Chapter 4: Generating Functions

This chapter looks at Probability Generating Functions (PGFs) for ***discrete*** random variables. PGFs are useful tools for dealing with sums and limits of random variables. For some stochastic processes, they also have a special role in telling us whether a process will *ever* reach a particular state.

By the end of this chapter, you should be able to:

- find the sum of Geometric, Binomial, and Exponential series;
- know the definition of the PGF, and use it to calculate the mean, variance, and probabilities;
- calculate the PGF for Geometric, Binomial, and Poisson distributions;
- calculate the PGF for a randomly stopped sum;
- calculate the PGF for first reaching times in the random walk;
- use the PGF to determine whether a process will *ever* reach a given state.

4.1 Common sums

1. Geometric Series

$$1 + r + r^2 + r^3 + \dots \;=\; \sum_{x=0}^{\infty} r^x \;=\; \frac{1}{1-r}, \qquad \textit{when } |r| < 1.$$

This formula proves that $\sum_{x=0}^{\infty} \mathbb{P}(X = x) = 1$ when $X \sim \text{Geometric}(p)$:

$$\begin{aligned}
\mathbb{P}(X = x) = p(1-p)^x \quad \Rightarrow \quad \sum_{x=0}^{\infty} \mathbb{P}(X = x) &= \sum_{x=0}^{\infty} p(1-p)^x \\
&= p \sum_{x=0}^{\infty} (1-p)^x \\
&= \frac{p}{1-(1-p)} \qquad \textit{(because } |1-p| < 1\textit{)} \\
&= 1.
\end{aligned}$$

2. Binomial Theorem *For any $p, q \in \mathbb{R}$, and integer n,*

$$(p+q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}.$$

Note that $\binom{n}{x} = \dfrac{n!}{(n-x)!\,x!}$ *(${}^nC_r$ button on calculator.)*

The Binomial Theorem proves that $\sum_{x=0}^{n} \mathbb{P}(X=x) = 1$ when $X \sim \text{Binomial}(n, p)$:
$\mathbb{P}(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$ *for* $x = 0, 1, \ldots, n$*, so*

$$\begin{aligned}\sum_{x=0}^{n} \mathbb{P}(X=x) &= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \Big(p + (1-p)\Big)^n \\ &= 1^n \\ &= 1.\end{aligned}$$

3. Exponential Power Series

For any $\lambda \in \mathbb{R}$*,* $\quad \displaystyle\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}.$

This proves that $\sum_{x=0}^{\infty} \mathbb{P}(X=x) = 1$ when $X \sim \text{Poisson}(\lambda)$:

$\mathbb{P}(X=x) = \dfrac{\lambda^x}{x!} e^{-\lambda}$ *for* $x = 0, 1, 2, \ldots$*, so*

$$\begin{aligned}\sum_{x=0}^{\infty} \mathbb{P}(X=x) = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \\ &= e^{-\lambda} e^{\lambda} \\ &= 1.\end{aligned}$$

Note: Another useful identity is: $\quad e^{\lambda} = \lim_{n \to \infty} \left(1 + \dfrac{\lambda}{n}\right)^n \quad$ *for* $\lambda \in \mathbb{R}$.

4.2 Probability Generating Functions

The **probability generating function (PGF)** is a useful tool for dealing with ***discrete*** random variables taking values $0, 1, 2, \ldots$. Its particular strength is that it gives us an easy way of characterizing the distribution of $X+Y$ when X and Y are independent. In general it is difficult to find the distribution of a sum using the traditional probability function. The PGF transforms a sum into a product and enables it to be handled much more easily.

Sums of random variables are particularly important in the study of stochastic processes, because many stochastic processes are formed from the sum of a sequence of repeating steps: for example, the Gambler's Ruin from Section 2.7.

The name *probability generating function* also gives us another clue to the role of the PGF. The PGF can be used to generate all the probabilities of the distribution. This is generally tedious and is not often an efficient way of calculating probabilities. However, the fact that it *can* be done demonstrates that *the PGF tells us everything there is to know about the distribution.*

Definition: Let X be a discrete random variable taking values in the non-negative integers $\{0, 1, 2, \ldots\}$. The **probability generating function (PGF)** of X is $G_X(s) = \mathbb{E}(s^X)$, *for all* $s \in \mathbb{R}$ *for which the sum converges.*

Calculating the probability generating function

$$G_X(s) = \mathbb{E}\left(s^X\right) = \sum_{x=0}^{\infty} s^x \mathbb{P}(X = x).$$

Properties of the PGF:

1. $G_X(0) = \mathbb{P}(X = 0)$:

$$\begin{aligned} G_X(0) &= 0^0 \times \mathbb{P}(X = 0) + 0^1 \times \mathbb{P}(X = 1) + 0^2 \times \mathbb{P}(X = 2) + \ldots \\ \therefore \quad G_X(0) &= \mathbb{P}(X = 0). \end{aligned}$$

2. $G_X(1) = 1$: $\quad G_X(1) = \sum_{x=0}^{\infty} 1^x \mathbb{P}(X = x) = \sum_{x=0}^{\infty} \mathbb{P}(X = x) = 1.$

Example 1: **Binomial Distribution**

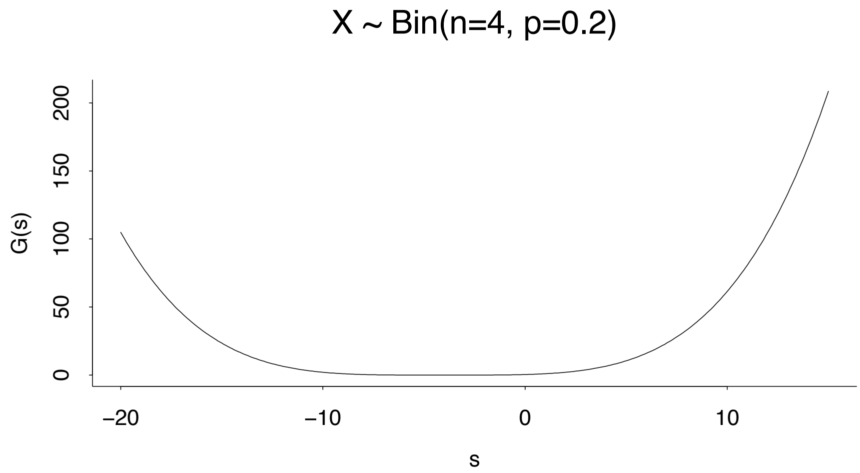
Let $X \sim \text{Binomial}(n, p)$, so $\mathbb{P}(X = x) = \binom{n}{x} p^x q^{n-x}$ for $x = 0, 1, \ldots, n$.

$$\begin{aligned} G_X(s) &= \sum_{x=0}^{n} s^x \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (ps)^x q^{n-x} \\ &= (ps+q)^n \quad \textit{by the Binomial Theorem: true for all } s. \end{aligned}$$

Thus $G_X(s) = (ps+q)^n$ *for all* $s \in \mathbb{R}$.

Check $G_X(0)$:

$$\begin{aligned} G_X(0) &= (p \times 0 + q)^n \\ &= q^n \\ &= \mathbb{P}(X = 0). \end{aligned}$$



Check $G_X(1)$:

$$\begin{aligned} G_X(1) &= (p \times 1 + q)^n \\ &= (1)^n \\ &= 1. \end{aligned}$$

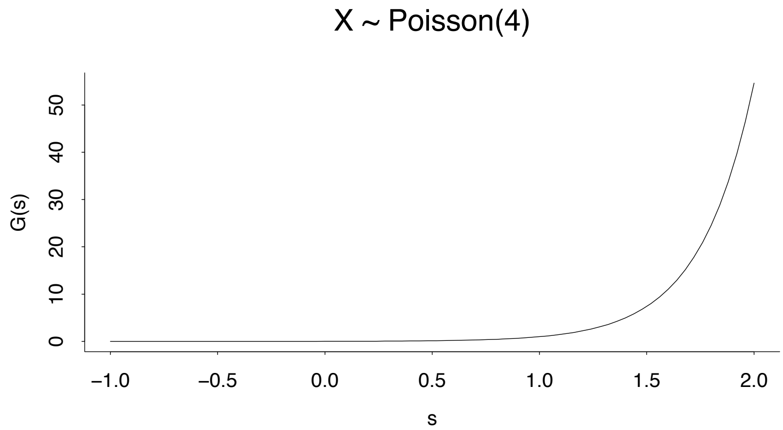
Example 2: Poisson Distribution

Let $X \sim \text{Poisson}(\lambda)$, so $\mathbb{P}(X = x) = \dfrac{\lambda^x}{x!}e^{-\lambda}$ for $x = 0, 1, 2, \ldots$.

$$G_X(s) = \sum_{x=0}^{\infty} s^x \frac{\lambda^x}{x!} e^{-\lambda} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda s)^x}{x!}$$

$$= e^{-\lambda} e^{(\lambda s)} \quad \textit{for all } s \in \mathbb{R}.$$

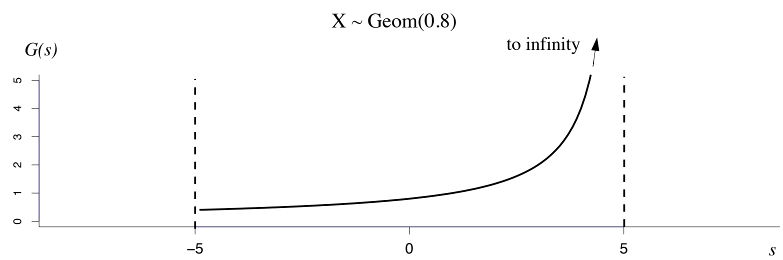
Thus $\quad G_X(s) = e^{\lambda(s-1)}$ *for all* $s \in \mathbb{R}$.



Example 3: Geometric Distribution

Let $X \sim \text{Geometric}(p)$, so $\mathbb{P}(X = x) = p(1-p)^x = pq^x$ for $x = 0, 1, 2, \ldots$, where $q = 1 - p$.

$$G_X(s) = \sum_{x=0}^{\infty} s^x p q^x$$



$$= p \sum_{x=0}^{\infty} (qs)^x$$

$$= \frac{p}{1 - qs} \quad \textit{for all } s \textit{ such that } |qs| < 1.$$

Thus $\quad G_X(s) = \dfrac{p}{1 - qs}$ *for* $|s| < \dfrac{1}{q}$.

4.3 Using the probability generating function to calculate probabilities

The probability generating function gets its name because the power series can be expanded and differentiated to reveal the individual probabilities. Thus, ***given only the PGF*** $G_X(s) = \mathbb{E}(s^X)$***, we can recover all probabilities*** $\mathbb{P}(X = x)$.

For shorthand, write $p_x = \mathbb{P}(X = x)$. Then

$$G_X(s) = \mathbb{E}(s^X) \; = \; \sum_{x=0}^{\infty} p_x s^x = p_0 + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + \ldots$$

Thus $p_0 = \mathbb{P}(X = 0) = G_X(0)$.

First derivative: $\quad G_X'(s) \; = \; p_1 + 2p_2 s + 3p_3 s^2 + 4p_4 s^3 + \ldots$

Thus $p_1 = \mathbb{P}(X = 1) = G_X'(0)$.

Second derivative: $\quad G_X''(s) \; = \; 2p_2 + (3 \times 2)p_3 s + (4 \times 3)p_4 s^2 + \ldots$

Thus $p_2 = \mathbb{P}(X = 2) = \frac{1}{2} G_X''(0)$.

Third derivative: $\quad G_X'''(s) \; = \; (3 \times 2 \times 1)p_3 + (4 \times 3 \times 2)p_4 s + \ldots$

Thus $p_3 = \mathbb{P}(X = 3) = \frac{1}{3!} G_X'''(0)$.

In general:

$$\boxed{p_n = \mathbb{P}(X = n) = \left(\frac{1}{n!}\right) G_X^{(n)}(0) = \left(\frac{1}{n!}\right) \frac{d^n}{ds^n} \left(G_X(s)\right)\bigg|_{s=0}.}$$

Example: Let X be a discrete random variable with PGF $G_X(s) = \frac{s}{5}(2+3s^2)$. Find the distribution of X.

$$\begin{aligned}
G_X(s) &= \frac{2}{5}s + \frac{3}{5}s^3 : & G_X(0) &= \mathbb{P}(X=0) = 0. \\
G'_X(s) &= \frac{2}{5} + \frac{9}{5}s^2 : & G'_X(0) &= \mathbb{P}(X=1) = \frac{2}{5}. \\
G''_X(s) &= \frac{18}{5}s : & \frac{1}{2}G''_X(0) &= \mathbb{P}(X=2) = 0. \\
G'''_X(s) &= \frac{18}{5} : & \frac{1}{3!}G'''_X(0) &= \mathbb{P}(X=3) = \frac{3}{5}. \\
G_X^{(r)}(s) &= 0 \; \forall r \geq 4 : & \frac{1}{r!}G_X^{(r)}(s) &= \mathbb{P}(X=r) = 0 \; \forall r \geq 4.
\end{aligned}$$

Thus

$$X = \begin{cases} 1 \textit{ with probability } 2/5, \\ 3 \textit{ with probability } 3/5. \end{cases}$$

Uniqueness of the PGF

The formula $p_n = \mathbb{P}(X=n) = \left(\frac{1}{n!}\right) G_X^{(n)}(0)$ shows that the whole sequence of probabilities $p_0, p_1, p_2, \ldots$ is determined by the values of the PGF and its derivatives at $s = 0$. It follows that the PGF specifies a ***unique*** set of probabilities.

Fact: If two power series agree on any interval containing 0, however small, then all terms of the two series are equal.

Formally: let $A(s)$ and $B(s)$ be PGFs with $A(s) = \sum_{n=0}^{\infty} a_n s^n$, $B(s) = \sum_{n=0}^{\infty} b_n s^n$. If there exists some $R' > 0$ such that $A(s) = B(s)$ for all $-R' < s < R'$, then $a_n = b_n$ for all n.

Practical use: If we can show that two random variables have the same PGF in some interval containing 0, then we have shown that *the two random variables have the same distribution.*

Another way of expressing this is to say that *the PGF of X tells us everything there is to know about the distribution of X.*

4.4 Expectation and moments from the PGF

As well as calculating probabilities, we can also use the PGF to calculate the moments of the distribution of X. The moments of a distribution are *the mean, variance, etc.*

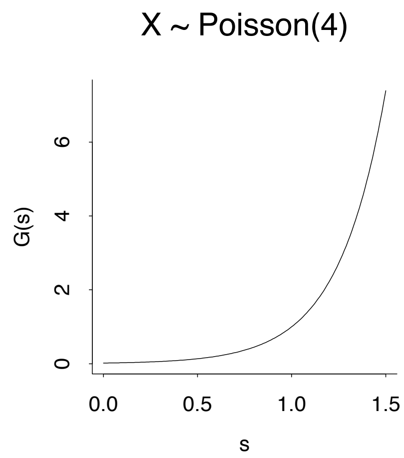
Theorem 4.4: Let X be a discrete random variable with PGF $G_X(s)$. Then:

1. $\mathbb{E}(X) = G'_X(1)$.

2. $\mathbb{E}\Big\{X(X-1)(X-2)\ldots(X-k+1)\Big\} = G_X^{(k)}(1) = \left.\dfrac{d^k\, G_X(s)}{ds^k}\right|_{s=1}$.
   (This is the kth factorial moment of X.)

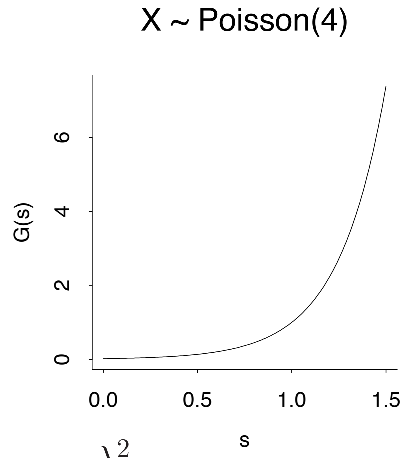
Proof: (Sketch: see Section 4.8 for more details)

1.
$$\begin{aligned} G_X(s) &= \sum_{x=0}^{\infty} s^x\, p_x, \\ \text{so} \quad G'_X(s) &= \sum_{x=0}^{\infty} x s^{x-1} p_x \\ \Rightarrow \quad G'_X(1) &= \sum_{x=0}^{\infty} x p_x = \mathbb{E}(X) \end{aligned}$$



2.
$$\begin{aligned} G_X^{(k)}(s) = \frac{d^k\, G_X(s)}{ds^k} &= \sum_{x=k}^{\infty} x(x-1)(x-2)\ldots(x-k+1)s^{x-k}p_x \\ \text{so} \quad G_X^{(k)}(1) &= \sum_{x=k}^{\infty} x(x-1)(x-2)\ldots(x-k+1)p_x \\ &= \mathbb{E}\Big\{X(X-1)(X-2)\ldots(X-k+1)\Big\}. \qquad \square \end{aligned}$$

Example: Let $X \sim \text{Poisson}(\lambda)$. The PGF of X is $G_X(s) = e^{\lambda(s-1)}$. Find $\mathbb{E}(X)$ and $\text{Var}(X)$.



Solution:

$$\begin{aligned} G'_X(s) &= \lambda e^{\lambda(s-1)} \\ \Rightarrow \quad \mathbb{E}(X) &= G'_X(1) = \lambda. \end{aligned}$$

For the variance, consider

$$\mathbb{E}\Big\{X(X-1)\Big\} = G''_X(1) = \lambda^2 e^{\lambda(s-1)}|_{s=1} = \lambda^2.$$

So

$$\begin{aligned} \mathit{Var}(X) &= \mathbb{E}(X^2) - (\mathbb{E}X)^2 \\ &= \mathbb{E}\Big\{X(X-1)\Big\} + \mathbb{E}X - (\mathbb{E}X)^2 \\ &= \lambda^2 + \lambda - \lambda^2 \\ &= \lambda. \end{aligned}$$

4.5 Probability generating function for a sum of independent r.v.s

One of the PGF's greatest strengths is that it turns a sum into a product:

$$\mathbb{E}\left(s^{(X_1+X_2)}\right) = \mathbb{E}\left(s^{X_1} s^{X_2}\right).$$

This makes the PGF useful for finding the probabilities and moments of *a sum of independent random variables.*

Theorem 4.5: Suppose that $X_1, \ldots, X_n$ are ***independent*** random variables, and let $Y = X_1 + \ldots + X_n$. Then

$$G_Y(s) = \prod_{i=1}^{n} G_{X_i}(s).$$

Proof:

$$\begin{aligned} G_Y(s) &= \mathbb{E}(s^{(X_1+\ldots+X_n)}) \\ &= \mathbb{E}(s^{X_1}s^{X_2}\ldots s^{X_n}) \\ &= \mathbb{E}(s^{X_1})\mathbb{E}(s^{X_2})\ldots\mathbb{E}(s^{X_n}) \end{aligned}$$

(because $X_1, \ldots, X_n$ are independent)

$$= \prod_{i=1}^{n} G_{X_i}(s). \quad \text{as required.} \quad \square$$

Example: Suppose that X and Y are independent with $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Find the distribution of $X+Y$.

Solution:

$$\begin{aligned} G_{X+Y}(s) &= G_X(s) \cdot G_Y(s) \\ &= e^{\lambda(s-1)}e^{\mu(s-1)} \\ &= e^{(\lambda+\mu)(s-1)}. \end{aligned}$$

But this is the PGF of the Poisson$(\lambda+\mu)$ distribution. So, by the uniqueness of PGFs, $X+Y \sim$ Poisson$(\lambda+\mu)$.

4.6 ~~Randomly stopped sum~~

~~Remember the randomly stopped sum model from Section 3.4. A random number N of events occur, and each event i has associated with it a cost or reward X_i. The question is to find the distribution of the total cost or reward: $T_N = X_1 + X_2 + \ldots + X_N$. T_N is called a *randomly stopped sum* because it has a random number of terms.~~

~~***Example:*** Cash machine model. N customers arrive during the day. Customer i withdraws amount X_i. The total amount withdrawn during the day is $T_N = X_1 + \ldots + X_N$.~~

$$
\begin{aligned}
&= \sum_{n=t}^{\infty} \binom{n}{t} p^t (1-p)^{n-t} (1-\theta)\theta^n \\
&= (1-\theta)\left(\frac{p}{1-p}\right)^t \sum_{n=t}^{\infty} \binom{n}{t} \Big[\theta(1-p)\Big]^n \qquad (\star\star) \\
&= \ldots ?
\end{aligned}
$$

~~As it happens, we can evaluate the sum in ($\star\star$) using the fact that Negative Binomial probabilities sum to 1. You can try this if you like, but it is quite tricky. [Hint: use the Negative Binomial ($t+1, 1-\theta(1-p)$) distribution.]~~

~~Overall, we obtain the same answer that $T \sim \text{Geometric}\left(\frac{1-\theta}{1-\theta+\theta p}\right)$, but hopefully you can see why the PGF is so useful.~~

~~*Without the PGF, we have two major difficulties:*~~

1. ~~*Writing down* $\mathbb{P}(T = t \,|\, N = n)$;~~
2. ~~*Evaluating the sum over* n *in* ($\star\star$).~~

~~For a general problem, both of these steps might be too difficult to do without a computer. The PGF has none of these difficulties, and even if $G_T(s)$ does not simplify readily, it still tells us everything there is to know about the distribution of T.~~

4.7 Summary: Properties of the PGF

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Used for:	Discrete r.v.s with values 0, 1, 2, ...	
Moments:	$\mathbb{E}(X) = G_X'(1)$	$\mathbb{E}\Big\{X(X-1)\ldots(X-k+1)\Big\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0)$	
Sums:	$G_{X+Y}(s) = G_X(s)G_Y(s)$ for independent X, Y	