Random Variables

Let $\boldsymbol{S}$ be the sample space associated with a given random experiment $\mathbf{E}$.

A real-valued function defined on $\boldsymbol{S}$ and taking values in $\boldsymbol{R}$ $(-\infty, \infty)$ is called a ***one-dimensional random variable***.

If the function values are ordered pairs of real numbers (*i.e.*, vectors in two-space), the function is said to be a ***two-dimensional random variable***.

More generally, an ***n***-dimensional random variable is simply a function whose domain is $\boldsymbol{S}$ and whose range is a collection of ***n***-tuples of real numbers (vectors in ***n***-space).

A real number $\boldsymbol{X}$ connected with the outcome of a random experiment $\boldsymbol{E}$.

For example, if $\boldsymbol{E}$ consists of two tosses the random variable which is the number of **heads** (0,1 or 2).

Outcome	:	*HH*	*HT*	*TH*	*TT*
Value of X	:	2	1	1	0

Thus to each outcome $\boldsymbol{\omega}$, there corresponds a real number $X(\boldsymbol{\omega})$.

Let us consider the probability space, the triplet $(\boldsymbol{S},\ \boldsymbol{B},\ \boldsymbol{P})$, where $\boldsymbol{S}$ is the sample space, *viz*, space of outcomes, $\boldsymbol{B}$ is the $\boldsymbol{\sigma}$- field of subsets in $\boldsymbol{S}$ and $\boldsymbol{P}$ is a probability function on $\boldsymbol{B}$.

Definition: A random variable is a function $X(\boldsymbol{\omega})$ with domain $\boldsymbol{S}$ and range $(-\infty, \infty)$ such that for every real number $\boldsymbol{a}$, the event $[\omega : X(\boldsymbol{\omega}) \leq a] \in \boldsymbol{B}$**.**

Example: $\boldsymbol{P}\{X \leq 1\} = \boldsymbol{P}\{HH, HT, TH\} = \boldsymbol{\frac{3}{4}}$

Note: One-dimensional *r.v.* will be denoted by capital letters, X, Y, Z….etc.

A typical outcome of the experiment will be denoted by $\boldsymbol{\omega}$ or *e*.

The values which *X*, *Y*, *Z*....*etc*, can assume are denoted by lower case letters., *x*, *y*, z…*etc*.

Discrete Random Variable

Let $\boldsymbol{X}$ be a finite random variable on a sample space $\boldsymbol{S}$, that is, $\boldsymbol{X}$ assigns only finite number or countably infinite number of values to $\boldsymbol{S}$.

$$Say, \quad R_X = \{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots, \infty\}$$

Example: 1. Marks obtained in a test.

2. Number of accidents per month.

3. Number of telephone calls per unit time.

4. Number of successes in $\boldsymbol{n}$-trails and so on.

Then, $\boldsymbol{X}$ induces a function $\boldsymbol{p}$ which assigns probabilities to the points in $\boldsymbol{R_X}$ as follows:

$$p(x_i) = p_X(x_i) = P(X = x_i) = P\{s \in S: X(s) = x_i\} \text{ for } i = 1, 2, \ldots\ldots, n$$

Probability Mass function: If $\boldsymbol{X}$ is a discrete random variable with distinct values $x_1, x_2, \dots \dots, x_n$ then the function $\boldsymbol{p_X(x)}$ is defined as :

$$p_X(x_i) = \begin{cases} P(X = x_i) = p_i\,, & if\ x = x_i \\ 0\,, & if\ x \neq x_i\ ;\, i = 1,2, \dots .\,, n \end{cases}$$

Is called the ***probability mass function*** of random variable $\boldsymbol{X}$.

The numbers $p(x_i); i = 1, 2, \dots ..$ must satisfy the following conditions:

(i) $p(x_i) \geq 0\ \forall\ i$

(ii) $\sum_{i=1}^{n} p(x_i) = 1$

Distribution Function

Definition: Let $\boldsymbol{X}$ be a random variable. The function $\boldsymbol{F}$ defined for all real $\boldsymbol{x}$ by

$$F_X(x) = F(x) = P(X \leq x) = P\{\omega : X(\omega) \leq x\}, -\infty < x < \infty$$

is called the *distribution function* or *cumulative distribution function* of r.v. ($\boldsymbol{X}$).

The domain of the distribution function is $(-\infty, \infty)$ and its range is [0,1]

Properties:

If $\boldsymbol{x}$ is a real number, the set of all $\boldsymbol{\omega}$ in $\boldsymbol{S}$ such that $\mathbf{X}(\boldsymbol{\omega}) = \boldsymbol{x}$ *is*, denoted by $\boldsymbol{X} = \boldsymbol{x}$.

1. $P(X = x) = P(\omega : X(\omega) = x)$

2. $P(X \le a) = P\{\omega : X(\omega) \in (-\infty, a]\}$

3. $P(a < X \le b) = P\{\omega : X(\omega) \in (a, b]\} = F(b) - F(a)$

$P(a \le X \le b) = P(X = a) + [F(b) - F(a)]$

$P(a \le X < b) = F(b) - F(a) - P(X = b) + P(X = a)$

$P(a < X < b) = F(b) - F(a) - P(X = b)$

4. $P(X = a \text{ or } X = b) = P\{(X = a) \cup (X = b)\}$

5. $P(X = a \text{ and } X = b) = P\{(X = a) \cap (X = b)\}$

Properties

If $\boldsymbol{F}$ is the *distribution function* of the *r.v.* $\boldsymbol{X}$ and if $a < b$, then

6. $P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$

7. (i) $0 \leq F(x) \leq 1$

   (ii) $F(x) \leq F(y)$ if $x < y$

8. $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$ and

   $F(\infty) = \lim_{x \to \infty} F(x) = 1$

Discrete Distribution Function:

A countable number of points $x_1, x_2, x_3, \ldots\ldots x_n$

$$p(x_i) \geq 0 \;\; \forall \;\; i,$$

$$\sum_{i=1}^{n} p(x_i) = 1 \;\; \text{such that}$$

$$F(x) = \sum_{i:\, x_i \leq x} p_i = \sum_{x_i} P(X = x_i)$$

If x_i is just integer i, so that $P(X = i) = p_i$; i=1, 2, 3,….. Then $\boldsymbol{F(x)}$ is a "***step function***" having jump $\boldsymbol{p}$ at i and being constant between each pair of integers.

Properties:

1. $p(x_j) = P(X = x_j) = F(x_j) - F(x_{j-1}),$ where F is the *d.f.* of X.

Example 1: A random variable X has the following probability function:

$\boldsymbol{x}$	0	1	2	3	4	5	6	7
$\boldsymbol{p}(\boldsymbol{x})$	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2$+k

(i) Find k, (ii) Evaluate $P(X < 6),\ P(X \geq 6)\ and\ P(0 < X < 5)$

(iii) Determine the distribution function of X.

Solution: (i) Since, $\sum_{i=1}^{n} P(X = x_i) = 1$

$$\sum_{i=0}^{7} P(X = x_i) = 1$$

$$\Longrightarrow \quad P(X = 0) + P(X = 1) + \cdots + P(X = 7) = 1$$

$$\Longrightarrow \quad 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$(10k - 1)(k + 1) = 0$$

$$\therefore \quad k = \frac{1}{10}$$

$(ii)\ P(X < 6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100}$

$= \frac{81}{100}$

$(iii)\ P(X \geq 6) = P(X=6) + P(X=7)$ or

$= 1 - P(X < 6) = 1 - \frac{81}{100} = \frac{19}{100}$

$(iv)\ P(0 < X < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10}$

X	$\mathbf{F_X(x) = P(X \leq x)}$
0	0
1	$k = \frac{1}{10}$
2	$3k = \frac{3}{10}$
3	$5k = \frac{5}{10}$
4	$8k = \frac{8}{10}$
5	$8k + k^2 = \frac{81}{100}$
6	$8k + 3k^2 = \frac{83}{100}$
7	$9k + 10k^2 = 1$

Example 2:

Two dice are rolled. Let X denote the random variable which counts the total number of points on the upturned faces, Construct a table giving the non-zero values of the $p.m.f$ and draw the probability chart. Also find the distribution function of X.

Solution: Let X = sum of the two points on the upturned faces

$$X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

Total possible outcomes when 2-dice are tossed = $6^2 = 36$

X	Possible Values
2	(1, 1)
3	(2, 1) (1, 2)
4	(2, 2) (1, 3) (3, 1)
5	(1, 4) (4, 1) (2, 3) (3, 2)
6	…
7	…

X	2	3	4	5	6	7	8	9	10	11	12
***P* (*X* = *x*)**	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Example 1: A random variable ***X*** has the following probability function:

$$P(X = x) = \begin{cases} \frac{x}{15} & ; \quad x = 1, 2, 3, 4, 5 \\ 0, & \quad elsewhere \end{cases}$$

Find (i) $P(X = 1 \; or \; 2)$, and (ii) $P\left\{\frac{1}{2} < X < \frac{5}{2} \mid X > 1\right\}$

Continuous Random Variable

Definition: A random variable X is said to be continuous if it can take all possible values between certain limits.

A continuous random variable is a r.v. that can be measured to any desired degree of accuracy.

Example: Age, height, weight etc.

Probability Density Function (p.d.f)

Definition: Consider the small interval $(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{dx})$ of length $\boldsymbol{dx}$ round the point $\boldsymbol{x}$. Let $\boldsymbol{f}(\boldsymbol{x})$ be any continuous function of $\boldsymbol{x}$ so that $\boldsymbol{f}(\boldsymbol{x})\boldsymbol{dx}$ represents the probability that $\boldsymbol{X}$ falls in the infinitesimal interval $(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{dx})$ **.**

$$P(x \leq X \leq x + dx) = f_X(x)dx$$

$$f_X(x) = \lim_{\delta x \to 0} \frac{P(x \leq X \leq x+\delta x)}{\delta x}$$

The probability for a variate value to line in the interval dx is $f(x)dx$ and hence the probability for a variate value to fall in the interval $[\alpha, \beta]$ is:

$$P(\alpha \ \leq X \leq \ \beta) = \int_{\alpha}^{\beta} f(x)dx$$

Continuous Distribution Function

Definition: If X is continuous *r.v.* with the *p.d.f.* $f(x)$, then the function

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt \ ; \ -\infty < x < \infty$$

is called the distribution function (*d.f.*) or sometimes the cumulative distribution function (*c.d.f.*) of the random variable X.

Properties of $\boldsymbol{p.d.f.}$:

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(iii) The probability $P(E)$ given by: $\int_E f(x) dx$ is well defined for any event E.

(iv) $P(X = c) = 0, \ \forall \ c$

(v) $P(\alpha \leq X \leq \beta) = P(\alpha \leq X < \beta) = P(\alpha < X \leq \beta) = P(\alpha < X < \beta)$

Properties: 1. $0 \leq F(x) \leq 1 \ ; -\infty < x < \infty$

2. $F'(x) = \frac{d}{dx} F(x) = f(x) \geq 0$

$\Rightarrow$ $F(x)$ is non-decreasing function of x.

3. $F(-\infty) = \lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{-\infty} f(x)\, dx = 0$

$F(+\infty) = \lim_{x \to \infty} F(x) = \lim_{x \to \infty} \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1$

4. $F(x)$ is continuous function of x on the right.

5. The discontinuities of $F(x)$ are at the most countable.

Example 3:

The diameter of an electric cable, say X, is assumed to be a continuous *r.v.* with *p.d.f* : $f(x) = 6x(1-x)\ ;\ 0 \le x \le 1$

(i) Check that $f(x)$ is p.d.f,

(ii) Determine a number b such that $P(X < b) = P(X > b)$.

Solution: ***(i)*** For $0 \leq x \leq 1$ **,** $f(x) \geq 0$

$$\int_0^1 f(x)dx = 6\int_0^1 x(1-x)dx$$

$$= 6\int_0^1 (x - x^2)dx = 6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 1$$

Hence $f(x)$ is the probability density function of random variable X.

$(\boldsymbol{ii})$ $$P(X < b) = P(X > b)$$

$$\Rightarrow \quad \int_0^b f(x)dx = \int_b^1 f(x)dx$$

$$\Rightarrow \quad 6\int_0^b x(1-x)dx = 6\int_b^1 x(1-x)dx$$

$$\Rightarrow \qquad \left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^b = \left|\frac{x^2}{2} - \frac{x^3}{3}\right|_b^1$$

$$\Rightarrow \qquad 4b^3 - 6b^2 + 1 = 0$$

$$(2b - 1)(2b^2 - 2b - 1) = 0$$

$\Rightarrow$ Hence $\quad b = \frac{1}{2}$

Example 4:

Let X be a continuous random variable with *p.d.f.* :

$$f(x) = \begin{cases} ax & ; \ 0 \leq x \leq 1 \\ a & ; \ 1 \leq x \leq 2 \\ -ax + 3a & ; \ 2 \leq x \leq 3 \\ 0 & ; \ elsewhere \end{cases}$$

(i) Determine the constant $\boldsymbol{a}$.

(ii) Compute $P(X \leq 1.5)$.

(i)

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\int_{0}^{1} ax\, dx + \int_{1}^{2} a\, dx + \int_{2}^{3} (-ax + 3a)\, dx = 1$$

(ii)

$$P(X \leq 1.5) = \int_{-\infty}^{1.5} f(x)dx$$

$$= \int_{0}^{1} ax\, dx + \int_{1}^{1.5} a\, dx$$

$$\Rightarrow \qquad a = \frac{1}{2}$$

Median: In case of continuous distribution, median is the point which divides the total area two equal parts. Thus if ***M*** is the median, then

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

Thus solving $\int_a^M f(x)\,dx = \frac{1}{2}$ or $\int_M^b f(x)\,dx = \frac{1}{2}$

for ***M***, we get the median value.

Mode: Mode is the value of x for which $f(x)$ is maximum. Mode is thus solution of

$f'(x) = 0$ and $f''(x) < 0$, provided it lies in [a, b].

Example:

Suppose that the life in hours of a certain part of radio tube is a continuous random variable X with the *p.d.f.* given by:

$$f(x) = \begin{cases} \frac{100}{x^2}, & x \geq 100 \\ 0, & elsewhere \end{cases}$$

(i) What is the probability that all of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation?

(ii) What is the probability that none of the of three original tubes will have to be replaced during that first 150 hours of operation?

(iii) What is the probability that a tube will last less than 200 hours if it is known that the tube is still function after 150 hours of service?

Example:

A petrol pump is supplied with petrol once a day. If its daily volume of sales (***X***) in thousands of litres is distributed by:

$$f(x) = 5(1 - x)^4 \; ; \quad 0 \le x \le 1$$

What must be the capacity of its tank in order that the probability that its supply will be exhausted in a given day shall be 0.01?

Example:

A bombing plane carrying three bombs flies directly above a railroad track. If a bomb falls within 40 meters of track, the track will be sufficiently damaged to disrupt the traffic. With a certain bomb site the points of impact of a bomb have the *p.d.f.*

$$f(x) = \begin{cases} \frac{100+x}{10000} & ; \ -100 \le x \le 0 \\ \frac{100-x}{10000} & ; \ 0 \le x \le 100 \\ 0 & ; \ elsewhere \end{cases}$$

Where x represents the vertical deviation (in meters) from the aiming point. Which is the track in this case. Find the distribution function.

If all the three bombs are used, **what is the probability that the track will be damaged**?

Two-dimensional Random Variables

Definition: Let X and Y be two r.v. defined on the same sample space S, then the function (X, Y) that assigns a point in $R^2 (= R \times R)$, is called a two-dimensional random variable.

When (X, Y) is a two-dimensional discrete r.v., the possible values of (X, Y) may be represented as (x_i, y_j), $i = 1, 2, \ldots\ldots, n;\ j = 1, 2, \ldots\ldots, m.$

If (X, Y) can assume all values in a specified region R in the *xy* –plane, (X, Y) is called a two-dimensional continuous R.V.

Joint Probability Function

Let X and Y be random variables on a sample space S with respective image sets $X(S) = \{x_1, x_2, \ldots \ldots, x_n\}$ and $Y(S) = \{y_1, y_2, \ldots \ldots, y_m\}$.

$$X(S) \times Y(S) = \{x_1, x_2, \ldots \ldots, x_n\} \times \{y_1, y_2, \ldots \ldots, y_m\}$$

Into a probability space by defining the probability of the ordered pair to be $\boldsymbol{P(X = x_i, Y = y_j)}$ or $\boldsymbol{p(x_i, y_j)}$.

The function $\boldsymbol{p}$ on $X(S) \times Y(S)$ is defined by:

$$p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$$ is called the

joint probability function of $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

$X \backslash Y$	y_1	y_2	y_3	$\cdots$	y_j	$\cdots$	y_m	Total
x_1	p_{11}	p_{12}	p_{13}	$\cdots$	p_{1j}	$\cdots$	p_{1m}	$p_{1\cdot}$
x_2	p_{21}	p_{22}	p_{23}	$\cdots$	p_{2j}	$\cdots$	p_{2m}	$p_{2\cdot}$
x_3	p_{31}	p_{32}	p_{33}	$\cdots$	p_{3j}	$\cdots$	p_{3m}	$p_{3\cdot}$
.	.	.	.	$\cdots$	.	$\cdots$	.	
.	.	.	.	$\cdots$	.	$\cdots$	.	$\vdots$
.	.	.	.	$\cdots$	.	$\cdots$	.	
x_i	p_{i1}	p_{i2}	p_{i3}	$\cdots$	p_{ij}	$\cdots$	p_{im}	$p_{i\cdot}$
.	.	.	.	$\cdots$	.	$\cdots$	.	
.	.	.	.	$\cdots$	.	$\cdots$	.	$\vdots$
.	.	.	.	$\cdots$	.	$\cdots$	.	
x_n	p_{n1}	p_{n2}	p_{n3}	$\cdots$	p_{nj}	$\cdots$	p_{nm}	$p_{n\cdot}$
Total	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	$\cdots$	$p_{\cdot j}$	$\cdots$	$p_{\cdot m}$	1

Definition

If (X, Y) is a two-dimensional discrete random variable, then the joint discrete function of X, Y also called the joint probability mass function of (X, Y) denoted by $\boldsymbol{p_{XY}}$ is defined as:

$$p_{XY}(x_i, y_j) = P(X = x_i, Y = y_j) \text{ for a value of } (x_i, y_j) \text{ of } (X, Y)$$

$$p_{XY}(x_i, y_j) = 0 \text{ , otherwise.}$$

$$\sum\sum p_{XY}(x_i, y_j) = 1,$$ where the summation is taken over all possible values of (X, Y).

Marginal Probability Function

$$p_X(x_i) = P(X = x_i)$$
$$= P(X = x_i \cap Y = y_1) + P(X = x_i \cap Y = y_2) + \ldots.. +$$
$$P(X = x_i \cap Y = y_m)$$
$$p_X(x_i) = p_{i1} + p_{i2} + \ldots + p_{ij} + \cdots + p_{im} = \sum_{j=1}^{m} p_{ij} = p_{i\cdot}$$

Also, $\sum_{i=1}^{n} p_{i\cdot} = p_{1\cdot} + p_{2\cdot} + \ldots + \cdots + p_{n\cdot} = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$

Similarly, $p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = p_{\cdot j}$

which is the marginal probability function of Y.

Conditional Probability Function

The conditional probability mass function of X, given $Y = y$ is defined as:

$$p_{X|Y}(x|y) = \frac{P(X=x, Y=y)}{P(Y=y)}, \text{ provided } P(Y = y) \neq 0$$

Similarly, $$p_{Y|X}(y|x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

A necessary and sufficient condition for the discrete random variables X and Y to be independent is:

$P\left(X = x_i, Y = y_j\right) = P\left(X = x_i\right) P\left(Y = y_j\right)$ for all values (x_i, y_j) of (X,Y).

Joint Probability Density Function

If (X, Y) is two-dimensional continuous RV such that,

$$P\left(x - \frac{dx}{2} \le X \le x + \frac{dx}{2} \; and \; y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right) = f(x, y)dxdy$$

Then $f(x, y)$ is called the joint pdf of (X, Y), provided $f(x, y)$ satisfies the following conditions:

(i) $f(x, y) \ge 0$, for all $(x, y) \in R,$ where R is the range space.

(ii) $\iint_R f(x, y)\ dx\ dy$ =1.

Two-dimensional distribution function

Definition: The distribution function of the two-dimensional random variable (X,Y) is a real valued function F defined for all real x and y by the relation:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = \sum_{y_j \leq y} \sum_{x_i \leq x} p_{ij}$$

In the continuous case,

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\, dx\, dy$$

Properties: $\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$

Marginal Distribution Functions:

$$F_X(x) = P(X \le x) = P(X \le x, Y < \infty) = \lim_{y \to \infty} F_{XY}(x, y) \;=\; F_{XY}(x, \infty)$$

Similarly, $F_Y(y) = P(Y \le y) = P(X < \infty, Y \le y) = \lim_{x \to \infty} F_{XY}(x, y) \;=\; F_{XY}(\infty, y)$

In case of joint discrete r.v., the marginal distribution functions are:

$$F_X(x) = \sum_y P(X \le x, Y = y) \text{ and } F_Y(y) = \sum_x P(X = x, Y \le y)$$

In case of Jointly continuous r.v.,

$$F_X(x) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \right\} dx \qquad F_Y(y) = \int_{-\infty}^{y} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \right\} dy$$

Marginal Density function:

$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$ - is the marginal density function of X.

$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$ - is the marginal density function of Y

$$P(a \le X \le b) = P(a \le X \le b, -\infty \le Y \le \infty)$$
$$= \int_a^b \left\{\int_{-\infty}^{\infty} f_{XY}(x, y)\, dy\right\} dx$$
$$= \int_a^b f_X(x)dx$$

$$P(a \le X \le b) = \int_c^d f_Y(y)dy$$

Independent RVs

If (X,Y) is two-dimensional independent RVs,

$$P(X = x,\ Y = y) = P(X = x) \cdot P(Y = y)$$ - Discrete

$$f(x, y) = f(x) \cdot f(y)$$ - Continuous

Example 10: For a bivariate probability distribution of (X, Y) given below,

X \ Y	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Find $P(X \leq 1)$

$P(Y \leq 3)$

$P(X \leq 1, Y \leq 3)$

$P(X \leq 1 | Y \leq 3)$

$P(Y \leq 3 | X \leq 1)$

$P(X + Y \leq 4)$

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= \sum_{j=1}^{6} P(X = 0, Y = j) + \sum_{j=1}^{6} P(X = 1, Y = j) = \frac{1}{4} + \frac{5}{8} = \frac{7}{8}$$

$$P(Y \leq 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= \sum_{i=0}^{2} P(X = i, Y = 1) + \sum_{i=0}^{2} P(X = i, Y = 2) + \sum_{i=0}^{2} P(X = i, Y = 3)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

$$P(X \leq 1, Y \leq 3) = \sum_{j=1}^{3} P(X = 0, Y = j) + \sum_{j=1}^{3} P(X = 1, Y = j)$$

$$= \frac{1}{32} + \frac{1}{4} = \frac{9}{32}$$

$$P(X \leq 1|Y \leq 3) = \frac{P(X\leq 1, Y\leq 3)}{P(Y\leq 3)} = \frac{9/32}{23/64} = \frac{18}{23}$$

$$P(Y \leq 3|X \leq 1) = \frac{P(X\leq 1, Y\leq 3)}{P(X\leq 1)} = \frac{9/32}{7/8} = \frac{9}{28}$$

$$P(X + Y \leq 4) = \sum_{j=1}^{4} P(X = 0, Y = j) + \sum_{j=1}^{3} P(X = 1, Y = j) + \sum_{j=1}^{2} P(X = 2, Y = j)$$

$$= \frac{3}{32} + \frac{1}{4} + \frac{1}{16} = \frac{13}{32}$$

Example 11

Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If ***X*** denotes the number of white balls drawn and ***Y*** denotes the number of red balls drawn. Find the joint probability distribution of (X, Y).

Solution:

$P(X = 0, Y = 0)$ = P(drawing 3 balls none of which is white or red)

= P(all the 3 balls are black)

$= \frac{4_{C_3}}{9_{C_3}} = \frac{1}{21}$

$P(X = 0, Y = 1)$ = P(drawing 1 red and 2 black balls) $= \frac{3}{14}$

X \ Y	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

Example 13

The joint pdf of a two-dimensional RV (X,Y) is given by

$$f(x,y) = xy^2 + \frac{x^2}{8} \quad ; \; 0 \le x \le 2, \; 0 \le y \le 1$$

Compute $P(X > 1), \; P(Y < \frac{1}{2}), \; P(X > 1|Y < \frac{1}{2}), \; P(X < Y)$ and $P(X + Y \le 1)$

Solution: Given, $f(x,y) = xy^2 + \frac{x^2}{8}$; $0 \le x \le 2,\ 0 \le y \le 1$

(i) $P(X > 1)$

$\because$ $F_X(x) = \mathrm{P}(\mathrm{X} \le x) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \right\} dx$

$$\mathrm{P}(\mathrm{X} > x) = \int_{x}^{\infty} \left\{ \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \right\} dx$$

$$P(X > 1) = \int_{1}^{2} \left\{ \int_{0}^{1} \left(xy^2 + \frac{x^2}{8} \right) dy \right\} dx$$

$$= \frac{19}{24}$$

$$(ii)\quad P\left(Y < \frac{1}{2}\right) = \int_0^{1/2}\left\{\int_0^2 f_{XY}(x,y)\,dx\right\}dy$$

$$= \int_0^{1/2}\left\{\int_0^2\left(xy^2 + \frac{x^2}{8}\right)dx\right\}dy$$

$$= \frac{1}{4}$$

$$(iii)\quad P\left(X > 1, Y < \frac{1}{2}\right) = \int_1^2\left\{\int_0^{\frac{1}{2}}\left(xy^2 + \frac{x^2}{8}\right)dy\right\}dx = \frac{5}{24}$$

$$(iv)\quad P(X > 1 |\, Y < \frac{1}{2}) = \frac{P\left(X>1 \cap Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)} = \frac{5/24}{1/4} = \frac{5}{6}$$

$$(v) \quad P(X < Y) = \int_0^1 \left\{\int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\right\} dy = \frac{53}{480}$$

$$(vi) \quad P(X + Y \leq 1) = \int_0^1 \left\{\int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\right\} dy$$

$$= \frac{13}{480}$$

Example 14

If X and Y are two random variables having joint density function:

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y) & ;\ 0 \leq x \leq 2,\ 2 \leq y \leq 4 \\ 0 & \end{cases}$$

Find the (i) Marginal density functions of X and Y

(ii) $P(X < 1 \cap Y < 3)$

(iii) $P(X + Y < 3)$ (iv) P(X<1 | Y<3)

Solution:

Given, $$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & ; \quad 0 \le x \le 2,\ 2 \le y \le 4 \\ 0 & \end{cases}$$

(i) Marginal density function of ***X*,**

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$f_X(x) = \int_2^4 \frac{1}{8}(6-x-y)dy = \frac{3-x}{4} \quad ; 0 \le x \le 2$$

Marginal density function of ***Y***,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

$$= \int_0^2 \frac{1}{8}(6-x-y)dx = \frac{5-y}{4} \quad ; 2 \le y \le 4$$

(ii) $P(X < 1 \cap Y < 3) = \int_{-\infty}^{1} \int_{-\infty}^{3} f(xy)dx\,dy$

$$= \int_0^1 \left\{ \int_2^3 \frac{1}{8}(6 - x - y)dy \right\} dx$$

$$P(X < 1 \cap Y < 3) = \frac{3}{8}$$

(iii) $P(X + Y < 3) = \int_0^1 \left\{ \int_2^{3-x} \frac{1}{8}(6 - x - y)dy \right\} dx$

$$= \frac{5}{24}$$

(iv) $P(X < 1 \mid Y < 3) = \frac{P(X<1 \cap Y<3)}{P(Y<3)} = \frac{3/8}{5/8} = \frac{3}{5}$

Example 15

Joint density function of X and Y is given by:

$$f(x, y) = 4xye^{-(x^2+y^2)} \quad ; \quad x \geq 0,\; y \geq 0$$

Test whether X and Y are independent.

Find the conditional density function of X given $Y = y$.