What does a nonabelian group sound like?

Harmonic Analysis on Finite Groups and DSP Applications

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Abstract

Underlying many digital signal processing (DSP) algorithms, in particular those used for digital audio filters, is the convolution operation, which is a weighted sum of translations $f(x-y)$. Most classical results of DSP are easily and elegantly derived if we define our functions on $\mathbb{Z}/n\mathbb{Z}$, the abelian group of integers modulo n. If we replace this underlying "index set" with a nonabelian group, then translation may be written $f(y^{-1}x)$, and the resulting audio filters arising from convolution naturally produce different effects that those obtained with ordinary (abelian group) convolution.

The goal of this project is to explore the idea of using the underlying finite group (i.e., the index set) as an adjustable parameter of a digital audio filter. By listening to samples produced using various nonabelian groups, we try to get a sense of the "acoustical characters" of finite groups.

1 Introduction

The *translation-invariance* of most classical signal processing transforms and filtering operations is largely responsible for their widespread use, and is crucial for efficient algorithmic implementation and interpretation of results [1].

DSP on *finite abelian groups* such as $\mathbb{Z}/n\mathbb{Z}$ is well understood and has great practical utility. Translations are defined using addition modulo n, and basic operations, including convolutions and Fourier expansions, are developed relative to these translations [2]. Recently, however, interest in the practical utility of *finite nonabelian groups* has grown significantly. Although the theoretical foundations of nonabelian groups is well established, application of the theory to DSP has yet to become common-place. A notable exception is [1], which develops theory and algorithms for indexing data with nonabelian groups, defining translations with a non-commutative group multiply operation, and performing typical DSP operations relative to these translations.

This paper describes the use of nonabelian groups for indexing one- and two-dimensional signals, and discusses some computational advantages and insights that can be gained from such an approach. A simple but instructive class of nonabelian groups is examined. When elements of such groups are used to index the data, and standard DSP operations are defined with respect to special group binary operators, more general and interesting signal transformations are possible.

1.1 Preview: Two Distinctions of Consequence

Abelian group DSP can be completely described in terms of a special class of signals called the *characters* of the group. (For $\mathbb{Z}/n\mathbb{Z}$, the characters are simply the exponentials.) Each character of an abelian group represents a one-dimensional translation-invariant subspace, and the set of all characters spans the space of signals indexed by the group; any such signal can be uniquely expanded as a linear combination over the characters.

In contrast, the characters of a nonabelian group G do not determine a basis for the space of signals indexed by G. However, a basis can be constructed by extending the characters of an abelian subgroup A of G, and then taking certain translations of these extensions. Some of the characters of A cannot be extended to characters of G, but only to proper subgroups of G. This presents some difficulties involving the underlying translation-invariant subspaces, some of which are now multi-dimensional. However, it also presents opportunities for alternative views of local signal domain information on these translation-invariant subspaces.

The other abelian/nonabelian distinction of primary importance concerns translations defined on the group. In the abelian group case, translations represent simple linear shifts in space or time. When nonabelian groups index the data, however, translations are no longer so narrowly defined.

List of Acronyms

DSP digital signal processing

References

[1] Myoung An and Richard Tolimieri. *Group Filters and Image Processing*. Psypher Press, Boston, 2003. URL: `http://prometheus-us.com/asi/algebra2003/papers/tolimieri.pdf`.

[2] Richard Tolimieri and Myoung An. *Time-Frequency Representations*. Birkhäuser, Boston, 1998.