Introduction to Moment Generating Functions (MGF)

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Common Sums: Geometric Series

One of the most common series we encounter is the geometric series. It is given by:

$$1 + r + r^2 + r^3 + \cdots = \sum_{x=0}^{\infty} r^x$$

The sum of this infinite geometric series is:

$$\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}, \quad \text{for } |r| < 1$$

This formula is useful for many applications in probability, statistics, and mathematical analysis.

Binomial Theorem

The Binomial Theorem allows us to expand powers of a binomial expression $(p+q)^n$ as a sum. For any $p, q \in \mathbb{R}$ and a non-negative integer n, we have:

$$(p+q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

Here, the binomial coefficient $\binom{n}{x}$ is given by:

$$\binom{n}{x} = \frac{n!}{(n-x)!x!}$$

The binomial theorem is used in many areas, such as probability, algebra, and combinatorics, for expanding expressions involving binomials.

Exponential Power Series

The exponential function can be represented as an infinite power series. For any $\lambda \in \mathbb{R}$, the series is given by:

$$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$$

This is known as the exponential power series, and it converges for all real values of λ.

Exponential Function as a Limit

Another useful identity for the exponential function is its expression as a limit:

$$e^{\lambda} = \lim_{n \to \infty} \left(1 + \frac{\lambda}{n}\right)^{n}$$

for $\lambda \in \mathbb{R}$.

Moments and Central Moments

Definition: The nth moment of a random variable X is defined as:

$$\mu_n = \mathbb{E}[X^n]$$

The nth central moment of X is defined as:

$$\mu'_n = \mathbb{E}[(X - \mathbb{E}[X])^n]$$

Examples:

- First moment: $\mathbb{E}[X]$ (the mean of X)
- Second central moment: $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$ (the variance of X)

Moment Generating Function (MGF)

Definition: The *moment generating function* (MGF) of a random variable X is a function $M_X(s)$ defined as:

$$M_X(s) = \mathbb{E}[e^{sX}]$$

Existence of MGF: The MGF exists if there is a positive constant a such that $M_X(s)$ is finite for all $s \in [-a, a]$.

Example: MGF of a Discrete Random Variable

Let X be a discrete random variable with the probability mass function (PMF):

$$P_X(k) = \begin{cases} \frac{1}{3} & \text{if } k = 1 \\ \frac{2}{3} & \text{if } k = 2 \end{cases}$$

The MGF of X is given by:

$$M_X(s) = \mathbb{E}[e^{sX}] = \sum_k e^{sk} P_X(k)$$

Substituting the values of k and $P_X(k)$:

$$M_X(s) = e^{s\cdot 1} \cdot \frac{1}{3} + e^{s\cdot 2} \cdot \frac{2}{3}$$

$$M_X(s) = \frac{1}{3}e^{s} + \frac{2}{3}e^{2s}$$

Example: MGF of Uniform(0,1) Random Variable

Let $Y \sim \text{Uniform}(0,1)$. The probability density function (PDF) of Y is:

$$f_Y(y) = 1, \quad 0 \leq y \leq 1$$

The MGF of Y is given by:

$$M_Y(s) = \mathbb{E}[e^{sY}] = \int_0^1 e^{sy} f_Y(y)\, dy = \int_0^1 e^{sy}\, dy$$

Evaluating the integral:

$$M_Y(s) = \left[\frac{e^{sy}}{s}\right]_0^1 = \frac{e^s - 1}{s}, \quad s \neq 0$$

For $s = 0$, we have:

$$M_Y(0) = \mathbb{E}[e^{0 \cdot Y}] = \mathbb{E}[1] = 1$$

Conclusion: The MGF $M_Y(s) = \frac{e^s - 1}{s}$ is well-defined for all $s \in \mathbb{R}$.

Why is the MGF Useful?

There are two primary reasons why the moment generating function (MGF) is a valuable tool in probability and statistics:

1. **MGF gives all moments of a random variable:** The MGF $M_X(s)$ is called the moment generating function because it allows us to compute all moments of the random variable X. The n-th moment is obtained by differentiating the MGF n times and evaluating it at $s = 0$.
2. **MGF uniquely determines the distribution:** If two random variables have the same MGF, they must have the same distribution. This means that the MGF, if it exists, fully characterizes the distribution of a random variable.

These properties make MGFs particularly useful in identifying distributions and analyzing sums of random variables.

Finding Moments from MGF

Finding Moments from the MGF:

Recall the Taylor series expansion for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

For e^{sX}, we can similarly write:

$$e^{sX} = \sum_{k=0}^{\infty} \frac{(sX)^k}{k!} = \sum_{k=0}^{\infty} \frac{X^k s^k}{k!}$$

Thus, the moment generating function (MGF) becomes:

$$M_X(s) = \mathbb{E}[e^{sX}] = \sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k] s^k}{k!}$$

Conclusion: The k-th moment $\mathbb{E}[X^k]$ is the coefficient of $\frac{s^k}{k!}$ in the Taylor series expansion of $M_X(s)$.

Example: Moments of Uniform(0,1) Using MGF

Let $Y \sim \text{Uniform}(0,1)$. To find $\mathbb{E}[Y^k]$ using the moment generating function $M_Y(s)$, recall from the earlier example:

$$M_Y(s) = \frac{e^s - 1}{s}$$

Now, expand $M_Y(s)$ using the Taylor series for e^s:

$$M_Y(s) = \frac{1}{s}\left(\sum_{k=0}^{\infty} \frac{s^k}{k!} - 1\right) = \frac{1}{s}\sum_{k=1}^{\infty} \frac{s^k}{k!}$$

Simplifying:

$$M_Y(s) = \sum_{k=1}^{\infty} \frac{s^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{1}{(k+1)} \frac{s^k}{k!}$$

From this series expansion, the coefficient of $\frac{s^k}{k!}$ is $\frac{1}{k+1}$.
Therefore, the k-th moment of Y is: $\mathbb{E}[Y^k] = \frac{1}{k+1}$

Moments from the MGF via Derivatives

Recall from calculus that the coefficient of $\frac{s^k}{k!}$ in the Taylor series expansion of the MGF $M_X(s)$ is obtained by taking the k-th derivative of $M_X(s)$ and evaluating it at $s = 0$.

Thus, we can write the k-th moment as:

$$\mathbb{E}[X^k] = \frac{d^k}{ds^k} M_X(s) \Big|_{s=0}$$

In general, the MGF $M_X(s)$ is expressed as:

$$M_X(s) = \sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k] s^k}{k!}$$

Therefore, all moments $\mathbb{E}[X^k]$ of the random variable X can be computed by differentiating the MGF $M_X(s)$ and evaluating the derivatives at $s = 0$:

$$\mathbb{E}[X^k] = \frac{d^k}{ds^k} M_X(s) \Big|_{s=0}$$

Example: MGF and Moments of Exponential Distribution

Let $X \sim \text{Exponential}(\lambda)$. The PDF of X is:

$$f_X(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0$$

where λ is the prarameter.
The MGF of X is given by:

$$M_X(s) = \mathbb{E}[e^{sX}] = \int_0^\infty \lambda e^{-\lambda x} e^{sx} dx$$

Simplifying the integrand:

$$M_X(s) = \int_0^\infty \lambda e^{-(\lambda - s)x} dx$$

Solving this integral:

$$M_X(s) = \left[\frac{-\lambda}{\lambda - s} e^{-(\lambda - s)x} \right]_0^\infty, \quad \text{for } s < \lambda$$

$$M_X(s) = \frac{\lambda}{\lambda - s}, \quad \text{for } s < \lambda$$

Moments of Exponential Distribution

The MGF $M_X(s)$ is:

$$M_X(s) = \frac{\lambda}{\lambda - s} = \frac{1}{1 - \frac{s}{\lambda}}$$

Using the geometric series expansion:

$$M_X(s) = \sum_{k=0}^{\infty} \left(\frac{s}{\lambda}\right)^k = \sum_{k=0}^{\infty} \frac{s^k}{\lambda^k} = \sum_{k=0}^{\infty} \frac{k!}{\lambda^k} \frac{s^k}{k!}$$

From the expansion, we identify the k-th moment $\mathbb{E}[X^k]$ as the coefficient of $\frac{s^k}{k!}$:

$$\mathbb{E}[X^k] = \frac{k!}{\lambda^k}, \quad \text{for } k = 0, 1, 2, \ldots$$

Example: MGF of Poisson Distribution

Let $X \sim \text{Poisson}(\lambda)$. The probability mass function (PMF) of X is:

$$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \text{for } k = 0, 1, 2, \dots$$

The moment generating function (MGF) of X is given by:

$$M_X(s) = \mathbb{E}[e^{sX}] = \sum_{k=0}^{\infty} e^{sk} \frac{e^{-\lambda}\lambda^k}{k!}$$

Simplifying this expression:

$$M_X(s) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^s)^k}{k!}$$

Using the Taylor series for e^x, we get:

$$M_X(s) = e^{-\lambda} e^{\lambda e^s} = e^{\lambda(e^s - 1)}$$

Example: MGF of Poisson Distribution

Therefore, the MGF of X is:

$$M_X(s) = e^{\lambda(e^s - 1)}, \quad \text{for all } s \in \mathbb{R}.$$

Theorem: Uniqueness of MGF

The Moment Generating Function (MGF) uniquely determines the distribution of a random variable. This is stated precisely in the following theorem:

Theorem

Consider two random variables X and Y. Suppose that there exists a positive constant c such that the MGFs of X and Y are finite and identical for all values of s in the interval $[-c, c]$. Then,

$$F_X(t) = F_Y(t), \quad \textit{for all } t \in \mathbb{R}.$$

This theorem tells us that if two random variables have the same MGF, their distributions are identical.

Example: Finding the Distribution Using MGF

Suppose we are given the moment generating function (MGF) of a random variable X as:

$$M_X(s) = \frac{2}{2-s}, \quad \text{for } s \in (-2, 2).$$

Solution:
We recognize that this is the MGF of an exponential distribution with rate parameter $\lambda = 2$. (Recall from the previous Example, the MGF of an exponential random variable with rate λ is $M_X(s) = \frac{\lambda}{\lambda - s}$.)
Thus, we conclude:

$$X \sim \text{Exponential}(2).$$

Therefore, X is an exponential random variable with parameter $\lambda = 2$.

Sum of Independent Random Variables

Suppose $X_1, X_2, \ldots, X_n$ are n independent random variables, and define the random variable:

$$Y = X_1 + X_2 + \cdots + X_n.$$

The moment generating function (MGF) of Y is:

$$M_Y(s) = \mathbb{E}[e^{sY}] = \mathbb{E}[e^{s(X_1+X_2+\cdots+X_n)}]$$

By the independence of $X_1, X_2, \ldots, X_n$, we have:

$$M_Y(s) = \mathbb{E}[e^{sX_1}]\mathbb{E}[e^{sX_2}]\cdots\mathbb{E}[e^{sX_n}]$$

This simplifies to:

$$M_Y(s) = M_{X_1}(s)M_{X_2}(s)\cdots M_{X_n}(s).$$

Conclusion: If $X_1, X_2, \ldots, X_n$ are independent random variables, the MGF of their sum is the product of their individual MGFs:

$$M_{X_1+X_2+\cdots+X_n}(s) = M_{X_1}(s)M_{X_2}(s)\cdots M_{X_n}(s).$$

MGF of Binomial Distribution

Suppose $X \sim \text{Binomial}(n, p)$. The moment generating function (MGF) of X can be found as follows:

Solution: A binomial random variable X can be viewed as the sum of n independent and identically distributed (i.i.d.) Bernoulli random variables X_i, where $X_i \sim \text{Bernoulli}(p)$. Thus:

$$X = X_1 + X_2 + \cdots + X_n$$

The MGF of X is:

$$M_X(s) = M_{X_1}(s) M_{X_2}(s) \cdots M_{X_n}(s) = (M_{X_1}(s))^n$$

For a Bernoulli random variable X_1:

$$M_{X_1}(s) = \mathbb{E}[e^{sX_1}] = pe^s + (1 - p)$$

Therefore:

$$M_X(s) = (pe^s + (1 - p))^n$$

Sum of Two Independent Binomial Random Variables

Prove that if $X \sim \text{Binomial}(m, p)$ and $Y \sim \text{Binomial}(n, p)$ are independent, then $X + Y \sim \text{Binomial}(m + n, p)$.

Solution: Compute the MGF of X and Y:

$$M_X(s) = (pe^s + (1-p))^m$$

$$M_Y(s) = (pe^s + (1-p))^n$$

Since X and Y are independent, the MGF of $X + Y$ is:

$$M_{X+Y}(s) = M_X(s) \cdot M_Y(s)$$

$$M_{X+Y}(s) = (pe^s + (1-p))^m \cdot (pe^s + (1-p))^n = (pe^s + (1-p))^{m+n}$$

This is the MGF of a Binomial $(m + n, p)$ random variable.
Thus, $X + Y \sim \text{Binomial}(m + n, p)$.