TRANSFORMATION OF FUNCTION OF A RANDOM VARIABLE

UNIVARIATE TRANSFORMATIONS

TRANSFORMATION OF RANDOM VARIABLES

- If X is an rv with cdf $F(x)$, then $Y=g(X)$ is also an rv.

- If we write $y=g(x)$, the function $g(x)$ defines a mapping from the original sample space of X, S, to a new sample space, Y, the sample space of the rv Y.

$$g(x): S \rightarrow Y$$

TRANSFORMATION OF RANDOM VARIABLES

- Let y=g(x) define a 1-to-1 transformation. That is, the equation y=g(x) can be solved uniquely: $x = g^{-1}(y)$
- Ex: Y=X-1 → X=Y+1 1-to-1
- Ex: $Y=X^2$ → X=± sqrt(Y) not 1-to-1
- When transformation is not 1-to-1, find disjoint partitions of S for which transformation is 1-to-1.

TRANSFORMATION OF RANDOM VARIABLES

If X is a discrete r.v. then S is countable. The sample space for Y=g(X) is $\mathcal{Y}=\{y: y=g(x), x\in S\}$, *also countable. The pmf for Y is*

$$f_Y(y) = P(Y=y) = \sum_{x\in g^{-1}(y)} P(X=x) = \sum_{x\in g^{-1}(y)} f(x)$$

Example

- Let X~GEO(p). That is, $f(x) = p(1-p)^{x-1} \text{ for } x = 1,2,3,\ldots$
- Find the p.m.f. of Y=X-1
- Solution: X=Y+1

$$f_Y(y) = f_X(y+1) = p(1-p)^y \text{ for } y = 0,1,2,\ldots$$

- P.m.f. of the number of failures before the first success
- Recall: X~GEO(p) is the p.m.f. of number of Bernoulli trials required to get the first success

Example

- Let X be an rv with pmf

$$p(x)=\begin{cases} 1/5, x=-2 \\ 1/6, x=-1 \\ 1/5, x=0 \\ 1/15, x=1 \\ 11/30, x=2 \end{cases}$$

$Let\ Y=X^2. \longrightarrow S=\{-2,-1,0,1,2\} \longrightarrow Y=\{0,1,4\}$

$$p(y)=\begin{cases} 1/5, y=0 \\ 7/30, y=1 \\ 17/30, y=4 \end{cases}$$

FUNCTIONS OF CONTINUOUS RANDOM VARIABLE

- Let X be an rv of the continuous type with pdf f. Let $y=g(x)$ be differentiable for all x and non-zero. Then, $Y=g(X)$ is also an rv of the continuous type with pdf given by

$$h(y)=\begin{cases} f(g^{-1}(y))\,\left|\dfrac{d}{dy}g^{-1}(y)\right| & for\ y\in\psi \\ 0 & o.w. \end{cases}$$

FUNCTIONS OF CONTINUOUS RANDOM VARIABLE

- **Example:** Let X have the density

$$f(x)=\begin{cases}1, & 0<x<1\\ 0, & \text{otherwise}\end{cases}$$

Let $Y=e^X$.

$$X=g^{-1}(y)=\log Y \rightarrow dx=(1/y)dy.$$

$$h(y)=1.\left|\frac{1}{y}\right|, 0<\log y<1$$

$$h(y)=\begin{cases}\frac{1}{y}, & 1<y<e\\ 0, & \text{otherwise}\end{cases}$$

FUNCTIONS OF CONTINUOUS RANDOM VARIABLE

- **Example:** Let X have the density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < x < \infty.$$

Let $Y=X^2$. Find the pdf of Y.

CDF method

- Example: Let $F(x) = 1 - e^{-2x}$ for $x > 0$

Consider $Y = e^{X}$. What is the p.d.f. of Y?

- Solution:

$$F_Y(y) = P(Y \le y) = P(e^{X} \le y) = P(X \le \ln y)$$

$$= F_X(\ln y) = 1 - y^{-2} \text{ for } y > 1$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = 2y^{-3} \text{ for } y > 1$$

CDF method

- Example: Consider a continuous r.v. X, and $Y=X^2$. Find p.d.f. of Y.
- Solution:

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = f_X(\sqrt{y})\frac{d}{dy}(\sqrt{y}) - f_X(-\sqrt{y})\frac{d}{dy}(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})]$$