5
Demand	4	3	4	4	15

Since the number of occupied cells = 6 = $m + n - 1$ and are also independent, there exists a non-degenerate basic feasible solution.

The initial transportation cost

$= 3 \times 2 + 3 \times 5 + 4 \times 4 + 7 \times 1 + 6 \times 3 + 6 \times 1 =$ ₹ 68

To find the optimal solution, applying the MODI method, we determine a set of numbers u_i and v_j for each row and column, such that $u_i + v_j = C_{ij}$ for each occupied cell. Since the third now has the maximum number of allocations, we give number $u_3 = 0$. The remaining numbers can be obtained as follows:

$$C_{31} = u_3 + v_1 = 7 = 0 + v_1 = 7 \Rightarrow v_1 = 7$$

$$C_{32} = u_3 + v_2 = 6 = 0 + v_2 = 6 \Rightarrow v_2 = 6$$

$$C_{33} = u_3 + v_3 = 6 = 0 + v_3 = 6 \Rightarrow v_3 = 6$$

NOTES

$C_{23} = u_2 + v_3 = 5 = u_2 + 6 = 5 \Rightarrow u_2 = -1$

$C_{24} = u_2 + v_4 = 4 = -1 + v_4 = 4 \Rightarrow v_4 = 5$

$C_{11} = u_1 + v_1 = 2 = u_1 + 7 = 2 \Rightarrow u_1 = -5$

We find the sum u_i and v_j for each empty cell and enter at the bottom left corner of the cell. Next, we find the net evaluation Δ_{ij} given by $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for each empty cell and enter at the bottom right corner of the cell.

Initial table

	D_1	D_2	D_3	D_4	u_i
O_1	[2] ③	[2] 1 1	[2] 1 1	[1] 0 1	u_1=–5
O_2	[10] 6 4	[8] 5 3	[5] ③	[4] ④	u_2=–1
O_3	[7] ①	[6] ③	[6] ①	[8] 5 3	u_3=0
	$v_1 = 7$	$v_2 = 6$	$v_3 = 6$	$v_4 = 5$	

Since all $\Delta_{ij} > 0$, the solution is optimum and unique. The solution is given by,

$X_{11} = 3; X_{23} = 3; X_{24} = 4, X_{31} = 1; X_{32} = 3; X_{33} = 1$

The total transportation cost

$= 2 \times 3 + 3 \times 5 + 4 \times 4 + 7 \times 1 + 6 \times 3 + 6 \times 1 =$ ₹ 68

Check Your Progress

1. What is a transportation problem?
2. Define feasible, basic, non-degenerate solutions of a transportation problem.
3. List the approaches used with transportation problems for determining the starting solution.
4. State the optimal solution to a transportation problem.
5. What is the purpose of the MODI method?
6. State the two conditions necessary for an alternate solution.

6.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A transportation problem deals with transportation of various quantities of a single homogeneous commodity initially stored at various origins to different destinations at the minimum cost.

NOTES

2. Definitions of feasible, basic, non-degenerate solutions of a transportation problem are as follows:
   - **Feasible solution:** Any set of non-negative allocations which satisfies the row and column sum (rim requirement) is called a feasible solution.
   - **Basic feasible solution:** A feasible solution is called a basic feasible solution if the number of non-negative allocation is equal to $m+n-1$, where m is the number of rows and n the number of columns in a transportation table.
   - **Non-degenerate basic feasible solution:** A feasible solution to a transportation problem containing m origins and n destinations is said to be non-degenerate, if it contains $m+n-1$ occupied cells and each allocation is in independent positions.
3. The approaches used with transportation problems for determining the starting solution are as follows:
   - North West Corner Rule
   - Least Cost Method (Matrix Minima)
   - Vogel's Approximation Method
4. An optimal solution to a transportation problem is one which minimizes the total transportation cost.
5. The purpose of the MODI method is to get the optimal solution of a transportation problem.
6. The two conditions necessary for an alternate solution are as follows:
   (i) The slope of the objective function should be similar as constraint which forms the edge of the feasible region.
   (ii) The constraint must be an active constraint, that is, in the path of optimal progress of the objective function.

6.4 SUMMARY

- The transportation problem (TP) is one of the subclasses of LPP (Linear Programming Problem) in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum.
- Any set of non-negative allocations ($X_{ij} > 0$) which satisfies the row and column sum (rim requirement) is called a feasible solution.
- A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to $m + n - 1$, where m is the number of rows and n the number of columns in a transportation table.

NOTES

- Any feasible solution to a transportation problem containing m origins and n destinations is said to be non-degenerate, if it contains $m+n-1$ occupied cells and each allocation is in independent positions.
- The allocations are said to be in independent positions if it is impossible to form a closed path. Closed path means by allowing horizontal and vertical lines and when all the corner cells are occupied.
- If a basic feasible solution contains less than $m + n - 1$ non-negative allocations, it is said to be degenerate basic feasible solution.
- An **optimal solution** is a feasible solution (not necessarily basic) which minimizes the total cost.
- The solution of a transportation problem (TP) can be obtained in two stages, namely initial solution and optimum solution.
- The cells in the transportation table can be classified into occupied cells and unoccupied cells. The allocated cells in the transportation table are called *occupied cells* and the *empty cells* in the transportation table are called *unoccupied cells*.
- Optimality test can be conducted to any initial basic feasible solution of a TP, provided such allocations has exactly $m+n-1$ non-negative allocations, where m is the number of origins and n is the number of destinations. Also, these allocations must be in independent positions. To perform this optimality test, modified distribution method (MODI) is used.
- MODI method is applied in order to determine the optimum solution. One can determine a set of numbers u_i and v_j for each row and column, with $u_i + v_j = C_{ij}$ for each occupied cell. To start with, we give $u_2 = 0$ as the second row has the maximum number of allocation.

6.5 KEY WORDS

- **Unbalanced Transportation:** The given TP is said to be unbalanced if the total supply is not equal to the total demand.
- **Vogel's Approximation Method (VAM):** It refers to iterative procedure calculated to find out the initial feasible solution of the transportation problem.
- **MODI Method:** The Modified Distribution Method or MODI is an efficient method of checking the optimality of the initial feasible solution.

6.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short Answer Questions

1. What are the numbers of non-basic variables for 4 rows and 5 columns?
2. While dealing with North West Corner rule, when does one move to the next cell in next column?
3. What is the coefficient of X_{ij} of constraints in a transportation problem?
4. When does degeneracy occur in an $m \times n$ transportation problem?

Long Answer Questions

1. What do you understand by transportation model?
2. Explain the following with examples:
   (*i*) North West Corner Rule
   (*ii*) Least Cost Method
   (*iii*) Vogel's Approximation Method
3. Explain degeneracy in a transportation problem. Describe a method to resolve it.
4. What do you mean by an unbalanced transportation problem? Explain the process of converting an unbalanced transportation problem into a balanced one.
5. Give the mathematical formulation of a transportation problem.
6. Write an algorithm to solve a transportation problem.

6.7 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 7 ASSIGNMENT PROBLEM

NOTES

Structure

7.0 INTRODUCTION

This unit deals with a very interesting method called the 'assignment technique', which is applicable to a class of very practical problems generally called 'assignment problems'.

The objective of assignment problems is to assign a number of origins (jobs) to the equal number of destinations (persons) at a minimum cost or maximum profit.

7.1 OBJECTIVES

After going through this unit, you will be able to:

- Define assignment problem and state mathematical formulation of the same
- Differentiate between transportation and assignment problems
- Derive the solution of an assignment problem using Hungarian Method
- Understand unbalanced assignment problem and modified matrix
- Assess maximization in assignment problem

7.2 ASSIGNMENT PROBLEM

Definition

Suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degrees of efficiency. Let c_{ij} be the cost if the ith person is assigned to the jth job. The problem is to find an assignment (which job should be assigned to which person, on a one to one basis) so that the total cost of performing all the jobs is minimum. Problems of this kind are known as assignment problems.

An assignment problem can be stated in the form of $n \times n$ cost matrix $[c_{ij}]$ of real numbers as given in the following table.

NOTES

$$
\begin{array}{cc}
 & \textbf{Jobs} \\
 & \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & j & \cdots & n \end{array} \\
\textbf{Persons} \begin{array}{c} 1 \\ 2 \\ 3 \\ \\ i \\ \\ n \end{array} &
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & \cdots & c_{1j} & \cdots & c_{1n} \\
c_{21} & c_{22} & c_{23} & \cdots & c_{2j} & \cdots & c_{2n} \\
c_{31} & c_{32} & c_{33} & \cdots & c_{3j} & \cdots & c_{3n} \\
c_{i1} & c_{i2} & c_{i3} & \cdots & c_{ij} & \cdots & c_{in} \\
\vdots & & & & & & \\
c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nj} & \cdots & c_{nn}
\end{bmatrix}
\end{array}
$$

7.2.1 Mathematical Formulation of an Assignment Problem

Mathematically, an assignment problem can be stated as,

Minimize $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$, where, $i = 1, 2, \ldots n$, and $j = 1, 2, \ldots n$

Subject to the restrictions,

$$x_{ij} = \begin{cases} 1, \text{ if the } i\text{th person is assigned } j\text{th job} \\ 0, \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ (one job is done by the } i\text{th person)}$$

$$\text{and } \sum_{j=1}^{n} x_{ij} = 1 \text{ (only one person should be assigned the } j\text{th job)}$$

where, x_{ij} denotes that the jth job is to be assigned to the ith person.

Difference between Transportation and Assignment Problems

Transportation problem	*Assignment problem*
1. Number of sources and destinations need not be equal. Hence, the cost matrix is not necessarily a square matrix.	Since assignment is done on a one to one basis, the number of sources and destinations are equal. Hence, the cost matrix must be a square matrix.
2. x_{ij}, the quantity to be transported from ith origin to jth destination can take any possible positive value, and it satisfies the rim requirements.	x_{ij}, the jth job is to be assigned to the ith person and can take either the value 1 or zero.
3. The capacity and the requirement value is equal to a_i and b_j for the ith source and jth destination ($i = 1, 2, \ldots m$; $j = 1, 2, \ldots n$).	The capacity and the requirement value is exactly one, i.e., for each source of each destination, the capacity and the requirement value is exactly one.
4. The problem is unbalanced if the total supply and total demand are not equal.	The problem is unbalanced if the cost matrix is not a square matrix.

7.2.2 Hungarian Method Algorithm

Solution of an assignment problem can be arrived at, by using the **Hungarian method**. The steps involved in this method are as follows.

NOTES

Step 1 Prepare a cost matrix. If the cost matrix is not a square matrix then add a dummy row (column) with zero cost element.

Step 2 Subtract the minimum element in each row from all the elements of the respective rows.

Step 3 Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus, obtain the modified matrix.

Step 4 Then, draw the minimum number of horizontal and vertical lines to cover all zeros in the resulting matrix. Let the minimum number of lines be N. Now there are two possible cases.

Case I If $N = n$, where n is the order of matrix, then an optimal assignment can be made. So make the assignment to get the required solution.

Case II If $N < n$, then proceed to step 5.

Step 5 Determine the smallest uncovered element in the matrix (element not covered by N lines). Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

Step 6 Repeat steps 3 and 4 until we get the case (i) of Step 4.

Step 7 (To make zero assignment) Examine the rows successively until a row-wise exactly single zero is found. Circle (O) this zero to make the assignment. Then mark a cross (×) over all zeros if lying in the column of the circled zero, showing that they cannot be considered for future assignment. Continue in this manner until all the zeros have been examined. Repeat the same procedure for columns also.

Step 8 Repeat step 6 successively until one of the following situation arises—

(*i*) If no unmarked zero is left, then the process ends or

(*ii*) If there lie more than one unmarked zero in any column or row, circle one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its row or column. Repeat the process until no unmarked zero is left in the matrix.

Step 9 Thus, exactly one marked circled zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked circled zeros will give the optimal assignment.

Example 7.1: Using the following cost matrix, determine (*a*) optimal job assignment (*b*) the cost of assignments.

$$\begin{array}{ccc} & & \textbf{Job} \\ & & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \textbf{Mechanic} & \begin{array}{c}A\\B\\C\\D\\E\end{array} & \begin{bmatrix}10 & 3 & 3 & 2 & 8\\ 9 & 7 & 8 & 2 & 7\\ 7 & 5 & 6 & 2 & 4\\ 3 & 5 & 8 & 2 & 4\\ 9 & 10 & 9 & 6 & 10\end{bmatrix}\end{array}$$

NOTES

Solution: Select the smallest element in each row and subtract this smallest element from all the elements in its row.

$$\begin{array}{cc} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}A\\B\\C\\D\\E\end{array} & \begin{bmatrix}8 & 1 & 1 & 0 & 6\\ 7 & 5 & 6 & 0 & 5\\ 5 & 3 & 4 & 0 & 2\\ 1 & 3 & 6 & 0 & 2\\ 3 & 4 & 3 & 0 & 4\end{bmatrix}\end{array}$$

Select the minimum element from each column and subtract from all other elements in its column. With this we get the first modified matrix.

$$\begin{array}{cc} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}A\\B\\C\\D\\E\end{array} & \begin{bmatrix}7 & 0 & 0 & 0 & 4\\ 6 & 4 & 5 & 0 & 3\\ 4 & 2 & 3 & 0 & 0\\ 0 & 2 & 5 & 0 & 0\\ 2 & 3 & 2 & 0 & 2\end{bmatrix}\end{array}$$

In this modified matrix we draw the minimum number of lines to cover all zeros (horizontal or vertical).

$$\begin{array}{cc} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}A\\B\\C\\D\\E\end{array} & \begin{bmatrix}7 & 0 & 0 & 0 & 4\\ 6 & 4 & 5 & 0 & 3\\ 4 & 2 & 3 & 0 & 0\\ 0 & 2 & 5 & 0 & 0\\ 2 & 3 & 2 & 0 & 2\end{bmatrix}\end{array}$$

Number of lines drawn to cover all zeros is $4 = N$.

The order of matrix is $n = 5$

Hence, $N < n$.

NOTES

Now we get the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding it to the element at the point of intersection of lines.

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 9 & 0 & \cancel{0} & 2 & 6 \\ 6 & 2 & 3 & 0 & 3 \\ 4 & \cancel{0} & 1 & \cancel{0} & 0 \\ 0 & \cancel{0} & 3 & \cancel{0} & \cancel{0} \\ 2 & 1 & 0 & \cancel{0} & 2 \end{bmatrix} \end{array}$$

Number of lines drawn to cover all zeros = N = 5

The order of matrix is n = 5.

Hence $N = n$. Now we determine the optimum assignment.

Assignment

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 9 & 0 & \cancel{0} & 2 & 6 \\ 6 & 2 & 3 & 0 & 3 \\ 4 & \cancel{0} & 1 & \cancel{0} & 0 \\ 0 & \cancel{0} & 3 & \cancel{0} & \cancel{0} \\ 2 & 1 & 0 & \cancel{0} & 2 \end{bmatrix} \end{array}$$

First row contains more than one zero. So proceed to the 2nd row. It has exactly one zero. The corresponding cell is (B, 4). Circle this zero thus, making an assignment. Mark (×) for all other zeros in its column. Showing that they cannot be used for making other assignments. Now row 5 has a single zero in the cell (E, 3). Make an assignment in this cell and cross the 2nd zero in the 3rd column.

Now row 1 has a single zero in the column 2, i.e., in the cell (A, 2). Make an assignment in this cell and cross the other zeros in the 2nd column. This leads to a single zero in column 1 of the cell (D, 1), make an assignment in this cell and cross the other zeros in the 4th row. Finally, we have a single zero left in the 3rd row, making an assignment in the cell (C, 5). Thus, we have the following assignment.

Optimal assignment and optimum cost of assignment.

Job	***Mechanic***	***Cost***
1	*D*	3
2	*A*	3
3	*E*	9
4	*B*	2
5	*C*	4
		₹ 21

Therefore, $1 \to D$, $2 \to A$, $3 \to E$, $4 \to B$, $5 \to C$, with minimum cost equal to ₹ 21.

Example 7.2: A company has 5 jobs to be done on five machines. Any job can be done on any machine. The cost of doing the jobs on different machines are given below. Assign the jobs for different machines so as to minimize the total cost.

	Machines				
Jobs	***A***	***B***	***C***	***D***	***E***
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	12	20

Solution: We form the first modified matrix by subtracting the minimum element from all the elements in the respective row, and the same with respective columns.

Step 1

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix} \end{array}$$

Since each column has the minimum element 0, we have the first modified matrix. Now we draw the minimum number of lines to cover all zeros.

Step 2

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix} \end{array}$$

Step 3

$$\begin{bmatrix} 5 & 0 & 5 & 10 & 8 \\ 0 & 6 & 12 & 0 & 0 \\ 11 & 8 & 0 & 3 & 0 \\ 0 & 6 & 1 & 2 & 4 \\ 3 & 5 & 3 & 0 & 5 \end{bmatrix}$$

Here, the smallest uncovered element is '3'. So '3' is added to all the junction points i.e., the prints of intersection of lines.

Number of lines drawn to cover zero is $N = 4 <$ the order of matrix $n = 5$.

We find the second modified matrix by subtracting the smallest uncovered element (3) from all the uncovered elements and adding to the element that is the point of intersection of lines.

NOTES

Step 4

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 5 & 0 & 5 & 10 & 8 \\ 0 & 6 & 12 & 0 & 0 \\ 11 & 8 & 0 & 3 & 0 \\ 0 & 6 & 1 & 2 & 4 \\ 3 & 5 & 3 & 0 & 5 \end{bmatrix} \end{array}$$

Number of lines drawn to cover all zeros = 5,

which is the order of matrix. Hence, we can form an assignment.

Assignment

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 5 & 0 & 5 & 10 & 8 \\ \cancel{0} & 6 & 12 & \cancel{0} & 0 \\ 11 & 8 & 0 & 3 & \cancel{0} \\ 0 & 6 & 1 & 2 & 4 \\ 3 & 5 & 3 & 0 & 5 \end{bmatrix} \end{array}$$

All the five jobs have been assigned to 5 different machines.

Here the optimal assignment is,

Job	***Machine***
1	B
2	E
3	C
4	A
5	D

Minimum (Total cost) = 8 + 12 + 4 + 6 + 12 = ₹ 42.

Example 7.3: Four different jobs can be done on four different machines and the take-down time costs are prohibitively high for change overs. The matrix below gives the cost in rupees for producing job *i* on machine *j*.

	Machines			
Jobs	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

How should the jobs be assigned to the various machines so that the total cost is minimized.

NOTES

Solution: We form a first modified matrix by subtracting the least element in the respective rows and respective columns.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{bmatrix} \end{array}$$

Since the third column has no zero element, we subtract the smallest element 4 from all the elements.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Now we draw minimum number of lines to cover all zeros.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Number of lines drawn to cover all zeros = 3, which is less than the order of matrix = 4.

Hence, we form the 2nd modified matrix, by subtracting the smallest uncovered element from all the uncovered elements and adding to the element that is at the point of intersection of lines.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 2 \\ 0 & 2 & 1 & 3 \\ 7 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 8 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$N = 3 < n = 4$ $\qquad$ $N = 4 = n = 4$

Hence, we can make an assignment.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} \cancel{0} & \cancel{0} & \cancel{0} & 0 \\ 5 & 0 & \cancel{0} & 2 \\ 0 & 1 & \cancel{0} & 2 \\ 8 & \cancel{0} & 0 & \cancel{0} \end{bmatrix} \end{array}$$

NOTES

Since no rows and no columns have single zero, we have a different assignment (Multiple solution).

Optimal assignment

Job	*Machine*
J_1	M_4
J_2	M_2
J_3	M_1
J_4	M_3

Minimum (Total cost)

$$6 + 5 + 4 + 8 = ₹\ 23$$

Alternate Solution

$$J_1 \to M_1; \quad J_2 \to M_2; \quad J_3 \to M_3; \quad J_4 \to M_4.$$

Minimum (Total cost)

$$5 + 5 + 10 + 3 = ₹\ 23.$$

Example 7.4: Solve the following assignment problem in order to minimize the total cost. The cost matrix given below gives the assignment cost when different operators are assigned to various machines.

Operators

$$\textbf{Machines}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} 30 & 25 & 33 & 35 & 36 \\ 23 & 29 & 38 & 23 & 26 \\ 30 & 27 & 22 & 22 & 22 \\ 25 & 31 & 29 & 27 & 32 \\ 27 & 29 & 30 & 24 & 32 \end{bmatrix} \end{array}$$

Solution: We form the first modified matrix by subtracting the least element from all the elements in the respective rows and then in the respective columns.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix} \end{array}$$

Since each column has the minimum element 0, the first modified matrix is obtained. We draw the minimum number of lines to cover all zeros.

The number of lines drawn to cover all zeros = 4 < the order of matrix = 5. Hence, we form the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element that is at the point of intersection of lines.

$$\begin{array}{c|ccccc} & I & II & III & IV & V \\ A & 5 & 0 & 5 & 10 & 8 \\ B & 0 & 6 & 12 & 0 & 0 \\ C & 11 & 8 & 0 & 3 & 0 \\ D & 0 & 6 & 1 & 2 & 4 \\ E & 3 & 5 & 3 & 0 & 5 \end{array}$$

$N = 5$, i.e., the number of lines drawn to cover all zeros = order of matrix. Hence, we can make an assignment.

$$\begin{array}{c|ccccc} & I & II & III & IV & V \\ A & 5 & 0 & 5 & 10 & 8 \\ B & \cancel{0} & 6 & 12 & \cancel{0} & 0 \\ C & 11 & 8 & 0 & 3 & \cancel{0} \\ D & 0 & 6 & 1 & 2 & 4 \\ E & 3 & 5 & 3 & 0 & 5 \end{array}$$

The optimum assignment is

Operators	***Machines***
I	D
II	A
III	C
IV	E
V	B

The optimum cost is given by

$$25 + 25 + 22 + 24 + 26 = ₹\ 122.$$

Unbalanced Assignment problem

Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e., the number of rows and columns are not equal. To make it balanced, we add a dummy row or dummy column with all the entries as zero.

Example 7.5: There are four jobs to be assigned to five machines. Only one job can be assigned to one machine. The amount of time in hours required for the jobs per machine are given in the following matrix.

	Machines				
Jobs	***A***	***B***	***C***	***D***	***E***
1	4	3	6	2	7
2	10	12	11	14	16
3	4	3	2	1	5
4	8	7	6	9	6

Find an optimum assignment of jobs to the machines to minimize the total processing time and also find out for which machine no job is assigned. What is the total processing time to complete all the jobs?

Solution Since the cost matrix is not a square matrix, the problem is unbalanced. We add a dummy job 5 with corresponding entries zero.

Modified Matrix

$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5 \end{array} & \left[\begin{array}{ccccc} 4 & 3 & 6 & 2 & 7 \\ 10 & 12 & 11 & 14 & 16 \\ 4 & 3 & 2 & 1 & 5 \\ 8 & 7 & 6 & 9 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

We subtract the smallest element from all the elements in the respective rows.

$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5 \end{array} & \left[\begin{array}{ccccc} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Since each column has minimum element as zero, we draw minimum number of lines to cover all zeros.

$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5 \end{array} & \left[\begin{array}{ccccc} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

The number of lines to cover all zeros = 4 < the order of matrix. We form the 2nd modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element at the point of intersection of lines.

$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5 \end{array} & \left[\begin{array}{ccccc} 2 & 0 & 3 & \cancel{0} & 4 \\ 0 & 1 & \cancel{0} & 4 & 5 \\ 3 & 1 & 0 & \cancel{0} & 3 \\ 3 & 1 & \cancel{0} & 4 & 0 \\ 1 & \cancel{0} & \cancel{0} & 1 & \cancel{0} \end{array}\right] \end{array}$$

Here the number of lines drawn to cover all zeros = 5 = Order of matrix. Therefore, we can make the assignment

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline 1 & 2 & 0 & 3 & 0 & 4 \\ 2 & 0 & 1 & 0 & 4 & 5 \\ 3 & 3 & 1 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 4 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \end{array}$$

Optimum assignment

1	B	3
2	A	10
3	C	2
4	D	6

For machine D, no job is assigned.

Optimum (minimum) cost = 3 + 10 + 1 + 6 = ₹ 20.

Example 7.6: A company has 4 machines to do 3 jobs. Each job can be assigned to only one machine. The cost of each job on each machine is given below. Determine the job assignments that will minimize the total cost.

Machine

$$\textbf{Job}\quad \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 18 \\ 10 & 15 & 19 & 22 \end{bmatrix} \end{array}$$

Solution: Since the cost matrix is not a square matrix, we add a dummy row D with all the elements 0.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 18 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Subtract the minimum element in each row from all the elements in its row.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 10 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

NOTES

Since each column has a minimum element 0, we draw minimum number of lines to cover all zeros.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 10 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\therefore$ The number of lines drawn to cover all zeros = 2 < the order of matrix, we form a second modified matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here, $N = 3 < n = 4$.

Again we subtract the smallest uncovered element from all the uncovered elements and add to the element at the point of intersection

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{bmatrix} \end{array}$$

Here, $N = 4 = n$. Hence, we make an assignment.

Assignment

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 4 \\ \cancel{0} & 0 & \cancel{0} & 1 \\ \cancel{0} & \cancel{0} & 0 & 3 \\ 9 & 4 & \cancel{0} & 0 \end{bmatrix} \end{array} \begin{array}{l} A \to W \\ D \to Z \\ B \to X \\ C \to Y \end{array} \quad \text{or} \quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 4 \\ \cancel{0} & \cancel{0} & 0 & 1 \\ \cancel{0} & 0 & \cancel{0} & 3 \\ 9 & 4 & \cancel{0} & 0 \end{bmatrix} \end{array} \begin{array}{l} A \to W \\ D \to Z \\ B \to Y \\ C \to X \end{array}$$

Since D is a dummy job, machine Z is assigned no job.

Therefore, optimum cost = 18 + 13 + 19 = ₹ 50.

Maximization in Assignment problem

In this, the objective is to maximize the profit. To solve this, we first convert the given profit matrix into the loss matrix by subtracting all the elements from the highest element. For this converted loss matrix we apply the steps in Hungarian method to get the optimum assignment.

NOTES

Example 7.7: The owner of a small machine shop has four mechanics available to assign jobs for the day. Five jobs are offered with expected profit for each mechanic on each jobs, which are as follows:

Job

$$\text{Mechanic}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 62 & 78 & 50 & 111 & 82 \\ 71 & 84 & 61 & 73 & 59 \\ 87 & 92 & 111 & 71 & 81 \\ 48 & 64 & 87 & 77 & 80 \end{bmatrix} \end{array}$$

By using the assignment method, find the assignment of mechanics to the job that will result in maximum profit. Which job should be declined ?

Solution: The given profit matrix is not a square matrix as the number of jobs is not equal to the number of mechanics. Hence, we introduce a dummy mechanic 5 with all the elements 0.

Job

$$\text{Mechanic}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 62 & 78 & 50 & 111 & 82 \\ 71 & 84 & 61 & 73 & 59 \\ 87 & 92 & 111 & 71 & 81 \\ 48 & 64 & 87 & 77 & 80 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now we convert this profit matrix into loss matrix by subtracting all the elements from the highest element 111.

Loss Matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 49 & 33 & 61 & 0 & 29 \\ 40 & 27 & 50 & 38 & 52 \\ 24 & 19 & 0 & 40 & 30 \\ 63 & 47 & 24 & 34 & 31 \\ 111 & 111 & 111 & 111 & 111 \end{bmatrix} \end{array}$$

We subtract the smallest element from all the elements in the respective rows.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 49 & 33 & 61 & 0 & 29 \\ 13 & 0 & 23 & 11 & 25 \\ 24 & 19 & 0 & 40 & 30 \\ 39 & 23 & 0 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

NOTES

Since each column has minimum element as zero, we draw minimum number of lines to cover all zeros.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 49 & 33 & 61 & 0 & 29 \\ 13 & 0 & 23 & 11 & 25 \\ 24 & 19 & 0 & 40 & 30 \\ 39 & 23 & 0 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here the number of lines drawn to cover all zeros = N = 4, is less than the order of matrix.

We form the 2nd modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element that is at the point of intersection of lines.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 49 & 40 & 68 & 0 & 29 \\ 6 & 0 & 23 & 5 & 18 \\ 17 & 19 & 0 & 33 & 23 \\ 32 & 23 & 0 & 3 & 0 \\ 0 & 7 & 7 & 0 & 0 \end{bmatrix} \end{array}$$

Here, $N = 5 = n$ (the order of matrix).

We make the assignment.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 49 & 40 & 68 & \boxed{0} & 29 \\ 6 & \boxed{0} & 23 & 5 & 18 \\ 17 & 19 & \boxed{0} & 33 & 23 \\ 32 & 23 & \cancel{0} & 3 & \boxed{0} \\ \boxed{0} & 7 & 7 & \cancel{0} & \cancel{0} \end{bmatrix} \end{array}$$

The optimum assignment is

Job	*Mechanic*
A	5
B	2
C	3
D	1
E	4

Since the 5th mechanic is a dummy, job *A* is assigned to the 5th mechanic, this job is declined.

The maximum profit is given by, 84 + 111 + 111 + 80 = ₹ 386.

NOTES

Example 7.8: A marketing manager has 5 salesmen and there are 5 sales districts. Considering the capabilities of the salesmen and the nature of districts, the estimates made by the marketing manager for the sales per month (in 1,000 rupees) for each salesman in each district would be as follows.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 32 & 38 & 40 & 28 & 40 \\ 40 & 24 & 28 & 21 & 36 \\ 41 & 27 & 33 & 30 & 37 \\ 22 & 38 & 41 & 36 & 36 \\ 29 & 33 & 40 & 35 & 39 \end{bmatrix} \end{array}$$

Find the assignment of salesmen to the districts that will result in the maximum sales.

Solution: We are given the profit matrix. To maximize the profit, first we convert it into a loss matrix, which can be minimized. To convert it into loss matrix, we subtract all the elements from the highest element 41. Subtract the smallest element from all the elements in the respective rows and columns, to get the first modified matrix.

Loss Matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 9 & 3 & 1 & 13 & 1 \\ 1 & 17 & 13 & 20 & 5 \\ 0 & 14 & 8 & 11 & 4 \\ 19 & 3 & 0 & 5 & 5 \\ 12 & 8 & 1 & 6 & 2 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 8 & 2 & 0 & 12 & 0 \\ 0 & 16 & 12 & 19 & 4 \\ 0 & 14 & 8 & 11 & 4 \\ 19 & 3 & 0 & 5 & 5 \\ 11 & 7 & 0 & 5 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 8 & 0 & 0 & 7 & 0 \\ 0 & 14 & 12 & 14 & 4 \\ 0 & 12 & 8 & 6 & 4 \\ 19 & 1 & 0 & 0 & 5 \\ 11 & 5 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

NOTES

We now draw minimum number of lines to cover all zeros.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 8 & 0 & 0 & 7 & 0 \\ 0 & 14 & 12 & 4 & 4 \\ 0 & 12 & 8 & 6 & 4 \\ 9 & 1 & 0 & 0 & 5 \\ 11 & 5 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$N = 4 < n = 5$

We subtract the smallest uncovered element from the remaining uncovered elements and add to the elements at the point of intersection of lines, to get the second modified matrix.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 9 & 0 & 1 & 8 & 0 \\ 0 & 13 & 12 & 4 & 3 \\ 0 & 11 & 8 & 6 & 3 \\ 9 & 0 & 0 & 0 & 4 \\ 11 & 4 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Again, $N = 4$ $n = 5$. Repeat the above step.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 12 & 0 & 1 & 8 & 0 \\ 0 & 10 & 9 & 1 & 0 \\ 0 & 8 & 5 & 3 & 0 \\ 12 & 0 & 0 & 0 & 4 \\ 14 & 4 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$N = 5 = n = 5$. Hence we make the assignment.

Assignment

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 12 & 0 & 1 & 8 & \cancel{0} \\ 0 & 10 & 9 & 1 & \cancel{0} \\ \cancel{0} & 8 & 5 & 3 & 0 \\ 12 & \cancel{0} & 0 & \cancel{0} & 4 \\ 14 & 4 & \cancel{0} & 0 & \cancel{0} \end{array}\right] \end{array}$$

Since no row or column has single zero, we get a multiple solution.

(*i*) The optimum assignment is:

$1 \rightarrow B,\ 2 \rightarrow A,\ 3 \rightarrow E,\ 4 \rightarrow C,\ 5 \rightarrow D.$

With maximum profit (38 + 40 + 37 + 41 + 35) = ₹ 191 (*ii*) The optimum assignment is:

$1 \rightarrow B,\ 2 \rightarrow A,\ 3 \rightarrow E,\ 4 \rightarrow D,\ 5 \rightarrow C.$

Maximum profit (38 + 40 + 37 + 36 + 40) = ₹ 191

NOTES

7.2.3 Routing Problem: The Travelling Salesman Problem

A network routing problem consists of finding an optimum route between two or more nodes in relation to total time, cost, or distance. Various constraints may exist, such as a prohibition on returning to a node already visited or a stipulation of passing through every node only once. Network routing problems commonly arise in communication and transportation systems. Delays that occur at the nodes (*e.g.,* railroad classification yards or telephone switchboards) may be a function of the loads placed on them and their capacities. Breakdowns may occur in either links or nodes. Much studied is the "traveling salesman problem," which consists of starting a route from a designated node that goes through each node (*e.g.,* city) only once and returns to the origin in the least time, cost, or distance. This problem arises in selecting an order for processing a set of production jobs when the cost of setting up each job depends on which job has preceded it.

Let us discuss the 'Traveling Salesman Problem'.

Assuming a salesman has to visit n cities. He wishes to start from a particular city, visit each city once and then return to his starting point. His objective is to select the sequence in which the cities are visited in such a way that his total travelling time is minimized.

To visit 2 cities (A and B, there is no choice. To visit 3 cities we have 2 possible routes. For 4 cities we have 3 possible routes. In general, to visit n cities there are $(n - 1)$ possible routes.

Mathematical Formulation

Let C_{ij} be the distance or time or cost of going from city i to city j. Let the decision variable X_{ij} be 1, if the salesman travels from city i to city j, otherwise let it be 0.

The objective is to minimize the travelling time.

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\ x_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^{n} X_{ij} = 1,\ i = 2 \ldots n.$$

$$\sum_{i=1}^{n} X_{ij} = 1,\ j = 2 \ldots n.$$

and subject to the additional constraint that X_{ij} is so chosen that, no city is visited twice before all the cities are visited.

NOTES

In particular, going from i directly to i is not permitted. This means $C_{ij} = \infty$, when $i = j$.

In the travelling salesman problem we cannot choose the element along the diagonal and this can be avoided by filling the diagonal with infinitely large elements.

The travelling salesman problem is very similar to the assignment problem except that in the former case, there is an additional restriction, that X_{ij} is so chosen that no city is visited twice before the tour of all the cities is completed.

Example 7.9: A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and then return to his starting point. Cost of going from one city to another is shown below. You are required to find the least cost route.

To City

$$\textbf{From City}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 4 & 10 & 14 & 2 \\ 12 & \infty & 6 & 10 & 4 \\ 16 & 14 & \infty & 8 & 14 \\ 24 & 8 & 12 & \infty & 10 \\ 2 & 6 & 4 & 16 & \infty \end{bmatrix} \end{array}$$

Solution: Treat the problem as an assignment problem and solve it using the same procedures. If the optimal solution of the assignment problem satisfies the additional constraint, then it is also an optimal solution of the given travelling salesman problem. If the solution to the assignment problem does not satisfy the additional restriction, then after solving the problem by assignment technique, we use the method of enumeration.

First we solve this problem as an assignment problem.

Subtract the minimum element in each row from all the elements in its row.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 2 & 8 & 12 & 0 \\ 8 & \infty & 2 & 6 & 0 \\ 8 & 6 & \infty & 0 & 6 \\ 16 & 0 & 4 & \infty & 2 \\ 0 & 4 & 2 & 14 & \infty \end{bmatrix} \end{array}$$

Subtract the minimum element in each column from all the elements in its column.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 2 & 8 & 12 & 0 \\ 8 & \infty & 2 & 6 & 0 \\ 8 & 6 & \infty & 0 & 6 \\ 16 & 0 & 4 & \infty & 2 \\ 0 & 4 & 2 & 14 & \infty \end{bmatrix} \end{array}$$

We have the first modified matrix. Draw minimum number of lines to cover all zeros.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 2 & 6 & 12 & 0 \\ 8 & \infty & 2 & 6 & 0 \\ 8 & 8 & \infty & 0 & 6 \\ 16 & 0 & 2 & \infty & 2 \\ 0 & 4 & 0 & 14 & \infty \end{bmatrix} \end{array}$$

$N = 4 < n = 5$. Subtract the smallest uncovered element from all the uncovered elements and add to the element that is at the point of intersection of lines. Hence, we get the 2nd modified matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 0 & 6 & 12 & 0 \\ 8 & \infty & 0 & 6 & 0 \\ 8 & 4 & \infty & 0 & 6 \\ 18 & 0 & 4 & \infty & 4 \\ 0 & 4 & 0 & 14 & \infty \end{bmatrix} \end{array}$$

$N = 5 = n = 5$. We make an assignment.

Assignment

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & \cancel{0} & 6 & 12 & 0 \\ 8 & \infty & 0 & 6 & \cancel{0} \\ 8 & 4 & \infty & 0 & 6 \\ 18 & 0 & 4 & \infty & 4 \\ 0 & 4 & \cancel{0} & 14 & \infty \end{bmatrix} \end{array} \quad \begin{array}{l} A \to E \\ B \to C \\ C \to D \\ D \to B \\ E \to A \\ A \to E \to A \end{array}$$

The salesman should go from A to E and then come back to A without covering B, C, D. But this is contradicting the constraint that no city is visited twice before all the cities are visited.

Hence, we obtain the next best solution by bringing the next minimum non-zero element namely 4.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 0 & 6 & 12 & 0 \\ 8 & \infty & 0 & 6 & \cancel{0} \\ 8 & 4 & \infty & 0 & 6 \\ 18 & \cancel{0} & 4 & \infty & 4 \\ 0 & 4 & \cancel{0} & 14 & \infty \end{bmatrix} \end{array}$$

$$A \to B,\ B \to C,\ C \to D,\ D \to E,\ E \to A$$

NOTES

Since all the cities have been visited and no city is visited twice before completing the tour of all the cities, we have an optimal solution for the travelling salesman.

The least cost route is $A \to B \to C \to D \to E \to A$.

Total cost = 4 + 6 + 8 + 10 + 2 = ₹ 30.

Example 7.10: A machine operator processes five types of items on his machine each week and must choose a sequence for them. The set-up cost per change depends on the items presently on the machine and the set-up to be made, according to the following table.

To Item

$$\textbf{From Item}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 4 & 7 & 3 & 4 \\ 4 & \infty & 6 & 3 & 4 \\ 7 & 6 & \infty & 7 & 5 \\ 3 & 3 & 7 & \infty & 7 \\ 4 & 4 & 5 & 7 & \infty \end{bmatrix} \end{array}$$

If he processes each type of item only once in each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

Solution: Reduce the cost matrix and make assignments in rows and columns having single row.

Modify the matrix by subtracting the least element from all the elements in its row and also in its column.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{bmatrix} \end{array}$$

Here, $N = 4$ $n = 5$, i.e., $N < n$.

Subtract the smallest uncovered element from all the uncovered elements and add to the element that is at the point of intersection of lines and get the reduced 2nd modified matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 0 & 2 & 0 & 1 \\ 0 & \infty & 1 & 0 & 1 \\ 1 & 0 & \infty & 1 & 0 \\ 0 & 0 & 3 & \infty & 5 \\ 0 & 0 & 0 & 4 & \infty \end{bmatrix} \end{array}$$

$N = 5 = n = 5$. We make the assignment.

Assignment

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & \cancel{0} & 2 & \cancel{0} & 1 \\ \cancel{0} & \infty & 1 & 0 & 1 \\ 1 & 0 & \infty & 1 & \cancel{0} \\ 0 & \cancel{0} & 3 & \infty & 5 \\ \cancel{0} & \cancel{0} & 0 & 4 & \infty \end{bmatrix} \end{array}\begin{array}{c} \\ A \to B \\ B \to D \\ C \to E \\ D \to A \\ E \to C \end{array}$$

We get the solution $A \to B \to D \to A$.

This schedule does not provide the required solution as each item is not processed only once in a week.

Hence, we make a better solution by considering the next smallest non-zero element by considering 1.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & \cancel{0} & 2 & \cancel{0} & 1 \\ \cancel{0} & \infty & \cancel{1} & 0 & \cancel{1} \\ 1 & 0 & \infty & 1 & \cancel{0} \\ 0 & \cancel{0} & 3 & \infty & 5 \\ \cancel{0} & \cancel{0} & 0 & 4 & \infty \end{bmatrix} \end{array}$$

$A \to E$, $E \to C$, $C \to B$, $B \to D$, $D \to A$,

i.e., $A \to E \to C \to B \to D \to A$.

The total set-up cost comes to ₹ 21.

Check Your Progress

1. State any one difference between transportation and assignment problems.
2. What is an unbalanced assignment problem?
3. What do you mean by routing problem?

7.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. The one main difference between transportation problem and assignment problem is that in transportation problem the number of sources and destinations need not be equal. Hence, the cost matrix is not necessarily a square matrix. While in assignment problem the assignment is done on a one to one basis, the number of sources and destinations are equal. Hence, the cost matrix must be a square matrix.
2. Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e., the number of rows and columns are not equal. To make it balanced, we add a dummy row or dummy column with all the entities as zero.
3. A network routing problem consists of finding an optimum route between two or more nodes in relation to total time, cost, or distance. Various constraints may exist, such as a prohibition on returning to a node already visited or a stipulation of passing through every node only once.

7.4 SUMMARY

- An assignment problem is one of the fundamental combinatorial optimization problems and helps to find a maximum weight identical in nature in a weighted bipartite graph.
- The solution of an assignment problem can be arrived at using the Hungarian method.
- An assignment problem is balanced if the cost matrix is a square matrix; otherwise, it is termed as unbalanced.
- To convert an unbalanced assignment problem into a balanced problem, dummy rows or columns are added with all entries as 0s.

7.5 KEY WORDS

- **Assignment problem:** It helps to find a maximum weight identical in nature in a weighted bipartite graph.
- **Unbalanced assignment problem:** Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix.
- **Square matrix:** In mathematics, a square matrix is a matrix with the same number of rows and columns.

NOTES

7.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. State the differences between a transportation problem and an assignment problem.
2. Give a mathematical formulation of the assignment problem.
3. How can you maximize an objective function in the assignment problem?

Long Answer Questions

1. Discuss the assignment problem with a suitable example.
2. Describe the algorithm for the solution of the assignment problem.
3. Explain the nature of *i* in travelling salesman problem and give its mathematical formulation.
4. Solve the following assignment problem.

$$(a)\quad \begin{array}{c} \\ I \\ II \\ III \\ IV \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 4 & 6 & 3 \\ 9 & 7 & 10 & 9 \\ 4 & 5 & 11 & 7 \\ 8 & 7 & 8 & 5 \end{bmatrix} \end{array} \qquad (b)\quad \text{Job}\ \begin{array}{c} \\ I \\ II \\ III \\ IV \end{array}\begin{array}{c} \textbf{Men} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 10 & 25 & 15 & 20 \\ 15 & 30 & 5 & 15 \\ 35 & 20 & 12 & 24 \\ 17 & 25 & 24 & 20 \end{bmatrix} \end{array}$$

5.

$$\text{Tasks}\ \begin{array}{c} \\ I \\ II \\ III \\ IV \\ V \end{array}\begin{array}{c} \textbf{Men} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & 3 & 2 & 8 & 8 \\ 2 & 4 & 3 & 1 & 5 \\ 5 & 6 & 3 & 4 & 6 \\ 3 & 1 & 4 & 2 & 2 \\ 1 & 5 & 6 & 5 & 4 \end{bmatrix} \end{array}$$

6. There are five jobs to be assigned, one each to 5 machines and the associated cost matrix is as follows.

$$\text{Job}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \textbf{Machine} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 11 & 17 & 8 & 16 & 20 \\ 9 & 7 & 12 & 6 & 15 \\ 13 & 16 & 15 & 12 & 16 \\ 21 & 24 & 17 & 28 & 26 \\ 14 & 10 & 12 & 11 & 15 \end{bmatrix} \end{array}$$

NOTES

7. A salesman has to visit five cities A, B, C, D and E. The distance (in hundred miles) between the five cities is as follows.

$$\text{From}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \text{To} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} - & 7 & 6 & 8 & 4 \\ 7 & - & 8 & 5 & 6 \\ 6 & 8 & - & 9 & 7 \\ 8 & 5 & 9 & - & 8 \\ 4 & 6 & 7 & 8 & - \end{bmatrix} \end{array} 0$$

If the salesman starts from city A and has to come back to his starting point, which route should he select so that the total distance travelled is minimum?

8. Determine the optimum assignment schedule for the following assignment problem. The cost matrix is given below.

$$\text{Job}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \text{Machine} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 11 & 17 & 8 & 16 & 20 & 15 \\ 9 & 7 & 12 & 6 & 15 & 13 \\ 13 & 16 & 15 & 12 & 16 & 8 \\ 21 & 24 & 17 & 28 & 2 & 15 \\ 14 & 10 & 12 & 11 & 15 & 6 \end{bmatrix} \end{array}$$

If the job C cannot be assigned to machine 6, will the optimum solution change?

7.7 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

BLOCK - II
INTEGER PROGRAMMING AND QUEUING CONCEPTS

NOTES

UNIT 8 INTEGER PROGRAMMING PROBLEM

Structure

8.0 INTRODUCTION

A linear programming problem in which all or some of the decision variables are constrained to assume non-negative integer values is called an *Integer Programming Problem* (IPP).

In a linear programming problem, if all variables are required to take integral values then it is called the *Pure* (*all*) *Integer Programming Problem* (Pure IPP).

If only some of the variables in the optimal solution of a LPP are restricted to assume non-negative integer values, while the remaining variables are free to take any non-negative values, then it is called a *Mixed Integer Programming Problem* (Mixed IPP).

Further, if all the variables in the optimal solution are allowed to take values 0 or 1, then the problem is called the *0–1 Programming Problem* or *Standard Discrete Programming Problem.*

The general integer programming problem is given by, Max $Z = CX$

Subject to the constraints,

$A\, x \leq b$

$x \geq 0$ and some or all variables are integers.

NOTES

8.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the importance of integer programming problems
- Discuss the applications and types of integer programming problem
- Explain the concept of mixed integer programming problem
- Describe the Branch and Bound Method

8.2 IMPORTANCE OF INTEGER PROGRAMMING PROBLEMS

In LPP, all the decision variables were allowed to take any non-negative real values as it is quite possible and appropriate to have fractional values in many situations. There are several frequently occurring circumstances in business and industry that lead to planning models involving integer-valued variables. For example, in production, manufacturing is frequently scheduled in terms of batches, lots or runs. In allocation of goods, a shipment must involve a discrete number of trucks or aircrafts. In such cases the fractional values of variables like 13/3 may be meaningless in the context of the actual decision problem.

Applications of Integer programming

Integer programming is applied in business and industry. All assignment and transportation problems are integer programming problems, as in the assignment and travelling salesmen problem, all the decision variables are either zero or one.

i.e., $x_{ij} = 0$ or 1

Other examples are capital budgeting and production scheduling problems. In fact, any situation involving decisions of the type 'either to do a job or not' can be viewed as an IPP In all such situations,

$x_{ij} = 1$, if the *j*th activity is performed,

0 if the *j*th activity is not performed.

In addition, allocation problems involving the allocation of men or machines give rise to IPP, since such commodities can be assigned only in integers and not in fractions.

Note: If the non-integer variable is rounded off, it violates the feasibility and there is no guarantee that the rounded off solution will be optimal. Due to these difficulties, there is a need for developing a systematic and efficient procedure for obtaining the exact optimal integer solution to such problems.

8.2.1 Types of Integer Programming Problem

There are two methods used to solve IPP, namely,

(*i*) Gomory's Cutting Plane Method

(*ii*) Branch and Bound Method (Search Method).

8.2.2 Gomory's All-IPP Method

A systematic procedure for solving pure IPP was first developed by R.E. Gomory, in 1956, which he later used to deal with the more complicated case of mixed integer programming problem. This method consists of first solving the IPP as an ordinary LPP by ignoring the restriction of integer values and then introducing a new constraint to the problem such that the new set of feasible solution includes all the original feasible integer solutions, but does not include the optimum non-integer solution initially found. This new constraint is called 'Fractional cut' or 'Gomorian constraint'. Then the revised problem is solved using the simplex method, till an optimum integer solution is obtained.

Search Method

This is an enumeration method in which all feasible integer points are enumerated. The widely used search method is the Branch and Bound method. It was developed in 1960, by A.H. Land and A.G. Doig. This method is applicable to both pure and mixed IPP. It first divides the feasible region into smaller subsets that eliminate parts containing no feasible integer solution.

8.2.3 Gomory's Fractional Cut Algorithm or Cutting Plane Method for Pure (All) IPP

Step 1 Convert the minimization IPP into an equivalent maximization IPP. Ignore the integrality condition.

Step 2 Introduce slack and/or surplus variables if necessary, to convert the given LPP in its standard form and obtain the optimum solution of the given LPP by using simplex method.

Step 3 Test the integrality of the optimum solution.

(*i*) If all $x_{Bi} \geq 0$ and are integers, an optimum integer solution is obtained.

(*ii*) If all $x_{Bi} \geq 0$ and at least one x_{Bi} is not an integer, then go to the next step.

Step 4 Rewrite each x_{Bi} as $x_{Bi} = [x_{Bi}] + f_i$ where x_{Bi} is the integral part of x_{Bi} and f_i is the positive fractional part of x_{Bi} $0 \leq f_i < 1$.

Choose the largest fraction of x_{Bi}'s, i.e., Choose max (f_i), if there is a tie, select arbitrarily. Let max $(f_i) = f_K$, corresponding to x_{BK} (the Kth row is called the 'source row').

Step 5 Express each negative fraction, if any, in the source row of the optimum simplex table as the sum of a negative integer and a non-negative fraction.

NOTES

Step 6 Find the fractional cut constraint (Gomorian Constraint)

From the source row $\sum_{j=1}^{n} a_{k_j} x_j = x_{Bi}$

i.e., $\sum_{j=1}^{n} ([a_{kj}] + f_{kj})\, x_j = [x_{BK}] + f_K$

in the form $\sum_{j=1}^{n} f_{kj} x_j \geq f_K \; - \sum_{j=1}^{n} f_{kj}\, x_j \leq -f_K$

or, $-\sum_{j=1}^{n} f_{kj} x_j + G_1 = -f_K$

where, G_1 is the Gomorian slack.

Step 7 Add the fractional cut constraint obtained in step (6) at the bottom of the simplex table obtained in step (2). Find the new feasible optimum solution using dual simplex method.

Step 8 Go to step (3) and repeat the procedure until an optimum integer solution is obtained.

Example 8.1: Find the optimum integer solution to the following LPP.

Max $Z = x_1 + x_2$

Subject to constraints, $3x_1 + 2x_2 \leq 5$

$x_2 \leq 2$

$x_1, x_2 \geq 0$ and are integers.

Solution: Introducing the non-negative slack variable $S_1, S_2 \geq 0$, the standard form of the LPP becomes,

Max $Z = x_1 + x_2 + 0S_1 + 0S_2$

Subject to, $3x_1 + 2x_2 + S_1 = 5$

$0x_1 + x_2 + S_2 = 2$

$x_1, x_2, S_1, S_2 \geq 0$

Ignoring the integrality condition, solve the problem by simplex method. The initial basic feasible solution is given by,

$$S_1 = 5 \text{ and } S_2 = 2.$$

Since all $Z_j - C_j \geq 0$ an optimum solution is obtained, given by

Max $Z = 7/3,\ x_1 = 1/3,\ x_2 = 2.$

To obtain an optimum integer solution, we have to add a fractional cut constraint in the optimum simplex table.

Since $x_B = 1/3$, the source row is the first row.

Expressing the negative fraction –2/3 as a sum of negative integer and positive fraction, we get

$$-2/3 = -1 + 1/3$$

NOTES

		C_j	1	1	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	$Min \frac{x_B}{x_1}$
← 0	S_1	5	3	2	1	0	5/3
0	S_2	2	0	1	0	1	—
	Z_j	0	0	0	0	0	
	$Z_j - C_j$		–1↑	–1	0	0	$Min \frac{x_B}{x_2}$
1	x_1	5/3	1	2/3	1/3	0	5/2 = 2 – 5
← 0	S_2	2	0	1	0	1	2/1 = 2
	Z_j	5/3	1	2/3	1/3	0	
	$Z_j - C_j$		0	–1/3↑	1/3	0	
1	x_1	1/3	1	0	1/3	–2/3	
1	x_2	2	0	1	0	1	
	Z_j	7/3	1	1	1/3	1/3	
	$Z_j - C_j$		0	0	1/3	1/3	

Since x_1 is the source row, we have,

$$1/3 = x_1 + 1/3\ S_1 - 2/3\ S_2$$

i.e.,
$$1/3 = x_1 + 1/3S_1 + (-1 + 1/3)\ S_2$$

The fractional cut (Gomorian) constraint is given by

$$1/3S_1 + 1/3S_2 \geq 1/3$$

$$\Rightarrow \quad -1/3S_1 - 1/3S_2 \leq -1/3$$

$$\Rightarrow \quad -1/3S_1 - 1/3\ S_2 + G_1 = -1/3$$

where, G_1 is the Gomorian slack. Add this fractional cut constraint at the bottom of the above optimal simplex table.

		C_j	1	1	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	G_1
1	x_1	1/3	1	0	1/3	–2/3	0
1	x_2	2	0	1	0	1	0
← 0	G_1	–1/3	0	0	–1/3	–1/3	1
	Z_j	7/3	1	1	1/3	1/3	0
	$Z_j - C_j$		0	0	1/3↑	1/3	0

We apply dual simplex method. Since $G_1 = -1/3$, G_1 leaves the basis. To find the entering variable we find,

$$\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{1/3}{-1/3}, \frac{1/3}{-1/3}\right\}$$

$$\text{Max}\ \{-1, -1\} = -1$$

NOTES

We choose S_1 as the entering variable arbitrarily.

	C_j		1	1	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	G_1
1	x_1	0	1	0	0	–1	1
1	x_2	2	0	1	0	1	1
0	S_1	1	0	0	1	1	–3
	Z_j	2	1	1	0	0	1
	$Z_j - C_j$		0	0	0	0	1

Since all $Z_j - C_j \geq 0$ and all $x_{Bi} \geq 0$, we obtain an optimal feasible integer solution.

$\therefore$ The optimum integer solution is,

Max $Z = 2$, $x_1 = 0$, $x_2 = 2$.

Example 8.2: Find an optimum integer solution to the following LPP.

Max $Z = x_1 + 2x_2$

Subject to the contraints,

$$2x_2 \leq 7$$
$$x_1 + x_2 \leq 7$$
$$2x_1 \leq 11$$
$$x_1, x_2 \geq 0 \text{ and } x_1, x_2 \text{ are integers.}$$

Solution: Introducing slack variables $S_1, S_2, S_3 \geq 0$, we get,

Max $Z = x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to,

$$2x_2 + S_1 = 7$$
$$x_1 + x_2 + S_2 = 7$$
$$2x_1 + S_3 = 11$$

Ignoring the integer condition, we get the optimum solution of the given LPP, with initial basic feasible solution as, $S_1 = 7$, $S_2 = 7$, $S_3 = 11$.

	C_j		1	2	0	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	$Min \frac{x_B}{x_2}$
←0	S_1	7	0	2	1	0	0	7/2 = 3.5
0	S_2	7	1	1	0	1	0	7/1 = 7
0	S_3	11	2	0	0	0	1	—
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		–1	–2↑	0	0	0	$Min \frac{x_B}{x_1}$
2	x_2	7/2	0	1	1/2	0	0	—
←0	S_2	7/2	1	0	–1/2	1	0	7/2 = 3.5
0	S_3	11	2	0	0	0	1	11/2 = 5.5
	Z_j	7	0	2	1	0	0	
	$Z_j - C_j$		–1↑	0	1	0	0	
2	x_2	7/2	0	1	1/2	0	0	
1	x_1	7/2	1	0	–1/2	1	0	
0	S_3	4	0	0	1	–2	1	
	Z_j	21/2	1	2	1/2	1	0	
	$Z_j - C_j$		0	0	1/2	1	0	

Since all $Z_j - C_j \geq 0$, an optimum solution is obtained which is given by,

$$\text{Max Z} = \frac{21}{2},\ x_1 = \frac{7}{2},\ x_2 = \frac{7}{2}$$

NOTES

Since the optimum solution obtained above is not an integer, we now select a constraint corresponding to

$$\text{Max } \{fi\} = \text{Max } \{f_1, f_2, f_3\}$$

$$x_1 = 7/2 = 3 + 1/2$$

$$x_2 = 7/2 = 3 + 1/2$$

$$S_3 = 4 = 4 + 0$$

$$\therefore \qquad \text{Max } \{f_i\} = \text{Max } \left(\frac{1}{2}, \frac{1}{2}, 0\right) = 1/2$$

Since the max fraction is same for both x_1 and x_2 rows, we choose x_1 row as the source row arbitrarily. From this row we have,

$$7/2 = x_1 + 0x_2 - 1/2\ S_1 + 1S_2 + 0S_3.$$

On expressing the negative fraction as a sum of negative integer and a positive fraction, we have,

$$3 + 1/2 = x_1 + 0x_2 + (-1 + 1/2)\ S_1 + 1S_2 + 0S_3$$

$\therefore$ The Gomorian constraint is given by,

$$1/2\ S_1 \geq 1/2$$

i.e., $$-1/2\ S_1 \leq -1/2 \Rightarrow -1/2\ S_1 + G_1 = -1/2$$

where, G_1 is the Gomorian slack. Adding this new constraint at the bottom of the above optimal simplex table, we get a new table.

		C_j	1	2	0	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	G_1
2	x_2	7/2	0	1	1/2	0	0	0
1	x_1	7/2	1	0	–1/2	1	0	0
0	S_3	4	0	0	1	–2	1	0
←0	G_1	–1/2	0	0	–1/2	0	0	1
	Z_j	21/2	1	2	1/2	1	0	0
	$Z_j - C_j$		0	0	1/2	1	0	0

We apply dual simplex method. Since $G_1 = -1/2$, G_1 leaves the basis. Entering variable is given by,

$$\text{Max} \left\{ \frac{Z_j - C_j}{a_{k_j}}, a_{k_j} < 0 \right\} = \text{Max} \left\{ \frac{1/2}{-1/2} \right\}$$

gives the non-basic variable S_1 to enter into the basis. Drop G_1 and introduce S_1.

NOTES

		c_j	1	2	0	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	G_1
2	x_2	3	0	1	0	0	0	1
1	x_1	4	1	0	0	1	0	–1
0	S_3	3	0	0	0	–2	1	2
0	S_1	1	0	0	1	0	0	–2
	Z_j	10	1	2	0	1	0	1
	$Z_j - C_j$		0	0	0	1	0	1

Since all $Z_j - C_j \geq 0$, an optimum solution has been obtained in integers. Hence, the integer optimum solution is given by,

$$\text{Max } Z = 10,\ x_1 = 4,\ x_2 = 3.$$

Example 8.3: Solve the following integer programming problem.

$$\text{Max } Z = 2x_1 + 20x_2 - 10x_3$$

Subject to,
$$2x_1 + 20x_2 + 4x_3 \leq 15$$
$$6x_1 + 20x_2 + 4x_3 = 20$$
$$x_1, x_2, x_3 \geq 0 \text{ and are integers.}$$

Solution: Introducing slack variable $S_1 \geq 0$ and an artificial variable $A_1 \geq 0$, the initial basic feasible solution is $S_1 = 15, A_1 = 20$. Ignoring the integer condition, solve the problem by simplex method.

$$\text{Max } Z = 2x_1 + 20x_2 - 10x_3 + 0S_1 - MA_1$$

Subject to,
$$2x_1 + 20x_2 + 4x_3 + S_1 = 15$$
$$6x_1 + 20x_2 + 4x_3 + A_1 = 20$$
$$x_1, x_2, x_3, S_1, A_1 \geq 0$$

		c_j	2	20	–10	0	–M	
C_B	B	x_B	x_1	x_2	x_3	S_1	A_1	Min $\frac{x_B}{x_2}$
←0	S_1	15	2	20	4	1	0	15/20 = 3/4
–M	A_1	20	6	20	4	0	1	20/20 = 1
	Z_j	–20M	–6M	–20M	–4M	0	–M	
	$Z_j - C_j$		–6M – 2	–20M – 20 ↑	–4M + 10	0	0	

		C_j	2	20	–10	0	–M	
C_B	B	x_B	x_1	x_2	x_3	S_1	A_1	Min x_B/x_1
20	x_2	3/4	1/10	1	$\frac{1}{5}$	1/20	0	$\frac{3}{4}\times 10 = \frac{15}{2} = 7.5$
← –M	A_1	5	4	0	0	–1	1	$\frac{5}{4} = 1.25$
	Z_j	15 – 5M	2 – 4M	20	4	1 + M	–M	
	$Z_j - C_j$		–4 — ↑	0	14	M + 1	0	
20	x_2	5/8	0	1	1/5	3/40	—	
2	x_1	5/4	1	0	0	–1/4	—	
	Z_j	15	2	20	4	1	—	
	$Z_j - C_j$		0	0	14	1		

Since all $Z_j - C_j \geq 0$, the solution is optimum but the variables are non-integer.

∴ The non-integer optimum solution is given by,

$$x_1 = 5/4,\ x_2 = 5/8,\ x_3 = 0,\ \text{Max } Z = 15$$

To obtain an integer optimum solution, we proceed as follows.

$$\text{Max } \{f_1 f_2\} = \text{Max } \{5/8,\ 1/4\} = 5/8$$

∴ The source row is the first row, namely, x_2 row. From this source row we have,

$$5/8 = 0x_1 + 1x_2 + (1/5)x_3 + (3/40)S_1.$$

The fractional cut constraint is given by,

$$(1/5)x_3 + (3/40)S_1 \geq 5/8$$

$$(-1/5)x_3 - (3/40)S_1 \leq -5/8 \Rightarrow (-1/5)x_3 - (3/40)S_1 + G_1 = 5/8$$

where, G_1 is the Gomorian slack.

		C_j	2	20	–10	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	G_1
20	x_2	$\frac{5}{8}$	0	1	1/5	3/40	0
2	x_1	$\frac{5}{4}$	1	0	0	–1/4	0
←0	G_1	$\frac{-5}{8}$	0	0	–1/5	–3/40	1
	Z_j	15	2	20	4	1	0
	$Z_j - C_j$		0	0	14	1 ↑	0

We apply dual simplex method. Since $G_1 = -5/8$, G_1 leaves the basis.

Also, $$\text{Max}\left\{\frac{Z_j - C_j}{a_i k},\ a_i k < 0\right\} = \text{Max}\left\{\frac{14}{-1/5}\ \frac{1}{-3/40}\right\} = \text{Max} -\frac{40}{3}$$

gives the non-basic variable S_1, this enters the basis.

NOTES

	C_j		2	20	–10	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	G_1
20	x_2	0	0	1	0	0	+ 1
2	x_1	10/3	1	0	2/3	0	–10/3
0	S_1	25/3	0	0	8/3	1	–40/3
	Z_j	20/3	2	20	4/3	0	40/3
	$Z_j - C_j$		0	0	34/3	0	40/3

Again since the solution is non-integer, we add one more fractional cut constraint.

$$\text{Max}\{f_i\} = \text{Max } \{ 0, 1/3, 1/3\}$$

Since the max fraction is same for both the rows x_1 and S_1, we choose S_1 arbitrarily.

∴ From the source row we have,

$$25/3 = 0x_1 + 0x_2 + (8/3)x_3 + 1S_1 - (40/3)G_1$$

Expressing the negative fraction as the sum of negative integer and positive fraction we have,

$$(8 + 1/3) = 0x_1 + 0x_2 + (2 + 2/3)x_3 + 1S_1 + (-14 + 2/3)G_1$$

The corresponding fractional cut is given by,

$$-2/3x_3 - 2/3\ G_1 + G_2 = -1/3.$$

Add this second Gomorian constraint at the bottom of the above simplex table and apply dual simplex method.

	C_j		2	20	–10	0	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	G_1	G_2
20	x_2	0	0	1	0	0	1	0
2	x_1	10/3	1	0	2/3	0	–10/3	0
0	S_1	25/3	0	0	8/3	1	–40/3	0
← 0	G_2	–1/3	0	0	–2/3	0	–2/3	1
	Z_j	20/3	2	20	4/3	0	40/3	0
	$Z_j - C_j$		0	0	34/3↑	0	40/3	0

Since G_2 = –1/3, G_2 leaves the basis. Also,

$$\text{Max}\left(\frac{Z_j - C_j}{a_i k}, a_i k < 0\right) = \text{Max}\left(\frac{34/3}{-2/3} - \frac{40/3}{-2/3} -\right) = -17$$

gives the non-basic variable x_3 which enters the basis. Using dual simplex method, introduce x_3 and drop G_2.

	C_j		2	20	–10	0	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	G_1	G_2
20	x_2	0	0	1	0	0	1	0
2	x_1	3	1	0	0	0	– 4	1
0	S_1	7	0	0	0	1	16	4
–10	x_3	1/2	0	0	1	0	1	–3/2
	Z_j	1	2	20	–10	0	2	17
	$Z_j - C_j$		0	0	0	0	2	17

Since the solution is still a non-integer, a third fractional cut is required. It is given from the source row (x_3 row) as,

$$-1/2 = -1/2\ G_2 + G_3$$

Insert this additional constraint at the bottom of the table, the modified simplex table is shown below.

		C_j	2	20	–10	0	0	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	G_1	G_2	G_3
20	x_2	0	0	1	0	0	1	0	0
2	x_1	3	1	0	0	0	–4	1	0
0	S_1	7	0	0	0	1	–16	4	0
–10	x_3	1/2	0	0	1	0	1	3/2	0
←0	G_3	–1/2	0	0	0	0	0	–1/2	1
	Z_j	1	2	20	–10	0	2	17	0
	$Z_j - C_j$		0	0	0	0	2	17↑	0

Using dual simplex method, we drop G_3 and introduce G_2.

		C_j	2	20	–10	0	0	0	0
G_B	B	x_B	x_1	x_2	x_3	S_1	G_1	G_2	G_3
20	x_2	0	0	1	0	0	0	0	0
2	x_1	2	1	0	0	0	–4	0	2
0	S_1	3	0	0	0	1	–16	0	8
–10	x_3	2	0	0	1	0	–1	0	–3
0	G_2	1	0	0	0	0	6	1	–2
	Z_j	–16	2	20	–10	0	2	0	34
	$Z_j - C_j$		0	0	0	0	2	0	34

Since all $Z_j - C_j \geq 0$ and also the variables are integers, the optimum integer solution is obtained and given by, $x_1 = 2, x_2 = 0, x_3 = 2$ and Max $Z = 16$.

Example 8.4: Solve the integer programming problem.

Max $Z = 7x_1 + 9x_2$

Subject to, $-x_1 + 3x_2 \leq 6$

$7x_1 + x_2 \leq 35$

$x_1, x_2 \geq 0$ are integers.

Solution: Introducing slack variables $S_1, S_2 \geq 0$, we get the standard form of LPP as,

Max $Z = 7x_1 + 9x_2 + 0S_1 + 0S_2$

Subject to, $-x_1 + 3x_2 + S_1 = 6$

$7x_1 + x_2 + S_2 = 35$

NOTES

Now ignoring the integer conditions, solve the given LPP by simplex method.

		c_j	7	9	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	$Min\ \frac{x_3}{x_2}$
←0	S_1	6	–1	3	1	0	6/3 = 2
0	S_2	35	7	1	0	1	35/1 = 35
	Z_j	0	0	0	0	0	
	$Z_j - C_j$		–7	– 9↑	0	0	
9	x_2	2	–1/3	1	1/3	0	–
←0	S_2	33	22/3	0	–1/3	1	$33\times\frac{22}{2}$
	Z_j	18	–3	9	3	0	
	$Z_j - C_j$		–10↑	0	3	0	
9	x_2	7/2	0	1	7/22	1/22	
7	x_1	9/2	1	0	–1/22	3/22	
	Z_j	63	7	9	28/11	15/11	
	$Z_j - C_j$		0	0	28/11	15/11	

Since all $Z_j - C_j \geq 0$, optimum solution is obtained as $x_1 = \frac{9}{2}$

$$x_2 = \frac{7}{2} \text{ and Max } Z = 63.$$

Since the optimum solution obtained above is not an integer solution, we select a constraint corresponding to,

$$\text{Max } \{f_i\} = \text{Max } \{f_1\, f_2\}$$

$$= \text{Max}\left\{\frac{1}{2},\frac{1}{2}\right\}\left[x_{B1} = \frac{7}{2} = [3] + \frac{1}{2},\, x_{B2} = \frac{9}{2} = [4] + \frac{1}{2}\right]$$

Since both the equations have the same value of f_i, either one of the two equations can be used. Let us consider the x_2 row as source row.

From x_2 row we have,

$$\frac{7}{2} = 0x_1 + x_2 + \frac{7}{22}S_1 + \frac{1}{22}S_2$$

There is no negative fraction.

The Gomorian constraint is given by,

$$\frac{7}{22}S_1 + \frac{1}{22}S_2 \geq \frac{1}{2}$$

i.e., $$\frac{7}{22}S_1 - \frac{1}{22}S_2 \leq -\frac{1}{2}$$

$\Rightarrow$ $$-\frac{7}{22}S_1 - \frac{1}{22}S_2 + G_1 = -\frac{1}{2}$$

NOTES

where, G_1 is the Gomorian slack. Adding this new constraint at the bottom of the above optimal simplex table, we have the new table.

		c_j	7	9	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	G_1
9	x_2	9/2	0	1	7/22	1/22	0
7	x_1	7/2	1	0	–1/22	3/22	0
0	G_1	–1/2	0	0	–7/22	–1/22	1
	Z_j	63	9	7	28/11	15/11	0
	$Z_j - C_j$		0	0	28/11↑	15/11	0

We apply dual simplex method, since G_1 = – 1/2, G_1 leaves the basis. Also,

$$\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{28}{11}}{-\frac{7}{22}}\ \frac{\frac{15}{11}}{-\frac{1}{22}}\right\}$$

$$= \text{MAX } (-8, -30) = -8$$

gives the non-basic variable S_1 to enter into the basis.

Applying dual simplex method, drop G_1 and introduce S_1.

		c_j	7	9	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	G_1
9	x_2	3	0	1	0	0	1
7	x_1	32/7	1	0	0	1/7	–1/7
0	S_1	11/7	0	0	1	+1/7	–22/7
	Z_j	59	7	9	0	1	8
	$Z_j - C_j$		0	0	0	1	8

The optimal solution obtained by dual simplex method as above is still a non-integer. Thus a new Gomory's constraint is to be reconsidered.

$$\text{Max } \{f_i\} = \text{Max}\left\{-, \frac{4}{7}, \frac{4}{7}\right\} = \frac{4}{7}$$

Choose the x_1 row as source row arbitrarily as both the fraction values are the same. From the source row we have,

$$\frac{4}{7} = 1x_1 + 0x_2 + 0S_1 + \frac{1}{7}S_2 + \frac{6}{7}G_1$$

There is no negative fraction in the source row.

The Gomory's constraint is given by,

$$\frac{1}{7}S_2 + \frac{6}{7}G_1 \geq \frac{4}{7} \quad \text{i.e.,} \quad \frac{1}{7}S_2 - \frac{6}{7}G_1 + G_2 = -\frac{4}{7}$$

NOTES

where, G_2 is the Gomorian slack. Adding this constraint in the above simplex table we get a modified table.

		C_j	7	9	0	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	G_1	G_2
9	x_2	3	0	1	0	0	1	0
7	x_1	32/7	1	0	0	1/7	–1/7	0
0	S_1	11/7	0	0	1	1/7	–22/7	0
0	G_2	–4/7	0	0	0	–1/7	–6/7	1
	Z_j	59	7	9	0	1	8	0
	$Z_j - C_j$		0	0	0	1↑	8	0

We again apply the dual simplex method.

Since $G_2 = -\frac{4}{7}$, G_2 leaves the basis. Also,

$$\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{1}{-\frac{1}{7}}, \frac{8}{-\frac{6}{7}}\right\}$$

$$= \text{Max}\ (-7, -9) = -7$$

gives the non-basic variable S_2 to enter into the basis.

		C_j	7	9	0	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	G_1	G_2
9	x_2	3	0	1	0	0	1	0
7	x_1	4	1	0	0	0	–1	1
0	S_1	1	0	0	1	0	–4	1
0	S_2	4	0	0	0	1	6	–7
	Z_j	55	7	9	0	0	2	7
	$Z_j - C_j$		0	0	0	0	2	7

Since all $Z_j - C_j \geq 0$ and also the solution is an integer, we obtain an optimum integer solution given by, $x_1 = 4$, $x_2 = 3$ and Max $Z = 55$.

8.2.4 Mixed Integer Programming Problem

In mixed IPP only some of the variables are restricted to integer values, while the other variables may take integer or other real values.

Mixed integer cutting plane procedure The iterative procedure for the solution of mixed integer programming problem is as follows.

Step 1 Reformulate the given LPP into a standard maximization form and then determine an optimum solution using simplex method.

Step 2 Test the integrality of the optimum solution.

(*i*) If all $x_{Bi} \geq 0$ ($i = 1, 2,... m$) and are integers, then the current solution is an optimum one.

(*ii*) If all $x_{Bi} \geq 0$ ($i = 1, 2,... m$) but the integer restricted variables are not integers, then go to the next step.

Step 3 Choose the largest fraction among those x_{Bi}, which are restricted to integers. Let it be $x_{Bk} = f_k$ (assume)

Step 4 Find the fractional cut constraints from the source row, namely Kth row.

From the source row,

$$\sum_{j=1}^{n} a_{kj}\, k_j = x_{Bk}$$

i.e,
$$\sum_{j=1}^{n} \left(\left[a_{kj}\right] + f_{ki}\right) r_j = [x_{BK}] + f_k$$

in the form
$$\sum_{j=1}^{n} f_{ki}\, x_j \geq f_k$$

i.e.,
$$\sum_{j \in j^+} f_{kj}\, x_j + \left(\frac{f_k}{f_{k-1}}\right) \sum_{j \in j^-} f_{kj}\, x_j \geq + f_k$$

$$-\sum_{j \in j^+} f_{kj}\, x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j \in j^-} f_{kj}\, x_j \leq -f_k$$

$$-\sum_{j \in j^+} f_{kj}\, x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j \in j^-} f_{kj}\, x_j + G_k = -f_k$$

where, G_k is Gomorian slack

$$\begin{matrix} j^+ = \\ j^{-1} = \end{matrix} \begin{bmatrix} j / f_{kj} \geq 0 \\ j / f_{kj} < 0 \end{bmatrix}$$

Step 5 Add this cutting plane generated in step K at the bottom of the optimum simplex table obtained in step 1. Find the new optimum solution using dual simplex method.

Step 6 Go to step 2 and repeat the procedure until all $x_{Bi} \geq 0$ ($i = 1, 2,... m$) and all restricted variables are integers.

Example 8.5:

Max $Z = x_1 + x_2$

Subject to, $3x_1 + 2x_2 \leq 5$

$x_2 \leq 2$

$x_1 + x_2 \geq 0$ and x_1 is an integer.

NOTES

Solution: Introducing slack variables $S_1, S_2 \geq 0$ the standard form of LPP is,

Max $Z = x_1 + x_2 + 0S_1 + 0S_2$

Subject to, $3x_1 + 2x_2 + S_1 = 5$

$$x_2 + S_2 = 2$$

$$x_1, x_2, S_1, S_2 \geq 0$$

Initial basic feasible solution,

$$S_1 = 5, S_2 = 2$$

Ignore the integer condition and solve the problem using simplex method, to obtain optimum solution.

		C_j	1	1	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	$Min \frac{x_B}{x_1}$
←0	S_1	5	3	2	1	0	5/3
0	S_2	2	0	1	0	1	—
	Z_j	0	0	0	0	0	
	$Z_j - C_j$		−1↑	−1	0	0	$Min \frac{x_B}{x_2}$
1	x_1	5/3	1	2/3	1/3	0	5/2
←0	S_2	2	0	1	0	1	5/2
	Z_j	5/3	1	2/3	1/3	0	
	$Z_j - C_j$		0	−1/3↑	1/3	0	
1	x_1	1/3	1	0	1/3	−2/3	
1	x_2	2	0	1	0	1	
	Z_j	7/3	1	1	1/3	1/3	
	$Z_j - C_j$		0	0	1/3	1/3	

Since all $Z_j - C_j \geq 0$, the current basic feasible solution is optimum. But x_1 is non-integer. From the source row (first row) we have,

$$1/3 = x_1 + 0\ x_2 + 1/3\ S_1 - 2/3\ S_2$$

The Gomorian constraint is given by,

$$\frac{1}{3}S_1 + \left(\frac{\frac{1}{3}}{\frac{1}{3}-1}\right)\left(\frac{-2}{3}\right)S_2 \geq \frac{1}{3}$$

$$\frac{1}{3}S_1 + \frac{1}{3}S_2 \geq \frac{1}{3} \quad \Rightarrow \quad -\frac{1}{3}S_1 - \frac{1}{3}S_2 \leq -\frac{1}{3}$$

$$\frac{-1}{3}S_1 - \frac{1}{3}S_2 + G_1 = -\frac{1}{3}$$

where, G_1 is the Gomorian slack.

Adding this Gomorian constraint at the bottom of the above simplex table, we have,

		C_j	1	1	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	G_1
1	x_1	1/3	1	0	1/3	–2/3	0
1	x_2	2	0	1	0	1	0
0	G_1	–1/3	0	0	–1/3	–1/3	1
	Z_j	7/3	1	1	1/3	1/3	0
	$Z_j - C_j$		0	0	1/3↑	1/3	0

NOTES

Using the dual simplex method, since $G_1 = –1/3 < 0$, G_1 leaves the basis. Also,

$$\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\}$$

$$\text{Max}\left\{\frac{\frac{1}{3}}{\frac{-1}{3}}, \frac{\frac{1}{3}}{\frac{-1}{3}}\right\} = \text{Max } \{-1, -1\} = -1$$

As this corresponds to both S_1 and S_2, we choose S_1 arbitrarily as the entering variable.

Drop G_1 and introduce S_1.

		C_j	1	1	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	G_1
1	x_1	0	1	0	0	– 1	1
1	x_2	2	0	1	0	1	0
0	S_2	1	0	0	1	1	– 3
	Z_j	2	1	1	0	0	1
	$Z_j - C_j$		0	0	0	0	1

Since all $Z_j – C_j \geq 0$ and all $x_{Bi} \geq 0$, the current solution is feasible and optimal.

The required optimal integer solution is given by,

$$x_1 = 0, x_2 = 2 \text{ and Max } Z = 2.$$

Example 8.6:

Max $Z = 4x_1 + 6x_2 = 2x_3$

Subject to,

$$4x_1 - 4x_2 \leq 5,$$
$$-x + 6x_2 \leq 5,$$
$$-x_1 + x_2 + x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0 \text{ and } x_1, x_3 \text{ are integers.}$$

Solution Introducing slack variables $S_1, S_2, S_3 \geq 0$ the standard form of LPP is,

Max $Z = 4x_1 + 6x_2 + 2x_3 + 0S_1 + 0S_2 + 0S_3$

NOTES

Subject to,
$$4x_1 - 4x_2 + S_1 = 5$$
$$-x_1 + 6x_2 + S_2 = 5$$
$$-x_1 + x_2 + x_3 + S_3 = 5$$

The initial basic feasible solution is given by $S_1 = 5$, $S_2 = 5$ and $S_3 = 5$. Ignoring the integer condition, the optimum solution of given LPP is obtained by the simplex method.

Since all $Z_j - C_j \geq 0$, the solution is optimum. But the integer constrained variables x_1 and x_3 are non-integer.

$$x_1 = 5/2 = 2 + 1/2$$
$$x_2 = 25/4 = 6 + 1/4$$

Max $(f_1\, f_3)$ = Max (1/2, 1/4) = 1/2

From the first row we have,

$$(2 + 1/2\ ) = x_1 + 0x_2 + 0x_3 + (3/10)\ S_1 + (1/5)\ S_2$$

		C_j	4	6	2	0	0	0	
C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	S_3	Min x_B/x_2
0	S_1	5	4	–4	0	1	0	0	—
←0	S_2	5	–1	6	0	0	1	0	5/6
0	S_3	5	–1	1	1	0	0	1	5/1
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$		–4	–6↑	–2	0	0	0	Min x_B/x_1
0	S_1	25/3	10/3	0	0	1	2/3	0	25/10
6	x_2	5/6	–1/6	1	0	0	1/6	0	—
0	S_3	25/6	–5/6	0	1	0	–1/6	1	—
	Z_j	5	–1	6	0	0	1	0	
	$Z_j - C_j$		–5↑	0	–2	0	1	0	Min x_B/x_3
4	x_1	5/2	1	0	0	3/10	1/5	0	—
6	x_2	5/4	0	1	0	1/20	1/5	0	25
←0	S_3	25/4	0	0	1	1/4	0	1	4/1
	Z_j	35/2	4	6	0	3/2	2	0	
	$Z_j - C_j$		0	0	–2↑	3/2	2	0	
4	x_1	5/2	1	0	0	3/10	1/5	0	
6	x_2	5/4	0	1	0	1/20	1/5	0	
2	x_3	25/4	0	0	1	1/4	0	1	
	Z_j	35/2	4	6	2	3/2	2	0	
	$Z_j - C_j$		0	0	0	3/2	2	0	

The Gomorian constraint is given by,

$$3/10\ S_1 + 1/5\ S_2 \geq 1/2$$
$$-3/10\ S_1 - 1/5\ S_2 \leq -1/2$$

i.e., $-3/10\ S_1 - 1/5\ S_2 + G_1 = -\ 1/2$, where G_1 is the Gomorian slack. Introduce this new constraint at the bottom of the above simplex table.

NOTES

Using dual simplex method, since $G_1 = -1/2 < 0$,

G_1 leaves the basis. Also,

$$\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{2}{\frac{-3}{10}}, \frac{2}{\frac{-1}{5}}\right\} = \text{Max}\left\{\frac{-20}{3}, -10\right\} = \frac{-20}{3}$$

		c_j	4	6	2	0	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	G_1
4	x_1	5/2	1	0	0	3/10	1/5	0
6	x_2	5/4	0	1	0	1/20	1/5	0
2	x_3	25/4	0	0	1	1/4	0	0
←0	G_1	–1/2	0	0	0	–3/10	–1/5	1
	Z_j	30	4	6	2	2	2	0
	$Z_j - C_j$		0	0	0	2↑	2	0

corresponding to S_1.Therefore, the non-basic variable S_1 enters the basics. Drop G_1 and introduce S_1.

		c_j	4	6	2	0	0	0	
C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	S_3	G_1
4	x_1	2	1	0	0	0	0	0	1
6	x_2	7/6	0	1	0	0	1/6	0	1/6
2	x_3	35/6	0	0	1	0	–1/6	1	5/6
0	S_1	5/3	0	0	0	1	2/3	0	–10/3
	Z_j	80/3	4	6	2	0	2/3	2	20/3
	$Z_j - C_j$		0	0	0	0	2/3	2	20/3

Since all $Z_j - C_j \geq 0$, the solution is optimum and also the integer restricted variable $x_3 = 35/6$ is not an integer, therefore, we add another Gomorian constraint

$$x_3 = 35/6 = 5 + 5/6$$

The source row is the third row.

From this row we have,

$$5 + \frac{5}{6} = 0x_1 + 0x_2 + x_3 + 0S_1 - \frac{1}{6}S_2 + S_3 + \frac{5}{6}G_1$$

The Gomorian constraint is given by,

$$\left(\frac{\frac{5}{6}}{\frac{5}{6} - 1}\right)\left(\frac{-1}{6}\right)S_2 + \frac{5}{6}G_1 \geq \frac{5}{6}$$

$$\Rightarrow \qquad \frac{5}{6}S_2 + \frac{5}{6}G_1 \geq \frac{5}{6}$$

$$\Rightarrow \qquad \frac{-5}{6}S_2 - \frac{5}{6}G_1 + G_2 \leq \frac{-5}{6}$$

NOTES

where, G_2 is the Gomorian slack.

Add this second cutting plane constraint at the bottom of the above optimum simplex table.

		C_j	4	6	2	0	0	0	0	0
C_B	B	x_B	x_1	x_2	x_3	S_3	S_1	S_2	G_1	G_2
4	x_1	2	1	0	0	0	0	0	1	0
6	x_2	7/6	0	1	0	0	0	1/6	1/6	0
2	x_3	35/6	0	0	1	1	0	–1/6	5/6	0
0	S_1	5/3	0	0	0	0	1	2/3	–10/3	0
$\leftarrow$ 0	G_2	–5/6	0	0	0	0	0	–5/6	–5/6	1
	Z_j	80/3	4	6	2	2	2/3	2/3	20/3	0
	$Z_j - C_j$		0	0	0	2	2/3	$\uparrow$2/3	20/3	0

Use dual simplex method. ($\because$ $G_2 = -5/6 < 0$)

G_2 leaves the basics.

Also, $\text{Max}\left\{\dfrac{Z_j - C_j}{a_{ik}}\; a_{ik} < 0\right\} = \text{Max}\left\{\dfrac{\frac{2}{3}}{\frac{-5}{6}}, \dfrac{\frac{20}{3}}{\frac{-5}{6}}\right\} = \text{Max}\left\{\dfrac{-4}{5}, -8\right\} = -\dfrac{4}{5},$

which corresponds to S_2.

Drop G_2 and introduce S_2.

Since all $Z_j - C_j \geq 0$ and also all the restricted variables x_1 and x_3 are integers, an optimum integer solution is obtained.

The optimum integer solution is,

$x_1 = 2, x_2 = 1, x_3 = 6$ and Max $Z = 26$

		C_j	4	6	2	0	0	0	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	S_3	G_1	G_2
4	x_1	2	1	0	0	0	0	0	1	0
6	x_2	1	0	1	0	0	0	0	0	1/5
2	x_3	6	0	0	1	0	0	1	1	–1/5
0	S_1	1	0	0	0	1	0	0	–4	4/5
0	S_2	1	0	0	0	0	1	0	1	–6/5
	Z_j	26	4	6	2	0	0	2	6	4/5
	$Z_j - C_j$		0	0	0	0	0	2	6	4/5

8.2.5 Branch and Bound Technique

This method is applicable to both, pure as well as mixed IPP. Sometimes a few or all the variables of an IPP are constrained by their upper or lower bounds. The most general method for the solution of such constrained optimization problems is called 'Branch and Bound method'.

This method first divides the feasible region into smaller subsets and then examines each of them successively, until a feasible solution that gives an optimal value of objective function is obtained.

Let the given IPP be,

Max $Z = CX$

Subject to, $Ax \leq b$

$X \geq 0$ are integers.

In this method, we first solve the problem by ignoring the integrality condition.

(*i*) If the solution is in integers, the current solution is optimum for the given IPP.

(*ii*) If the solution is not in integers, say one of the variable X_r is not an integer, then $x_r^* < x_r < x^*_{r+1}$ where x^*_r, x^*_{r+1} are consecutive non-negative integers.

Hence, any feasible integer value of x_r must satisfy one of the two conditions.

$$x_r \leq x_r^* \text{ or } x_r \geq x^*_{r+1}.$$

These two conditions are mutually exclusive (both cannot be true simultaneously). By adding these two conditions separately to the given LPP, we form different sub-problems.

Sub-problem 1	Sub-problem 2
Max $Z = Cx$	Max $Z = Cx$
Subject to, $Ax \leq b$	Subject to, $Ax \leq b$
$x_r \leq x^*_r$	$x_r \geq x^*_{r+1}$
$x \geq 0.$	$x \geq 9.$

Thus, we have branched or partitioned the original problem into two sub-problems. Each of these sub-problems is then solved separately as LPP.

If any sub-problem yields an optimum integer solution, it is not further branched. But if any sub-problem yields a non-integer solution, it is further branched into two sub-problems. This branching process is continued until each problem terminates with either an integer optimal solution or there is an evidence that it cannot yield a better solution. The integer-valued solution among all the sub-problems, which gives the most optimal value of the objective function is then selected as the optimum solution.

Note: For minimization problem, the procedure is the same except that upper bounds are used. The sub-problem is said to be fathomed and is dropped from further consideration if it yields a value of the objective function lower than that of the best available integer solution and it is useless to explore the problem any further.

NOTES

Example 8.7: Use branch and bound technique to solve the following:

Max $Z = x_1 + 4x_2$

Subject to, $2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_1, x_2 \ge 0$ and are integers.

Solution: Ignoring the integrality condition we solve the LPP,

Max $Z = x_1 + 4x_2$

Subject to, $2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_{1,}\ x_2 \ge 0$

Introducing slack variables $S_1, S_2 \ge 0$, the standard form of LPP becomes,

Max $Z = x_1 + 4x_2 + 0S_1 + 0S_2$

Subject to, $2x_1 + 4x_2 + S_1 = 7$

$5x_1 + 3x_2 + S_2 = 15$

		C_j	1	4	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	Min x_B/x_2
←0	S_1	7	2	4	1	0	7/4
0	S_2	15	5	3	0	1	15/3 = 5
	Z_j	0	0	0	0	0	
	$Z_j - C_j$		–1	–4↑	0	0	
4	x_2	7/4	$\frac{1}{2}$	1	1/4	0	
0	S_2	$\frac{39}{4}$	$\frac{7}{2}$	0	–3/4	1	
	Z_j	7	2	4	1	0	
	$Z_j - C_j$		1	0	1	0	

Since all $Z_j - C_j \ge 0$, an optimum solution is obtained.

$x_1 = 0,\ x_2 = 7/4$ and Max $Z = 7$

Since $x_2 = \frac{7}{4}$, this problem should be branched into two sub-problems.

For $x_2 = \frac{7}{4},\ 1 < x_2 < 2 = x_2 \le 1,\ x_2 \ge 2$

Applying these two conditions separately in the given LPP we get two sub-problems.

Sub-problem (1) Sub-problem (2)

Max $Z = x_1 + 4x_2$ Max $Z = x_1 + x_2$

Subject to, $2x + 4x_2 \leq 7$ Subject to, $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$ $5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$ $x_2 \geq 2$

$x_1, x_2 \geq 0$ $x_1, x_2 \geq 0$

NOTES

Sub-Problem (1)

		C_j	1	4	0	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	$Min \frac{x_B}{x_2}$
0	s_1	7	2	4	1	0	0	7/4
0	s_2	15	5	3	0	1	0	15/3
←0	s_3	1	0	1	0	0	1	1/1
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		–1	–4↑	0	0	0	$Min \frac{x_B}{x_1}$
←0	s_1	3	2	0	1	0	–4↑	3/2
0	s_2	12	5	0	0	1	–3	12/5
0	x_2	1	0	1	0	0	1	
	Z_j	4	0	4	0	0	4	
	$Z_j - C_j$		–1↑	0	0	0	4	

		C_j	1	4	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	S_3
1	x_1	3/2	1	0	1/2	0	–2
0	s_2	9/2	0	0	–5/2	1	7
4	x_2	1	0	1	0	0	1
	Z_j	11/2	1	4	1/2	0	2
	$Z_j - C_j$		0	0	1/2	0	2

Since all $Z_j - C_j \geq 0$, the solution is optimum, given by $x_1 = 3/2$

$x_2 = 1$, and Max $Z = 11/2$

Since $x_1 = 3/2$ is not an integer, this sub-problem is branched again.

Sub-Problem (2)

Max $Z = x_1 + 4x_2$

Subject to,
$$2x_1 + 4x_2 \leq 7$$
$$5x_1 + 3x_2 \leq 15$$
$$x_2 \geq 2$$
$$x_1, x_2 \geq 0.$$

NOTES

		C_j	1	4	0	0	0	$-M$	
C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	A_1	Min x_B/x_2
←0	S_1	7	2	4	1	0	0	0	7/4
0	S_2	15	5	3	0	1	0	0	15/3
$-M$	A_1	2	0	1	0	0	-1	1	2/1
	Z_j	$-2M$	0	$-M$	0	0	M	$-M$	
	$Z_j - C_j$		-1	$-M-4$	0	0	M	0	
4	x_2	7/4	1/2	1	1/4	0	0	0	
0	S_2	39/4	7/2	0	$-3/4$	1	0	0	
$-M$	A_1	1/4	$-1/2$	0	$-1/4$	0	-1	1	
	Z_j	$7-\frac{5M}{4}$	$2+\frac{M}{2}$	4	$1+\frac{M}{4}$	0	M	$-M$	
	$Z_j - C_j$		$\frac{M}{2}+1$	0	$\frac{M}{4}+1$	0	M	0	

Since all $Z_j - C_j \geq 0$, but an artificial variable A_1 is in the basis at positive level, there exists no feasible solution. Hence, this sub-problem is dropped.

In sub-problem (1) Since, $x_1 = 3/2$

we have, $1 \leq x_1 \leq 2$

$= x_1 \leq 1, \quad x_1 \geq 2$

Applying these two conditions separately in the sub-problem (1), we get two sub-problems.

Sub-problem (3)

Max $Z = x_1 + 4x_2$

Subject to, $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

$x_1 \leq 1$

$x_1, x_2 \geq 0.$

Sub-problem (4)

Max $Z = x_1 + x_2$

Subject to, $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

$x_1 \geq 2$

$x_1, x_2 \geq 0$

Sub-Problem (3)

Since all $Z_j - C_j \geq 0$, an optimum solution is obtained. It is given by, $x_1 = 1$, $x_2 = 1$ and Max $Z = 5$. Since this solution is integer-valued this sub-problem cannot be branched further. The lower bound of the objective function is 5.

NOTES

		C_j	1	4	0	0	0	0	
C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	S_4	$Min \frac{x_B}{x_2}$
0	S_1	7	2	4	1	0	0	0	7/4
0	S_2	15	5	3	0	1	0	0	15/3
←0	S_3	1	0	1	0	0	1	0	1/1
0	S_4	1	1	0	0	0	0	1	—
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$		–1	–4↑	0	0	0	0	$Min \frac{x_B}{x_1}$
0	S_1	3	2	0	1	0	–4	0	3/2
0	S_2	12	5	0	0	1	–3	0	12/5
4	x_2	1	0	1	0	0	1	0	—
←0	S_4	1	1	0	0	0	0	1	1/1
	Z_j	4	0	4	0	0	4	0	
	$Z_j - C_j$		–1↑	0	0	0	4	0	

		C_j	1	4	0	0	0	0
C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	S_4
0	S_1	1	0	0	1	0	–4	–2
0	S_2	7	0	0	0	1	–3	–5
4	x_2	1	0	1	0	0	1	0
1	x_1	1	1	0	0	0	0	1
	Z_j	5	1	4	0	0	4	1
	$Z_j - C_j$		0	0	0	0	4	1

Sub-Problem (4)

Max $Z = x_1 + 4x_2$

Subject to,
$$2x_1 + 4x_2 \leq 7$$
$$5x_1 + 3x_2 \leq 15$$
$$x_2 \leq 1$$
$$x_1 \geq 2$$
$$x_1,\ x_2 \geq 0$$

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by,

$$x_1 = 2, \quad x_2 = 3/4$$

NOTES

	C_B	x_B	B	x_1	x_2	S_1	S_2	S_3	S_4	A_1	$Min \frac{x_B}{x_1}$
			c_j	1	4	0	0	0	0	$-M$	
	0	7	S_1	2	4	1	0	0	0	0	7/2
	0	15	S_2	5	3	0	1	0	0	0	15/5
	0	1	S_3	0	1	0	0	1	0	0	—
←	$-M$	2	A_1	1	0	0	0	0	–1	1	2/1
		$-2M$	Z_j	$-M$	0	0	0	0	M	$-M$	
		—	$Z_j - C_j$	$-M-1$	–4	0	0	0	M	0	

C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	S_4	A_1	$Min\ x_B/x_2$
	c_j		1	4	0	0	0	0	$-M$	
←0	S_1	3	0	4	1	0	0	2	–2	3/4
0	S_2	5	0	3	0	1	0	5	–5	5/3
0	S_3	1	0	1	0	0	1	0	0	1/1
1	x_1	2	1	0	0	0	0	–1	1	—
	Z_j	2	1	0	0	0	0	–2	1	
	$Z_j - C_j$		0	–4	0	0	0	–2	$1 + M$	
4	x_2	3/4	0	1	1/4	0	0	1/2	—	
0	S_2	11/4	0	0	–3/4	1	0	7/2	—	
0	S_3	1/4	0	0	–1/4	0	1	–1/2	—	
1	x_1	2	1	0	0	0	0	–1	—	
	Z_j	5	1	4	1	0	0	1	—	
	$Z_j - C_j$		0	0	1	0	0	1	—	

Since $x_2 = 3/4$ is not an integer, this sub-problem is branched further.

In sub-problem (4) since $x_2 = 3/4$, $0 \le x_2 \le 1$

$$= x_2 \le 0, \quad \text{or} \quad x_2 \ge 1$$

Applying these two conditions in the sub-problem (4)

We get two sub-problems.

Sub-problem (5)

Max $Z = x_1 + 4x_2$

Subject to,$2x + 4x_2 \le 7$

$$5x_1 + 3x_2 \le 15$$
$$x_2 \le 1$$
$$x_1 \ge 2$$
$$x_2 \le 0$$
$$x_1, x_2 \ge 0$$

Sub-problem (6)

Max $Z = x_1 + x_2$

Subject to, $2x_1 + 4x_2 \le 7$

$$5x_1 + 3x_2 \le 15$$
$$x_2 \le 1$$
$$x_1 \ge 2$$
$$x_2 \ge 1$$
$$x_1, x_2 \ge 0.$$

NOTES

Sub-Problem (5)

	C_B	B	x_B	x_1	x_2	S_1	S_2	S_3	S_4	A_1	S_5	Min x_B/x_1
		C_j	1	4	0	0	0	0	$-M$	0		
	0	S_1	7	2	4	1	0	0	0	0	0	7/2
	0	S_2	15	5	3	0	1	0	0	0	0	15/5
	0	S_3	1	0	1	0	0	1	0	0	0	—
←	$-M$	A_1	2	1	0	0	0	0	–1	1	0	2/1
	0	S_5	0	0	1	0	0	0	0	0	1	—
		Z_j	$-2M$	$-M$	0	0	0	0	M	$-M$	0	
		$Z_j - C_j$		$-M-1$	–4	0	0	0	M	0	0	Min x_B/x_2
	0	S_1	3	0	4	1	0	0	2	—	0	3/4
	0	S_2	5	0	3	0	1	0	5	—	0	5/3
	0	S_3	1	0	1	0	0	1	0	—	0	1/1
1	x_1	2	1	0	0	0	0	–1	—	0	—	
←	0	S_5	0	0	1	0	0	0	0	—	1	0/1
		Z_j	2	1	0	0	0	0	–2	—	0	
		$Z_j - C_j$		0	–4↑	0	0	0	–2	—	0	Min x_B/S_4
	0	S_1	3	0	0	1	0	0	2	—	0	3/2
←	0	S_2	5	0	0	0	1	0	5	—	0	1
	0	S_3	1	0	0	0	0	1	0	—	–1	—
	1	x_1	2	1	0	0	0	0	–1	—	0	—
	4	x_2	0	0	1	0	0	0	0	—	1	—
		Z_j	2	1	4	0	0	0	–1	—	0	
		$Z_j - C_j$		0	0	0	0	0	–1↑	—	0	
	0	S_1	1	0	0	1	–2/5	0	0	—	0	
	0	S_4	1	0	0	0	1/5	0	1	—	0	
	0	S_3	1	0	0	0	0	1	0	—	0	
	1	x_1	3	1	0	0	1/5	1	0	—	0	
	4	x_2	0	0	1	0	0	0	0	—	1	
		Z_j	3	1	4	0	3/5	0	0		4	
		$Z_j - C_j$		0	0	0	3/5	0	0	—	4	

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = 3$, $x_2 = 0$ and Max $Z = 3$. This sub-problem yields an optimum integer solution. Hence, this sub-problem is dropped.

Sub-Problem (6)

Max $Z = x_1 + 4x_2$

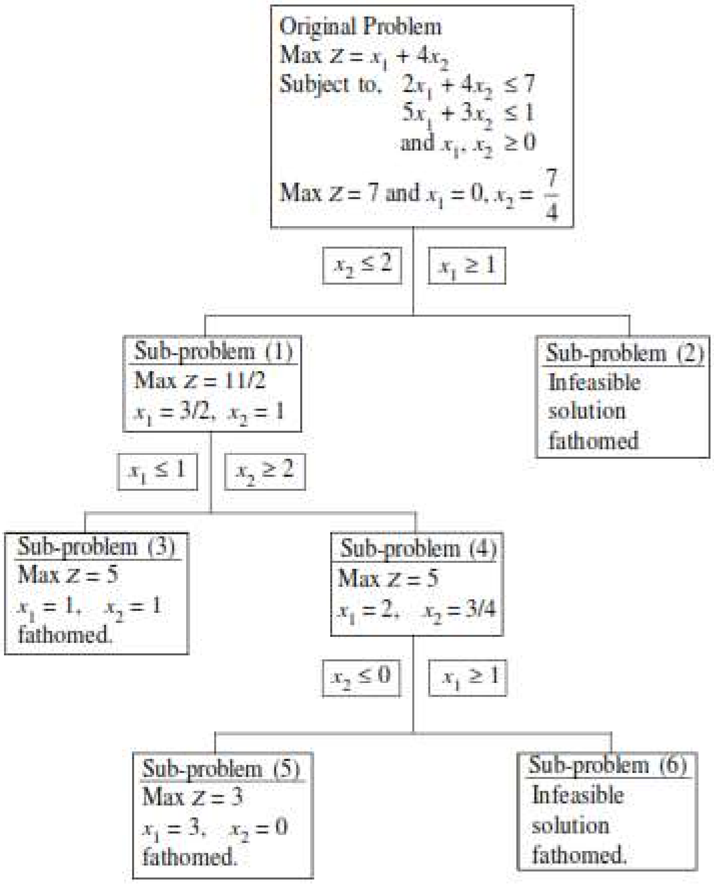
Subject to,

$$2x_1 + 4x_2 \leq 7$$
$$5x_1 + 3x_2 \leq 15$$
$$x_2 \leq 1$$
$$x_1 \geq 2$$
$$x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

NOTES

This sub-problem has no feasible solution. Hence, this sub-problem is also fathomed.

Original Problem



Among the available integer-valued solutions, the best integer solution is given by sub-problem (3).

$\therefore$ The optimum integer solution is,

$$\text{Max } Z = 5,\ x_1 = 1 \text{ and } x_2 = 1.$$

The best available integer optimal solution is,

$$\text{Max } Z = 5,\ x_1 = 1 \text{ and } x_2 = 1.$$

Check Your Progress

1. What do you mean by integer programming problem?
2. Define a pure integer programming problem.
3. Define a mixed integer programming problem.
4. Differentiate between pure and mixed IPP.
5. Mention a few applications of IPP.
6. Name the two methods used in solving IPP.

8.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. An LPP in which some or all the variables in the optimal solution are restricted to assume non-negative integer values is called an integer programming problem.
2. In a LPP, if all the variables in the optimal solution are restricted to assume non-negative integer value, then it is called a pure IPP.
3. In an LPP, if only some of the variables in the optimal solution are restricted to assume non-negative integer values, while the remaining variables are free to take any non-negative values, then it is called a mixed integer programming problem.
4. In a pure IPP all the variables in the optimal solution are restricted to assume non-negative integer values. Whereas in mixed IPP, only some of the variables in the optional solution are restricted to assume non-negative integer values.
5. Some applications of IPP are as follows:
   (i) In product mix problem
   (ii) Sequencing and routing decisions
   (iii) All allocation problems involving the allocation of goods, men and machine.
6. The two methods used in solving IPP are as follows:
   (i) Cutting methods (Gomany's cutting plane algorithm)
   (ii) Search method (Branch and Bound Technique)

8.4 SUMMARY

- A linear programming problem in which all or some of the decision variables are constrained to assume non-negative integer values is called an *Integer Programming Problem* (IPP).
- In a linear programming problem, if all variables are required to take integral values then it is called the *Pure* (*all*) *Integer Programming Problem* (Pure IPP).
- If only some of the variables in the optimal solution of a LPP are restricted to assume non-negative integer values, while the remaining variables are free to take any non-negative values, then it is called a *Mixed Integer Programming Problem* (Mixed IPP).
- If all the variables in the optimal solution are allowed to take values 0 or 1, then the problem is called the *0–1 Programming Problem* or *Standard Discrete Programming Problem.*

NOTES

- In LPP, all the decision variables were allowed to take any non-negative real values as it is quite possible and appropriate to have fractional values in many situations. There are several frequently occurring circumstances in business and industry that lead to planning models involving integer-valued variables.
- Integer programming is applied in business and industry. All assignment and transportation problems are integer programming problems, as in the assignment and travelling salesmen problem, all the decision variables are either zero or one.
- There are two methods used to solve IPP, namely Gomory's Cutting Plane Method and Branch and Bound Method (Search Method).
- Cutting method is a systematic procedure of solving pure IPP was first developed by R.E. Gomory, in 1956, which he later used to deal with the more complicated case of mixed integer programming problem.
- Cutting Method consists of first solving the IPP as an ordinary LPP by ignoring the restriction of integer values and then introducing a new constraint to the problem such that the new set of feasible solution includes all the original feasible integer solutions, but does not include the optimum non-integer solution initially found. This new constraint is called 'Fractional cut' or 'Gomorian constraint'.
- Search method is an enumeration method in which all feasible integer points are enumerated. The widely used search method is the Branch and Bound method. It was developed in 1960, by A.H. Land and A.G. Doig. This method is applicable to both pure and mixed IPP. It first divides the feasible region into smaller subsets that eliminate parts containing no feasible integer solution.
- In mixed IPP only some of the variables are restricted to integer values, while the other variables may take integer or other real values.
- Branch and Bound method is applicable to both, pure as well as mixed IPP. Sometimes a few or all the variables of an IPP are constrained by their upper or lower bounds. The most general method for the solution of such constrained optimization problems is called 'Branch and Bound method'.
- If any sub-problem yields an optimum integer solution, it is not further branched. But if any sub-problem yields a non-integer solution, it is further branched into two sub-problems.
- The integer-valued solution among all the sub-problems, which gives the most optimal value of the objective function is then selected as the optimum solution.

8.5 KEY WORDS

- **Integer Programming Problem (IPP):** It refers to a mathematical optimization or feasibility program in which some or all of the variables are restricted to be integers.
- **Cutting-plane method:** It refers to any of a variety of optimization methods that iteratively refine a feasible set or objective function by means of linear inequalities, termed cuts.
- **Branch and Bound Method:** It refers a systematic method for solving optimization problems.

8.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What are the applications of integer programming?
2. What is Standard Discrete Programming Problem?
3. Name the widely used search method and when was it developed.
4. Write an informative note on mixed integer programming problem.
5. What is branch and bound technique?

Long Answer Questions

1. Explain Gomarian constraint and also explain its geometrical interpretation.
2. Why not round off the optimum values instead of resorting to integer programming? Explain.
3. Discuss the importance of IPP.

8.7 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 9 INFINITE QUEUING MODELS

NOTES

Structure

9.0 INTRODUCTION

A flow of customers from finite/infinite population towards the service facility forms a *queue* (waiting line), on account of a lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, waiting time is required either for the service facilities or for the customer's arrival.

The arriving unit that requires some service to be performed is called *customer*. The customer may be persons, machines, vehicles, etc. Queue (waiting line) stands for the number of customers waiting to be serviced. This does not include the customer being serviced. The process or system that provides services to the customer is termed as service channel or service facility.

In this unit, you will learn about the fundamentals of queuing theory and models.

9.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the basic concepts of queuing theory and models
- Classify queuing models into finite and infinite models
- Explain the operating characteristics and constituents of a queuing system

9.2 QUEUING THEORY AND MODELS

NOTES

Queuing theory is the mathematical study of the congestion and delays of waiting in line. Queuing theory (or "queueing theory") examines every component of waiting in line to be served, including the arrival process, service process, number of servers, number of system places and the number of "customers" (which might be people, data packets, cars, etc.). As a branch of operations research, queuing theory can help users make informed business decisions on how to build efficient and cost-effective workflow systems. Real-life applications of queuing theory cover a wide range of applications, such as how to provide faster customer service, improve traffic flow, efficiently ship orders from a warehouse and the design of telecommunications systems, from data networks to call centres.

Queues happen when resources are limited. In fact, queues make economic sense; no queues would equate to costly overcapacity. Queuing theory helps in the design of balanced systems that serve customers quickly and efficiently but do not cost too much to be sustainable. All queuing systems are broken down into the entities queuing for an activity.

At its most elementary level, queuing theory involves the analysis of arrivals at a facility, such as a bank or fast food restaurant, then the service requirements of that facility, e.g., tellers or attendants. By applying queuing theory, a business can develop more efficient queuing systems, processes, pricing mechanisms, staffing solutions and arrival management strategies to reduce customer wait times and increase the number of customers that can be served.

Kendall's notation for representing Queueing Models

Generally, queueing model may be completely specified in the following symbol form $(a/b/c)$: (d/e) where,

a = probability law for the arrival (inter-arrival) time

b = probability law according to which the customers are being served

c = number of channels (or service stations)

d = capacity of the system, i.e., the maximum number allowed in the system (in service and waiting)

e = queue discipline

9.2.1 Classification of Queueing Models into Finite and Infinite Models

The queueing models are classified as follows:

Model I: (M/M/1): (∞ /FCFS): This denotes Poisson arrival (exponential inter-arrival), Poisson departure (exponential service time), Single server, Infinite capacity and First come first served service discipline. The letter M is used due to Markovian property of exponential process.

Model II: (M/M/1): (N/FCFS): In this model, the capacity of the system is limited (finite), say *N*. Obviously, the number of arrivals will not exceed the number *N* in any case.

NOTES

Model III: Multiservice Model (M/M/S):(∞/FCFS): This model takes the number of service channels as *S*.

Model IV: (M/M/S): (N/FCFS): This model is essentially the same as model II, except the maximum number of customers in the system is limited to *N,* where, ($N > S$).

Of these four models, Model II and Model IV are **finite models**, whereas Models I and III are **infinite models**. we will discuss these models in subsequent units.

Check Your Progress

1. What is the queuing theory?
2. State some real-life applications of queuing theory.
3. How is a business benefitted by queuing theory?

9.3 OPERATING CHARACTERISTICS AND CONSTITUENTS OF A QUEUEING SYSTEM

In daily life, customers wait in lines for different purposes such as for service at a bank ATM counter, for purchasing rail / air tickets, submitting telephone / electricity bills, etc. In factories, jobs wait in lines to be worked in different machines and machines wait their turn to be overhauled, trucks wait to unload / load their cargo, ships wait at docks for unloading / loading, etc. Planners analyse service requirements and establish service facilities keeping in view the customers' requirements and satisfaction level. In other words, the ultimate goal is to achieve an economic balance between the cost of service and the cost associated with the waiting for that service.

The central problem in waiting line situation is a trade-off decision. The manager weighs the added cost of providing more rapid service against the cost of waiting. Queue discipline is a priority rule for determining the order of the service to customers in a waiting line to have better overall performance.

A queuing system depends upon the source population and customer arrival pattern. Arrivals in a service system may be drawn from finite population when limited number is involved or from infinite population when a large number is involved. Any increase or reduction does not have any effect on population size. A constant arrival distribution is periodic with exactly the same time between arrivals. However, random or variable distributions are common.

NOTES

Operating characteristics of a queuing system are as follows:

- Queue length
- Number of customers in the system
- Waiting time in a queue
- Waiting time in system
- Service facility utilization

A queueing system can be completely described by,

(*i*) Input (arrival pattern)

(*ii*) Service mechanism (service pattern)

(*iii*) Queue discipline

(*iv*) Customer's behaviour

9.3.1 The Input (Arrival Pattern)

Input describes the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random fashion, which is not worth predicting. Thus, the arrival pattern can be described in terms of probabilities, and consequently, the probability distribution for inter-arrival times (the time between two successive arrivals) must be defined. We deal with those queueing systems in which the customers arrive in Poisson fashion. The mean arrival rate is denoted by λ.

9.3.2 The Service Facility

This means, the arrangement of service facility to serve customers. If there is an infinite number of servers, then all the customers are served instantaneously on arrival, and there will be no queue.

If the number of servers is finite then the customers are served according to a specific order, with service time as a constant or random variable. Distribution of service time that is important in practice is the *negative exponential distribution*. The mean service rate is denoted by m.

9.3.3 The Queue Discipline

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are:

— First come first served (FCFS)
— First in first out (FIFO)
— Last in first out (LIFO)
— Selection for service in random order (SIRO).

There are various other disciplines according to which a customer is served in preference over the other. Under priority discipline, the service is of two types, namely pre-emptive and non-pre-emptive. In pre-emptive system, the high priority customers are given service over the low priority customers; in non-pre-emptive system, a customer of low priority is serviced before a customer of high priority. In the case of parallel channels 'fastest server rule' is adopted.

NOTES

9.3.4 Customer's Behaviour

The customers generally behave in the following four ways:

(*i*) ***Balking*** A customer who leaves the queue because the queue is too long and he has no time to wait or does not have sufficient waiting space.

(*ii*) ***Reneging*** This occurs when a waiting customer leaves the queue due to impatience.

(*iii*) ***Priorities*** In certain applications some customers are served before others, regardless of their arrival. These customers have priority over others.

(*iv*) ***Jockeying*** Customers may jockey from one waiting line to another. This is most common in a supermarket.

Transient and steady states A system is said to be in a *transient state* when its operating characteristics are dependent on time.

A steady state system is the one in which the behaviour of the system is independent of time. Let $P_n(t)$ denote the probability that there are n customers in the system, at time t. Then in steady state,

$$\lim_{t \to \infty} p_n(t) = p_n \quad \text{(independent of } t\text{)}$$

$$\Rightarrow \qquad \frac{dp_n(t)}{dt} = \frac{dp_n}{dt}$$

$$\Rightarrow \qquad \lim_{t \to \infty} p_n'(t) = 0$$

***Traffic intensity* (*or utilization factor*)** An important measure of a simple queue is its traffic intensity given by,

$$\text{Traffic intensity } \rho = \frac{\text{Mean arrival rate}}{\text{Mean service rate}} = \frac{\lambda}{\mu}$$

The unit of traffic intensity is *Erlang*.

9.3.5 How to Manage Queues?

Some useful suggestions for managing queues are as follows:

- Determine an acceptable waiting time of customers
- Divert customers' attention while they are waiting
- Keep customers informed about difficulty when waiting time is longer than normal
- Segment customers
- Train servers to be polite and friendly
- Encourage customers to come during slack periods
- Take a long-term perspective towards reduction in queues.

Check Your Progress

4. State the factors governing a queuing system.
5. How are customers served if the number of servers is finite?
6. When is a system said to be in a transient state?

NOTES

9.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Queuing theory is the mathematical study of the congestion and delays of waiting in line.
2. Real-life applications of queuing theory cover a wide range of applications, such as how to provide faster customer service, improve traffic flow, efficiently ship orders from a warehouse and the design of telecommunications systems, from data networks to call centres.
3. By applying queuing theory, a business can develop more efficient queuing systems, processes, pricing mechanisms, staffing solutions and arrival management strategies to reduce customer wait times and increase the number of customers that can be served.
4. A queuing system depends upon the source population and customer arrival pattern.
5. If the number of servers is finite then the customers are served according to a specific order, with service time as a constant or random variable.
6. A system is said to be in a *transient state* when its operating characteristics are dependent on time.

9.5 SUMMARY

- As a branch of operations research, queuing theory can help users make informed business decisions on how to build efficient and cost-effective workflow systems.
- Queuing theory helps in the design of balanced systems that serve customers quickly and efficiently but do not cost too much to be sustainable.
- Queuing theory involves the analysis of arrivals at a facility, such as a bank or fast food restaurant, then the service requirements of that facility, e.g., tellers or attendants.
- The central problem in waiting line situation is a trade-off decision. The manager weighs the added cost of providing more rapid service against the cost of waiting.

NOTES

- Arrivals in a service system may be drawn from finite population when limited number is involved or from infinite population when a large number is involved.
- Operating characteristics of a queuing system are as follows:
  - o Queue length
  - o Number of customers in the system
  - o Waiting time in a queue
  - o Waiting time in system
  - o Service facility utilization
- Distribution of service time that is important in practice is the *negative exponential distribution*. The mean service rate is denoted by *m*.

9.6 KEY WORDS

- **Queuing theory:** It examines every component of waiting in line to be served, including the arrival process, service process, number of servers, number of system places and the number of 'customers' (which might be people, data packets, cars, etc.).
- **Queue discipline:** It is a priority rule for determining the order of the service to customers in a waiting line to have better overall performance.
- **Steady state system:** It refers to the condition in which the behaviour of the system is independent of time.

9.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. State the Kendall's notation for representing queuing models.
2. What are the operating characteristics of a queuing system?
3. Provide some useful suggestions for managing queues.

Long Answer Questions

1. Classify the various queuing models and state their important characteristics.
2. Discuss the various components of a queuing system.
3. Write short notes on: (i) Customer's behaviour, (ii) transient and steady states and (iii) Traffic intensity.

9.8 FURTHER READINGS

NOTES

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

NOTES

UNIT 10 MATHEMATICAL ANALYSIS OF QUEUING THEORY

Structure

10.0 INTRODUCTION

As you studied in the last unit, queuing theory is the study of queues or waiting lines. One can derive a few analysis using queuing theory like expected waiting time in the queue, the average time in the system, the assumed que length, the number of customers served at one time, the probability of balking customers. In this unit, you will be studying about the mathematical analysis of queuing theory, properties of queuing theory, single channel queuing model and multiple channel queuing model. In addition to this, you will also learn about the Erlang Family of Distribution of Service Times and will examine the applications and limitations of the queuing model.

10.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand mathematical analysis of queuing theory
- Discuss the properties of queuing system
- Analyse the single channel queuing model and multiple channel queuing model
- Examine the Erlang Family of Distribution of Service Times
- Discuss the applications and limitations of queuing model

10.2 MODEL I: (M/M/I) (∞/FCFS) (BIRTH AND DEATH MODEL)

NOTES

To obtain the steady state equations: The probability that there will be n units ($n > 0$) in the system at time ($t + \Delta t$), may be expressed as the sum of three independent compound probabilities by using the fundamental properties of probability, Poisson arrivals and exponential service times.

The following are the three cases:

Time (t) No. of units	*Arrival*	*Service*	*Time (t + Δt) No. of units*
$n - 1$	0	0	n
$n - 1$	1	0	n
$n + 1$	0	1	n

Now, by adding the above three independent compound probabilities, we obtain the probability of n units in the system at time ($t + \Delta t$).

$$P_n(t + \Delta t) = P_n(t)\,(1 - (\lambda + \mu)\,\Delta t) + P_{n-1}(t)\,\lambda\,\Delta t + P_{n+1}(t)\mu\,\Delta t + 0(\Delta t)$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -(\lambda + \mu)\,P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\underset{\Delta t \to 0}{Lt}\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \underset{\Delta t \to 0}{Lt}\left[-(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0\Delta(t)}{\Delta t}\right.$$

$$\frac{dP_n(t)}{dt} = -(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

where, $\qquad n > 0 \quad \left(\because \underset{\Delta t \to 0}{Lt}\frac{0\Delta(t)}{\Delta t} = 0\right)$

In the steady state,

$$P_n(t) \to 0,\ P_n(t) = P_n$$

$$0 = -(\lambda + \mu)P_n + \lambda P_{n-1} + \mu P_{n+1} \qquad (1)$$

In a similar fashion, the probability that there will be n units (i.e., $n = 0$) in the system at time ($t + \Delta t$), will be the sum of the following two independent probabilities.

(*i*) Probability (that there is no unit in the system at time t and no arrival in time Δt)

$$= P_0(t)\,(1 - \lambda\,\Delta t)$$

(*ii*) Probability (that there is one unit in the system at time t, one unit serviced in Δt and no arrival in Δt)

$$= P_i(t)\mu\,\Delta t\,(1 - \lambda\,\Delta t)$$
$$= P_i(t)\mu\,\Delta t - 0(\Delta t)$$

NOTES

Adding these two probabilities,

$$P_0(t-\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)\mu\Delta t + 0\,(\Delta t)$$

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\underset{\Delta t\to 0}{Lt}\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) \quad \text{for } n = 0$$

$$\frac{dP_0(t)}{d(t)} = -\lambda P_0(t) + \mu P_1(t)$$

Under steady state, we have,

$$0 = -\lambda P_0 + \mu P_1 \qquad (2)$$

Equations (1) and (2) are called *steady state difference equations* for this model.

From (2), $$P_1 = \frac{\lambda}{\mu}P_0$$

From (1), $$P_2 = \frac{\lambda}{\mu}P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Generally, $$P_n = \left(\frac{\lambda}{\mu}\right)^\eta P_0$$

Since, $$\sum_{n=0}^{\infty} P_n = 1$$

$\Rightarrow$ $$P_0 + \frac{\lambda}{\mu}P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \ldots = 1$$

$$P_0\left[1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\ldots\right] = 1$$

i.e., $$P_0\left(\frac{1}{1-\frac{\lambda}{\mu}}\right) = 1$$

Since, $\frac{\lambda}{\mu} < 1$, sum of infinite G.P. is valid.

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho$$

Also, $$P_n = \left(\frac{\lambda}{\mu}\right)^\eta P_0 = \left(\frac{\lambda}{\mu}\right)^\eta\left(1-\frac{\lambda}{\mu}\right)$$

$$P_n = \rho^n(1-\rho)$$

Measures of Model I

NOTES

1. Expected (average) number of units in the system L_S.

$$L_S = \sum_{n=1}^{\infty} nP_n$$

$$= \sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\left(1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+...\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-2}$$

$$= \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\rho}{1-\rho}, \ \rho = \frac{\lambda}{\mu} < 1.$$

$$L_S = \frac{\rho}{1-\rho}$$

2. Expected (average) queue length L_q.

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{\rho^2}{1-\rho}$$

$$L_q = \frac{\rho}{1-\rho}$$

3. Expected waiting line in the queue,

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{1-\rho}$$

4. Expected waiting line in the system,

$$W_S = W_q + \frac{1}{\mu}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu-\lambda}$$

$$W_S = \frac{1}{\mu-\lambda}$$

5. Expected waiting time of a customer who has to wait $(w/w > 0)$

$$(W/W > 0) = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}$$

NOTES

6. Expected length of non-empty queue,

$$(L/L > 0) = \frac{\mu}{\mu - \lambda} = \frac{1}{1-\rho}$$

7. Probability of queue size $\geq N = \rho^N$

$$= \int_t^\infty \rho(\mu - \lambda)e^{-(\mu-\lambda)\omega} d\omega$$

8. Probability of waiting time in the queue $\geq t$
9. Probability (waiting time in the system $\geq t$)

$$= \int_t^\infty \rho(\mu - \lambda)e^{-(\mu-\lambda)\omega} d\omega$$

10. Traffic intensity, $\rho = \frac{\lambda}{\mu}$

Inter-relationship between L_S, L_q, W_S, W_q (Littles formula)

We know,

$$L_S = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu - \lambda}$$

$$W_S = \frac{1}{\mu - \lambda}$$

$$\therefore \quad L_s = \lambda\ W_s$$

Similarly, $L_q = \lambda W_q$ hold in general,

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$W_S = \frac{1}{\mu - \lambda}$$

$$\therefore \quad W_S - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\mu - (\mu - \lambda)}{\mu(\mu - \lambda)} = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$\therefore \quad W_q = W_S - \frac{1}{\mu}$$

Multiplying both sides by λ, we have,

$$\lambda W_q = \lambda\left(W_S - \frac{1}{\mu}\right)$$

$$L_q = \lambda\ W_S - \frac{\lambda}{\mu} = L_S - \frac{\lambda}{\mu}$$

$$L_q = L_S - \frac{\lambda}{\mu}$$

Example 10.1: A T.V. mechanic finds that the time spent on his jobs has an exponential distribution with mean 30 minutes, if he repairs sets in the order in which they come in. If the arrival of sets is approximately Poisson with an average rate of 10 per eight-hour day, what is the mechanic's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution:

Here, $\mu = 1/30, \quad \lambda = \frac{10}{8 \times 60} = \frac{1}{48}$

Expected number of jobs are,

$$L_S = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$$

$$= \frac{\frac{1}{48}}{1/30 - 1/48} = 1 \ \ 2/3 \text{ jobs.}$$

Since the fraction of the time the machanic is busy equals to $\frac{\lambda}{\mu}$' the number of hours for which the repairman remains busy in an eight-hour day,

$$= 8\left(\frac{\lambda}{\mu}\right) = 8 \times \frac{30}{48} = 5 \text{ hours}$$

Therefore, the time for which the machanic remains idle in an eight-hour day = (8 – 5) hours = 3 hours.

Example 10.2: At what average rate must a clerk at a supermarket work, in order to insure a probability of 0.90 that the customers will not have to wait longer than 12 minutes? It is assumed that there is only one counter, to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

Solution:

Here, $\lambda = \frac{15}{60} = \frac{1}{4}$ customer/minute $\mu = ?$

Prob. (waiting time ≥ 12) = 1 – 0.9 = 0.10

$$\therefore \quad \int_{12}^{\infty} \lambda\left(1-\frac{\mu}{\lambda}\right) e^{-(\mu-\lambda)\omega} d\omega = 0.1$$

$$\lambda\left(1-\frac{\mu}{\lambda}\right)\left(\frac{e^{-(\mu-\lambda)\omega}}{-(\mu-\lambda)}\right)_{12}^{\infty} = 0.1$$

$$\frac{\lambda}{\mu}(e^{-12(\mu-\lambda}) = 0.10$$

$$e^{(3-12\mu)} = 0.4\mu$$

$$\frac{1}{\mu} = 2.48 \text{ minutes per service.}$$

NOTES

Example 10.3: Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean three minutes.

(*i*) What is the probability that a person arriving at the booth will have to wait?

(*ii*) What is the average length of the queue that forms from time to time?

(*iii*) The telephone department will install a second booth when convinced that an arrival would have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth?

Solution:

Given, $\lambda = 1/10, \quad \mu = 1/3$

(*i*) Probability $(w > 0) = 1 - P_0 = \frac{\lambda}{\mu} = \frac{1}{10} \times \frac{3}{1} = 3/10 = 0.3$

(*ii*) $(L/L > 0) = \frac{\mu}{\mu - \lambda} = \frac{1/3}{1/3 - 1/10} = 1.43$ persons

(*iii*) $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$

Since, $W_q = 3, \mu = \frac{1}{3}, \lambda = \lambda'$ for second booth,

$$3 = \frac{\lambda'}{\frac{1}{3}\left(\frac{1}{3} - \lambda'\right)} \Rightarrow \lambda' = 0.16$$

Hence, increase in the arrival rate = 0.16 – 0.10 = 0.06 arrival per minute.

Example 10.4: As in example 10.3, in a telephone booth with Poisson arrivals spaced 10 minutes apart on the average and exponential call length averaging three minutes.

(*i*) What is the probability that an arrival will have to wait for more than 10 minutes before the phone becomes free?

(*ii*) What is the probability that it will take him more than 10 minutes in total to wait for the phone and complete his call?

(*iii*) Estimate the fraction of a day that the phone will be in use.

(*iv*) Find the average number of units in the system.

Solution: Given, $n\lambda = 0.1$ arrival/minute

$\mu = 0.33$ service/minute

(*i*) Probability (waiting time ≥ 10) $= \int_{10}^{\infty} \left(1 - \frac{\lambda}{\mu}\right) \lambda e^{-(\mu-\lambda)\omega} d\omega$

$$= -\frac{\lambda}{\mu}\left(e^{-(\mu-\lambda)\omega}\right)_{10}^{\infty}$$

$$= 0.3\, e^{-2.3} = 0.03$$

(*ii*) Probability (waiting time in the system ≥ 10)

$$= \int_{10}^{\infty} (\mu - \lambda)_e^{-(\mu-\lambda)\omega} \, d\omega$$

$$= e^{-10(\mu-\lambda)} = e^{-2.3} = 10$$

(*iii*) The fraction of a day that the phone will be busy = traffic intensity

$$\rho = \frac{\lambda}{\mu} = 0.3.$$

(*iv*) Average number of units in the system,

$$L_S = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{10}}{\frac{1}{3} - \frac{1}{10}} = 3/7 = 0.43 \text{ customer.}$$

Example 10.5: Customers arrive at a one-window drive-in bank according to Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean five minutes. The space in front of the window including that for the serviced car can accommodate a maximum of three cars. Others can wait outside this space.

(*i*) What is the probability that an arriving customer can drive directly to the space in front of the window?

(*ii*) What is the probability that an arriving customer will have to wait outside the indicated space?

(*iii*) How long is an arriving customer expected to wait before starting service?

Solution: Given, $\lambda = 10$ per hour

$$\mu = \frac{1}{5} \times 60 = 12 \text{ per hour}$$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{12}$$

(*i*) The probability that an arriving customer can drive directly to the space in front of the window,

$$P_0 + P_1 + P_2 = P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 \quad \text{or} \quad 1 - P^x$$

$$= P_0 \left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right) \quad \because \quad P_0 = 1 - \frac{\lambda}{\mu}$$

$$= \left(1 - \frac{10}{12}\right)\left(1 + \frac{10}{12} + \frac{100}{144}\right) = 0.42$$

(*ii*) Probability that an arriving customer will have to wait outside the indicated space,

$$S = 1 - 0.42 = 0.58$$

(*iii*) Average waiting time of a customer in a queue,

$$= \frac{\lambda}{\mu}\frac{1}{\mu-\lambda} = \frac{10}{12}\left(\frac{1}{12-10}\right) = \frac{5}{12}$$

$$= 0.417 \text{ hours.}$$

Example 10.6: In a supermarket, the average arrival rate of customers is 10 every 30 minutes, following Poisson process. The average time taken by a cashier to list and calculate the customer's purchase is two and a half minutes following exponential distribution. What is the probability that the queue length exceeds six?

What is the expected time spent by a customer in the system?

Solution:

$$\lambda = \frac{10}{30} \text{ per minute}$$

$$\mu = \frac{1}{2.5} \text{ per minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{3}}{1/2.5} = 0.8333$$

(*i*) The probability of queue size $> 6 = \rho^6$

$$W_S = \frac{1}{\mu-\lambda} = (0.8333)^6 = 0.3348.$$

(*ii*) $$W_S = \frac{L_S}{\lambda} = \frac{\frac{\rho}{1-\rho}}{\lambda} = \frac{0.833}{1-0.8333}\times 3$$

$$= 14.96 \text{ minutes.}$$

Example 10.7: On an average, 96 patients per 24-hour day require the service of an emergency clinic. Also, on an average a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic ₹ 100 per patient treated, to obtain an average servicing time of 10 minutes and thus, each minute of decrease in this average time would cost ₹ 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ patients?

Solution:

Given, $$\lambda = \frac{96}{24\times 60} = \frac{1}{15} \text{ patient/minute}$$

$$\mu = \frac{1}{10} \text{ patient/minute}$$

Average number of patients in the queue,

$$L_q = \frac{\lambda}{\mu} - \frac{\lambda}{\lambda - \mu}$$

$$= \frac{\left(\frac{1}{15}\right)^2}{\left(\frac{1}{10} - \frac{1}{15}\right)\frac{1}{10}} = 1\frac{1}{3} \text{ patients}$$

But, $L_q = 1\frac{1}{3}$ is reduced to $L_q{}' = 1/2$

$\therefore$ Substituting $L_q{}' = 1/2, \lambda' = \lambda = \frac{1}{15}$ in the formula

$$L_q{}' = \frac{\lambda^{12}}{\mu'(\mu' - \lambda')}$$

$$\frac{1}{2} = \frac{\left(\frac{1}{15}\right)^2}{\mu'(\mu' - 1/15)} \Rightarrow \mu' = 2/15 \text{ patients/minute}$$

Hence, the average rate of treatment required is, $\frac{1}{\mu'} = 7.5$ minutes. Decrease in time required by each patient

$$= 10 - \frac{15}{2} = \frac{5}{2} \text{ minutes}$$

$\therefore$ The budget required for each patient

$$= 100 + \frac{5}{2} \times 10 = ₹\ 125$$

So, in order to get the required size of the queue, the budget should be increased from ₹ 100 to ₹ 125 per patient.

Example 10.8: In a public telephone booth, the arrivals on an average are 15 per hour. A call on an average takes three minutes. If there is just one phone, find (*i*) the expected number of callers in the booth at any time (*ii*) the proportion of the time, the booth is expected to be idle?

Solution:

Given, $\lambda = 15$ per hour

$$\mu = \frac{1}{3} \times 60 \text{ per hour}$$

(*i*) Expected length of the non-empty queue

$$= \frac{\mu}{\mu - \lambda} = \frac{20}{20 - 15} = 4$$

(*ii*) The service is busy $= \frac{\lambda}{\mu} = \frac{15}{20} = \frac{3}{4}$

$\therefore$ the booth is expected to be idle for $1 - \frac{3}{4} = \frac{1}{4}$ hours = 15 minutes.

NOTES

Example 10.9: In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that inter-arrival time and service time distribution follows an exponential distribution with an average of 30 minutes, calculate the following.

(*i*) The mean queue size.

(*ii*) The probability that queue size exceeds 10.

(*iii*) If the input of the train increases to an average of 33 per day, what will be the changes in (*i*) and (*ii*)?

Solution:

Given,
$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ trains/minute}$$
$$\mu = \frac{1}{30} \text{ trains/minute}$$
$$\rho = \frac{\lambda}{\mu} = \frac{30}{48} = \frac{5}{8}$$

(*i*) $L_S = \frac{\rho}{1-\rho} = \frac{5/8}{1-5/8} = \frac{5/8}{5/8} = \frac{5}{3}$ = 1.66 trains = 2 trains (app.)

(*ii*) $P(\geq 10) = (0.75)^{10} = 0.056$

(*iii*) When the input increases to 33 trains per day,

we have, $\lambda = \frac{33}{60 \times 24} = \frac{1}{480}, \mu = \frac{1}{30}$ trains/min

$\Rightarrow$ $L_S = \frac{\rho}{1-\rho}$ where, $\rho = \frac{\lambda}{\mu} = \frac{11}{16}$

$$L_S = \frac{11/16}{5/16} = \frac{11}{5} = 2.1 \text{ trains}$$

also, $\rho(\geq 10) = \rho^{10} = \left(\frac{11}{16}\right)^{10} = 0.1460.$

10.3 MODEL III: (MULTISERVICE MODEL) (M/M/S): (∞/FCFS)

When there are n units in the system, difference equations may be obtained in the following two situations.

(*i*) if $n \leq s$, all the customers may be served simultaneously. There will be no queue. $(s-n)$ number of servers may remain idle and then,
$$\mu n = n\mu,\ n = 0, 1, 2, \ldots S;$$

(*ii*) if $n \geq s$, all the servers are busy, and the maximum number of customers waiting in queue will be $(n-s)$, then, $\mu_n = s_\mu$

Also, $\lambda_n = \lambda[n = 0, 1, 2 \ldots]$

The steady state difference equations are,

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda\Delta t) + P_1(t)\mu\Delta t + 0(\Delta t)$$

$$\text{for } n = 0$$

$$P_n(t + \Delta t) = P_n(t)[1 - (\lambda + n\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)(n + 1)\mu\Delta t + 0(\Delta t) \text{ for } n = 1, 2 \ldots S - 1$$

$$P_n(t + \Delta t) = P_n(t)[1 - (\lambda + S\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)S\mu\Delta t + 0(\Delta t) \text{ for } n = S, S + 1, S + 2 \ldots$$

Now, dividing these equations by Δt and taking limit as $\Delta t \to 0$, the difference equations are,

$$P_0^1(t) = -\lambda P_0(t) + \mu P_1(t) \text{ for } n = 0$$

$$P_n^1(t) = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n + 1)\mu P_{n+1}(t)$$

for, $n = 1, 2 \ldots S - 1.$

$$P_n^1(t) = -(\lambda + S\mu)P_n(t) + \lambda P_{n-1}(t) + S\mu P_{n+1}(t)$$

for, $n = S, S + 1, S + 2 \ldots$

Considering the case of steady state, i.e., when $t \to \infty, P_n(t) \to P_n$ and hence, $P_n{}^1(t) \to 0$ for all n, above equations become,

$$O = -\lambda\, \rho_0 + \mu P_1 \text{ for } n = 0 \quad (1)$$

$$O = -(\lambda + n\mu)P_n + \lambda P_{n-1} + (n + 1)\mu P_{n+1} \quad (2)$$

$$\text{for, } 1 \leq n \leq S - 1$$

$$O = -(\lambda + S\mu)P_n + \lambda P_{n-1} + S\mu P_{n+1} \text{ for } n \leq S \quad (3)$$

Here, $P_0 = P_0$ initially,

$$P_1 = \frac{\lambda}{\mu} P_0 \text{ from (1)}$$

$$P_2 = \frac{\lambda}{2\mu} P_1 = \frac{\lambda^2}{2!\mu^2} P_0 \text{ (Put } n = 1 \text{ in (2))}$$

$$P_3 = \frac{\lambda}{3\mu} P_2 = \frac{\lambda^3}{3!\mu^3} P_0 \text{ (Put } n = 2 \text{ in (2))}$$

In general,

$$P_n = \frac{\lambda}{n\mu} P_{n-1} = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$1 \leq n \leq S$$

$$P_S = \frac{\lambda}{S\mu} P_{S-1} = \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \cdot P_0$$

NOTES

$$P_{S+1} = \frac{\lambda}{S\mu} P_S = \frac{1}{S}\frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^{S+1} \cdot P_0$$

$$P_{S+2} = \frac{\lambda}{S\mu} P_{S+1} = \frac{1}{S^2}\frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^{S+2} .P_0$$

In general, $$P_n = P_{S+(n-S)} = \frac{1}{S^{n-S}}\frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad n \geq S$$

Now, find P_n using the fact,

$$\sum_{n=0}^{\infty} P_n = 1$$

i.e., $$\sum_{n=0}^{S-1} P_n + \sum_{n=S}^{\infty} P_n = 1$$

This gives the steady state distribution of arrivals (n) as,

$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0,\ n = 0, 1, 2 \ldots S-1 \\ \frac{1}{S!}\frac{1}{S^{n-S}}\left(\frac{\lambda}{\mu}\right)^n P_0, \text{ if } n = S, S+1 \ldots \end{cases}$$

Measures of Model III

1. Length of the queue,

$$L_q = P_s \frac{P}{(1-P)^2}$$

where, $$P_S = \frac{\left(\frac{\lambda}{\mu}\right)^S P_0}{S!}$$

2. Length of the system,

$$L_S = \frac{\lambda}{\mu} + L_q$$

3. Waiting time in the queue,

$$W_q = \frac{L_q}{\lambda}$$

4. Waiting time in the system,

$$W_S = \frac{L_s}{\lambda}$$

5. The mean number of individuals who actually wait is given by,

$$L(L>0) = \frac{1}{1-\rho}$$

NOTES

6. The mean waiting time in the queue for those who actually wait is given by $W(W > 0)$,

$$= \frac{1}{S\mu - \lambda}$$

7. Probability $(W > 0) = \dfrac{P_s}{1-\rho}$

8. Probability that there will be someone waiting $= \dfrac{P_s\,\rho}{1-\rho}$

9. Average number of idle servers $= S -$ (average number of customers served)

10. Efficiency of M/M/S model $= \dfrac{\text{average number of customers served}}{\text{total number of customers served}}$

Example 10.10: A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean four minutes and if people arrive in a Poisson fashion at the counter, at the rate of 10 per hour, then calculate,

(*i*) the probability of having to wait for service.
(*ii*) the expected percentage of idle time for each girl.
(*iii*) if a customer has to wait, find the expected length of his waiting time.

Solution:

(*i*) Probability of having to wait for service is,

$$P(W > 0) = \frac{\left(\dfrac{\lambda}{\mu}\right)^S}{S!(1-\rho)} P_0$$

$$\lambda = \frac{1}{6},\ \mu = 1/4,\ S = 2$$

$$\rho = \frac{\lambda}{\mu s} = \frac{1}{3}$$

$$\rho_0 = \left[\sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!(1-\rho)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{\left(2\cdot\dfrac{1}{3}\right)^n}{n!} + \frac{\left(2\cdot\dfrac{1}{3}\right)^2}{2!(1-1/3)}\right]^{-1}$$

$$= 1 + 2/3 + \frac{\dfrac{4}{9}}{2\times 2/3} = \frac{1}{2}$$

$$\text{Thus, probability } (W > 0) = \frac{(4/6)^2\cdot\dfrac{1}{2}}{2!(1-1/3)} = 1/6$$

NOTES

(*ii*) The fraction of the time the service remains busy,

(i.e., traffic intensity) is given by, $\rho = \frac{\lambda}{S\mu} = \frac{1}{3}$

$\therefore$ The fraction of the time the service remains idle is,

$$\left(1-\frac{1}{3}\right)=\frac{2}{3} = 67\% \text{ (approximately)}$$

(*iii*) $(W/W>0) = \frac{1}{1-\rho}\cdot\frac{1}{S\mu} = \frac{1}{1-\frac{1}{3}}\cdot\frac{1}{2\times\frac{1}{4}} = 3$ minutes.

Example 10.11: A petrol station has two pumps. The service time follows the exponential distribution with mean four minutes and cars arrive for service in a Poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service. What proportion of time do the pumps remain idle?

Solution:

Given, $S = 2,\ \lambda = 10$ per hour

$$\mu = \frac{1}{4} \text{ per minute} = \frac{60}{4} = 15 \text{ per hour}$$

$$\rho = \frac{\lambda}{S\mu} = \frac{10}{2\times 15} = \frac{1}{3}$$

(*i*) Probability $(W > 0) = \frac{P_S}{1-\rho},\ P_S = \frac{\left(\frac{\lambda}{\mu}\right)^S}{S!}P_0$

where, $$P_0 = \left[\sum_{n=0}^{S-1}\frac{(S\rho)^n}{n!}\,\frac{(S\rho)^S}{S!(1-\rho)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1}\frac{(\rho S)^n}{n!}\,\frac{(\rho S)^2}{2!(1-\rho)}\right]^{-1}$$

$$= \left[1+\frac{(\rho S)^1}{1!}+\frac{(\rho S)^2}{2!(1-\rho)}\right]^{-1}$$

$$= \left[1+\frac{1}{3}\times 2+\frac{\left(\frac{1}{3}\cdot 2\right)}{2!\left(1-\frac{1}{3}\right)}\right]^{-1} = \frac{1}{2}$$

NOTES

$$P(W > 0) = \frac{P_S}{1-\rho} = \frac{1}{1-\rho}\frac{\left(\frac{\lambda}{\mu}\right)^S}{S!}P_0$$

$$= \frac{1}{1-\frac{1}{3}}\left[\frac{\left(\frac{2}{3}\right)^2\frac{1}{2}}{2!}\right] = \frac{1}{9}\times\frac{3}{2} = 0.167 \text{ (approximately)} .$$

(*ii*) The duration of time for which the pumps are busy $= \frac{\lambda}{S\mu} = \frac{1}{3}$

$\therefore$ The duration of time for which the pumps remain idle

$$= 1 - \frac{1}{3} = \frac{2}{3} = 67\% \text{ (approximately)}$$

Example 10.12: A telephone exchange has two long distance operators. The telephone company finds that during the peak load, long-distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately distributed with mean length five minutes.

(*i*) What is the probability that a subscriber will have to wait for his long-istance call during the peak hours of the day?

(*ii*) If the subscribers will wait and be serviced in turn, what is the expected waiting time?

Establish the formula used.

Solution:

Here, $\lambda = \frac{15}{60} = \frac{1}{4}, \mu = \frac{1}{5}, S = 2$

$\therefore$ $\rho = \frac{\lambda}{\mu S} = \frac{5}{8}$

First, calculate $P_0 = \left[\sum_{n=0}^{S-1}\frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!(1-\rho)}\right]^{-1}$

$$= \left[\sum_{n=0}^{1}\frac{(5/4)^n}{n!} + \frac{(5/4)^2}{2!(1-5/8)}\right]^{-1}$$

$$= \frac{1}{1+5/4+(5/4)^2\cdot\frac{1}{2}\cdot 8/3} = 3/13$$

NOTES

(i) $$P(W>0) = \frac{\left(\frac{\lambda}{\mu}\right)^S}{S!(1-\rho)} \cdot P_0$$

$$= \frac{\left(\frac{5}{4}\right)^2 \cdot 3/13}{2!(1-5/8)} = 25/52 = 0.48$$

(ii) $$W_q = L_q/\lambda$$

$$= \frac{1}{\lambda} \frac{\rho(S\rho)^S}{S!(1-\rho)^2} P_0$$

$$= 4 \frac{5/8 \cdot (5/4)^2}{2!(1-5/8)^2} \cdot \frac{3}{13} = \frac{125}{39} = 3.2 \text{ minutes.}$$

Example 10.13: Four counters are being run on the frontier of a country to check the passports and necessary papers of the tourists. The tourists choose any counter at random. If the arrival at the frontier is Poisson at the rate λ and the service time is exponential with parameter λ/2, what is the steady state average queue at each counter?

Solution:

Here, $S = 4$, $\lambda = \lambda$, $\mu = \lambda/2$

$$\rho = \frac{\lambda}{\mu S} = \frac{1}{2}$$

$$P_0 = \left[\sum_{n=0}^{3} \frac{2^n}{n!} + \frac{4^4}{4!} \frac{\left(\frac{1}{2}\right)^4}{(1-1/2)} \right]^{-1}$$

$$= \left[\frac{20}{0!} + \frac{21}{1!} + \frac{22}{2!} + \frac{23}{3!} + \frac{256}{24} \times \frac{1}{8} \right]^{-1}$$

$$= [1 + 2 + 2 + 8/6 + 4/3]^{-1} = [5 + 8/3]^{-1} = 3/23$$

But, the expected queue length λq is,

$$= \frac{(\lambda/\mu)^S}{S!} \frac{\rho}{1-\rho^2} P_0 = \frac{2^4}{4!} \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} \cdot \frac{3}{23} = 4/23$$

10.4 PROPERTIES OF QUEUING SYSTEM

The essential characteristics or properties of a queuing system are as follows:

- **Arrival Pattern:** It expresses the way of arrival of customers at the service facility. These arrivals may occur in batches as opposed to one at a time.

- **Service Pattern:** This mechanism describes the way in which service is being provided. The units are served either singly or in batches. Service time is a term used for the time required for servicing a unit or a batch.
- **Queue Discipline:** It refers to the mode which is used to select the customers for service after the formation of queue. First in first out (FIFO) and last in first out (LIFO) are the common discipline.
- **Service Channels:** There may be several service channels in a queuing system to provide service. The arrangement of these service channels may be parallel or in series or both. The arrangement is basically depend on the design of the system's service mechanism.

10.5 SINGLE AND MULTIPLE CHANNEL QUEUING MODEL

Single Channel Queuing Model

It is the one of the naivest queuing model and also the common one. Following assumptions are made in the model:

- In this model, the arriving customers are served as First in First Out (FIFO) basis.
- The rate of arrival is constant and does not change with time.
- Arrivals follow Poisson's distribution and are not of infinite population.
- All the customers are served as per their different service time requirements and are independent of each other.
- Negative exponential probability distribution is used to describe service time.

Multiple Channel Queuing Model

The multiple channel queuing model exhibits a variety of queue disciplines. In this model, the possibilities of unusual customer behaviour is realized frequently. Classical multiple-channel models primarily undertake the operation of a FIFO rule, as compared to LIFO and SIRO rules. They follow the principle that all arriving customers would join a single queue and this queue would serve all service-channels with customers being drawn from it according to some clearly defined rule of the FIFO, LIFO or SIRO type.

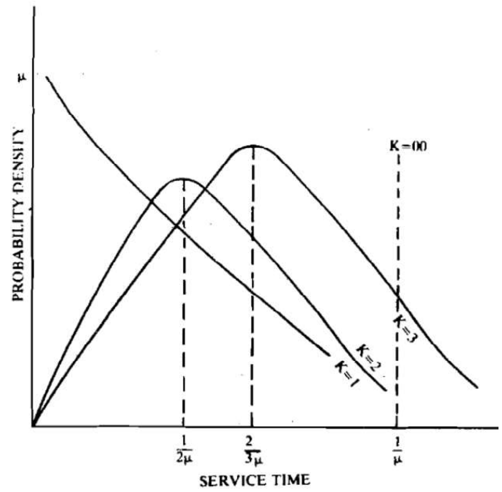
10.6 ERLANG FAMILY OF DISTRIBUTION OF SERVICE TIMES

There are number of phases comprised in many queuing models. For instance, a number of sequential steps are required to repair or revamp a certain machine.

The time taken at each phase of repair is a random variable following an exponential distribution with parameter μ. Let there be k phases, the service time of the ith phase being S_i (i = 1,2, . . ., k). The total service time S can be expressed as

$$S = S_1 + S_2 + \ldots + S_k.$$

It is also assumed that S_i (i = 1,2, . . ., k) are independently distributed. Hence, the probability distribution of S is a Gamma distribution with parameter k and p. The Gamma distribution is also a member of Erlang family of distribution and is denoted by Ek when k exponential variables are involved.



10.6.1 Applications of Queuing Theory

- The queuing theory techniques can help in determining suitable number and type of service facilities to be provided to different types of customers.
- Queuing theory is helpful in obtaining complete details concerning setting up workstations, requirement of manpower and the people who would visit that domain. Though, all these details are based on the concept of probability.
- The theory helps in developing a scientific acknowledgement of the problem which further helps in making optimal usage of available facilities which also reduces the total waiting time.

10.6.2 Limitations of Queuing Theory

Although, the queuing theory offers a scientific method of understanding the queues and solving the queues' problems, still the theory has certain limitations which are stated as follows:

- Mathematical distributions are only a close approximation of the behaviour of customers, the time between their arrival and service time is required by each customer.
- Many real life queuing problems are complex situation are very difficult to use the queuing theory technique.

- In industry and service, many situations are multi-channel queuing problems. When a customer has been attended to and the service provided, it may still have to get some other service from another service and may have to fall in queue once again. Here, the departure of one channel queue becomes the arrival of the channel queue. In such situations, the problem becomes still more difficult to analyse.

Check Your Progress

1. What is arrival pattern?
2. How the units are served in the service pattern?
3. What do you understand by service channels?

10.7 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. It expresses the way of arrival of customers at the service facility. These arrivals may occur in batches as opposed to one at a time.
2. In service pattern, the units are served either singly or in batches. Service time is a term used for the time required for servicing a unit or a batch.
3. There may be several service channels in a queuing system to provide service. The arrangement of these service channels may be parallel or in series or both. The arrangement is basically depend on the design of the system's service mechanism.

10.8 SUMMARY

- Arrival pattern, service pattern, queue discipline, service channels etc. are some of the properties of queuing system.
- Single channel queuing model is the one of the naivest queuing model and also the common one.
- The multiple channel queuing model exhibits a variety of queue disciplines. In this model, the possibilities of unusual customer behaviour is realized frequently. Classical multiple-channel models primarily undertake the operation of a FIFO rule, as compared to LIFO and SIRO rules.
- Multiple channel queuing model follow the principle that all arriving customers would join a single queue and this queue would serve all service-channels with customers being drawn from it according to some clearly defined rule of the FIFO, LIFO or SIRO type.

- The queuing theory techniques can help in determining suitable number and type of service facilities to be provided to different types of customers.
- Queuing theory helps in developing a scientific acknowledgement of the problem which further helps in making optimal usage of available facilities which also reduces the total waiting time.

10.9 KEY WORDS

- **Queuing Theory:** It refers to the mathematical study of waiting lines or queues.
- **Queue Discipline:** It refers to the rules a company or other organization follows to process orders, inventory, or other things that it receives.
- **Erlang Distribution:** It refers to a two-parameter family of the continuous probability distributions with support.

10.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a brief note on single channel queuing model?
2. What is queue discipline?
3. Give an overview of multiple channel queuing model.
4. What is Model I and Model III of the queuing model?

Long Answer Questions

1. Discuss the properties of queuing model.
2. What assumptions are made in single channel queuing model?
3. Analyse the Erlang Family of Distribution of Service Times.
4. What are the applications and limitations of the queuing theory? Discuss.

10.11 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 11 FINITE QUEUING MODELS

NOTES

Structure

11.0 INTRODUCTION

Finite queuing system is the combination of the queue and the service channels. Under this model, it is assumed that the total number of customers that can be present in the system is limited to some maximum number. That is to say that the size of the queue is finite and it is limited by the maximum number in the system less than the number of servers. In this unit, you will study about finite queuing models and the concept of simulation.

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the measures of finite queuing models
- State the concept of simulation
- Discuss the types and examples of simulation
- Summarize the concept of random variable

11.2 MODEL II (M/M/1): (N/FCFS)

This model differs from model I in the sense that, the maximum number of customers in the system is limited to N. Therefore, the difference equations of model I are valid for this model as long as $n < N$. Arrivals will not exceed N in any case. The various measures of this model are,

1. $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$ where, $\rho = \dfrac{\lambda}{\mu}\left(\dfrac{\lambda}{\mu} > 1 \text{ is allowed}\right)$

2. $P_N = \dfrac{1-\rho}{1-\rho^{N+1}}\rho^n$, for $n = 0, 1, 2, \ldots N$

3. $L_q = \lambda/\mu$

4. $L_S = \rho_0 \sum_{n=0}^{N} n\rho^n$

NOTES

5. $L_q = L_s - \dfrac{\lambda}{\mu}$

6. $W_S = L_S/\lambda$

7. $W_q = L_q/\lambda$

Example 11.1: In a railway marshalling yard, goods trains arrive at the rate of 30 trains per day. Assume that the inter-arrival time follows an exponential distribution and the service time is also to be assumed as exponential with mean of 36 minutes.Calculate,

(*i*) the probability that the yard is empty.

(*ii*) the average queue length, assuming that the line capacity of the yard is nine trains.

Solution: for model II

Given,
$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \quad \mu = \frac{1}{36} \text{ trains per minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{36}{48} \times 0.75$$

(*i*) The probability that the queue is empty is given by,

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} \text{ where } N = 9$$

$$= \frac{1-0.75}{1-(0.75)^{9+1}} = \frac{0.25}{0.90} = 0.28.$$

(*ii*) Average queue length is given by,

$$L_s = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n$$

$$= \frac{1-0.75}{1-(0.75)^{10}} \sum_{n=0}^{9} n(0.75)^n$$

$$= 0.28 \times 9.58 = 3 \text{ trains.}$$

11.3 MODEL IV (M/M/S): (M/FCFS)

This model is essentially the same as model III, except that the maximum number of customers in the system is limited to N, where $N > S$ (S = Number of channels)

$$\therefore \qquad \lambda_n = \begin{cases} \lambda, & 0 \le n \le N \\ 0, & n \ge N \end{cases}$$

$$\mu_n = \begin{cases} n\mu, \ 0 \le n \le S \\ S\mu, \ S \le n \le N \end{cases}$$

$$P_n = \begin{Bmatrix} \dfrac{1}{n}\left(\dfrac{\lambda}{\mu}\right)^n P_0, \ 0 \le n \le S \\ \dfrac{1}{S^{n-S}S!}\left(\dfrac{\lambda}{\mu}\right)^n P_0, \ S \le n \le N \end{Bmatrix}$$

NOTES

$$P_0 = \left[\sum_{n=0}^{S-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=C}^{N}\frac{1}{S^{n-S}S!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$$

$$L_q = \frac{(S\rho)^C \rho}{S!(1-\rho)^2}[1-\rho^{N-S+1}-(1-\rho)(N-S+1)\rho^{N-S}]P_0$$

$$L_S = L_q + S - P_0\sum_{n=0}^{S-1}\frac{(S-n)(S\rho)^n}{n!}$$

$$W_S = \frac{L_S}{\lambda'}, \text{ where } \lambda' = \lambda\,(1-P_N)$$

$$W_q = W_S - \frac{1}{\mu}$$

Example 11.2: A barber shop has two barbers and three chairs for customers. Assume that the customers arrive in a Poisson fashion at a rate of five per hour and each barber services customers according to an exponential distribution with mean 15 minutes. Further, if a customer arrives and there are no empty chairs in the shop, he will leave. What is the expected number of customers in the shop?

Solution:

Here, $S = 2, N = 3, \lambda = \dfrac{5}{60} = \dfrac{1}{12}$ customer/min

$$\mu = \frac{1}{15}/\text{min}$$

$$P_0 = \left[\sum_{n=0}^{2-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=2}^{3}\frac{1}{2^{n-2}}\frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$$

$$= \left[1 + 1.\frac{5}{4} + \frac{1}{2!}\left(\frac{5}{4}\right)^2 + \frac{1}{2.2!}\left(\frac{5}{4}\right)^3\right]^{-1}$$

$$= \left[1 + \frac{5}{4} + \frac{1}{2}(5/4)^2 + \frac{1}{4}(5/4)^3\right]^{-1}$$

$$= \frac{256}{901} = 0.28$$

NOTES

$$P_n = \begin{cases} \dfrac{1}{n!}(5/4)^n \times 0.28 & 0 \le n < 2 \\ \dfrac{1}{2^{n-2}2!}(5/4)^n \times 0.28 & 2 \le n \le 3 \end{cases}$$

$$= \frac{1}{n!}(1.25)^n \times 0.28 \qquad 0 \le n < 2$$

$$= \frac{1}{2^{n-2}2!}(1.25)^n \times 0.28 \qquad 2 \le n \le 3$$

$$L_S = L_q + S - P_0 \sum_{n=0}^{S-1} \frac{(S-n)\left(\dfrac{\lambda}{\mu}\right)^n}{n!}$$

$$= \sum_{n=2}^{3}(n-2)P_n + 2 - P_0 \sum_{n=0}^{2-1} \frac{(2-n)(1.25)^n}{n!}$$

$$= P_3 + 2 - 3.2\,P_0$$

$$= \left[\frac{1}{2.2!}(1.25)^3 \times 0.28\right] + 2 - 3.2 \times 0.28 = 1.226 \text{ customers.}$$

11.4 METHODOLOGY TO SIMULATION

Simulation is a representation of reality through the use of a model or other device, which will react in the same manner as reality under a given set of conditions.

Simulation is also defined as the use of a system model that has the designed characteristics of reality, in order to produce the essence of an actual operation.

11.4.1 Types of Simulation

Simulation is mainly of two types.

(*i*) Analog (environmental) simulation
(*ii*) Computer (system) simulation

Some examples of simulation models are given below:

(*i*) Testing an aircraft model in a wind tunnel.
(*ii*) Children cycling in a park with various signals and crossings—to model a traffic system.
(*iii*) Planetorium.

To determine the behaviour of a real system in actual environment, a number of experiments are performed on simulated models either in the laboratories or on the computer itself.

Random Variable

The random variable is a real-valued function, defined over a sample space associated with the outcome of a conceptual chance experiment. Random variables are classified according to their probability density function.

NOTES

Random number: It refers to a uniform random variable or a numerical value assigned to a random variable, following uniform probability density function. In other words, it is a number in a sequence of numbers, whose probability of occurrence is the same as that of any other number in that sequence.

Pseudo-random numbers: Random numbers are called pseudo-random numbers when they are generated by some deterministic process, but have already qualified the pre-determined statistical test for randomness.

The methodology, procedure and application of simulation have been discussed in the next unit.

Check Your Progress

1. What are the two types of simulation?
2. What do you mean by random variable?
3. Define random number.
4. Which numbers are called pseudo-random numbers?

11.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Simulation is mainly of two types: analog (environmental) simulation and computer (system) simulation.
2. The random variable is a real-valued function, defined over a sample space associated with the outcome of a conceptual chance experiment. Random variables are classified according to their probability density function.
3. Random number is a number whose probability of occurrence is the same as that of any other number in the collection.
4. Random numbers are called pseudo-random numbers when they are generated by some deterministic process, but have already qualified the pre-determined statistical test for randomness.

11.6 SUMMARY

- Model II (M/M/1) differs from model I in the sense that, the maximum number of customers in the system is limited to *N*. Therefore, the difference

NOTES

equations of model I are valid for this model as long as $n < N$. Arrivals will not exceed N in any case.

- The various measures of model II are as follows:

1. $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$, where $\rho = \dfrac{\lambda}{\mu}\left(\dfrac{\lambda}{\mu} > 1 \text{ is allowed}\right)$
2. $P_N = \dfrac{1-\rho}{1-\rho^{N+1}}\,\rho^n$ for $n = 0\ 1, 2, ..., N$.
3. $L_q = \dfrac{\lambda}{\mu}$
4. $L_S = \rho_0 \sum_{n=0}^{N} n\rho^n$
5. $L_q = L_S - \dfrac{\lambda}{\mu}$
6. $W_S = L_S/\lambda$
7. $W_q = L_q/\lambda$

- Model IV is essentially the same as model II, except that the maximum number of customers in the system is limited to N, where N>S (S=Number of channels).
- The various measures of model IV are as follows:

1. $\lambda_n = \begin{cases} \lambda, & 0 \le n \le N \\ 0, & n \ge N \end{cases}$
2. $\mu_n = \begin{cases} n\mu, & 0 \le n \le S \\ S\mu, & S \le n \le N \end{cases}$
3. $P_n = \begin{cases} \dfrac{1}{n}\left(\dfrac{\lambda}{\mu}\right)^n P_0, \ 0 \le n \le S & \dfrac{1}{S^{n-S}}\,S!\left(\dfrac{\lambda}{\mu}\right)^n P_0, S \le n \le N \end{cases}$
4. $P_0 = \left[\sum_{n=0}^{s-1} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n + \sum_{n=C}^{N} \dfrac{1}{S^{n-s}S!}\left(\dfrac{\lambda}{\mu}\right)^n\right]^{-1}$
5. $L_q = \dfrac{(S\rho)^c\ \rho}{S!(1-\rho)^2}\left[1-\rho^{N-S+1} - (1-\rho)(N-S+1)\ \rho^{N-S}\right] P_0$
6. $L_S = L_q + S - P_0 \sum_{n=0}^{S-1} \dfrac{(S-n)\ (SP)^n}{n!}$
7. $w_S = \dfrac{L_S}{\lambda'}$ where $\lambda' = \lambda(1-P_N)$
8. $w_q = w_S - \dfrac{1}{\mu}$

- Simulation is a representation of reality through the use of a model or other device, which will react in the same manner as reality under a given set of conditions.

- Simulation is also defined as the use of a system model that has the designed characteristics of reality, in order to produce the essence of an actual operation.
- Simulation is mainly of two types: analog (environmental) simulation and computer (system) simulation.
- To determine the behaviour of a real system in actual environment, a number of experiments are performed on simulated models either in the laboratories or on the computer itself.
- The random variable is a real-valued function, defined over a sample space associated with the outcome of a conceptual chance experiment. Random variables are classified according to their probability density function.
- Random number refers to a uniform random variable or a numerical value assigned to a random variable, following uniform probability density function. In other words, it is a number in a sequence of numbers, whose probability of occurrence is the same as that of any other number in that sequence.
- Random numbers are called pseudo-random numbers when they are generated by some deterministic process, but have already qualified the pre-determined statistical test for randomness.

11.7 KEY WORDS

- **Simulation:** It refers to an approximate imitation of the operation pf a process or system.
- **Pseudo-random numbers:** These numbers are generated by computers. These are not truly random, they just follow certain deterministic formula and thence it looks random.
- **Analog Simulation:** It refers to the representation of physical systems and phenomena by variables such as translation, rotation, resistance, and voltage.
- **Computer Simulation:** It refers to a computer program that attempts to simulate an abstract model of a particular system.

11.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Define simulation.
2. Give some examples of simulation models.
3. List the various measures used in Model II.

NOTES

Long Answer Questions

1. Let there be an automobile inspection site with three inspection stalls. Assume that cars wait in such a way that when a stall becomes vacant, the car at the head of the pulls up to it. The station can accommodate four cars at the most, to be waiting (seven in station) at one time. The arrival pattern is Poisson with a mean of one car every minute during peak hours. The service time is exponential with mean six minutes. Find the average number of cars in the site during peak hours, the average waiting time and the average number of cars per hour that cannot enter the station because of full capacity.
2. Customers arrive at a one-window drive-in bank according to Poisson distribution with mean 10 per hour. Service tome per customer is exponential with mean 5 minutes. The car space in front of the window, including that for the service, can accommodate a maximum of 3 cars. Other cars can wait outside this space.
   (a) What is the probability that an arriving customer can drive directly to the space in front of the window?
   (b) What is the probability that an arriving customer can drive directly to the space in front of the window?
   (c) How long is an arriving customer expected to wait before starting the service?

11.9 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

NOTES

BLOCK - III
SIMULATION GAME THEORY CONCEPTS

UNIT 12 SIMULATION: MONTE CARLO METHOD

Structure

12.0 INTRODUCTION

As already discussed in the previous unit, simulation is a technique which is used in an attempt to duplicate the features, appearance and characteristics of a real system. In this unit, you will learn more about simulation technique and Monte Carlo method. The unit covers the importance and uses of simulation technique and areas of application of Monte Carlo simulation. In addition, the unit also discusses about the applications of simulation, advantages and disadvantages of simulation.

12.1 OBJECTIVES

After going through this unit, you will be able to:

- Assess the importance and uses of simulation technique
- Discuss the Monte Carlo Method
- Describe the application of simulation
- Discuss the advantages and disadvantages of simulation

12.2 IMPORTANCE AND USES OF SIMULATION TECHNIQUE

NOTES

Monte Carlo simulation is often used by modern management when it cannot use other techniques. There are many industrial problems which defy mathematical solutions. The reason is that either they are too complicated or that the data cannot be expressed in mathematical terms. In such cases, it is still possible to reach valid conclusions by using the Monte Carlo technique. A considerable help is thus obtained at practically no cost in taking decisions concerning the functioning of a business system. The data and conclusions can be obtained through simulation of an actual operation on the basis of its own past working. It paves the way for predicting the changes in its behaviour and the result is evaluated from innovations that we want to introduce.

By using a fresh series of random numbers at the appropriate junctures, we can also examine the reactions of the simulated model just as if the same alterations had actually been made in the system itself. Monte Carlo simulation, therefore, provides a tool of knowing in advance as to whether or not the expense to be incurred or the investment to be made in making the changes is envisaged. Through this technique you can introduce the innovations on a piece of paper, examine their effects and then may decide to adopt or not to adopt such innovations in the functioning of real system. The usefulness of simulation lies in the fact that it allows us to experiment with a model of the system rather than the actual system; in case we are convinced about the results of our experiments, we can put the same into practice. Thus the effect of the actual decisions are tested in advance through the technique of simulation by resorting to the study of the model representing the real life situation or the system.

The main purpose of simulation in management is to provide feedback, which is vital for the learning process. It creates an atmosphere in which managers play a dynamic role by enriching their experience through involvement in reckoning with actual conditions through experimentation on paper. The technique permits trying out several alternatives as the entire production for service process can be worked out on paper, without dislocating the system in any way. Thus, Monte Carlo technique transforms the manager from a blind folded driver of an automobile, reacting to instructions of a fellow passenger to one who can see fairly, clearly, where he is going.

12.2.1 Areas of Application of Monte Carlo Simulation

Monte Carlo simulation has been applied to a wide diversity of problems ranging from queuing process, inventory problem, risk analysis concerning a major capital investment such as the introduction of a new product, expansion of the capacity and many other problems. *Budgeting* is another area where simulation can be very useful. In fact, the system of flexible budgeting is an exercise in simulation.

Simulation can as well be used for preparing the master budget through functional budgets.

Over and above, the greatest contribution of simulation is in the analysis of complex systems. Many real-world problems involve systems made up of many component parts that are interrelated. The system may be dynamic and changing over time and may involve probabilistic or uncertain events. Simulation is the only technique for quantitative analysis of such problems.

12.3 MONTE CARLO SIMULATION

Monte Carlo methods are basically the algorithms used in the computation of result to be calculated from repeated random sampling. These methods help in computerized calculations because these can perform repeated computation using random or pseudo-random numbers. It is also used when it is not feasible to compute correct result with a deterministic algorithm. Monte Carlo simulation methods are used to study systems having degrees of freedom and in the situations when there is significant ambiguity in inputs, for example, calculating risk factor in a business.

Various simulation models, based on the principle of similitude (such as model of aeroplanes initiating flight conditions in a wind tunnel) have been in use for a long time. But Monte Carlo simulation is a recent operations research innovation. The novelty lies in making use of pure chance to contact a simulated version of the process under analysis, in exactly the same way as pure chance operates the original system under working conditions. Only models under uncertainty can be evaluated using Monte Carlo technique.

'Monte Carlo' is the code name given by John von Neumann and S.M. Ulam to the technique of solving problems though it is too expensive for experimental solutions and too complicated for analytical treatment.

Monte Carlo method is not one single method. It involves various widely-used classes of approaches to follow a specific model. Using it we can,

- Define a domain with feasible inputs
- Randomly generate inputs from the domain
- Perform deterministic computation with the inputs
- Combine the results of the personal computations into the final result

Monte Carlo methods are used to solve various mathematical problems based on sampling experiments in statistics using the sequences of random numbers for simulation and are termed as statistical simulation methods. Thus, Monte Carlo method is not one single method but it is a collection of various methods and is basically used to perform similar procedure. Some of these methods are discussed here with the help of solved examples.

The Monte Carlo simulation technique can as well be used to solve probabilistic problems. Suppose, we are to evaluate the probability *P* that a tank will be knocked out by either a first or second shot from an antitank gun assumed to possess a constant kill probability of 1/2. The probability analysis will say that the chance of tank being knocked out by either a first or second shot from an antitank gun is 1/2+1/2 (1–1/2) =3/4. But we can also work out this probability by simulating each round of the antitank gun by the flip of a coin through Monte Carlo simulation technique.

Since the probability of a 'head' is the same as that of a kill, we may call it a hit when the coin turns up a head and otherwise a miss. If we flip the coin a large number of times, the value of P may be calculated by merely counting the number of times a head turns up at least in two successive throws and then dividing this number by the total pairs of throws of the coin. Monte Carlo method in this simple case, is indeed a poor substitute for the theoretical probability analysis. But many real-life systems are so complicated that even the well-defined probability analysis very often fails but such situations can be handled by Monte Carlo simulation, particularly the Monte Carlo technique that provides the simplest possible solutions for queuing problems. Problems of corporate planning, inventory control, capital investment, consumer behaviour and quality control can also be handled through simulation.

Monte Carlo simulation uses random number tables to reproduce on paper the operation of any given system under its own working conditions. This technique is used to solve problems that depend upon probability where formulation of mathematical model is not possible. It involves first, the determination of the probability distribution of the concerned variables and then sampling from this distribution by means of random numbers to obtain data. It may, however, be emphasized here that the probability distributions to be used should closely resemble the real world situation.

One should always remember that simulation is not a perfect substitute but rather an alternative procedure for evaluating a model. Analytical solution produces the optimal answer to a given problem, while Monte Carlo simulation yields a solution which should be very close to the optimal but not necessarily the exact correct solution. Monte Carlo Simulation solution converge with to the optimal solution as the number of simulated trials goes to infinity.

In context of Queuing Theory: Monte Carlo simulation which uses random number tables can better be illustrated by considering any concrete operation subject to chance. Let us take the arrival of scooters at a service station. First of all, observe the actual arrival of scooters on number of days, say for five days, then put this information in the following two ways:

- Group the number of scooters arriving every hour, say between 7–8, 8–9 a.m. and so on till 4–5 p.m. (assuming the working hours of the service station is from 7 a.m. to 5 p.m. and also assuming that no scooter arrives before 7 a.m. nor any scooter after 5 p.m.). Work out the mean number of scooters arriving during 7–8 a.m., 8–9 a.m., 9–10 a.m. and so on. Let us suppose, we get the following information:

NOTES

Table 12.1 *Mean Number of Scooters Arriving per Hour*

7-8 a.m.	5.6
8-9 a.m.	5.4
9-10 a.m.	3.4
10-11 a.m.	3.6
11-12 noon	2.0
12-1 p.m.	3.0
1-2 p.m.	4.0
2-3 p.m.	6.0
3-4 p.m.	3.0
4-5 p.m.	4.0

- Obtain the deviation of actual arrivals during a particular hour from the corresponding mean and do it for all the hours from 7 a.m. to 5 p.m. There will thus be 10 sets of 5 deviations (because of the observation on 5 days) from each of the 10 mean hourly arrivals and then prepare a frequency distribution of such deviations is to be prepared. Suppose we get the frequency distribution of such deviations as under:

Table 12.2 *Random Number Allotment*

Deviation Nos.	***Frequency from Mean***	***Percentage***	***Probability Frequency***	***Random Allotted***
+4.0	2	4	0.04	00–03
+3.5	3	6	0.06	04–09
+3.0	5	10	0.10	10–19
+2.5	4	8	0.08	20–27
+2.0	6	12	0.12	28–39
+1.5	2	4	0.04	40–43
+1.0	3	6	0.06	44–49
+0.5	4	8	0.08	50–57
–0.5	3	6	0.06	58–63
–1.0	2	4	0.04	64–67
–1.5	4	8	0.08	68–75
–2.0	2	4	0.04	76–79
–2.5	3	6	0.06	80–85
–3.0	1	2	0.02	86–87
–3.5	2	4	0.04	88–91
–4.0	4	8	0.08	92–99
Total	50	100	1.00	

In the last column of the above table, we have allotted 100 random numbers from 00 to 99, both inclusive according to the percentage frequency of the deviation or the probability distribution of deviations. Thus deviation to the extent of +4.0 from the mean, having a frequency of 2 out of 50 (and as such 4 out of 100) or a probability of 0.04, has been allotted 4 random numbers, 00 to 03. The next deviation of +3.5 with a probability of 0.06 has the next 6 random numbers, 04 to 09, allotted to it. The same treatment has been done to all the remaining deviations. The last deviation – 4.0 with a probability of 0.08 has been allotted to the last 8 random numbers, from 92 to 99.

The object of doing all this is to derive, by simulation, the actual number of scooters that may be expected to arrive during any given hour. Suppose we want to know the expected number of scooters arriving on a particular day during the hour 8 to 9 a.m. The table giving the mean number of scooters arrival shows that the mean arrival during this hour is 5.4. If we can ascertain the deviation of the actual arrivals from the mean, we can easily work out the actual number of scooters arrived. To do so, we look at the table of random numbers and select any two-digit number at random. Suppose, the random number so selected is 84 corresponding to which the value of the deviation of actual arrivals from the mean as per the above table is –2.5. In other words the actual arrivals for the hour 8-9 a.m. will be (5.4) – (2.5) = 2.9 (or approximately 3), so that we may say that 3 scooters will arrive between 8-9 a.m. on the day in question. The underlying rationale of this simulation procedure is that every deviation from its corresponding mean has the same chance of occurring by random number selection as in the actual case because each deviation has as many random numbers allotted to it as its frequency percentage in the general pool of all deviations as stated above.

We can go a step further and using the random number technique can even simulate and tell the actual arrival time of the scooter coming to the service station during any particular hour. If we are satisfied to note the arrival time correct to within, say, 5 minutes the required numbers of minutes past the hour can take a value only in one of the 12 intervals (0-5, 6-10, 11-15, ..., 56 to 60 minutes) into which any hour can be divided. Keeping this in view and the actual observations for all 5 days under consideration we can prepare a frequency table showing how many scooters arrive within 5 minutes, how many within 6–10 minutes past the hour and so on for the remaining intervals into which we choose to split the hour. Let the observed information for 5 days period on this basis be put as shown in Table 12.3.

Random numbers in the last column of the Table 12.3 have been allotted in a similar manner as we did in an earlier table. We have already seen that 3 scooters arrive during the hour 8–9 a.m. and now we want to know the actual time of their coming to service station.

Table 12.3 *Number of Scooters Arriving Within the Number of Minutes Past the Hour*

Deviation Nos.	*Frequency from Mean*	*Percentage*	*Probability Frequency*	*Random Allotted*
0-5 mts.	20	10	0.10	00-09
6-l0 mts.	30	15	0.15	10-24
11-15 mts.	10	5	0.05	25-29
16-20 mts.	40	20	0.20	30-49
21-25 mts.	16	8	0.08	50-57
26-30 mts.	14	7	0.07	58-64
31-35 mts.	18	9	0.09	65-73
36-40 mts.	12	6	0.06	74-79
41-45 mts.	16	8	0.08	80-87
46-50 mts.	14	7	0.07	88-94
51-55 mts.	6	3	0.03	95-97
56-60 mts.	4	2	0.02	98-99
Total	200	100	1.00	

NOTES

For this purpose we pick a two-digit random number from the table of random numbers and let us say it is 25. A reference to the above table shows that this number occurs in the range of 25–29 which belongs to the interval 11–15 minutes and this means that the first of the 3 scooters arriving between 8–9 a.m. arrives at 15 minutes past 8 a.m. Similarly picking two more random numbers viz., 36 and 96 we find from the above table that they are related to intervals 16–20 minutes and 51–55 minutes respectively. Thus the second scooter arrives at 20 minutes past 8 a.m. and the third scooter arrives at 55 minutes past 8 a.m.

Proceeding in a similar manner we can also make a frequency table showing the number of scooters serviced within intervals of varying magnitudes. Each of these intervals can then be allotted a set of 100 random numbers (i.e., 00 to 99) according to the probability distribution to provide a basis for simulating the pattern of available service.

The above is an example of how arrival as well as service pattern in a queuing process may be derived by Monte Carlo simulation. Remember that there is no regularity either in the arrival of the scooters or in rendering service and because of this there may be times when scooters have to wait for service while at other times the service attendant may remain idle. If in such a case we want to add one more service point to the service station then certainly we would first like to assess whether the same would be economical or not. Simulation technique can assist us in the matter. How does simulation help can better be made clear by the following example:

NOTES

Table 12.4 *Simulation Worksheet for Arrival Time, Service Time and Waiting Time*

Random Number	Time Till Next Arrival (Minutes)	Arrival Time (a.m.)	Service Begins (a.m.)	Random Number	Service Time (Minutes)	Service Ends (a.m.)	Waiting Time Attendant (Minutes)	Waiting Time Customers (Minutes)	Length of the Line
44	5	10.05	10.05	50	5	10.10	5	–	–
84	6	10.11	10.11	95	7	10.18	1	–	1
82	6	10.17	10.18	58	5	10.23	–	1	1
50	5	10.22	10.23	44	5	10.28	–	1	–
83	6	10.28	10.28	77	6	10.34	–	–	1
40	5	10.33	10.34	11	4	10.38	–	1	–
96	8	10.41	10.41	08	3	10.44	3	–	–
88	7	10.48	10.48	38	5	10.53	4	–	1
16	4	10.52	10.53	87	6	10.59	–	1	1
16	4	10.56	10.59	45	5	11.04	–	3	–
97	8	11.04	11.04	09	3	11.07	–	–	–
92	7	11.11	11.11	99	7	11.18	4	–	1
39	5	11.16	11.18	81	6	11.24	–	2	1
33	5	11.21	11.24	97	7	11.31	–	3	1
83	6	11.27	11.31	30	4	11.35	–	4	1
42	5	11.32	11.35	36	5	11.40	–	3	1
16	4	11.36	11.40	75	6	11.46	–	4	1
07	4	11.40	11.46	72	6	11.52	–	6	1
77	6	11.46	11.52	79	6	11.58	–	6	1
66	6	11.52	11.58	83	6	12.04	–	6	–
20	112				107		17	41	13

Example 12.1: A firm has a single channel service station with following empirical data available to its management:

(*i*) The mean arrival rate is 6.2 minutes.

(*ii*) The mean service time is 5.5 minutes.

(*iii*) The arrival and service time probability distributions are as follows:

Arrivals (Minutes)	*Probability*	*Service Time (Minutes)*	*Probability*
3-4	0.05	3-4	0.10
4-5	0.20	4-5	0.20
5-6	0.35	5-6	0.40
6-7	0.25	6-7	0.20
7-8	0.10	7-8	0.10
8-9	0.05	8-9	0.00
	1.00		1.00

The queuing process begins at 10.00 a.m. and proceeds for nearly 2 hours. An arrival goes to the service facility immediately if it is empty otherwise it will wait in a queue. The queue discipline is, First come first served.

If the attendant's wage is ₹ 8 per hour and the customer's waiting time cost ₹ 9 per hour, would it be an economical proposition to engage second attendant? Answer on the basis of Monte Carlo simulation technique. You may use the figures based upon the simulated period for 2 hours.

Solution: From the given probability distributions of arrivals and service times, first of all we allot the random numbers to the various intervals. This has been done as follows:

Arrivals (Minutes)	*Probability Nos.*	*Random Allotted (Minutes)*	*Service Time*	*Probability Allotted*	*Random Nos.*
3-4	0.05	00-04	3-4	0.10	00-09
4-5	0.20	05-24	4.5	0.20	10-29
5-6	0.35	25-59	5-6	0.40	30-69
6-7	0.25	60-84	6-7	0.20	70-89
7-8	0.10	85-94	7-8	0.10	90-99
8-9	0.05	95-99	8-9	0.00	

These allotted random numbers now become the basis for generating arrival and service times in conjunction with a table of random numbers. A simulation worksheet as shown on the next page has been developed in the following manner:

As the given mean arrival rate is 6.2 minutes which means that in an hour approximately 10 units arrive and as such in a simulation period for 2 hours about

20 units are expected to arrive. Hence, the number of arrivals for our simulation exercise is 20.

A table of random numbers (given in table 12.4) is used for developing the simulation worksheet. The first random number for arrival time is 44. This number lies between 25–59 and this indicates a simulated arrival time of 5 minutes. All simulated arrival and service times have been worked out in a similar fashion.

The next step is to list the arrival time in the appropriate column. The first arrival comes in 5 minutes after the starting time. It means the attendant waited for 5 minutes. This has been shown under the column 'Waiting Time: Attendant'. The simulated service time for the first arrival is 5 minutes which results in the service being completed by 10.10 a.m. The next arrival comes at 10.11 a.m. which indicates that no one has waited in queue but the attendant has waited for 1 minute from 10.10 a.m. to 10.11 a.m. The service time ends at 10.18 a.m. But the third arrival comes at 10.17 a.m. and the service of the second continues upto 10.18 a.m., hence the third arrival has to wait in the queue. This is shown in the column 'Waiting Time: Customer' of the simulation worksheet. One customer waiting in queue is shown in the column—'Length of the Line.' The same procedure has been followed throughout the preparation of the simulation worksheet.

The following information can be derived from the above stated simulation worksheet.

1. Average length of queue:

$$= \frac{\text{No. of customers in line}}{\text{No. of arrivals}} = \frac{13}{20} = 0.65$$

2. Average waiting time of customer before service:

$$= \frac{\text{Customer waiting time}}{\text{No. of arrivals}} = \frac{41}{20} = 2.05 \text{ minutes}$$

3. Average service time:

$$= \frac{\text{Total service time}}{\text{No. of arrivals}} = \frac{107}{20} = 5.35 \text{ minutes}$$

4. Time a customer spends in the system:

= Average service time + Average waiting time before service

= 5.35 + 2.05 = 7.40 minutes.

Simulation worksheet developed above also states that if one more attendant is added then there is no need for a customer to 'Wait in Queue'. But the cost of having one more attendant in addition to the existing one is to be compared with the cost of one attendant and the customer waiting time. This can be worked out as under:

Two Hour Period	Cost with	
	One Attendant	Two Attendants
Customer waiting time		
(41 mts. × ₹ 9 per hour)	₹ 6.15	nil
Attendant's cost		
(2 hours × ₹ 8 per hour)	₹ 16	₹ 32
Total	₹ 22.15	₹ 32

If the above analysis is based on simulation for a period of 2 hours and is representative of the actual situation, then it can be concluded that the cost with one attendant is lower than with two attendants. Hence, it would not be an economical proposition to engage as additional attendant.

12.4 APPLICATION OF SIMULATION

Simulation is used in almost all fields. Some of the areas where simulation is used are as follows:

- Manufacturing Applications
- Semiconductor Manufacturing
- Construction Engineering and project management
- Business Process Simulation
- Healthcare
- Logistics, Supply Chain and Distribution Application
- Risk Analysis
- Computer Simulation
- Transportation modes and Traffic

12.4.1 Advantages and Disadvantages of Simulation

Main advantages of simulation are as follows:

- Simulation helps to study the behaviour of a system without building it.
- The results of simulation are accurate in general, in contrast to analytical model.
- It also helps to find an unexpected phenomenon, behaviour of the system.
- The process of simulation also eases the procedure of 'What-If' analysis.

Main disadvantages of simulation are as follows:

- To build a simulation model is a bit expensive.
- In addition to this, it is also expensive to conduct simulation.
- At times, it is difficult to interpret the results of the simulation.

NOTES

Check Your Progress

1. What is the main purpose of simulation in management?
2. List the problems where Monte Carlo simulation is applied.
3. What is the greatest contribution of simulation?
4. Who introduced the name 'Monte Carlo'?
5. State one of the uses of the Monte Carlo methods?

12.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The main purpose of simulation in management is to provide feedback which is vital for the learning process. It creates an atmosphere in which managers play a dynamic role by enriching their experience through involvement in reckoning with actual conditions through experimentation on paper.
2. Monte Carlo simulation has been applied to a wide diversity of problems ranging from queuing process, inventory problem, risk analysis concerning a major capital investment such as the introduction of a new product, expansion of the capacity and many other problems. *Budgeting* is another area where simulation can be very useful. In fact, the system of flexible budgeting is an exercise in simulation. Simulation can as well be used for preparing the master budget through functional budgets.
3. The greatest contribution of simulation is in the analysis of complex systems.
4. 'Monte Carlo' is the code name given by John von Neumann and S.M.Ulam.
5. Monte Carlo methods are used to solve various mathematical problems based on sampling experiments in statistics using the sequences of random numbers for simulation.

12.6 SUMMARY

- Monte Carlo simulation is often used by modern management when it cannot use other techniques.
- The main purpose of simulation in management is to provide feedback, which is vital for the learning process. It creates an atmosphere in which managers play a dynamic role by enriching their experience through involvement in reckoning with actual conditions through experimentation on paper.

NOTES

- The technique permits trying out several alternatives as the entire production for service process can be worked out on paper, without dislocating the system in any way.
- Monte Carlo technique transforms the manager from a blind folded driver of an automobile, reacting to instructions of a fellow passenger to one who can see fairly, clearly, where he is going.
- Monte Carlo simulation has been applied to a wide diversity of problems ranging from queuing process, inventory problem, risk analysis concerning a major capital investment such as the introduction of a new product, expansion of the capacity and many other problems. *Budgeting* is another area where simulation can be very useful. In fact, the system of flexible budgeting is an exercise in simulation. Simulation can as well be used for preparing the master budget through functional budgets.
- The greatest contribution of simulation is in the analysis of complex systems. Many real-world problems involve systems made up of many component parts that are interrelated. The system may be dynamic and changing over time and may involve probabilistic or uncertain events. Simulation is the only technique for quantitative analysis of such problems.
- Monte Carlo methods are basically the algorithms used in the computation of result to be calculated from repeated random sampling. These methods help in computerized calculations because these can perform repeated computation using random or pseudo-random numbers. It is also used when it is not feasible to compute correct result with a deterministic algorithm.
- 'Monte Carlo' is the code name given by John von Neumann and S.M. Ulam to the technique of solving problems though it is too expensive for experimental solutions and too complicated for analytical treatment.
- Monte Carlo methods are used to solve various mathematical problems based on sampling experiments in statistics using the sequences of random numbers for simulation and are termed as statistical simulation methods.
- Monte Carlo simulation uses random number tables to reproduce on paper the operation of any given system under its own working conditions. This technique is used to solve problems that depend upon probability where formulation of mathematical model is not possible.
- Monte Carlo simulation which uses random number tables can better be illustrated by considering any concrete operation subject to chance.
- Simulation is used in almost all fields. Some of the areas where simulation is used are manufacturing applications, semiconductor manufacturing, business process simulation healthcare, computer simulation etc.
- Simulation helps to study the behaviour of a system without building it and the results of simulation are accurate in general, in contrast to analytical model.

NOTES

12.7 KEY WORDS

- **Monte Carlo Simulation:** It refers to a method that is used to model the probability of different outcomes in a process that cannot easily be predicted due to the intervention of random variables.
- **Sampling:** It refers to the process of selecting a representative group from the population under study. The target population is the total group of individuals from which the sample might be drawn.

12.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. State the areas of application of Monte Carlo simulation.
2. List the areas where simulation technique is used.
3. What are the advantages and disadvantages of the simulation?

Long Answer Questions

1. Discuss Monte Carlo simulation with reference to queuing theory.
2. The following data is observed in a tea serving counter. The arrival is for one minute interval.

No. of Persons Arriving	0	1	2	3	4	5
Probability	0.05	0.15	0.40	0.20	0.15	0.05

The service is taken as 2 person for one-minute interval. Using the following random numbers simulate for 15-minutes period.

09, 54, 94, 01, 80, 73, 20, 26, 90, 79, 25, 48, 99, 25, 89

Calculate also the average number of persons waiting in queue per minute.

12.9 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 13 PROJECT SCHEDULING AND PERT-CPM

Structure

13.0 INTRODUCTION

Network scheduling is a technique used for planning and scheduling large projects, in the fields of construction, maintenance, fabrication and purchasing of computer systems, etc. It is a method of minimizing the trouble spots such as production, delays and interruptions, by determining critical factors and co-ordinating various parts of the overall job.

There are two basic planning and control techniques that utilize a network to complete a predetermined project or schedule. These are Programme Evaluation Review Technique (PERT) and Critical Path Method (CPM). A project is defined as a combination of interrelated activities, all of which must be executed in a certain order for its completion.

The work involved in a project can be divided into three phases, corresponding to the management functions of planning, scheduling and controlling.

Planning: This phase involves setting the objectives of the project as well as the assumptions to be made. It also involves the listing of tasks or jobs that must be performed in order to complete a project under consideration. In this phase, in addition to the estimates of costs and duration of the various activities, the manpower, machines and materials required for the project are also determined.

NOTES

Scheduling: This consists of laying the activities according to their order of precedence and determining the following:

(*i*) The start and finish times for each activity

(*ii*) The critical path on which the activities require special attention.

(*iii*) The slack and float for the non-critical paths.

Controlling: This phase is exercised after the planning and scheduling. It involves the following:

(*i*) Making periodical progress reports

(*ii*) Reviewing the progress

(*iii*) Analyzing the status of the project

(*iv*) Making management decisions regarding updating, crashing and resource allocation, etc.

13.1 OBJECTIVES

After going through this unit, you will be able to:

- Differentiate between PERT and CPM
- Identify the components of PERT/CPM network
- Explain the precedence relationship in PERT/CPM
- Apply PERT for project management

13.2 PERT/CPM NETWORK COMPONENTS AND PRECEDENCE RELATIONSHIP

Program Evaluation Review Technique (PERT) and Critical Path Method (CPM) are two planning and control techniques for keeping a project schedule on track to complete within the scheduled time. Network scheduling is a technique used for planning and scheduling large projects in the field of construction, maintenance, fabrication, purchasing computer system, etc. PERT is a probabilistic method where the activity times are represented by a probability distribution.

Critical path method (CPM) is a graphical technique for planning and scheduling of projects. This technique involves the preparation of the network in the form of arrow diagram and its analysis to indicate the critical path. It has the potential for scheduling of a task in minimum time and/or cost in accordance with specified constraints. After preparing the network diagram and indicating the times for each activity, you can mention the various possible paths for determining the critical path. The critical path being the longest path can easily be found out from the possible paths as the one taking the maximum time in the completion of the project.

NOTES

13.2.1 Basic Difference between PERT and CPM

Although PERT and CPM were developed separately and originally there were many differences between the two, over time these project planning techniques have become almost identical. CPM also is a display of a complex project as a network, with one "time estimate" used for each step in the project, as compared to PERT where three "time estimates" are used. In this technique the times and cost of activities are known. It accommodates situations in which sets of standardized activities are a part of the project for which the times for completion of these activities can be more accurately calculated. However, since there is only one time estimate, which may or may not be accurate, the probability for completing the project on time is difficult to compute. In all other aspects PERT and CPM one similar.

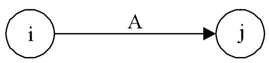
13.2.2 Basic Terms

To understand the network techniques, one should be familiar with a few basic terms of which both CPM and PERT are special applications.

Network It is the graphic representation of logically and sequentially connected arrows and nodes, representing activities and events in a project. Networks are also called arrow diagrams.

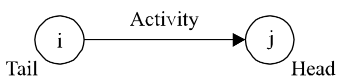
Activity An activity represents some action and is a time consuming effort necessary to complete a particular part of the overall project. Thus, each and every activity has a point of time where it begins and a point where it ends.

It is represented in the network by an arrow,

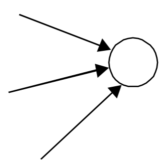


Here *A* is called the activity.

Event The beginning and end points of an activity are called events or nodes. Event is a point in time and does not consume any resources. It is represented by a numbered circle. The head event called the *j*th event always has a number higher than the tail event, which is also called the *i*th event.

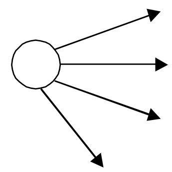


Merge and burst events It is not necessary for an event to be the ending event of only one activity as it can be the ending event of two or more activities. Such an event is defined as a merge event.



If the event happens to be the beginning event of two or more activities, it is defined as a burst event.

NOTES



Preceding, succeeding and concurrent activities Activities that must be accomplished before a given event can occur, are termed as *preceding activities.*

Activities that cannot be accomplished until an event has occurred, are termed as *succeeding activities.*

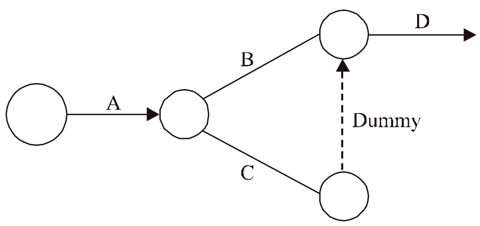
Activities that can be accomplished concurrently, are known as *concurrent activities.*

This classification is relative, which means that one activity can be preceding to a certain event, and the same activity can be succeeding to some other event or it may be a concurrent activity with one or more activities.

Dummy activity Certain activities, which neither consume time nor resources but are used simply to represent a connection or a link between the events are known as dummies. It is shown in the network by a dotted line. The purpose of introducing dummy activity is:

(*i*) To maintain uniqueness in the numbering system, as every activity may have a distinct set of events by which the activity can be identified.

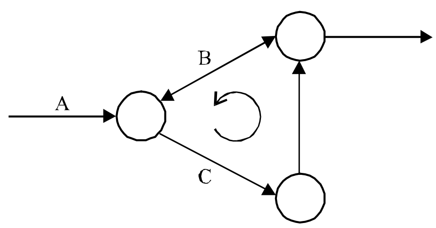
(*ii*) To maintain a proper logic in the network.



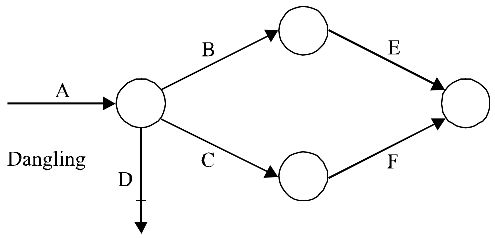
13.2.3 Common Errors

Following are the three common errors in a network construction:

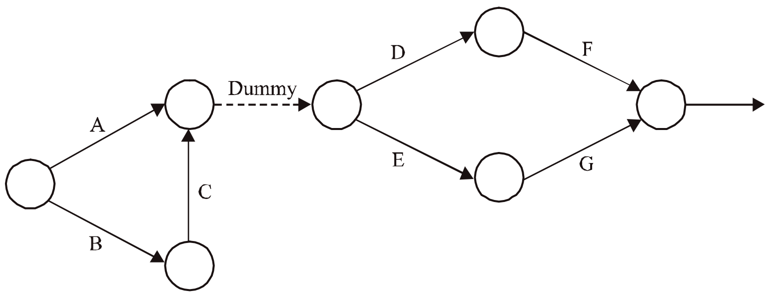
Looping (cycling) In a network diagram, a looping error is also known as cycling error. Drawing an endless loop in a network is known as error of looping. A loop can be formed if an activity is represented as going back in time.



Dangling To disconnect an activity before the completion of all the activities in a network diagram, is known as dangling.



Redundancy If a dummy activity is the only activity emanating from an event and can be eliminated, it is known as redundancy.



13.2.4 Rules of Network Construction

There are a number of rules in connection with the handling of events and activities of a project network that should be followed.

(*i*) Try to avoid arrows that cross each other.
(*ii*) Use straight arrows.
(*iii*) No event can occur until every activity preceding it has been completed.
(*iv*) An event cannot occur twice, i.e., there must be no loops.
(*v*) An activity succeeding an event cannot be started until that event has occurred.
(*vi*) Use arrows from left to right. Avoid mixing two directions, vertical and standing arrows may be used if necessary.
(*vii*) Dummies should be introduced only if it is extremely necessary.
(*viii*) The network has only one entry point called the start event and one point of emergence called the end or terminal event.

13.2.5 Numbering the Events (Fulkerson's Rule)

After the network is drawn in a logical sequence, every event is assigned a number. The number sequence must be such so as to reflect the flow of the network. In numbering the events, the following rules should be observed.

(*i*) Event numbers should be unique.
(*ii*) Event numbering should be carried out on a sequential basis, from left to right.
(*iii*) The initial event, which has all outgoing arrows with no incoming arrow is numbered as 1.

NOTES

(*iv*) Delete all arrows emerging from all the numbered events. This will create at least one new start event, out of the preceding events.

(*v*) Number all new start events 2, 3 and so on. Repeat this process until the terminal event without any successor activity is reached. Number the terminal node suitably.

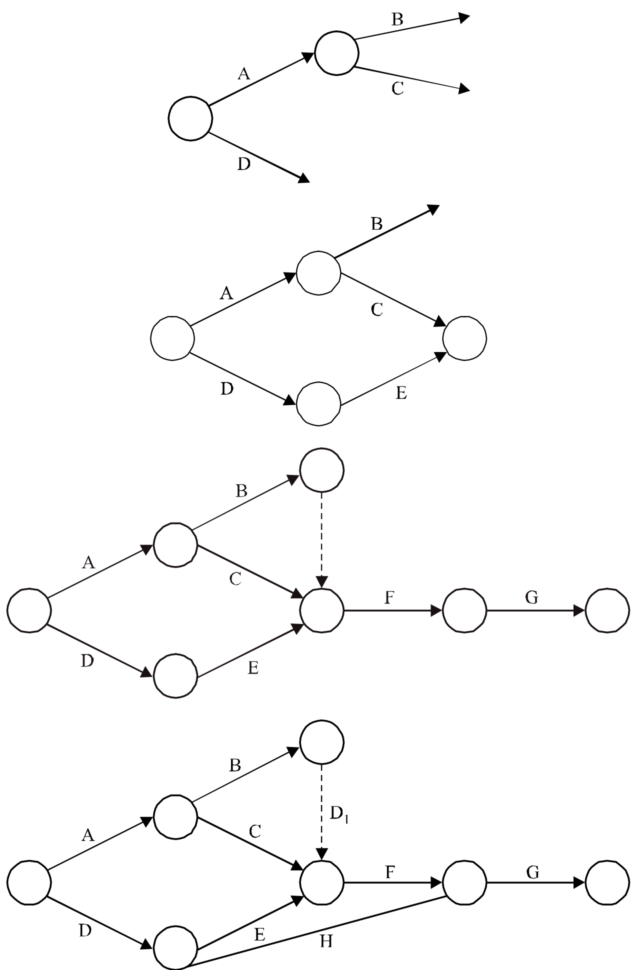
Note: The head of an arrow should always bear a number higher than the one assigned to the tail of the arrow.

13.2.6 Construction of Network

Example 13.1 Construct a network for the project whose activities and precedence relationships are as given below:

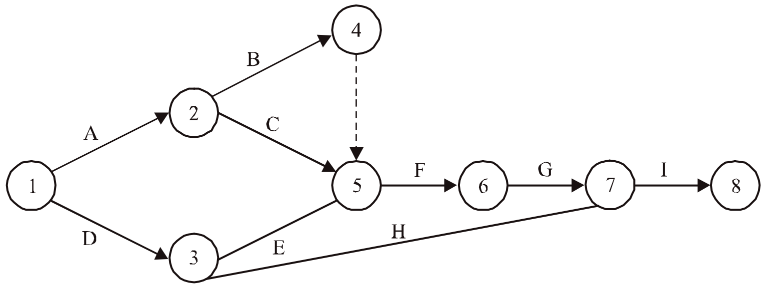
Activities	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Immediate Predecessor	–	*A*	*A*	–	*D*	*B*, *C*, *E*	*F*	*D*	*G*, *H*

Solution From the given constraints, it is clear that A and D are the starting activities and I the terminal activity. B and C are starting with the same event and are both the predecessors of the activity F. Also, E has to be the predecessor of both F and H. Hence, we have to introduce a dummy activity.



D_1 is the dummy activity.

Finally, we have the following network.

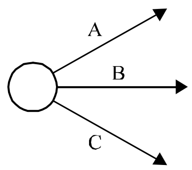


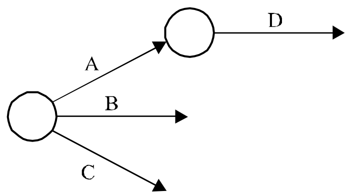
NOTES

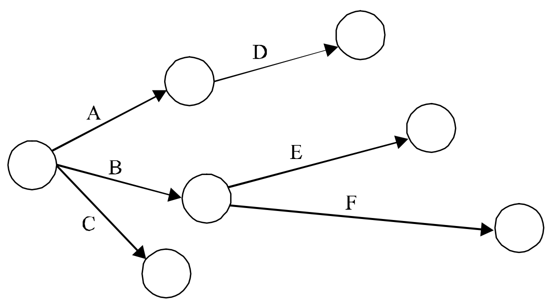
Example 13.2 Construct a network for each of the projects whose activities and their precedence relationships are given below.

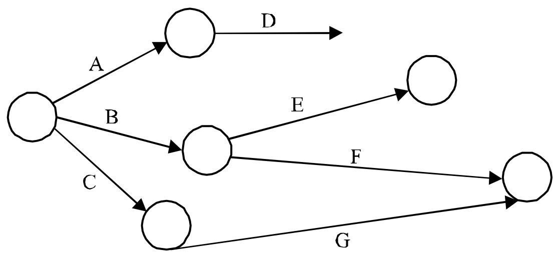
Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
Predecessor	–	–	–	*A*	*B*	*B*	*C*	*D*	*E*	*H*, *I*	*F*, *G*

Solution *A*, *B* and *C* are the concurrent activities as they start simultaneously. *B* becomes the predecessor of activities *E* and *F*. Since the activities *J* and *K* have two preceding activities, a dummy may be introduced (if possible).

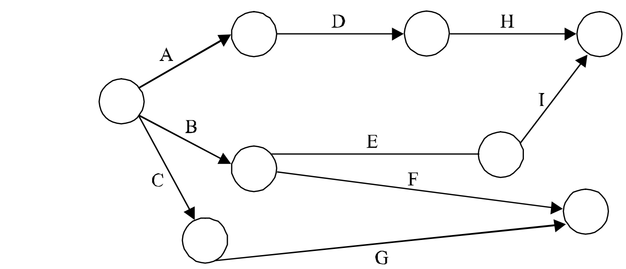


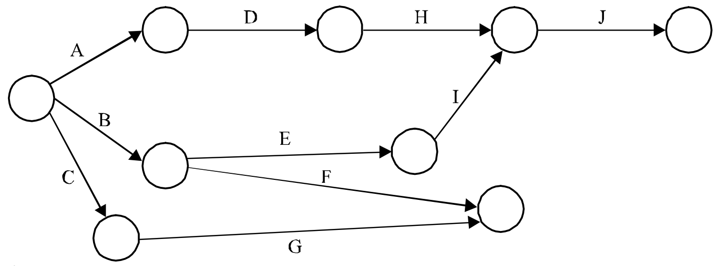




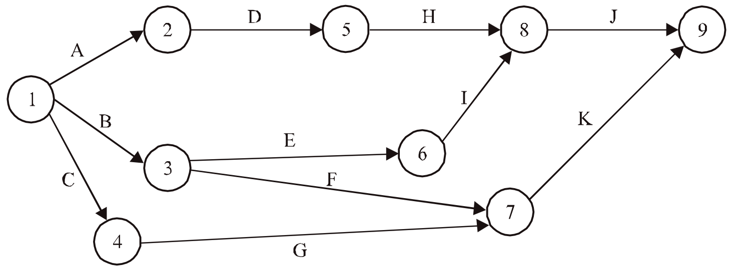


NOTES





Finally we have,

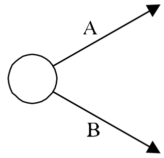


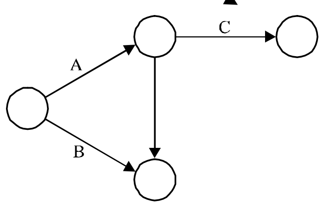
Example 13.3 $A < C, D, I$; $B < G, F$; $D < G, F$; $F < H, K$; $G, H < J$; $I, J, K < E$

Solution Given $A < C$, which means that C cannot be started until A is completed. That is, A is the preceding activity to C. The above constraints can be given in the following table.

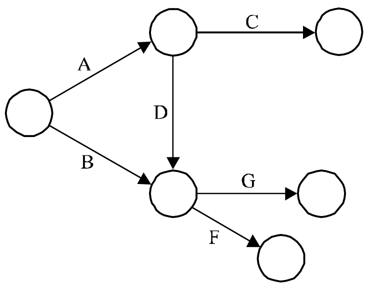
Activity	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***	***I***	***J***	***K***
Predecessor	–	–	*A*	*A*	*I*, *J*, *K*	*B*, *D*	*B*, *D*	*F*	*A*	*G*, *H*	*F*

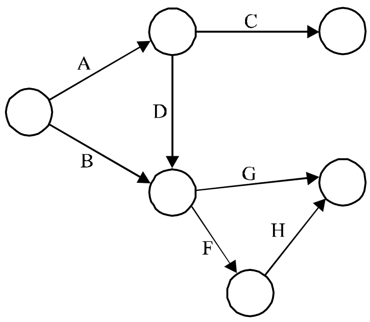
A and *B* are the starting activities, and *E* is the terminal activity.

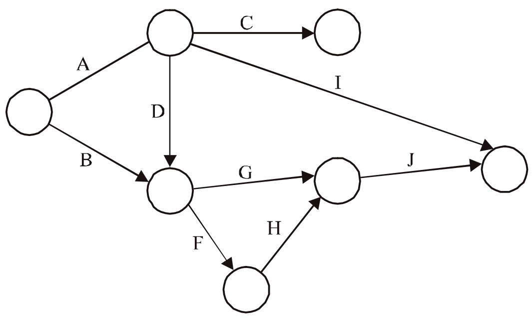


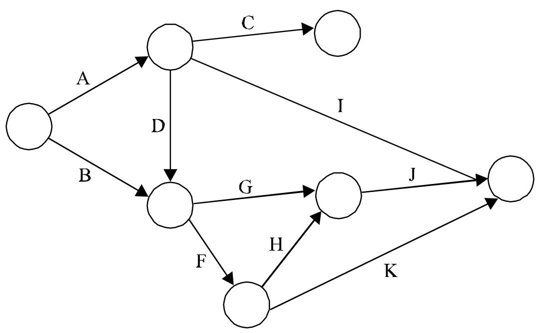


NOTES

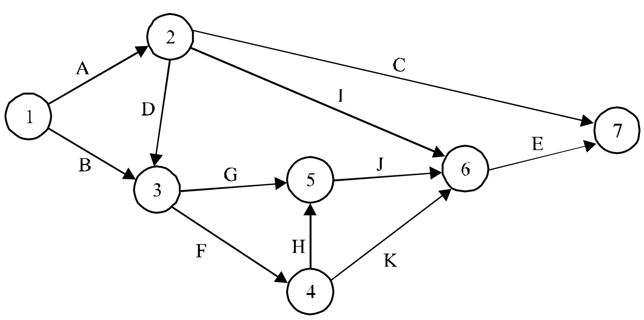








Finally, we have,



NOTES

13.2.7 Time Analysis

Once the network of a project is constructed, the time analysis of the network becomes essential for planning various activities of the project. Activity time is a forecast of the time an activity is expected to take from its starting point to its completion (under normal conditions).

We shall use the following notation for basic scheduling computations.

(i, j) = Activity (i, j) with tail event i and head event j

T_{ij} = Estimated completion time of activity (i, j)

ES_{ij} = Earliest starting time of activity (i, j)

EF_{ij} = Earliest finishing time of activity (i, j)

LS_{ij} = Latest starting time of activity (i, j)

LF_{ij} = Latest finishing time of activity (i, j).

The basic scheduling computation can be put under the following three groups.

13.2.8 Forward Pass Computations (For Earliest Event Time)

Before starting computations, the occurrence time of the initial network event is fixed. The forward pass computation yields the earliest start and the earliest finish time for each activity (i, j) and indirectly the earliest occurrence time for each event namely E_i. This consists of the following three steps:

Step 1 The computations begin from the start node and move towards the end node. Let zero be the starting time for the project.

Step 2 Earliest starting time $(ES)_{ij} = E_i$ is the earliest possible time when an activity can begin, assuming that all of the predecessors are also started at their earliest starting time. Earliest finish time of activity (i, j) is the earliest starting time + the activity time.

$$(EF)_{ij} = (ES)_{ij} + t_{ij}$$

Step 3 Earliest event time for event j is the maximum of the earliest finish time of all the activities, ending at that event.

$$E_j = \max_i \; (E_i + t_{ij})$$

The computed 'E' values are put over the respective rectangles representing each event.

13.2.9 Backward Pass Computations (For Latest Allowable Time)

The latest event time (L) indicates the time by which all activities entering into that event must be completed without delaying the completion of the project. These can be calculated by reversing the method of calculations used for the earliest event time. This is done in the following steps.

Step 1 For ending event assume, $E = L$.

Step 2 Latest finish time for activity (i, j) is the target time for completing the project,

$$(LF_{ij}) = L_j$$

Step 3 Latest starting time of the activity (i, j) = latest completion time of (i, j) – the activity time

$$LS_{ij} = LF_{ij} - t_{ij}$$
$$= L_j - t_{ij}$$

Step 4 Latest event time for event i is the minimum of the latest start time of all activities originating from the event.

$$L_i = \underset{j}{\text{Min}}(L_j - t_{ij})$$

The computed 'L' values are put over the respective triangles representing each event.

13.2.10 Determination of Floats and Slack Times

Float is defined as the difference between the latest and the earliest activity time.

Slack is defined as the difference between the latest and the earliest event time.

Hence, the basic difference between the slack and float is that slack is used for events only; whereas float is used for activities.

There are mainly three kinds of floats as given below.

Total float It refers to the amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time, without affecting the overall project duration time.

Mathematically, the total float of an activity (i,j) is the difference between the latest start time and the earliest start time of that activity.

Hence, the total float for an activity (i,j) denoted by $(TF)_{ij}$ is calculated by the formula,

$(TF)_{ij}$ = (Latest start – Earliest start) for activity (i,j)

$$\text{i.e., } (TF)_{ij} = (LS)_{ij} - (ES)_{ij}$$
$$\text{or } (TF)_{ij} = (L_j - E_i) - t_{ij}$$

where, E_i and L_j are the earliest time and latest time for the tail event i and head event j and t_{ij} is the normal time for the activity (i,j). This is the most important type of float as it concerns the overall project duration.

Free float The time by which the completion of an activity can be delayed beyond the earliest finish time, without affecting the earliest start of a subsequent succeeding activity.

NOTES

Mathematically, the free float for activity (i, j) denoted by $(FF)_{ij}$ can be calculated by the formula,

$$FF_{ij} = (E_j - E_i) - t_{ij}$$
$$(FF)_{ij} = \text{Total float} - \text{Head event slack}$$

Head event slack $= L_j - E_j$

This float is concerned with the commencement of the subsequent activity.

The free float can take values from zero up to total float, but it cannot exceed total float. This float is very useful for rescheduling an activity with minimum disruption in earlier plans.

Independent float The amount of time by which the start of an activity can be delayed, without affecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time.

Mathematically, independent float of an activity (i, j) denoted by $(IF)_{ij}$ can be calculated by the formula,

$$IF_{ij} = (E_j - L_i) - t_{ij}$$

or

$$(IF)_{ij} = \text{Free Float} - \text{Tail event slack}$$

where tail event slack is given by,

$$\text{Tail event slack} = L_i - E_i$$

The negative independent float is always taken as zero. This float is concerned with prior and subsequent activities.

$$IF_{ij} \leq FF_{ij} \leq TF_{ij}$$

Note: (*i*) If the total float TF_{ij} for any activity (i, j) is zero, then such an activity is called *critical activity*.

(*ii*) The float can be used to reduce project duration. While doing this, the float of not only that activity, but that of other activities will also change.

Critical activity An activity is said to be critical if a delay in its start cause a further delay in the completion of the entire project.

Critical path The sequence of critical activities in a network is called the critical path. It is the longest path in the network, from the starting event to the ending event and defines the minimum time required to complete the project. In the network it is denoted by a double line and identifies all the critical activities of the project. Hence, for the activities (i, j) to lie on the critical path, following conditions must be satisfied.

(*a*) $ES_i = LF_i$

(*b*) $ES_j = LF_j$

(*c*) $ES_j - ES_i = LF_j - LF_i = t_{ij}$

ES_i and ES_j are the earliest start and finish time of the events *i* and *j*.

LF_i and LF_j are the latest start and finish time of the events *i* and *j*.

NOTES

13.3 CRITICAL PATH METHOD (CPM)

The iterative procedure of determining the critical path (CP) is as follows:

Step 1 List all the jobs and then draw an arrow (network) diagram. Each job is indicated by an arrow with the direction of the arrow showing the sequence of jobs. The length of the arrows has no significance. The arrows are placed based on the predecessor, successor and concurrent relation within the job.

Step 2 Indicate the normal time (t_{ij}) for each activity (i, j) above the arrow, which is deterministic.

Step 3 Calculate the earliest start time and the earliest finish time for each event and write the earliest time E_i for each event *i* in the $\square$. Also calculate the latest finish and latest start time. From this we calculate the latest time L_j for each event *j* and put it in the Δ.

Step 4 Tabulate the various times, namely, normal time, earliest time and latest time on the arrow diagram.

Step 5 Determine the total float for each activity by taking the difference between the earliest start and the latest start time.

Step 6 Identify the critical activities and connect them with the beginning and the ending events in the network diagram by double line arrows. This gives the critical path.

Step 7 Calculate the total project duration.

Note: The earliest start and finish time of an activity, as well as the latest start and finish time of an activity are shown in the table. These are calculated by using the following hints.

To find the earliest time, we consider the tail event of the activity. Let the starting time of the project namely $ES_i = 0$. Add the normal time with the starting time, to get the earliest finish time. The earliest starting time for the tail event of the next activity is given by the maximum of the earliest finish time for the head event of the previous activity.

Similarly, to get the latest time, we consider the head event of the activity.

The latest finish time of the head event of the final activity is given by the target time of the project. The latest start time can be obtained by subtracting the normal time of that activity. The latest finish time for the head event of the next activity is given by the minimum of the latest start time for the tail event of the previous activity.

NOTES

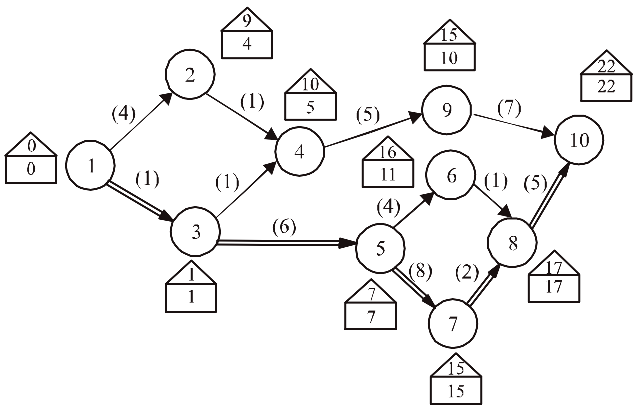
Example 13.4 A project schedule has the following characteristics.

Activity	*1–2*	*1–3*	*2–4*	*3–4*	*3–5*	*4–9*	*5–6*	*5–7*	*6–8*	*7–8*	*8–10*	*9–10*
Time (days)	4	1	1	1	6	5	4	8	1	2	5	7

From the above information, you are required to:

1. Construct a network diagram.
2. Compute the earliest event time and latest event time.
3. Determine the critical path and total project duration.
4. Compute total and free float for each activity.

Solution First we construct the network with the given constraints (here we get it by just connecting the event numbers).



The following table gives the critical path as well as total and free floats calculation.

Activity	*Normal time*	*Earliest*		*Latest*		*TF*	*FF*
		Start	*Finish*	*Start*	*Finish*		
1 – 2	4	0	4	5	9	5	5 – 5 = 0
1 – 3	1	0	1	0	1	0	0
2 – 4	1	4	5	9	10	5	0
3 – 4	1	1	2	9	10	8	3
3 – 5	6	1	7	1	7	0	0
4 – 9	5	5	10	10	15	5	0
5 – 6	4	7	11	12	16	5	0
5 – 7	8	7	15	7	15	0	0
6 – 8	1	11	12	16	17	5	0
7 – 8	2	15	17	15	17	0	0
8 –10	5	17	22	17	22	0	0
9 – 10	7	10	17	15	22	5	5

The earliest and latest calculations are shown below.

NOTES

Forward pass calculation In this we estimate the earliest start (Es_i) and finish times (Es_j). The earliest time for the event i is given by,

$$E_i = \underset{i}{\text{Max}}\ (ES_i + t_{ij})$$
$$ES_1 = 0 = E_1 = 0$$
$$E_2 = ES_2 = ES_1 + t_{12} = 0 + 4 = 4$$
$$E_3 = ES_3 = ES_1 + t_{13} = 0 + 1 = 1$$
$$E_4 = ES_4 = \text{Max}\ (ES_3 + t_{34}, ES_2 + t_{24})$$
$$= \text{Max}\ (1 + 1, 4 + 1) = 5$$
$$E_5 = (E_3 + t_{35}) = 1 + 6 = 7$$
$$E_6 = E_5 + t_{56} = 7 + 4 = 11$$
$$E_7 = E_5 + t_{57} = 7 + 8 = 15$$
$$E_8 = \text{Max}\ (E_6 + t_{68}, E_7 + t_{78})$$
$$= \text{Max}\ (11 + 1, 15 + 2) = 17$$
$$E_9 = E_4 + t_{49} = 5 + 5 = 10$$
$$E_{10} = \text{Max}\ (E_9 + t_{9,10}, E_8 + t_{8,\,10})$$
$$= \text{Max}\ (10 + 7, 17 + 5) = 22.$$

Backward pass calculation In this we calculate the latest finish and the latest start time. The latest time L for an event i is given by $L_i = \underset{j}{\text{Min}}\ (LF_j - t_{ij})$, where, LF_j is the latest finish time for the event j, t_{ij} is the normal time of the activity.

$$L_{10} = 22$$
$$L_9 = L_{10} - t_{9,\,10} = 22 - 7 = 15$$
$$L_8 = L_{10} - t_{8,\,10} = 22 - 5 = 17$$
$$L_7 = L_8 - t_{7,\,8} = 17 - 2 = 15$$
$$L_6 = L_8 - t_{6,\,8} = 17 - 1 = 16$$
$$L_5 = \text{Min}\ (L_6 - t_{5,\,6}, L_7 - t_{5,\,7})$$
$$= \text{Min}\ (16 - 4, 15 - 8) = 7$$
$$L_4 = L_9 - t_{4,\,9} = 15 - 5 = 10$$
$$L_3 = \text{Min}\ (L_4 - t_{3,\,4}, L_5 - t_{3,\,5})$$
$$= \text{Min}\ (10 - 1, 7 - 6) = 1$$
$$L_2 = L_4 - t_{2,\,4} = 10 - 1 = 9$$
$$L_1 = \text{Min}\ (L_2 - t_{12}, L_3 - t_{13}) = \text{Min}\ (9 - 4, 1 - 1) = 0.$$

These calculations are shown in the above table.

NOTES

To find the TF (Total Float) Considering the activity 1 – 2, *TF* of (1 – 2) Latest start Earliest start

5 0 5

Similarly *TF* (2 – 4) *LS* *ES*

9 4 5

Free float *TF* Head event slack.

Consider the activity 1 – 2

FF of 1 – 2 *TF* of 1 – 2 Slack for the head event 2

5 (9 – 4) (from the figure for event 2)

5 5 0

FF of 2 – 4 *TF* of 2 – 4 Slack for the head event 4

5 (10 – 5) 5 5 0

Like this we calculate the *TF* and *FF* for the remaining activities.

From the above table we observe that the activities 1 – 3, 3 – 5, 5 – 7, 7 – 9, 8 – 10 are the critical activities as their total float is 0.

Hence, we have the following critical path.

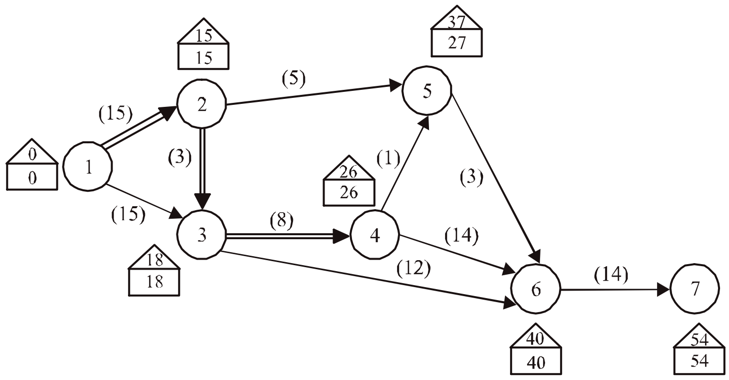
1 – 3 – 5 – 7 – 8 – 10, with the total project duration of 22 days.

Example 13.5 A small maintenance project consists of the following jobs, whose precedence relationships are given below.

Job	***1–2***	***1–3***	***2–3***	***2–5***	***3–4***	***3–6***	***4–5***	***4–6***	***5–6***	***6–7***
Duration (days)	15	15	3	5	8	12	1	14	3	14

1. Draw an arrow diagram representing the project.
2. Find the total float for each activity.
3. Find the critical path and the total project duration.

Solution



NOTES

Forward pass calculation In this we estimate the earliest start and the earliest finish time ES_j given by,

$ES_j = \underset{i}{\text{Max}}\ (ES_i + t_{ij})$ where, ES_i is the earliest start time and t_{ij} is the normal time for the activity (i, j).

$$ES_1 = 0$$
$$ES_2 = ES_1 + t_{15} = 0 + 15 = 15$$
$$ES_3 = \text{Max}\ (ES_2 + t_{23}, ES_1 + t_{13})$$
$$= \text{Max}\ (15 + 3, 0 + 15) = 18$$
$$ES_4 = ES_3 + t_{34} = 18 + 8 = 26$$
$$ES_5 = \text{Max}\ (ES_2 + t_{25}, ES_4 + t_{45})$$
$$= \text{Max}\ (15 + 5, 26 + 1) = 27$$
$$ES_6 = \text{Max}\ (ES_3 + t_{36}, ES_4 + t_{46}, ES_5 + t_{56})$$
$$= \text{Max}\ (18 + 12, 26 + 14, 27 + 3)$$
$$= 40$$
$$ES_7 = ES_6 + t_{67} = 40 + 14 = 54.$$

Backward pass calculation In this we calculate the latest finish and latest start time LF_i, given by $LF_i = \underset{i}{\text{Min}}(LF_j - t_{ij})$ where, LF_j is the latest finish time for the event j

$$LF_7 = 54$$
$$LF_6 = LF_7 - t_{67} = 54 - 14 = 40$$
$$LF_5 = LS_6 - t_{56} = 40 - 3 = 37$$
$$LF_4 = \text{Min}\ (LS_5 - t_{45}, LS_6 - t_{46})$$
$$= \text{Min}\ (37 - 1, 40 - 14) = 26$$
$$LF_3 = \text{Min}\ (LF_4 - t_{34}, LF_6 - t_{36})$$
$$= \text{Min}\ (26 - 8, 40 - 12) = 18$$
$$LF_2 = \text{Min}\ (LF_5 - t_{25}, LF_3 - t_{23})$$
$$= \text{Min}\ (37 - 5, 18 - 3) = 15$$
$$LF_1 = \text{Min}\ (LF_3 - t_{13}, LF_2 - t_{12})$$
$$= \text{Min}\ (18 - 15, 15 - 15) = 0$$

NOTES

The following table gives the calculations for critical path and total float.

Activity	Normal time	Earliest		Latest		Total float
		Start	Finish	Start	Finish	$LF_j - ES_j$ or $LF_i - ES_i$
		ES_i	ES_j	LF_i	LF_j	
1 – 2	15	0	15	0	15	0
1 – 3	15	0	15	3	18	3
2 – 3	3	15	18	15	18	0
2 – 5	5	15	20	32	37	17
3 – 4	8	18	26	18	26	0
3 – 6	12	18	30	28	40	10
4 – 5	1	26	27	36	37	10
4 – 6	14	26	40	26	40	0
5 – 6	3	27	30	37	40	10
6 – 7	14	40	54	40	54	0

From the above table, we observe that the activities 1—2, 2—3, 3—4, 4—6, 6—7 are the critical activities and the critical path is given by, 1—2—3—4—6—7

The total time taken for project completion is 54 days.

Example 13.6 Tasks A, B, ... H, I constitute a project. The notation $X < Y$ means that the task X must be completed before Y is started. With the notation,

$$A < D, A < E, B < F, D < F, C < G, C < H, F < I, G < I$$

Draw a graph to represent the sequence of tasks and find the minimum time of completion of the project, when the time (in days) of completion of each task is as follows.

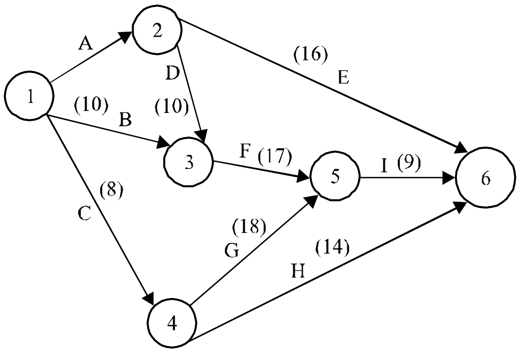
The above constraints can be given as in the following table:

Task	A	B	C	D	E	F	G	H	I
Time (days)	8	10	8	10	16	17	18	14	9

Solution The above constraints are given in the following table:

Activity	A	B	C	D	E	F	G	H	I
Preceding Activity	–	–	–	A	A	B, D	C	C	F, G

NOTES



Time calculation Using forward and backward pass calculation, we first estimate the earliest and the latest time for each event.

$$ES_1 = E_1 = 0$$

$$E_2 = E_1 + t_{12} = 0 + 8 = 8$$

$$E_3 = \text{Max}\ (E_1 + E_{13}, E_2 + t_{23})$$
$$= \text{Max}\ (0 + 10, 8 + 10) = 18$$

$$E_4 = E_1 + t_{14} = 0 + 8 = 8$$

$$E_5 = \text{Max}\ (E_3 + t_{35}, E_4 + t_{45})$$
$$= \text{Max}\ (18 + 17, 8 + 18) = 35$$

$$E_6 = \text{Max}\ (E_2 + t_{26}, E_4 + t_{46}, E_5 + t_{56})$$
$$= \text{Max}\ (8 + 16, 8 + 14, 35 + 9) = 44$$

The value of the latest time can now be obtained.

$L_6 = E_6 = 44$ (Target completion time for the project)

$$L_5 = L_6 - t_{56} = 44 - 9 = 35$$

$$L_4 = \text{Min}\ (L_6 - t_{46}, L_5 - t_{45})$$
$$= \text{Min}\ (44 - 14, 35 - 18) = 17$$

$$L_3 = L_5 - t_{35} = 35 - 17 = 18$$

$$L_2 = \text{Min}\ (L_6 - t_{26}, L_3 - t_{23})$$
$$= \text{Min}\ (44 - 16, 18 - 10 = 8$$

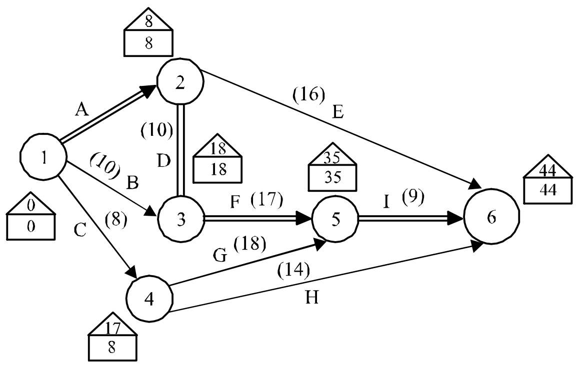
$$L_1 = \text{Min}\ (L_4 - t_{14}, L_3 - t_{13}, L_2 - t_{12})$$
$$= \text{Min}\ (17 - 8, 18 - 10, 8 - 8) = 0.$$

NOTES

To evaluate the critical events, all these calculations are put in the following table.

Task	Normal Time/days	Earliest		Latest			Floats	
		Start	Finish	Start	Finish	TF	FF	IF
A 1–2	8	0	8	0	8	0	0 – 0 = 0	0 – 0 = 0
B 1–3	10	0	10	8	18	8	8 – 0 = 8	8 – 0 = 8
C 1–4	8	0	8	9	17	9	9 – 9 = 0	0 – 0 = 0
D 2–3	10	8	18	8	18	0	0 – 0 = 0	0 – 0 = 0
E 2–6	16	8	24	28	44	20	20 – 0 = 20	20 – 0 = 20
F 3–5	17	18	35	18	35	0	0 – 0 = 0	0 – 0 = 0
G 4–5	18	8	26	17	35	9	9 – 0 = 9	9 – 9 = 0
H 4–6	14	8	22	30	44	22	22 – 0 = 22	22 – 9 = 13
I 5–6	9	35	44	35	44	0	0 – 0 = 0	0 – 0 = 0

The above table shows that the critical events are the tasks 1 – 2, 2 – 3, 3 – 5, 5 – 6 as their total float is zero.



The critical path is given by 1–2–3–5–6 or *A–D–F–I*, with the total project duration as 44 days.

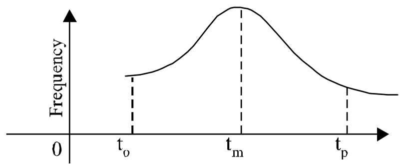
13.4 PROJECT MANAGEMENT-PERT

The network methods discussed so far may be termed as deterministic, since estimated activity times are assumed to be known with certainty. However, in the research project or design of a gear box or a new machine, various activities are based on judgement. It is difficult to obtain a reliable time estimate due to the changing technology since time values are subject to chance variations. For such cases, where the activities are non-deterministic in nature, PERT was developed. Hence, PERT is a probabilistic method, where the activity times are represented by a probability distribution. This distribution of activity times is based on three different time estimates made for each activity, which are as follows:

(*i*) Optimistic time estimate

(*ii*) Most likely time estimate

(*iii*) Pessimistic time estimate



Time distribution curve

NOTES

Optimistic time estimate It is the smallest time taken to complete the activity, if everything goes well. There is very little chance that an activity can be completed in a time less than the optimistic time. It is denoted by t_o or a.

Most likely time estimate It refers to the estimate of the normal time the activity would take. This assumes normal delays. It is the mode of the probability distribution. It is denoted by t_m or m.

Pessimistic time estimate It is the longest time that an activity would take, if everything goes wrong. It is denoted by t_p or b. These three time values are shown in the following figure.

From these three time estimates, we have to calculate the expected time of an activity. It is given by the weighted average of the three time estimates,

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

[β distribution with weights of 1, 4 and 1, for t_o, t_m and t_p estimates respectively.]

Variance of the activity is given by,

$$\sigma^2 = \left[\frac{t_p - t_o}{6}\right]^2$$

The expected length (duration), denoted by to T_c of the entire project is the length of the critical path, i.e., the sum of the t_c's of all the activities along the critical path.

The main objective of the analysis through PERT is to find the completion for a particular event within the specified date T_s, given by $P(Z \le D)$ where,

$$D = \frac{\text{Due date – Expected date of completion}}{\sqrt{\text{Project variance}}}$$

Here, Z stands for standard normal variable.

PERT Procedure

Step 1 Draw the project network.

Step 2 Compute the expected duration of each activity using the formula,

$$t_e = \frac{t_o + 4tm + tp}{6}$$

Also calculate the expected variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$ of each activity.

Step 3 Compute the earliest start, earliest finish, latest start, latest finish and total float of each activity.

NOTES

Step 4 Find the critical path and identify the critical activities.

Step 5 Compute the project length variance σ^2, which is the sum of the variance of all the critical activities and hence, find the standard deviation of the project length σ.

Step 6 Calculate the standard normal variable $Z = \frac{T_s - T_e}{\sigma}$, where T_s is the scheduled time to complete the project.

T_e = Normal expected project length duration.

σ = Expected standard deviation of the project length.

Using the normal curve, we can estimate the probability of completing the project within a specified time.

Example 13.7 The following table shows the jobs of a network along with their time estimates.

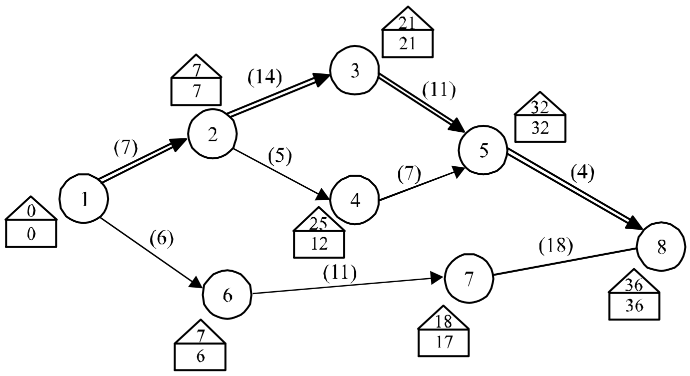
Job	*1–2*	*1–6*	*2–3*	*2–4*	*3–5*	*4–5*	*6–7*	*5–8*	*7–8*
a (days)	1	2	2	2	7	5	5	3	8
m (days)	7	5	14	5	10	5	8	3	17
b (days)	13	14	26	8	19	17	29	9	32

Draw the project network and find the probability of the project completing in 40 days.

Solution First we calculate the expected time and standard deviation for each activity.

Activity	$t_e = \frac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1–2	$\frac{1 + (4 \times 7) + 13}{6} = 7$	$\left(\frac{13 - 1}{6}\right)^2 = 4$
1–6	$\frac{2 + (4 \times 5) + 14}{6} = 6$	$\left(\frac{14 - 2}{6}\right)^2 = 4$
2–3	$\frac{2 + (4 \times 14) + 26}{6} = 14$	$\left(\frac{26 - 2}{6}\right) = 16$
2–4	$\frac{2 + (5 \times 4) + 8}{6} = 5$	$\left(\frac{8 - 2}{6}\right)^2 = 1$
3–5	$\frac{7 + (4 \times 10) + 19}{6} = 11$	$\left(\frac{19 - 7}{6}\right)^2 = 4$
4–5	$\frac{5 + (5 \times 4) + 17}{6} = 7$	$\left(\frac{17 - 5}{6}\right)^2 = 4$
6–7	$\frac{5 + (8 \times 4) + 29}{6} = 11$	$\left(\frac{29 - 5}{6}\right)^2 = 16$
5–8	$\frac{3 + (3 \times 4) + 9}{6} = 4$	$\left(\frac{9 - 3}{6}\right) = 1$
7–8	$\frac{8 + (4 \times 17) + 32}{6} = 18$	$\left(\frac{32 - 8}{6}\right)^2 = 16$

NOTES



Expected project duration = 36 days

Critical path 1–2–3–5–8

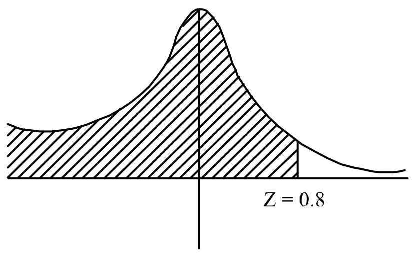
Project length variance, $\sigma^2 = 4 + 16 + 4 + 1 = 25$

$\sigma = 5$

The probability that the project will be completed in 40 days is given by,

$$P(Z \le D)$$

$$D = \frac{T_s - T_e}{\sigma} = \frac{40 - 36}{5} = \frac{4}{5} = 0.8$$



Area under the normal curve for the region $Z \le 0.8$

$$P(Z \le 0.8)$$

$= 0.5 + \phi(0.8)$ $\quad$ [$\phi(8) = 0.2881$ (from table)]

$= 0.5 + 0.2881$ $\quad = 0.7881$

$= 78.81\%$

Conclusion If the project is performed 100 times, under the same conditions, there will be 78.81 occasions for this job to be completed in 40 days.

Example 13.8 A small project is composed of seven activities, whose time estimates are listed in the table as follows:

NOTES

Activity	Estimated duration (weeks)		
	Optimistic (a)	Most likely (m)	Pessimistic (b)
1–2	1	1	7
1–3	1	4	7
2–4	2	2	8
2–5	1	1	1
3–5	2	5	14
4–6	2	5	8
5–6	3	6	15

You are required to:

1. Draw the project network.
2. Find the expected duration and variance of each activity.
3. Calculate the earliest and latest occurrence for each event and the expected project length.
4. Calculate the variance and standard deviations of project length.
5. What is the probability that the project will be completed,
   (*i*) 4 weeks earlier than expected?
   (*ii*) Not more than 4 weeks later than expected?
   (*iii*) If the project's due date is 19 weeks, what is the probability of meeting the due date?

Solution The expected time and variance of each activity is computed as shown in the table below:

Activity	a	m	b	$t_e = \dfrac{a+4t_m+b}{6}$	$s^2 = \left(\dfrac{b-a}{6}\right)^2$
1–2	1	1	7	2	1
1–3	1	4	7	4	1
1–4	2	2	8	3	1
2–5	1	1	1	1	0
3–5	2	5	14	6	4
4–6	2	5	8	5	1
5–6	3	6	15	7	4

The earliest and the latest occurrence time for each is calculated as below:

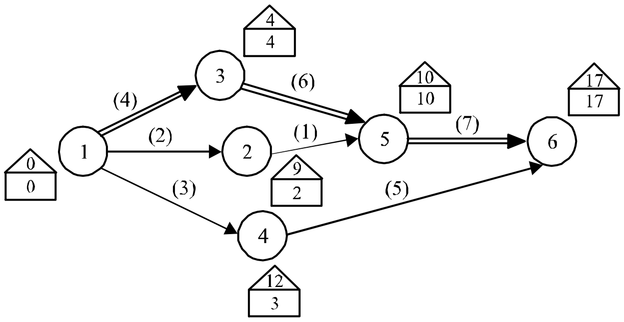
$$E_1 = 0;$$
$$E_2 = 0 + 2 + 2$$
$$E_3 = 0 + 4 = 4$$
$$E_4 = 0 + 3 = 3$$
$$E_5 = \text{Max}\ (2 + 1, 4 + 6) = 10$$
$$E_6 = \text{Max}\ (10 + 7, 3 + 5) = 17.$$

NOTES

To determine the latest expected time, we start with E_6 being the last event and move backwards subtracting t_e from each activity. Hence, we have,

$$L_6 = E_6 = 17$$
$$L_5 = L_6 - 7 = 17 - 7 = 10$$
$$L_4 = 17 - 5 = 12$$
$$L_3 = 10 - 6 = 4$$
$$L_2 = 10 - 1 = 9$$
$$L_1 = \text{Min}\,(9 - 2, 4 - 4, 12 - 3) = 0$$

Using the above information we get the following network, where the critical path is shown by the double-line arrows.



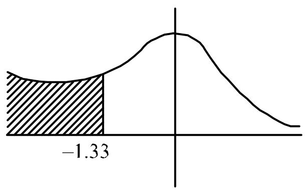
We observe the critical path of the above network as $1 \rightarrow 3 \rightarrow 5 \rightarrow 6$.

The expected project duration is 17 weeks, i.e., $T_e = 17$ weeks.

The variance of the project length is given by,

$$\sigma^2 = 1 + 4 + 4 = 9.$$

(*i*) The probability of completing the project within 4 weeks earlier than expected is given by,



$$P(Z \le D), \text{ where } D = \frac{T_s - T}{s}$$

$$D = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

$$D = \frac{17 - 4 - 17}{3} = \frac{12 - 17}{3} = \frac{-4}{3}$$
$$= -1.33$$

$$\therefore \quad P(Z \le -1.33) = 0.5 - \phi(1.33)$$
$$= 0.5 - 0.4082 \text{ (from the table)}$$
$$= .0918 = 9.18\%.$$

NOTES

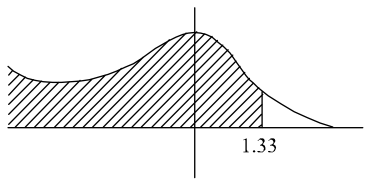
Conclusion If the project is performed 100 times under the same conditions, then there will be 9 occasions for this job to be completed in 4 weeks earlier than expected.

(*ii*) The probability of completing the project not more than 4 weeks later than expected is given by,

$$P(Z \le D), \text{ where}$$
$$D = \frac{T_s - T_e}{\sigma}$$

Here, $T_s = 17 + 4 = 21$

$$0 = \frac{21 - 17}{3} = \frac{4}{3} = 1.33$$
$$P(Z \le 1.33)$$
$$= 0.5 + \phi(1.33)$$
$$= 0.5 + 0.4082 \text{ (from the table)}$$
$$= 0.9082 = 90.82\%.$$



Conclusion If the project is performed 100 times under the same conditions, then there will be 90.82 occasions when this job will be completed not more than 4 weeks later than expected.

(*iii*) The probability of completing the project within 19 weeks, is given by,

$$P(Z \le D), \text{ where } D = \frac{19 - 17}{3} = \frac{2}{3} \quad [\because T_s = 19]$$
$$= 0.666.$$
$$\text{i.e., } P(Z \le 0.666) = 0.5 + \phi(0.666)$$
$$= 0.5 + 0.2514 \text{ (from the table)}$$
$$= 0.7514 = 75.14\%.$$

Conclusion If the project is performed 100 times, under the same conditions, then there will be 75.14 occasions for this job to be completed in 19 weeks.

NOTES

Example 13.9 Consider the following project.

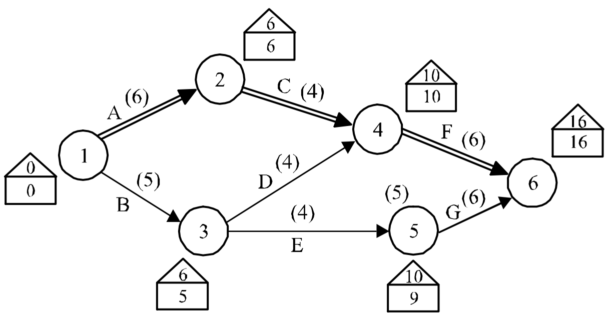
Activity	*Time estimate in weeks* t_o	t_m	t_p	*Predecessor*
A	3	6	9	None
B	2	5	8	None
C	2	4	6	*A*
D	2	3	10	*B*
E	1	3	11	*B*
F	4	6	8	*C, D*
G	1	5	15	*E*

Find the path and standard deviation. Also find the probability of completing the project by 18 weeks.

Solution First we calculate the expected time and variance of each activity as in the following table.

Activity	t_o	t_m	t_p	$t_e = \dfrac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\dfrac{t_p - t_o}{6}\right)^2$
A	3	6	9	$\dfrac{3+(4\times6)+9}{6}=6$	$\left(\dfrac{9-3}{6}\right)^2 = 1$
B	2	5	8	$\dfrac{2+(4\times5)+8}{6}=5$	$\left(\dfrac{8-2}{6}\right)^2 = 1$
C	2	4	6	$\dfrac{2+(4\times4)+6}{6}=4$	$\left(\dfrac{6-2}{6}\right)^2 = 0.444$
D	2	3	10	$\dfrac{2+(4\times3)+10}{6}=4$	$\left(\dfrac{10-3}{6}\right)^2 = 1.777$
E	1	3	11	$\dfrac{1+(4\times3)+11}{6}=4$	$\left(\dfrac{11-1}{6}\right)^2 = 2.777$
F	4	6	8	$\dfrac{4+(4\times6)+8}{6}=6$	$\left(\dfrac{8-4}{6}\right)^2 = 0.444$
G	1	5	15	$\dfrac{1+(4\times5)+15}{6}=6$	$\left(\dfrac{15-1}{6}\right)^2 = 5.444$

We construct the network with the help of the predecessor relation given in the data.



NOTES

Critical path is 1–2–4–6 or A–C–F.

The project length = 18 weeks.

Project length variance, $\sigma^2 = 1 + 0.444 + 0.444 = 1.888$

Standard deviation, $\sigma = 1.374$

The probability of completing the project in 18 weeks is given by,

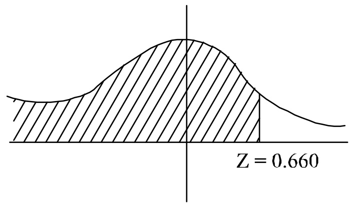
$P(Z \le D)$, where $D = \frac{T_s - T_e}{\sigma}$

$$T_s = 18;\ T_e = 16;\ \sigma = 1.374$$

$$D = \frac{18-16}{1.374} = 1.4556$$

$$P(Z \le D) = P(Z \le 1.4556) = 0.5 + \phi(1.4456)$$

$$= 0.5 + 0.4265 = 0.9265 = 92.65\%.$$



Conclusion If the project is performed 100 times under the same conditions, then there will be 92.65 occasions when this job will be completed by 18 weeks.

Check Your Progress

1. Define the preceding, succeeding and concurrent activities.
2. What is the purpose of introducing dummy activity?
3. What does the latest event time (L) indicate?
4. How is the critical path denoted in a network?

13.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Activities that must be accomplished before a given event can occur, are termed as *preceding activities.* Activities that cannot be accomplished until an event has occurred, are termed as *succeeding activities.* Activities that can be accomplished concurrently, are known as *concurrent activities.*

2. The purpose of introducing dummy activity is:
   (*i*) To maintain uniqueness in the numbering system, as every activity may have a distinct set of events by which the activity can be identified.
   (*ii*) To maintain a proper logic in the network.
3. The latest event time (L) indicates the time by which all activities entering into that event must be completed without delaying the completion of the project.
4. In the network critical path is denoted by a double line and identifies all the critical activities of the project.

13.6 SUMMARY

- Network is the graphic representation of logically and sequentially connected arrows and nodes, representing activities and events in a project. Networks are also called arrow diagrams.
- An activity represents some action and is a time consuming effort necessary to complete a particular part of the overall project. Thus, each and every activity has a point of time where it begins and a point where it ends.
- The beginning and end points of an activity are called events or nodes. Event is a point in time and does not consume any resources. It is represented by a numbered circle. The head event called the jth event always has a number higher than the tail event, which is also called the ith event.
- It is not necessary for an event to be the ending event of only one activity as it can be the ending event of two or more activities. Such an event is defined as a merge event.
- If the event happens to be the beginning event of two or more activities, it is defined as a burst event.
- A certain event, and the same activity can be succeeding to some other event or it may be a concurrent activity with one or more activities.
- Certain activities, which neither consume time nor resources but are used simply to represent a connection or a link between the events are known as dummies. It is shown in the network by a dotted line.
- In a network diagram, a looping error is also known as cycling error. Drawing an endless loop in a network is known as error of looping. A loop can be formed if an activity is represented as going back in time.
- After the network is drawn in a logical sequence, every event is assigned a number. The number sequence must be such so as to reflect the flow of the network.

NOTES

- Once the network of a project is constructed, the time analysis of the network becomes essential for planning various activities of the project. Activity time is a forecast of the time an activity is expected to take from its starting point to its completion (under normal conditions).
- The basic difference between the slack and float is that slack is used for events only; whereas float is used for activities.
- The sequence of critical activities in a network is called the critical path. It is the longest path in the network, from the starting event to the ending event and defines the minimum time required to complete the project.

13.7 KEY WORDS

- **Network:** It is the graphic representation of logically and sequentially connected arrows and nodes representing activities and events in a project.
- **Activity:** An activity represents some action and is a time consuming effort, necessary to complete a particular part of the overall project.
- **Event:** The beginning and end points of an activity are called events or nodes.
- **Dummy activity:** Certain activities, which neither consume time nor resources but are used simply to represent a connection or a link between the events, are known as dummies.
- **Float:** It is defined as the difference between the latest and the earliest activity time.
- **Slack:** It is defined as the difference between the latest and the earliest event time.
- **Critical activity:** An activity is said to be critical if a delay in its start will cause a further delay in the completion of the entire project.
- **Critical path:** The sequence of critical activities in a network is called the critical path.
- **Optimistic time estimate (t_o):** It is the smallest time taken to complete the activity if everything goes well.
- **Most likely time estimate (t_m):** It refers to the estimate of the normal time the activity would take.
- **Pessimistic time estimate (t_p):** It is the longest time that an activity would take if everything goes wrong.

13.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short Answer Questions

1. What do you understand by merge and burst events?
2. What are the common errors in a network construction?
3. State the rules that need to be followed in numbering the events.
4. What is the difference between free float and independent float?
5. What is critical path? Mention the conditions that must be satisfied for the activities to lie on the critical path.

Long Answer Questions

1. The following table gives the activities and duration of a construction project.

Activity	*1–2*	*1–3*	*2–3*	*2–4*	*3–4*	*4–5*
Duration (days)	20	25	10	12	6	10

   (*i*) Draw the network for the project

   (*ii*) Find the critical path.

2. A small project consists of 11 activities A, B, C ... K. According to the precedence relationship A and B can start simultaneously, given $A < C, D, I$; $B < G, F$; $D < G, F$; $F < H, K$; $G, H < J$; $I, J, K < E$. The duration of the activities are as follows.

Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
Duration (days)	5	3	10	2	8	4	5	6	12	8	9

   Draw the network of the project. Summarise the CPM calculations in a tabular form computing the total and free floats of activities as well as determine the critical path.

3. Draw the network and determine the critical path for the given data. Also calculate all the floats involved in CPM.

Jobs	*1–2*	*1–3*	*2–4*	*3–4*	*3–5*	*4–5*	*4–6*	*5–6*
Duration	6	5	10	3	4	6	2	9

NOTES

4. A small maintenance project consists of the following 12 jobs.

Jobs	***1–2***	***2–3***	***2–4***	***3–4***	***3–5***	***4–6***	***5–8***	***6–7***	***6–10***	***7–9***	***8–9***	***9–10***
Duration (days)	2	7	3	3	5	3	5	8	4	4	1	7

Draw the arrow network of the project. Summarise CPM calculations in a tabular form, calculating the three types of floats and hence determine the critical path.

5. Consider the following data for activities in a given project.

6. The data for a small PERT project is as given below, where a represents optimistic time, m the most likely time and b the pessimistic time. Estimates (in days) of the activities A, B ... J, K are given in the table.

Activity	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***	***I***	***J***	***K***
a	3	2	6	2	5	3	3	1	4	1	2
m	6	5	12	5	11	6	9	4	19	2	4
b	5	14	30	8	17	15	27	7	28	9	12

A, B and C can start simultaneously; A £ D, I; $B < G, F$; $D < G, F$; $C < E$; $E < H, K$; $F < H, K$; $G, H < J$.

(*i*) Draw the arrow network of the project.

(*ii*) Calculate the earliest and the latest expected times to each event and find the critical path.

(*iii*) What is the probability that the project will be completed 2 days later than expected?

7. The three estimates for the activities of a project are given below:

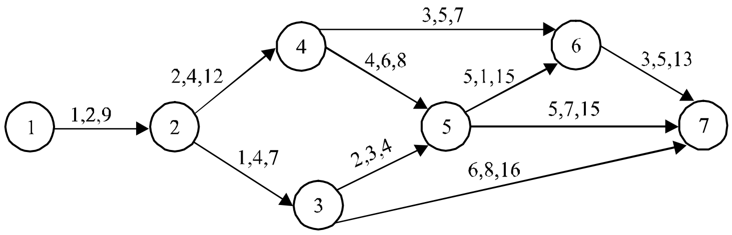
Activity	***Estimate duration (days)***		
	a	***m***	***b***
1–2	5	6	7
1–3	1	1	7
1–4	2	4	12
2–5	3	6	15
3–5	1	1	1
4–6	2	2	8
5–6	1	4	7

Draw the project network. Find out the critical path and duration of the project. What is the probability that the project will be completed at least 5 days earlier than expected?

What is the probability that the project will be completed by 22 days?

8. Consider the network shown in the figure below. The estimate t_o, t_m and t_p are shown in this order for each of the activities, on top of the arcs denoting the respective activities.

   Find the probability of completing the project in 25days.



9. A project is represented by the network shown below and has the following table:

Task	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Least time	5	18	26	16	15	6	7	7	3
Greatest time	10	22	40	20	25	12	12	9	5
Most likely time	8	20	33	18	20	9	10	8	4

   Determine the following:

   (*i*) Expected time of tasks and their variance.

   (*ii*) The earliest and the latest expected time to reach each mode.

   (*iii*) The critical path.

   (*iv*) The probability of completing the project within 41.5 weeks.

10. Consider a project having the following activities and their time estimates.

    Draw an arrow diagram for the project. Identify the critical path and compute the expected completion time. What is the probability that the project will require at least 75 days?

Activity	*Predecessor*	t_o	t_m	t_p
			days	
A	–	2	4	6
B	A	8	12	16
C	A	14	16	30
D	B	4	10	16
E	C, B	6	12	18
F	E	6	8	22
G	D	18	18	30
H	F, G	8	14	32

13.9 FURTHER READINGS

NOTES

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.

UNIT 14 GAME THEORY: COMPETITIVE SITUATIONS

NOTES

Structure

14.0 INTRODUCTION

Competition is the watch word of modern life. We say that a competitive solution exists, if two or more individuals make decisions in a situation that involves conflicting interests, and in which the outcome is controlled by the decision of all the concerned parties. A competitive situation is called a *game*. The term *game* represents a conflict between two or more parties. A situation is termed a game when it possesses the following properties.

(*i*) The number of competitors is finite.

(*ii*) There is a conflict in interests between the participants.

(*iii*) Each of the participants has a finite set of possible courses of action.

(*iv*) The rules governing these choices are specified and known to all players. The game begins when each player chooses a single course of action from the list of courses available to him.

(*v*) The outcome of the game is affected by the choices made by all the players.

(*vi*) The outcome for all specific set of choices, by all the players, is known in advance and numerically defined.

The outcome of a game consists of a particular set of courses of action undertaken by the competitors. Each outcome determines a set of payments (positive, negative or zero), one to each competitor.

NOTES

14.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss the characteristics of game theory and competitive situations
- Explain the maximin-minimax principle
- Describe the dominance theory

14.2 BASIC CONCEPTS OF GAME THEORY

The term '*strategy*' is defined as a complete set of plans of action specifying precisely what the player will do under every possible future contigency that might occur during the play of the game, i.e., strategy of a player is the decision rule he uses for making a choice, from his list of courses of action. Strategy can be classified as:

(*i*) Pure strategy

(*ii*) Mixed strategy

A strategy is called *pure* if one knows in advance of the play that it is certain to be adopted, irrespective of the strategy the other players might choose.

The optimal strategy mixture for each player may be determined by assigning to each strategy, its probability of being chosen. The strategy so determined is called *mixed strategy* because it is a probabilistic combination of the available choices of strategy. Mixed strategy is denoted by the set, $S = \{X_1, X_2...X_n\}$ where, X_j is the probability of choosing the course j such that $X_j > 0, j = 1, 2...n$ and $X_1 + X_2 + ... X_n = 1$. It is evident that a pure strategy is a special case of mixed strategy.

In the case where all but one X_j is zero, a player may be able to choose only n pure strategy, but he has an infinite number of mixed strategies to choose them.

Pay-off

Pay-off is the outcome of playing the game. A pay-off matrix is a table showing the amount received by the player named at the left hand side after all possible plays of the game. The payment is made by the player named at the top of the table.

If a player *A* has *m* courses of action and player *B* has *n* courses, then a payoff matrix may be constructed using the following steps.

(*i*) Row designations for each matrix are the courses of action available to *A*.

(*ii*) Column designations for each matrix are the courses of action available to *B*.

(*iii*) With a two-person zero-sum game, the cell entries in *B*'s payoff matrix will be the negative of the corresponding entries in *A*'s payoff matrix and the matrices will be as shown below.

NOTES

Player B

$$\text{Player } A \begin{bmatrix} 1 & 2 & 3 & \ldots & j & \ldots & n \\ a_{11} & a_{12} & a_{13} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2j} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3j} & \ldots & a_{3n} \\ \vdots & & & \ldots & & \ldots & \\ m & a_{m2} & a_{m3} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}$$

A's payoff matrix.

Player B

$$\text{Player } A \quad \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & \ldots & j & \ldots & n \end{array} \\ \begin{bmatrix} -a_{11} & -a_{12} & a_{13} & \cdots & -a_{1j} & \cdots & -a_{1} \\ -a_{21} & -a_{22} & a_{23} & \cdots & -a_{2j} & \cdots & a_{2} \\ \vdots & & & & & & \\ -a_{i1} & -a_{i2} & -a_{i3} & \cdots & -a_{1j} & \cdots & a_{in} \\ \vdots & & & & & & \\ -a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

14.2.1 Types of Games and Their Characteristics

(*i*) ***Two-person games and n-person games*** In two-person games, the players may have many possible choices open to them for each play of the game but the number of players remain only two. Hence, it is called a *two-person game*. In case of more than two persons, the game is generally called *n*-person game.

(*ii*) ***Zero-sum game*** A *zero-sum game* is one in which the sum of the payment to all the competitors is zero, for every possible outcome of the game is in a game if the sum of the points won, equals the sum of the points lost.

(*iii*) ***Two-person zero-sum game*** A game with two players, where the gain of one player equals the loss of the other, is known as a *two-person zero-sum game*. It is also called a *rectangular game* because their payoff matrix is in the rectangular form. The characteristics of such a game are:

(*a*) Only two players participate in the game.

(*b*) Each player has a finite number of strategies to use.

(*c*) Each specific strategy results in a payoff.

(*d*) Total payoff to the two players at the end of each play is zero.

NOTES

14.2.2 The Maximin-Minimax Principle

This principle is used for the selection of optimal strategies by two players. Consider two players A and B. A is a player who wishes to maximize his gains, while player B wishes to minimize his losses. Since A would like to maximize his minimum gain, we obtain for player A, the value called *maximin value* and the corresponding strategy is called the *maximin strategy.*

On the other hand, since player B wishes to minimize his losses, a value called the *minimax value*, which is the minimum of the maximum losses is found. The corresponding strategy is called the *minimax strategy*. When these two are equal (maximin value = minimax value), the corresponding strategies are called *optimal strategies* and the game is said to have a *saddle point*. The value of the game is given by the saddle point.

The selection of maximin and minimax strategies by A and B is based upon the so-called maximin-minimax principle, which guarantees the best of the worst results.

Saddle point A saddle point is a position in the payoff matrix where, the maximum of row minima coincides with the minimum of column maxima. The payoff at the saddle point is called the *value* of the game.

We shall denote the maximin value by $\underline{\gamma}$, the minimax value of the game by $\overline{\gamma}$ and the value of the game by γ.

Note:

(*i*) A game is said to be fair if,

maximin value = minimax value = 0, i.e., if $\overline{\gamma} = \underline{\gamma} = 0$

(*ii*) A game is said to be strictly determinable if,

maximin value = minimax value $\neq 0$. $\underline{\gamma} = \gamma = \overline{\gamma}$.

Example 14.1: Solve the game whose payoff matrix is given by,

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \overset{\text{Player } B}{\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}}$$

Solution:

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \overset{\text{Player } B}{\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}} \begin{array}{c} \text{Row minima} \\ 1 \\ -4 \\ -1 \end{array}$$

Column maxima 1 5 1

$$\text{Maxi (minimum)} = \text{Max } (1, -4, -1) = 1$$

$$\text{Mini (maximum)} = \text{Min } (1, 5, 1) = 1.$$

i.e., Maximin value $\underline{\gamma} = 1 = $ Minimax value $\overline{\gamma}$

NOTES

$\therefore$ Saddle point exists. The value of the game is the saddle point, which is 1. The optimal strategy is the position of the saddle point and is given by, (A_1, B_1).

Example 14.2: For what value of 1, is the game with the following matrix strictly determinable?

$$\text{Player } B$$

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array}$$

Solution Ignoring the value of l, the payoff matrix is given by,

$$\text{Player } B$$

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array} \begin{array}{c} \text{Row minima} \\ 2 \\ -7 \\ -2 \end{array}$$

Column maxima -1 6 2

The game is strictly determinable if,

$$\underline{\gamma} = \gamma = \overline{\gamma}. \text{ Hence, } \underline{\gamma} = 2, \ \overline{\gamma} = -1$$

$$\Rightarrow \qquad -1 \leq \lambda \leq 2.$$

Example 14.3: Determine which of the following two-person zero-sum games are strictly determinable and fair. Given the optimum strategy for each player in the case of strictly determinable games.

(*i*) Player *B*

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{array}$$

(*ii*) Player *B*

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix} \end{array}$$

$$\text{Maxi (minimum)} = \underline{\gamma} = \text{Max } (-5, -7) = -5$$

$$\text{Mini (maximum)} = \overline{\gamma} = \text{Min } (-5, 2) = -5$$

NOTES

Solution:

(*i*)

$$\begin{array}{c|cc|c} & \multicolumn{2}{c}{\text{Player } B} & \\ & B_1 & B_2 & \text{Row minima} \\ \text{Player } A \quad A_1 & -5 & 2 & -5 \\ A_2 & -7 & -4 & -7 \\ \text{Column maxima} & -5 & 2 & \end{array}$$

Since $\underline{\gamma} = \overline{\gamma} = -5 \neq 0$, the game is strictly determinable. There exists a saddle point $= -5$. Hence, the value of the game is -5. The optimal strategy is the position of the saddle point given by, (A_1, B_1).

(*ii*)

$$\begin{array}{c|cc|c} & \multicolumn{2}{c}{\text{Player } B} & \\ & B_1 & B_2 & \text{Row minimum} \\ \text{Player } A \quad A_1 & 1 & 1 & 1 \\ A_2 & 4 & -3 & -3 \\ \text{Column maximum} & 4 & 1 & \end{array}$$

Maxi (minimum) $= \underline{\gamma} = \text{Max}(1, -3) = 1$.

Mini (maximum) $= \overline{\gamma} = \text{Min}(4, 1) = 1$.

Since $\underline{\gamma} = \overline{\gamma} = 1 \neq 0$, the game is strictly determinable. Value of the game is 1. The optimal strategy is, (A_1, B_2).

Example 14.4: Solve the game whose payoff matrix is given below.

$$\begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}$$

Solution:

$$\begin{array}{c|ccccc|c} & B_1 & B_2 & B_3 & B_4 & B_5 & \text{Row minima} \\ A_1 & -2 & 0 & 0 & 5 & 3 & -2 \\ \text{Player } A \quad A_2 & 3 & 2 & 1 & 2 & 2 & 1 \\ A_3 & -4 & -3 & 0 & -2 & 6 & -4 \\ A_4 & 5 & 3 & -4 & 2 & -6 & -6 \\ \text{Column maxima} & 5 & 3 & 1 & 5 & 6 & \end{array}$$

Maxi (minimum) $= \underline{\gamma} = \text{Max}(-2, 1, -4, -6) = 1$.

Mini (maximum) $= \overline{\gamma} = \text{Min}(5, 3, 1, 5, 6) = 1$.

Since, $\underline{\gamma} = \overline{\gamma} = 1$, there exists a saddle point. Value of the game is 1. The position of the saddle point is the optimal strategy and is given by, $[A_2, B_3]$.

14.2.3 Games without Saddle Points (Mixed Strategies)

A game without saddle point can be solved by various solution methods.

NOTES

2 × 2 Games Without Saddle Point

Consider a 2 × 2 two-person zero-sum game without any saddle point, having the payoff matrix for player *A* as,

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies,

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ and } S_B \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

where, $$p_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})},\ p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})},\ q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$$

The value of the game $(\gamma) = \dfrac{a_{11}\,a_{22} - a_{12}\,a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$.

Example 14.5: Solve the following payoff matrix. Also determine the optimal strategies and value of the game.

$$\begin{array}{c} \\ A \end{array}\begin{array}{c} B \\ \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Solution:

$$\begin{array}{c} \\ A \end{array}\begin{array}{c} B \\ \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}. \text{ Let this be,}$$

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}. \text{ The optimum mixed strategies,}$$

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

where, $$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22)} - (a_{12} + a_{21)}} = \frac{4 - 3}{(5 + 4) - (1 + 3)} = \frac{1}{5}$$

$$p_2 = 1 - p_1 \Rightarrow p_2 = 1 - \frac{1}{5} = \frac{4}{5}$$

NOTES

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22)} - (a_{12} + a_{21)}} = \frac{4-1}{(5+4)-(1+3)} = \frac{3}{5}$$

$$q_2 = 1 - q_1 \Rightarrow q_2 = 1 - \frac{3}{5} = \frac{2}{5}$$

value of game, $\gamma = \dfrac{(5 \times 4) - (1 \times 3)}{(5+4)-(1+3)} = \dfrac{17}{5}$

∴ The optimum mixed strategies,

$$S_A = \left(\frac{1}{5}, \frac{4}{5}\right); S_B = \left(\frac{3}{5}, \frac{2}{5}\right)$$

$$\text{Value of game} = \frac{17}{5}.$$

Example 14.6: Solve the following game and determine its value.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

Solution: It is clear that the payoff matrix does not possess any saddle point. The players will use mixed strategies. The optimum mixed strategy for the players are,

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}, p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}, q_1 + q_2 = 1.$$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4-(-4)}{4+4-(-4-4)} = \frac{8}{16} = \frac{1}{2}$$

$$p_2 = 1 - p_1 \Rightarrow p_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4-(-4)}{4+4-(-4-4)} = \frac{8}{16} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

The optimum strategy is, $S_A = \left(\dfrac{1}{2}, \dfrac{1}{2}\right); S_B = \left(\dfrac{1}{2} \;\; \dfrac{1}{2}\right)$

The value of the game is, $\gamma = \dfrac{a_{22}\,a_{11} - a_{12}\,a_{21}}{(a_{22} + a_{11}) - (a_{12} + a_{21})}$

$$= \frac{(4 \times 4) - [-4 \times (-4)]}{(4+4) - [-4 + (-4)]} = 0.$$

Example 14.7: In a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails and losses 1/2 unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game to A.

Solution: The payoff matrix for the player A is given by,

$$\text{Player } A \quad \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \end{array}$$

Let this be,

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array}.$$

The optimum mixed strategies, $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}, p_1 + p_2 = 1$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}, q_1 + q_2 = 1.$$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0 - \left(-\frac{1}{2}\right)}{1 + 0 - \left(-\frac{1}{2} - \frac{1}{2}\right)}$$

$$= \frac{\frac{1}{2}}{2} = \frac{1}{4}$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{4} = \frac{3}{4}.$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0 - \left(-\frac{1}{2}\right)}{1 + 0 - \left(-\frac{1}{2} - \frac{1}{2}\right)} = \frac{1}{4}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\text{Value of the game} = \frac{1 \times 0 - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{1 + 0 - \left(-\frac{1}{2} - \frac{1}{2}\right)} = \frac{\frac{-1}{4}}{2}$$

$$\gamma = -\frac{1}{8}$$

NOTES

$\therefore$ The optimum mixed strategy is given by,

$$S_A = \left(\frac{1}{4}, \frac{3}{4}\right); S_B = \left(\frac{1}{4}, \frac{3}{4}\right)$$

and the value of game is –1/8.

Graphical Method for 2 × *n* or *m* × 2 Games

Consider the following 2 × n games.

$$\begin{array}{cc} & B \\ A & \begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{bmatrix} B_1 & B_2 & \dots & B_n \\ a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix} \end{array}$$

Let the mixed strategy for player A be given by,

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ such that, } p_1 + p_2 = 1, p_{1,}\ p_2 \geq 0$$

Now, for each of the pure strategies available to B, expected payoff for player A would be as follows.

B's pure move	A's expected payoff $E(p)$
B_1	$E_1\ (p) = a_{11}\, p_1 + a_{21}\, p_2$
B_2	$E_2\ (p) = a_{12}\, p_1 + a_{22}\, p_2$
B_n	$E_n\ (p) = a_{1n}\, p_1 + a_{2n}\, p_2$

The player B would like to choose that pure move B_j against S_A for which $E_j\,(p)$ is a minimum for $j = 1, 2 \dots n$. Let us denote this minimum expected payoff for A by,

$$\gamma = \text{Min } (E_j(p))\ j = 1, 2 \dots n.$$

The objective of player A is to select P_1 and P_2 in such a way that, γ is as large as possible. This may be done by plotting the straight lines,

$$Ej(p) = a_{1j}\, p_1 - a_{2j}\, p_2 = (a_{1j} - a_{2j})\, p_1 + a_{2j}$$
$$j = 1, 2.\dots\ n.$$

as linear functions of P_1.

The highest point on the lower boundary of these lines will give the maximum value among the minimum expected payoffs on the lower boundary (lower envelope) as well as the optimum value of probability P_1 and P_2.

Now the two strategies of player B corresponding to the lines that pass through the maximin point can be determined. It helps in reducing the size of the game to (2 × 2).

Similarly, we can treat m × 2 games in the same way and get the minimax point, which will be the lowest point on the upper boundary (upper envelope).

NOTES

Example 14.8: Solve the following 2 × 3 game graphically.

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{bmatrix} \end{array}$$

Solution: Since the problem does not possess any saddle point, let player A play by the mixed strategy

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ with, } p_2 = 1 - p_1$$

against player B.

A's expected payoff against B's pure move is given by,

B's pure move	A's expected payoff $E(p_1)$
B_1	$E_1(p_1) = p_1 + 8(1 - p_1) = -7p_1 + 8$
B_2	$E_2(p_1) = 3p_1 + 5(1 - p_1) = 2p_1 + 5$
B_3	$E_3(p_1) = 11P_1 + 2(1 - P_1) = 9p_1 + 2.$

These expected payoff equations are then plotted as functions of P_1 as shown in the Fig. 14.1, which shows the payoffs of each column represented as points on two vertical axes 1 and 2, of unit distance apart.

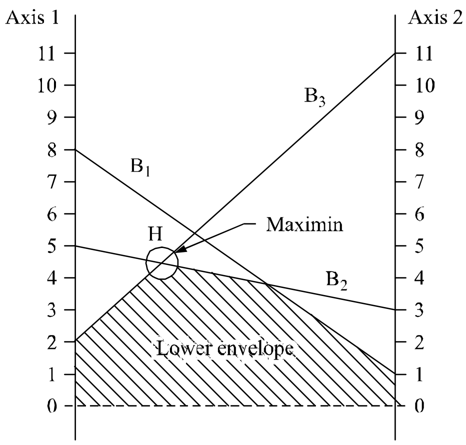


Fig. 14.1

Now, since player A wishes to maximize his minimum expected payoff, we consider the highest point of intersection H on the lower envelope of A's expected payoff equation. The lines B_2 and B_3 passing through H define the relevant moves that B_2 and B_3 alone need to play. The solution to the original 2 × 3 game reduces to,

$$\begin{array}{cc} & \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 3 & 11 \\ 5 & 2 \end{bmatrix} \end{array}$$

NOTES

The optimum strategy for A and B is given by,

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & q_1 & q_2 \end{pmatrix} q_1 + q_2 = 1$$

$$p_1 = \frac{2-5}{3+2-(11+5)} = \frac{-3}{-11} = \frac{3}{11}$$

$$p_2 = 1 - p_1 = 1 - \frac{-3}{-11} = \frac{3}{11}$$

$$q_1 = \frac{2-11}{-11} = \frac{-9}{-11} = \frac{9}{11}$$

$$q_2 = 1 - \frac{9}{11} = \frac{2}{11}$$

$A_1\ A_2\ B_1\ B_2\ B_3$

$$S_A = \begin{pmatrix} \frac{3}{11} & \frac{8}{11} \end{pmatrix} \text{and } S_B \begin{pmatrix} 0 & \frac{9}{11} & \frac{2}{11} \end{pmatrix}$$

Value of the game, $\gamma = \dfrac{6-11}{-11} = \dfrac{49}{11}$.

Example 14.9: Solve the game problem graphically.

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

Solution: The given problem does not possess any saddle point. Therefore, let player B play by the mixed strategy

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix} \text{ with, } q_2 = 1 - q_1$$

against player A.

The expected payoff equations are plotted in the Fig. 14.2 with two axes I and II vertically at unit distance apart.

NOTES

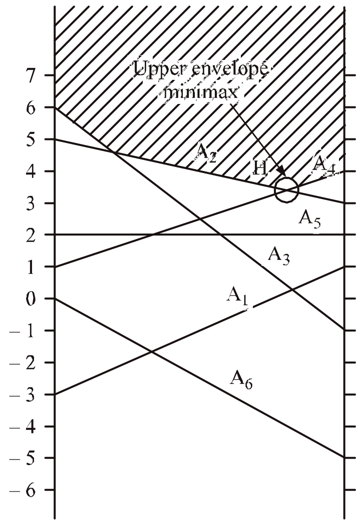


Fig. 14.2

Since player B wishes to minimize his maximum expected payoff, we consider the lowest point of the upper boundary of B's expected payoff equation. The point H (intersection of lines A_2 and A_4) represents the minimax expected value of the game for player B. Hence, the solution to the original 6×2 game reduces to the 2×2 payoff matrix.

$$\text{Player } B$$

$$\text{Player } A \quad \begin{array}{c} \\ A_2 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} 3 & 5 \\ 4 & 1 \end{pmatrix} \end{array}$$

The optimal mixed strategy for A and B is given by,

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & p_1 & 0 & p_2 & 0 & 0 \end{pmatrix} \text{ with, } p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix} q_1 + q_2 = 1$$

$$P_1 = \frac{1-4}{3+1-(5+4)} = \frac{-3}{-5} = \frac{3}{5}$$

$$P_2 = 1 - \frac{3}{5} = \frac{2}{5}$$

$$q_1 = \frac{1-5}{-5} = \frac{-4}{-5} = \frac{4}{5}$$

$$q_2 = 1 - q_1 = 1 - \frac{4}{5} = \frac{1}{5}$$

$$S_A = \begin{pmatrix} A_1, & A_2, & A_3, & A_4, & A_5, & A_6 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 \end{pmatrix}$$

NOTES

$$S_B = \begin{pmatrix} B_1 & B_2 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

Value of the game, $$\gamma = \frac{3\times 1 - 5\times 4}{(3+1)-(5+4)} = \frac{3-20}{-5} = \frac{-17}{-5}$$

$$\gamma = -\frac{17}{5}$$

14.3 DOMINANCE PROPERTY

Sometimes it is observed that one of the pure strategies of either player is always inferior to at least one of the remaining. The superior strategies are said to dominate the inferior ones. In such cases of dominance, we reduce the size of the payoff matrix by deleting those strategies, which are dominated by others. The general rules for dominance are:

(*i*) If all the elements of a row, say kth row, are less than or equal to the corresponding elements of any other row, say rth row, then kth row is dominated by the rth row.

(*ii*) If all the elements of a column, say kth column, are greater than or equal to the corresponding elements of any other column, say rth column, then the kth column is dominated by the rth column.

(*iii*) Dominated rows and columns may be deleted to reduce the size of the payoff matrix as the optimal strategies will remain unaffected.

(*iv*) If some linear combinations of some rows dominate ith row, then the ith row will be deleted. Similar arguments follow for columns.

Example 14.10: Using the principle of dominance, solve the following game.

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

Solution: In the given payoff matrix, all the elements in the third column are greater than or equal to the corresponding elements in the first column. Therefore, column three is dominated by first column. Delete column three. The reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 3 & -2 \\ -1 & 4 \\ 2 & 2 \end{bmatrix} \end{array}$$

Since no row (or column) dominates another row (or column). The 3×2 game can now be solved by the graphical method. Since player B wishes to minimize his

NOTES

maximum loss, we find the lowest point of the upper boundary. The expected payoff equations are then plotted as shown in Fig. 14.3.

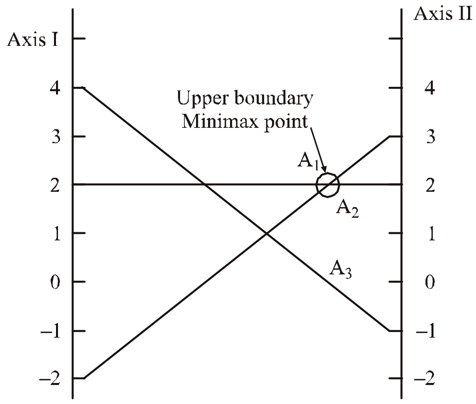


Fig. 14.3

The lowest point in the upper boundary is given by the intersection of lines A_1 and A_2. The solution in the original game is reduced to a 2 × 2 matrix.

$$\begin{array}{c} \\ A_1 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix} \end{array}$$

The optimum strategy for A and B is given by,

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & 0 & p_2 \end{pmatrix} p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix} q_1 + q_2 = 1$$

$$p_1 = \frac{2-2}{3+2-(-2+2)} = 0$$

$$p_2 = 1 - p_1 = 1 - 0 = 1$$

$$q_1 = \frac{2-2}{3+2-(-2+2)} = \frac{4}{5}$$

$$q_2 = 1 - q_1 = 1\frac{4}{5} = \frac{1}{5}$$

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

$$\text{Value of the game, } \gamma = \frac{3 \times 2 + (-2) \times (2)}{3+2-(-2+2)} = \frac{2}{5}.$$

NOTES

Example 14.11: Solve the following game.

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix} \end{array}$$

Solution: Since all the elements in the third row are less than or equal to the corresponding elements of the second row, therefore, the third row is dominated by the second row. Delete this dominated row. The reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \end{bmatrix} \end{array}$$

The elements of the third column are greater than or equal to the corresponding elements of the first column, which give that column third is dominated by column one. This dominated column is deleted and the reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix} \end{array}$$

The reduced payoff matrix is a 2×2 matrix. The optimal strategy for players A and B is given by,

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & 0 \end{pmatrix} p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & 0 \end{pmatrix} q_1 + q_2 = 1$$

$$p_1 = \frac{2-6}{2+1-(7+6)} = \frac{-4}{-10} = \frac{2}{5}$$

$$p_2 = 1 - \frac{2}{5} = \frac{3}{5}$$

$$q_1 = \frac{2-6}{2+1-(7+6)} = \frac{-5}{-10} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

Value of the game, $\gamma = \dfrac{2 \times 1 - 7 \times 6}{2+1-(7+6)} = \dfrac{-40}{10} = 4.$

The optimal strategy is given by,

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ \frac{2}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Value of the game is, $\gamma = 4.$

NOTES

Example 14.12: Is the following two-person zero-sum game stable? Solve the game.

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 5 & -10 & 9 & 0 \\ 6 & 7 & 8 & 1 \\ 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{bmatrix} \end{array}$$

Solution Since the game has no saddle point, it is not stable. All the elements of the first row and the second row are ≤ to the corresponding elements of the third row. Hence, these two rows are dominated rows. Deleting these two rows from the payoff matrix, the reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{bmatrix} \end{array}$$

In this modified payoff matrix, we observe that all the elements of the second column are ≥ to the corresponding elements of the fourth column. Hence, this dominated column (2nd column) is deleted from the payoff matrix. The reduced payoff matrix is given by,

$$\text{Player } A \begin{pmatrix} 8 & 15 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

Now we observe that no row or column dominates another row or column. However, we note that a convex combination of 2nd and 3rd columns is given by,

$$15 \times \frac{1}{2} + 1 \times \frac{1}{2} = 8 \leq 8$$

$$-1 \times \frac{1}{2} + 4 \times \frac{1}{2} = \frac{3}{2} \leq 3$$

and hence the elements of the first column are greater than or equal to the corresponding elements of this combination. Deleting this dominated column, the reduced payoff matrix is given by,

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{pmatrix} 15 & 1 \\ -1 & 4 \end{pmatrix} \end{array}$$

$$S_A = \begin{bmatrix} A_3 & B_4 \\ p_1 & p_2 \end{bmatrix}, p_1 + p_2 = 1$$

$$p_1 = \frac{4 - (-1)}{15 + 4 - (1 + 1 - 1)} = \frac{5}{19}$$

$$p_2 = 1 - \frac{5}{19} = \frac{14}{19}$$

$$S_B = \begin{bmatrix} B_3 & B_4 \\ q_1 & q_2 \end{bmatrix} q_1 + q_2 = 1$$

$$q_1 = \frac{4-1}{19} = \frac{3}{19}$$

$$q_2 = 1 - \frac{3}{19} = \frac{19}{16}$$

NOTES

The optimum strategy of the given payoff matrix is given by,

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & \frac{5}{19} & \frac{14}{19} \end{pmatrix} \quad S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & \frac{3}{19} & \frac{16}{19} \end{pmatrix}$$

and the value of game is, $\frac{(4 \times 15) - (1 \times -1)}{19} = \frac{61}{19}$.

Check Your Progress

1. How is the optimal strategy mixture determined by each player?
2. What are the characteristics of two-person zero-sum game?
3. What do you understand by the value of game?

14.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The optimal strategy mixture for each player may be determined by assigning to each strategy, its probability of being chosen. The strategy so determined is called *mixed strategy* because it is a probabilistic combination of the available choices of strategy.
2. The characteristics of two-person zero-sum game are the following:
   (*a*) Only two players participate in the game.
   (*b*) Each player has a finite number of strategies to use.
   (*c*) Each specific strategy results in a payoff.
   (*d*) Total payoff to the two players at the end of each play is zero.
3. The payoff at the saddle point is called the *value* of the game.

14.5 SUMMARY

- The term '*strategy*' is defined as a complete set of plans of action specifying precisely what the player will do under every possible future contigency that might occur during the play of the game, i.e., strategy of a player is the decision rule he uses for making a choice, from his list of courses of action. Strategy can be classified as pure strategy and mixed strategy.

NOTES

- Pay-off is the outcome of playing the game. A pay-off matrix is a table showing the amount received by the player named at the left hand side after all possible plays of the game. The payment is made by the player named at the top of the table.
- In two-person games, the players may have many possible choices open to them for each play of the game but the number of players remains only two. Hence, it is called a two-person game. In case of more than two persons, the game is generally called *n*-person game.
- A zero sum game is one in which the sum of the payment to all the competitors is zero for every possible outcome of the game is in a game if the sum of the points won equals the sum of the points lost.
- A game with two players, where the gain of one player equals the loss of the other is known as a two-person zero sum game. It is also called a rectangular game because the pay-off matrix is in a rectangular form.
- A saddle point is a position in the pay-off matrix, where the maximum of row minima coincides with the minimum of column maxima. The pay-off at the saddle point is called the 'value of the game'.
- It is observed that one of the pure strategies of either player is always inferior to at least one of those remaining. The superior strategies are said to dominate the inferior ones. In such cases of dominance, we reduce the size of the pay-off matrix by deleting those strategies which are dominated by others.
- The maximin-minimax principle is used for the selection of optimal strategies by two players.

14.6 KEY WORDS

- **Pure strategy:** A strategy is called pure if one knows in advance of the play that it is certain to be adopted, irrespective of the strategy the other players might choose.
- **Zero-sum game:** A zero-sum game is one in which the sum of the payment to all the competitors is zero, for every possible outcome of the game is in a game if the sum of the points won, equals the sum of the points lost.
- **Two-person zero-sum game:** A game with two players, where the gain of one player equals the loss of the other, is known as a two-person zero-sum game.
- **Saddle point**: A saddle point is a position in the payoff matrix where, the maximum of row minima coincides with the minimum of column maxima.

14.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short Answer Questions

1. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games.

(*i*) Player B

$$\text{Player } A \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}$$

(*ii*) Player B

$$\text{Player } A \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}$$

2. For a game with the following payoff matrix,

Player A

$$\text{Player } B \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix}$$

determine the best strategies as well as the value of the game for players A and B. Is this game (*i*) fair, (*ii*) strictly determinable?

3. Determine the optimal minimax strategies for each player in the following game.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} -5 & 2 & 0 & 7 \\ 5 & 6 & 4 & 8 \\ 4 & 0 & 2 & -3 \end{bmatrix} \end{array}$$

Long Answer Questions

1. For a game with the following payoff matrix, determine the optimal strategy and the value of the game.

(*i*) B

$$A\begin{pmatrix} 6 & -3 \\ -3 & 3 \end{pmatrix}$$

(*ii*) B

$$A\begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}$$

2. Two players, A and B match coins. If the coins match, then A wins two units of value. If coins do not match, then B wins two units of value.

   Determine the optimum strategies for the players and the value of the game.

3. Consider a 'modified form' of matching-based coins game problem. The matching player is paid ₹ 800 if both the coins turn heads and Re. 1 if both the coins turn tails. The non-matching player is paid ₹ 300 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?

4. Solve the following problems graphically.

   (*i*) $$\begin{array}{cc} & \text{Player B} \\ \text{Player } A & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

   (*ii*) $$\begin{array}{cc} & \text{Player B} \\ \text{Player } A & \begin{pmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{pmatrix} \end{array}$$

   (*iii*) $$\begin{array}{cc} & \text{Player B} \\ \text{Player } A & \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

   $$S_B = \left[\begin{bmatrix} B_1 & B_2 \\ \frac{5}{17}, & \frac{12}{17} \end{bmatrix}; \gamma = \frac{73}{17}\right]$$

   (*iv*) $$\begin{array}{cc} & \text{Player B} \\ \text{Player } A & \begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & 6 \end{bmatrix} \end{array}$$

   $$S_B = \left[\begin{pmatrix} B_1 & B_2 \\ \frac{11}{20} & \frac{9}{20} \end{pmatrix}; \gamma = \frac{23}{30}\right]$$

5. The companies *A* and *B* are competing for the same product. Their different strategies are given in the following payoff matrix.

   $$\begin{array}{cc} & \text{Company B} \\ \text{Company } A & \begin{pmatrix} 4 & -3 & 3 \\ -3 & 1 & -1 \end{pmatrix} \end{array}$$

   Determine the best strategies for the two companies.

6. Using dominance, solve the payoff matrix, given by,

NOTES

$$(i)\ \text{Player } A \quad \overset{\text{Player } B}{\begin{bmatrix} 2 & -2 & 4 & 1 \\ 6 & 1 & 12 & 3 \\ -3 & 2 & 0 & 6 \\ 2 & -3 & 7 & 7 \end{bmatrix}}$$

$$(ii)\ \text{Player } A \quad \overset{\text{Player } B}{\begin{bmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix}}$$

$$(iii)\ \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 4 & 2 & -4 & -6 \\ 8 & 6 & 8 & -4 & 0 \\ 10 & 2 & 4 & 0 & 12 \end{bmatrix} \end{array}$$

7. The following matrix represents the payoff to P_1 in a rectangular game between two persons P_1 and P_2.

$$P_1 \overset{P_2}{\begin{bmatrix} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix}}$$

By the notion of dominance, reduce the game to a 2×4 game and solve it graphically.

14.8 FURTHER READINGS

Sharma, A. 2009. *Operations Research.* New Delhi: Himalaya Publishing House.

Singh, J. K. 2009. *Business Mathematics*. New Delhi: Himalaya Publishing House.

Kalavathy, S. 2010. *Operations Research with C Programmes,* 3rd edition. New Delhi: Vikas Publishing House Pvt. Ltd.

Kothari, C. R. 1992. *An Introduction to Operational Research.* New Delhi: Vikas Publishing House Pvt. Ltd.