What does a nonabelian group sound like?

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Background and prior work. Underlying many digital signal processing (DSP) algorithms, in particular those used for digital audio filters, is the convolution operation. This operation acts on a signal, $f(x)$, and can be viewed as a weighted sum of translations $f(x-y)$. Most classical results of dsp are easily and elegantly derived if we define our functions on $\mathbb{Z}/n$, the abelian (or commutative) group of integers modulo n. (See, for example, [6].) The term *abelian* here refers to the fact the basic group operation is addition (modulo n) which is a commutative operation (i.e., $x + y = y + x$).

If we replace the underlying "index set" with a *nonabelian* (or noncommutative) group—where the group operation is now multiplication, xy, and where in general it is not the case that $xy = yx$—then instead of translation of functions $f(x - y)$, we have a *generalized translation* $f(xy^{-1})$. If we carry out convolution using this generalized translation, the resulting audio filters will naturally produce different effects than those obtained with ordinary (abelian group) convolution.

One of the faculty mentors for the proposed project, Dr. DeMeo (Mathematics), initiated research in this direction in 2004 and presented some preliminary findings at the International Symposium on Musical Acoustics. Similar ideas have been successfully applied to two dimensional image data as well as to other areas of engineering. (See, for example, [1] and [7].) However, to date the application of nonabelian groups to audio signal processing has yet to be fully explored. Thus there are a number of fundamental open questions in this area that we hope to answer.

Research question. If the underlying index set of a digital audio filtering algorithm is modified to use various nonabelian groups (instead of the commonly used abelian group), how does this change the behavior of the filter and the resulting audio output?

Project goals and objectives. We propose to explore the idea of using the underlying finite group (i.e., the index set) as an adjustable parameter of a digital audio filter. By listening to samples produced using various nonabelian groups, we hope to get a sense of the "acoustical characters" of finite groups. We will attempt to associate these acoustical features with various mathematical properties of the groups, and develop a classification scheme that might be useful to practitioners in audio signal processing and computer music composition.

There are some basic classes of nonabelian groups that have been studied by mathematicians for more than 100 years. For example, the symmetric and alternating groups, semidirect product groups, solvable groups, nilpotent groups, etc. These classes are now well understood and catalogued in [2], and billions of examples are readily available from the SmallGroups library of the GAP Computer Algebra System [5]. We seek to discover which mathematical features of a group can be used to describe how a given dsp algorithm based on that group will behave.

- *Goal:* Develop mathematical the theory necessary to provide sonic characterizations of nonabelian groups.
- *Objective:* Find a short list of nonabelian groups that are useful for nonabelian group audio filters and effects processors, prove their effectiveness both mathematically and experimentally, and document the results.
- *Objective:* Implement a software program that takes audio as input and allows the user to apply filters corresponding to specific nonabelian groups to acheive different effects.

Project significance. The proposed research introduces the novel concept of giving sonic characteristics to mathematical groups. There is also a possibility of applications in digital audio engineering, especially if certain nonabelian groups are shown to produce interesting audio effects.

Methodology. We will replace the index set $\mathbb{Z}/n$ with various finite nonabelian groups. In the beginning, the simplest examples of nonabelian groups (such as semidirect product groups), will be constructed "by hand"using GAP's `SemidirectProduct` function. Groups with a more complicated structure will be selected from GAP's vast SmallGroups library using various selection criteria. For each of the groups tested, we implement the convolution function using the generalized (nonabelian) translation $f(xy^{-1})$ in place of oridinary translation $f(x-y)$ used in classical convolution. (Here, the group multiplication is replacing the traditional addition modulo n as the group operation. We are simply generalizing the underlying translation operation, and the result is a generalized convolution.)

Both GAP and Matlab will be used for much of the initial testing and prototyping. Matlab provides easy methods for constructing WAV files "from scratch" with its wavwrite() function. Additionally, Myoung An (a colleague of Dr. DeMeo) has provided us with the Matlab code she and Richard Tolimieri developed for their work in image processing, in which they applied nonabelian group filters to the processing of 2D digital images. This code will be a valuable resource as we seek to apply similar ideas to audio signal processing.

As the project progresses, we will likely use the JavaSound library and implement our generalized DSP algorithms in Java. JavaSound provides methods for reading and altering wav files frequencies and sound intensity levels, which will prove useful when we apply our generalized DSP algorithms to more complex sounds.

Project timeline.
October 2013–December 2013: Become more familiar with current DSP algorithms and gain further knowledge of group theory, and its role in classical DSP implementations.

December 2013–March 2014: Write code to implement DSP algorithms with a selection of nonabelian groups and group operations.

January 2014–May 2014: Identify specific characteristics of groups that make them more (or less) useful as an index set on which to define DSP operations like convolution. Gather and analyze results, and write up reports.

June 2014–October 2014: Submit manuscript to an academic journal. Prepare for and attend conferences.

Anticipated results, final products, and dissemination. By the end of the Spring 2014 semester, I expect to have written Matlab programs to test the results of the modified DSP implementations described above. I also expect to have developed a Java software program which allows easy application of nonabelian group filters through a graphical user interface. I hope that the results will prove interesting and have practical applications for computer music composition.

The abstract for this project has already been accepted for presentation at the Joint Mathematics Meetings in Baltimore in 2014. In addition, I will submit our work to the International Computer Music Conference (ICMC) and International Symposium on Musical Accoustics (ISMA), as well as the 14th International Conference on New Interfaces for Musical Expression (NIME). Previous, similar works by my faculty mentor Dr. DeMeo have been accepted at both ICMC and ISMA (see [4] and [3]), so we have high expectations for this project. Furthermore, I will send in the formal paper to several Mathematical and Music scholarly journals. Of course, I will also present the results of our work at Discovery Day 2014.

TODO: add something about Bain's interest/collaboration!

Personal statement. I have never had considerable ability with music, but I have always been fascinated by its intersection with my favorite subject, mathematics. This project piqued my interest because it allows me the rare opportunity to contribute to both fields. I believe that I have developed the necessary skills to succeed in this project through my past mathematical research projects and my current internship developing Java applications at a local software company. A Magellan Grant would allow me the fiscal freedom to dedicate time to engaging in research and possibly traveling to share my findings with international audiences.

References

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