Vector Spaces

1. Let S be a subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$. Show that S forms a subspace of $\mathbb{R}^3$.

2. Let S be a subset defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$. Show that S forms a subspace of $\mathbb{R}^3$.

3. Let S be the subset defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$. Examine S forms a subspace of $\mathbb{R}^3$ or not.

4. Let V be a vector space over a field $\mathbb{F}$ and let $\alpha \in V$. Show that the set $W = \{c\alpha : c \in \mathbb{F}\}$ forms a subspace of V.

5. Let V be a vector space over a field $\mathbb{F}$ and let $\alpha, \beta \in V$. Show that the set $W = \{c\alpha + d\beta : c, d \in \mathbb{F}\}$ forms a subspace of V.

6. Prove that the set $C[a, b]$ of all real valued continuos functions defined on the closed interval $[a, b]$ forms a real vector space if
   (a) addition is defined by $(f + g)(x) = f(x) + g(x), f, g \in C[a, b]$.
   (b) multiplication by a real number r is defined by $(r.f)(x) = rf(x), f \in C[a, b]$.

7. Prove that the subset $D[a, b]$ of all real valued differentiable functions defined on $[a, b]$ is a subspace of $C[a, b]$.

8. Examine if the set S is a subspace of the vector space $\mathbb{R}_{2\times 2}$, where
   (a) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times 2} : a + b = 0 \right\}$;
   (b) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times 2} : det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$;
   (c) S is the set of all 2×2 real diagonal matrices;
   (d) S is the set of all 2×2 real symmetric matrices;
   (e) S is the set of all 2×2 real skew symmetric matrices;
   (f) S is the set of all 2×2 real upper triangular matrices;
   (g) S is the set of all 2×2 real lower triangular matrices;

9. Show that the set S is a subspace of the vector space $C[0,1]$, where

   (a) $S = \{f \in C[0,1] : f(0) = 0\}$.

   (b) $S = \{f \in C[0,1] : f(0) = 0, f(1) = 0\}$.

10. In $\mathbb{R}^3, \alpha = (4,3,5), \beta = (0,1,3), \gamma = (2,1,1), \delta = (4,2,2)$. Examine if

    (a) α is a linear combination of β and γ.

    (b) β is a linear combination of γ and δ.

11. Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1,2,3), \beta = (3,1,0)$. Examine if $\gamma = (2,1,3), \delta = (-1,3,6)$ are in the subspace.

12. Let $S = \{\alpha, \beta, \gamma\}, T = \{\alpha, \beta, \alpha+\beta, \beta+\gamma\}$ be subsets of a real vector space V. Show that $L(S) = L(T)$, where $L(S)$ stands for linear span of the set of elements of S.

13. Examine if the set of vectors $\{(2,1,1), (1,2,2), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

14. Prove that the set of vectors $\{(1,2,2), (2,1,2), (2,2,1)\}$ is linearly independent in $\mathbb{R}^3$.

15. Show that the set $E = \{\epsilon_1 = (1,0,\ldots,0), \ldots, \epsilon_n = (0,0,\ldots,1)\}$ is a basis of $\mathbb{R}^n$.

16. Prove that the set $S = \{(1,0,1), (0,1,1), (1,1,0)\}$ is a basis of $\mathbb{R}^3$.

17. Find a basis for the vector space $\mathbb{R}^3$ that contains the vectors $(1, 2, 0), (1, 3, 1)$.

18. Prove that the set $S = \{(2, 1, 1,), (1, 2, 1), (1, 1, 2)\}$ is a basis of $\mathbb{R}^3$.

19. Find a basis and the dimension of the subspace of the subspace W of $\mathbb{R}^3$ where $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.

20. Find a basis and the dimension of the subspace W of $\mathbb{R}^3$ where $W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 0, 2x + y + 3z = 0\}$.

21. S and T are subspaces of the vector space $\mathbb{R}^4$ given by
$S = \{(x, y, z, w) \in \mathbb{R}^4 : 2x + y + 3z + w = 0\}$,
$T = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y + z + 3w = 0\}$.
Find dim $(S \cap T)$.

22. Extend the set $\{(1, 1, 1, 1), (1, -1, 1, -1)\}$ to a basis of $\mathbb{R}^4$.

23. Find the co-ordinate vector of $\alpha = (1, 3, 1)$ relative to the ordered basis $B = (\alpha_1 = (1, 1, 1), \alpha_2 = (1, 1, 0), \alpha_3 = (1, 0, 0))$ of $\mathbb{R}^3$.

24. Determine k so that the set S is linearly dependent in $\mathbb{R}^3$, where
$S = \{(1, 2, 1), (k, 3, 1), (2, k, 0)\}$.

Linear Transformations

1. Let T be defined in R^2. Verify the following are linear transformations:

(i) $T(x, y) = (x + y, x)$

(ii) $T(x, y) = (x - y, x + y)$

(iii) $T(x, y) = (y, x)$

(iv) $T(x, y) = (xy, x + y)$

(v) $T(x, y) = (x^2, x)$

(vi) $T(x, y) = (x + 1, y)$

(vii) $T(x, y) = (x, y - 1)$

(viii) $T(x, y) = (1, x)$

(ix) $T(x, y) = (0, x)$

(x) $T(x, y) = (0, y^2)$

2. Let $T: R^3 \to R^2$ be a linear transformation defined by $T(x, y, z) = (z, x + y)$ with respect to the bases $\alpha = \{(1,1,0), (1,0,2), (0,1,2)\}$ and $\beta = \{(1,2), (2,3)\}$ respectively. Find $[T]_{\alpha}^{\beta}$.

3. Let $T: R^2 \to R^3$ be a linear transformation defined by $T(x, y) = (x - y, y, x + y)$ with respect to the bases $\beta = \{(1,2), (2,3)\}$ and $\alpha = \{(1,1,0), (1,0,2), (0,1,2)\}$ respectively. Find $[T]_{\beta}^{\alpha}$.

4. Let $T: R^3 \to R^2$ be a linear transformation with $T(1,1,2) = (1,2), T(1,0,3) = (0,5), T(2,6,1) = (5,1)$ with respect to the standard bases. Find the linear transformation and its matrix representation.

5. Let $T: P_3(t) \to P_3(t)$ be a linear transformation defined by $T(f(x)) = f'(x)$ with respect to the bases $\beta = \{2, t + 2, 8 + 2t - t^2, t^3 + 5\}$ and $\alpha -$ standard basis respectively. Find $[T]_{\beta}^{\alpha}$.