One Sphere in Stokes flow

Taylor Series:- $\underbrace{u^\infty(x)}_{\text{velocity field without any disturbance due to particle.}} = \overbrace{u^\infty(x_0)}^{U^\infty} + \nabla u^\infty(x_0)\cdot(x - x_0) + \ldots\ldots$

{Neglecting the higher order terms in the series, assuming $(x-x_0) \to$ small}.

we get, $\underset{\sim}{u}^\infty(x) = U^\infty + \Omega^\infty\cdot x + E^\infty\cdot x$

i.e. $\nabla u^\infty = \Omega^\infty + E^\infty$

(Rate of rotation) $\Omega^\infty_{ij} = \frac{1}{2}\left[\frac{\partial u_i^\infty}{\partial x_j} - \frac{\partial u_j^\infty}{\partial x_i}\right]$

(Rate of strain) $E^\infty_{ij} = \frac{1}{2}\left[\frac{\partial u_i^\infty}{\partial x_j} + \frac{\partial u_j^\infty}{\partial x_i}\right]$

$$\underset{\sim}{\omega}_i^\infty = -\frac{1}{2}\epsilon_{ijk}\,\Omega^\infty_{jk} = \frac{1}{2}(\nabla\times u^\infty)_i = \frac{1}{2}\tilde{\omega}_i^\infty$$

$\underset{\sim}{\omega}_i^\infty \Rightarrow$ Rotation vector (Pseudo vector).

$\tilde{\omega}_i^\infty \Rightarrow$ Vorticity.

Pseudo vector:- A quantity represented by a vector but in which there is an arbitrary choice of one, from a possible two directions.

Note:- Because we consider stokes flow, we may apply linearity to consider separately each of the three problems →
i) a particle in uniform translation.
ii) a particle in pure rotation.
iii) a particle in pure straining.

And then superimpose ~~the~~ the solutions to obtain the result of stated case.

Rotation

$$\boxed{\underset{\sim}{u}(x) = \underset{\sim}{u}^{act}(x) - \underset{\sim}{u}^{\infty}(x)}$$

$u^{act}(x) \longrightarrow$ the velocity (actual) at which the particle moves.

$\underset{\sim}{\omega}^{\infty} \times \underset{\sim}{X} = u^{\infty}(x) \longrightarrow$ the velocity imposed in the fluid for moving the particle.

$u(x) \rightarrow$ disturbance velocity.

Similarly, disturbance pressure field :-

$$\boxed{p(x) = p^{act}(x) - p^{\infty}(x)}$$

Now, Governing equations:-

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$$

$$\mu \nabla^2 \underset{\sim}{u} = \underset{\sim}{\nabla} p$$

B-C :-

Pseudo vector

$\underset{\sim}{u} = -\omega^{\infty} \times \underset{\sim}{X}$ at $r = |\underset{\sim}{X}| = a$

$\underset{\sim}{u} \longrightarrow 0$ as $r = |\underset{\sim}{X}| \longrightarrow \infty$
$p \longrightarrow 0$

{ Decaying vector Harmonics. }

Note :- The apparent rotation of the particle seen in the velocity B-C, is the deviation in the particle rotation rate (which is zero because its fixed) from bulk rotation far away.

By the theorem of uniqueness of stokes flow, ~~[illegible]~~ if we have found a soln. to the stokes eqns. which satisfies the B-C of a problem, one has found the soln.

Now, velocity is a true vector and pressure is a scalar but the rotation vector ω^{∞} is a pseudo vector so, as in B-C we have to do vector (or cross) product in order to obtain a true vector.

Decaying vector Harmonics:- As we say, $\nabla^2 p = 0$

$\nabla^2 \underset{\sim}{\omega} = 0$

Hence, pressure is a vector Harmonics which decays to zero at far away from the particle.

For spherical symmetric ϕ_1, $\frac{1}{r}$ is the soln Hence it must satisfy,

$$\nabla^2 \left(\frac{1}{r}\right) = 0$$

$r \to$ radial distence

$\underset{\sim}{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\underset{\sim}{x}$ is coordinate

$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$r = \sqrt{x_i^2}$ in Einstein Notation.

$$\phi_1 = \frac{1}{r}; \quad \phi_2 = \underset{\sim}{\nabla}\left(\frac{1}{r}\right) = \frac{-\underset{\sim}{x}_i}{r^3};$$

$$\phi_3 = \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5}; \ldots$$

These are termed as spherical solid Harmonics.

Soln for disturbance pressure:- Since pressure is a real scalar which must be linear in $-\omega^\infty$, ~~...~~ Hence we can see that the only scalar which can be formed using a product of a harmonic function and ω^∞ is

$$p(x) \propto \frac{\omega_i^\infty x_i}{r^3}$$

Even this has a problem, because the pseudo vector ω^∞ appears and the convention used for its direction is used only once in this expression; therefore the coefficient of this form for the pressure must be zero, implying that there is no pressure induced by fixing a sphere in the rotating fluid at $Re = 0$,

Hence, disturbence pressure field : $p = 0$.

Now, Because the pressure is zero, the disturbance velocity also satisfies the laplace eqn. It should be a true vector and linear in $-\omega^\infty$

$$\underset{\sim}{u}(x) = \lambda_1 \underset{\sim}{\omega}^\infty \times \frac{\underset{\sim}{x}}{r^3}$$

Applying B-C:-

$$\underset{\sim}{u} = -\underset{\sim}{\omega}^{\infty} \times \underset{\sim}{x} \quad \text{at } r = |\underset{\sim}{x}| = a$$

$$-\underset{\sim}{\omega}^{\infty} \times \underset{\sim}{x} = \lambda_1 \, \underset{\sim}{\omega}^{\infty} \times \frac{\underset{\sim}{x}}{a^3}$$

$$\boxed{\lambda_1 = -a^3}$$

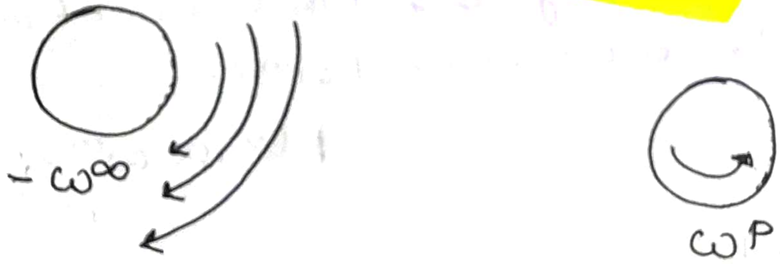
So, $$\boxed{\underset{\sim}{u}(x) = -\underset{\sim}{\omega}^{\infty} \times \underset{\sim}{x} \left(\frac{a}{r}\right)^3}$$

$$\boxed{p(x) = 0}$$

If particle rotates at ω^P and there is no induced pressure, then fluid velocity r (substituting $\underset{\sim}{\omega}^P = -\underset{\sim}{\omega}^{\infty}$) comes out to be

$$\boxed{\underset{\sim}{u}(x) = \underset{\sim}{\omega}^P \times \underset{\sim}{x} \left(\frac{a}{r}\right)^3}$$

Hence, we conclude that there is no pressure induced by the presence of the sphere in this rotating fluid, and the velocity retains the symmetry of the B-C on the sphere and decays as r^{-2}.



Translation

This problem represents the motion of a particle moving at $-\underset{\sim}{U}^{\infty}$ and thus is basic to such problems as sedimentation.

G.E:-

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u}=0$$

$$\mu\nabla^2\underset{\sim}{u}=\underset{\sim}{\nabla}p$$

Boundary condⁿ:- $\underset{\sim}{u}=-\underset{\sim}{U}^{\infty}$ at $r=|\underset{\sim}{x}|=a$

$\underset{\sim}{u}\to 0$

$p\to 0$ as $r=|\underset{\sim}{x}|\to\infty$

For disturbance pressure field:-

The pressure should be linear in $-\underset{\sim}{U}^{\infty}$. Hence the only way is to obtain a scalar with U^{∞} taking dot product with vector Harmonic solⁿ: $\phi_2=\frac{\underset{\sim}{x}}{r^3}$.

$$p(x)=\lambda_1\,\underset{\sim}{U}^{\infty}\cdot\frac{\underset{\sim}{x}}{r^3}$$

$$p(x)=\lambda_1\,\frac{U_j^{\infty}x_j}{r^3}$$

Now, as per Homogenous Stokes equation, $\mu\nabla^2\underset{\sim}{u}=\underset{\sim}{\nabla}p$, the solⁿ of velocity has two parts $u^{(P)}$ (particular solⁿ) and $u^{(h)}$ (homogenous solⁿ).

$$u^{(P)}=\frac{cp}{\mu}\underset{\sim}{x}$$

Where, $c=\frac{1}{2}$ $\Rightarrow$ $\mu\nabla^2\left(\frac{cp\underset{\sim}{x}}{\mu}\right)=\underset{\sim}{\nabla}p$

$$\boxed{u^{(P)}=\frac{p}{2\mu}\underset{\sim}{x}}$$

$c\underset{\sim}{\nabla}(\underset{\sim}{\nabla}px+p)=\underset{\sim}{\nabla}p$

$c(\nabla^2px+\underset{\sim}{\nabla}p+\underset{\sim}{\nabla}p)=\underset{\sim}{\nabla}p$

$c(\cancel{\nabla^2 px}^{\,0})=\underset{\sim}{\nabla}p$

$+\,c(2\underset{\sim}{\nabla}p)$

$c=1/2.$

For $u^{(h)}$:- Since the velocity is a vector, linear in $\underset{\sim}{U}^{\infty}$, there are two ways to build it. One is by forming the product of $\underset{\sim}{U}^{\infty}$ with the scalar Harmonic and the other is by contracting U^{∞} with second-rank tensor Harmonic.

$$\boxed{u^{(h)} = \lambda_2 \underset{\sim}{U}^{\infty}\frac{1}{r} + \lambda_3\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)\cdot\underset{\sim}{U}^{\infty}}$$

To find λ_2 and λ_3

we use, $\nabla\cdot\underset{\sim}{u} = 0$

$$0 = \frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(u^{(h)} + \frac{p}{2\mu}x_i\right)$$

$$u_i = \frac{\lambda_1}{2\mu}\frac{U_j^{\infty}x_j}{r^3}x_i + \lambda_2\frac{U_i^{\infty}}{r} + \lambda_3 U_j\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)$$

$$u_i = \frac{\lambda_1}{2\mu}x_i U_j^{\infty}\phi_j^{(1)} + \lambda_2 U_i^{\infty}\phi^{(0)} + \lambda_3 U_j\phi_{ij}^{(2)}$$

$$\frac{\partial u_i}{\partial x_i} = \frac{\lambda_1}{2\mu}\left[\delta_{ii}U_j^{\infty}\phi_j^{(1)} + x_iU_j^{\infty}\frac{\partial\phi_j^{(1)}}{\partial x_i}\right] + \lambda_2 U_i^{\infty}\frac{\partial(\phi^{(0)})}{\partial x_i}$$
$$+ \cancel{\lambda_3 U_j\frac{\partial}{\partial x_i}(\phi_{ij}^{(2)})} \to 0$$

Now, $\frac{\partial}{\partial x_j}(\phi_{ij}^{(2)}) = 0$

as, $$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}\left(\frac{\partial\phi^{(0)}}{\partial x_j}\right)\right) = \frac{\partial}{\partial x_i}\left(\underbrace{\nabla^2\phi^{(0)}}_{=0}\right) = 0$$

$$\frac{\partial u_i}{\partial x_i} = \frac{\lambda_1}{2\mu}\left[3U_j^{\infty}\phi_j^{(1)} + x_iU_j^{\infty}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)\right]$$
$$+ \lambda_2 U_i^{\infty}\left(\frac{-x_i}{r^3}\right)$$

$$\frac{\partial u_i}{\partial x_i} = -\lambda_2 \frac{U_j^\infty x_i}{r^3} + \frac{\lambda_1}{2\mu}\left[3U_j^\infty \cancel{\left(\frac{+x_i}{r^3}\right)} + \frac{U_j^\infty x_i}{r^3} - \cancel{\frac{3x_i^2 x_j U_j^\infty}{r^5}}\right]$$

$$\frac{\partial u_i}{\partial x_i} = -\lambda_2 \frac{U_j^\infty x_i}{r^3} + \frac{\lambda_1}{2\mu}\left[\frac{U_j^\infty x_i}{r^3}\right]$$

and ~~also~~ since, $\frac{\partial u_i}{\partial x_i} = 0$

$$\boxed{\frac{\lambda_1}{2\mu} = \lambda_2}$$

Now, Applying Boundary conditions:-

$u = -U^\infty$ at $r = |\underset{\sim}{x}| = a.$

$$u_i = \lambda_2\left[\frac{U_j^\infty}{r} + \frac{U_j^\infty x_j x_i}{r^3}\right] + \lambda_3 U_j^\infty\left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right]$$

$$u_i = U_i^\infty\left[\frac{\lambda_2}{r} + \frac{\lambda_3}{r^3}\right] + U_j^\infty x_i x_j\left[\frac{\lambda_2}{r^3} - \frac{3\lambda_3}{r^5}\right]$$

$$\underset{\sim}{u} = \underset{\sim}{U}\left[\frac{\lambda_2}{r} + \frac{\lambda_3}{r^3}\right] + \underset{\sim}{x}(\underset{\sim}{U}\cdot\underset{\sim}{x})\left[\frac{\lambda_2}{r^3} - \frac{3\lambda_3}{r^5}\right]$$

At $r = |\underset{\sim}{x}| = a$, $u = -U^\infty$

i.e.

$$\frac{\lambda_2}{a} + \frac{\lambda_3}{a^3} = -1$$

$$\frac{\lambda_2}{a^3} - \frac{3\lambda_3}{a^5} = 0$$

$$\frac{\lambda_2}{\cancel{a^3}} = \frac{3\lambda_3}{a^2}$$

$$\lambda_2 = \frac{3}{a^2}\lambda_3$$

$$\frac{3\lambda_3}{a^3} + \frac{\lambda_3}{a^3} = -1$$

$$\lambda_3 = \frac{-a^3}{4} \quad \Big| \quad \lambda_2 = -\frac{3a}{4} \quad \Big| \quad \lambda_1 = \frac{-3\mu a}{2}$$

Hence, $$\boxed{u_i = -\frac{3a}{4} U_j^{\infty}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right) - \frac{3a^3}{4} U_j^{\infty}\left(\frac{\delta_{ij}}{3r^3} - \frac{x_i x_j}{r^5}\right)}$$

$$\boxed{P - P_\infty = -\frac{3\mu a}{2} \frac{U_j^{\infty} x_j}{r^3}}$$

Note:- A result of considerable importance is that the disturbance fields decay very slowly from the translating sphere, as r^{-2} for the pressure and as r^{-1} for the dominant portion of the velocity. The velocity also includes a portion that decays more rapidly as r^{-3}.

Two velocity fields are plotted in Mathematica
i.) (Disturbance streamlines for a translating sphere).
ii.) Full streamlines for a particle fixed in uniform stream.

Straining

The third basic case we examine is that of a particle fixed in the straining flow $E^{\infty} \cdot x$. Because of symmetry, we note that fixing the particle in this flow doesnot require the exertion of either force or torque.

Governing eqn. is the same, $\nabla \cdot u = 0$
$$\mu \nabla^2 u = \nabla p$$

Only the boundary conditions has changed.

B-Cs:- disturbance velocity $u = -\underset{\approx}{E}^{\infty} \cdot x$ at $r = |x| = a$

$u \to 0$
$p \to 0$ as $r = |x| \to \infty$.

The complexity of the problem increases because the solution is now linear in the second rank tensor $\underset{\approx}{E}^{\infty}$.

For disturbance pressure:-

Since, Pressure is a scalar and should be linear in the rate of strain so, $p = C_0 \underset{\approx}{E}^{\infty} \cdot \left(\frac{1}{r}\right) + C_1 \underset{\approx}{E}^{\infty} \cdot \nabla\left(\frac{1}{r}\right) + C_2 \underset{\approx}{E}^{\infty} : \nabla\nabla\left(\frac{1}{r}\right)$

tensor $\to 0$, vector $\to 0$, scalar

So,

$$p(x) = \lambda_1 E_{ij}^{\infty} \partial_j \partial_i \left(\frac{1}{r}\right)$$

$$p(x) = \lambda_1 E_{ij}^{\infty} \left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right)$$

$$p(x) = \lambda_1 \left(\frac{E_{ii}^{\infty}}{r^3} - \frac{3x_i x_j E_{ij}^{\infty}}{r^5}\right)$$

$$E_{ii}^{\infty} = \nabla \cdot \underset{\sim}{u}^{\infty} = 0$$

$$\boxed{p(x) = \lambda_1 \frac{x_i E_{ij}^{\infty} x_j}{r^5}}$$

Absorbed -3 in λ_1 as constant.

For disturbance velocity :-

Particular solution :-

$$\underset{\sim}{u}_i^{(p)} = \frac{p}{2\mu} \underset{\sim}{x}$$

$$\boxed{u_i^{(p)} = \frac{\lambda_1}{2\mu} x_i \frac{x_j E_{jk}^{\infty} x_k}{r^5}}$$

Homogenous solution:-

$$\underset{\sim}{u}_i^{(h)} = \lambda_2 \underset{\approx}{E}^{\infty} \cdot \underset{\sim}{\nabla}\left(\frac{1}{r}\right) + \lambda_3 \underset{\approx}{E}^{\infty} : \underset{\sim}{\nabla}\underset{\sim}{\nabla}\underset{\sim}{\nabla}\left(\frac{1}{r}\right)$$

$$\underset{\sim}{\nabla}\underset{\sim}{\nabla}\underset{\sim}{\nabla}\left(\frac{1}{r}\right) = \frac{x_i x_j x_k}{r^7} - \frac{x_i \delta_{jk} + x_j \delta_{ik} + x_k \delta_{ij}}{5r^5}$$

$$u_i^{(h)} = \lambda_2 E_{ij}^{\infty} \frac{x_j}{r^3} + \lambda_3 E_{jk}^{\infty} \left(\frac{x_i x_j x_k}{r^7} - \frac{x_i \delta_{jk} + x_j \delta_{ik} + x_k \delta_{ij}}{5r^5}\right)$$

$$u_i^{(h)} = \lambda_2 E_{ij}^{\infty} \frac{x_j}{r^3} + \lambda_3 \left(\frac{0 + x_j E_{ij}^{\infty} + x_k E_{ik}^{\infty}}{r^5} - \frac{5 x_i x_j E_{jk}^{\infty} x_k}{r^7}\right)$$

-5 is absorbed in λ_3.

$$u_i^{(h)} = \lambda_2 E_{ij}^{\infty} \frac{x_j}{r^3} + \lambda_3 E_{jk}^{\infty} \left(\frac{\delta_{ij} x_k + \delta_{ik} x_j}{r^5} - \frac{5 x_i x_j E_{jk}^{\infty} x_k}{\cdots}\right.$$

Hence, $u_i = u_i^{(P)} + u_i^{(h)}$

$$u_i = \frac{\lambda_1}{2\mu} x_i \frac{x_j E_{jk}^{\infty} x_k}{r^5} + \lambda_2 E_{ij}^{\infty}\frac{x_j}{r^3} + \lambda_3 E_{jk}^{\infty}\left(\frac{\delta_{ij}x_k + \delta_{ik}x_j}{r^5} - \frac{5x_i x_j x_k}{r^7}\right)$$

~~$\nabla \cdot u = 0$~~

$$\underset{\sim}{u} = \frac{\lambda_1}{2\mu}\underset{\sim}{x}\frac{(\underset{\approx}{E}^{\infty} : \underset{\sim}{x}\underset{\sim}{x})}{r^5} + \lambda_2 \frac{\underset{\approx}{E}^{\infty}\cdot \underset{\sim}{x}}{r^3} + \lambda_3\left(\frac{2\underset{\approx}{E}^{\infty}\cdot\underset{\sim}{x}}{r^5} - \frac{5\underset{\sim}{x}(\underset{\approx}{E}^{\infty} : \underset{\sim}{x}\underset{\sim}{x})}{r^7}\right)$$

Using Boundary conditions:-

$$\underset{\sim}{u}\Big|_{r=a} = -\underset{\approx}{E}^{\infty}\cdot\underset{\sim}{x}\Big|_{r=a}$$

$$-(E_{ij}^{\infty} n_j)\, a = \frac{\lambda_1}{2\mu} n_i\left(\frac{E_{jk}^{\infty} n_j n_k}{a^2}\right) + \lambda_2 \frac{E_{ij}^{\infty} n_j}{a^2} + \lambda_3\left(\frac{2E_{ij}^{\infty} n_j}{a^4} - \frac{5n_i(E_{jk}^{\infty} n_j n_k)}{a^4}\right)$$

$$-(E_{ij}^{\infty} n_j)\, a = (E_{ij}^{\infty} n_j)\left(\frac{\lambda_2}{a^2} + \frac{2\lambda_3}{a^4}\right) + n_i(E_{jk}^{\infty} n_j n_k)\left(\frac{\lambda_1}{2\mu a^2} - \frac{5\lambda_3}{a^4}\right)$$

So,

$$\frac{\lambda_2}{a^2} + \frac{2\lambda_3}{a^4} = -a \quad \text{—(I)}$$

3 unknowns
2 eqns.
So, we need one more eqn. lets use $\nabla\cdot u = 0$

$$\frac{\lambda_1}{2\mu a^2} - \frac{5\lambda_3}{a^4} = 0 \quad \text{—(II)}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = \partial_i\left[\frac{\lambda_1}{2\mu} x_i \frac{x_j E_{jk}^{\infty} x_k}{r^5} + \lambda_2 E_{ij}^{\infty}\frac{x_j}{r^3} + \lambda_3 \left(\frac{2E_{ij}^{\infty} x_j}{r^5} - \frac{5x_i x_j x_k E_{jk}^{\infty}}{r^7}\right)\right.$$

$$0 = \frac{\lambda_1 E^\infty_{jk}}{2\mu}\left[\partial_i\left(\frac{x_i x_j x_k}{r^5}\right)\right] + \lambda_2 E^\infty_{ij}\left(\partial_i\left(\frac{x_j}{r^3}\right)\right) + 2\lambda_3 E^\infty_{ij}\left[\partial_i\left(\frac{x_j}{r^5}\right)\right] - 5\lambda_3 E^\infty_{jk}\left[\partial_i\left(\frac{x_i x_j x_k}{r^7}\right)\right]$$

$$0 = \frac{\lambda_1 E^\infty_{jk}}{2\mu}\left[\frac{3x_j x_k + \delta_{ji} x_k x_i + \delta_{ki} x_j x_i}{r^5} - \frac{5x_j x_k}{r^5}\right] + \lambda_2 E^\infty_{ij}\left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right] + 2\lambda_3 E^\infty_{ij}\left[\frac{\delta_{ij}}{r^5} - \frac{5x_i x_j}{r^7}\right] - 5\lambda_3 E^\infty_{jk}\left[\frac{3x_j x_k + \delta_{ji} x_k x_i + \delta_{ki} x_j x_i}{r^7} - \frac{7x_j x_k}{r^7}\right]$$

$$0 = \frac{\lambda_1}{2\mu} E^\infty_{jk} x_j x_k \left[\cancel{\frac{3+1+1}{r^5} - \frac{5}{r^5}} \to 0\right] + \lambda_2 E^\infty_{ij} x_i x_j \left[0 - \frac{3}{r^5}\right] + 2\lambda_3 E^\infty_{ij} x_i x_j \left[0 - \frac{5}{r^7}\right] - 5\lambda_3 E^\infty_{jk} x_j x_k \left(\frac{3+1+1}{r^7} - \frac{7}{r^7}\right)$$

In vector form all the terms are same as $\underset{\approx}{E}^\infty : \underset{\sim}{x}\,\underset{\sim}{x}$

so,

$$0 = -\frac{3\lambda_2}{r^5} + \left(-\frac{10\lambda_3}{r^7} + \frac{10\lambda_3}{r^7}\right)$$

λ_3 cancels out to be zero & hence $\boxed{\lambda_2 = 0}$ for satisfying this eqn.

from (1),

$$\lambda_3 = -\frac{a^5}{2}.$$

$$\frac{\lambda_1}{2\mu a^2} = -\frac{5a}{2}$$

$$\lambda_1 = -5\mu a^3$$

Hence, $\lambda_1 = -5\mu a^3$

$\lambda_2 = 0$

$\lambda_3 = -a^5/2.$

So, ~~as~~ . we get,

$$\boxed{u_i = \frac{-5a^3}{2} x_i \frac{(x_j E^{\infty}_{jk} x_k)}{r^5} - \frac{a^5}{2} E^{\infty}_{jk}\left[\frac{\delta_{ij} x_k + \delta_{ik} x_j}{r^5} - \frac{5 x_i x_j x_k}{r^7}\right]}$$

and pressure as:-

$$\boxed{P(x) = -5\mu a^3 \frac{x_i E^{\infty}_{ij} x_j}{r^5}}$$

$\underset{\approx}{E}^{\infty} : \underset{\sim}{x}\underset{\sim}{x}$

Note :- The pressure decays as r^{-3} and the velocity is seen to decay as r^{-2} with a more rapidly decaying portion going as r^{-4}. It is seen that pressure field comes as quadrupolar form.

(*) The disturbance pressure changes sign as we pass quadrant to quadrant being positive in the quadrants where the fluid motion is toward the body ($x<0, y>0$ or $x>0, y<0$) and negative in the other two quadrants where the flow moves away from the surface.

Hydrodynamic force, Torque and Streslet

⊛ Force ⊛

⊛ A simple interpretation of total hydrodynamic force is that it is a sum of differential forces $\underset{\approx}{\sigma}\cdot \underset{\sim}{n}\, ds$ on the particle surface, where $\underset{\approx}{\sigma}\cdot \underset{\sim}{n}$ is called the traction vector.

For a spherical particle of radius a, moving with velocity $\underset{\sim}{U}^{\infty}$, the hydrodynamic force on the particle is given by :-

$$\boxed{F^{h} = 6\pi \mu a \underset{\sim}{U}^{\infty}}$$

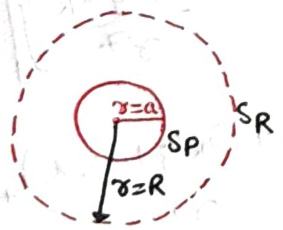
Calculation of hydrodynamic force for a single isolated translating particle in an infinite fluid :-

We have already found out the velocity field (disturbance) u_i as follows,

$$u_i = \underbrace{\frac{3a}{4} U_j^{\infty}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right)}_{\text{This has an order of } (1/r)} + \underbrace{\frac{3a^3}{4} U_j^{\infty}\left(\frac{\delta_{ij}}{3r^3} - \frac{x_i x_j}{r^5}\right)}_{\text{This has an order of } (1/r^3)}$$

Force, $F_i = \int_{r=a} \sigma_{ik} n_k \, ds$

Since, $\underset{\approx}{\sigma}$ is symmetric so from stokes flow $\nabla\cdot\underset{\approx}{\sigma} = 0$



$$F_i = \int_{\substack{r=a\\ S_p}} \sigma_{ik} n_k \, ds = \int_{\substack{r=R\\ S_R}} \sigma_{ik} n_k \, ds$$

Where, $S_p \rightarrow$ Surface of the particle of radius 'a'.

$S_R \rightarrow$ Surface of an imaginary sphere of radius $r=R$.

$$F_i = \int_{r=a} \sigma_{ik} n_k \, ds$$

$$\sigma_{ik} = -p\,\delta_{ik} + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$$\frac{\partial u_i}{\partial x_k} = \frac{3a}{4} U_j^\infty \left(-\frac{\delta_{ij} x_k}{r^3} + \frac{\delta_{ik} x_j + \delta_{jk} x_i}{r^3} - \frac{3 x_i x_j x_k}{r^5} \right)$$

$$+ \frac{U_j^\infty a^3}{4} \left(\frac{-3\delta_{ij} x_k}{r^5} - \frac{3[\delta_{ik} x_j + \delta_{jk} x_i]}{r^5} + \frac{15 x_i x_j x_k}{r^7} \right)$$

$$\frac{\partial u_i}{\partial x_k} = \frac{3a}{4} U_j^\infty \left(\frac{\delta_{ik} x_j + \delta_{jk} x_i - \delta_{ij} x_k}{r^3} - \frac{3 x_i x_j x_k}{r^5} \right)$$

$$+ \frac{3 U_j^\infty a^3}{4} \left(\frac{-\delta_{ij} x_k - \delta_{ik} x_j - \delta_{jk} x_i}{r^5} + 5 \frac{x_i x_j x_k}{r^7} \right)$$

$$\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} = 2 \times \text{symmetric parts of } \frac{\partial u_i}{\partial x_k}$$

$$\left[\text{Symmetric part} = \frac{1}{2}\left(\nabla v + \nabla v^T\right) \right]$$

$$\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} = \frac{3a}{2} U_j^\infty \left[\frac{\delta_{ik} x_j}{r^3} - \frac{3 x_i x_j x_k}{r^5} \right] + \frac{3}{2} a^3 U_j^\infty \left[\frac{-\delta_{ij} x_k - \delta_{ik} x_j - \delta_{jk} x_i}{r^5} + 5 \frac{x_i x_j x_k}{r^7} \right]$$

$$\sigma_{ik} = -p\,\delta_{ik} + \mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

$$\sigma_{ik} = -\frac{3\mu a}{2} \cancel{\frac{U_m^\infty x_m}{r^3} \delta_{ik}} + \mu \left[\frac{3a}{2} U_j^\infty \left\{ \cancel{\frac{\delta_{ik} x_j}{r^3}} - \frac{3 x_i x_j x_k}{r^5} \right\} + \frac{3}{2} a^3 U_j^\infty \left\{ \frac{-\delta_{ij} x_k - \delta_{ik} x_j - \delta_{jk} x_i}{r^5} + 5 \frac{x_i x_j x_k}{r^7} \right\} \right]$$

same

same

$$\sigma_{ik} = \frac{3\mu a}{2} U_j^\infty \left[-\frac{3 x_i x_j x_k}{r^5} + a^2 \left\{ -\frac{(\delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i)}{r^5} + 5 \frac{x_i x_j x_k}{r^7} \right\} \right]$$

$$\sigma_{ik}\Big|_{\substack{r=a \\ x_i = a n_i}} = \frac{3\mu a}{2} U_j^\infty \left[-\frac{3 n_i n_j n_k}{a^2} + a^2 \left\{ -\frac{(\delta_{ij} n_k + \delta_{ik} n_j + \delta_{jk} n_i)}{a^4} + \frac{5 n_i n_j n_k}{a^4} \right\} \right]$$

$$= \frac{3\mu}{2a} U_j^\infty \left[-3 n_i n_j n_k - (\delta_{ij} n_k + \delta_{ik} n_j + \delta_{jk} n_i) + 5 n_i n_j n_k \right]$$

$$\sigma_{ik}\Big|_{r=a} = \frac{3\mu}{2a} U_j^\infty \left[2n_in_jn_k - \left(\delta_{ij}n_k + \delta_{ik}n_j + \delta_{jk}n_i\right)\right]$$

$$\sigma_{ik}n_k\Big|_{r=a} = \frac{3\mu}{2a} U_j^\infty \left[2n_in_jn_kn_k - \delta_{ij}(n_kn_k) - \delta_{ik}n_jn_k - \delta_{jk}n_in_k\right]$$

$$\sigma_{ik}n_k\big|_{r=a} = \frac{3\mu}{2a} U_j^\infty \left[2n_in_j - \delta_{ij} - 2n_in_j\right]$$

$$\boxed{\sigma_{ik}n_k\Big|_{r=a} = -\frac{3\mu}{2a} U_i^\infty}$$

Note:- The force per unit area doesnot vary as you change angular position on the sphere. Thus, there is no force tending to deform the sphere. (Happens only in stokes flow for spherical particles).

$$F_i = \int -\frac{3\mu}{2a} U_i^\infty ds = \frac{3\mu U_i^\infty}{\cancel{2a}} \times \cancel{4}^{2}\pi a^{\cancel{2}}$$

$$\boxed{F_i = -6\pi\mu a U_i^\infty}$$

~~Method-II:~~ $F_i = \int_{r=R} \sigma_{ik}n_k ds$

~~In this approach we will exploit the feature of stokes flow that force is transmitted unchanged across the fluid and thus evaluate the surface integral at large distance $R >> a$.~~

Now,

$$u_i = \frac{3U_j^\infty a}{4}\left(\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right) + \frac{U_j^\infty a^3}{4}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)$$

$$u_i = \frac{3\times 2\pi\mu U_j^\infty a}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right) + \frac{6\pi\mu a U_j^\infty a^2}{4(6\pi\mu)}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)$$

$$u_i = \frac{F_j}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right) + \frac{F_j a^2}{24\pi\mu}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)$$

$$\boxed{u_i = \frac{F_j}{8\pi\mu}\left[\left(\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right) + \frac{a^2}{3}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right)\right]}$$

$$\boxed{P = \frac{F_i x_i}{4\pi r^3}}$$

where, $F_i = 6\pi\mu a U_i^{\infty}$

If we let $a \to 0$ we have,

$$\boxed{\begin{aligned} u_i &= \frac{F_j}{8\pi\mu}\left[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right] \\ P &= \frac{1}{4\pi} F_i \frac{x_i}{r^3} \end{aligned}}$$

$\longrightarrow$ Stokeslet.

Sol^n of stokes eqn, with a point forcing of strength F_i.

$$-\mu\nabla^2 \underset{\sim}{u} + \underset{\sim}{\nabla} p = \underset{\sim}{F}\,\delta(\underset{\sim}{x})$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0.$$

** Torque **

The hydrodynamic torque resulting from the fluid traction on the surface is written :-

$$\underset{\sim}{T}^h = \int_{S_P} \underset{\sim}{x} \times \underset{\approx}{\sigma}\cdot\underset{\sim}{n}\, ds$$

We see that it is a first moment of the traction distribution.

(*) To obtain an unique result, this expression should be independent of the origin. However, if we shift the origin by an arbitrary vector, say x_0, we find :-

$$T^h = \int_{S_P} \underset{\sim}{x} \times \underset{\approx}{\sigma}\cdot\underset{\sim}{n}\, ds = \int_{S_P} (\underset{\sim}{x} - \underset{\sim}{x}_0) \times \underset{\approx}{\sigma}\cdot\underset{\sim}{n}\, ds + x_0 \times \int_{S_P} \underset{\approx}{\sigma}\cdot\underset{\sim}{n}\, ds$$

$$T^h = \int_{S_P} (\underset{\sim}{x} - \underset{\sim}{x}_0) \times \underset{\approx}{\sigma}\cdot\underset{\sim}{n}\, ds + \underset{\sim}{x}_0 \times \underset{\sim}{F}^h$$

For a sphere of radius a held fixed in an ambient rotational flow $\underset{\sim}{\omega}^{\infty} \times \underset{\sim}{x}$, the hydrodynamic torque on the body is :-

$$\boxed{T_i^h = 8\pi\mu a^3 \omega_i^{\infty}.}$$

... single isolated rotating sphere in an infinite fluid :-

As we know, The velocity field :-

$$u_i = \left(\frac{a}{r}\right)^3 \epsilon_{ijk} \Omega_j x_k$$

Lets see if its satisfies incompressibility condition,

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial}{\partial x_i}\left[\left(\frac{a}{r}\right)^3 \epsilon_{ijk}\Omega_j x_k\right] = a^3 \epsilon_{ijk}\Omega_j \frac{\partial}{\partial x_i}\left[\frac{x_k}{r^3}\right]$$

$$= a^3 \epsilon_{ijk}\Omega_j \left[\frac{\delta_{ik}}{r^3} - \frac{3x_i x_k}{r^5}\right]$$

$$= a^3 \Omega_j \left[\frac{\epsilon_{ijk}\delta_{ik}}{r^3} - \frac{3\epsilon_{ijk} x_i x_k}{r^5}\right]$$

$$= a^3 \Omega_j \left[\cancel{\frac{\epsilon_{iji}}{r^3}}^{\to 0} - \cancel{\frac{3\epsilon_{ijk} x_i x_k}{r^5}}^{\to 0}\right]$$

symmetric
Anti symmetric
contradiction.

Hence, $\boxed{\nabla \cdot \underset{\sim}{u} = 0}$

Now, In order to find Torque on the sphere. we need to find stress on the surface of the sphere.

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\frac{a}{r}\right)^3 \epsilon_{imn}\Omega_m x_n\right] = a^3 \epsilon_{imn}\Omega_m \frac{\partial}{\partial x_j}\left[\frac{x_n}{r^3}\right]$$

$$\frac{\partial u_i}{\partial x_j} = a^3 \Omega_m \left[\frac{\delta_{jn}\epsilon_{imn}}{r^3} - \frac{3\epsilon_{imn} x_j x_n}{r^5}\right]$$

$$\frac{\partial u_j}{\partial x_i} = a^3 \Omega_m \epsilon_{jmn}\left(\frac{\delta_{in}}{r^3} - \frac{3x_i x_n}{r^5}\right)$$

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\sigma_{ij} = \mu a^3 \epsilon_{imn} \Omega_m \left\{ \frac{\delta_{jn}}{r^3} - \frac{3x_j x_n}{r^5} \right\} + \mu a^3 \epsilon_{jmn} \Omega_m \left\{ \frac{\delta_{in}}{r^3} - \frac{3x_i x_n}{r^5} \right\}$$

$$\sigma_{ij}\big|_{r=a} = \mu \epsilon_{imn} \Omega_m \left\{ \delta_{jn} - 3n_j n_n \right\} + \mu \epsilon_{jmn} \Omega_m \left\{ \delta_{in} - 3n_i n_n \right\}$$

$$= \mu \Omega_m \left[\epsilon_{imj} - 3\epsilon_{imn} n_j n_n + \epsilon_{jmi} - 3\epsilon_{jmn} n_i n_n \right]$$

$$= \mu \Omega_m \left[\cancel{\epsilon_{imj} + \epsilon_{jmi}} - 3n_n \left(\epsilon_{imn} n_j + \epsilon_{jmn} n_i \right) \right]$$

$$\sigma_{ij}\big|_{r=a} = -3\mu \Omega_m n_n \left(\epsilon_{imn} n_j + \epsilon_{jmn} n_i \right)$$

Torque:- $$\mathcal{L}_i = \int_{r=a} \left(\underset{\sim}{x} \times (\underset{\approx}{\sigma} \cdot \underset{\sim}{n}) \right)_i ds$$

$$\mathcal{L}_i = \int_{r=a} \epsilon_{ijk} x_j (\underset{\approx}{\sigma} \cdot \underset{\sim}{n})_k \, ds$$

$$\mathcal{L}_i = \int_{r=a} \epsilon_{ijk} a n_j \sigma_{kp} n_p \, a^2 d\Omega$$

$ds = a^2 d\Omega$

$\Omega \to$ solid Angle.

$$\mathcal{L}_i = a^3 \int_{r=a} \epsilon_{ijk} \sigma_{kp} n_j n_p \, d\Omega$$

$$\mathcal{L}_i = a^3 \int_{r=a} \epsilon_{ijk} \left(-3\mu \Omega_m n_n \left(\epsilon_{kmn} n_p + \epsilon_{pmn} n_k \right) \right) n_j n_p d\Omega$$

$$\mathcal{L}_i = -3\mu a^3 \Omega_m \int_{r=a} \epsilon_{ijk} \left(\epsilon_{kmn} n_p n_n + \epsilon_{pmn} n_k n_n \right) n_j n_p d\Omega$$

$$\mathcal{L}_i = -3\mu a^3 \Omega_m \int_{r=a} \left(\epsilon_{ijk} \epsilon_{kmn} n_p n_n + \epsilon_{ijk} \epsilon_{pmn} n_k n_n \right) n_j n_p d\Omega$$

$$\mathcal{L}_i = -3\mu a^3 \Omega_m \int_{r=a} \left(\epsilon_{ijk} \epsilon_{kmn} n_n n_j \, d\Omega + \epsilon_{ijk} \epsilon_{pmn} n_k n_n n_j n_p \, d\Omega \right)$$

$$\mathcal{L}_i = -3\mu a^3 \Omega_m \int_{r=a} (\epsilon_{ijk}\epsilon_{kmn} n_n n_j) d\Omega$$

$$\mathcal{L}_i = -3\mu a^3 \Omega_m \int_{r=a} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) n_n n_j \, d\Omega$$

$$\mathcal{L}_i = -3\mu a^3 \Omega_m \int_{r=a} (\delta_{im} - n_i n_m) \, d\Omega$$

$$\mathcal{L}_i = -3\mu a^3 \Omega_m \left[\delta_{im}\int_{r=a} d\Omega - \int_{r=a} n_i n_m \, d\Omega\right]$$

$$\mathcal{L}_i = -3\mu a^3 \Omega_m \left[\delta_{im}\, 4\pi - \frac{4\pi}{3}\delta_{im}\right]$$

$$\boxed{\mathcal{L}_i = -8\pi\mu a^3 \Omega_i}$$ Torque.

Rotlet:-

$$u_i = \epsilon_{ijk}\Omega_j x_k \left(\frac{a}{r}\right)^3$$

$$\boxed{u_i = \frac{\epsilon_{ijk}\mathcal{L}_j x_k}{8\pi\mu r^3}}$$

decays as r^{-2}

** Stresslet **

Torque is only a part of the first moment of the force distribution, torque and stresslet together form the complete first moment of the force distribution.

$$\boxed{M_{ij} = \int_{S_p} \sigma_{ik} n_k x_j \, ds}$$ 1st moment of force distribution.

$M_{ij} = S_{ij} + A_{ij}$ (Symmetric & Antisymmetric portions).

S_{ij} → Stresslet (Symmetric portion)

Stresslet.

$$\boxed{S_{ij} = \frac{1}{2}\int_{S_p} [\sigma_{ik} x_j + \sigma_{jk} x_i] n_k \, ds.}$$

It is more different and complicated for a deformable particle where the stresslet involves the surface velocity.

$$A_{ij} = \frac{1}{2}\int_{S_p} (\sigma_{ik} x_j - \sigma_{jk} x_i) n_k \, ds = -\frac{1}{2}\epsilon_{ijk}\mathcal{L}_k$$

The torque is a pseudo vector because it requires application of a convention in the definition of its direction. The stresslet is the result of the resistance of the rigid particle to a straining motion, which provides a clue to its relevance.

The stresslet is not needed in the equations of motion for a particle. It nonetheless has a very important role in suspension mechanics, as it describes the added stress associated with the particles in a suspension.

For a sphere of radius a in a straining flow $E^{\infty}.x$ for which we obtained the soln,

$$\boxed{S_{ij} = \frac{20\pi}{3}\mu a^3 E_{ij}^{\infty}}$$

Since isolated stresslet has a linear reln. with rate of strain, this implies a Newtonian contribution to the bulk stress by the particles in the dilute limit.

Known as Einstein viscosity, $\mu_E(\phi) = \mu\left[1+\left(\frac{5}{2}\right)\phi\right]$

$\phi \rightarrow$ particle volm fraction.

This result is giving the effective viscosity of a dilute $(\phi<<1)$ suspension of spheres.

Faxen's Laws for the Sphere

The linear relations between the motion (of the particle and fluid) and the force and its moments are known as Faxen's Laws and contain additional pieces owing to the curvature of the flow $\nabla^2 u^{\infty}$.

Here x=0 implies that the ambient fields are evaluated at the positions occupied by the centre of the particle.

$$\underset{\sim}{F} = 6\pi\mu a\left[\left(1+\frac{a^2}{6}\nabla^2\right)\underset{\sim}{u}^{\infty}(x=0) - \underset{\sim}{U}^P\right]$$

$$\underset{\sim}{T} = 8\pi\mu a^3\left[\underset{\sim}{\omega}^{\infty}(x=0) - \underset{\sim}{\omega}^P\right]$$

$$\underset{\sim}{S} = \frac{20}{3}\pi\mu a^3\left(1+\frac{a^2}{10}\nabla^2\right)\underset{\approx}{E}^{\infty}(x=0)$$

Here $\underset{\sim}{\omega}^{\infty}$ is the rotation vector and $\underset{\approx}{E}^{\infty}$ is the rate of strain of the ambient flow and therefore can vary with position.

A.) Calculation of Stresslet (Method-I) Entire Calculation:-

As we know, the velocity field and pressure is as follows:-

velocity,

$$\boxed{u_i = -\frac{5a^3}{2} x_i \left(\frac{x_j E^\infty_{jk} x_k}{r^5}\right) - \frac{a^5}{2} E^\infty_{jk} \left[\frac{\delta_{ij} x_k + \delta_{ik} x_j}{r^5} - \frac{5 x_i x_j x_k}{r^7}\right]}$$

Pressure,

$$\boxed{P(x) = -5\mu a^3 \frac{x_i E^\infty_{ij} x_j}{r^5}}$$

Now, Stresslet $\Rightarrow$ $S_{ij} = \frac{1}{2} \int_{\substack{S_P \\ (r=a)}} \left[\sigma_{ik} x_j + \sigma_{jk} x_i\right] n_k \, dS$

& $\sigma_{ik} = -P + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$

So starting by finding the gradients of velocity fields,

Pathway $\rightarrow$ $\nabla \underset{\sim}{u} \longrightarrow \nabla u + \nabla \underset{\sim}{u}^T \longrightarrow \underset{\approx}{E} \longrightarrow \underset{\approx}{\sigma} \longrightarrow \underset{\approx}{\sigma} \cdot \underset{\sim}{x} \rightarrow$ Stresslet

$$\frac{\partial u_i}{\partial x_m} = -\frac{5a^3}{2} E^\infty_{jk} \left[\delta_{im}\left(\frac{x_j x_k}{r^5}\right) + x_i \left(\frac{\delta_{jm} x_k + \delta_{km} x_j}{r^5} - \frac{5 x_k x_j x_m}{r^7}\right)\right]$$

$$- \frac{a^5}{2} E^\infty_{jk} \left[\frac{\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm}}{r^5} - 5\left\{\frac{\left(\delta_{ij} x_k x_m + \delta_{ik} x_j x_m + \delta_{im} x_j x_k + \delta_{jm} x_i x_k + \delta_{km} x_i x_j\right)}{r^7} + 7\frac{x_m x_j x_k x_i}{r^9}\right\}\right]$$

$$\frac{\partial u_m}{\partial x_i} = -\frac{5a^3}{2} E^\infty_{jk} \left[\delta_{mi}\left(\frac{x_j x_k}{r^5}\right) + x_m \left(\frac{\delta_{ji} x_k + \delta_{ki} x_j}{r^5} - 5\frac{x_i x_j x_k}{r^7}\right)\right]$$

$$- \frac{a^5}{2} E^\infty_{jk} \left[\frac{\delta_{mj}\delta_{ki} + \delta_{mk}\delta_{ji}}{r^5} - 5\left\{\frac{\left(\delta_{mj} x_k x_i + \delta_{mk} x_j x_i + \delta_{mi} x_j x_k + \delta_{ji} x_m x_k + \delta_{ki} x_m x_j\right)}{r^7} - 7\frac{x_i x_j x_k x_m}{r^9}\right\}\right]$$

$$\sigma_{im} = \cancel{-\frac{5\mu a^3 E^{\infty}_{pq} x_p x_q}{r^5}\,\delta_{im}} - \frac{5\mu a^3 E^{\infty}_{jk}}{2}\Bigg[\cancel{(\delta_{im}+\delta_{mi})\frac{x_j x_k}{r^5}} + \quad {}^{im\,x}$$

$$\frac{\delta_{jm} x_k x_i + \delta_{km} x_i x_j + \delta_{ji} x_k x_m + \delta_{ki} x_m x_j}{r^5} - 10\frac{x_i x_j x_k x_m}{r^7}$$

$$-2a^2\Bigg\{\frac{\delta_{ij}\delta_{km}+\delta_{ik}\delta_{jm}}{5r^5} - \left(\frac{\delta_{ij} x_k x_m + \delta_{ik} x_j x_m + \delta_{im} x_j x_k + \delta_{km} x_i x_j}{r^7}\right) + 7\frac{x_i x_j x_k x_m}{r^9}\Bigg\}\Bigg]\Bigg]$$

$$\sigma_{lm} = -\frac{5\mu a^3 E^{\infty}_{jk}}{2}\Bigg[\frac{\delta_{jm} x_k x_l + \delta_{km} x_l x_j + \delta_{jl} x_k x_m + \delta_{kl} x_m x_j}{r^5} - 10\frac{x_l x_j x_k x_m}{r^7}$$

$$-2a^2\Bigg\{\frac{\delta_{lj}\delta_{km}+\delta_{lk}\delta_{jm}}{5r^5} - \left(\frac{\delta_{jl} x_k x_m + \delta_{lk} x_j x_m + \delta_{lm} x_j x_k + \delta_{km} x_l x_j}{r^7}\right) + 7\frac{x_l x_j x_k x_m}{r^9}\Bigg\}\Bigg]\Bigg]$$

$$\sigma_{im} x_l = -\frac{5\mu a^3 E^{\infty}_{jk}}{2}\Bigg[\frac{\delta_{jm} x_i x_k x_l + \delta_{km} x_i x_j x_l + \delta_{ji} x_k x_m x_l + \delta_{ki} x_m x_j x_l}{r^5}$$

$$-10\frac{x_i x_j x_k x_l x_m}{r^7} - 2a^2\Bigg\{\frac{\delta_{ij}\delta_{km} x_l + \delta_{ik}\delta_{jm} x_l}{5r^5} - \left(\frac{\delta_{ij} x_k x_m x_l + \delta_{ik} x_j x_m x_l + \delta_{im} x_j x_k x_l + \delta_{km} x_i x_j x_l}{r^7}\right) + 7\frac{x_i x_j x_k x_m x_l}{r^9}\Bigg\}\Bigg]\Bigg]$$

$$\sigma_{lm} x_i = -\frac{5\mu a^3 E^{\infty}_{jk}}{2}\Bigg[\frac{\delta_{jm} x_i x_k x_l + \delta_{km} x_i x_j x_l + \delta_{jl} x_i x_k x_m + \delta_{ki} x_m x_j x_l}{r^5}$$

$$-10\frac{x_i x_j x_k x_l x_m}{r^7} - 2a^2\Bigg\{\frac{\delta_{lj}\delta_{km} x_i + \delta_{lk}\delta_{jm} x_i}{5r^5} - \left(\frac{\delta_{jl} x_i x_k x_m + \delta_{lk} x_i x_j x_m + \delta_{lm} x_i x_j x_k + \delta_{km} x_i x_j x_l}{r^7}\right) + 7\frac{x_i x_j x_k x_l x_m}{r^9}\Bigg\}\Bigg]$$

$$\sigma_{im} x_l + \sigma_{lm} x_i \Rightarrow -\frac{5\mu a^3 E^{\infty}_{jk}}{2} \Bigg[\frac{2(\delta_{jm} x_i x_k x_l + \delta_{km} x_i x_j x_l) + \delta_{jl} x_i x_k x_m + \delta_{ji} x_k x_m x_l + \delta_{ki} x_m x_j x_l + \delta_{kl} x_i x_j x_m}{r^5}$$

$$- 20 \frac{x_i x_j x_k x_l x_m}{r^7} - 2a^2 \Big\{ \frac{\delta_{jl}\delta_{km} x_i + \delta_{lk}\delta_{jm} x_i + \delta_{ij}\delta_{km} x_l + \delta_{ik}\delta_{jm} x_l}{5r^5}$$

$$- \frac{(\delta_{ij} x_k x_m x_l + \delta_{ik} x_j x_m x_l + \delta_{im} x_j x_k x_l + \delta_{km} x_i x_j x_l + \delta_{jl} x_i x_k x_m + \delta_{lk} x_i x_j x_m + \delta_{lm} x_i x_j x_k + \delta_{km} x_i x_j x_l)}{r^7}$$

$$+ 14 \frac{x_i x_j x_k x_l x_m}{r^9} \Big\} \Bigg]$$

$$(\sigma_{im} x_l + \sigma_{lm} x_i) n_m = -\frac{5\mu a^3 E^{\infty}_{jk}}{2} \Big[2(\delta_{jm}\, n_i n_k n_l n_m + \delta_{km}\, n_i n_j n_l n_m) + \delta_{jl} n_i n_k n_m n_m + \delta_{ji} n_k n_m n_m n_l + \delta_{ki} n_m n_j n_l n_m + \delta_{kl} n_i n_j n_m n_m$$

$$- 20 n_i n_j n_k n_l n_m n_m - \frac{2}{5}\big(\delta_{jl}\delta_{km} n_i n_m + \delta_{lk}\delta_{jm} n_i n_m + \delta_{ij}\delta_{km} n_l n_m + \delta_{ik}\delta_{jm} n_l n_m \big)$$

$$- 2\big(\delta_{ij} n_k n_m n_l n_m + \delta_{ik} n_j n_m n_l n_m + \delta_{im} n_j n_k n_l n_m + \delta_{km} n_i n_j n_l n_m + \delta_{jl} n_i n_k n_m n_m + \delta_{lk} n_i n_j n_m n_m + \delta_{lm} n_i n_j n_k n_m + \delta_{km} n_i n_j n_l n_m \big) + 28 n_i n_j n_k n_l n_m n_m \Big]$$

$$S_{il} = \frac{1}{2} \int_{r=a} \left[\sigma_{im} x_l + \sigma_{lm} x_i \right] n_m \, dS$$

$$S_{il} = -\frac{5\mu a^3 E_{jk}^{\infty}}{4} \int_{r=a} \Big[4 n_i n_j n_k n_l + \left(\delta_{jl} n_i n_k + \delta_{ij} n_k n_l + \delta_{ki} n_j n_l + \delta_{kl} n_i n_j\right) - \frac{2}{5}\left(\delta_{jl} n_i n_k + \delta_{lk} n_i n_j + \delta_{ij} n_l n_k + \delta_{ik} n_l n_j\right)$$

$$- 2\left(\delta_{ij} n_k n_l + \delta_{ik} n_j n_l + \delta_{jl} n_i n_k + \delta_{lk} n_i n_j\right)$$

$$- 8 n_i n_j n_k n_l - 20 n_i n_j n_k n_l + 28 n_i n_j n_k n_l \Big] d\Omega$$

$$S_{il} = -\frac{5\mu a^3 E_{jk}^{\infty}}{4} \int_{r=a} \Big[4 n_i n_j n_k n_l + \left(1-2-\frac{2}{5}\right)\left(\delta_{jl} n_i n_k + \delta_{ij} n_k n_l + \delta_{ki} n_j n_l + \delta_{kl} n_i n_j\right)\Big] d\Omega$$

$$S_{il} = -\frac{5\mu a^3 E_{jk}^{\infty}}{4} \Big[4 \times \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\overset{0}{\delta_{jk}}\right) - \frac{7}{5} \times \frac{4\pi}{3}\left(\delta_{jl}\delta_{ik} + \delta_{ij}\delta_{kl} + \delta_{ki}\delta_{jl} + \delta_{kl}\delta_{ij}\right)\Big]$$

$$S_{il} = -\frac{5\mu a^3 E_{il}^{\infty}}{4} \Big[4 \times \frac{4\pi}{15}(2) - \frac{7}{5} \times \frac{4\pi}{3} \times 2 \times 2 \Big]$$

$$S_{il} = -\frac{\cancel{5}\mu a^3 E_{il}^{\infty}}{\cancel{4}} \times \frac{\cancel{4}\pi}{\cancel{15}_3} (8 - 28)$$

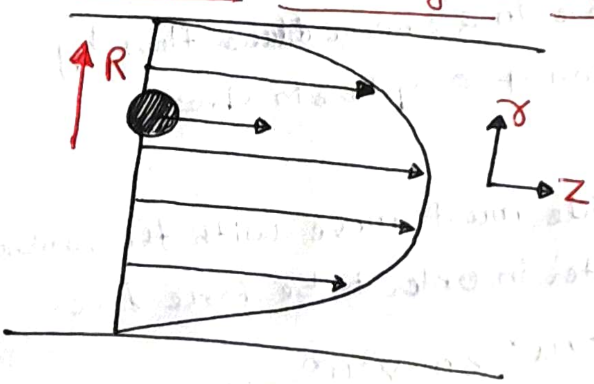
$$\boxed{S_{il} = \frac{+20\mu a^3 E_{il}^{\infty}}{3}}$$

Note: When there is a curvature of the flow $\nabla^2 u^\infty$, ~~as advertised~~. there is an additional term in each of the Faxen's Laws for the force and stresslet but not for the torque. This is associated with the lack of such a finite-size piece in the flow induced by a rotating sphere. (as the more rapidly decaying portions are associated with the particle size and as a consequence are often termed the "finite size pieces" of the flow).

- We recover the results presented in the above section for a constant ∇u^∞.

- Unlike the force and torque, the Faxen's Law for the stresslet doesnot contain a difference between fluid and particle variables, but simply the fluid variable E^∞; this is a result of the fact that a rigid particle doesnot experience strain.

Example for showing the influence of curvature terms.



$r \rightarrow$ radial distance

$R \rightarrow$ tube radius.

When the particle is neutrally buoyant then, $F=0$

i.e.

$$\left(1+\frac{a^2}{6}\nabla^2\right)\underset{\sim}{u}^\infty(x=0) = \underset{\sim}{U}^P$$

It shows that the particle must lag the fluid velocity at its centre by an amount which scales as $(a/R)^2$

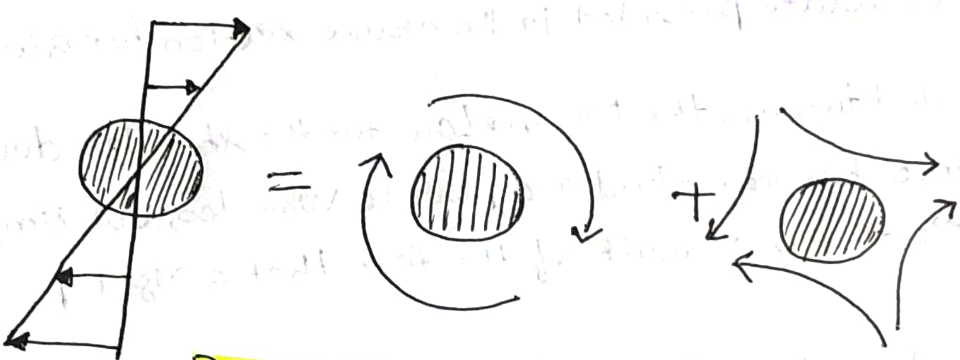
Since, the sphere is torque free i.e. $\underset{\sim}{\omega}^\infty = \underset{\sim}{\omega}^P$

Here, $\boxed{\omega_\theta^P = \dfrac{r\, U_{max}}{R^2}}$

A Sphere in Simple Shear flow

~~Let us consider a spherical particle translating with velocity $\underset{\sim}{U}^P$ and rotating with angular velocity $\underset{\sim}{\Omega}^p$ in an ambient linear flow $\underset{\sim}{u}^\infty$ =~~

Considering a freely mobile particle immersed in simple shear flow, of the form $\underset{\sim}{u}^\infty = (\dot{\gamma} y, 0, 0)$. By freely mobile we mean that the sphere has no external influence upon it and also experiences no hydrodynamic force or torque.



Decomposition of a sphere in a shear flow by a sphere in a rotation + a sphere in strain.

According to the Faxen's Laws, a particle must move with the ambient velocity which would be found at its center in order to be force free.

Since, Simple shear has constant $\nabla \underset{\sim}{u}^\infty$ so $\nabla^2 \underset{\sim}{u}^\infty = 0$.

Note :- ⊛ we are interested only in disturbance flow created by the sphere, so we take the origin to be at the center of the sphere.

Let a be the radius of sphere,

G.E :- $\mu^2 \nabla^2 \underset{\sim}{u} = \underset{\sim}{\nabla} p$; $\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$

B-C :- $\underset{\sim}{u}\big|_{r=a} = \underset{\sim}{\omega}^P \times \underset{\sim}{x}$

$\underset{\sim}{u} - \underset{\sim}{u}^\infty \longrightarrow 0$ as $r \longrightarrow \infty$.

If the particle rotation rate is $\omega^P = \omega^\infty$, then by Faxen's Laws for the torque we can say that $\underset{\sim}{T} = 0$

Since the particle exerts no Force and Torque on the fluid and we also know that the freely rotating sphere embedded in a solid-body rotation creates no disturbance, the disturbance flow generated by a sphere due to only its resistance to the straining component of the shearing flow.

Thus we sum $\underset{\sim}{u}^{\infty}$ with the disturbance flow field generated by a sphere in straining flow, to the total velocity field :-

$$\boxed{u_i = u_i^{\infty} - \frac{5a^3}{2} \frac{x_i (x_j E_{jk}^{\infty} x_k)}{r^5} - \frac{a^5}{2} E_{jk}^{\infty} \left[\frac{\delta_{ij} x_k + \delta_{ik} x_j}{r^5} - \frac{5 x_i x_j x_k}{r^7} \right]}$$

from this we can deduce that the velocity at the surface is the sum of the ambient shear flow $\underset{\sim}{u}^{\infty} = (\dot{\gamma} y, 0, 0)$ and $-\underset{\approx}{E}^{\infty} \cdot \underset{\sim}{x}_i$

Note that the shear flow can be written as $(\dot{\gamma} y, 0, 0) = (\underset{\approx}{E}^{\infty} + \underset{\approx}{\Omega}^{\infty}) \cdot \underset{\sim}{x}$

the surface velocity is given by $\underset{\approx}{\Omega}^{\infty} \cdot \underset{\sim}{x} = \underset{\sim}{\omega}^{P} \times \underset{\sim}{x}$, the solid body rotation required.

This leads us to expect a rotationally dominated motion in the vicinity of the rotating sphere.