What does a nonabelian group sound like?

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Background and prior work. Underlying many digital signal processing (DSP) algorithms, in particular those used for digital audio filters, is the convolution operation. This operation acts on a signal, $f(x)$, and can be viewed as a weighted sum of translations, $f(x-y)$. Most classical results of DSP are easily and elegantly derived if we define our functions on $\mathbb{Z}/n$, the abelian (or commutative) group of integers modulo n (see [6]). The term *abelian* here refers to the fact that the basic group operation is addition (modulo n) which is a commutative operation (i.e., $x+y=y+x$).

If we replace this "index set" (the set on which functions are defined) with a *nonabelian* group—where the group operation is now multiplication, xy—then instead of the usual translation, $f(x-y)$, we have a *generalized translation*, $f(xy^{-1})$. If we carry out convolution using this generalized translation, the resulting audio filters will naturally produce different effects than those obtained with ordinary (abelian group) convolution.

One of my faculty mentors, Dr. DeMeo, initiated research based on these ideas in 2004 and presented some preliminary findings at the International Symposium on Musical Acoustics (see [4], which received a "best paper" award). Similar ideas have been successfully applied to two dimensional image data as well as to other areas of engineering (see [1] and [7]). However, to date the application of nonabelian groups to audio signal processing seems relatively unexplored, and there are a number of fundamental open questions in this area that we hope to answer.

Research question. If the underlying index set of a digital audio filtering algorithm is modified to use various nonabelian groups (instead of the commonly used abelian group), how does this change the behavior of the filter and the resulting audio output?

Project goals and objectives. We propose to explore the idea of using the underlying finite group (i.e., the index set) as an adjustable parameter of a digital audio filter. By listening to samples produced using various nonabelian groups, we hope to get a sense of the "acoustical characters" of finite groups. We will attempt to associate these acoustical features with various mathematical properties of the groups, and develop a classification scheme that might be useful to practitioners in audio signal processing and computer music composition.

- *Goals:* Develop the mathematical theory necessary to provide sonic characterizations of nonabelian groups. Discover which mathematical features of a group can be used to describe how a given DSP algorithm based on that group will behave. Produce computer software that allows users to process and manipulate musical signals using nonabelian group filters.

- *Objective 1:* Develop an understanding of the basic math underlying signal processing algorithms in general and convolution in particular and show mathematically what effects the use of a nonabelian group index set will have on the convolution operation.

- *Objective 2:* Find a short list of nonabelian groups that are useful for nonabelian group audio filters and effects processors, prove their effectiveness both mathematically and experimentally, and document these discoveries.

- *Objective 3:* Implement a software program that takes an audio signal as input and allows the user to apply filters corresponding to specific nonabelian groups to achieve different effects.

Project significance. The proposed research introduces the novel concept of describing acoustical properties of mathematical groups. This may be interesting to some mathematicians, and we will present preliminary results of this work at the Mathematical Association of America's special session, "At the Intersection of Mathematics and the Arts." (Our abstract has already been accepted.) Of greater significance, however, will be the impact this research has on applications in digital audio engineering, especially if certain nonabelian groups are shown to produce interesting audio effects.

Methodology. We will conduct controlled experiments with very simple sound signals at first (sine waves and linear chirps), and filter these signals using standard convolution. Then we will filter the original signals using a generalized (nonabelian) convolution, substituting the underlying index set with various groups from the wide variety of nonabelian groups available in the SmallGroups library of GAP [5].

When we replace the index set $\mathbb{Z}/n$ with various finite nonabelian groups, in the beginning, the simplest examples of nonabelian groups (such as semidirect product groups), will be constructed "by hand"using GAP's `SemidirectProduct` function. Groups with a more complicated structure will be selected from GAP's vast SmallGroups library using various selection criteria. For each of the groups tested, we will implement the convolution function using the generalized (nonabelian) translation $f(xy^{-1})$ in place of ordinary translation $f(x - y)$ used in classical convolution.

After completing these controlled experiments, we will analyze the results to compare the effects of the choice of group on the resulting convolution filter. Finally, we will attempt to make a connection between the mathematical properties of the group and the acoustical properties of the resulting convolution.

Both GAP and Matlab will be used for much of the initial prototyping and testing. Matlab provides easy methods for constructing wav files "from scratch" with its `wavwrite()` function. Additionally, Myoung An (a colleague of Dr. DeMeo) has provided us with the Matlab code that she and Richard Tolimieri developed for their work in image processing, where they applied nonabelian group filters to the processing of 2D digital images. This code will be a valuable resource as we seek to apply similar ideas to audio signal processing.

As the project progresses, we will likely use the JavaSound library and implement our generalized DSP algorithms in Java. JavaSound provides methods for reading and altering wav files frequencies and sound intensity levels, which will prove useful when we apply our generalized DSP algorithms to more complex sounds.

Project timeline.
October 2013–December 2013: Become more familiar with music analysis/synthesis and DSP algorithms, and gain further knowledge of group theory and its role in classical DSP implementations.

January 2014–April 2014: Write code to implement algorithms for general nonabelian group DSP. Identify specific characteristics of groups that make them more (or less) useful as an index set on which to define DSP operations like convolution.

May 2014–October 2014: Gather and analyze results, and write up reports. Submit manuscript to an academic journal. Prepare for and attend conferences.

Anticipated results, final products, and dissemination. By the end of the Spring 2014 semester, I expect to have written Matlab programs to test the results of the modified DSP implementations described above. I also expect to have developed a Java software program which allows easy application of nonabelian group filters through a graphical user interface. I hope that the results will prove interesting and have practical applications for computer music composition.

The abstract for this project has already been accepted for presentation at the Joint Mathematics Meetings in Baltimore in 2014. In addition, I will submit the work to the International Computer Music Conference (ICMC), the International Symposium on Musical Acoustics (ISMA), and the 14th International Conference on New Interfaces for Musical Expression (NIME). Previous work on topics related to this proposal by my faculty mentors have been accepted at both ICMC and ISMA, so we have high expectations for this project. I will write up a formal article describing the research and results and submit the manuscript to at least one scholarly journal in mathematics or music. Finally, if my project proposal is accepted and I become a Magellan Scholar, I will be honored to present the work at Discovery Day 2014.

Personal statement. I have never had considerable ability with music, but I have always been fascinated by its intersection with my favorite subject, mathematics. This project piqued my interest because it allows me the rare opportunity to contribute to both fields. I believe that I have developed the necessary skills to succeed in this project through my past mathematical research projects and my current internship developing Java applications at a local software company. A Magellan Grant would allow me the fiscal freedom to dedicate time to engaging in research and possibly traveling to share my findings with international audiences.

References

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