# MATRIX ALGEBRA

# What is it?

- Matrix algebra is a means of making calculations upon arrays of numbers (or data).
- Most data sets are matrix-type

## Why use it?

- Matrix algebra makes *mathematical expression and computation* easier.
- It allows you to get rid of cumbersome notation, concentrate on the concepts involved and understand where your results come from.

# 1.1 Matrices

Consider the following set of equations:

$$\begin{cases} x+y=7, \\ 3x-y=5. \end{cases}$$

It is easy to show that $x = 3$ and $y = 4$.

How about solving

$$\begin{cases} x+y-2z=7, \\ 2x-y-4z=2, \\ -5x+4y+10z=1, \\ 3x-y-6z=5. \end{cases}$$

**Matrices can help…**

# Definitions - scalar

- a scalar is a number
  - (denoted with regular type: 1 or 22)

# DEFINITIONS - VECTOR

- Vector: a single row or column of numbers
  - denoted with **bold small letters**
  - Row vector

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

  - Column vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

# Definitions - Matrix

- A system of *m n* numbers arranged in the form of an ordered set of *m* rows, each consisting of an ordered set of *n* numbers, is called an *m* x *n* matrix
- If there are m rows and n columns in the array, the matrix is said to be of order *m* x *n* or *(m,n)* or *m by n*
- A matrix is an array of numbers

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- Denoted with a **bold Capital letter**
- All matrices have an order (or dimension):

that is, the number of rows × the number of columns. So, **A** is 2 by 3 or (2 × 3).

# 1.1 Matrices

In the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

- numbers $a_{ij}$ are called *elements*. First subscript indicates the row; second subscript indicates the column. The matrix consists of $mn$ elements
- It is called "the $m \times n$ matrix $A = [a_{ij}]$" or simply "the matrix $A$" if number of rows and columns are understood.

# Types of Matrices

- Row Matrix: An $m$ x $n$ matrix is called row matrix if m = 1. Ex: A = $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$
- Column Matrix: An $m$ x $n$ matrix is called row matrix if n = 1. Ex: A = $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$
- Square Matrix: A square matrix is a matrix that has the same number of rows and columns i.e. if $m = n$. Ex: A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

# Types of Matrices

- Zero Matrix: A matrix each of whose elements is zero & is called a zero matrix. It is usually denoted by "O". It is also called *"Null Matrix"*

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$$

# Types of Matrices

- **Diagonal Matrix:** A square matrix with its all non *diagonal elements* as zero. i.e if $A = [a_{ij}]$ is a diagonal matrix, then $a_{ij} = 0$ whenever $i \neq j$. ***Diagonal elements*** are the $a_{ij}$ elements of the square matrix A for which $i = j$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

# Types of Matrices

- Diagonal elements are said to constitute the **main diagonal** or **principal diagonal** or simply a **diagonal**.
- The diagonals which lie on a line perpendicular to the diagonal are said to constitute **secondary diagonal.**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Here main diagonal consists of 1 & 4 and secondary diagonal consists of 2 & 3

# Types of Matrices

- Scalar Matrix: It's a diagonal matrix whose all elements are equal.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- Unit Matrix: It's a scalar matrix whose all diagonal elements are equal to unity. It is also called a Unit Matrix or Identity Matrix. It is denoted by $I_n$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Types of Matrices

- Triangular Matrix: If every element above or below the diagonal is zero, the matrix is said to be a triangular matrix.

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$ Upper Triangular Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & -6 & 3 \end{bmatrix}$$ Lower Triangular Matrix

# Equality of Matrices

- Two matrices A & B are said to be equal iff:
  1. A and B are of the same order
  2. All the elements of A are equal as that of corresponding elements of B
- Two matrices A = [aij] & B = [bij] of the same order are said to be equal if aij = bij

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

If A & B are equal, then

$x=1,\ y=2,\ z=3,\ w=4$

# Equality of Matrices (Problems for Practice)

Q1: If $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$; find x,y,z,w.

Q2: If $\begin{bmatrix} x & 3x-y \\ 2x+z & 3y-w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$; find x,y,z,w.

Q3: If $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$; find a,b.

# Trace of a Matrix

- In a square matrix A, the sum of all the diagonal elements is called the trace of A. It is denoted by tr A.
  - Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 6 & 1 \end{bmatrix}$ tr A = 1+4+1 = 6
  - Ex: If $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ tr B = 1+4 = 5

# Operations on Matrices

- Addition/Subtraction
- Scalar Multiplication
- Matrix Multiplication

# Addition and Subtraction

- Two matrices may be added (or subtracted) iff they are the same order.

- Simply add (or subtract) the corresponding elements. So, **A + B = C**

# Addition and Subtraction (cont.)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

- Where

$$\begin{aligned} a_{11} + b_{11} &= c_{11} \\ a_{12} + b_{12} &= c_{12} \\ a_{21} + b_{21} &= c_{21} \\ a_{22} + b_{22} &= c_{22} \\ a_{31} + b_{31} &= c_{31} \\ a_{32} + b_{32} &= c_{32} \end{aligned}$$

# Addition / Subtraction (Problems for Practice)

Q1: If $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & -1 \end{bmatrix}$ and $B \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$

find A+B, A-B.

Q2: If $A = \begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{bmatrix}$

find A+B, A-B.

# Scalar Multiplication

- To multiply a scalar times a matrix, simply multiply each element of the matrix by the scalar quantity

$$k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

- Ex: If A = $\begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix}$, then

$$10\text{A} = \begin{bmatrix} 30 & 80 & 110 \\ 60 & -30 & 80 \end{bmatrix}$$

# PROBLEMS FOR PRACTICE

Q1: If $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & -1 \end{bmatrix}$ and B $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$

find 5A+2B.

Q2: If $A = \begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{bmatrix}$

find 7A - 5B.

Q3: If $A = \begin{bmatrix} 2 & -2 & 7 \\ 4 & 6 & 3 \end{bmatrix}$; find matrix X such that X+A=O where O is a null matrix.

# Problems for Practice

Q4: If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 \\ 5 & 3 \end{bmatrix}$

Show that 5(A+B) = 5A + 5B.

Q5: If $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -2 & 2 & 3 \end{bmatrix}$

find a 2 x 4 matrix "X" such that A - 2X = 3B.

Q6: If $X+Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ Find X&Y.

Q7: Find additive inverse of $\begin{bmatrix} 5 & 10 & 9 \\ 1 & -3 & \\ -2 & & \end{bmatrix}$

# MATRIX MULTIPLICATION

- If $A = [a_{ij}]$ is a $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then $AB$ is defined as a $m \times n$ matrix $C = AB$, where $C = [c_{ij}]$ with

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$$ for $1 \le i \le m,\ 1 \le j \le n$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $C = AB$.
Evaluate $c_{21}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \qquad c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22$$

# Rule of Matrix Multiplication

- Multiplication or Product of two matrices A & B is possible iff the number of columns of A is equal to the number of rows of B.
- The rule of the multiplication of the matrices is row-column wise (→↓).
- The first row of AB is obtained by multiplying the $1^{st}$ row of A with $1^{st}$, $2^{nd}$ & $3^{rd}$ column of B.
- The second row of AB is obtained by multiplying the $2^{nd}$ row of A with $1^{st}$, $2^{nd}$ & $3^{rd}$ column of B.
- The third row of AB is obtained by multiplying the $3^{rd}$ row of A with $1^{st}$, $2^{nd}$ & $3^{rd}$ column of B.

# Matrix Multiplication

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$, Evaluate $C = AB$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1 \times (-1) + 2 \times 2 + 3 \times 5 = 18 \\ c_{12} = 1 \times 2 + 2 \times 3 + 3 \times 0 = 8 \\ c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22 \\ c_{22} = 0 \times 2 + 1 \times 3 + 4 \times 0 = 3 \end{cases}$$

$$C = AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 22 & 3 \end{bmatrix}$$

# Problems for Practice

Q1: If A = $\begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 9 \\ 0 & 1 \\ 6 & 9 \end{bmatrix}$. Find AB.

Q2: If A = $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

Show that AB is a null matrix & BA is not a null matrix.

Q3: If A = $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$ Find a & b such that AB = BA.

Q4: If A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, C = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Show that A(BC) = (AB)C

## Problems for Practice

Q5: Find "x" such that :

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$$

Q6: If A = $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 2 & -1 & 0 \end{bmatrix}$ . Find AB & BA if exists.

# Problems for Practice

Q7: A factory produces three items A, B and C. Annual sales are given below:

| City | Products | | |
|---|---|---|---|
| | A | B | C |
| Delhi | 5000 | 1000 | 20000 |
| Mumbai | 6000 | 10000 | 8000 |

If the unit price of the items are Rs. 2.50/-, Rs. 1.25/- and Rs. 1.50/- respectively, find the total revenue in each city with the help of matrices.

# Problems for Practice

Q8: If $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 7 \\ 6 & 4 & 8 \end{bmatrix}$ Find $A^2 + 7A + 3I$

Q9: If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ Prove that $A^2 = 4A + 5I$

# Properties of Matrices

Matrices $A$, $B$ and $C$ are conformable,

- $A + B = B + A$ (commutative law)
- $A + (B + C) = (A + B) + C$ (associative law)
- $\lambda(A + B) = \lambda A + \lambda B$, where $\lambda$ is a scalar (distributive law)

# Properties of Matrices

Matrices $A$, $B$ and $C$ are conformable,

- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $A(BC) = (AB)\,C$

However

- $AB \neq BA$ in general
- $AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$
- $AB = AC$ NOT necessarily imply $B = C$

# Transpose of a Matrix

- The matrix obtained by interchanging the rows and columns of a matrix $A$ is called the transpose of $A$ (written as $A^T$ or $A`$ ).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The transpose of $A$ is $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- For a matrix $A = [a_{ij}]$, its transpose $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.

# Practice Problems

Q1: If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$

Find $A' + B'$, $(A+B)'$, $A'B'$

Q2: Verify that $(AB)' = B'A'$ if

If $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}$

Q3: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Show that $(ABC)' = C'B'A'$

# Symmetric & Skew Symmetric Matrices

- A matrix $A$ such that $A^T = A$ is called symmetric, i.e., $a_{ji} = a_{ij}$ for all $i$ and $j$.
- $A + A^T$ must be symmetric. Why?

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

- A matrix $A$ such that $A^T = -A$ is called skew-symmetric, i.e., $a_{ji} = -a_{ij}$ for all $i$ and $j$.
- $A - A^T$ must be skew-symmetric. Why?

# Practice Problems

Q1: Express $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ as a sum of symmetric & skew symmetric matrix.

Q2: If A = $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, then prove that

i) A+A′ is a symmetric matrix

ii) A - A′ is a skew symmetric matrix

iii) AA′ & A′A are symmetric matrices

Q3: Express $\begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix}$ as a sum of symmetric & skew symmetric matrix.

# 1.5 Determinants

## Determinant of order 2

Consider a $2 \times 2$ matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- Determinant of $A$, denoted $|A|$, is a number and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

# 1.5 Determinants

Determinant of order 2

- easy to remember (for order 2 only)..

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

\- +

Example: Evaluate the determinant: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

# Practice Problems

Q1: Find the determinant of :

i) $\begin{bmatrix} 8 & 9 \\ 1 & 7 \end{bmatrix}$

ii) $\begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix}$

# 1.5 Determinants of order 3

Consider an example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Its determinant can be obtained by:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 3(-3) - 6(-6) + 9(-3) = 0$$

You are encouraged to find the determinant by using other rows or columns

# Practice Problems

Find the value of

i) $\begin{vmatrix} 3 & -5 & 4 \\ 7 & 6 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

ii) $\begin{vmatrix} 1 & 4 & 7 \\ -2 & 3 & 4 \\ 1 & 4 & -4 \end{vmatrix}$

# 1.5 Determinants

The following properties are true for determinants of *any* order.

1. If every element of a row (column) is zero, e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1 \times 0 - 2 \times 0 = 0$, then $|A| = 0$.

2. $|A^T| = |A|$

   determinant of a matrix = that of its transpose

3. $|AB| = |A||B|$

# 1.3 Types of matrices

Orthogonal matrix

- A matrix $A$ is called orthogonal if $AA^T = A^TA = I$, i.e., $A^T = A^{-1}$

Example: prove that $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal.

Since, $A^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$. Hence, $AA^T = A^TA = I$.

Can you show the details?

We'll see that orthogonal matrix represents a rotation in fact!

# 1.4 Properties of matrix

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^T = A$ and $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

# Application of Matrices
## (Method of Solving a System of Linear Equations)

- Cramer's Rule

Q1:

# Practice Problems – Cramer's Rule / Determinant Method

Q1: Solve: $2x + 3y = 5$, $3x - 2y = 1$

Q2: Solve: $x + 3y = 2$, $2x + 6y = 7$

Q3: Solve: $2x + 7y = 9$, $4x + 14y = 18$

Q4: Solve: $x + y + z = 20$, $2x + y - z = 23$, $3x + y + z = 46$

Q5: Solve: $2x - 3y - z = 0$, $x + 3y - 2z = 0$, $x - 3y = 0$

Q6: Solve: $x + 4y - 2z = 3$, $3x + y + 5z = 7$, $2x + 3y + z = 5$

Q7: Solve: $x - y + 3z = 6$, $x + 3y - 3z = -4$, $5x + 3y + 3z = 10$

# Practice Problems – Cramer's Rule / Determinant Method

Q8: Find the cost of sugar and wheat per kg if the cost of 7 kg of sugar and 3 kg of wheat is Rs. 34 and cost of 3 kg of sugar and 7 kg of wheat is Rs. 26.

Q9: The perimeter of a triangle is 45 cm. The longest side exceeds the shortest side by 8 cm and the sum of lengths of the longest & shortest side is twice the length of the other side. Find the length of the sides of triangle.

Q10: The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second & third number to three times the first number, we get 12. Find the numbers.

# Introduction

- Cramer's Rule is a method for solving linear simultaneous equations. It makes use of determinants and so a knowledge of these is necessary before proceeding.

- Cramer's Rule relies on determinants

# Coefficient Matrices

- You can use determinants to solve a system of linear equations.
- You use the coefficient matrix of the linear system.
- <u>**Linear System**</u> <u>**Coeff Matrix**</u>

$$ax+by=e$$
$$cx+dy=f$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

# Cramer's Rule for 2x2 System

- Let A be the coefficient matrix
- **Linear System** **Coeff Matrix**

$$ax+by=e$$
$$cx+dy=f$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- If detA $\neq$ 0, then the system has exactly one solution:

and $$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$$

# Key Points

- The denominator consists of the coefficients of variables (x in the first column, and y in the second column).

- The numerator is the same as the denominator, with the constants replacing the coefficients of the variable for which you are solving.

# Example - Applying Cramer's Rule on a System of Two Equations

Solve the system:

- 8x+5y= 2
- 2x-4y= -10

The coefficient matrix is: $\begin{bmatrix} 8 & 5 \\ 2 & -4 \end{bmatrix}$ and $\begin{vmatrix} 8 & 5 \\ 2 & -4 \end{vmatrix} = (-32) - (10) = -42$

So:

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42}$$

and

$$y = \frac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42}$$

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42} = \frac{-8-(-50)}{-42} = \frac{42}{-42} = -1$$

$$y = \frac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42} = \frac{-80-4}{-42} = \frac{-84}{-42} = 2$$

Solution: (-1,2)

# Applying Cramer's Rule on a System of Two Equations

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

$$D_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D}$$

$$\begin{cases} 2x - 3y = -16 \\ 3x + 5y = 14 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 3 & 5 \end{vmatrix} = (2)(5) - (-3)(3) = 10 + 9 = 19$$

$$D_x = \begin{vmatrix} -16 & -3 \\ 14 & 5 \end{vmatrix} = (-16)(5) - (-3)(14) = -80 + 42 = -38$$

$$D_y = \begin{vmatrix} 2 & -16 \\ 3 & 14 \end{vmatrix} = (2)(14) - (3)(-16) = 28 + 48 = 76$$

$$x = \frac{D_x}{D} = \frac{-38}{19} = -2 \quad y = \frac{D_y}{D} = \frac{76}{19} = 4$$

# EVALUATING A 3X3 DETERMINANT
## (EXPANDING ALONG THE TOP ROW)

- Expanding by Minors (little 2x2 determinants)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 3 \\ 1 & 2 & 3 \end{vmatrix} = (1)\begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} - (3)\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (1)(-6) \;\; -(3)(3) \;\; +(-2)(4)$$

$$= -6 \quad -9 \quad -8 = -23$$

# Using Cramer's Rule to Solve a System of Three Equations

Consider the following set of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

# Using Cramer's Rule to Solve a System of Three Equations

The system of equations above can be written in a matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

# Using Cramer's Rule to Solve a System of Three Equations

Define

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; and \; [B] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $D \neq 0$, then the system has a unique solution as shown below (Cramer's Rule).

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

# Using Cramer's Rule to Solve a System of Three Equations

where

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{vmatrix} \qquad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{12} & b_2 & a_{23} \\ a_{13} & b_3 & a_{33} \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ a_{13} & a_{32} & b_3 \end{vmatrix}$$

# Example 1

Consider the following equations:

$$2x_1 - 4x_2 + 5x_3 = 36$$

$$-3x_1 + 5x_2 + 7x_3 = 7$$

$$5x_1 + 3x_2 - 8x_3 = -31$$

$$[A][x] = [B]$$

where

$$[A] = \begin{bmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{bmatrix}$$

# Example 1

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad and \quad [B] = \begin{bmatrix} 36 \\ 7 \\ -31 \end{bmatrix}$$

$$D = \begin{vmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{vmatrix} = -336$$

$$D_1 = \begin{vmatrix} 36 & -4 & 5 \\ 7 & 5 & 7 \\ -31 & 3 & -8 \end{vmatrix} = -672$$

# Example 1

$$D_2 = \begin{vmatrix} 2 & 36 & 5 \\ -3 & 7 & 7 \\ 5 & -31 & -8 \end{vmatrix} = 1008$$

$$D_3 = \begin{vmatrix} 2 & -4 & 36 \\ -3 & 5 & 7 \\ 5 & 3 & -31 \end{vmatrix} = -1344$$

$$x_1 = \frac{D_1}{D} = \frac{-672}{-336} = 2$$

$$x_2 = \frac{D_2}{D} = \frac{1008}{-336} = -3$$

$$x_3 = \frac{D_3}{D} = \frac{-1344}{-336} = 4$$

# Cramer's Rule - 3 x 3

- Consider the 3 equation system below with variables $x$, $y$ and $z$:

$$a_1x + b_1y + c_1z = C_1$$
$$a_2x + b_2y + c_2z = C_2$$
$$a_3x + b_3y + c_3z = C_3$$

# Cramer's Rule - 3 x 3

- The formulae for the values of $x$, $y$ and $z$ are shown below. Notice that all three have the same denominator.

$$x = \frac{\begin{vmatrix} C_1 & b_1 & c_1 \\ C_2 & b_2 & c_2 \\ C_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & C_1 & c_1 \\ a_2 & C_2 & c_2 \\ a_3 & C_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & C_1 \\ a_2 & b_2 & C_2 \\ a_3 & b_3 & C_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

# Example 1

- Solve the system : $3x - 2y + z = 9$
- $x + 2y - 2z = -5$

$x + y - 4z = -2$

$$x = \frac{\begin{vmatrix} 9 & -2 & 1 \\ -5 & 2 & -2 \\ -2 & 1 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{-23}{-23} = 1$$

$$y = \frac{\begin{vmatrix} 3 & 9 & 1 \\ 1 & -5 & -2 \\ 1 & -2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{69}{-23} = -3$$

# Example 1

$$z = \frac{\begin{vmatrix} 3 & -2 & 9 \\ 1 & 2 & -5 \\ 1 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix}} = \frac{0}{-23} = 0$$

The solution is

(1, -3, 0)

# Cramer's Rule

- Not all systems have a definite solution. If the determinant of the coefficient matrix is zero, a solution cannot be found using Cramer's Rule because of division by zero.
- When the solution cannot be determined, one of two conditions exists:
  - The planes graphed by each equation are parallel and there are no solutions.
  - The three planes share one line (like three pages of a book share the same spine) or represent the same plane, in which case there are infinite solutions.

# Minors & Cofactors

Q1: Calculate the minors of all the elements of a given matrix:

$$\begin{bmatrix} 1 & 4 & 7 \\ -2 & 3 & 4 \\ 1 & 4 & -4 \end{bmatrix}$$

Q2: Calculate the cofactors of all the elements of a given matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 4 & 6 & 5 \end{bmatrix}$$

# Adjoint of a Matrix

- It is defined as the transpose of cofactor matrix.

Q1: Find the adjoint of $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

Q2: Find the adjoint of $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

- Imp. Result: $A(\text{Adj. } A) = (\text{Adj. } A).A = |A|.\, I$

Q3: Verify: $A(\text{Adj. } A) = (\text{Adj. } A).A = |A|.\, I_3$

if $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

# Inverse of a Matrix

- Let A be any square matrix of order n. The n-square matrix B of the same order is called the inverse of A if AB = BA = I.
- It is denoted by $A^{-1}$ or $B = A^{-1}$
- The necessary & sufficient condition for finding inverse is that the matrix must be a ***non-singular matrix i.e. its determinant is not equal to zero.***
- $A^{-1} = \frac{Adj.A}{|A|}; \ |A| \neq 0$

# Inverse of a Matrix

Q1: Find the inverse of $\begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix}$ & $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Q2: If A = $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ & B = $\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}$

Verify that $(AB)^{-1} = B^{-1} A^{-1}$

# Matrix Inversion Method - To Solve Simultaneous Equations

Q1: $x + 2y = 4,\ 2x + 5y = 9$

Q2: $x + 2y + z = 7,\ x + 3z = 11,\ 2x - 3y = 1$

Q3: $5x - 6y + 4z = 15,\ 7x + 4y - 3z = 19,\ 2x + y + 6z = 46$

Q4: $2x - y + 3z = 1,\ x + 2y - z = 2,\ 5y - 5z = 3$

Q5: $2x - y + z = 4,\ x + 3y + 2z = 12,\ 3x + 2y + 3z = 16$

Q6: $x - y + 3z = 6,\ x + 3y - 3z = -4,\ 5x + 3y + 3z = 10$

Q7: $x + y + 3z = 6,\ x - 3y - 3z = -4,\ 5x - 3y + 3z = 8$

# Important Questions – Inverse

Q1: If A = $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, Show that $A^2 - 4A + 7I = O$ and hence deduce $A^{-1}$.

Q2: If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, Show that $A^2 - 5A + 7I = O$ and hence find $A^{-1}$.

Q3: If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, Find $A^{-1}$. Also deduce that $A^2 - 4A + 5I = O$.

# 1.3 Types of matrices

## The inverse of a matrix

- If matrices $A$ and $B$ such that $AB = BA = I$, then $B$ is called the inverse of $A$ (symbol: $A^{-1}$); and $A$ is called the inverse of $B$ (symbol: $B^{-1}$).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Show $B$ is the the inverse of matrix $A$.

Ans: Note that $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Can you show the details?