Probability & Distribution Models

Code: PMDS502L

Digital Assignment 1

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Digital Assignment - 1

1) $$f_x(x_1, x_2) = \begin{cases} c(x_1 + 2x_2); \text{if } x_1 = 1, 2 \ \& \ x_2 = 1, 2 \\ 0, \text{ ow.} \end{cases}$$

a) $$c(1+2) + c(1+4) + c(2+2) + c(2+4) = 1$$

or, $3c + 5c + 4c + 6c = 1$

or, $\boxed{c = \frac{1}{18}}$

b) marginal p.m.f of x_1,

$x_1 = 1$: $\frac{1}{18}(1+2) + \frac{1}{18}(1+4)$

$= \frac{1}{6} + \frac{5}{18} = \frac{3+5}{18} = \frac{8}{18} = \frac{4}{9}$

$x_1 = 2$: $\frac{1}{\cancel{18}_9} \times \cancel{4}^2 + \frac{1}{\cancel{18}_3} \times \cancel{6} = \frac{2}{9} + \frac{1}{3} = \frac{2+3}{9} = \frac{5}{9}$

marginal p.m.f of x_2,

$x_2 = 1$: $\frac{1}{18}(1+2) + \frac{1}{18}(2+2)$

$= \frac{\cancel{3}}{\cancel{18}_6} + \frac{\cancel{4}^2}{\cancel{18}_9} = \frac{3+4}{18} = \frac{7}{18}$

$x_2 = 2$: $\frac{1}{18}(1+4) + \frac{1}{18}(2+4)$

$= \frac{5}{18} + \frac{6}{18} = \frac{11}{18}$

c) conditional varience of x_2 given $x_1 = x_1$ $x_1 = 1, 2$

$\therefore x_1 = 1$ $P(x_2 | x_1 = 1) = \frac{\frac{1}{18}(1+2)}{4/9} = \frac{1}{\cancel{6}_2} \times \frac{\cancel{9}^3}{4}$

$= \frac{3}{8}$

$P(x_2 | x_1 = 2) = \frac{\frac{1}{18}(1+4)}{4/9} = \frac{5}{\cancel{18}_2} \times \frac{\cancel{9}}{4}$

$= \frac{5}{8}$

$x_1 = 2$

$$P(x_2 = 1 \mid x_1 = 2) = \frac{\frac{1}{18}(2+2)}{5/9} = \frac{\cancel{4}^{2}}{\cancel{18}_{2}} \times \frac{9}{5} = \frac{2}{5}$$

$$P(x_2 = 2 \mid x_1 = 2) = \frac{\frac{1}{18}(2+4)}{5/9} = \frac{\cancel{6}^{3}}{\cancel{18}} \times \frac{9}{5} = \frac{3}{5}$$

$$\therefore E(x_2 \mid x_1 = 1) = 1 \cdot \frac{3}{8} + 2 \cdot \frac{5}{8} = \frac{3}{8} + \frac{10}{8} = \frac{13}{8}$$

$$E(x_2^2 \mid x_1 = 1) = 1 \cdot \frac{3}{8} + 4 \cdot \frac{5}{8} = \frac{3}{8} + \frac{20}{8} = \frac{23}{8}$$

$$E(x_2 \mid x_1 = 2) = 1 \cdot \frac{2}{5} + 2 \cdot \frac{3}{5} = \frac{2}{5} + \frac{6}{5} = \frac{8}{5}$$

$$E(x_2^2 \mid x_1 = 2) = 1 \cdot \frac{2}{5} + 4 \cdot \frac{3}{5} = \frac{2}{5} + \frac{12}{5} = \frac{14}{5}$$

$\therefore$ Var of x_2 given $x_1 = 1$

$$Var(x_2 \mid x_1 = 1) = E(x_2^2 \mid x_1 = 1) - E(x_2 \mid x_1 = 1)^2$$

$$= \frac{23}{8} - \left(\frac{13}{8}\right)^2$$

$$= \frac{23}{8} - \frac{169}{64}$$

$$= \frac{15}{64}$$

Var of x_2 given $x_1 = 2$

$$Var(x_2 \mid x_1 = 2) = E(x_2^2 \mid x_1 = 2) - E(x_2 \mid x_1 = 2)^2$$

$$= \frac{14}{5} - \frac{64}{25}$$

$$= \frac{70 - 64}{25} = \frac{6}{25}$$

d)

x \ y	1	2
1	3/18	5/18
2	4/18	6/18

i) $P(x_1 < \frac{x_2}{3})$

if $x_2 = 1$ if $x_2 = 2$

$P(x_1 < \frac{1}{3})$ $P(x_1 < \frac{2}{3})$

ii) $P(x_1 = x_2)$

$\therefore$ if $x_1 = 1$ then $x_2 = 1$

$x_1 = 2$ then $x_2 = 2$

$\therefore P(x_1 = x_2) = f_x(1,1) + f_x(2,2)$

$= \frac{1}{18}(1+2) + \frac{1}{18}(2+4)$

$= 3/18 + 6/18 = \frac{9}{18} = \frac{1}{2}$

iii) $P(x_1 \geq \frac{x_2}{2})$

if $x_2 = 1$ then $x_1 = 1$ or 2

if $x_2 = 2$ then $x_1 = 1$ or 2

$\therefore P(x_1 \geq \frac{x_2}{2}) = f(1,1) + f(1,2) + f(2,1) + f(2,2)$

$= 1$

iv) $P(x_1 + x_2 \leq 3)$

if $x_1 = 1$ then $x_2 = 1$ or 2

if $x_1 = 2$ then $x_2 = 1$

$\therefore P(x_1 + x_2 \leq 3)$

$\Rightarrow f(1,1) + f(1,2) + f(2,1)$

$= \frac{3}{18} + \frac{5}{18} + \frac{4}{18}$

$= \frac{12}{18} = \frac{4}{6} = \frac{2}{3}$

f) $f(x_1, x_2)$

$\Rightarrow f(1,1) = \frac{3}{18} \neq f(1) \cdot f(1) = \frac{8}{18} \times \frac{7}{18} = \frac{56}{324}$

$f(1,2) = \frac{5}{18} \neq f(1) \cdot f(2) = \frac{8}{18} \times \frac{11}{18} = \frac{88}{324}$

$f(x_1, x_2) \neq f(x_1) \cdot f(x_2)$

$\therefore$ x_1 & x_2 are ~~inde~~ not independent.

e) $\operatorname{cov}(x,y) = \dfrac{E(x_1 x_2) - E(x_1) E(x_2)}{\sqrt{\operatorname{var}(x_1)} \cdot \sqrt{\operatorname{var}(x_2)}}$

$E(x_1 x_2) = \sum\sum x_1 x_2 \, f(x_1, x_2)$

$= \frac{1}{18}(1 \cdot 3 + 2 \cdot 5 + 2 \cdot 4 + 4 \cdot 6)$

$= \dfrac{3+10+8+24}{18} = \dfrac{45}{18} = \dfrac{5}{2}$

$E(x_1) = 1 \cdot \frac{4}{9} + 2 \cdot \frac{5}{9} = \frac{4}{9} + \frac{10}{9} = \frac{14}{9}$

$E(x_2) = 1 \cdot \frac{7}{18} + 2 \cdot \frac{11}{18} = \frac{29}{18}$

$E(x_1^2) = 1 \cdot \frac{4}{9} + 4 \cdot \frac{5}{9} = \frac{4}{9} + \frac{20}{9} = \frac{\cancel{24}\,8}{\cancel{9}\,3} = \frac{8}{3} = \frac{16}{6}$

$E(x_2^2) = 1 \cdot \frac{7}{18} + 4 \cdot \frac{11}{18} = \frac{51}{18} = \frac{17}{6}$

$\therefore \operatorname{corr}(x,y) = \dfrac{\frac{5}{2} - \left(\frac{28}{18} \cdot \frac{29}{18}\right)}{\sqrt{\frac{16}{6} - \left(\frac{14}{9}\right)^2} \sqrt{\frac{17}{6} - \left(\frac{29}{18}\right)^2}}$

$= \dfrac{\frac{5}{2} - \frac{812}{324}}{\sqrt{\frac{16}{6} - \frac{196}{81}} \sqrt{\frac{17}{6} - \frac{841}{324}}}$

$= \dfrac{\frac{810 - 812}{324}}{\sqrt{\frac{216 - 196}{81}} \sqrt{\frac{918 - 841}{324}}}$

$= -\frac{2}{324} \Big/ \sqrt{\frac{20}{81}} \sqrt{\frac{77}{324}} = -0.025$

2)

$$f(x_1,x_2,x_3) = \frac{1}{2\pi^{3/2}} e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)} \cdot (1+x_1x_2x_3) \cdot e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)}$$

$$f = \frac{1}{2\pi^{3/2}} e^{-(x_1^2+x_2^2 x_3^2)} (1+x_1x_2x_3)$$

$$f(x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_2x_3)\, dx_2\, dx_3$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi^{3/2}} e^{-(x_1^2+x_2^2)} \underbrace{\int_{-\infty}^{\infty} (1+x_1x_2x_3)\, e^{-x_3^2}\, dx_3}_{I_3}$$

$$I_3 = \int_{-\infty}^{\infty} e^{-x_3^2}\, dx_3 + x_1x_2 \int_{-\infty}^{\infty} x_3^1\, e^{-x_3^2}\, dx$$

$$= \sqrt{\pi} + 0 = \sqrt{\pi}$$

$$I_1 = \frac{\pi^{1/2}}{(2\pi)^{3/2}} \cdot e^{-x_1^2} \int_{-\infty}^{\infty} e^{-x_2^2}\, dx_2$$

$$= \frac{\pi e^{-x_1^2}}{(2\pi)^{3/2}} = f(x_1)$$

Similarly,

$$\therefore f(x_2) = \frac{\pi e^{-x_2^2}}{(2\pi)^{3/2}} \qquad f(x_3) = \frac{\pi e^{-x_3^2}}{(2\pi)^{3/2}}$$

$$\therefore f(x_1)\cdot f(x_2)\cdot f(x_3) = \frac{e^{-(x_1^2+x_2^2+x_3^2)}}{2\pi\sqrt{2\pi}} \neq f(x_1,x_2,x_3)$$

$\therefore$ x_1, x_2, x_3 are not independent.

b) Pairwise: $f(x_1, x_2) = \int_{-\infty}^{\infty} f \, dx_3$

$$f(x_1, x_2) = \frac{1}{(2\pi)^{3/2}} e^{-x_1^2 - x_2^2} \underbrace{\int_{-\infty}^{\infty} e^{-x_3^2} (1 + x_1 x_2 x_3) \, dx_3}_{I}$$

$$I = \int_{-\infty}^{\infty} e^{-x_3^2} dx_3 + x_1 x_2 \int_{-\infty}^{\infty} x_3^1 e^{-x_3^2} dx_3 = \sqrt{\pi} + 0 = \sqrt{\pi}$$

So, $f(x_1, x_2) = \frac{\sqrt{\pi}}{(2\pi)^{3/2}} e^{-(x_1^2 + x_2^2)}$

Similarly:

$$f(x_2, x_3) = \frac{\sqrt{\pi}}{(2\pi)^{3/2}} e^{-(x_2^2 + x_3^2)}$$

$$f(x_1, x_3) = \frac{\sqrt{\pi}}{(2\pi)^{3/2}} \cdot e^{-(x_1^2 + x_3^2)}$$

$$f(x_1, x_2) \cdot f(x_2, x_3) \cdot f(x_1, x_3) \cancel{\neq f(x_1, x_2, x_3)}$$

$$= \frac{(\sqrt{\pi})^3}{(2\pi^{3/2})^3} \cdot e^{-x_1^2 - x_2^2 - x_2^2 - x_3^2 - x_1^2 - x_3^2}$$

$$= \frac{\pi\sqrt{\pi}}{(2\pi^{3/2})^2} \cdot e^{-2(x_1^2 + x_2^2 + x_3^2)}$$

$$\neq f(x_1, x_2, x_3)$$

$\therefore$ x_1, x_2 & x_3 are not pairwise independent.

3)

$E(x) = E(y) = 0$

$E(x^2) = E(y^2) = 2$

$corr(x, y) = 1/3$

$Z_1 = \frac{x}{3} + \frac{2y}{3}, \quad Z_2 = \frac{2x}{3} + \frac{y}{3}$

$$corr(Z_1, Z_2) = \frac{cov(Z_1, Z_2)}{\sqrt{Var(Z_1)}\sqrt{var(Z_2)}}$$

$cov(Z_1, Z_2) = cov\left(\frac{x}{3} + \frac{2y}{3}, \frac{2x}{3} + \frac{y}{3}\right)$

$cov(Z_1, Z_2) = \frac{1}{9} cov(x, 2x) + \frac{2}{9} cov(x, y)$

$+ \frac{2}{9} cov(y, x) + \frac{4}{9} cov(y, y)$

$\therefore cov(Z_1 Z_2) = \frac{2}{9} var(x) + \frac{4}{9} cov(x, y) + \frac{4}{9} var(y)$

$Var(x) = E(x^2) - (E(x))^2 \qquad var(y) = E(y^2) - (E(y))^2$

$= 2 - 0 \qquad = 2 - 0$

$= 2 \qquad = 2$

$\therefore cov(x, y) = corr(x, y) \cdot \sqrt{Var(x)} \cdot \sqrt{Var(y)}$

$= 1/3 \cdot \sqrt{2} \cdot \sqrt{2}$

$= 2/3$

$\therefore cov(Z_1 Z_2) = \frac{2}{9} \cdot 2 + \frac{4}{9} \times \frac{2}{3} + \frac{4}{9} \cdot 2$

$= \frac{4}{9} + \frac{8}{27} + \frac{8}{9}$

$= \frac{12 + 8 + 24}{27} = \frac{44}{27}$

Varience of Z1,

$Var(Z_1) = var\left(\frac{x}{3} + \frac{2y}{3}\right)$

$\therefore var(Z_1) = \frac{1}{9} var(x) + \frac{4}{9} var(y) + \frac{4}{9} cov(x, y)$

$\therefore Var(Z_1) = \frac{2}{9} + \frac{8}{9} + \frac{8}{27} = \frac{38}{27}$

varience of Z_2,

$$var(Z_2) = var\left(\frac{2x}{3} + \frac{y}{3}\right)$$

$$var(Z_2) = \frac{4}{9} var(x) + \frac{1}{9} var(y) + \frac{4}{9} cov(x, y)$$

$$= \frac{8}{9} + \frac{2}{9} + \frac{8}{27} = \frac{34}{27}$$

$$\therefore corr(Z_1, Z_2) = \frac{\frac{44}{27}}{\sqrt{\frac{34}{27}} \cdot \sqrt{\frac{34}{27}}} = \frac{44}{34} \text{ Ans}$$