PMDS504L: Stationary Time Series Models

Stationary Models and the Autocorrelation Function

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Stationary Models and the Autocorrelation Function

Definition of Stationarity:

- A time series $\{X_t\}$ is said to be **stationary** if its statistical properties remain unchanged over time. That is, shifting the time series forward or backward does not alter its overall behavior.
- A time series $\{X_t\}$ is stationary if its statistical properties remain unchanged under time shifts.

Mean and Covariance Functions of a Time Series

Let $\{X_t\}$ be a time series with $E(X_t^2) < \infty$.

Mean Function

The **mean function** of $\{X_t\}$ is given by:

$$\mu_X(t) = E(X_t).$$

Covariance Function

The **covariance function** of $\{X_t\}$ is defined as:

$$\gamma_X(r, s) = \text{Cov}(X_r, X_s) = E[(X_r - \mu_X(r))(X_s - \mu_X(s))],$$

for all integers r and s.

Definition of Weak Stationarity

A time series $\{X_t\}$ is **weakly stationary** if:

1. **Constant Mean:** The expected value of the series is independent of time:

$$E(X_t) = \mu, \quad \forall t.$$

2. **Constant Covariance Structure:** The covariance between two points in time depends only on their time difference (lag h), not on their absolute positions:

$$\gamma_X(t, t+h) = \mathrm{Cov}(X_t, X_{t+h}) = E[(X_t - \mu)(X_{t+h} - \mu)] = \gamma_X(h).$$

Strict vs. Weak Stationarity

Strict Stationarity:

- A time series $\{X_t\}$ is **strictly stationary** if the joint distribution of $(X_1, ..., X_n)$ is the same as $(X_{1+h}, ..., X_{n+h})$ for all h, n.
- This implies weak stationarity if $E[X_t^2] < \infty$.

Weak Stationarity:

- Only first- and second-order moments are invariant over time.
- Most practical applications assume weak stationarity.

Autocovariance Function (ACVF)

Definition:

- The autocovariance function of a stationary time series $\{X_t\}$ at lag h is:

$$\gamma_X(h) = \text{Cov}(X_{t+h}, X_t) = E[(X_{t+h} - \mu_X)(X_t - \mu_X)]$$

- $\gamma_X(h)$ depends only on h, not t.

Autocorrelation Function (ACF)

Definition:

- The autocorrelation function (ACF) at lag h is:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

- Measures the strength of linear dependence between X_t and X_{t+h}.
- $-1 \leq \rho_X(h) \leq 1$.

Properties of Autocovariance and ACF

- $\gamma_X(0)$ is the variance of X_t, i.e., $\gamma_X(0) = \mathrm{Var}(X_t)$.
- $\gamma_X(h)$ is symmetric:

$$\gamma_X(h) = \gamma_X(-h)$$

- ACF values closer to 1 indicate strong correlation.
- ACF helps in identifying patterns such as trends and seasonality.

Autocorrelation Function

- Stationary models are essential for time series analysis.
- Weak stationarity ensures constant mean and time-invariant covariance.
- ACF provides insights into dependency structures.
- Understanding these concepts aids in model selection for forecasting.

Stationarity of a Time Series

Definition: The stationarity of a time series refers to the consistency of its **statistical properties** over time.

Key Idea: A stationary time series exhibits a stable **probability distribution** over time, meaning that statistical properties do not change.

There are two main types of stationarity:

- **Strict Stationarity**: The probability distribution remains constant over time.
- **Weak Stationarity**: Defined based on the first two moments (mean and covariance).

Weak Stationarity Conditions

A time series y_t is considered **weakly stationary** if:

1. **Constant Mean**: The expected value does not change over time.

$$E(y_t) = \mu, \quad \forall t$$

2. **Time-Invariant Autocovariance**: The autocovariance function depends only on the time gap (lag h), not on the actual time t.

$$\gamma_y(h) = \text{Cov}(y_t, y_{t+h})$$

Interpretation: If the mean and autocovariance structure remain the same over time, the time series can be considered stationary.

Determining Stationarity

Visual Inspection:

- Take arbitrary "snapshots" of the time series at different points in time.
- If the series exhibits **similar behavior** across different time periods, it is likely stationary.

Determining Stationarity

Autocorrelation Function (ACF):

- A **strong and slowly decaying** ACF suggests **non-stationarity**.
    - If the ACF values remain **large and decrease slowly** over increasing lags, the series likely exhibits **non-stationarity**.
    - This behavior suggests **long-term dependencies** and possible trends in the data.
    - Non-stationary series often require **differencing** to remove trends or seasonality before modeling.
- If ACF values **decline rapidly**, the series is likely **stationary**.
    - If the ACF **drops quickly** (e.g., approaching zero within a few lags), it indicates a **stationary time series**.
    - This suggests that observations are not strongly dependent on past values beyond a short range.
    - Such series are stable over time, making them suitable for direct modeling without transformation.

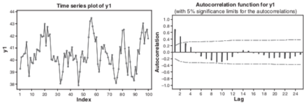
Determining Stationarity

- **Example:**
  - **Non-Stationary Series:** A **random walk** (where each value is the previous value plus a random change) has an ACF that decays very slowly.
  - **Stationary Series:** A **white noise** process has an ACF that quickly drops to near zero after lag 0.
- By analyzing the ACF plot, we can determine whether a series needs **transformation or differencing** before applying different statistical models.

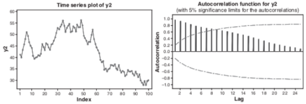
Formal Tests:

- **Augmented Dickey-Fuller (ADF) Test**
- **KPSS Test** (Kwiatkowski-Phillips-Schmidt-Shin)
- **Phillips-Perron (PP) Test**

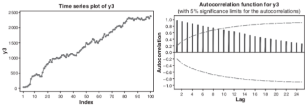
Examples of Stationary and Nonstationary Time Series



(a) $y_{1,t} = 10 + 0.75 y_{1,t-1} + \varepsilon_t$



(b) $y_{2,t} = 2 + 0.95 y_{2,t-1} + \varepsilon_t$



(c) $y_{3,t} = 20 + y_{3,t-1} + \varepsilon_t$

- The stationary series exhibits **constant mean and variance**.
- The nonstationary series shows **a trend or changing variance over time**.

Conclusion

- Stationary time series can be represented using infinite moving averages.
- Wold's decomposition helps in understanding nondeterministic weakly stationary processes.
- Autocovariance functions describe dependencies within the time series.

References

This presentation is adapted from:

- Montgomery, Douglas C., Cheryl L. Jennings, and Murat Kulahci. Introduction to time series analysis and forecasting. John Wiley & Sons, 2015.

Thank You!

Thank you for your attention!