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Enhancement of Heat/Mass Transfer around a Sphere in Weakly Viscoelastic Flows

Abstract: We review how weak fluid elasticity (modeled by a second-order fluid, SOF) alters convection around a neutrally buoyant sphere. In an SOF the stress is $\tau = -pI + 2\eta D + \psi_1 A_2 + \psi_2 A_1$ (with ψ_1,ψ_2 the normal-stress coefficients and A_1,A_2 the first two Rivlin–Ericksen tensors). For small Deborah number (De) one expands the velocity field as **u** = **u**^(0)+De**u**^(1). The leading-order (Newtonian) flow around a freely-rotating sphere in shear has *closed* streamlines, limiting convection – Nu becomes O(1) at high Péclet number (Pe) [1]. Adding weak elasticity (SOF) breaks fore–aft symmetry: finite-De streamlines "spiral out" and approach open-streamline behavior [2]. Ganesh & Koch (2007) analytically found for large Pe, small De that

$$Nu = 0.478\,[Pe\,De\,(1+\lambda)^2(1+\varepsilon)]^{1/3},$$

where $\varepsilon = \psi_2/\psi_1$ and λ parameterizes the flow (λ=0 for simple shear) [3]. In simple shear (λ=0) this reduces to

$$Nu = 0.478\,(Pe\,De)^{1/3}\left[\frac{0.5+\psi_2/\psi_1}{1+\psi_2/\psi_1}\right]^{1/3},$$

with Deborah number De = $(\psi_1+\psi_2)$\,$\dot{\gamma}/\eta$ (η = solvent viscosity) [4]. Thus, in a weakly elastic shear flow the Nusselt number grows as (Pe·De)^1/3 multiplied by a factor depending on ψ_2/ψ_1.

Introduction: Heat/mass transfer from suspended particles is crucial in polymer processing, food sterilization, and microfluidics. In Newtonian Stokes flow around a freely-rotating sphere, convection is trapped by closed streamlines, so even at large Pe the Nusselt number saturates at a finite value (no boundary-layer growth) [1]. By contrast, if the particle is held fixed in a uniform flow or if the flow has no closed loops, Nu grows as Pe^m (e.g. ~Pe^1/2 in classic boundary-layer scaling) [5]. Here we examine the effect of small viscoelasticity (second-order fluid) on these phenomena. We outline the SOF constitutive model (ψ_1,ψ_2 definitions, De scaling), present Ganesh & Koch's analytical correlation for Nu, compare to Newtonian limits, survey experiments/simulations of viscoelastic convective transport, and discuss extensions (Oldroyd-B, FENE-P) and applications.

Theory (Second-Order Fluid Model): A second-order fluid is the first nonlinear correction to a Newtonian (zero Reynolds) fluid, capturing elasticity for De≪1. Its stress is

$$\tau = -pI + 2\eta D + \alpha_1 A_2 + \alpha_2 A_1,$$

or equivalently using normal-stress coefficients ψ_1,ψ_2 ($\psi_1 = -2\alpha_1$, $\psi_2 = -2\alpha_2$). For simple shear (rate $\dot{\gamma}$), the first normal stress difference $N_1 = \psi_1\dot{\gamma}^2 > 0$, second $N_2 = \psi_2\dot{\gamma}^2 < 0$, so typically $\psi_2/\psi_1 < 0$. The Deborah number De is the ratio of polymer relaxation time to flow time (e.g. De ~ $\psi_1\dot{\gamma}/\eta$). Expanding **u** = **u**^(0) + De **u**^(1) yields a known Stokes (u^(0)) plus an O(De) viscoelastic correction [6] [4]. In planar linear flows (parameter λ measures extension vs. vorticity), the Newtonian (De=0) flow around a freely-rotating sphere has a closed region of streamlines except when λ=1 (pure extension) [7]. In simple shear (λ=0), Newtonian streamlines are fore–aft symmetric and closed in an annulus [8]. Finite De adds a disturbance velocity **u**^(1) with a weak out-of-plane component; this breaks symmetry so that even initially closed trajectories spiral outward [2]. Ganesh & Koch proved that for Pe≫1 and De≪1 this creates a thin thermal boundary layer of thickness ~(Pe·De)^{-1/3}, leading to the above Nu~(Pe·De)^{1/3} scaling.

Analytical Nusselt Correlation (Ganesh–Koch 2007): Using matched asymptotics, Ganesh & Koch found for a torque-free sphere in a general linear flow:

$$Nu \;=\; 0.478\,[Pe\,De\,(1+\lambda)^2(1+\varepsilon)]^{1/3},$$

valid for Pe·De≪1/De, De≪1 [3]. Here $\varepsilon = \psi_2/\psi_1$ encapsulates fluid rheology, and λ=0 for simple shear flow. In shear (λ=0) this simplifies to the formula above. Physically, increasing De (with fixed Pe) shrinks the boundary layer and boosts convective transport. The bracket factor [(0.5+ε)/(1+ε)]^{1/3} shows that if ψ_2<0 (ε negative), the enhancement is somewhat reduced. For a fluid with ψ_2=0 (upper-convected Maxwell), ε=0 and the factor is (0.5)^{1/3}≈0.79.

Newtonian Comparison (Closed vs. Open Streamlines): In the Newtonian case (De=0), Frankel & Acrivos (1968) and subsequent studies showed that in shear flows Nu→const (O(1)) as Pe→∞ [1]. For example, they found Nu plateaus (~5) rather than growing with Pe. This is because fluid parcels circulate indefinitely around the sphere and a finite thermal resistance remains. In contrast, open flows (no closed loops) admit a Leveque-type boundary layer. For a fixed sphere in uniform flow or drops in extensional flows, classical theory gives Nu~Pe^1/2 for large Pe [5]. Krishnamurthy & Subramanian (2018) showed in such linear flows that Nusselt ∝Pe^{1/2} when streamlines cross from front to back [5].

Weak elasticity effectively *opens* the closed-streamline region. Ganesh & Koch illustrate numerically that a tracer starting in the closed annulus will "spiral out" rather than remain trapped [2]. This allows downstream fluid to reach previously-insulated regions, yielding Nu~(Pe·De)^{1/3} instead of const. Thus the SOF result bridges the two Newtonian limits: for fixed De, as Pe increases Nu grows as Pe^{1/3}; for fixed large Pe, as De→0 Nu→const (the Newtonian plateau).

Literature Review – Shear & Extensional Flows: Several studies (theoretical, numerical and experimental) have explored viscoelastic effects on transport around particles. Ganesh & Koch's analytical model is a cornerstone for weak elasticity. In shear flows, experiments show viscoelastic "negative wakes" (flow reversal) even at Re≈0 (e.g. falling spheres) – polymer tension pulls fluid behind the sphere [9]. Numerical studies (e.g. Pimenta & Alves 2021) confirm that elasticity reduces the drag and creates a slender heated wake. Pimenta & Alves found for a FENE-P/Oldroyd-B fluid past a hot sphere that Nu increases modestly with Wi at first, but then decreases towards or above the Newtonian value at larger Wi [9]. They report "heat transfer enhancement… for the range tested" but with Nu eventually decreasing as De grows [9]. This agrees qualitatively with SOF predictions and illustrates the role of shear-thinning: fully nonlinear models (FENE-P) have decreasing effective λ (stretch limit) which can mitigate further Nu gains.

In extensional (hyperbolic) flows or around drops, streamlines are inherently open. Krishnamurthy & Subramanian (2018) analyzed polymeric drops in such flows and found Nu~Pe^{1/2} (with elastic effects simply altering prefactors) [5]. There are fewer studies of spheres in pure extension, but one expects viscoelastic fluids to reduce extensional viscosity (Trouton effect) and thus modestly alter the boundary layer. Overall, studies indicate that even weak elasticity significantly raises Nu/Sh compared to Newtonian, but the enhancement typically grows only as a fractional power of De or Wi, and can saturate if polymer stretch becomes limited.

Extensions to Oldroyd-B/FENE-P Fluids: Going beyond SOF, realistic polymeric fluids (Oldroyd-B, FENE-P) introduce finite extensibility and shear-thinning. In these models the effective Deborah (Weissenberg) number $Wi = \lambda\dot{\gamma}$ can be large. Initially increasing Wi breaks streamlines and enhances transport (via the SOF mechanism). However, at moderate-to-large Wi, nonlinear effects (e.g. coil-stretch, FENE finite extensibility) kick in. Some numerical studies report optimal Wi for maximum Nu: beyond this, polymer strain hardening increases flow resistance and can reduce transport. For example, in unbounded shear, energy dissipation by stretched polymers can thicken the thermal boundary layer. Elastic instabilities also emerge at higher Wi: even flows that were stable at small De become time-dependent. Indeed, experiments and simulations show that above a critical Wi one observes elastic instabilities and "elastic turbulence" – chaotic flow states that dramatically enhance mixing [10]. Although elastic turbulence can boost mixing in bulk flows, it is less clear in simple shear – some theoretical work suggests only subcritical (finite-perturbation) instabilities in rectilinear flows [11]. Nonetheless, in curved-streamline systems (e.g. Taylor–Couette, flow past cylinders), elastic turbulence leads to near-homogeneous mixing. Future modeling efforts must incorporate these nonlinear behaviors: for instance, solving Oldroyd-B or Giesekus equations around a sphere to capture negative wakes, transient effects and instabilities.

Results and Visualizations: The analytical SOF results predict a clear trend: Nu ~ (Pe·De)^{1/3}. If one plots Nu vs (Pe·De) (on log–log axes), theory gives a straight line of slope 1/3, in contrast to the flat line (slope 0) for a Newtonian closed-streamline case. Unfortunately, high-fidelity simulation data for validation are scarce. (Ganesh & Koch did compare with their computed streamlines and temperature fields, confirming the scaling.) One conceptual illustration is that of streamlines in the flow-gradient plane: in the Newtonian limit they are closed loops around the sphere, whereas at finite De they become spirals ending downstream [2]. Although we lack published plots of Nu(Pe,De), the SOF correlation implies that contours of constant Nu would curve in the Pe–De plane as $De \sim Nu^3/Pe$.

Numerical Codes: There are few open-source codes tailored to this niche. General CFD tools (OpenFOAM with rheoTool, FEniCS scripts, or commercial software) can simulate Oldroyd-B or FENE-P flows, but examples of sphere heat-transfer are not widely published. One related resource is *rheoTool* (an OpenFOAM library) for viscoelastic flow, which could be adapted. For low-Re Stokes flows, one could also implement the SOF boundary-value problem in COMSOL or MATLAB by perturbation (using the expressions for **u**^(1) from Ganesh & Koch as known inputs). We are not aware of a publicly available notebook for Ganesh–Koch's problem specifically.

Implications for Applications: These findings have practical relevance. In microfluidics, viscoelastic mixing is a hot topic: even very slow flows can achieve rapid mixing if polymers are present [12] [10]. Zizzari *et al.* (2020) demonstrated that adding elastic micromotors in a microreactor cut mixing times drastically [12] – a direct analog of breaking closed-streamlines. For particle heating/cooling (e.g. in reactors or polymerization), enhancing convective heat transfer means more uniform temperature fields. In industrial polymer processing (extrusion, emulsification), understanding how additives change particle heat/mass transfer can inform design: e.g. polymer suspensions in lubrication (continuously stirred tank reactors) or biomedical flows (blood cells in viscoelastic plasma). We note, however, that strong elasticity can also cause particle migration (viscoelastic lift) and inhomogeneity in suspensions, which interacts with heat transfer.

Future Work: Connecting SOF theory to fully nonlinear models is an open challenge. One route is to use the SOF predictions as a baseline for small-De flows, then extend asymptotics or numerical continuation to higher Wi. Experimentally, validation is needed: e.g. measure heat transfer from small heated beads in dilute polymer solutions (varying Wi) to test the (Pe·De)^{1/3} law. Another direction is to study unsteady and finite-Re effects: while De→0 theory is steady, real polymers often exhibit elastic instabilities at higher Wi, which could further augment (or degrade) mixing. Finally, these insights should feed back to engineering: designing mixers or heat exchangers that exploit elasticity (elastic turbulence mixers are already used). Ultimately, bridging the gap between idealized SOF theory and "real" fluid (FENE-P/Oldroyd-B) in complex geometries will enable optimized processes in microfluidics, chemical reactors, and polymeric systems.

References: Ganesh & Koch (2007) derive the SOF Nusselt scaling [3] [4]. Classical Newtonian results are reviewed by Frankel & Acrivos and by Krishnamurthy & Subramanian [1] [5]. Pimenta & Alves (2021) provide simulations for FENE-P flows [9]. Zizzari *et al.* (2020) experimentally demonstrated elastic mixing in microfluidics [12]. Van Buel & Stark (2022) review elastic turbulence and mixing [10] [11]. (Additional sources: Acrivos 1971; Cox & Mason 1968; Pakdel & McKinley 1996; Leal & Hinch 1971; Larson 1988 – for background on viscoelastic flows and instabilities.)

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