Probability Theory

1. ***Random Experiment***: If in each trail an experiment conducted under identical conditions, the outcome is not unique, but may be one of the possible outcomes, then such an experiment is called a *random experiment*.

   Example: Throwing a die, a pack of cards.

2. ***Sample space***: The set of all possible outcomes of an experiment. It is denoted by **"*S*".**

3. ***Trial***: Any particular performance of a random experiment is called a ***trial*** and outcome or combination of outcomes are termed as ***events***.

   Example: Tossing of a coin: {H, T}

4. ***Exhaustive events or cases***: The total number of possible outcomes of a random experiments is known as the exhaustive events or cases.

5. ***Favourable Events or Cases***: The number of cases favourable to an event in a trail is the number of outcomes which entail the happening of the event.

Example: In drawing a card from a pack of cards the number of cases favourable to drawing of

an ace is 4,
for drawing a spade card is 13 and
for drawing a red card is 26.

6. ***Mutually Exclusive events:*** Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes the happening of all the others, *i.e.*, no two or more of them can happen simultaneously in the same trail.

7. ***Equally likely events:*** Outcomes of trail are said to be equally likely if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the others.

8. ***Independent Events:*** Several events are said to be independent if the happening (non-happening) of an event is not affected by the supplementary knowledge concerning the occurrence of any number of remaining events.

Mathematical Definition

If a random experiment or trail results in '$\boldsymbol{n}$' exhaustive, mutually exclusive and equally likely outcomes (or cases), out of which '$\boldsymbol{m}$' are favourable to the occurrence of an event $\boldsymbol{E}$, then the probability '$\boldsymbol{p}$' of occurrence of $\boldsymbol{E}$, denoted by $\boldsymbol{P}(\boldsymbol{E})$, is given by:

$$p = P(E) = \frac{Number\ of\ favourable\ cases}{Total\ number\ of\ exhaustive\ cases} = \frac{m}{n}$$

(i) Since $m \geq 0, n \geq 0 \ \ and\ m \leq n$

$P(E) \geq 0 \ \ and\ \ P(E) \leq 1 \quad \Rightarrow 0 \leq P(E) \ \leq 1$

(ii) $P(E) + P(\bar{E}) = 1$

Statistical Definition

If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it being assumed that the limit is finite and unique.

If in **N** – trials an event **E-** happened **M** – times, then the probability of the happening of E is given by

$$\boldsymbol{P(E) = \lim_{N \to \infty} \frac{M}{N}}$$

Axiomatic Probability

$P(A)$ is the probability function defined on a $\boldsymbol{\sigma}$ $-$field $\boldsymbol{B}$ of events if the following properties or axioms hold:

1. For each $A \in \mathbf{B}$, $P(A)$ is defined, is real and $P(A) \geq 0$. (Axiom of non-negativity)
2. $P(S) = 1$ (Axiom of certainty)
3. If $\{A_n\}$ is any finite or infinite sequence of disjoint events in $\boldsymbol{B}$, then

   $$P(\cup_{i=1}^{n} Ai) = \sum_{i=1}^{n} P(A_i)$$ (Axiom of additivity)

4. The set function ***P*** defined on $\boldsymbol{\sigma}$ $-$field ***B*,** taking its values in the real line and satisfying the above three axioms is called the ***probability measure***.

Algebra of Events

For events A and B,

(i) $A \cup B = \{\omega \in S: \omega \in A \ or\ \omega \in B\}$

(ii) $A \cap B = \{\omega \in S: \omega \in A \ and\ \omega \in B\}$

(iii) $\bar{A} = \{\omega \in S: \omega \notin A\}$

(iv) $A - B = \{\omega \in S: \omega \in A \ but\ \omega \notin B\}$

(v) $A \subset B \ for\ every\ \omega \in A,\ \omega \in B$

(vi) $A = B$ if and only if A and B have same elements, $i.e., A \subset B$ and B $\subset A$

(vii) A and B disjoint events (mutually exclusive) $\Longrightarrow A \cap B = \Phi\ (null\ set)$

(viii) $A \cup B$ can be denoted by $A + B$ if A and B are disjoint events.

(ix) A Δ B denotes those ω belonging to exactly one of A and B, *i.e.*,

$$A \,\Delta\, B = \bar{A}\, B \cup A\, \bar{B} = \bar{A}\, B + A\, \bar{B} \quad \text{(disjoint events)}$$

Theorems on Probability of Events

Addition Theorem of Probability for two events:

$$\mathrm{P}(\mathrm{A} \cup \mathrm{B}) = \mathrm{P}(\mathrm{A}) + \mathrm{P}(\mathrm{B}) - \mathrm{P}(\mathrm{A} \cap \mathrm{B})$$

Addition Theorem of Probability for n-events:

$$\mathrm{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum\sum P(A_i \cap A_j) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots\ A_n)$$

Example

A class in Mathematics subject consists of 6 men and 4 women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same score (a) how many different rankings are possible? (b) If all rankings are considered equally likely, what is the probability that women receive the top 4 scores?

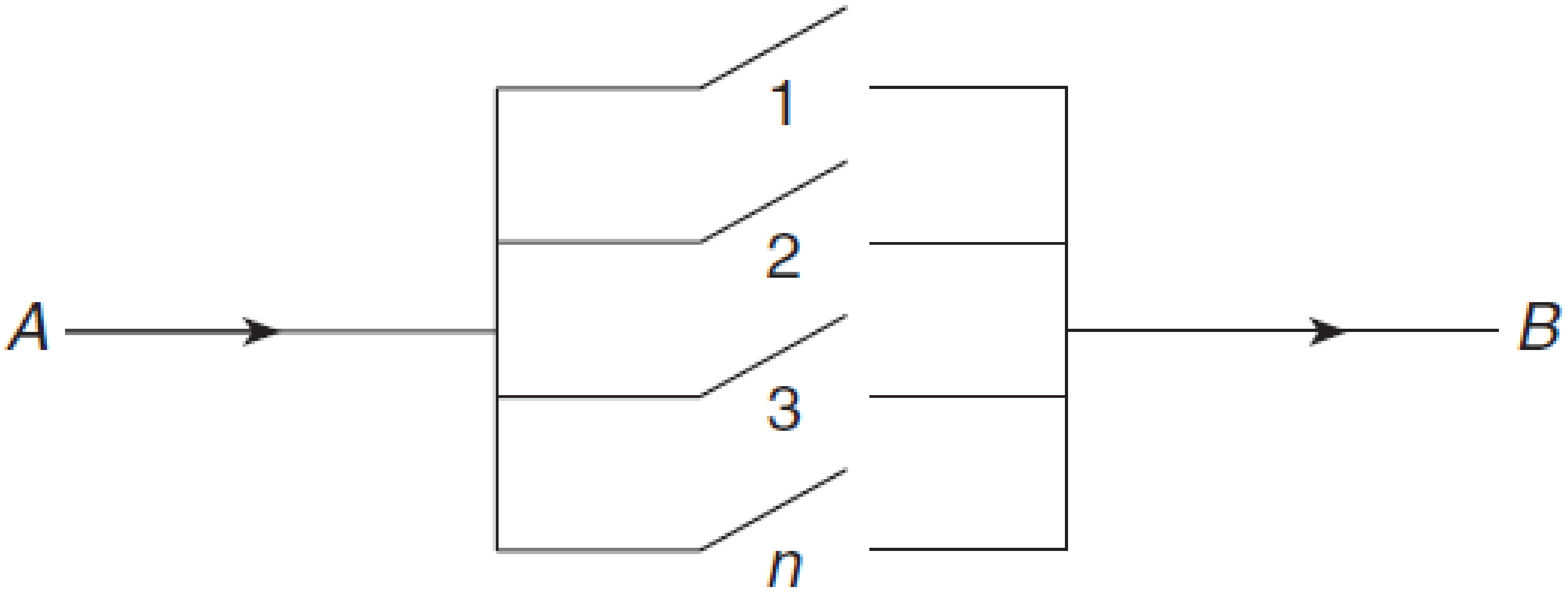
Solution:

Example:

A box containing 6-white and 5 black balls. If 3 balls are drawn randomly from the box, what is the probability that one of the drawn balls is white and the other two are black balls?

Example

A system composed of $\boldsymbol{n}$ separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component i, independent of other components, functions with probability $\mathbf{p}_{\boldsymbol{i}}$, $i = 1, \ldots, n$, what is the probability the system functions?



Infinite Sample Spaces

Countably Infinite Sample Spaces:

Example

A coin is tossed until a head appears. Let $\boldsymbol{n}$ – be the number of times the coin is tossed. Find the probability that (i) n – is at most 3, (ii) n – is even.

Uncountable Spaces

Example

A point is chosen at random inside a rectangle measuring 3 by 5 in . Find the probability $\boldsymbol{p}$ that the point is at least 1 in from the edge.

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad ; \quad P(B) > 0$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad ; \quad P(A) > 0$$

If A and B are independent, $P(A|B) = P(A)$

$$P(B|A) = P(B)$$

Multiplication theorem of Probability :

$$P(A \cap B) \ = \ P(B) \cdot P(A|B)$$

$$P(B \cap A) \ = \ P(A) \cdot P(B|A)$$

For n events, $P(A_1 \cap A_2 \cap \cdots \ldots . A_n) = P(A_1)\ P(A_2|\ A_1) \times$

$$\mathrm{P}(A_3|\ A_1 \cap A_2) \ldots \ldots \times P(A_n|\ A_1 \cap A_2 \cap \cdots \ldots . A_{n-1})$$

Example

A box contains **4** bad and **6** good tubes. **Two** are selected from the box at a time. One of them tested and found to be good. What is the probability that the other one is also **good**.

Example

Urn 1 contains one white and two black marbles, urn 2 contains one black and two white marbles, and urn 3 contains three black and three white marbles. A die is rolled. If a 1, 2, or 3 shows up, urn 1 is selected; if a 4 shows up, urn 2 is selected; and if a 5 or 6 shows up, urn 3 is selected. A marble is then drawn at random from the selected urn. Let A be the event that the marble drawn is white. If U, V, W, respectively, denote the events that the urn selected is 1, 2, 3, then

Bayes Theorem of Probability:

Theorem: If $\boldsymbol{E_1, E_2, \ldots\ldots, E_n}$ are mutually disjoint events with $\boldsymbol{P(E_i) \neq 0}$, $(i = 1, 2, \ldots.n)$, then for any arbitrary event $\boldsymbol{A}$ which is a subset of $\bigcup_{\boldsymbol{i=1}}^{\boldsymbol{n}}(\boldsymbol{E_i})$ such that $\boldsymbol{P(A) > 0}$, we have

$$P(E_i|A) = \frac{P(E_i)\, P(A|E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A|E_i)} \;\; ; \;\; i = 1,2, \ldots \ldots, n$$

Example

The contents of 3-urns I, II and III are as follows:

I - *3* red and 5 white marbles

II – 2 *red and* 1 *white marbles*

III - 2 *red and* 3 *white marbles*.

A box is selected at random, and a marble is randomly drawn from the urn. If the marble is red, find the probability that it came from ***urn I***.

Example

In answering a question on a multiple-choice test, a student either knows the answer or she guesses. Let $\boldsymbol{p}$ be the probability that she knows the answer and $(\mathbf{1}-\boldsymbol{p})$the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{\mathbf{1}}{\mathbf{5}}$, where $\mathbf{5}$ is the number of multiple-choice alternatives. What is the probability that a student knew the answer to a question given that she answered it correctly?

Example

A laboratory blood test is **99 percent** effective in detecting a certain disease when it is, in fact, present. However, the test also yields a “false positive” result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability **0.01**, the test result will imply he or she has the disease). If **0.5** percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?

Example

An urn contains **5** defective (that immediately fail when put in use), **10** partially defective (that fail after a couple of hours of use) and **25** acceptable transistors. A transistor is chosen at random from the urn and put into use. If it does not immediately fail, what is the probability it is acceptable?