The Geometry Of Musical Logarithms

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La géometrie permet de penser le discontinu et le rend susceptible d'une analyse à la limite de la continuité, tandis que le nombre intervient pour compter des éléments apparament continus, et les rend alors discrets. [Frederic de Buzon, 1985, 119]

Abstract

The aim of this essay is to create a geometrical bridge between music theory and mathematics of the early 17th century by studying and comparing diagrams which directly or indirectly refer to what we call mathematical logarithms.[2]
The focus is on the relationships between ratios of positive integer numbers referring to sounds and related concepts of perception. The relationship between frequencies and pitch perception is a paradigmatic case of the Weber-Fechner law of psychophysics[3], stating that equal frequency ratios are perceived as equally sized musical intervals. The Weber-Fechner law maintains that many perceptual phenomena are logarithmic by their very nature.
The circular diagrams studied in this text are by Descartes (1618), Robert Fludd (1618) and Jost Bürgi (1620). Descartes's diagrams have recently attracted the attention of several authors from different fields.[4] A second type of geometric diagrams related to musical arithmetic is looked at in the final section of this article.

Introduction

Surveys on the history of mathematical logarithms[5] usually mention that Archimedes was close to the invention of logarithms. The related concepts and calculation techniques involving negative and fractional powers were redeveloped and advanced in the middle of the 16th century by Christoff Rudolf and Michael Stifel. The Dutch Simon Stevin propagated the decimal number system and decimal fractions towards the end of the 16th century. This laid the ground for the logarithmic calculation techniques invented by John Napier and Jost Bürgi towards the turn of the century, which were of high use in astronomy.
By the end of the year 1618, René Descartes offers his manuscript *Compendium Musicæ* to Isaac Beeckman as a New Year's gift. This early text by Descartes is first published in 1650, shortly after Descartes's death. According to H. Floris Cohen the content of this brief treatise is retrospective rather than innovative.[6] This would be true if it were taken only as a theory of how contemporary music is structured or how new music is to be composed. As a compendium it is certainly not supposed to develop or propagate a new theory of music. However, the *Compendium Musicæ* contains some intriguing diagrams that use the circle as a metaphor for the octave similarity in combination with a logarithmic representation of musical ratios.

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[2] In the near future, this topic, much hated by young students, might vanish from the curricula of our schools as obsolete due to the computational power of computers.
[3] Ernst Heinrich Weber (1795–1878); Gustav Theodor Fechner (1801–1887)
[4] Gage 1995, 171-176, 231-235; Gouk 1999, 113-153; Wardhaugh, 2008; Wardhaugh 2013, xxi-xxxii; Muzzulini 2012. A comprehensive theory of scientific diagrams, 'diagrammatology', is still under construction, cf. Mersch (2006, 2012), Krämer (2012).
[5] Voellmy 1948, Waldvogel 2014, 91-92
[6] He calls it "Zarlino *more geometrico*", Cohen 1984, 163

In 1618, the renowned publishing house of de Bry in Frankfurt published the second tractate of the first volume of *Utriusque cosmi …*, the encyclopaedic opus magnum by the English physician and philosopher Robert Fludd with some illustrations by Matthäus Merian.[7] This part of *Utriusque cosmi* contains the *Templum Musicœ*.[8] On the one hand, the diagrams and illustrations in Robert Fludd's entire work visualize human knowledge and science in a way that has not been seen before. On the other hand, Fludd's music theory is based on Neoplatonism and a Pythagorean mode of thinking, which at its most modern refers to the ars nova/ars subtilior-period of the 14th century. Fludd is also an adherent of the Ptolemaic cosmology, according to which the sun and the other planets circulate around the earth. His interest in and defence of the Rosicrucian movement might have been detrimental for a positive reception of his work on the continent by the philosophers of the new age of mechanization Kepler, Mersenne and Gassendi.[9]

Fludd expounds the tone system in a way that appears outdated if compared to Descartes's "Zarlinoism". Fludd defends the ratio 81 : 64, the ditonus, of the Pythagorean tone system, while Descartes constructs the diatonic scale with consonant thirds in the ratio 5 : 4 as propagated by Gioseffo Zarlino in the second half of the 16th century. However, Fludd seems to be mathematically more up-to-date than the 22 years old Descartes in his *Compendium Musicœ*.[10]

In 1619, Johannes Kepler publishes *Harmonice mundi*.[11] The musical section of this work uses classical geometry in order to determine the consonant intervals. Kepler claims a correspondence between the family of regular polygons that are constructible by ruler and compasses and Zarlino's system of consonances. Thus, constructability acts as a natural selection criterion. By assuming that the highest constructible prime number division of the circle is by 5, Kepler gets a three-dimensional generator system defined by the formula $2^k \cdot 3^l \cdot 5^m \quad k \in \mathbb{N}_0, \quad l,m \in \{0,1\}$. By varying the values of the parameters *k*, *l*, and *m* in the formula, the numbers of the vertices of constructible regular polygons are obtained. This set of numbers is open only in the 'octave dimension' 2^k corresponding to bisection of angles, which is always possible. In the 19th century however, Carl Friedrich Gauss proves that the regular polygons whose number of vertices are prime numbers of the format $2^k + 1$, as for example 17 and 257, are also constructible with ruler and compasses, which renders Kepler's musical universe potentially higher dimensional. In the Appendix of *Harmonice mundi*, Kepler criticizes Fludd's *Utriusque cosmi*, which marks the beginning of a long controversy.[12]

In 1620, *Arithmetische und geometrische Progreß-Tabulen* by Jost Bürgi are printed in Prague. Their title page, a synopsis of his number tables in form of a circular diagram, is in its nature very similar to Descartes's circular diagrams. At that time Bürgi seems to have been using his tables for more than ten years.[13] Since Bürgi's *Progreß-Tabulen* have survived in only a few printed copies, it is conceivable that this "publication" in 1620 was merely a test print. The long war period of the Thirty Years' War in Germany, 1618–1648, prevented Bürgi's work from becoming more generally known.[14]

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[7] Only few of them, the more artistic ones, are signed.

[8] Fludd's tractate 2 has been shown at the Frankfurt bookfairs in spring 1618 where Kepler has seen it, cf. Hauge 2011, 22 (FN 80)

[9] Cf. Max Caspar in KGW VI, 513-521

[10] At least Descartes does not show in the *Compendium* whether he is familiar with the theory of algebraic equations, fractional powers and decimal fractions. On the other hand, it is also unclear, how much of higher algebra Fludd actively mastered that is described in his compilation of contemporary science.

[11] Johanes Kepler 1619 in KGW VI, München 1940, 7-377

[12] KGW VI, 373-377. Kepler's approach will not be discussed any further here, cf. Cohen 1984, 13-34

[13] Staudacher 2013, 197

[14] Staudacher 2013, 203–204

René Descartes's musical diagrams

In the *Compendium Musicæ*, the young René Descartes (1596–1650) recapitulated the state of diatonic musical thinking as propagated by Gioseffo Zarlino[15] in the 16th century. Completed by the end of 1618, the manuscript *Compendium Musicæ* is a New Year's gift to his friend Isaac Beeckman (1588–1637). The Compendium's *Prænotanda* present a system of aesthetic principles underlying the organisation of the horizontal time and the vertical pitch/frequency domain of music theory. Descartes accepts only small integer ratios as fundamental and "understandable by the senses" - seing and hearing - and he pointedly argues in favour of arithmetic against geometric division or ratios.[16]

He illustrates the two ways of dividing ratios with line segments 2 : 3 : 4 against 2 : $\sqrt{8}$: 4, i.e., by a visual analogy in one dimension (see Fig 1). Where 3 is the arithmetic mean of the outer terms 2 and 4 because of $3=\frac{2+4}{2}$, the geometric mean of the outer terms 2 and 4 is $\sqrt{8}$ because of $\sqrt{8}=\sqrt{2\cdot 4}$.

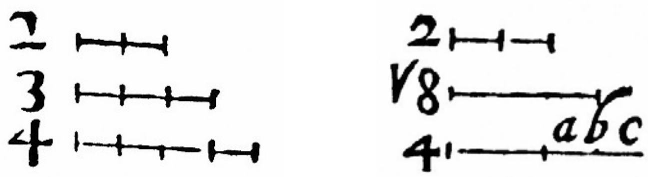


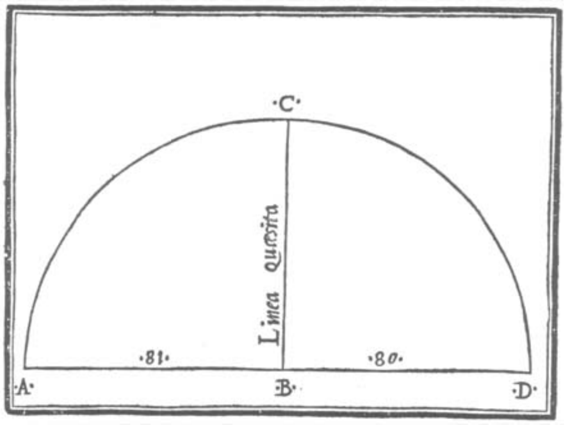
Fig. 1 Arithmetic and geometric division of the octave according to Descartes (Compendium musicæ, Prænotanda).

The word *geometric* in *geometric mean* is motivated by the geometry problem of transforming a given rectangle into a square of equal area; in Descartes's example, turning the rectangle having sides 2 and 4 into a square of equal area 8. The sides of such a square must be precisely $\sqrt{8}$ units long. The word *mean* in the notion 'geometric mean' is justified by the fact that the squareroot of the product of two positive numbers is always between the two factors ($0<a<b \quad \Rightarrow \quad a<\sqrt{ab}<b$).

In combining Euclid's altitude theorem with Thales' theorem, the geometric mean can be found using a straight-edge/compass-construction, known also by music theorists. The construction given by Fogliano 1529 shows the construction of the geometric mean of the numbers 80 and 81 (see Fig. 2a). In other words, it serves to determine geometrically the square root of $80\cdot 81=6480$, resulting algebraically in $80.4984...$ This number is very close to the arithmetic mean $\frac{1}{2}(80+81)=80.5$. Therefore, this example would not be given in a modern geometry text book in order to explain the difference between the two ways of averaging numbers, because a rectangle with sides of 80 and 81 is almost a square. Descartes's example is more convincing in this respect, because $\sqrt{8}\approx 2.8284$ compared with $\frac{1}{2}\cdot(2+4)=3$ yields a visible relative deviation of 6.1%. Note that in the construction of the geometric mean the arithmetic mean appears as the radius of the circle (see Fig. 2b).

[15] Zarlino 1558, [3]1573; 1571

[16] Cohen 1984, 161-179; Muzzulini 2006, 35-37; Muzzulini 2012



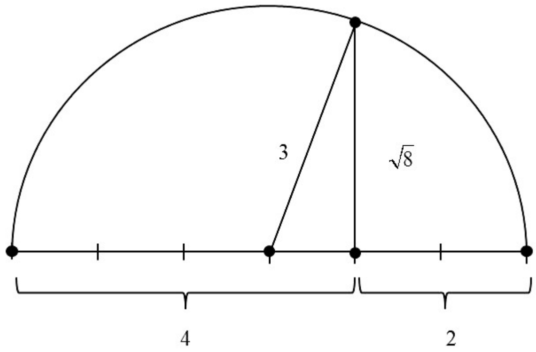
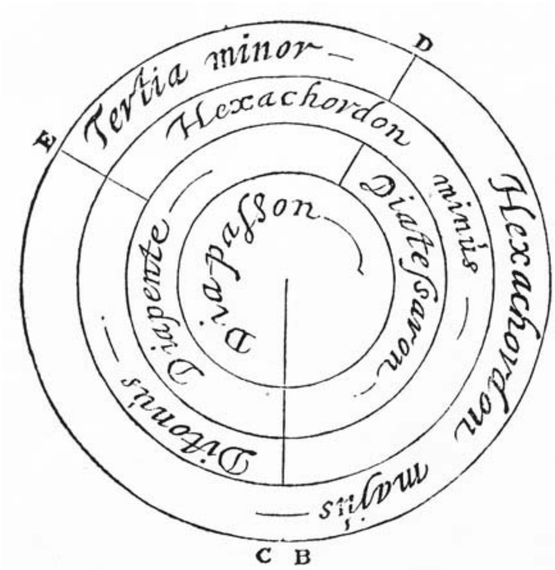


Fig. 2a: Construction of the geometric mean by Fogliano (1529).
Fig. 2b Descartes's example in the light of Fogliano's construction.

Both examples are tied to music theoretical questions in the pitch domain. Fogliano's construction bisects the socalled *syntonic comma*, 81 : 80, into two equal musical intervals. By equal *musical intervals* we mean equal *frequency ratios*. The bisection of the syntonic comma can be used in order to divide the third 5 : 4 into two equal whole tones by lowering and increasing respectively the major and minor tones, 9 : 8 and 10 : 9, by half a syntonic comma. This procedure leads to whole tones in the ratio $\sqrt{5:4}:1=\sqrt{5}:2\approx 1.1180:1$. Such tempered whole tones are used in meantone tempered tuning systems. The ratio of the syntonic comma can also be expressed as the intervallic *difference* between the major and the minor tone, i.e., by the *quotient* of the corresponding ratios: $\frac{9}{8}:\frac{10}{9}=\frac{9}{8}\cdot\frac{9}{10}=\frac{81}{80}=81:80$.
Descartes's juxtaposition bisects the musical octave 2 : 1 *arithmetically* into a fifth 3 : 2 and a fourth 4 : 3 and *geometrically* into two equally sized musical intervals, tritones or diminished fifths of the irrational ratio $\sqrt{2}:1$.[17]
Descartes's proportion 2 : $\sqrt{8}$: 4, is part of the infinite geometric progression $1,\sqrt{2},2,\sqrt{8},4,\sqrt{32},\ldots$ with the common ratio $\sqrt{2}$, defining an arbitrary long sequence of semi-octaves. Similarly, the geometric progression having the common factor $\sqrt[12]{2}$ defines the 12-tempered equal tuning. Descartes's aesthetic principles exclude equal temperament rigorously as not understandable by the sense of hearing. His *Compendium Musicæ* develops the tone system with three congruent diatonic hexachords separated by Pythagorean fifths 3 : 2. This results in a range of two diatonic major scales centred on f and c having the pitch classes c, d, e, f, g, a, b-flat, b (see Fig3d). Thereby, the leading notes on the seventh degree of the major scales, e and b, are a by-product of the neighbouring hexachords.
Furthermore, Descartes's derivation leads to ambiguous values for the tones d (320 vs. 324) and g (480 vs. 486 where c = 360). The two values of these "mobile tones", d and g, differ by a syntonic comma and are clearly distinguished in the drawings.[18] The number values in all the continued proportions given by Descartes represent string lengths at the monochord (or time periods), not frequencies. This is because the frequency interpretation of pitch is not yet well established at that time.[19]

[17] Classical music theory distinguishes between harmonic and arithmetic division. For example 3 : 4 : 5 is harmonic, but 1/3 : 1/4 : 1/5, the ratio of its reciprocals is harmonic, a distinction Descartes does not make.
[18] The musical context decides which one of the ambiguous tones is to be used, cf. Muzzulini 2012, 694–698
[19] Exceptions are Benedetti (1585) and Beeckman (1614), who independently developed a pulse theory of sound, JIB I: 56-57 (fol. 24v, 1614), cf. Cohen 1984, 75-78, 94-97, 127-147; see also the section on 'Beeckman and Stevin' below.



3a

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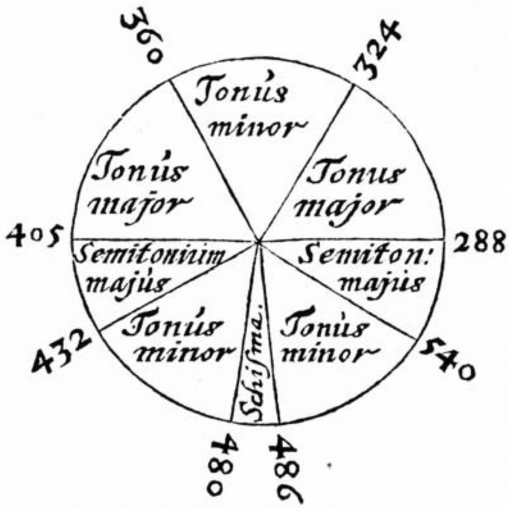

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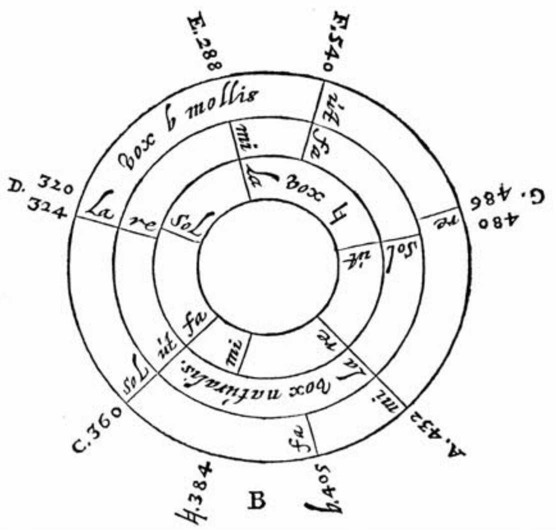

3d

Fig. 3 Descartes's circles.
Fig 3a The consonant intervals within the octave (Diapason).
Fig 3b The same diagram in Beeckman's copy of Descartes's manuscript: The angle for the minor third seems to be much bigger than 90° because the divider line at D does not pass through the clearly indicated centre of the circles, which does not permit measuring the related angles accurately.
Fig. 3c. The diatonic major scale, starting at ut = 540 in clockwise direction with an ambiguous tone (486/480), separated by a syntonic comma, which Descartes calls *Schisma*. The diagram is completely symmetric about the bisector of the syntonic comma. The radii defining the tritone and the diminished fifth (at 405 and 288) are hardly distinguishable from a straight diameter.
Fig. 3d. The three hexachords from F (540), C(360) and G(480) have congruent angles, each given with relative solmization. The leading notes B quadratum (384) and E (288) are a byproduct of the hexachords (ut, re, mi, fa, sol, la). There are two ambiguous pitch classes at G and D. Note again that the diminished fifths (540-384 and 405-288) 'mi contra fa' (the devil in music) are on diameters of the circle.

The comparison of arithmetic and geometric ratios is carried out by Descartes in a continuous "geometric" context, since the geometric and arithmetic ratios are both visualized by ratios of lengths of line segments. In other words, the discrete integer numbers are understood a priori as a part of a comprehensive continuum. There is no other way to compare arithmetic ratios with geometric ratios in general. A physically consistent interpretation of the diagrams is by ratios of lengths of equally tensed strings as at the Monochord.

It is essential for Descartes's paradigmatic choice of numbers that the geometric ratio is an irrational one. In other words, the pair 8 : 13 : 18 (arithmetic progression with common difference $d = 5$) and 8 : 12 : 18 (geometric progression with common ratio $r = 3/2$) would not have served him to reject geometrical ratios in such a rigorous way, because the latter is musically meaningful as a dissonance – two fifths forming a ninth –, whereas the former, due to the prime number 13, does not play a role in classical Western music theory at all. It must be emphasized that the embedding of the musical arithmetic into a geometric setting is essential for Descartes's reasoning and didactical presentation of his thoughts.

What does the young Descartes know about logarithms?

The eye-catching feature of Descartes's *Compendium Musicæ* is its geometrical modes of visualization, especially the use of the circle for visualizing the octave similarity. Descartes seems to be the first to express the octave systematically as a full 360° angle. However, Robert Fludd, who uses the circle very frequently in his illustration, is very close to such an interpretation in his *Templum Musicæ* published in 1618 (see Fig. 6a below).
The diagrams in the first printed Latin editions of Descartes's *Compendium Musicæ* (1550 and 1556), are rather accurate in the following sense: equal musical intervals, i.e., equal number ratios, are represented by equal circular sectors, so that the full octave corresponds to the full circle of 360°. Furthermore, the ratios between the different intervals (major tone, minor tone, semitone, etc.) are maintained in the ratios of angles.

However, an analysis of the angles in the diagrams reveals that the ratio of the tritone is usually equal to 180°, and some of the minor thirds are very close to the 12-tempered 90°. Some of the minor tones are even greater than major tones. An essential feature of Descartes's diagrams is their symmetry, which was deliberately abandoned in the early English edition.[20]
Unfortunately, the manuscript of the *Compendium Musicæ* is lost. Several manuscript copies are still extant, the earliest of them was made for Isaac Beeckman about 1628.[21]
Assuming that Descartes's original drawings were as accurate as the printed ones, one could conclude that he had a feeling for logarithms at a time when they have just been made public. Where could he have got the necessary mathematical knowledge from?[22]
In 1614, the Scot mathematician John Napier published his first tables, *Mirifici logarithmorum descriptio*. However, these tables were of direct use in astronomy, not in musical arithmetic.[23] And the circular diagram by Jost Bürgi, the title page of his *Arithmetische und geometrische Progress-Tabulen* (1620, see below Fig. 4b), which is much easier to understand, had not yet been published when Descartes composed the *Compendium Musicæ*. Although Bürgi's tables were finished in 1609 or even earlier[24], it can be excluded that Descartes knew of Bürgi's tables and calculation techniques, because Bürgi kept them secret.

Although Descartes and Bürgi stayed at least once in the same town at the same time,[25] there is no evidence that a meeting or an exchange of ideas ever took place, as one could suspect when comparing their diagrams.

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[20] Wardhaugh 2008, Section 3. Descartes; Wardhaugh 2013, xxxi-xxxii
[21] Buzon 2012, 22-23, 47; Wardhaugh 2008, Section 3, Descartes.
[22] The mathematical und musical formation that Descartes obtained at the Jesuits' College at Flèche is discussed in Gaukroger 1995, 55–59 and in Schneider 2008.
[23] Sonar 2011, 296-301
[24] Staudacher 2013, 197; Waldvogel 2014, 89
[25] Staudacher 2013, 137

According to Isaac Beeckman's scientific diary *Loci communes* ..., Descartes added a short passage, referring to Beeckman's pulse theory to the *Compendium* in 1619 following Beeckman's suggestion.[26] Is it therefore conceivable that Descartes made or reworked the drawings at a later date? Benjamin Wardhaugh suggests that the accurate circular diagrams in the printed Latin Compendium could have been either the fruit of a much later collaboration with Constantin Huygens (about 1635) or been influenced by Descartes's meeting with Johannes Faulhaber in 1620.[27] The former can be excluded because all the diagrams appear in Beeckman's copy. And the latter is highly improbable with respect to the meticulous working style of Beeckman. He certainly would have mentioned it in his diary if Descartes had changed the manuscript a second time. Concerning Faulhaber, it is generally assumed that Descartes not only benefitted mathematically, but that he was also influenced by Faulhaber's interest in natural magic and Rosicrucian ideas.[28]

Jost Bürgi's mathematical diagram

In his recent and comprehensive study *Jost Bürgi, Kepler und der Kaiser* (2013), Fritz Staudacher highlights that Kepler, when he secretly edited the *Arithmetica Bürgii* in 1603, was certainly familiar with some of Bürgi's calculation techniques. He conjectures that there must have been an agreement of silence between Kepler and Bürgi (as between Tycho Brahe and Kepler). In his own writings, Kepler referred to Bürgi's results only when they were already published by Bartholomäus Pitiscus (1561-1613).[29]

From a mathematical point of view, the title page of Bürgi's *Arithmetische und geometrische Progress-Tabulen* (see Fig. 4) is equivalent to Descartes's musical diagrams.
Bürgi's diagram closes the circle at 10, which is to be identified with 1, whereas Descartes's diagrams close at 2. The use of the circle and its closure at 10 is due to the decimal number system and shows that Bürgi had a deep and thorough understanding of logarithms and of the decimal number system.[30] Bürgi based his tables on powers of 1.0001. In other words, they are not base 10 exponential tables. The closing of the circle at 10 means that multiplications and power computations between decimal numbers can be done with a relative precision determined by the number of digits given in the table.
The closing at the octave in Descartes's diagrams, however, seems to indicate base 2 and fractional powers. The octave similarity results in the identification 1 = 2 = 4 = 8 = … so that the transformation from the frequency domain onto the interval class domain can be mathematically described by

$$pitch \sim \log_2\left(frequency\right)$$

$$pitchclass \sim \mathrm{mod}_1\left(\log_2\left(frequency\right)\right)\cdot 360°$$

as shown in Fig. 5.

[26] JIB I, 269 (Fol. 108r :2 ET 10 JANVIER 1619); Cohen 1984, 188–190 ; Gaukroger 1995, 78 ; Arnăutu (2013)
[27] Wardhaugh (2013, xxxi)
[28] When comparing Faulhaber 1622 with Stifel 1544, Stevin 1585, Zarlino 1571/1573, Fludd 1618 (UCH Vol I, Tract. II, Part I-II) we wonder what kind of higher mathematics Descartes might have learned from Faulhaber at that time. Later, in the early 1630s Faulhaber publishes tables of logarithms and trigonometric functions and makes logarithms better known in Germany.
[29] Staudacher 2013, 197; Kepler's redaction of Bürgi's text is given and discussed in List & Bialas 1973.
[30] cf. Waldvogel 2014, 79

Fig. 4. Jost Buergi, Arithmetische und geometrische Progress-Tabulen (1620)

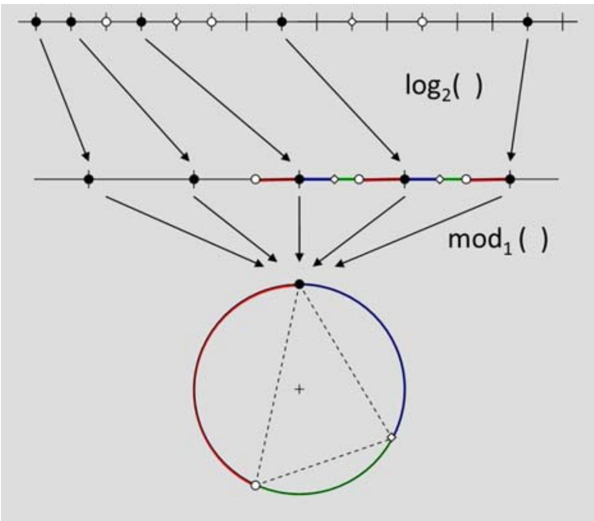


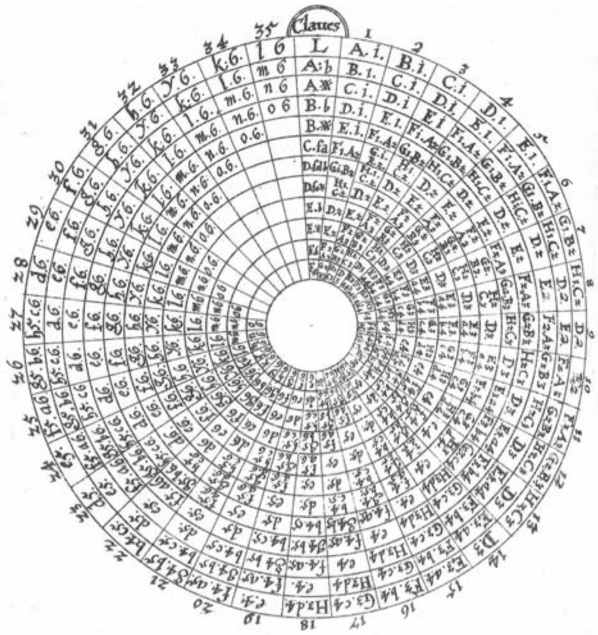
Fig. 5 Pitch classes as a composite mathematical transformation. The first transformation, base 2 logarithm, maps frequency onto pitch, the second modulo 1 maps pitch onto pitch classes. The triangle corresponds to a major triad (4 : 5 :6).

The circle metaphor realizes an implicit “geometric reduction” by powers of two. In the surrounding text Descartes insists that music theory can be reduced into an octave in order to study consonant intervals and scales.[31]
Remarkably, Descartes never comments on the angles in his illustrations. And he does not highlight the origin or originality of his illustrations.[32]
Eventually, it must be emphasized that the octave reduction as a geometric reduction principle seems to contradict Descartes’s rejection of geometric ratios at the most fundamental level. And the use of angles to measure musical intervals is clearly conceived from a continuous point of view.[33]

Robert Fludd’s circular diagrams

The circle, “the perfect shape” with its infinity of inner symmetries, is present in every topic Robert Fludd studies – from divine numbers to the colours of urine. We will focus on two examples, taken from the *Templum musicæ* and from *De Numero et Numeratione*.[34]

The first drawing (see Fig. 6a resembles Descartes’s diagrams in many ways: It is a circular arrangement, it consists of concentric circles, and it is about music. Furthermore, it uses a “logarithmic” presentation by equating musical intervals with distances. At first sight, the diagram is confusing, since it is overcrowded with letters and numbers, and its meaning does not leap to the eye. The analysis is supported either by comparing it to other pictures of the same chapter or by picking the related information from the text.



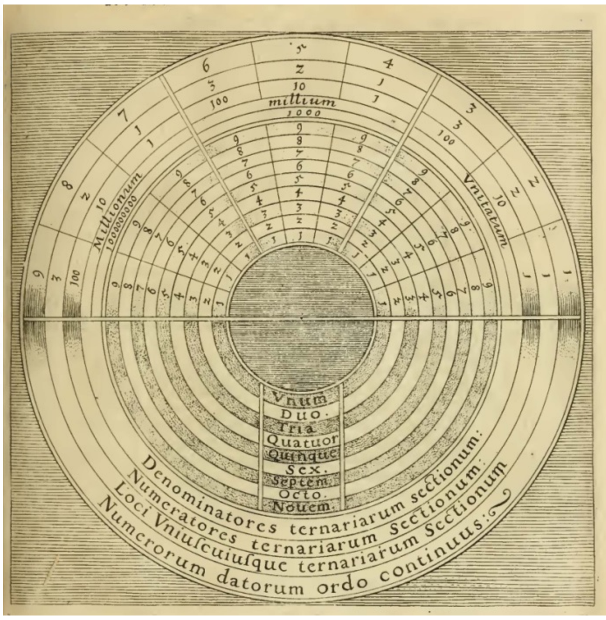


Fig. 6a Transposition circle for the lute (barbitum) (UCH Vol I, Tract II Part II Lib VI, De Instrumentis Musicis vulgariter notis, 232).
Fig. 6b Numerationis Speculum (UCH Vol I, Tract II Part I Lib I, 9, De Numero et Numeratione).

[31] Descartes *Compendium*: Brockt 1978, 12-22; De Buzon 2012, 68-81.
[32] There is no priority dispute about diagrams at that time.
[33] The full circle line corresponds to the continuous unit interval of real numbers [0;1).
[34] UCH Vol I, Tract II Part II Lib VI, *De Instrumentis Musicis vulgariter notis*, 232; UCH Vol I, Tract II Part I Lib I, *De Numero et Numeratione*, 9. Other examples are divine numbers, Coss, colours, colours of urine, multiplication and power tables, astrology.

The diagram displays three octaves of the chromatic scale within twelve concentric rings. The letters and indices are not used to indicate tone names and octave position. Rather, they alphabetically number the frets and indicate the strings on the lute ('barbitum') with its lowest 1 and highest 6. Both dimensions, the radial and the angular, display chromatic scales. The radial scale covers one octave whereas the circular scale covers three octaves, and the semitones of both scales are positioned equidistantly. It follows from the surrounding text that the vertical scale on top is meant to be a spinning pointer, so that the drawing represents a mechanical tool to be used by lute players, in order to transpose their part quickly if necessary.[35] While Descartes's labels and tone names refer to the borders between the circular sectors so that they indicate points on the circle line, Fludd's designations are in the centre of the cells. The tuning of the lute can be guessed by looking at the cells containing two designations. It turns out to be G-C-F-A-D-G. This is consistent with Fludd's drawing of the fretboard of the lute with the notes indicated by their position in the stave (see Fig 7).[36]

The musical universe of Fludd operates within three octaves. The three octaves are a recurring metaphysical idea manifest in many of his drawings. However, the range of the musical instrument exceeds three octaves by two semitones, of which the last ones are displayed in the inner rings only. The radial chromatic scale, the pointer, covers the last sector at position 36 holding m_6, n_6, o_6. Each chromatic step is 10° of the full rotation of 360°. Thus, one octave comprises a 120° angle, where for Descartes it is 360°. The number 3 is a holy number for Fludd because it is the first that has a beginning, a middle part and an end. In the same mode of thinking, the perfect division in the Middle Ages is ternary and not binary.[37]

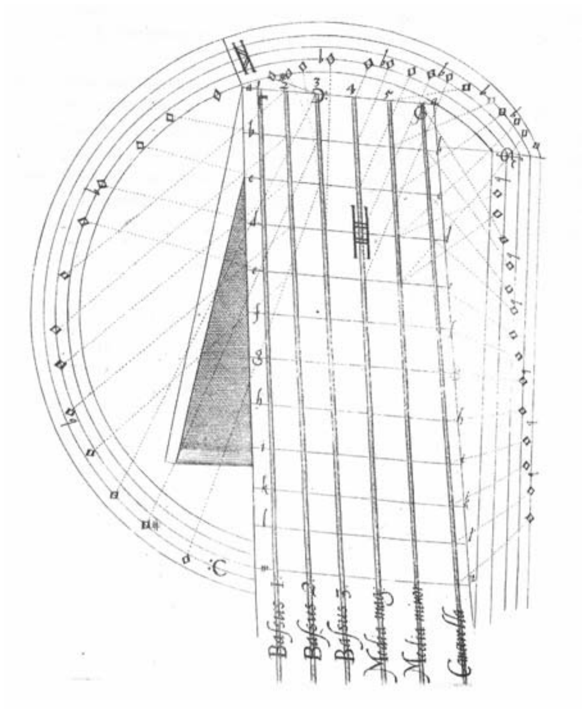


Fig. 7 Fretboard of the barbitum. This picture confirms the tuning derived from Fig. 6a (UCH Vol I, Tract II Part II Lib VI, De Instrumentis Musicis vulgariter notis, 230).

[35] Fludd uses the word rota (wheel): "Tunc convertendo rotam L invenio a.6. in loco ejusdem spharae. 25. & sub ipso in orbe. A.re.c.6.", UCH Vol I, Tract II Part II Lib VI, 232 (= Hauge 2011, 188)
[36] UCH Vol I, Tract II Part II Lib VI, 230
[37] UCH Vol II, Tract I, Sect I, Lib 1, *De numeris divinis*, 26, 35-36

The second drawing (Fig. 6b) is a mathematical diagram taken from *De Numero et Numeratione*.[38] In this part of his encyclopaedia, Fludd summarizes the comparatively young 'Coss', the theory of polynomials and equations, where he explicitly recommends Stifel for further reading.[39] The diagram explains the naming principles of the decimal numbers within a logarithmic presentation giving the powers of 10 in anticlockwise direction. Each of the nine digits 1-9 of the outermost circle gets the same angle (20°). The second circle groups the 9 sectors as 3 times 1-2–3, which indicates the three positions Ones, Tens and Hundreds within each of the three groups Units, Thousands and Millions. Since the numbering starts at 1 and not at 0, the digits in the first two rows are not power indices, they just indicate the position of a digit counted from the right to the left. The number of digits in a place value system is a raw logarithmic measure of a number: An integer number x with 5 decimal digits, for example, has a base 10 logarithm between 4 and 5 ($4 \le \log_{10}(x) < 5$), which is equivalent to $10,000 \le x < 100,000$ (see Fig. 8).

8	7	6	5	4	3	2	1	0
10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0
100	10	1	100	10	1	100	10	1
10^2	10^1	10^0	10^2	10^1	10^0	10^2	10^1	10^0
Millionum			*Millium*			*Vnitatem*		
1 000 000			1000			1		
10^6			10^3			10^0		

Fig. 8 Fludd's Speculum rewritten in modern power notation, which makes the law $10^n 10^m = 10^{n+m}$ evident.

Notice that 0 does not occur as a number or digit of its own right in the *Speculum*, the digits in the inner circles run from 1 to 9. Zero is not yet a full-fledged number – it is merely an articulation sign indicating an empty position. Forming numbers from the *Speculum* is combinatorics: picking from each sector a digit or an articulator. Thereby, every positive integer number less than 1,000 Million can be obtained.
The lower semicircle in Fludd's diagram contains only the labels of the circles, but has otherwise no topological meaning related to the number system. However, it could symmetrically hold the related decimal fractions in clockwise direction.

Actually, zero (the devil in numbers) would disturb the "perfect order" of three times three in the two aspects, digit and position, of numbers. Again as in the transposition circle of the lute, the closing of the system with nine digit numbers is not well motivated as the choice of the circular arrangement per se. As a mechanical tool, however, it is easier to be realized with rotating pointers than with straight sliders.[40]

The diagram seems to allude to Ramon Llull's (~1232-1316) concentric circles, which served a mechanical combinatorial device.[41] However, in order to turn Fludd's *Speculum* into a proper mechanical tool, it would have been better to interchange the roles of the radial and the angular dimension, so that the individual numbers could be read off in radial direction.

[38] UCH Vol I, Tract II, Part I, Lib I, *De Numero et Numeratione*, 9
[39] UCH Vol I, Tract II, Part I, Lib III, *De Arithmetica Cossica, Epilogus*, 79. Stifel (1544) treats musical intervals and scales in a way very similar to Zarlino.
[40] For the history of logarithmic straight and circular slide rules, cf. Cajori 1909, 1920
[41] Ramon Llull 1305, Ars Brevis, Fig. IV. The author owes the conjecture of a possible link between Llull's and Descartes's diagrams to a conversation with Angela Lohri (Vienna).

To sum up, the transposition circle (Fig. 6a), as a two-dimensional matrix arrangement of chromatic scales with equal semitones, is a double logarithmic representation in polar coordinates, whereas the *Numerationis Speculum* (Fig. 6b) combines a radial linear dimension with a logarithmic angular dimension.
A closer inspection of Fludd's many mathematical diagrams reveals that he is systematically searching for adequate visualising techniques. He explicitly reflects on the graphical means to support the performance of memory in other parts of the *Utriusque cosmi*.[42]

What is possible without logarithms as developed by Napier and Briggs?

Descartes's *Compendium Musicæ* seems to show that he is familiar with geometric sequences, even for irrational base, i.e., fractional powers. He certainly knows that multiplying ratios corresponds to adding musical intervals. This is evident from the numbers added to the circular diagrams and from the continued proportions defining the tone system. However, this is standard in music theory since Boethius and well known in the 16th century.[43] In *Géometrie* (1637) Descartes refers to the problem – posed by Pappus of Alexandria (c. 290 – c. 350 AD) – of determining several intermediate proportional numbers between two given numbers.[44] In order to solve this problem, Descartes depicts a mechanical instrument that permits drawing the corresponding *loci*, i.e., the graphs of power functions, if a unit length is defined in the geometric plane (see Fig. 9).[45]

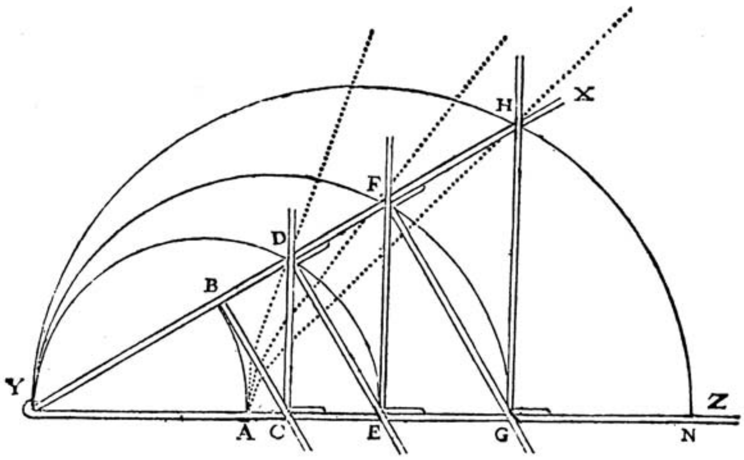


Fig. 9 Descartes's instrument (mesolabe compasses) for constructing geometric progressions.

For each opening of the legs, the lengths of the line segments between the legs defined by the system of moveable rulers are kept in a constant ratio. This property follows from the array of similar right triangles. Therefore, these segments form geometric progressions. Changing the opening of this compasses alters the base (> 1 = YA). This instrument, called mesolabe compasses, can be used in two ways so that not only integer powers, but also fractional powers can be constructed. With many and sufficiently long rulers, Descartes's instrument admits – in principle - the construction of arbitrary fractional powers of any real base greater than 1. These considerations lead Descartes to question the traditional notion of *mechanical curves*. The main ideas for Descartes's analytical geometry were developed in the 1620s, soon

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[42] UCH Vol I, Tract II Part I Lib X, *De Arithmetica Memoriali*, 153–158; Vol II, Tract I, Sect II, Port 3, Lib 2, *De Animæ memorativæ scientia*, 47–70
[43] Fogliano 1529, Stifel 1544; Zarlino 1558, 1571
[44] Descartes, *Géometrie*, Annexe to *Discours de la méthode*…, Leiden 1637, 306 ; cf. Gaukroger 1985, 93-99
[45] Descartes 1637, 318, 370-371

after the completion of the *Compendium Musicæ*.[46] Descartes in 1637 does not give a hint that his instrument could be used to mechanically determine the frets of lutes, and he did never publish on music theory either.[47]

Comparing the octave indirectly with the syntonic comma

In 1581, Vincenzo Galilei remarks that 18 : 17 provides an excellent superparticular approximation of the semitone in equal temperament: $\left(18/17\right)^{12} \approx 1.9856 \approx 2$. Comparing this with $\left(17/16\right)^{12} \approx 2.0699$ and $\left(19/18\right)^{12} \approx 1.9133$ shows that $18/17$ is indeed the best "semitone" of the form $n+1/n$ to approximate 12-tempered equal tuning. Already, Ptolemy knew the relationship $18/17 < \sqrt{9/8} < 17/16$, which holds true because the middle term is the geometric mean of the outer terms, see Fig. 10.[48]

The ratio 18 : 17 was used to determine the positions of the frets in lutes, where, by practical reasons, equal temperament was earlier accepted as a compromise.[49] Nevertheless, the prime number 17 does not play a fundamental role in Western music theory. In 1619, Kepler calculated the related string lengths and compared them with his own chromatic scale.[50] However, he could have picked these values directly from Stevin's tables of interest.[51]

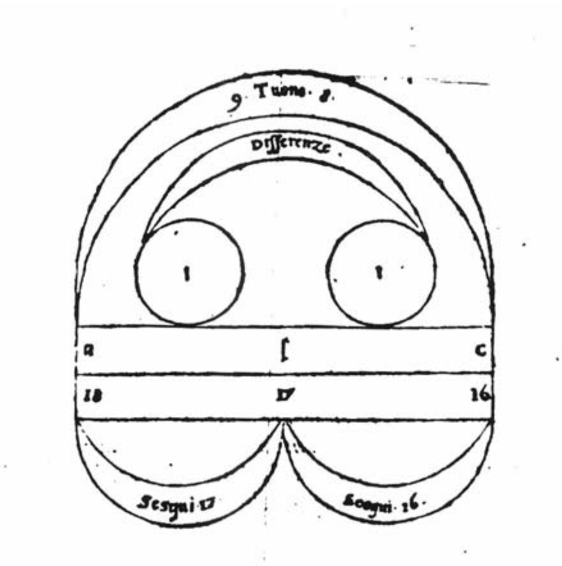

Fig. 10 Division of the whole tone 9 : 8 into semi-tones 17 :16 and 18 : 17 (Zarlino 1571)

Comparing the syntonic comma 81 : 80 with the semitone $18/17 \approx 1.05882$ leads to about 9 syntonic "semi-commas" per semitone 18 : 17, or approximately $9/2 \cdot 12 = 54$ syntonic commas per octave. The value correct to four decimal places is 55.7976, in other words, about 55.8

[46] Gaukroger, 99-103

[47] However, in his correspondence with Mersenne music theoretical questions are addressed frequently, for example overtones in 1633, AT t. I 267-268, cf. Muzzulini 2006, 126-129.

[48] Forster 2010, 354

[49] By pressing down a string of a lute, its tension is slightly increased, so that the tuning by semitones sized 18 : 17 approximates 12-tempered tuning more accurately, cf. Forster 2010, 357.

[50] Cohen 1984, 68, KGW 5, 143

[51] *Tafel van Interest van den penninck 17*, 1582, PWS II, 75

syntonic commas are equal to an octave, which results in an angle of $^{360°}/_{55.8} = 6.45°$ for the syntonic comma. Notice that Descartes calls the syntonic comma *Schisma*.[52]
Determining the Pythagorean tone 9 : 8 as approximately 9 syntonic commas and the octave as approximately 6 Pythagorean tones also gives approximately 54 syntonic commas per octave. Even with the rough estimate of the syntonic comma by 6° (resulting in an octave of 60 syntonic commas) one could construct Descartes's diagrams with more accuracy than the ones of Beeckman's copy and the early Latin printed editions.

The syntonic comma has two properties the Pythagorean comma is lacking: It is a superparticular ratio and it has a simple finite decimal representation 1.0125. Therefore, the syntonic comma can be taken as a convenient *unit interval* in order to measure the size of the other intervals, i.e., to determine their ratios as powers of the ratio of the unit interval in decimal representation.
Comparing the syntonic comma with the semitone (16 : 15) and with the major and minor tones (9 : 8 and 10 : 9) in this way gives estimates of the angles for these intervals. The technique to estimate intervals as the Pythagorean third (81 : 64) and the Pythagorean comma (531441 : 524288) by superparticular ratios is already applied by Boethius, and the more sophisticated measuring of intervals in terms of Pythagorean commas is done by Stifel (1544) and Faber Stapulensis (1496, 1551).[53] Without using decimal fractions, the use of superparticular ratios as means of comparison is a convenient approach, because the relative interval size of superparticular ratios can be seen immediately in the numbers involved, which is not possible for numbers with varying differences.

Beeckman and Stevin – rationalizing the chromatic scale

It goes without saying that also the 12-tempered values, multiples of 30°, could have served as points of reference for Descartes's angle calculations. As mentioned before, some of the minor thirds in Descartes's diagrams are very close to 90°.

Simon Stevin gives accurate numerical values for the ratios of the 12-tempered equal tuning in *Vande Spiegheling der Singconst*. The negative feedback by the organist Abraham Verheyen in about 1608 might have prevented Stevin from publishing this text.[54]

In 1624, Beeckman borrows Stevin's manuscript from Stevin's widow[55] and in his diary he mentions Stevin's description of the fifth as the 12th part of seven octaves by $\sqrt[12]{128}$.[56]
Beeckman, however, has known Stevin's Mathematical Memoirs (1605/08) earlier. He has been referring to Stevin's writings from 1612 onwards, with respect to music theory and geometric division of musical ratios in 1613/1614 and in 1618[57]. Beeckman, after having mentioned that Stevin tries to make the halftones equally sized, presents the continued proportions of the chromatic and diatonic scales shown in Fig 11a.

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[52] The angles for the 'Schismata' in the early French and Latin edition vary between 5° and 14°. See Buzon 2012: 100-101, 104-105 and Fig. 3 for reproductions of some of the diagrams.
[53] Boethius, MUS. III. 12, 286: "In qua numerorum proportione sit comma et quoniam in ea, quae maior sit quam .LXXV. ad .LXXIIII. minor quam .LXXIIII. ad .LXXIII."; Stifel 1544, Arithmetica I; no pagination. between 72 and 76(?); Faber Stapulensis 1551, Tomus II, 35
[54] Cohen 1984, 61–63
[55] Later on, Stevin's manuscript was in the hands of Constantin Huygens and eventually published in 1884, cf. Waard in JIB II, 292, JIB II, Appendix (fol. 228r-228v), 403-405.
[56] JIB II, 291-292 (fol. 194r: 16.–24. June 1624, Flemish), cf. Cohen 1984, 185
[57] JIB I, Fol. 14r [Juillet 1613-Avril 1614] 29 ; JIB I, Fol. 74v [Avril-25 Juin 1618], 180-181

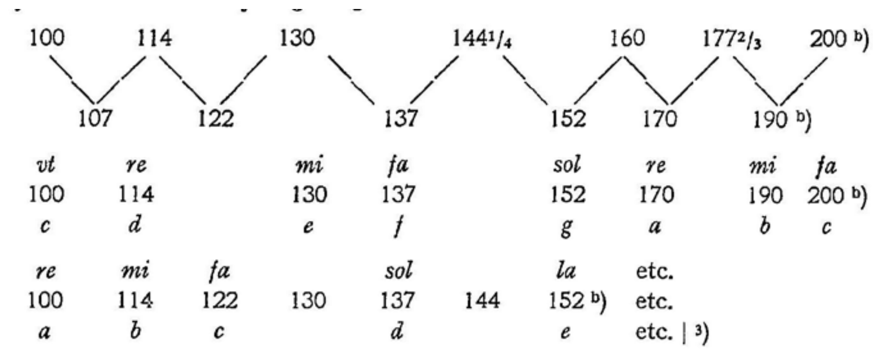

Fig. 11a Isaac Beeckman's chromatic and diatonic scale (Beeckman 1614, JIB I, Fol. 14r [Juillet 1613-Avril 1614] 29).

+7	+7	+8	+8	+7	+7.25	+7.75	+8	+10	+7.67	+12.33	+10

$100 \mapsto 107 \mapsto 114 \mapsto 122 \mapsto 130 \mapsto 137 \mapsto 144.25 \mapsto 152 \mapsto 160 \mapsto 170 \mapsto 177.67 \mapsto 190 \mapsto 200$

Fig. 11b The differences between the numbers would form a monotone increasing sequence if Beeckman's continuous proportion provided an accurate rationalisation of the 12-tempered equal tuning.

A mere superficial inspection of the sequence of numbers on top of Fig10a reveals that it cannot be the best rationalization of the equally tempered scale within the implicitly indicated precision of thirds and quarters of the unity. The outer terms 100 and 200 are chosen with respect to the decimal number system and not by musical considerations. As a geometric sequence with common ratio $r > 1$ the difference between consecutive terms should be monotone increasing. Instead, the differences shown in Fig. 10b are obtained.

It can hardly have been Beeckman's intention to find a rationalization of the 12-tempered equal tuning, rather to define a compromise of the chromatic scale with improved semitones and tones. Under this premise it turns out that harmonic considerations to create good major and minor triads must have played a subordinate role in Beeckman's rationalization.

The best approximation of the equally tempered chromatic scale with one decimal place is given in Table 1 column B. These values fully agree with Stevin's numbers scaled down to Beeckman's range.

	A	B	A / B	A / B [syntonic commas]	Guessed rationalization			
c = ut	100	100	1.0	0	1	1		
	107	105.9	1.01039	0.832	16/15	1.14 8/7		
d = re	114	112.2	1.01604	1.281	16/15			
	122	118.9	1.02607	2.072	16/15	1.1404 8/7		
e = mi	130	126.0	**1.03175**	**2.516**	16/15			
f = fa	137	133.5	1.02622	2.083	20/19			
	144.25	141.4	1.02016	1.607	10/9	$(10/9)^3$ = 1.372		
g = sol	152	149.8	1.01469	1.174				
	160	158.7	1.00819	0.657	10/9	190/137 =1.387	5 / 4	
a = re	170	168.2	1.0107	0.857				

	177.67	178.2	**0.99703**	**-0.239**	10/9			
b = mi	190	188.8	1.00636	0.510				
c = fa	200	200	1.0	0	20/19			
				octave	1.968	1.985		

A: Beeckman/Stevin 1614; B: Stevin, *Vande Spiegheling*, scaled down from 5000/10000 to 100/200 (1 d.p.).

Table 1 Analysis of the chromatic scale given by Beeckman (see Fig. 11).

Checking the guessed diatonic scale leads to

$$\left(\tfrac{10}{9}\right)^3 \cdot \left(\tfrac{16}{15}\right)^4 \cdot \left(\tfrac{20}{19}\right)^2 = 1.968\ldots \qquad = 1 \text{ octave minus } 1.314 \text{ syntonic commas}$$

$$\left(\tfrac{10}{9}\right)^3 \cdot \left(\tfrac{8}{7}\right)^2 \cdot \left(\tfrac{20}{19}\right)^2 = 1.985\ldots \qquad = 1 \text{ octave minus } 0.597 \text{ syntonic commas}$$

Compared with the numbers given by Beeckman, Stevin's values between 5000 and 10,000 in *Vande Spiegheling* are excellent approximations of the 12-tempered tuning. According to H. Floris Cohen, Beeckman was not only against equal temperament but also against mobilizing the second step in the diatonic scale by a syntonic comma as it was suggested by Descartes.[58]

The fact that Beeckman is informed about Stevin's "12-tempered ideas" already in 1614 and the detailed information on tunings he gives in earlier entries of the same year to his diary make clear that he is familiar with musical interval computation several years before he meets Descartes. In a remarkably modern way, Beeckman bases his numbers on frequencies and not, as then still common, on string lengths as Stevin, Kepler and Descartes.

In 1616, Beeckman writes that multiplying ratios corresponds to adding musical intervals and he distinguishes between the Pythagorean and the syntonic thirds explicitly: "Verus enim ditonus est 80/64, id est 5/4, eorum verò 81/64 a duplicatâ ratione 9/8."[59] Notice the collocation 'duplicatâ ratione': doubling a ratio is squaring its fraction. The traditional Latin term 'ditonus' for the major third makes a clear statement about interval size. It is conceivable that Beeckman influenced Descartes's interval calculations in 1618, which resulted in the circular diagrams, but Descartes could have also learnt these basic facts about musical intervals from studying Zarlino at Flèche in the years before his friendship with Beeckman.

Alternative approaches based on syntonic major thirds

In *Vande Spiegheling der Singconst*, Simon Stevin claims that the octave consists of 6 equal whole tones or equivalently of 5 whole tones and 2 semitones, which seems to imply that the octave consists of 12 equal semitones.[60] And he gives "experimental evidence" by listening to two harpsichords, tuned a tritone apart.[61]

Likewise, the octave would also consist of three major thirds. According to H. Floris Cohen, Abraham Verheyen observes (~1608): "Finding, in repeating the experiment with the two harpsichords, that the addition of two augmented fourths does *not* yield a pure octave, he reasons that a simpler test would be to listen whether the addition of three major thirds would (as this would equally follow from Stevin's division)."[62]

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[58] Cohen 1984, 152; 156; FN 109, 281; FN 114, 281 ; see also Fig. 3 above.

[59] JIB I: fol. 40r [6 Février-23 Décembre 1616] 88-89, in the same time he also quotes Faber Stapulensis, JIB I: fol. 38v [<Mars 1615> -6 Février 1616], 84.

[60] cf. Cohen, 1984, 51-53

[61] Cohen 1984, 51

[62] Cohen 1984, 62; FN 46, p. 265-266

Three major thirds of 5 : 4 give 125 : 64. So three pure major thirds compared with the octave are in the ratio 125 : 128, a ratio called *lesser diesis*.
The 12-tempered major third equals 120°, and since $125:128 = 0.9766... \approx 0.975 = 117:120$, by linear approximation, the angle of the major third is close to 117° ($^{125}\!/_{128} \cdot 120° = 117.188°$).
The correct angle of the major third 5 : 4 is 115.89°.[63] The next step would be to divide the major third – in the musically correct ratio – into a major and a minor tone.

Given the approximation 6.5° for the syntonic comma 81 : 80, the angles constituting the diatonic scales can be calculated very accurately and quickly. The ratio between two syntonic commas 6561 : 6400 and the diesis 128 : 125 is only $^{32805}\!/_{32768} = 1.00113$. So the major third 5 : 4 can be approximated by $120° - \frac{2}{3} \cdot 6.5° = 115.67°$ (correct 115.89°). Then, the minor tone 10 : 9 is the major third minus the syntonic comma divided by two, which equals 54.6° (correct 54.7°), and the major tone 9 : 8 becomes equal to 61.1° (correct 61.2°). Finally, the semitone 16 : 15 results in $\frac{1}{2} \cdot \left(360 - 3 \cdot 61.1 - 2 \cdot 54.6\right)° = 33.75°$ (correct 33.52°).

Comparing the tritone and the diminished fifth

In order to make a drawing of the diatonic scale for didactical purposes, a crucial point is to visually distinguish the major from the minor tones and to make the tritone consisting of two major tones and one minor tone different from 180°. The latter requirement is due to the fact that the diminished fifth (64 : 45) and the tritone (45 : 32) are different intervals:

$$^{64}\!/_{45} : {}^{45}\!/_{32} = 2048:2025 = 1.01136$$

The best and very accurate superparticular approximation of this intervallic difference is 89 : 88, which is slightly less than a syntonic comma. In other words, the difference of the triton from the 180°-angle must be less than half the angle of the syntonic comma: The correct values are indeed 177.07° for the augmented fourth 45 : 32 and 182.93° for the diminished fifth 64 : 45. Descartes's diagrams do not clearly distinguish between the tritone and the diminished fifths.
We conclude that, no table of logarithms is needed to find the angles in Descartes's diagrams in a way that satisfies the eye. However, in order to construct a "circular musical slide rule" with astronomical precision, Bürgi's *Progress Tabulen* would be very convenient…

Interlude: A Rosicrucian link?

In 1620, Descartes visited Johann Faulhaber (1580–1635) in order to study with him. Faulhaber, the founder of a mathematical school at Ulm (1600), published *Arithmetischer Wegweiser zu der hochnutzlichen freyen Rechenkunst* (1614), which mathematically refers back to Michael Stifel. Faulhaber was interested in alchemy and also in the Rosicrucian Society. "On 21 January 1618 he wrote to Rudolph von Bünau: '... I am not sparing any efforts in inquiring about the commendable Rosicrucian Society...' "[64]
Apparently, in 1619, Descartes planned to write a book provisionally titled *The Thesaurus of Polybius Cosmopolitanus* and to dedicate it to the Rosicrucians. According to an extant copy of its summary, which is of similar content as Rule 4 of Descartes's *Regulæ ad directionem ingenii*, its intention was creating a new science that merges algebra with geometry.[65] At that

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63 Such a linearization would not work at all for minor thirds.
64 http://www.encyclopedia.com/doc/1G2-2830901390.html [140825]: "Faulhaber, Johann." Complete Dictionary of Scientific Biography. 2008. Encyclopedia.com. 25 Aug. 2014 <http://www.encyclopedia.com>.
65 AT X, 371-378

time Descartes was fascinated by the compasses, later on described in the *Géometrie* (1637), see Fig. 9.[66]
Seemingly, Descartes had also tried to find out about the Rosicrucian Society without success[67] and in this he was in good company with another Rosicrucian exponent, Robert Fludd. Gary L. Stewart claims that not only Faulhaber but also Descartes and Beeckman were members of the secretive order of the Rosicrucians, however he conceals that no hard facts such as member cards or lists of members have survived.[68]

Kepler as well as Mersenne and Gassendi argued against Robert Fludd's Neoplatonism. Descartes, however, remains silent about this issue. We wonder whether Descartes has seen the first parts of Robert Fludd's *Utriusque cosmi* (published in 1617 and in spring 1618) during his stay at Breda. The Kepler/Fludd exchange beginning immediately after its publication must have been of public interest. Kepler's quick response can be partly explained by the fact that he was directly asked by the editor of his *Harmonices mundi* to comment on Fludd.[69] It is conceivable that Descartes had seen the diagrams and taken inspiration from the many circular arrangements in the *Templum Musicæ* or also from the parts dealing with arithmetic. It is also possible that Kepler took inspiration from one of Fludd's diagrams (see below).[70]

Measuring the octave in terms of superparticular ratios

The calculations given above to estimate the angles of the musical intervals in Descartes's circular diagrams, were carried out with superparticular ratios: The idea was to express the octave 2 : 1 or other interval ratios as powers of superparticular ratios. It is clear that there is no such exact representation in the form $\left({}^{n+1}\!/\!{}_{n}\right)^{k} = 2$ for integer values of *k* and $n > 1$, since *n* and $n + 1$ are relatively prime numbers. However, there are some good approximations already for small values of *n*, for example $\left({}^{12}\!/\!{}_{11}\right)^{8}$, $\left({}^{18}\!/\!{}_{17}\right)^{12}$ and $\left({}^{9}\!/\!{}_{8}\right)^{6}$ in descending order of relative accuracy. The equation $2 = \left({}^{81}\!/\!{}_{80}\right)^{55.798\ldots}$ determines the angle of the smallest interval of Descartes's circular diagrams. Since the syntonic comma is a very small musical interval, even the rough estimate $2 \approx \left({}^{81}\!/\!{}_{80}\right)^{54}$ results in angles that satisfy the eye.

Jost Bürgi's calculation techniques

In his *Progress-Tabulen*, Jost Bürgi used the base 1.0001 in order to create a fine grained table of powers, of which the numerical values of the powers run from 1 to 10.
The numbers 1.1, 1.01, 1.001, 1.0001, … are of the format $b_k = 1 + 10^{-k}$. Integer powers of numbers in this format can be evaluated with the aid of the power series of the exponential function or, more elementary, by using the binomial formula. Because the tables are given in finite precision, with finitely many decimal digits, the practical question for Bürgi is, how these powers can be computed efficiently and with limited precision by hand, so that the

[66] Gaukroger 1995, 99-103
[67] In 1624 Nicolaes Wassenar claimed in *Historich Verhal*, that Descartes was a Rosicrucian, cf. Gary L. Stewart (1986) <http://www.crcsite.org/affiliation.htm> [140826]
[68] Gary L. Stewart 1986, loc. cit.
[69] Kepler KGW 6, 373-457; 513-517 (Nachbericht, Max Caspar), Fludd 1623, *Monochordum Mundi*
[70] By the way, Faulhaber was in contact with Kepler…

results are correct to the number of digits required. The well-defined limited precision makes Bürgi's and Napier's tables interesting from the perspective of modern numerical analysis.[71]

Linearization, interpolation

The black values listed in Bürgi's table form a geometric sequence, which can also be viewed as a regularly sampled continuous exponential function $f_k(x) = (b_k)^x$. Bürgi used $k = 4$.
The graphs of exponential functions are concave-up and not straight lines. Nevertheless, the first values $f_k(n) = (b_k)^n, \quad n = 0,1,2,...$ rounded to k decimal places form a finite arithmetic sequence with the common difference $d_k = 10^{-k}$. Some powers for Bürgi's base $b_4 = 1.0001$ are shown in Table 2.

1.0001^0	= 1	= 1.0000
1.0001^1	= 1.0001	= 1.0001
1.0001^2	= 1.0002000	= 1.0002
1.0001^3	= 1.000300030001	= 1.0003
1.0001^4	= 1.0004000600040001	= 1.0004
…		
1.0001^{100}	= 1.01004966…	= 1.0100
$1.0001^{\mathbf{101}}$	= 1.01015067…	= 1.0**102**

Table 2 Bürgi's geometric/arithmetic progression

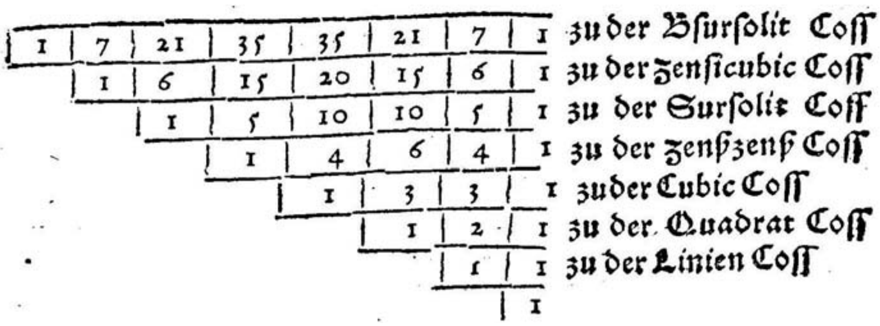

Fig. 12 Pascal's Triangle according to Michael Stifel (1544)

The second equality sign in each row of Table 1 means "is equal to … when rounded to 4 decimal places". As can be seen in the first five lines of Table 2 the decimal representations contain the numbers of Pascal's triangle (see Fig. 12) filled up with zeroes. So the decimal representation is correct to four decimal places, as long as the higher order coefficients $\binom{n}{2}, \binom{n}{3},...,$ of Pascal's triangle do not interfere with $\binom{n}{1}$. In four digits precision, linear extrapolation with slope 1.0001 works fine up to and including $n = 100$. For $n = 101$, however, the rounding up in the last significant figure breaks the regular pattern. Actually, just the second coefficient $\binom{n}{2} = \frac{1}{2}n\cdot(n-1)$ is responsible for the rounding up at the fourth decimal place for $n = 101$:

$$\binom{100}{2} = \frac{100\cdot(100-1)}{2} = 4950 \text{ but } \binom{101}{2} = \frac{101\cdot(101-1)}{2} = 5050$$

[71] Jörg Waldvogel has shown that Bürgi's table contains no systematic mistakes, see Waldvogel 2014, 104-115.

The first rounds down the second up. The interference of the third binomial coefficient at $n = 100$ becomes visible only in the 7th and 8th decimal places because of

$$\binom{100}{3} = \frac{100\cdot(100-1)\cdot(100-2)}{3\cdot 2\cdot 1} = \underline{161}700 \text{ and } \binom{101}{3} = \frac{101\cdot(101-1)\cdot(101-2)}{3\cdot 2\cdot 1} = \underline{166}650.$$

Indeed 4950 + 16 = 4966 and 5050 + 17 = 5067, i.e., using the first three terms of the binomial development gives results that are accurate to 8 decimal places for all values of $n = 0, \ldots, 100$.

For varying k the limit for linear extrapolation with k accurate digits is approximately $10^{k/2}$. With $k = 3$, one gets 31.62 and with $k = 4$ one gets 100 as the upper limit of linear interpolation (with the gradient $b_k - 1$), etc. This procedure permits calculating the first powers of b^k.

However, the digits of $(1.0001)^{10000} = 2.7181459268252\ldots$, have no resemblance with the numbers in Pascal's triangle anymore and this irregular number[72] is still far from 10: Solving $1.0001^n = 10$ for n gives 23027.0022. In other words, Bürgi's table has as much as 23027 entries.

Since the binomial coefficients can be read from Pascal's triangle, the values of the powers 1.0001^k can be calculated directly at a precision of 8 decimal places "quickly" by hand, without determining them iteratively. The iterative step from k to $k + 1$ involves a simple addition of the decimal representation of 1.0001^k and its shift by four decimal places in order to obtain the decimal representation of 1.0001^{k+1}.

In order to fill the gaps, Bürgi must have calculated suitable intermediate values very accurately and made use of interpolation in a clever not fully documented manner. One could wonder whether Bürgi was familiar with Pascal's triangle.[73] Indeed, it was known in the 16th century, by Nicolo Tartaglia (1523), Girolamo Cardano (1539) and Michael Stifel (1554), and as derived from figurate numbers it can be traced back to Greek antiquity.[74]

How Bürgi could have calculated Descartes's angles

From a musician's point of view Bürgi's type of base is a superparticular ratio ($\frac{k+1}{k} = 1 + \frac{1}{k}$), but for values of k ($k = 10, 100, 1000, 10000, \ldots$), which do not have any direct meaning in Western music theory.

If he had had to calculate the angles for Descartes's circular diagrams, Bürgi could have created a new table with the base 1.0125 = 81/80 with black values running from 1 to 2 and corresponding red values from 0 to 55.798…

However, the selection of the base is not really decisive, because base transformation allows switching between different bases:

$$\log_2(x) = \frac{\ln(x)}{\ln(2)} = \frac{\log_{1.0001}(x)}{\log_{1.0001}(2)}.$$

In other words, if one logarithm or exponential function is known, all of them are known. Therefore, Bürgi could have used his own diagram or his complete *Progress-Tabulen* in order to calculate Descartes's angles. To illustrate this, an estimation of the angle for the major third 5 : 4 is determined by using Bürgi's diagram (see Fig. 4 above): The black number closest to the ratio 5/4 = 1.250 nearly in the middle of the black numbers 1221… and 1284… of which

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[72] It is an approximation of Euler's number e.

[73] Blaise Pascal (1623-1662)

[74] Milan Milanovic 2004, Edwards, 2013

the related red values are 20,000 and 25,000. So the arithmetic mean 22,500 is taken as a rough estimate.[75] The black value next to 2 = 2.000 (corresponding to the octave) is 2013…, of which the red value is 70,000. So the ratio of the two estimated red values gives the ratio between the angle of the major third and the angle of the octave. This leads to $^{22500}\!/_{70000}\cdot 360° = 115.71°$ for the angle of the major third (correct 115.89°).

Octagesimal and heptadecimal number systems

The more digits a number system has, the fewer calculations are needed to obtain accurate approximations for powers. Therefore, the Babylonian number system with base 60, which has survived partially in astronomy, quickly provides good approximations for iterative tasks as extracting square roots.

In the following, some of the above calculations for Descartes circular diagrams are done again by using base 80 and base 17 number systems.

Inspired by the angle notation of the hexagesimal system with separators °,', " (grades, minutes, seconds) we apply an ad hoc notational convention to separate the digits, running from 0 to 79 in decimal notation. The first digit followed by two colons is followed by the fractional part in the octagesimal number system where octagesimal digits are separated by single colons. For example 1::25:3 stands for $1+{}^{25}\!/_{80}+{}^{3}\!/_{6400}$ and the syntonic comma is represented by 1::1 because of $1+{}^{1}\!/_{80}={}^{81}\!/_{80}$.

The ninth row in Pascal's triangle reads 1, 9, 36, 84, … Therefore,

$$\left({}^{81}\!/_{80}\right)^{9}\approx 1::9:36:84=1::9:37:4\approx 1::9:37=1+{}^{9}\!/_{80}+{}^{37}\!/_{6400}=\tfrac{6400+9\cdot 80+37}{6400}=\tfrac{7157}{6400}=1.11828...$$

This value is between the whole tones ${}^{10}\!/_{9}=1.11111...$ and ${}^{9}\!/_{8}=1.125$, and very close to two Galilean semitones $\left({}^{18}\!/_{17}\right)^{2}=\tfrac{324}{289}=1.12111...$.

Likewise, in the heptadecimal number system and with row 12 of Pascal's triangle we can control quickly Vincenzo Galilei's approximation of the octave by twelve semitones sized 18 : 17:

$$\left({}^{18}\!/_{17}\right)^{12}\approx 1::12:66:220\approx 1::12:66+13=1::12:79=1::16:11=1+{}^{16}\!/_{17}+{}^{11}\!/_{289}=\tfrac{572}{289}\approx 1.980$$

So again, as in a previous section, the octave measures roughly $9\cdot 6=54$ syntonic commas, which results in the angle 6.67° for the syntonic comma.

Alternatively, Pascal's triangle could have been written down to sufficiently many rows to do the measuring of the musical intervals by syntonic commas once and for ever, which would have revealed that a little bit less than 56 syntonic commas, leading to an angle of 6.43°, would fill the octave.[76]

Musical power tables derived from the tetraktys

We claim that in music theory logarithmic thinking is a standard since Pythagoras's time. Kepler gives a concise summary of Pythagorean music theory.[77] The (*first*) *tetraktys* is usually depicted in triangular form as shown in Fig. 13.

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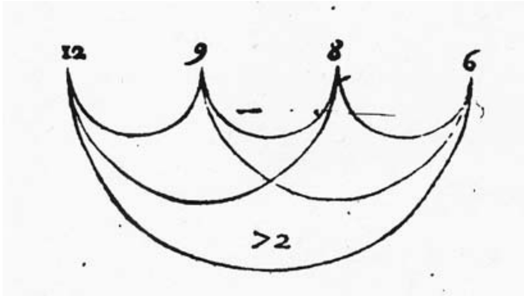
[75] Bürgi usually omits the (obvious) decimal points.

[76] The attentive reader will have noticed that these calculations are the same as the ones needed to calculate the doubling time of capital with compound interest. By this reason, Stifel mentions in one of his final examples in Rudolff/Stifel (1554) the usefulness of the higher dimensional Coss for 'Wucher' problems and Stevin writes a popular book on compound interest (Stevin 1582).

[77] Kepler, 1619 Lib. III, KGW 6, 95-101

1
1 1
1 1 1
1 1 1 1

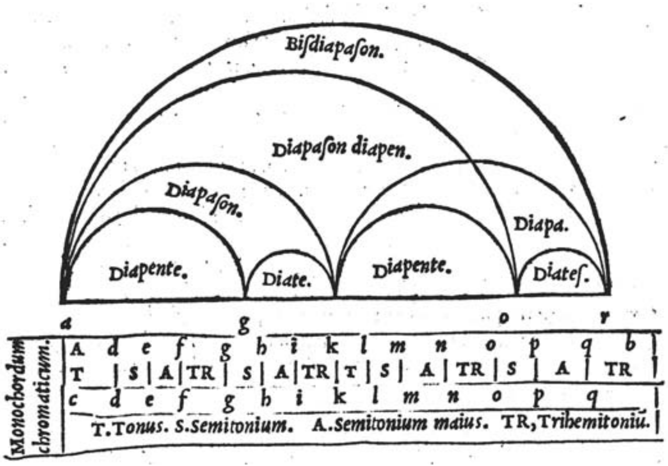
13a



13b

1
2 3
4 6 9
8 12 18 27

13c



13d

Fig. 13a The Pythagorean tetraktys as a triangular number: $1 + 2 + 3 + 4 = 10$.

Fig. 13b Division of the octave (6 : 8 : 9 : 12), the tetractys of the second kind, according to Zarlino. A detail of the famous "School of Athens" (1509–1511) by Raphael displays the diagrams of Fig. 13a and 13b.

Fig. 13d Power table $2^k \cdot 3^m$ for positive integers *k* and *m* ("filled lambda-tetractys").

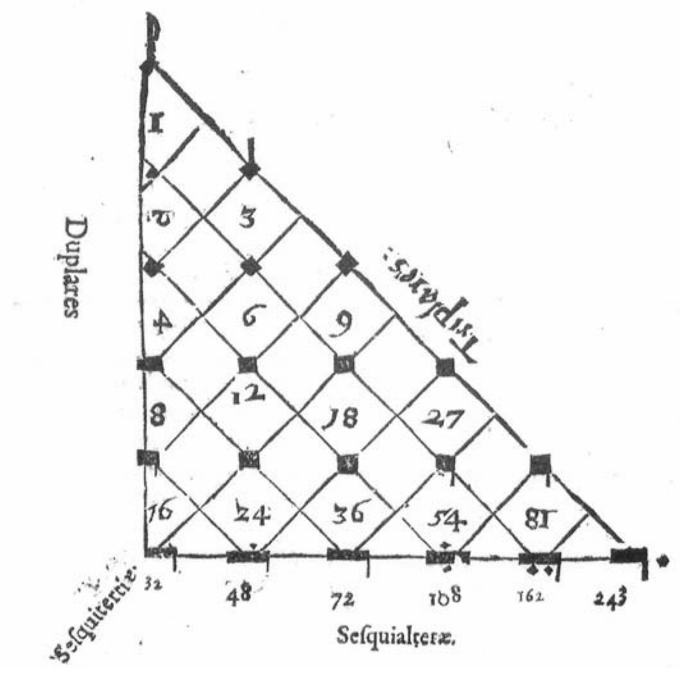
Fig. 13d Division of the double octave (2 : 3 : 4 : 6 : 12) and the Pythagorean chromatic genus (Faber Stapulensis 1551, 32v)

The pattern shown in Fig. 13a can be interpreted as the number 10 given as a figurate triangular number $10 = 1 + 2 + 3 + 4$. The number pairs from the constituents 1, 2, 3, 4 viewed as musical ratios form the primary Pythagorean consonances. The smallest of these intervals, the ratio 4 : 3, the *sesquitertia*, the fourth, is used to subdivide the octave into two fourths separated by a major tone, which results in the continuous proportion 6 : 8 : 9 : 12, which is also called the *tetraktys of the second kind* (see. Fig. 13b). The ratio 9 : 8 formed by the middle terms, a major tone, is not among the primary Pythagorean consonances, but one of the primary melodic intervals. In the next step of the derivation, the fourths (8 : 6 and 12 : 6) are divided into two major tones and a semitone 256 : 243. Eventually, the Pythagorean diatonic scale consists of five major tones 9 : 8 and two semitones 256 : 243 (see also Fig. 17a below). The Pythagorean chromatic genus and a tetraktys of the second kind spanning two octaves by Faber Stapulensis (1496/1551) is shown in Fig. 13d. Faber Stapulensis is quoted by Beeckman as early as 1616[78].

The intervals of the Pythagorean tone system can be expressed as products of integer powers of the form $2^k \cdot 3^m, \quad k, m \in \mathbb{Z}$. It is therefore straightforward to visualize the Pythagorean ratios as a part of the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ by interpreting the power indices as coordinates in a two dimensional setting. The first quadrant of this Cartesian product (the

[78] JIB, I, Fol. 38v, [[MARS 1615]-6 FÉVRIER 1616], 84. In 1630, in a letter to Marin Mersenne, Descartes judged Beeckman's knowledge of music theory to be very poor by saying that he never surpassed what he has learnt from Faber Stapulensis, cf. Buzon 1985, 99.

subset $\mathbb{N}\times\mathbb{N}\subset\mathbb{Z}\times\mathbb{Z}$) is found in musical treatises of about 1400, as triangles with a lattice structure (see Fig. 13c, 14a). The diagram is also displayed by Kepler[79], who could have picked it from Fludd's *Templum Musicæ* (1618).



14a

14b

Fig. 14a Robert Fludd's triangle (UCH Vol I, Tract II Part II Lib IV, De Temporibus Musicis, 204)
Fig. 14b Arithmetic triangles by Boethius (Mh 1)

These diagrams were used to illustrate the combinations of binary and ternary durations in the *ars nova*/*ars subtilior* period and they were rather popular from the late fourteenth into the sixteenth century in British sources.[80] Robert Fludd, in *Templum Musicæ,* has not only copied the diagram from Johannes Torkesey *Declaratio et Scuti* but also copied from its text.[81]
The same type of diagram is already used in a Boethius *Arithmetic* copy of the 10th century, where the underlying number pairs are 2/3, 3/4 and 5/6 are in a superparticular ratio (see Fig. 14b). The numbers in such a diagram are all different, if and only if the two base numbers are relatively prime numbers.[82]

In Fludd's diagram (Fig. 14a), the numbers on parallel lines connecting grid points form geometric progressions with the common ratios 2, 3, 3/2 and 4/3. The labels added to the triangle make clear that Fludd is aware of this fact. He indicates clearly the directions of the Duplares (2 : 1), Triplares (3 : 1), Sesquialteræ (3 : 2) and Sesquitertiæ (4 : 3). Torkesey's diagram, as given by Willelmus[83], contains the words sesquialtera and sesquitertia as well, but the directionality of the ratios is less clear, however this fundamental property of the diagram is recognized in the explanation by Willelmus.[84]

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[79] Kepler 1619, Lib III, KGW: 94-95
[80] Reany & Gilles, 1966, 9; Koehler 1990, Band 1, 46-51, Band 2, 1-3
[81] Reany & Gilles, 1966, 57
[82] Boethius (Mh 1) states that such diagrams can be used to construct arbitrarily long continued proportions for superparticular ratios $(n+1) : n$. Boethius, Inst. Mus. II, 8, 234–235.
[83] Reany & Gilles 1966, 28. We have not checked the manuscript against the reprint.
[84] Koehler 1990, 51

Self-similarity

The complete infinite power table is self-similar, since every cell can be taken as the root of a new diagram which produces exactly the same number ratios. By dividing the respective numbers by the value of the new root the old diagram is obtained (see Fig. 15).

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & & \boxed{3} & & & \\
 & & 4 & & \boxed{6} & & \boxed{9} & & \\
 & 8 & & \boxed{12} & & \boxed{18} & & \boxed{27} & \\
16 & & \boxed{24} & & \boxed{36} & & \boxed{54} & & \boxed{81}
\end{array}
\xrightarrow{\square/3}
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & & \boxed{1} & & & \\
 & & 4 & & \boxed{2} & & \boxed{3} & & \\
 & 8 & & \boxed{4} & & \boxed{6} & & \boxed{9} & \\
16 & & \boxed{8} & & \boxed{12} & & \boxed{18} & & \boxed{27}
\end{array}
$$

Fig. 15 Power tables as a self-similar structure

Mathematical background

Let us assume now that the grid is given in rectangular form with powers of 2 displayed horizontally and powers of 3 displayed vertically including negative powers as shown in Fig. 16. In this representation, the original triangle is contained in the first quadrant of positive integer numbers. The originally horizontally arranged numbers are now found in the diagonals from bottom-right to top-left, e.g., 16 – 24 – 36 – 54 – 81. Note that opposite quadrants contain reciprocal numbers: The product of a pair of numbers mirror symmetric about the centre 1 always equals 1.

This grid can be extended in all directions to infinity in an obvious way. If a straight line hits two points of the grid, it automatically contains infinitely many equidistant grid points, thereby defining geometric progressions on the underlying numbers. For an arbitrary rational slope y/x there are such lines and each rational slope represents a different rational multiplication factor between nearest grid points, i.e., a different musical interval.

Because 2 and 3 are relatively prime numbers, a power of 2 never equals a power of 3. The line through $2^{19} = 524,288$ and $3^{12} = 531,441$ represents a ratio close to 1, the Pythagorean comma ($531441/524288 \approx 1.0136$). Obviously, there exists a line ℓ_1 through the origin with an *irrational* slope close to $-12/19 \approx -0.6315$ that represents the common ratio 1 exactly.[85] The correct value for this slope is $-\log_3 2 \approx -0.6309$.[86] The corresponding line through the centre 1 separates the numbers greater than 1 from those less than 1: the grid points above the line ℓ_1 belong to numbers greater than 1 and the ones below ℓ_1 to numbers less than 1.

Since a geometric sequence with common factor 1 is a constant sequence, the entire line ℓ_1 through 1 can be said to represent the number 1. Taken together, the lines parallel to ℓ_1 partition the plane, where each of them represents a different positive real number. Therefore, there is a monotone one-to-one relationship between this bundle of parallel lines and the positive real numbers.

---

[85] Stifel spells out the 16 digits numbers of a better approximation of the unison, a Pythagorean semitone 256 : 243 minus 3 Pythagorean commas, the 'recisum tertium' $2^{65} : 3^{41}$ [= 1.0115], saying that $2^{84} : 3^{53}$, a semitone minus 4 commas would fall below 1 [0.9979]: Stifel, 1544, Lib. I, s. p.

[86] A less accurate value visible in the table is -2/3 = -0.6667, which corresponds to the line connecting 1 with 9/8, the line of the sesquioctavæ.

$y \backslash x$	−4	−3	−2	−1	0	1	2	3	4
4	$\frac{81}{16}$	$\frac{81}{8}$	$\frac{81}{4}$	$\frac{81}{2}$	$\underline{81}$	162	324	648	1296
3	$\frac{27}{16}$	$\frac{27}{8}$	$\frac{27}{4}$	$\frac{27}{2}$	27	$\underline{54}$	108	216	432
2	$\frac{9}{16}$	$\boxed{\frac{9}{8}}$	$\frac{9}{4}$	$\frac{9}{2}$	9	18	$\underline{36}$	72	144
1	$\frac{3}{16}$	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	3	6	12	$\underline{24}$	48
0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\boxed{1}$	2	4	8	$\underline{16}$
−1	$\frac{1}{48}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{16}{3}$
−2	$\frac{1}{144}$	$\frac{1}{72}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\boxed{\frac{8}{9}}$	$\frac{16}{9}$
−3	$\frac{1}{432}$	$\frac{1}{216}$	$\frac{1}{108}$	$\frac{1}{54}$	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{8}{27}$	$\frac{16}{27}$
−4	$\frac{1}{1296}$	$\frac{1}{648}$	$\frac{1}{324}$	$\frac{1}{162}$	$\frac{1}{81}$	$\frac{2}{81}$	$\frac{4}{81}$	$\frac{8}{81}$	$\frac{16}{81}$

Fig. 16 Two dimensional Pythagorean power grid.

Moreover, it can be shown that to any parallel to ℓ_1 there are arbitrarily close grid points. Put it differently, the grid points representing numbers of the format $2^x \cdot 3^y$ form a *dense* subset of the positive real numbers.[87] In the following, we call this set of numbers the *Pythagorean number system*.

Whereas the full table with all the four quadrants provides an immediate and unique interpretation of a point as a positive rational number, the ratios of numbers studied in traditional music theory are obtained by picking ordered pairs of numbers from the triangular diagram and using the first as the numerator and the second as the denominator. By allowing negative powers, the second number picked can always be 1. This is equivalent to working with position vectors originating at 1. If a vector in the grid – corresponding to a musical interval – is to be expressed in its lowest terms, the representative beginning at 1 can be taken. Negative power indices in one dimension were introduced and systematically studied by Michael Stifel.[88] Stifel also gives an example with fractional powers, by “Volo multiplicare 81/16 per 3/4” done (in modern notation) as $\left({}^{81}\!/_{16}\right)^{3/4} = \sqrt[4]{\left({}^{81}\!/_{16}\right)^{3}} = \sqrt[4]{{}^{531441}\!/_{4096}} = {}^{27}\!/_{8}$.[89]

The fact that the Pythagorean number system is dense in the positive real numbers means that it could be used to approximate any real number with arbitrary precision. Or starting with a given frequency and adding Pythagorean fifths 3 : 2 reduced modulo octave repeatedly will fill Descartes’s octave circle more and more, and eventually gives the illusion of a continuous circle line. Chaining only six fifths defines the Pythagorean diatonic scale (see Fig. 17).

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[87] The proof given in Mazzola 1990, 304, 311 can be adapted. Seemingly, the music theoretical implications of these mathematical facts have not been stated before Mazzola.

[88] Stifel 1544, Lib III, 249v

[89] Stifel 1544, Lib I, 53r. Notice the word “multiplicare” which refers to powers, not to simple multiplication.

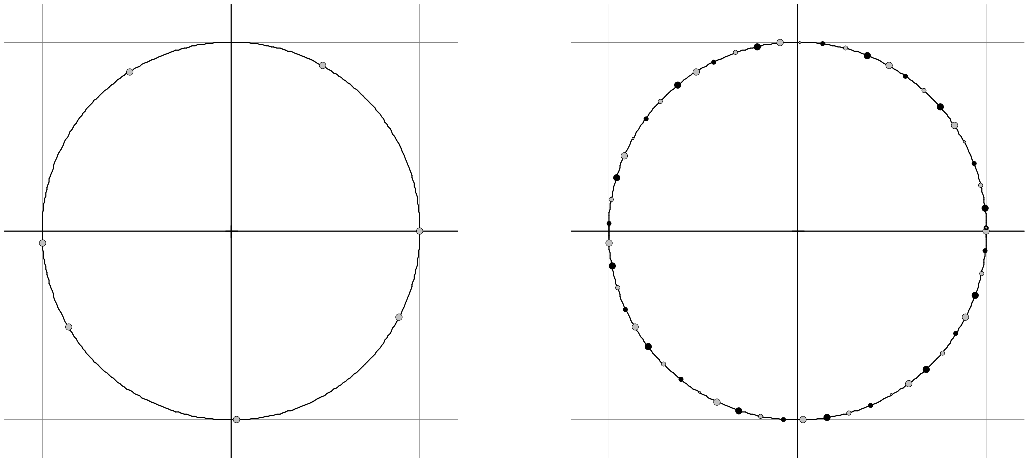


Fig. 17a Diatonic Pythagorean scale generated by 6 Pythagorean fifths 3 : 2.
Fig. 17b With 52 Pythagorean fifths an almost regular polygon is obtained.

The interpretation of the above table as a subset of the Cartesian plane is actually performed by a logarithmic scaling, using base-2 logarithms in the x-direction and base-3 logarithms in the y-direction. The mapping

$$\mathbb{R}\times\mathbb{R}\to\mathbb{R}^{+}: \quad (x,y)\mapsto 2^{x}\cdot 3^{y}$$

is many-to-one. Its restriction to integer pairs

$$\mathbb{Z}\times\mathbb{Z}\to P\subset\mathbb{Q}^{+}: \quad (x,y)\mapsto 2^{x}\cdot 3^{y}$$

is one-to-one, and the range of this restriction is a subset of the rational numbers that is even dense in the positive real numbers.

Multiplying numbers corresponds to vector addition in the power grid. Dividing numbers corresponds to vector subtraction. Integer powers of numbers correspond with integer multiples of position vectors. In other words, the power grid defining the Pythagorean tone system permits a reduction of the arithmetic operation complexity by one degree. It is a kind of dense two-dimensional table of logarithms. The work that would have to be done to turn it into a two-dimensional slide rule for computations in the decimal numbers system, would be to provide an appropriate decimal scale, in the direction perpendicular to a moveable ruler in the direction of ℓ_1.

Syntonic tone system and higher dimensional grids

Similar grids can be created for any pair of relatively prime numbers. Musically meaningful in Western music theory are the primes 3 and 5 to represent the fifth and the major third of the syntonic tuning system. Such grids were studied by Rameau (1726) and Euler (1739) in order to describe the ratios between the pitch classes of just tuning systems in geometrical terms. To create the scale within an octave the powers $3^{x}\cdot 5^{y}$ are reduced into values between 1 and 2 by adding or subtracting one or several octaves, i.e., by multiplying these numbers by suitable powers of 2. In other words, numbers of the form $2^{x}\cdot 3^{y}\cdot 5^{z}$ are studied, formally by embedding

$$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{Q}: \quad (x,y,z)\mapsto 2^{x}\cdot 3^{y}\cdot 5^{z}$$

This structure can be viewed as a three-dimensional grid of numbers. The diminished fifth 45/32, for example, is expressed as $2^{-5}\cdot 3^{2}\cdot 5^{1}$ and corresponds to the vector $[-5;2;1]^{T}$ in

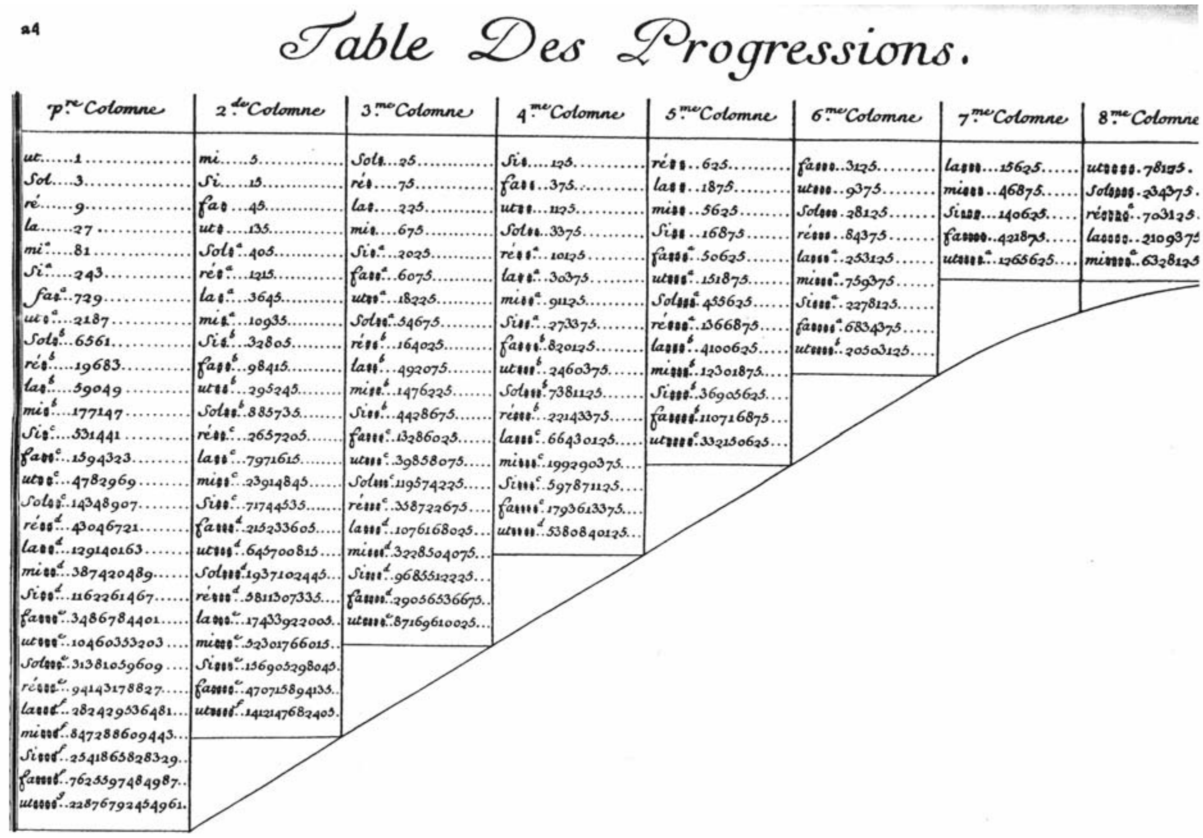
$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. In this three-dimensional setting, points of equal pitch are represented on parallel planes.[90]

Rameau's grid (see Fig. 18) is obtained by neglecting the octave information x:

$$\begin{array}{llcl} \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{Q}: & (x,y,z) & \mapsto & 2^x\cdot 3^y\cdot 5^z \\ & \downarrow & & \downarrow \\ \mathbb{Z}\times\mathbb{Z}\to\mathbb{Q}: & (0,y,z) & \mapsto & 3^y\cdot 5^z \end{array}$$

In this pitch grid the consonant major and minor triads form right-angled triangles with legs of unit length. Each interval class (interval modulo octave) corresponds to a vector in the grid. For instance, the augmented forth is the vector $[2;1]^T$, and the syntonic comma is given by the vector $[4;-1]^T$.

Likewise, a set of "*n* pairwise relatively prime numbers" translates into "*n* linearly independent space directions". Therefore, some authors as Christiaan Huygens in the 17th and Martin Vogel in the 20th century[91] have opted for an additional musical dimension for powers of 7. Kepler, in possession of Gauss's result, however, would have taken 17 instead of 7 …



24

Table Des Progressions.

p.re Colomne	2.de Colomne	3.me Colomne	4.me Colomne	5.me Colomne	6.me Colomne	7.me Colomne	8.me Colomne
ut … 1	mi … 5	Sol♯ … 25	Si♯ … 125	ré♯♯ … 625	fa♯♯♯ … 3125	la♯♯♯ … 15625	ut♯♯♯♯ … 78125
Sol … 3	Si … 15	ré♯ … 75	fa♯♯ … 375	la♯♯ … 1875	ut♯♯♯ … 9375	mi♯♯♯ … 46875	Sol♯♯♯♯ … 234375
ré … 9	fa♯ … 45	la♯ … 225	ut♯♯ … 1125	mi♯♯ … 5625	Sol♯♯♯ … 28125	Si♯♯♯ … 140625	ré♯♯♯♯^{a} … 703125
la … 27	ut♯ … 135	mi♯ … 675	Sol♯♯ … 3375	Si♯♯ … 16875	ré♯♯♯ … 84375	fa♯♯♯♯ … 421875	la♯♯♯♯ … 2109375
mi^{a} … 81	Sol♯^{a} … 405	Si♯^{a} … 2025	ré♯♯^{a} … 10125	fa♯♯♯^{a} … 50625	la♯♯♯^{a} … 253125	ut♯♯♯♯^{a} … 1265625	mi♯♯♯♯^{a} … 6328125
Si^{a} … 243	ré♯^{a} … 1215	fa♯♯^{a} … 6075	la♯♯^{a} … 30375	ut♯♯♯^{a} … 151875	mi♯♯♯^{a} … 759375		
fa♯^{a} … 729	la♯^{a} … 3645	ut♯♯^{a} … 18225	mi♯♯^{a} … 91125	Sol♯♯♯^{a} … 455625	Si♯♯♯^{a} … 2278125		
ut♯^{a} … 2187	mi♯^{a} … 10935	Sol♯♯^{a} … 54675	Si♯♯^{a} … 273375	ré♯♯♯^{a} … 1366875	fa♯♯♯♯^{a} … 6834375		
Sol♯^{b} … 6561	Si♯^{b} … 32805	ré♯♯^{b} … 164025	fa♯♯♯^{b} … 820125	la♯♯♯^{b} … 4100625	ut♯♯♯♯^{b} … 20503125		
ré♯^{b} … 19683	fa♯♯^{b} … 98415	la♯♯^{b} … 492075	ut♯♯♯^{b} … 2460375	mi♯♯♯^{b} … 12301875			
la♯^{b} … 59049	ut♯♯^{b} … 295245	mi♯♯^{b} … 1476225	Sol♯♯♯^{b} … 7381125	Si♯♯♯^{b} … 36905625			
mi♯^{b} … 177147	Sol♯♯^{b} … 885735	Si♯♯^{b} … 4428675	ré♯♯♯^{b} … 22143375	fa♯♯♯♯^{b} … 110716875			
Si♯^{c} … 531441	ré♯♯^{c} … 2657205	fa♯♯♯^{c} … 13286025	la♯♯♯^{c} … 66430125	ut♯♯♯♯^{c} … 332150625			
fa♯♯^{c} … 1594323	la♯♯^{c} … 7971615	ut♯♯♯^{c} … 39858075	mi♯♯♯^{c} … 199290375				
ut♯♯^{c} … 4782969	mi♯♯^{c} … 23914845	Sol♯♯♯^{c} … 119574225	Si♯♯♯^{c} … 597871125				
Sol♯♯^{c} … 14348907	Si♯♯^{c} … 71744535	ré♯♯♯^{c} … 358722675	fa♯♯♯♯^{c} … 1793613375				
ré♯♯^{d} … 43046721	fa♯♯♯^{d} … 215233605	la♯♯♯^{d} … 1076168025	ut♯♯♯♯^{d} … 5380840125				
la♯♯^{d} … 129140163	ut♯♯♯^{d} … 645700815	mi♯♯♯^{d} … 3228504075					
mi♯♯^{d} … 387420489	Sol♯♯♯^{d} … 1937102445	Si♯♯♯^{d} … 9685512225					
Si♯♯^{d} … 1162261467	ré♯♯♯^{d} … 5811307335	fa♯♯♯♯^{d} … 29056536675					
fa♯♯♯^{e} … 3486784401	la♯♯♯^{e} … 17433922005	ut♯♯♯♯^{e} … 87169610025					
ut♯♯♯^{e} … 10460353203	mi♯♯♯^{e} … 52301766015						
Sol♯♯♯^{e} … 31381059609	Si♯♯♯^{e} … 156905298045						
ré♯♯♯^{e} … 94143178827	fa♯♯♯♯^{e} … 470715894135						
la♯♯♯^{f} … 282429536481	ut♯♯♯♯^{f} … 1412147682405						
mi♯♯♯^{f} … 847288609443							
Si♯♯♯^{f} … 2541865828329							
fa♯♯♯♯^{f} … 7625597484987							
ut♯♯♯♯^{g} … 22876792454961							

Fig. 18 Syntonic grid of pitch classes (Rameau 1726) combining powers of 3 (fifths) vertically and powers of 5 (major thirds) horizontally.

[90] For details see Mazzola 1990, 63-84. Descartes (1618) and Kepler (1619) were aware of the fundamental importance of the first three prime numbers for constructing tone systems.
[91] Vogel 1975

Conclusions

Two types of geometric logarithmic representation of frequency ratios used in the early 17th century are in the centre of this essay. The common property of the related diagrams is the use of spatial distance for measuring musical intervals. Whereas a two dimensional "Cartesian representation" used by Robert Fludd within a discrete straight line coordinate system can be traced back to Boethius, Descartes's representation of musical interval classes and pitch classes on a circle seems to have no early forerunners, but it bears apparent similarities to one of Robert Fludd's musical diagrams and to a mathematical diagram by Jost Bürgi. The latter was printed two years after Descartes composed his *Compendium musicœ*, whereas the former had already occurred in print eight months before.

It is argued here that thinking in musical intervals and scales is genuinely logarithmic and pre-dates the invention of mathematical logarithms as calculation techniques by the end of the 16th century in Scotland and Switzerland by John Napier and Jost Bürgi. The equivalency of adding musical intervals and multiplying their frequency ratios is a music theoretical truism, which manifests in the traditional Latin terms ditonus, tritonus, bisdiapason, etc. In traditional music theory and arithmetic, the standard operation on pairs of ratios is multiplication and not addition. This is state of the art already in Boethius's reception of Greek music theory and arithmetic as handed down through the Middle Ages.

The most remarkable element of Descartes diagrams is not the spatial representation of musical interval size per se, but its use within a circular topology that captures the octave similarity as a perceptual phenomenon. Descartes gives a representation which does not only visualize pitch classes as locations on the circle line and intervals as central angles of circular sectors. The diagrams also illustrate clearly that the set of consonances of Zarlino's senario is closed under octave addition as well as under octave complements. Moreover, the transposition of scales by multiples of fifths is understood as rotation about the centre of the circle. The model is also suited to show the potential infinity of the syntonic diatonic tone systems mentioned in Descartes's *Compendium Musicœ*.

One of the forerunners of modern power calculus and logarithms, Michael Stifel (1544), gives also a detailed introduction into musical arithmetic, in which the size of the Pythagorean comma and semitone are compared in a way similar to our tentative "reconstruction" of Descartes considerations concerning the Syntonic comma, which are needed in order to determine the angles in the circular diagrams. An estimation of the size of the Pythagorean comma $3^{12} : 2^{19}$ [= 1.01364] as being between the superparticular ratios 75 : 74 [= 1.01351] and 74 : 73 [= 1.01370] was already stated by Boethius.[92]

Zarlino is familiar with Stifel's musical arithmetic.[93] And Zarlino must have been part of Descartes's education at the Jesuits's college at Flèche.[94]
Stifel's arithmetic is used by Robert Fludd to explain the 'Coss', the comparatively young theory of polynomials and higher order equations in one variable, and it serves Faulhaber's arithmetic with examples.
Descartes visited Faulhaber in the year after the completion of the *Compendium*, however, Faulhaber's propagation of (Brigg's) logarithms in Germany takes place much after his

[92] Boethius, Inst Mus. III. 12, 286
[93] According to Cristiano Forster, Zarlino criticized Stifel sharply in 1571 and plagiarizes Stifel in 1573, Forster 2010, 378.
[94] Descartes's Compendium Musicae refers to Zarlino in the context of the modal scales. For Descartes's mathematical formation, cf. Sasaki 2003, 13-14. 'Descartes and Jesuit Mathematical Education'. For the state of mathematics in Germany at the time of Descartes's stay in southern Germany, see Schneider 2008.

meeting with Descartes. We do not believe that Faulhaber was particularly important for Descartes mathematical formation and we could not find out whether Descartes knew Fludd's relevant texts and pictures when he composed the *Compendium*. We are convinced that Beeckman's knowledge of music theory and mathematics in the years before he met Descartes would have been more than sufficient to construct Descartes's diagrams.

Whereas Fludd's use of the circle is an inherent archetypal feature of his unified philosophy, Kepler uses the circle to justify the set of Zarlino's consonances in an entirely geometric way – without any direct reference to perception or contemporary physics. The set of these consonances corresponds with the then constructible regular polygons, where constructible means "by straight edge and ordinary compasses". A comparable symbolic approach by Kepler in astronomy, exploiting the platonic solids in order to quantitatively describe the orbits of the planets around the sun, proved successful in the sense of fitting well with the observed phenomena.

The obvious generalization into a three dimensional setting does not seem to be practicable on a two dimensional paper surface. By this reason, in the pitch grids of the 18th century by Rameau and Euler built from the same principle as Boethius's triangles, the octaves are reduced to points. In other words, the grid points now symbolize classes of tones with unspecified octave. An appropriate geometrical visualisation of the syntonic tone system taking account of octave cycles in both, the fifths and the thirds dimensions, on a torus would obscure the relationships between the pitches in just tuning more than to make them evident.[95]

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Abbreviations:

AT t. x.	Œuvre complète de Descartes, Charles Adam et Paul Tannery, Paris 1897-1913
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JIB	De Waard, C. (ed.) Journal tenu par Isaac Beeckman de 1604 à 1634, 4 Vol., 1939-1953
KGW VI	Johannes Kepler. Gesammelte Werke. Band VI. Harmonice Mundi. Max Caspar (Hg.). Beck'sche Verlagsbuchandlung, München 1940
PWS II	The Principal Works of Simon Stevin, Vol II, Mathematics, D.J. Struik, C.V. Swets & Zeitlinger, Amsterdam 1958

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[95] Roger Shepard proposes a four dimensional pitch model (double helix on a helical cylinder), cf. Shepard 2001, 163.

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