Part 1

Problem 1. (3 points) *Marbles* is a solitaire game played on an undirected graph G, where each vertex has zero or more marbles. A single move in this game consists of removing two marbles from a vertex v and adding one marble to an arbitrary neighbor of v. Note that the vertex v must have at least two marbles on it before the move.

The **Marbles Elimination** problem asks, given a graph $G = (V, E)$ and a marble count $p(v)$ for each vertex v, whether is there a sequence of valid moves that removes all but one marble.

Prove that determining the answer to the **Marbles Elimination** question is as hard as finding a Hamiltonian cycle in a graph, i.e, a cycle that visits every vertex exactly once.

In other words, describe a reduction from the Hamiltonian cycle problem to the problem of **Marbles Elimination**.

Solution

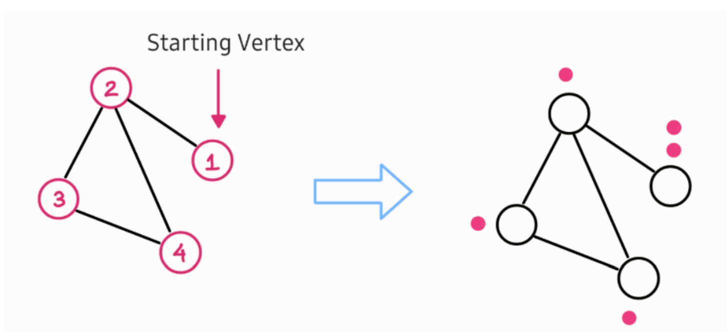
Problem – Marbles Elimination, that is, given a graph G = (V, E) and a marble count p(v) for each vertex v, whether is there a sequence of valid moves that removes all but one marble.

To Prove – Marbles Elimination question is as hard as finding a Hamiltonian cycle in a graph.

Let G with n vertices be the graph given in an undirected Hamiltonian Path Problem. The reduction from Hamiltonian Path Problem to Marbles Elimination problem can be deduced using the following strategy –

Let that starting vertex of Marbles Elimination problem be v_1. Let p(v) be the number of marbles at vertex v. The number of marbles in each vertex can be formulated as

- If vertex v is the starting vertex, then p(v) = 2
- Otherwise, p(v) = 1



Since there are n possible starting vertices, it's sufficient to check if there's a Hamiltonian Path for each of these n possible starting vertices.

Lemma –

There is a Hamiltonian Path if and only if any of these Marbles Elimination problems is true.

<u>Claim 1 – There exist a sequence of Marbles Elimination moves removing all but one marble if and only if there exists a Hamiltonian Path.</u>

Let us assume that there exist a Hamiltonian Path $v_1 \rightarrow v_2 \rightarrow .... \rightarrow v_n$. Now, let's consider the following marble elimination moves (v_1, v_2) (v_2, v_3) ... (v_{n-1}, v_n).

Now, it can be observed that for any v_i where $1\leq i < n$, since (v_i, v_{i+1}) is in the Hamiltonian Path, then (v_i, v_{i+1}) must be an edge in G.

Since (v_i, v_{i+1}) is the first marble elimination move that takes away marbles from v_i, and since v_i starts with at least one marble, then v_i has all its starting marbles when attempt to do that marble elimination move. In the case that i = 1, then v_i has enough marbles to do the move since it starts with two marbles.

In the case that $1 < i < n$, then (v_{i-1}, v_i) was the previous marble elimination move, so v_i just gained a marble and started with one marble. So, v_i has at least two marbles and has enough to marbles to make the elimination move.

After this sequence, it can be observed that v_n has not lost any marble, so it still has its starting marble, and it also just gained a marble from the move (v_{n-1}, v_n), so it has two marbles.

Since the graph started with n+1 marbles, and we made n-1 moves, there are only two marbles left, and v_n has both of them. Then if we simply add the move (v_n, v_{n-1}), which is valid since v_n has two marbles and the edge (v_n, v_{n-1}) exists since (v_{n-1}, v_n) was a valid move.

Now the graph has one marble left. So, there is indeed a sequence of marble elimination moves removing all but one marble. Thus, Claim 1 is valid.

<u>Claim 2 – If there exist a sequence of Marbles Elimination moves removing all but one marble, then there exists a Hamiltonian Path</u>

Since we start with n+1 marbles, and each move removes one marble, this sequence must have exactly n moves. It can be noted that after we have made zero moves, the only vertex with at least two marbles is v_1 by construction, which has exactly two.

Now, suppose after we have made k-1 moves, where $0 \leq k-1 < n-1$, there is exactly one vertex v_k that has exactly two marbles. Then since $k-1 < n-1$, we still have more moves in our sequence. Since only v_k has two marbles, our k^{th} move must be from v_k to some other vertex, call it v_{k+1}.

In the case that v_{k+1} has no marbles, then no vertices will have two marbles after this move, so we can't make moves, but since $k < n$ we still have moves and that's a contradiction.

Now, by hypothesis, v_{k+1} has less than two marbles, so v_{k+1} had one marble and after the k^{th} move it has two marbles. So, after k moves, there is exactly one vertex, v_{k+1}, with exactly two marbles.

Thus, the first n-1 moves can be written as the sequence moves (v_1, v_2) (v_2, v_3) ... (v_{n-1}, v_n).

Since this removes n-1 marbles, there are 2 marbles remaining after this sequence. Then one of the marbles must be at v_n since that was the last move in the subsequence. Since there is still one more move, one of the vertices must have two marbles, and since v_n already has one, it must have the other. So v_n is the only vertex with marbles after the first n-1 moves. And, since v_n has exactly 2 marbles, the n^{th} (final) move should be from v_n to some other vertex, that results in a configuration where there is only one marble remaining.

Since each of the other n-1 vertices start with a marble and end with none after the first n-1 moves, there must be some move that removes their marble. Since there are exactly n-1 moves in the beginning, there is a bijection between those n-1 vertices and the first n-1 moves corresponding to which vertex a move removes marbles from. So, all the vertices $v_1 \ldots v_n$ are distinct.

Then consider the path $v_1 \ldots v_n$, which visits each of the n vertices exactly once. This is indeed a valid path since for $1 \leq i < n-1$, since v_i, v_{i+1} was a marble elimination move, then v_i, v_{i+1} must be an edge. So, this is a Hamiltonian Path. Thus, Claim 2 is valid.

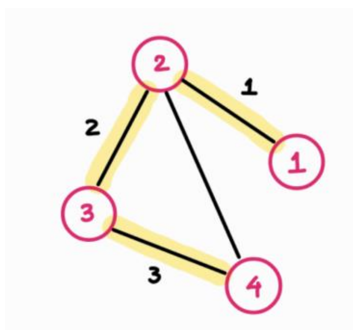
Claim 1 and Claim 2 contribute to the proof of the lemma that there is a Hamiltonian Path if and only if any of these Marbles Elimination problems is true.

From the reduction, running the Marble Elimination algorithm n times, once for each start vertex, so this only contributes a polynomial factor. Additionally, to modify the graph each time, simply label each vertex in constant time, which takes linear time to do so. Thus, reduction from Hamiltonian Path problem to Marbel Elimination takes polynomial time.

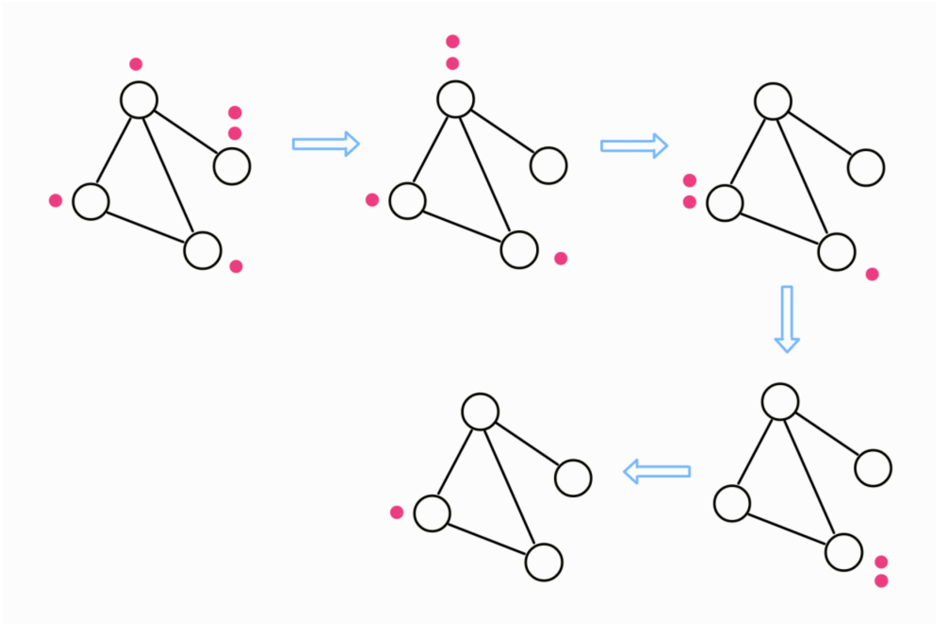
We know Hamiltonian Path problem is NP-Hard, therefore **Marble Elimination problem is also NP-Hard**. Now since Hamiltonian Path problem is as hard as Hamiltonian Cycle problem, the Marble Elimination problem is also as hard and Hamiltonian Cycle problem.

Example

The following graph shows a valid Hamiltonian path.

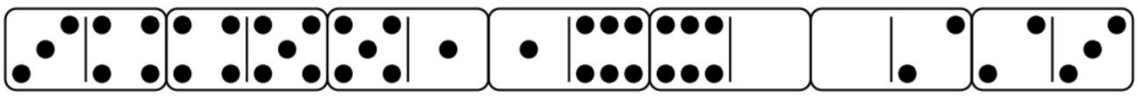


The corresponding sequence for the game play of the Marble Elimination can be formulated as –



Problem 2. Total: 4

A domino is a 1×2 rectangle divided into two squares, each of which is labeled with an integer. In a legal arrangement of dominos, the dominos are lined up end-to-end so that the numbers on adjacent ends match. The following is an example:



Determine the complexity of the two problems given below. Hint. Try and relate these problems to edge and vertex paths/walks in a suitably defined graph. Note that a closed Euler walk can be found in polynomial time, while the problem of finding a Hamiltonian cycle is NP-complete.

(2.a) (2 points) Given an arbitrary bag D of dominos, is there a legal arrangement of all the dominos in D?

(2.b) (2 points) Given an arbitrary bag D of dominos, is there a legal arrangement of a subset of dominos from D in which every integer between 1 and n appears exactly twice?

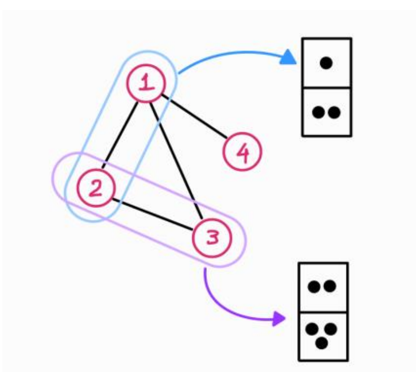
Solution (2.a)

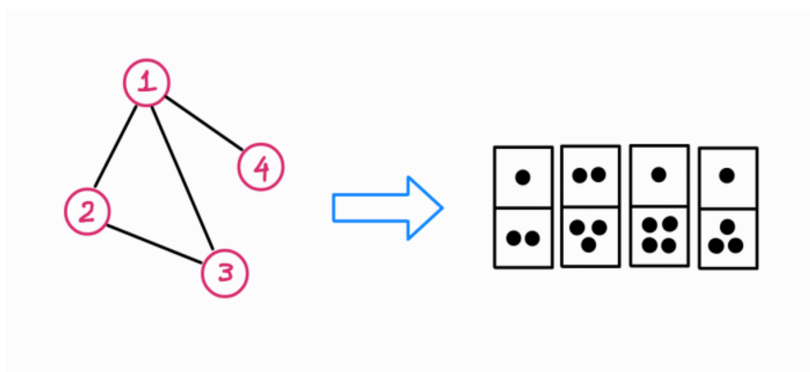
Problem – Given an arbitrary bag D of dominos, is there a legal arrangement of all the dominos in D.

Lemma – The problem is equivalent to finding a Eulerian path or Eulerian cycle in the graph.

The Eulerian Cycle problem can be reduced to the given problem using the following strategy –

Let's represent the set of dominos as a graph G, where each unique integer on a domino corresponds to a vertex (v_x) in the graph, and each domino represents an edge (v_p, v_q) connecting the two integers (vertices) it contains. This graph is undirected because a domino can be flipped.





<u>Claim 1 – There exists a legal arrangement of all the dominos in D if and only if there exists a Eulerian path in the graph.</u>

Let Ω = (v_1, v_2) → (v_2, v_3) → ... → (v_{n-1}, v_n) be a valid Eulerian path in the graph. Since each vertex v_x represents a face of a domino, then, any two adjacent edges, say (v_a, v_b) → (v_b, v_c) share a common vertex v_b, which indeed means the numbers on adjacent ends of the dominos must match. And, since each edge in a Eulerian path is visited exactly once, then each domino is lined up exactly once in the sequence. Therefore, Ω is a valid arrangement.

Thus, Claim 1 is correct.

Claim 2 – If there exists a legal arrangement of all the dominos in D, then there exists a Eulerian path in the graph.

Let Θ = $(v_1, v_2) \rightarrow (v_2, v_3) \rightarrow \ldots \rightarrow (v_{n-1}, v_n)$ be a valid arrangement of the Dominos in D. Since each face v_x represents a vertex in G, then, any two adjacent dominos, say $(v_a, v_b) \rightarrow (v_b, v_c)$ share a common face v_b, which indeed means that they share a common vertex v_b in G, thus (v_a, v_b) and (v_b, v_c) are adjacent edges in G. And, since each domino in D can be used exactly once, then each edge in G is visited exactly once. Therefore, Θ is an Eulerian path.

Thus, Claim 2 is correct.

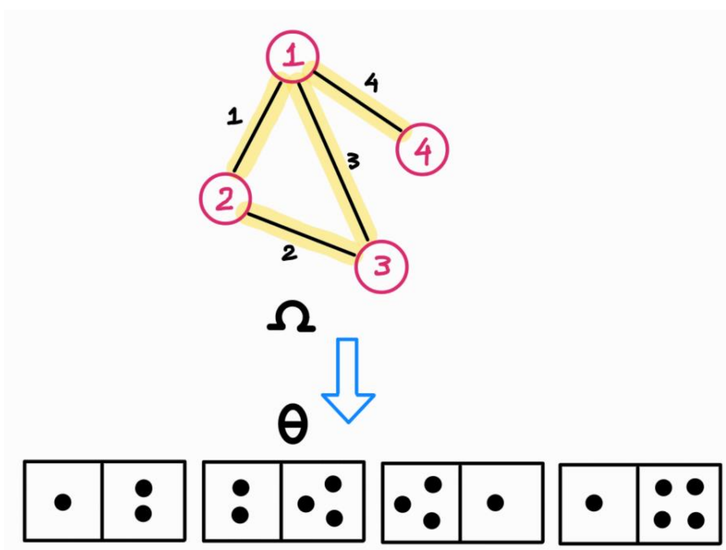
The correctness of Claim 1 and Claim 2 contributes to proof of the Lemma that the problem is equivalent to finding a Eulerian path or Eulerian cycle in the graph.

Now reduction of the Eulerian path problem to the given problem clearly takes polynomial time as we are just looking at each edge in the graph and adding a domino into D based on its two ends. This could be achieved by a single pass over all the edges in G, which will take polynomial time.

Since finding a Eulerian path or cycle is a problem that can be solved in polynomial time. **Therefore, the given problem can also be solved in polynomial time.**

Example

The following graph has a valid Eulerian path Ω which can be reduced to a valid arrangement of the Dominos Θ.



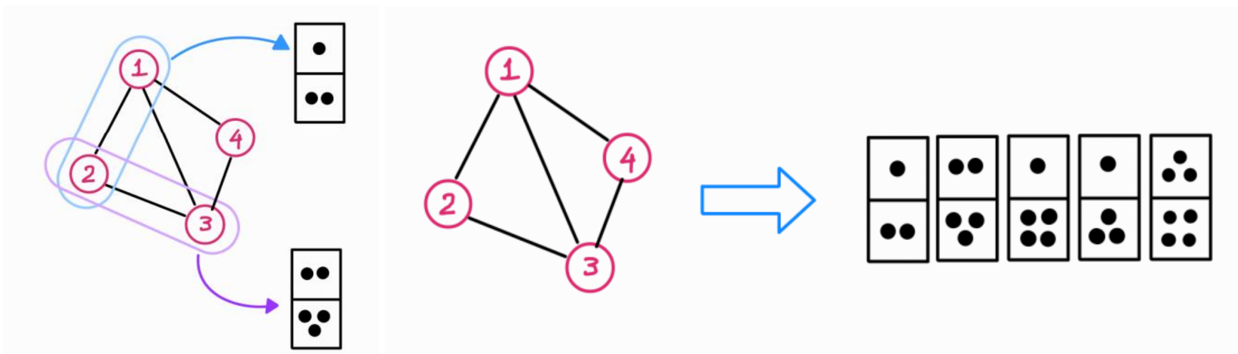
Solution (2.b)

Problem – Given an arbitrary bag D of dominos, is there a legal arrangement of a subset of dominos from D in which every integer between 1 and n appears exactly twice.

Lemma – The problem is equivalent to finding a Hamiltonian cycle in the graph.

The Hamiltonian Cycle problem can be reduced to the given problem using the following strategy –

Let's represent the set of dominos as a graph G, where each unique integer on a domino corresponds to a vertex (v_x) in the graph, and each domino represents an edge (v_p, v_q) connecting the two integers (vertices) it contains. This graph is undirected because a domino can be flipped.



Claim 1 – There exists a legal arrangement of a subset of dominos from D in which every integer between 1 and n appears exactly twice if and only if there exists a Hamiltonian cycle in the graph.

Let Ω = (v_1, v_2) → (v_2, v_3) → … → (v_{n-1}, v_n) → (v_n, v_1) be a valid Hamiltonian cycle in the graph. Since each vertex v_x represents a face of a domino, then, any two adjacent edges, say (v_a, v_b) → (v_b, v_c) share a common vertex v_b, which indeed means the numbers on adjacent ends of the dominos must match.

And, since each vertex in a Hamiltonian cycle is visited exactly once, then each integer, say v_b, appears exactly twice in D, once with the previous edge, say (v_a, v_b), and once with the next edge, say (v_b, v_c).

Also, since Ω is a Hamiltonian cycle, then every vertex in the graph is covered by Ω. Which in-turn means that every integer in the set of D has been explored.

It can be observed that all edges might not be present in Ω, and no edge is repeated in Ω whatsoever.

Therefore, Ω is a valid subset of dominos from D in which every integer between 1 and n appears exactly twice. Thus, Claim 1 is correct.

Claim 1 – If there exists a legal arrangement of a subset of dominos from D in which every integer between 1 and n appears exactly twice, then there exists a Hamiltonian cycle in the graph.

Let Θ = (v_1, v_2) → (v_2, v_3) → … → (v_{n-1}, v_n) → (v_n, v_1) be a valid arrangement of the subset of Dominos in D in which every integer between 1 and n appears exactly twice. Since each face v_x represents a vertex in G, then, any two adjacent dominos, say (v_a, v_b) → (v_b, v_c) share a common face v_b, which indeed means that they share a common vertex v_b in G, thus (v_a, v_b) and (v_b, v_c) are adjacent edges in G.

Now, each face/integer v_x appears exactly twice in Θ, which mean that each vertex v_x appears exactly once in G. Also, Θ includes all the integers in D, which means that every vertex v_x is also visited. Therefore, Θ is also a Hamiltonian cycle.

Thus, Claim 2 is correct.

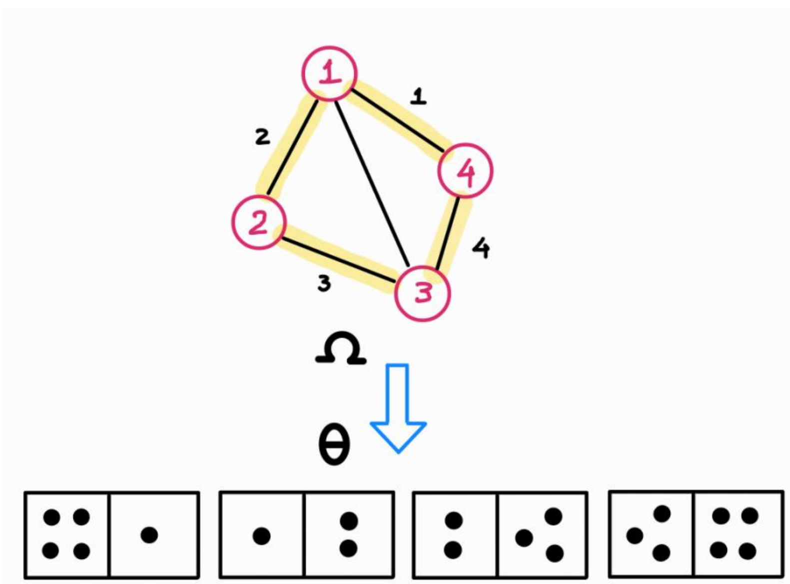
The correctness of Claim 1 and Claim 2 contributes to proof of the Lemma that problem is equivalent to finding a Hamiltonian cycle in the graph.

Now reduction of the Hamiltonian cycle problem to the given problem clearly takes polynomial time as we are just looking at each edge in the graph and adding a domino into D based on its two ends. This could be achieved by a single pass over all the edges in G, which will take polynomial time.

Since finding a Hamiltonian cycle is a problem is NP-Hard, **the given problem is also NP-Hard**.

Example

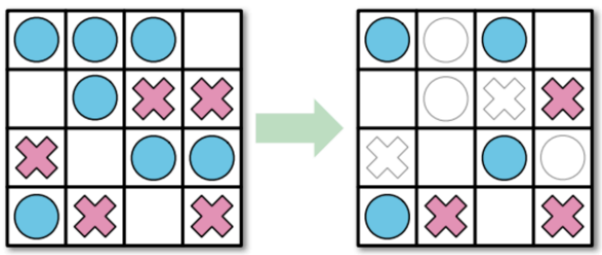
The following graph has a valid Hamiltonian cycle Ω which can be reduced to a valid arrangement of the subset of Dominos Θ where every integer between 1 and n appears exactly twice.



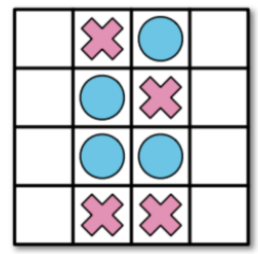
Problem 3. (5 points) Consider the following solitaire game. The puzzle consists of an $n \times m$ grid of squares, where each square may be empty, occupied by a red token, or occupied by a blue token. The goal of the puzzle is to remove some of the given tokens so that the remaining tokens satisfy two conditions:

1. every row contains at least one token, and
2. no column contains tokens of both colors.

For some initial configurations of tokens, reaching this goal is impossible.



A solvable puzzle and one of its many solutions.



An unsolvable puzzle.

Prove that it is NP-hard to determine, given an initial configuration of red and blue tokens, whether the puzzle can be solved.

(Hint: reduce from 3SAT.)

Solution

Problem – Given an initial configuration of red and blue stones, whether this puzzle can be solved.

To Prove – The problem is NP- Hard

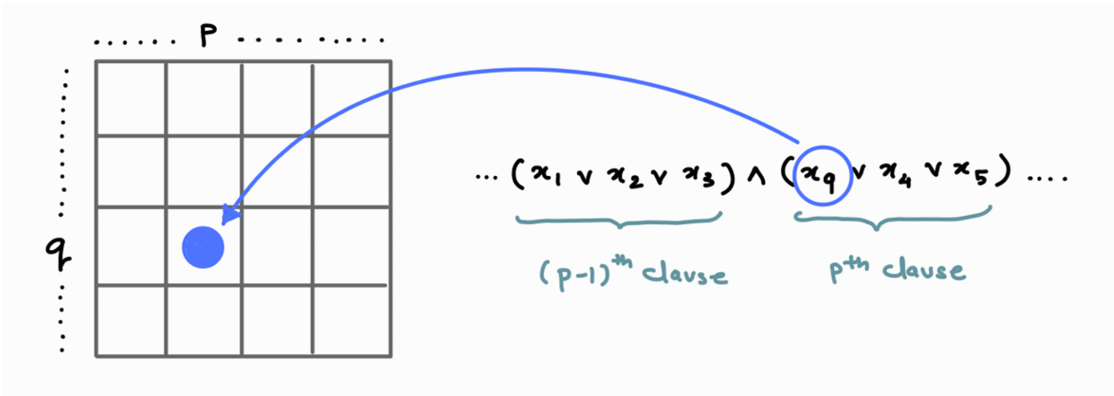
To prove that the given problem is NP-Hard, it is sufficient to deduce a polynomial time reduction from a known NP-hard problem to the given problem statement.

Lemma – The given problem is as hard as 3SAT problem.

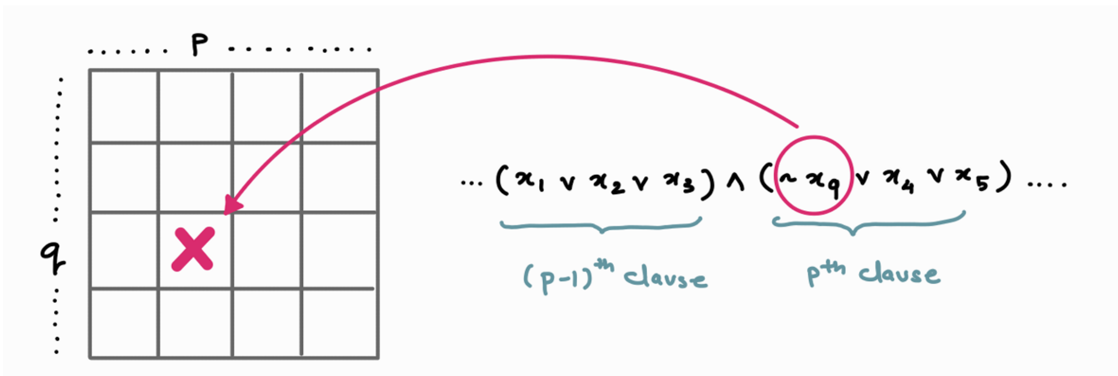
Let ϕ denote a 3CNF Boolean equation with m variables and n clauses. The reduction from ϕ to the given problem can be deduced using the following strategy –

Let us consider the size of the board is n x m. At the position (p, q) the stones should be placed using the following rules –

1. If the variable x_q appears in the p^{th} clause of ϕ, a blue stone should be placed at (p, q) position of the board.



2. If the negated variable $\sim x_q$ appears in the p^{th} clause of ϕ, a red stone should be placed at (p, q) position of the board.



3. Otherwise, the (p, q) position in the board remains blank.

Claim 1 – The puzzle is solvable if and only if ϕ is satisfiable.

Let's consider ϕ is satisfiable. Also, let's consider an arbitrary satisfying assignment. Then, for each index in column q, if x_q is True, then remove all red stones from column q. Otherwise, if x_q is False, remove all blue stones from column q.

It can be observed that, since every variable appears in at least one clause, each column now contains stones of one color only (if any), since a literal say x_i can either be a True or a False, not both, at the same time. Also, each clause of ϕ must contain at least one True literal, and thus, each row still must contain at least one stone. We can conclude that the puzzle is satisfiable.

The observations prove Claim 1.

Claim 2 – If the puzzle is solvable, then ϕ is satisfiable.

Let's assume the puzzle is solved. For each literal x_q, assign the value to x_q depending on the color of the stones left in column q.

If column q contains blue stones, set x_q = True. If column q contains red stones, set x_q = False. Otherwise, if the column is empty, set any arbitrary assignment to x_q.

It can be observed that, since each row still has at least one stone, so each clause of Φ contains at least one True literal, so this assignment makes Φ = True. We conclude that Φ is satisfiable.

The observations prove Claim 2.

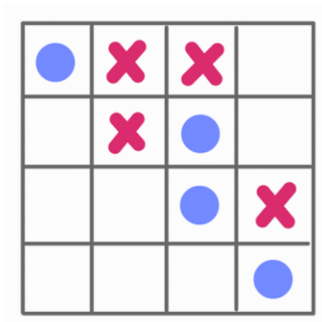
Claim 1 and Claim 2 contribute to the proof of the Lemma that the given problem is as hard as 3SAT problem. The reduction requires only polynomial time since we are just looking the clauses and placing the stones in the board accordingly, which can be achieved by only a single pass over the sequence of the clauses.

Now, since 3SAT problem is NP-Hard, **the given problem is also NP-Hard.**

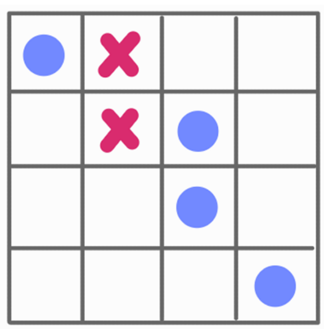
Example

Let $\Phi = x_1 \wedge (\sim x_1 \vee \sim x_2) \wedge (\sim x_1 \vee x_2 \vee x_3) \wedge (\sim x_3 \vee x_4)$

On reducing Φ to the given puzzle, the board obtained looks like the following.



Now, it can be observed that, x_1 = *True*, x_2 = *False*, x_3 = *True*, x_4 = *True* is a valid assignment and satisfies Φ. Thus, on updating the board, the board looks like the following.



It can be observed that the board is also solved, with each column containing only single-color stone, and each row has at least one stone.

Problem 4. (3 points) Betal has picked up a part-time job that involves ferrying n deceased people across a river for a smooth transition into the next phase. Certain pairs of these people are sworn enemies, who cannot be taken together on the ferry because when on water, they can in fact get into a deadly fight, and this will be a distraction to Betal who will be navigating rough waters. It is safe to leave them on the shore because corpses don't fight on land.

The ferry has unlimited capacity, but Betal has limited time.

Prove that it is NP-hard to decide whether Betal can ferry all n people across the river safely in at most k rounds. The input for Betal's problem consists of the integers k and n and an n-vertex graph G describing the pairs of enemies. The output is either TRUE or FALSE.

Solution

To Prove – It is NP-hard to decide whether Betal can ferry all n people across the river safely in at most k rounds.

To prove that the given problem is NP-Hard, it is sufficient to deduce a polynomial time reduction from a known NP-hard problem to the given problem statement.

Lemma – The given problem is as hard as Minimum Graph Coloring Problem.

Given a graph G with n vertices. The reduction from the minimum graph coloring to the given problem can be accomplished by the following strategy –

Assign each vertex v_x, $1 \leq x \leq n$ to each deceased person. And, if there exists a dispute between two people say, v_a and v_b, there exists an edge (v_a, v_b) in G.

Claim 1 – Betal can take all the people across the river in k rounds if and only if the graph is k colorable.

Given the graph is k colorable. Now, it means that the vertices of the graph can be colored using at most k colors in such a way that no two adjacent vertices have the same color.

Now, since two enemies share a common edge between them, they are neighbors in G. Thus, no two people who are enemies of each other could be assigned with a same color.

This means, that Betal can take all the people who are assigned with a same color in G together across the river, since none of them are enemies of each other.

This, he must repeat for k time, taking all the people who are assigned with same color across the river at a time. Therefore, Betal can take all the people across the river in at most k round.

Thus, Claim 1 is true.

Claim 2 – If Betal can take all the people across the river in k rounds, then the graph is k colorable.

Let v_{11}, v_{12}, ... v_{1x} be the group of people Betal takes in the 1st round. It is certain that these people do not share an edge between them. Thus, all these people can be assigned the same color, say k_1.

Now, let v_{21}, v_{22}, ... v_{2y} be the group of people Betal takes in the 2nd round. It is certain that these people also do not share an edge between them. Thus, all these people can be assigned the same color, say k_2, which is different from the color k_1.

Now, let v_{k1}, v_{k2}, ... v_{kz} be the group of people Betal takes in the k^{th} round. It is certain that these people also do not share an edge between them. Thus, all these people can be assigned a same color, say k_n that is different from all the previously assigned colors (k_1, k_2 ... k_n).

After k rounds, Betal has taken all the people across the river, thus no vertex in the graph remains uncolored. Also, from all k rounds, it is for sure that no two enemies share the same color. Thus, the graph has been colored in such a way that each vertex has been colored in such a way that no two neighboring vertices have been assigned the same color.

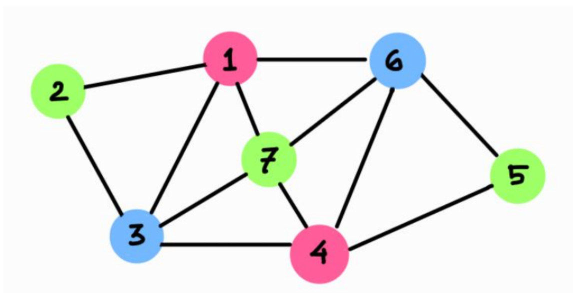
Therefore, the graph is k colorable. Thus, Claim 2 is true.

Claim 1 and Claim 2 contribute to the proof of the Lemma that the given problem is as hard as Minimum Graph Coloring Problem. The reduction requires only polynomial time since we are just looking at the pair of enemies and adding edge between them, which can be achieved by only a single pass over all the pair of enemies, which will take polynomial time.

Now, since Minimum Graph Coloring Problem is NP-Hard, **the given problem is also NP-Hard.**

Example

Given the following graph is 3 colorable. That is, the graph requires at least 3 colors to color each of its vertices such that no two neighboring vertices are assigned a same color.



Now, considering each vertex denotes a person, and an edge between two vertices exists only if they are enemies. Then Betal can take all the people across the river in 3 rounds. In the 1^{st} round, he will take all people marked as **red** across the river. In the 2nd round, he might take all the people who are marked as **blue** across the river. And in the third round he needs to take all the people who are marked as **green** across the river. Since, in all three rounds, no two people share an edge between them, they are not enemies and they do not engage into fights. Thus, all reach the other side safely (expecting that corpses don't fight on land).

Part 2

Problem 1. (2 points) A positional game consists of a set X of positions and a family $W_1, W_2, \ldots, W_m \subseteq X$ of winning sets (Tic-Tac-Toe has 9 positions corresponding to the 9 boxes, and 8 winning sets corresponding to the three rows, three columns, and two diagonals). Two players alternately choose positions; a player wins when they collect a winning set.

Suppose that each winning set has size at least a and each position appears in at most b winning sets (in Tic-Tac-Toe $a = 3$ and $b = 4$). Prove that Player 2 can force a draw if $a \geqslant 2b$.

Hint: Form a bipartite graph G with bipartition (X, Y) where $Y = \{W_1, W_2, \ldots, W_m\} \cup \{W_1', W_2', \ldots, W_m'\}$. Try to add edges in such a way that a matching in this graph will help you argue a draw. Appeal to Hall's Theorem to argue the existence of a matching.

Solution 1

Let's consider a subset Y' of Y and define S as the collection of all edges adjacent to any vertex within Y'. Given that every vertex in X connects to at least a vertex in Y', the set S should have at least the size of a times the cardinality of Y', that is |S| ≥ a|Y'|.

Conversely, each vertex in X is connected to no more than 2b vertices in Y'. Thus, 2b times the cardinality of set N(Y'), representing neighbors of any subset Y' of Y, is at-most |S|, or 2b|N(Y')| ≤ |S|.

Combining these we get a|Y'| ≤ 2b|N(Y')| and with the condition that 'a' is greater than or equal to '2b', we derive N(Y') is at least as large as the cardinality of Y', that is |N(Y')| ≥ |(Y')|. By Hall's Theorem, we can conclude that there is a matching M that includes every vertex of Y.

Now, for each index 'i' ranging from 1 to m, let's identify Q_i as the pair of vertices in X that are connected to both W_i and W_i'. The collection of sets Q_1 through Q_m forms a disjoint two-element subset of X, and each W_i contains its respective Q_i.

A strategy that ensures at least a draw for the second player – Whenever the first player picks a vertex, if that vertex is in Q_i for some 'i', the second player should respond by choosing the remaining vertex in Q_i if it's still available. If not, the second player can make any move. It's clear through induction that after each move by the second player, there's no set Q_i from which the first player has chosen a vertex, and the second player hasn't. This implies the first player cannot select both elements from any Q_i, and thus cannot cover all elements of any W_i. As a result, the first player cannot win when the second player follows this strategy.

Problem 2. Total: 8

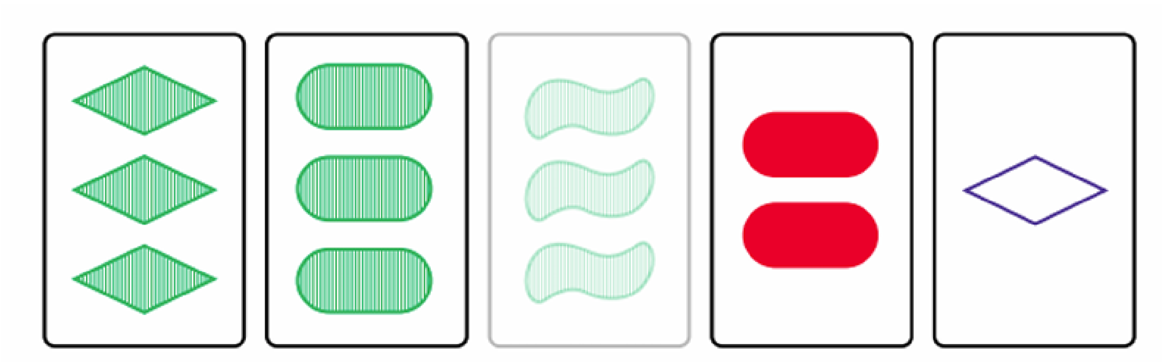
Recall that, in the game of SET, for any pair of cards there exists exactly one missing card that completes the pair to a Set. Any single card, however, serves as missing card for many (in the standard deck, 40) different pairs.

We consider a variation called *twin set*. A twin set is a collection of **four cards** that from two Sets with the same missing card. In other words, four cards form a twin set if there exists a fifth card X and a partition of the four cards into two pairs of cards such that each pair, when augmented with X, forms a Set.

For example, consider the four cards:

- Three Green Shaded Diamonds (A)
- Three Green Shaded Ovals (B)
- Two Red Solid Ovals (P)
- One Purple Empty Diamond (Q)

as shown below:



This is a twin set; the joint is the card X: `Three Green Shaded Squiggles`. Note that ABX is a Set, as well as PQX.

The following questions are about properties of twin set. Assume that you are playing with the standard deck of 81 cards with no duplicates.

(2.a) (1 point) Show that a twin set never contains three cards that are already a Set.

Solution (2.a)

Let the cards ϕ = {A, B, C, D} form a twin set.

Let us assume that the twin set indeed contains three cards that are already a set. Let it be the cards {A, B, C}. Now let us assume that there exists a firth card E, which when added to ϕ, makes two Sets.

Now, the newly formed Sets could be one of the following –

ABE and CDE ACE and BDE BCE and ADE

Note that for either of the cases to be true, either ABE or ACE or BCE should form a set.

But ABE cannot form a set since ABC is already a set, which violates the unique completion rule. For the same reason, neither ACE nor BCE can form a set.

Thus, our initial assumption is wrong. Therefore, a twin set cannot contain three cards that are already a set.

(2.b) (1 point) Given any three cards that do not form a Set, there are *exactly* three other cards that will form a twin set with those three.

Solution (2.b)

Let {A, B, C} be the given cards that do not form a Set.

Now, let us assume that there is a card X_1 that forms a set with A and B. That is {A, B, X_1} is a set. Now from the unique completion rule, we know that there exists a card Y_1 that makes a set with X_1 and C. Thus, {X_1, C, Y_1} is a set. Apparently, {A, B, C, Y_1} is a twin set (with X_1 as joint).

Also, let us assume that there is a card X_2 that forms a set with A and C. That is {A, C, X_2} is a set. Now from the unique completion rule, we know that there exists a card Y_2 that makes a set with X_2 and B. Thus, {X_2, B, Y_2} is a set. Apparently, {A, B, C, Y_2} is a twin set (with X_2 as joint).

Also, let us assume that there is a card X_3 that forms a set with B and C. That is {B, C, X_3} is a set. Now from the unique completion rule, we know that there exists a card Y_3 that makes a set with X_3 and A. Thus, {X_3, A, Y_3} is a set. Apparently, {A, B, C, Y_3} is a twin set (with X_3 as joint).

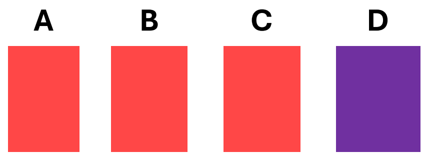
For the fourth card Y_4 to exist such that {A, B, C, Y_4} is a twin set, there must exist a corresponding joint card X_4. Now, the card X_4 is supposed to form a set with either AB, or BC, or AC. This is however not possible as there exists a card X_1 that forms a set with AB, X_2 with AC and X_3 with BC. Thus, the existence of X_4 will violate the unique completion rule.

Therefore, given the three cards, **there are exactly three other cards that will form a twin set with those three cards.**

(2.c) (1 point) If there are three of one and one of another, then it's not a twin set. For example, if three cards are red and the fourth one isn't, then it's not a twin set.

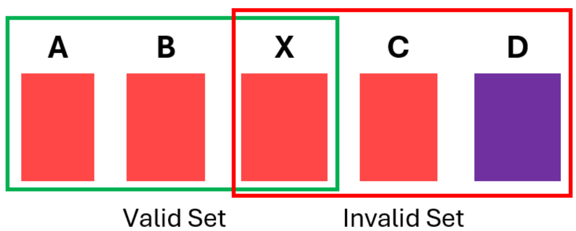
Solution (2.c)

Let us assume that there exists a twin set {A, B, C, D} such that three cards satisfy the same property while the fourth doesn't. As given, let's take color as the property. Let A, B and C be red, while D is some other color, say purple.

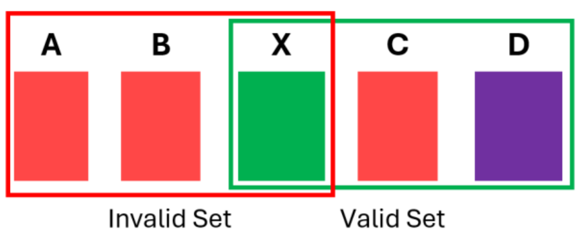


Now, let X be the joint card for the twin set {A, B, C, D}. Now, splitting {A, B, C, D} into two will result in two cards on each side. Now, no matter what we do, two red cards will always end up on the same side (by pigeonhole principal).

Now for X to make a set with the cards on the left (A and B), since both are red, X needs to be red. But, doing so, it won't form a set with the cards on the right (C and D), since C is red, and D is purple.



For X to make a set with the cards on the right (C and D), since both of their colors are different, X needs to have a different color (Green). But, doing so, it won't form a set with the cards on the right (A and B), since both are red.



Thus, both the choices contradict each other. Therefore, our assumption was wrong. This is true for the other properties as well. This proves that **if there are three of one and one of another, then it's not a twin set**.

(2.d) (1 point) Given any twin set, show that the joint is unique.

Solution (2.d)

Let us assume that {A, B, C, D} is the given twin set. Let X be the joint. This means that {A, B, X} is a set and {X, C, D} is a set.

Now let us assume that there exists a card Y, that is not X, and is the joint for the twin set {A, B, C, D}. Clearly, Y cannot split {A, B, C, D} as AB and CD, because that will infer {A, B, Y} and {Y, C, D} are set, which violates the unique completion rule. Thus, Y needs to spit {A, B, C, D} to some other pair, say, AC and BD.

We need to show either {A, C, Y} or {Y, B, D} is not a set.

Let us assume that {A, C, Y} or {Y, B, D} are valid sets.

Let K_h be the vector representing the characteristics of the card K (for eg. $K_h = [1, 2, 1, 1]^T$). Now, from the rules of set, we know that for {A, B, X} to be a set,

$(A_h + B_h + X_h) \bmod 3 = \phi$, where $\phi = [0, 0, 0, 0]^T$ (*Finding 1*)

Similarly,

$(X_h + C_h + D_h) \bmod 3 = \phi$ (*Finding 2*)
$(A_h + C_h + Y_h) \bmod 3 = \phi$ (*Finding 3*)
$(Y_h + B_h + D_h) \bmod 3 = \phi$ (*Finding 4*)

From Findings 1 and 3, we get $(B_h + X_h) \bmod 3 = (C_h + Y_h) \bmod 3$ (*Finding 5*)
From Findings 2 and 4, we get $(X_h + C_h) \bmod 3 = (Y_h + B_h) \bmod 3$ (*Finding 6*)

From Findings 5 and 6, we get, $(X_h + C_h - Y_h + X_h) \bmod 3 = (C_h + Y_h) \bmod 3$
Or, $(2X_h) \bmod 3 = (2Y_h) \bmod 3$

Now, this is only possible if and only if X and Y have exact feature set. But this contradicts our assumption that there exists some Y that is different from X and forms a twin set as a joint in {A, B, C, D}. Thus, **given any twin set, the joint is unique.**

(2.e) (2 points) How many twin sets are there in a standard deck of SET?

Solution (2.e)

The twin set consists of 4 cards. Considering the standards deck, for the first card, we have a total of 81 choices.

81			

For the second card, we have a total of 80 choices.

81	80		

Now, we know from (2.a) that a twin set must not contain a set within itself. Thus, for the third card we have a total of 78 choices (79 minus one card that makes the set with the first two cards).

81	80	78	

Form (2.b) we know for the fourth card we have only 3 choices.

81	80	78	3

Now, we know, these 4 cards can arrange themselves in 4! (24) Ways, so the total calculation comes out to be **(81 x 80 x 78 x 3) / (24) = 63180.**

Thus, the total twin sets in a standard deck is **63180**.

(2.f) (2 points) Let X be any three cards that do not form a Set. Let Y be the three cards that each create a twin set with X. Let Z be the three cards that are the joints of those three twin sets. Then show that any card in Z creates a twin set when joined with the three cards in Y (and the joint is in X), and any card in X creates a twin set when joined with the three cards in Z (and the joint is in Y). Also, those nine cards form a "plane".

Solution (2.f)

Let X = {A, B, C} be three cards that do not form a Set. Let Y = {D, E, F} be the three cards that each create a twin set with X. Let Z = {P, Q, R} be the three cards that are the joints of those three twin sets.

This means,

- {A, B, C, D} is a twin set and P is a joint such that {A, B, P} and {P, C, D} are sets. (*Finding 1*)
- {A, C, B, E} is a twin set and Q is a joint such that {A, C, Q} and {Q, B, E} are sets (*Finding 2*)
- {B, C, A, F} is a twin set and R is a joint such that {B, C, R} and {R, A, F} are sets. (*Finding 3*)

To prove: <u>Any card in Z creates a twin set when joined with three cards in Y (X as joint).</u>

Claim 1: {D, P, E, F} is a twin set with C as joint.

Clearly, with C as joint, {D, P, C} is a set (from Finding 1). We need to show that {C, E, F} is also a set.

Let K_h be the vector representing the characteristics of the card K (for eg. $K_h = [1, 2, 1, 1]^T$). Now, from the rules of set, we know that for {A, C, Q} to be a set,

$(A_h + C_h + Q_h) \bmod 3 = \phi$, where $\phi = [0, 0, 0, 0]^T$ (*Finding 4*)

Similarly,

$(Q_h + B_h + E_h) \bmod 3 = \phi$ (*Finding 5*)
$(B_h + C_h + R_h) \bmod 3 = \phi$ (*Finding 6*)
$(R_h + A_h + F_h) \bmod 3 = \phi$ (*Finding 7*)

Now, from *Finding 4 and 7,* we see $(C_h + Q_h) \bmod 3 = (R_h + F_h) \bmod 3$ (*Finding 8*)
And, from *Finding 5 and 6,* we see $(Q_h + E_h) \bmod 3 = (C_h + R_h) \bmod 3$ (*Finding 9*)

From *Finding 8 and 9,* we can infer $(C_h + C_h + R_h - E_h) \bmod 3 = (R_h + F_h) \bmod 3$
Or $(2C_h) \bmod 3 = (F_h + E_h) \bmod 3$
Or $(3C_h) \bmod 3 = (F_h + E_h + C_h) \bmod 3$

Now, clearly, $(3C_h) \bmod 3 = \phi$.
Therefore, $(F_h + E_h + C_h) \bmod 3 = \phi$. Thus, {C, E, F} is a set. (*Finding 10*)

Also, we observe from Finding 10 (*a generalized form*), if $\{T_1, T_2, T_3\}$ is a SET, $\{T_3, T_4, T_5\}$ is a SET, $\{T_2, T_4, T_6\}$ is a SET and $\{T_6, T_1, T_7\}$ is a SET, then, (T_2, T_5, T_7) is also a SET. (*Finding 11*)

This proves Claim 1.

Claim 2: {E, Q, D, F} is a twin set with B as joint.

Clearly, with B as joint, {E, Q, B} is a set (from Finding 2). Now, it can be proved that {B, D, F} is also a set using Findings 1, 3 and 11. Thus, Claim 2 is true.

Claim 3: {F, R, D, E} is a twin set with A as joint.

Clearly, with A as joint, {F, R, A} is a set (from Finding 3). Now, it can be proved that {A, D, E} is also a set using Findings 1, 2 and 11. Thus, Claim 3 is true.

The correctness of Claims 1, Claim 2 and Claim 3 proves that **any card in Z creates a twin set when joined with three cards in Y (X as joint).**

To prove: <u>Any card in X creates a twin set when joined with three cards in Z (Y as joint).</u>

From the previous proof we have,

- {D, P, E, F} is a twin set and C is the joint, thus, {D, P, C} and {C, E, F} are sets. (*Finding 12*)
- {E, Q, D, F} is a twin set and B is the joint, thus, {E, Q, B} and {B, D, F} are sets (*Finding 13*)
- {F, R, D, E} is a twin set and A is the joint, thus, {F, R, A} and {A, D, E} are sets. (*Finding 14*)

Claim 1: {P, Q, R, A} is a twin set with F as joint.

Clearly, with B as joint, {F, R, A} is a set (from Finding 14). Now, it can be proved that {P, Q, F} is also a set using Findings 12, 13 and 11. Thus, Claim 1 is true. (*Finding 15*)

Claim 2: {Q, R, P, C} is a twin set with D as joint.

Clearly, with B as joint, {D, P, C} is a set (from Finding 12). Now, it can be proved that {Q, R, D} is also a set using Findings 13, 14 and 11. Thus, Claim 2 is true. (*Finding 16*)

Claim 2: {R, P, Q, B} is a twin set with E as joint.

Clearly, with B as joint, {E, Q, B} is a set (from Finding 13). Now, it can be proved that {R, P, E} is also a set using Findings 12, 14 and 11. Thus, Claim 3 is true. (*Finding 17*)

The correctness of Claims 1, Claim 2 and Claim 3 proves that **any card in X creates a twin set when joined with three cards in Z (Y as joint).**

To show that the nine cards form a "plane" or "magic square".

While there can be more than one possible magic square that are possible with these nine cards. One such magic square is as follows.

A	B	P
F	E	C
R	Q	D

Rows –

{A, B, P} is a SET (from *Finding 1)*

{F, E, C} is a SET (from *Finding 12)*

{R, Q, D} is a SET (from *Finding 16)*

Columns –

{A, F, R} is a SET (from *Finding 14)*

{B, E, Q} is a SET (from *Finding 13)*

{P, C, D} is a SET (from *Finding 12)*

Diagonals –

{A, E, D} is a SET (from *Finding 14)*

{P, E, R} is a SET (from *Finding 17)*

This proves that the nine cards form a “plane”.

Problem 3. Total: 6

In this exercise, we will try to count the number of configurations of a tic tac toe board discounting symmetries.

(3.a) (2 points) In how many ways can a 3×3 grid be filled with five X marks and four 0 marks so that each grid location is filled with exactly one mark?

Note: if one configuration P can be obtained as a rotation or a reflection of another configuration Q, then P and Q count as distinct configurations.

However, the X and 0 marks are indistingushable, for example, they are not labeled by the turn in which they were played.

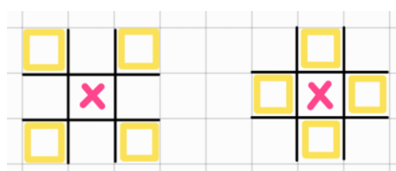
Solution (3.a)

Number of ways a 3 × 3 grid be filled with five X marks and four O marks so that each grid location is filled with exactly one mark, considering rotation or reflection as distinct configurations, can be obtained as ${}^{9}C_{5}$ = 9! / (5! X 4!) = **126**.

(3.b) (2 points) How many configurations have a X in the center that are draw configurations? Identify the rotations and reflections in your enumeration.

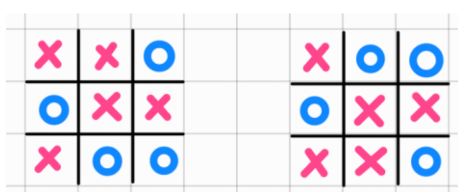
Solution (3.b)

With X in the center. We are supposed to put 4 more X(s) and 4 O(s) such that the configuration is a draw.

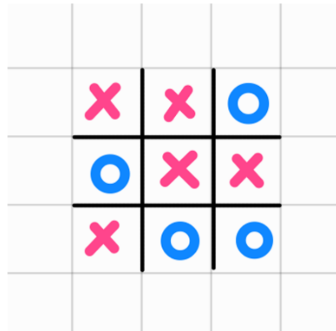


Now, out of the 8 void spaces, we can at **most put two X(s) at the corner**, otherwise, it will be winning for X and they should not be on the opposite corners. Similarly, we can **at-most put two X(s) on the side**, and both should not be on the opposite side.

From the two observations we can infer that we need to put **exactly two X(s) on corner** (non-opposite) and **exactly two X(s) at the sides** (non-opposite). The following figure displaces the possible configurations.



Apparently, both of them are reflections of each other. Thus, **considering reflection and rotation configs are non-distinct**, there is only <u>**one configuration**</u> with X at center and resulting in a draw.

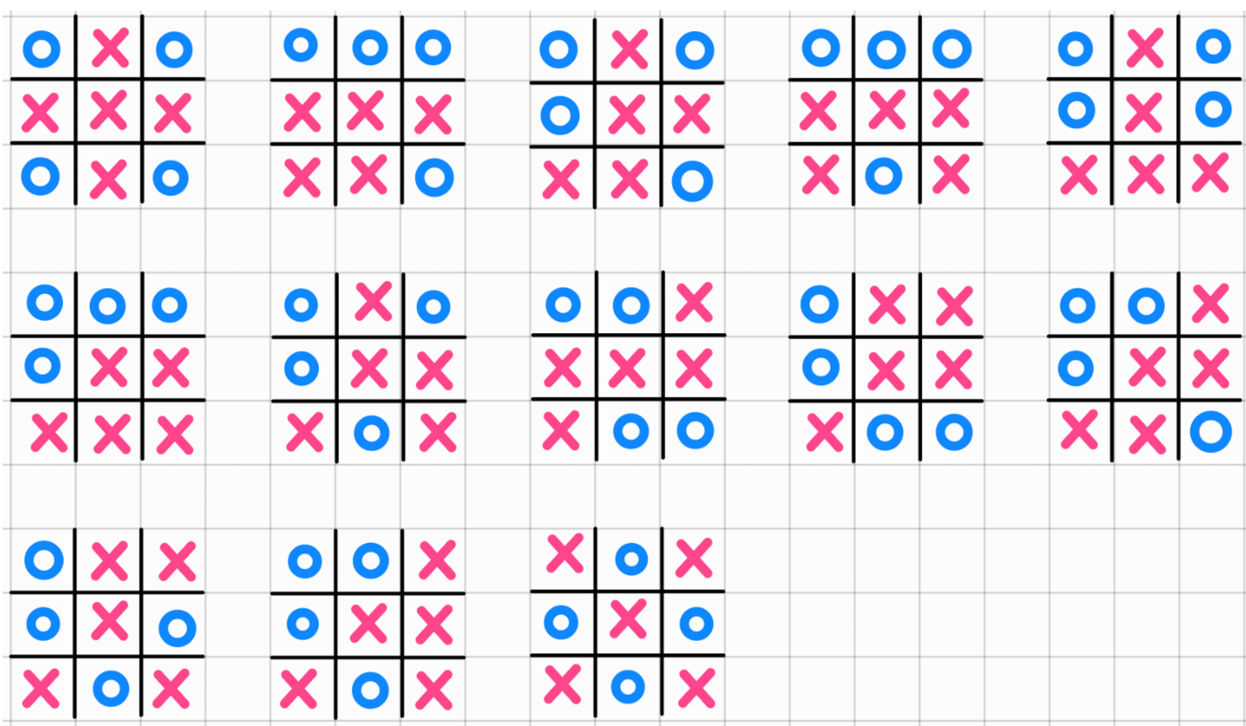


All the other configurations can be obtained by rotation and reflection of this configuration.

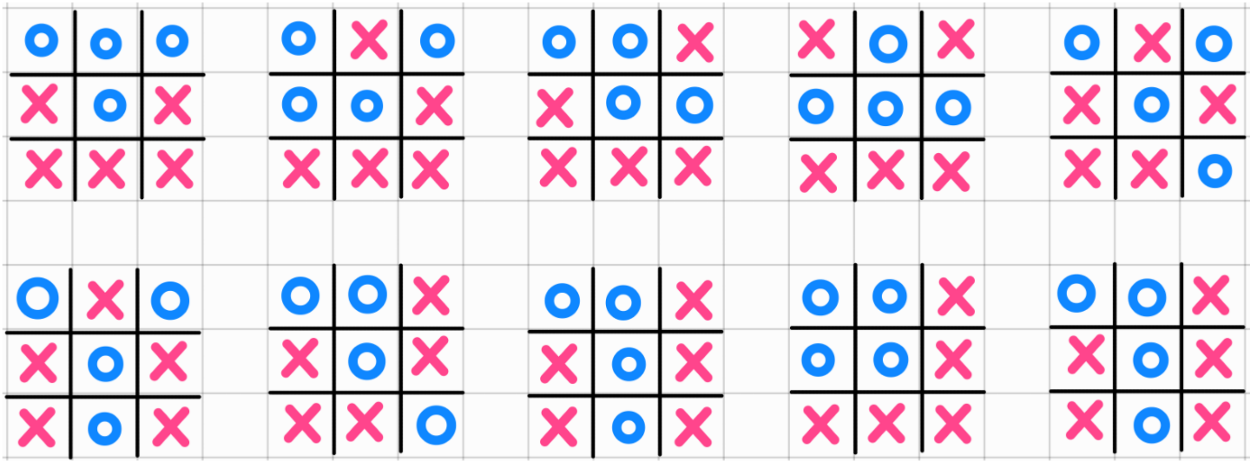
(3.c) (2 points) Now, consider that if one configuration P can be obtained as a rotation or a reflection of another configuration Q, then P and Q **do not count** as distinct configurations.
In how many ways can a 3×3 grid be filled with five X marks and four 0 marks so that each grid location is filled with exactly one mark?

Solution (3.c)

Considering rotation or reflection **do not count as distinct** configurations, then with X at the center there are a total of 13 unique configurations.



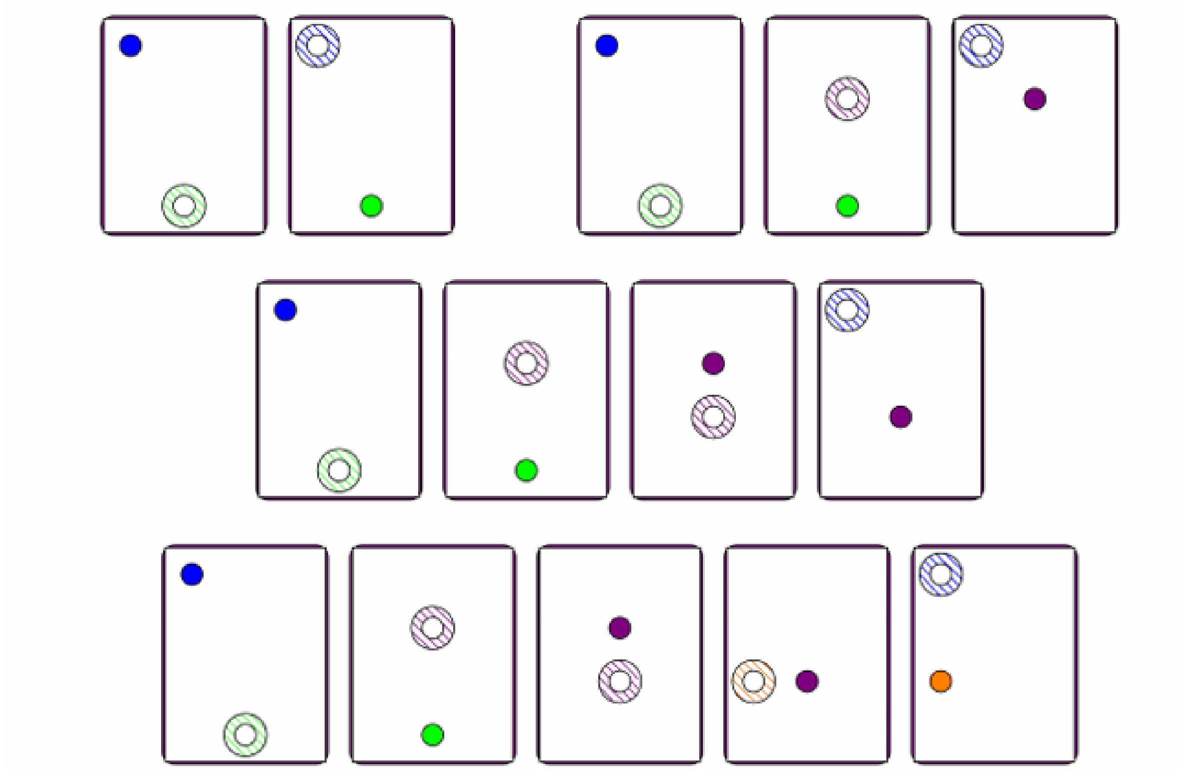
With O at the center, there are a total of 10 unique configurations.



Thus, out of 126 configurations, there are a total of **23 unique configurations** using 5 X(s) and 4 O(s). All the other configurations are either rotation or reflection of these 23 configurations.

Problem 4. (2 points) In this question, we consider a card game called SWISH.

There are 60 transparent cards in the commercial version of SWISH. Those cards are made up of three columns and four rows, they are obtained by placing a point in each of the four possible positions (accounting for symmetries), and then a circle in each of the other possible positions. For the points in the left column, the circle can be in 11 positions. For the points in the middle column, due to axial symmetry, the circle can be in 7 positions. Note that this only generates 36 cards, but there are 24 cards which are duplicated, reaching a total of 60 cards.



The generalized version of SWISH is played on cards of height h and width w. Cards can have one or several symbols, which can be points or circles. For a given card C, we denote by $C[a][b]$ the spot in row a and column b. Other than that, the generalized version is played the exact same way as the board game version: from a set $\mathscr{C}$ of cards, the players try to create a swish, that is, a subset $\mathscr{S} \subseteq \mathscr{C}$ such that every card is in the same orientation, every point meets a circle, every circle meets a point, and no two points or two circles meet. The cards can still be flipped or rotated, which can also be seen as applying axial (vertical or horizontal) or central symmetry.

Since the cards are drawn from the deck at random, the players cannot anticipate what is going to come next. Hence, we will assume that they will try to maximize their given score at each round of the game. Thus, the question that we ask is the following: given a set of cards, can we find a swish that is as large as possible? This optimization question leads to the following decision problem:

Instance. A set $\mathscr{C}$ of cards, an integer k.

Question. Is there a swish $\mathscr{S} \subseteq \mathscr{C}$ such that $|\mathscr{S}| \geqslant k$?

Show that this problem can be solved in polynomial time if every card has at most one symbol.

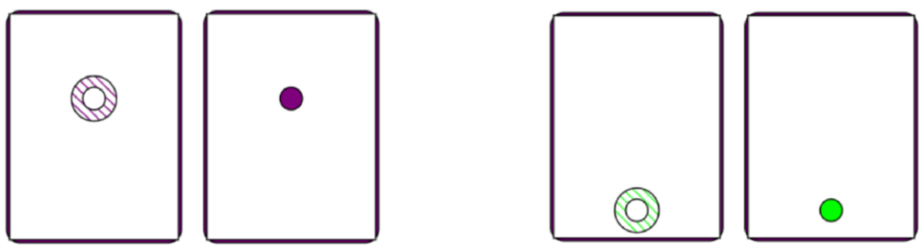
Solution 4

The process is as follows –

Start by grouping the cards into sets of duplicates. Two cards are considered duplicates if they become identical through either axial or central rotation.

For any group of duplicates larger than four, discard cards until only four remain, as a maximum of four duplicates can be utilized in a single swish.

Next, create a compatibility graph G: represent each card as a vertex, and draw an edge between two cards, C_i and C_j, if one has a dot at position 'a' and the other has a circle at the same position.



The goal is then to find the largest matching M in G; if the size of M is at least k, the answer is YES; if not, it's NO.

This method is efficient because once a card is matched, it cannot be matched again unless its duplicate is used, which is manageable since each card contains only one symbol. Trimming duplicates can be done quickly with a hash table, constructing the graph and finding the maximum matching are both tasks that can be completed in polynomial time.

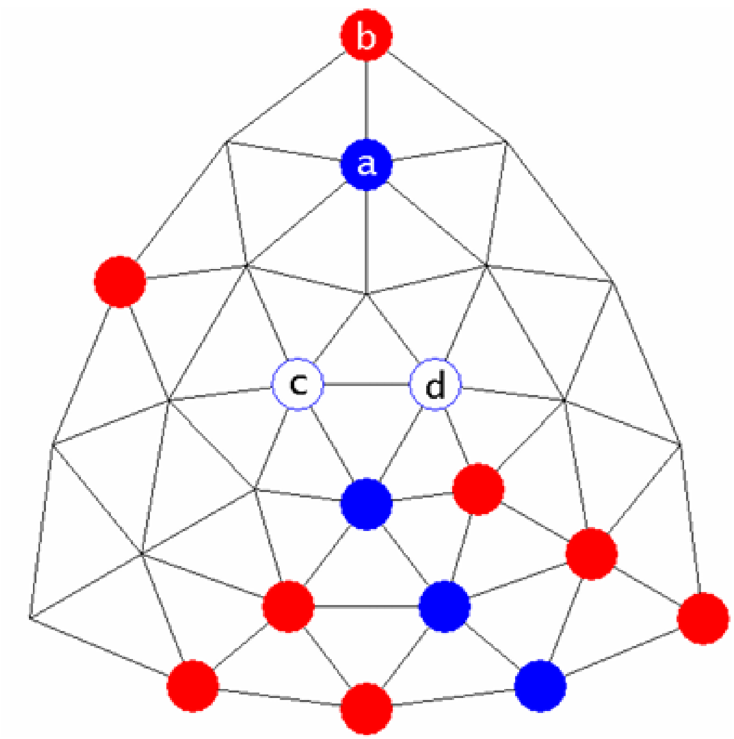
This proves that the problem can be solved in polynomial time if every card has at most one symbol.

Problem 5. (2 points) The game of Y is a connection game invented by Craige Schenstead and Charles Titus. In its original form, it is played on a triangular grid of hexagons. There are two players, who have one colour each, and a move consists of placing a piece of your colour in one of the hexagons on the board. The winner is the first player to complete a chain connecting all three sides of the board.

The inventors tried out a number of alternative playing grids, and eventually concluded that the most suitable one is the following "bent" version. The pieces are placed on the intersections (like in Go).

In the game below, the unlabeled locations show a partially played game of Y. Although blue has made only three turns, we think blue has a pretty good position. Our regular players, Lata and Raj, are in the commentary box today. Lata feels that the blue player should play on the location labeled a and predicts that red will respond with b and then blue can win by playing c or d. Raj feels that blue should play on one of c or d first (it does not matter which), and no matter what red does after this, blue will be able to pull off a win.

Which of these are valid arguments, and why?

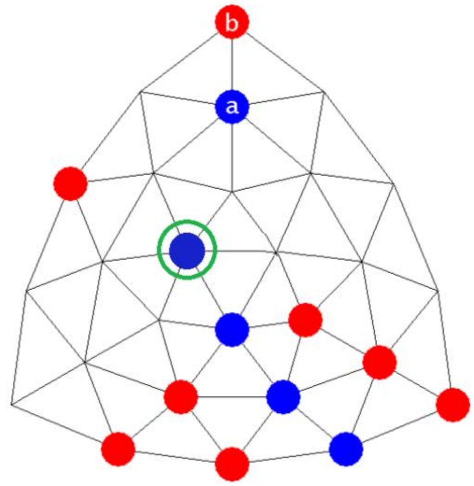


Solution 5

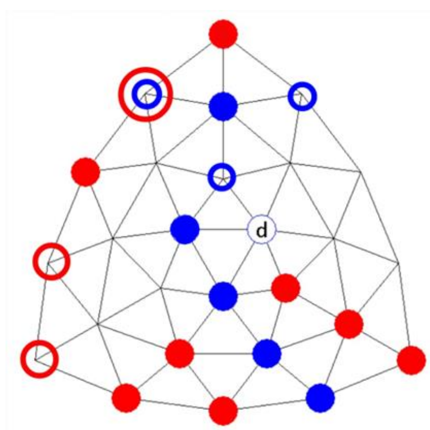
Let us simulate the prediction made by each of them.

As per Lata, blue should play on the location labelled as a. And then red counters the move by playing at location b. Next, as per Lata, blue should win by playing at either of the location c or d.

Let us assume blue played his/her move at the location c.

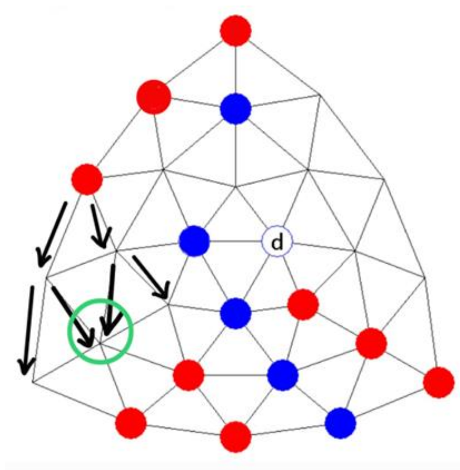


At this point, the minimum number of moves required by blue to win 3 is 3, marked as unfilled blue circles image below. Also, the minimum required moves for red to win is 3, marked as unfilled red circles in the image below.

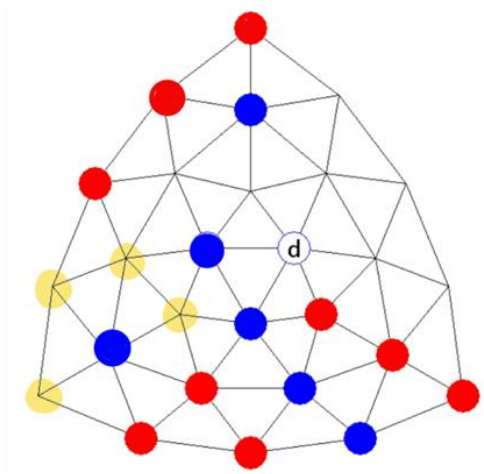


So red will choose to put the mark at overlapping position to mess up blue's winning strategy.

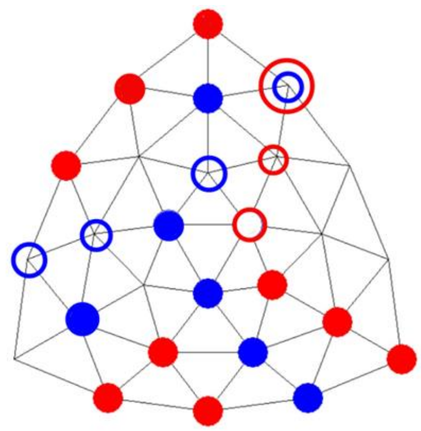
This will open up new winning possibilities for red, and as blue there is only a single position where it can play to not to lose, marked as green in the image below.



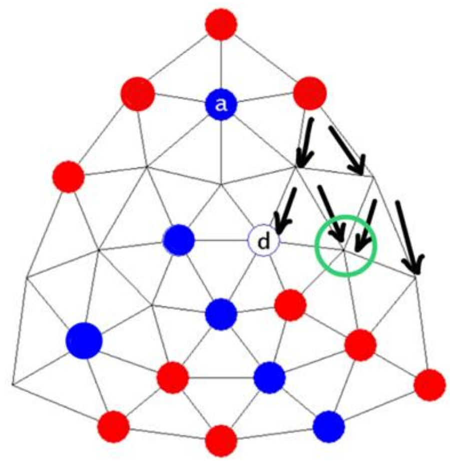
Blue will play its next move at the green marked position. This will however block any further moves by red. That is, for each move played by red in any of the yellow circle, blue has a countermove to cut off the red's spread. Thus in its next move, there is no point for red to play at the positions marked as yellow in the following image.



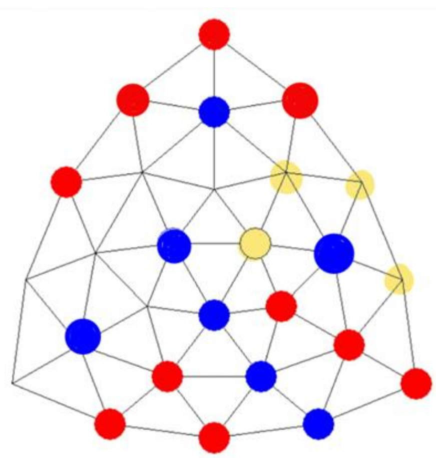
Now, blue needs at least 4 more moves to win from this point. While red needs only 3 more. Red will choose to play at a position at overlapping winning for blue, for messing up with blue's winning strategy.



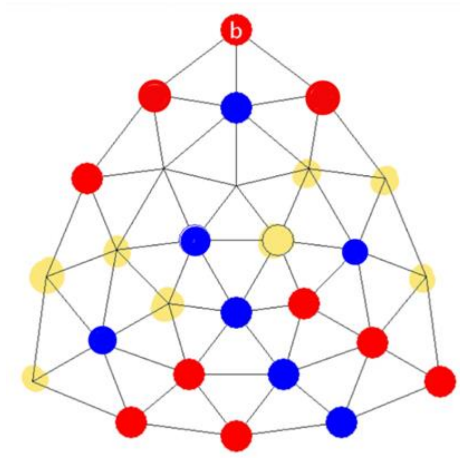
This will open up new winning positions for red again. And as blue, there is only a single position that can be played to not to lose, marked as green in the image below.



Playing so, blue now safekeeps all the positions marked with yellow, and any move that red plays on any yellow position, blue has a countermove to cut up the reds spread.



At this point, blue safekeeps all the points marked with yellow in the image below, and for each of red's move, blue has a counter-move.

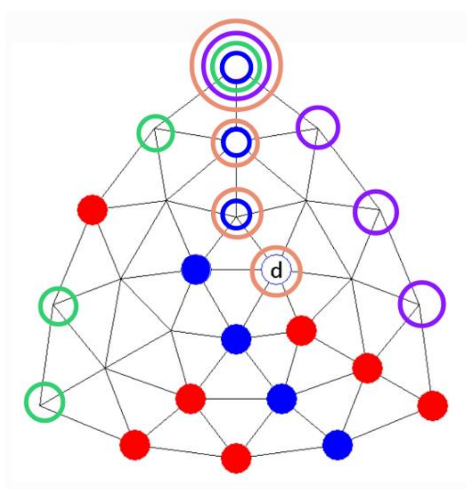


Since red cannot win this game, as blue has a countermove for each move played by red, we conclude that blue will win and **Lata was right about her prediction.**

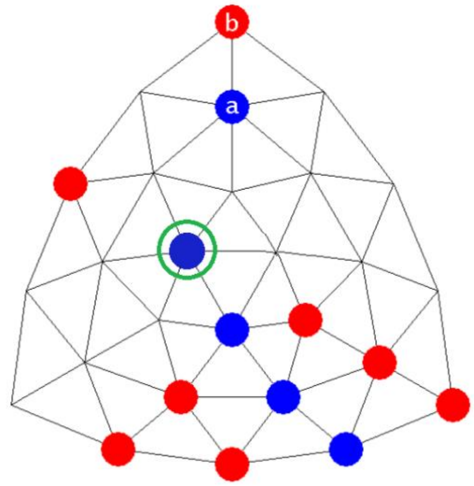
Now, this is true even if red had played d rather than c in the second move.

Now, **coming to Raj's prediction**. Let's suppose positions a and b are empty and blue play's at position c.

At red's move, we can see, red's needs a minimum of four more moves to win. And whatever winning path red has in his/her mind, either green, or violet or orange (image below), they all have in common the position 'b'. So, to maximize his/her chance of winning it's logical for red to play at position 'b' (topmost position).



After red plays at 'b', blue just needs to play at position 'a', and as we have proved that this configuration (image below) with red to move is a wining for blue (from Lata's argument), we can conclude that blue will win from this point and Raj's argument is also correct.

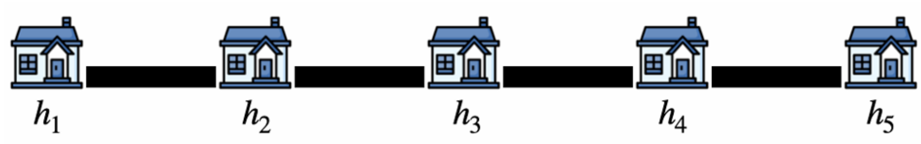


Thus, both Lata's and Raj’s argument are correct, and indeed blue packs a win!

Part 3

Problem 1.

The city of CandyLand has a row of n houses $\{h_1, \ldots, h_n\}$ which are joined by $n-1$ roads, where the i^{th} road connects h_i to h_{i+1}, as shown in the picture below (with $n = 5$).



Candyland is being attacked by an persistent and omnipresent Demon, who picks a road to attack at the start of every hour. Lata wants to save Candyland with the help of her police friends. She is allowed to position at most one guard at a house. Whenever the Demon attacks a road, at least one guard must cross the road to repair it. Our guards have limited energy, and are able to cross at most one road within the span of one hour.

The following scenarios can occur when the Demon attacks the road h_ih_{i+1}:

If there are guards in *both* houses h_i and h_{i+1}. In this scenario, the attack can be defended by the two guards swapping their positions.

If there is a guard in *exactly one of* the houses h_i and h_{i+1}, say h_i. In this scenario, the attack can be defended by the guard in h_i crossing the road and moving her position to house h_{i+1}.

If there is no guard in *either of* the houses h_i and h_{i+1}. In this scenario, the Demon succeeds and Candyland is destroyed :(

For example, if $n = 3$ and Lata positions a guard in h_2 and leaves h_1 and h_3 empty, then the demon can win by attacking h_2h_3 (which forces the guard in h_2 to move to h_3) and then attacking h_1h_2. However, if Lata positions two guards, one in h_1 and one in h_3, then Lata can defend the Demon's attacks forever.

The **defense budget** of Candyland is the minimum number of guards Lata needs to deploy to defend Candyland against the Demon forever.

The **election budget** of Candyland is the minimum number of guards Lata needs to deploy to defend Candyland against the Demon for 24 hours.

(1.a) (1 point) If Candyland has $n = 420$ houses, what is its defense budget?

Solution 1.a

For n = 420, defense budget is **419**

(1.b) (1 point) If Candyland has $n = 420$ houses, what is its election budget?

Solution 1.b

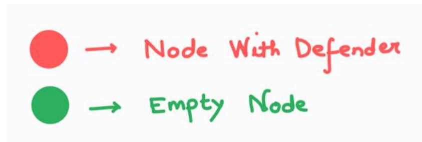
For n = 420, election budget is **233**

(1.c) (2 points) If Candyland has $n = N$ houses, what is its defense budget as a function of N? Justify your answer.

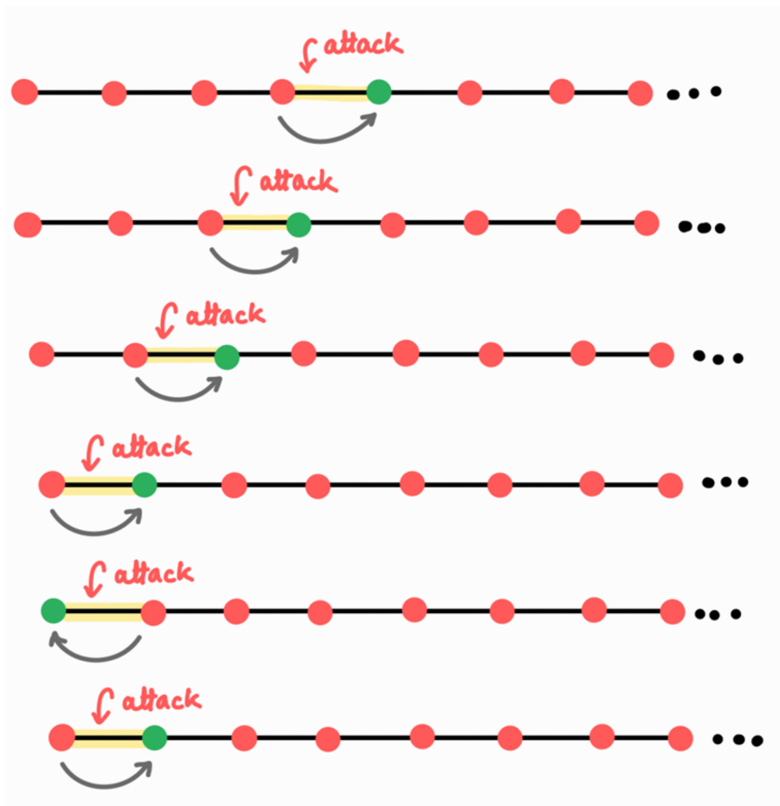
Solution 1.c

With n = N, the defense budget can be given as D(N) = N – 1.

This can be proved from the following inference. Consider the following notation –



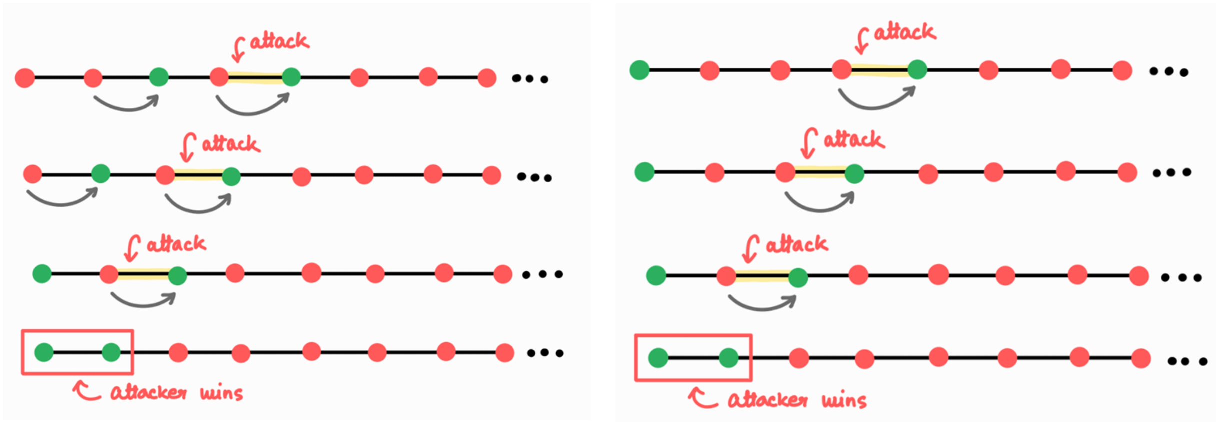
With, N – 1 guard, Candyland can be defended forever as shown below.



Now, it is clear that N – 1 guard can protect Candyland from the Demons forever. Now we need to verify if there exists any number less than N – 1 that can protect Candyland forever from the Demons.

With even N – 2 guards, the attackers have a strategy to destroy Candyland as shown in the images below.

Instance 1 Instance 2

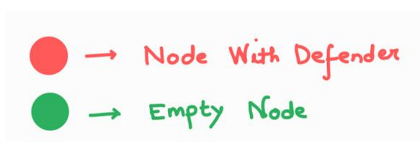


Thus, N – 1 is indeed the minimum number of guards required to protect Candyland from the Demons forever. Therefore, Defense Budget **D(N) = N – 1.**

(1.d) (2 points) If Candyland has $n = N$ houses, what is its election budget as a function of N? Justify your answer.

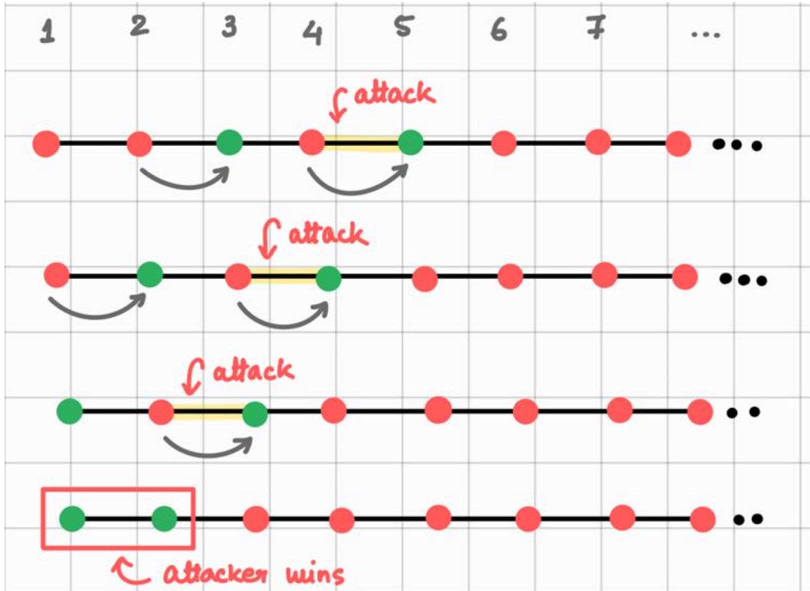
Solution 1.d

Consider the following notation –



One efficient way to protect Candyland with the minimum number of guards is to place them in alternating houses as shown below.

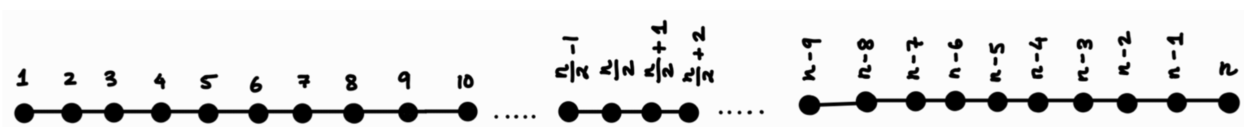
The number of guards required for this is **floor(N/2).** But this configuration will only protect Candyland for 1 hour. Also, we can infer, that if there exists an adjacent empty nodes (unguarded house) at positions p and p + 1, the attackers can corner them in at-least p attacks, and in the next, that is $(p + 1)^{th}$ attack, they can destroy Candyland, as shown in the image below.



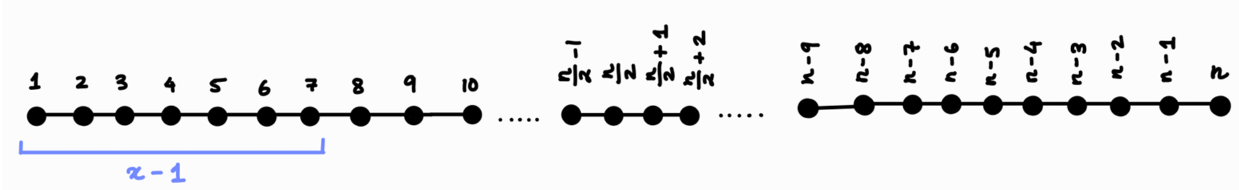
Here, the empty nodes were in the positions 3 and 5, and attacker can corner the void nodes in 3 moves and in the 4^{th} attack, they win.

Thus, protecting Candyland for 24 hours will mean protecting it from 24 attacks. This can be achieved as the following –

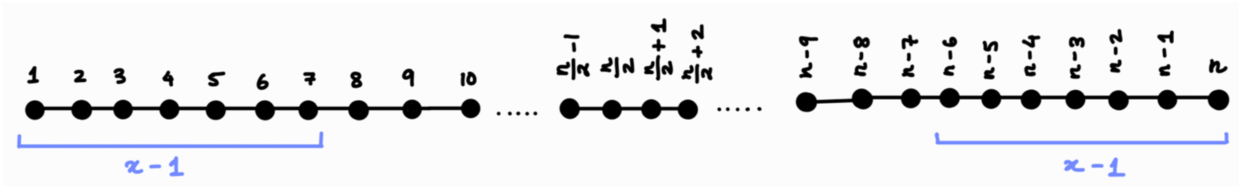
Let us assume that the *n* nodes/houses are in line as follows –



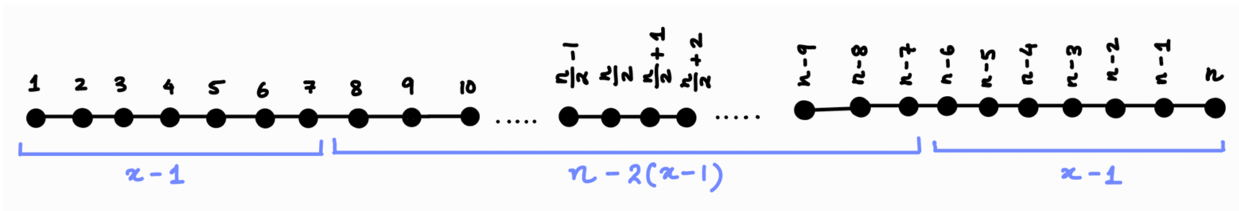
Let us assume we need to protect it from a total of *x* attacks. So, all the first ***x-1*** positions are to assign with guards, and the first unguarded position is *x*. This will ensure that it takes at least *x* moves for the attacker to corner the voids.



Now, this has to be done from the other end as well, since the map is symmetric. This results in placing of more ***x-1*** guards to the map.



Finally, in the remaining ***n - 2(x - 1)*** houses in the middle, the guards could be placed optimally, that is, $\boldsymbol{floor}\left(\frac{n-2(x-1)}{2}\right)$



Therefore, the election budget E(N) can be given as

$$\mathrm{E(N)} \;=\; 2(\mathrm{x}-1) \;+\; \mathrm{floor}\left(\frac{\mathrm{N} \;-\; 2(\mathrm{x} \;-\; 1)}{2}\right)$$

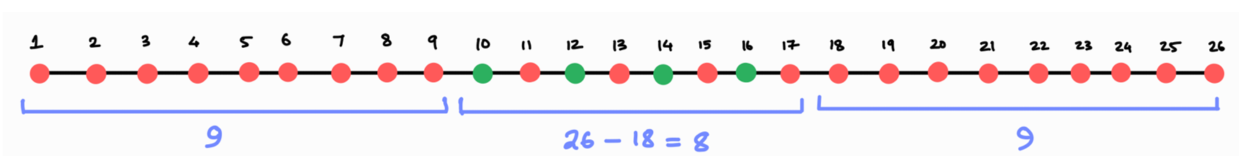
Since, *x* = *24*, we get

$$\mathbf{E(N)} \;=\; \mathbf{46} \;+\; \mathbf{floor}\left(\frac{\mathbf{N} \;-\; \mathbf{46}}{\mathbf{2}}\right)$$

Example –

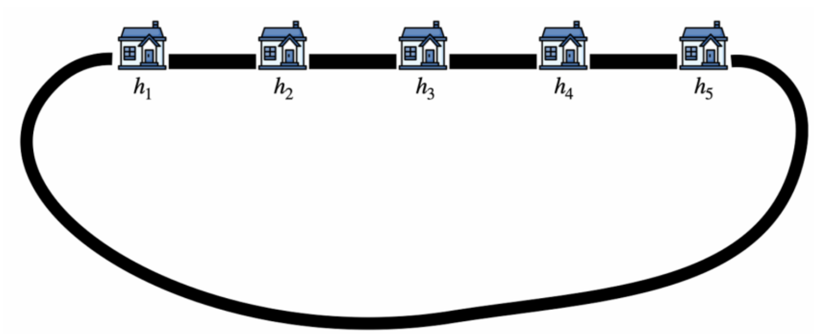
Suppose there are *N* = *26* houses, and we need to protect them for *x* = *10* hours.

Plugging the values to the equation, we get, the minimum number of guards required = 22.



(1.e) (1 point) Lata's friend, Raj, works in Lalaland, which also has n houses $\{\ell_1, \ldots, \ell_n\}$ and an identical Demon perpetually attacking it, and with the same rules of attack and defense as Candyland.

However, there is one minor difference in the road network: in Lalaland there are n roads, where the i^{th} road connects ℓ_i to ℓ_{i+1} for $1 \leqslant i \leqslant n-1$ (as before), and additionally, the n^{th} road connects ℓ_n and ℓ_1, as shown below (for $n = 5$).



If Lalaland has $n = 420$ houses, what is its defense budget?

Solution 1.e

For n = 420, the defense budget for Lalaland is **210.**

(1.f) (1 point) If Lalaland has $n = 420$ houses, what is its election budget?

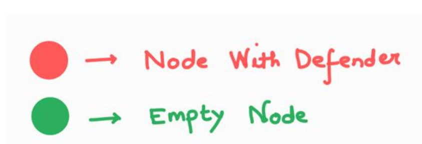
Solution 1.f

For n = 420, the election budget for Lalaland is **210.**

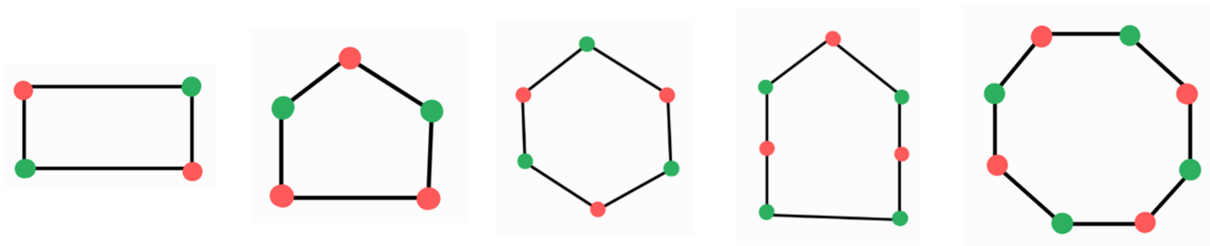
(1.g) (2 points) If Lalaland has $n = N$ houses, what is its defense budget as a function of N? Justify your answer.

Solution 1.g

Consider the following notation –

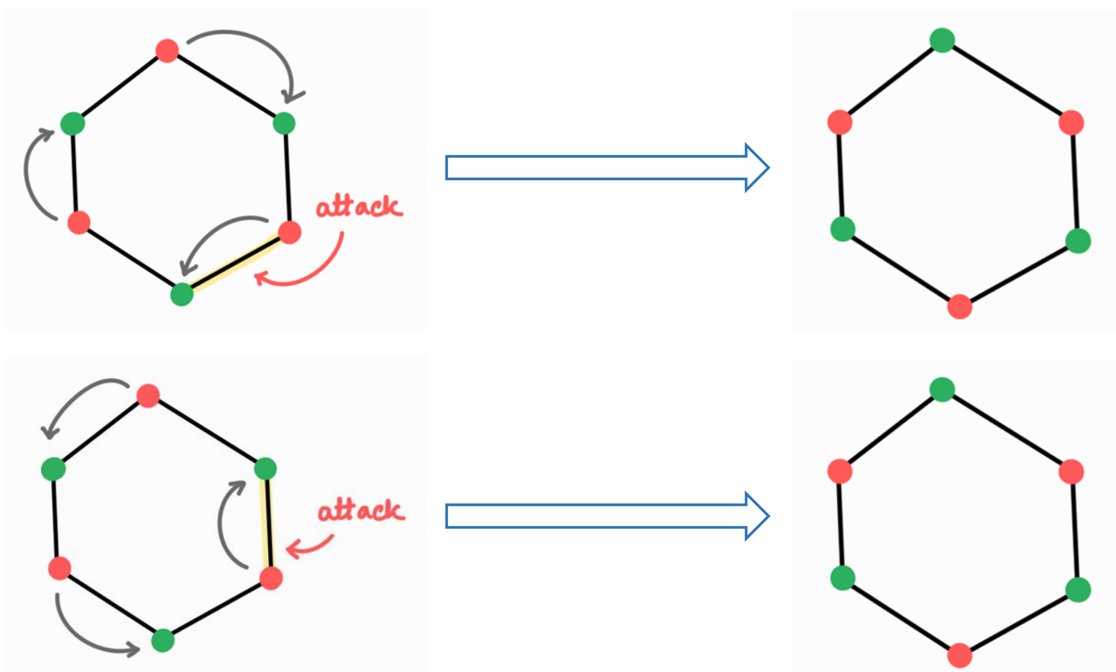


Lalaland can be visualized as a cyclic graph. The minimum number of guards required to protect Lalaland with *n* = *N* houses for 1 hour can be given as ***ceil(N/2)***. This is evident from the examples below.



n = 4 ***n = 5*** ***n = 6*** ***n = 7*** ***n = 8***

Now, it can be proved that ***ceil(N/2)*** number of guards are enough to protect Lalaland from the attacks forever. This can be proved from the following instances.



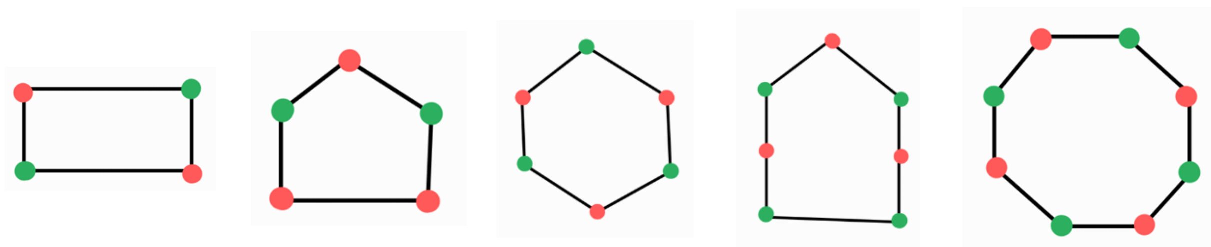
Winning strategy for the defender – Any attack from the attacker can be countered by moving all the guards to the same direction, which results in rotationally equivalent configuration. Thus, protecting Lalaland forever.

Therefore, for Lalaland, defense budget $\mathbf{D(N)} = \mathbf{ceil}\left(\frac{\mathbf{N}}{\mathbf{2}}\right)$

(1.h) (2 points) If Lalaland has $n = N$ houses, what is its election budget as a function of N? Justify your answer.

Solution 1.h

The election budget for Lalaland is the same as the defense budget. This is clear from the figure below.

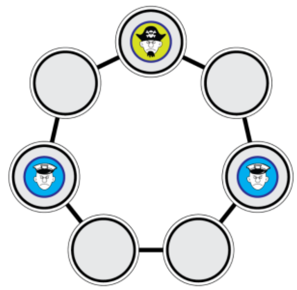


***n* = 4** ***n* = 5** ***n* = 6** ***n* = 7** ***n* = 8**

Removing even a single defender from any of the graph above will make Lalaland undefended. Thus, the minimum number of guards required to protect Lalaland with *n* = *N* houses for 24 hours can be given as ***ceil(N/2)***.

Therefore, for Lalaland, election budget $\mathbf{E(N)} = \mathbf{ceil}\left(\frac{\mathbf{N}}{\mathbf{2}}\right)$

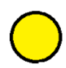
Problem 2. (3 points) Lata and Raj play a board game called *Pirate Hunters*. At each move, one of the policemen (but not both) moves to a neighboring place. In the next move, the pirate, who is faster and always jumps for two places. Policemen always move to an unoccupied place – they cannot move to a place occupied by the pirate or his colleague policeman. The game is finished when the pirate is forced to jump onto one of the policemen… which would be now (see the picture), except that it is currently the policemen turn. To win, the policemen must force the pirate into this position when it is the pirate's turn.



Lata, who plays the pirate is quite skilled at evading being captured. You are smart as well, though. If you help Raj play a perfect game, how many moves will she make before the pirate is caught?

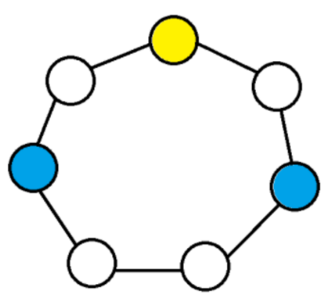
Solution 2

Consider the following notation –

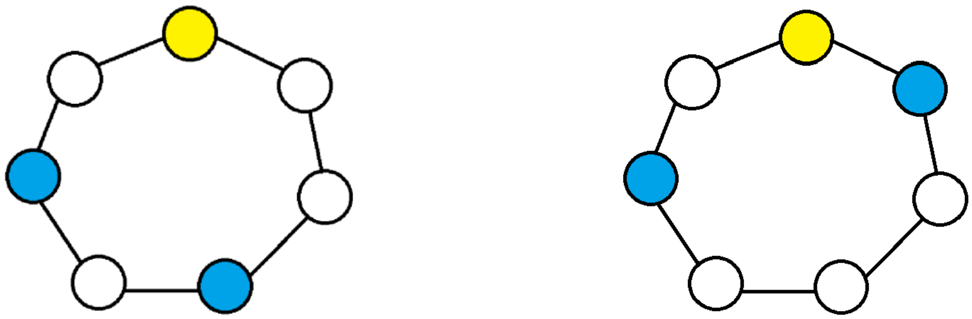
 Pirate

 Policemen

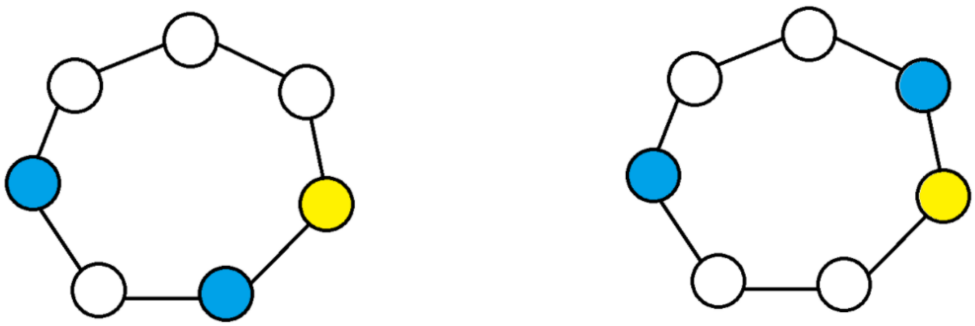
Let us assume that Raj manages to win the game. This means that the configuration of the board is as exactly as shown in the image below (which is indeed the only winning configuration for Raj) and it's Lata's chance to play.



If we trace back a move from Raj, that is undo the move Raj last played, the only two unique board configurations before Raj plays his winning move are as follows –

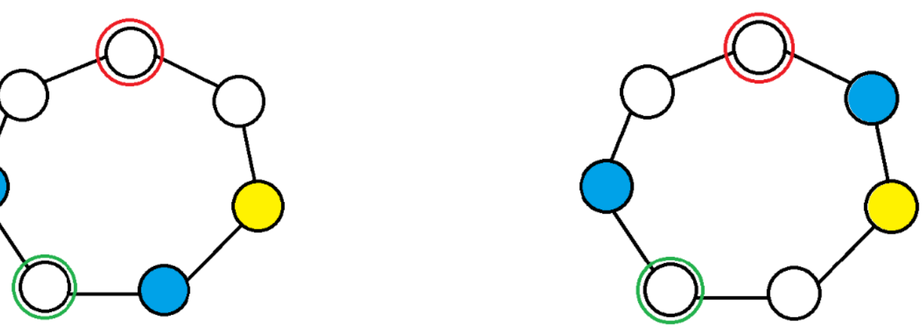


All the other possible configurations are symmetrically equivalent to these two configurations. Now, if we trace back one more move, that is, undo the move that Lata has played that has resulted in any of the board configurations listed above, we get the followings –



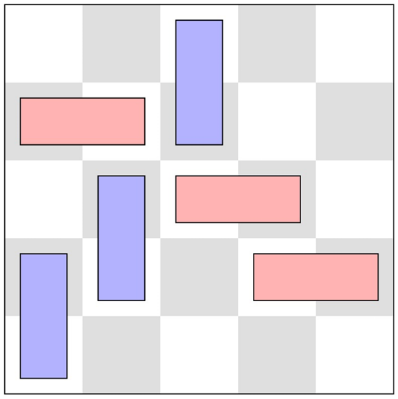
All the other possible configurations are symmetrically equivalent to these two configurations.

Now, considering Lata is skilled enough to know that moving at the position marked in red below will result in a defeat for her, she would never move her piece to that position, rather she will move her piece to green position. Also, whenever this kind of situation will arrive, Lata will always have an option, that is, there will always be a green location for her to move if she plays optimally.



Thus, **if Lata plays optimally, Raj will never be able to catch the pirate.**

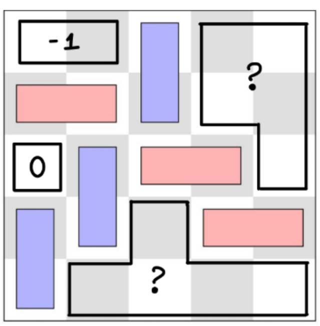
Problem 3. (2 points) What is the game value of the following domineering position? Justify your answer.



Solution 3

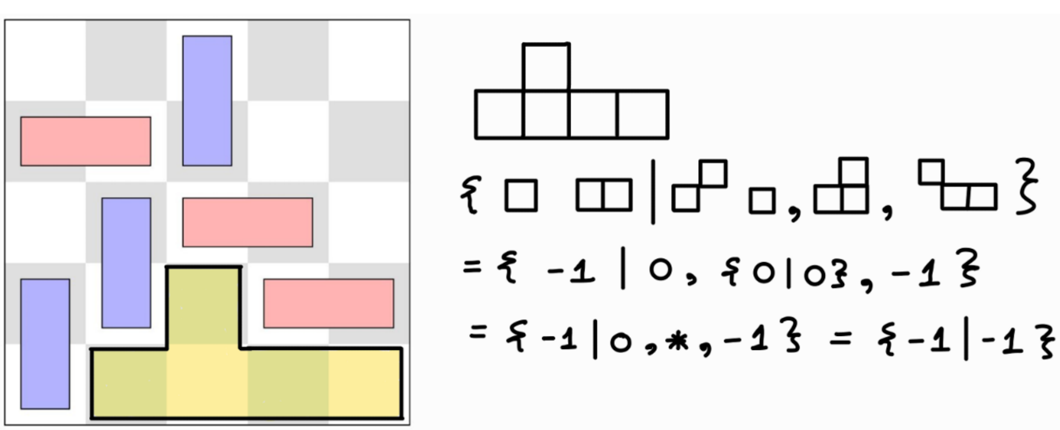
Considering the following notation,

The value of the game is the sum of the values of each segment.

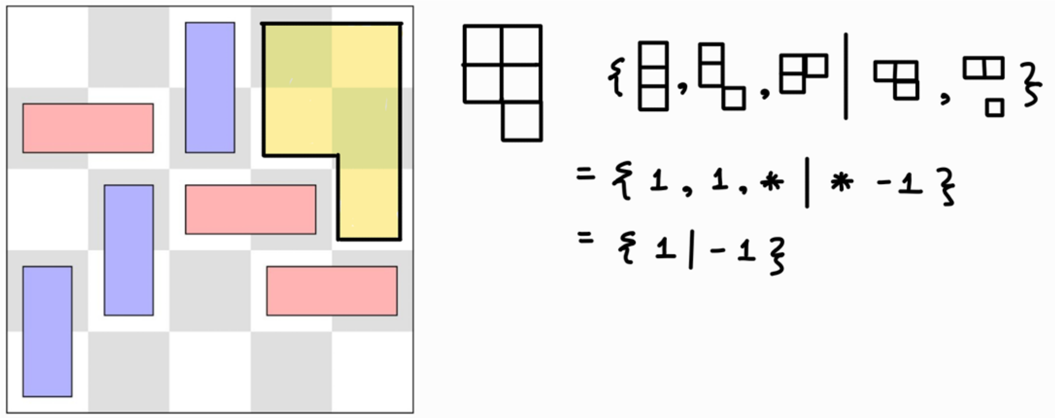


The given game consists of four segments. The value of the two of the segments are known to be 0 and -1. The value of the other two segments can be computed considering one at a time.

The value of the bottom segment can be computed as follows,



The value of the rightmost segment can be computed as follows,



Now, combining the value of all four segments we get the value of the board, that is,

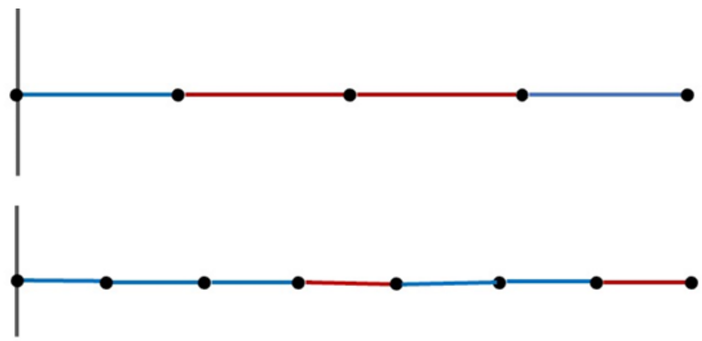
Value of the game = 0 + (-1) + {-1 | -1} + {+1 | -1}

Therefore, **the value of the game board is (-1) + {-1 | -1} + {+1 | -1}.**

Problem 4.

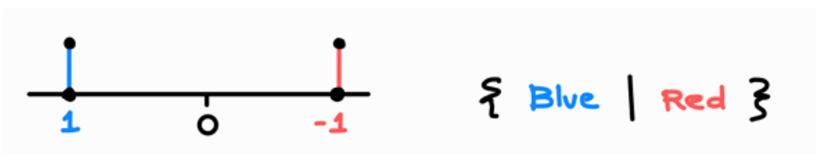
Read this article on Hackenbush.

(4.a) (2 points) Evaluate the values of the following Hackenbush games.

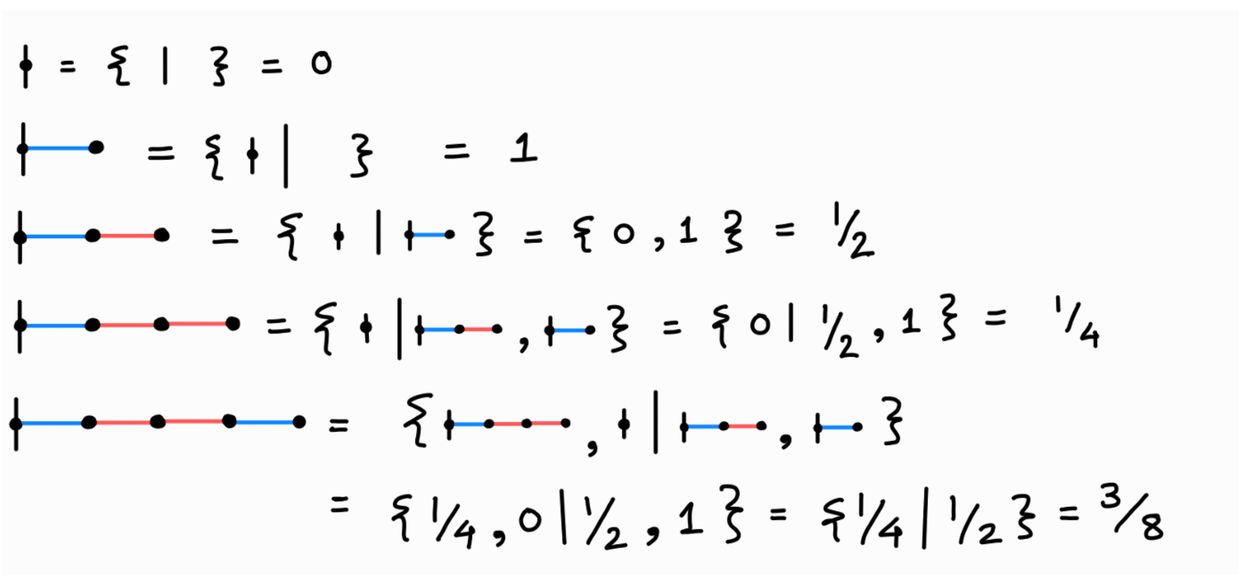


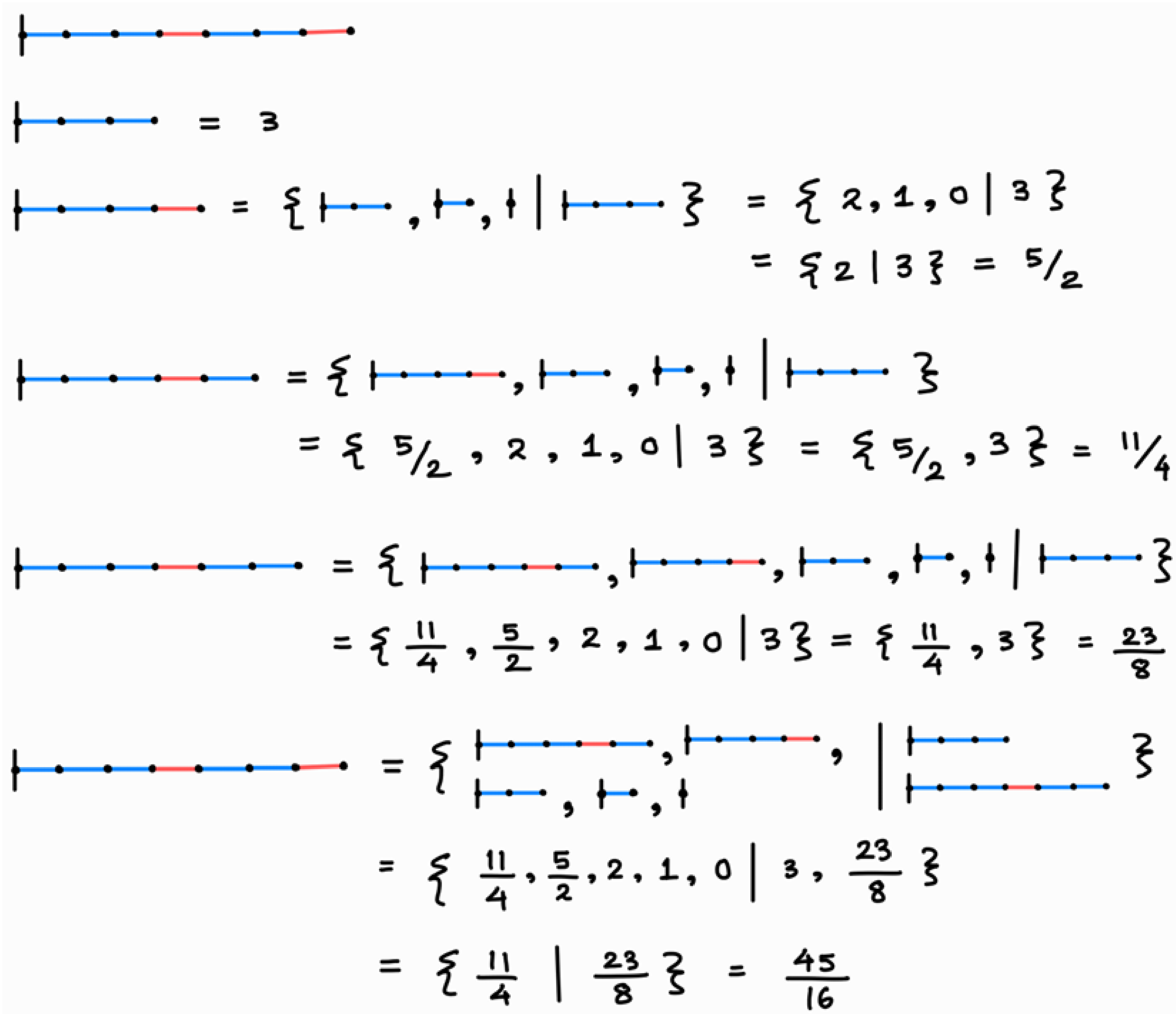
Solution 4.a

Considering the following notation,



The value of each Hackenbush pile can be obtained by bottom-up fashion.





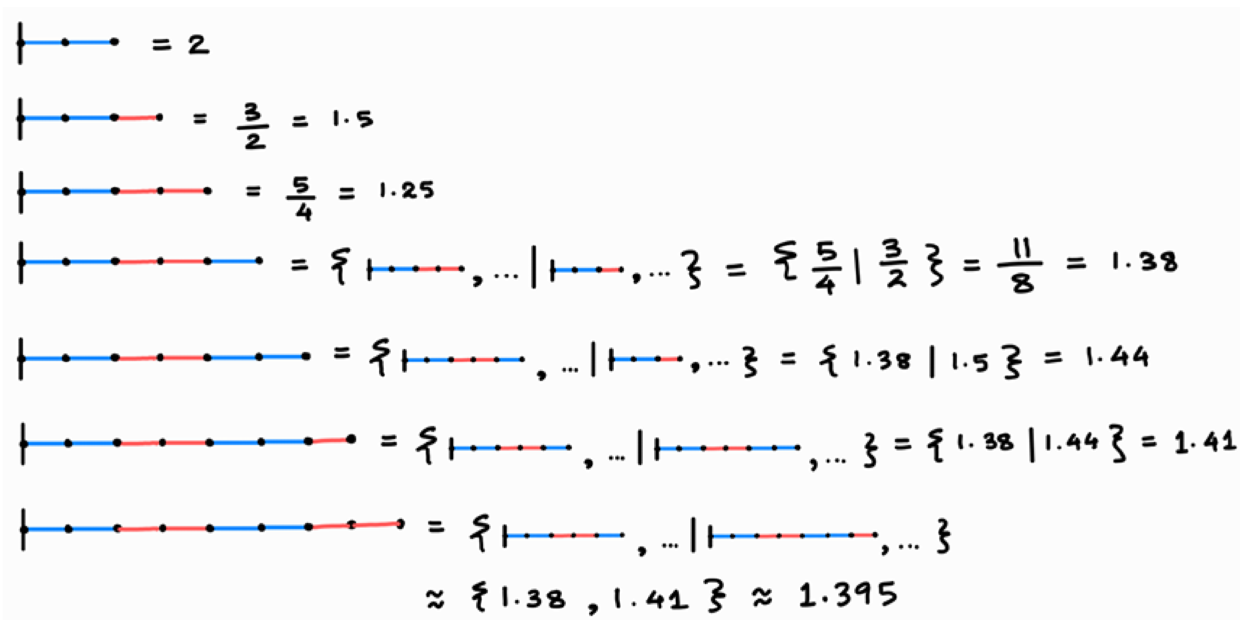
The combined value of the given Hackenbush piles can be obtained by summing the value of each pile, that is, **(3/8) + (45/16) = (51/16) = 3.187**

(4.b) (2 points) Evaluate the value of the following (infinite) Hackenbush game.

Solution 4.b

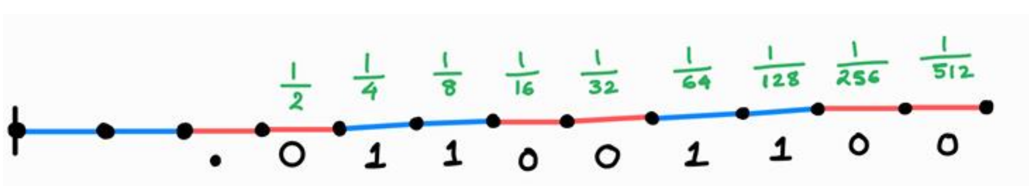
Considering the following notation,

The value of the Hackenbush pile can be obtained by bottom-up fashion.



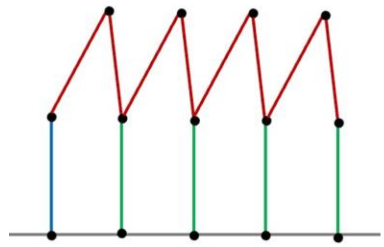
Therefore, **the value of the given infinite Hackenbush pile is approximately 1.4.**

Alternatively, the value of the Hackenbush pile can also be computed using the idea that the pile resembles its value in binary.



We know the value of the pile should be between 1 and 2. On computing, the value comes out to be **1.399**, which is approximately 1.4. Thus, **the value of the given infinite Hackenbush pile is 1.4.**

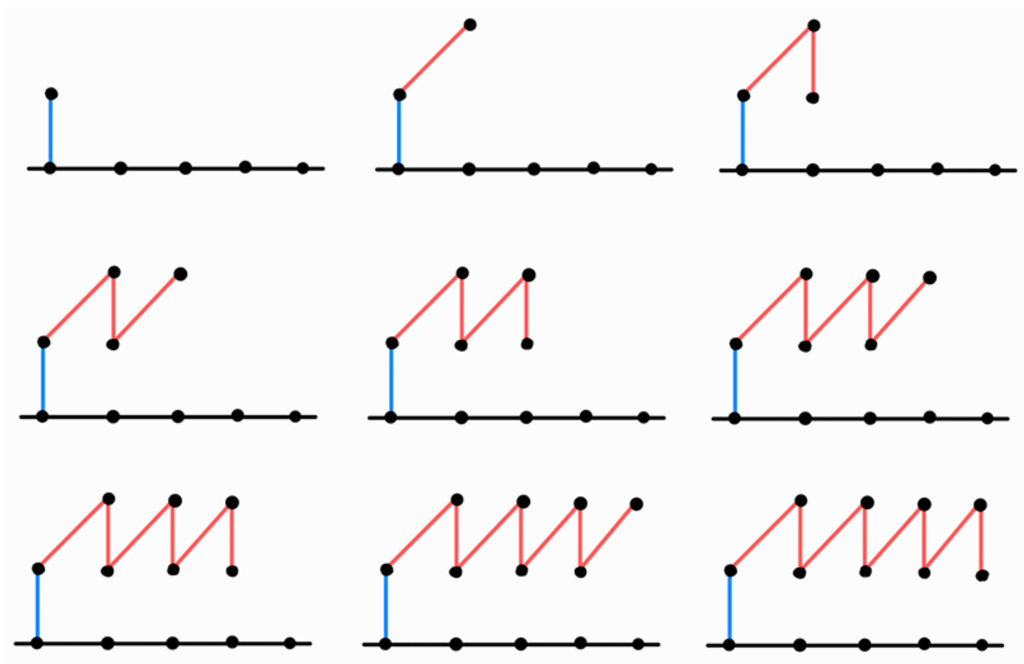
(4.c) (2 points) Who wins this Hackenbush game? On the bottom row, the left-most segment is blue and all others are green; on the top row, all segments are red.



Solution 4.c

For the given game, Blue has a winning strategy.

The following are the substates of the game that are for sure winning for blue as its values are greater than 0.



Winning strategy for Blue – It does not matter who starts the game, blue should target all the **green segments** first and should play the blue segment only if no green segment is remaining. Doing so, it is guaranteed that the game will end up as one of the nine states listed above, regardless of whatever move red chooses to play. And for sure, it's a winning for blue.

Therefore, **Blue wins the game regardless of whoever starts**.

Problem 5. (2 points) Find the value of the following Toads-Frogs game: T□FF

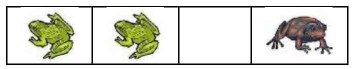
Solution

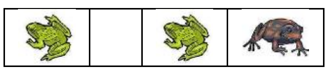
Given the game state,

Let us assume that the game has ended, the end sate of the game will look like the following,

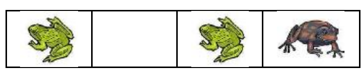
 = 0

We can evaluate the game in the reverse order.

 = {  | }
= { 0 | } = 1

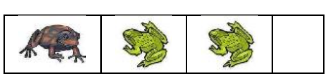
 = { |  }
= { | 0 } = -1

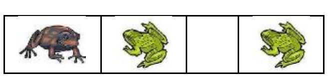
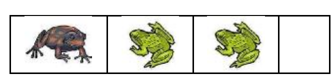
 = { |  }
= { | 1 } = 0

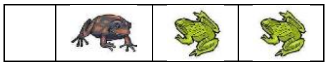
 = {  | }
= { -1 | } = 0

 = {  |  }
= { 0 | 0 } = *

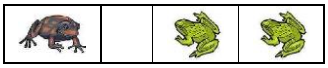
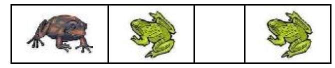
 = { |  }
= { | 0 } = -1

 = 0

 = {  |  }
= { -1 | 0 } = -1/2 = - 0.5

 = { |  }

= { | * } = 0

 = {  |  }

= { 0 | - 0.5 }

Thus, the game value of the given game state is { 0 | – 0.5 }