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Original Article

Revisited Typical Integrals and their Evaluations

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Abstract - Some typical integrals are innovated and in course of evaluating these integrals two unusual integrals $\int \frac{dx}{1+x^4}$ and $\int \frac{dx}{x\sqrt{1+x^4}}$ are encountered. A number of such integrals are evaluated by parts. In some integrals the integrands are inverse circular functions. In some integrals the integrands are Logarithmic functions. The most of these integrals are converted into the foregoing types of integrals on simplification and elaborations to underscore the final results .Also evaluated are the integrals: $\int \frac{dx}{x^4+x^2+1}, \int \frac{x^2dx}{x^4+x^2+1}, \int \frac{x^2dx}{x^4-x^2+1}, \int \frac{dx}{(2+x^2)^2+5}, \int \log(x^4+4x^2+9)dx, \int \frac{\log(x^4+4x^2+9)}{x^2}dx$, *etc.*

Keywords - Integrals, Partial, Fraction, Parts, Evaluation, Solution.

1. Introduction

As far as the introduction is concerned the above six vital integrals and some other integrals are evaluated herein without direct application of any textbook formulae. That way eighteen integrals are innovated and evaluated mostly by parts using “dx “ as the second part. The last seven integrals are evaluated by integration by partial fractions . However, this type of integrals are neither found in any textbooks of Integral Calculus [1] nor have been published elsewhere. SN Maitra[2], the present author, thought of this type of integrals and evaluated them in close form.The constants of integrations are kept understood.

$$\int tan^{-1x^2}\, dx = xtan^{-1x^2} - 2\int \frac{x^2dx}{1+(x^2)^2} = xtan^{-1x^2} - 2I_1 \tag{1}$$

(Integrating by parts)

Where $I_1 = \int \frac{x^2dx}{1+x^4} = \int \frac{dx}{x^2+\frac{1}{x^2}} = \int \frac{dx}{(x+\frac{1}{x})^2-2} = \frac{1}{2}\int \{\frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2-2} + \frac{1+\frac{1}{x^2}}{(x-\frac{1}{x})^2+2}\}dx$

$$=\frac{1}{2}\int\{\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2} + \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2}\}dx = \frac{1}{2}\{\frac{1}{2\sqrt{2}}log\{(x^2-\sqrt{2}x+1)/(x^2+\sqrt{2}x+1)\} + \frac{1}{\sqrt{2}}tan^{-1\frac{x^2-1}{\sqrt{2}x}}\} \tag{2}$$

$$\int tan^{-1\frac{1}{x^2}}\, dx = xtan^{-1\frac{1}{x^2}} - \int \frac{x\frac{-2}{x^3.}dx}{1+\frac{1}{x^4}} = xtan^{-1\frac{1}{x^2}} + 2I_2 \tag{3}$$

Where $I_2 = \int \frac{x^2dx}{1+x^4} = dx = I_1$

$\int \frac{dx}{sinx+sin^3x}$

I=$\int \frac{dx}{sinx(1+sin^2x)}$ (Multiplying num and deno by sinx)

=$\int \frac{sinxdx}{sin^2x(1+sin^2x)}$=$\int \frac{-d(cosx)}{(1-cos^2x)(2-cos^2x)}$ =$\int\{\frac{1}{(2-cos^2x)} - \frac{1}{(1-cos^2x)}\}$d(cosx)

=$\frac{1}{2\sqrt{2}} log \frac{\sqrt{2}+cosx}{\sqrt{2}-cosx}$-log$\frac{1+cosx}{1-cosx}$

$$\int \frac{xdx}{(1+x^2)(1+x^4)} \tag{4}$$

(putting y=x^2so that dy=2xdx)

= $\frac{1}{2}\int \frac{dy}{(1+y)(1+y^2)}$ =$\frac{1}{2}\int\{\frac{1}{(1+y)} + \int \frac{(1-y)}{(1+y^2)}\}dy$

=$\frac{1}{2}$ {log(1+y)--$\frac{1}{2} log(1+y^2) + tan^{-1y}$}

=$\frac{1}{2}$ {log(1+x^2)- $\frac{1}{2} log(1+x^4) + tan^{-1x^2}$}

1.1. Problem No 1 and its Evaluation

$$\int \frac{dx}{x^4+x^2+1} = =\frac{1}{2}\int \frac{\{(x^2+1)-(x^2-1)\}dx}{x^4+x^2+1} = \frac{1}{2}\int \frac{\{(x^2+1)-(x^2-1)\}dx}{x^2(x^2+\frac{1}{x^2}+1)} \tag{5}$$

=$\frac{1}{2}\int \frac{\{\left(1+\frac{1}{x^2}\right)-(\left(1-\frac{1}{x^2}\right)\}dx}{(x^2+\frac{1}{x^2}+1)}$ =$\frac{1}{2}\{\int\{\frac{\left(1+\frac{1}{x^2}\right)dx}{(x-\frac{1}{x})^2+3} - \int \frac{\left(1-\frac{1}{x^2}\right)dx}{(x+\frac{1}{x})^2-1)}\}$ =$\frac{1}{2}\{\int \frac{d\left(1-\frac{1}{x}\right)}{(x-\frac{1}{x})^2+3} - \int \frac{d\left(1+\frac{1}{x}\right)}{(x+\frac{1}{x})^2-1)}\}$

$$=\frac{1}{2}\{\frac{1}{\sqrt{3}} tan^{-1} \frac{x^2-1}{x\sqrt{3}}-\frac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}\} \tag{6}$$

1.2. Problem No 2 and its Evaluation

$$\int \frac{x^2dx}{x^4+x^2+1} = =\frac{1}{2}\int \frac{(x^2+1)+(x^2-1)dx}{x^4+x^2+1} = \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)dx}{x^2(x^2+\frac{1}{x^2}+1)} \tag{7}$$

=$\frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)+((1-\frac{1}{x^2})dx}{(x^2+\frac{1}{x^2}+1)}$ =$\frac{1}{2}\{\int \frac{\left(1+\frac{1}{x^2}\right)dx}{(x-\frac{1}{x})^2+3}+\int \frac{\left(1-\frac{1}{x^2}\right))dx}{(x+\frac{1}{x})^2-1)}\}$ =$\frac{1}{2}\{\int \frac{d\left(1-\frac{1}{x}\right)}{(x-\frac{1}{x})^2+3}+\int \frac{d\left(1+\frac{1}{x}\right))dx}{(x+\frac{1}{x})^2-1)}\}$

$$=\frac{1}{2}\{\frac{1}{\sqrt{3}} tan^{-1} \frac{x^2-1}{x\sqrt{3}}+\frac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}\} \tag{8}$$

1.3. Problem No 3 and its Evaluation

$$\int \frac{dx}{x^4-x^2+1} = =\frac{1}{2}\int \frac{(x^2+1)-(x^2-1)dx}{x^4-x^2+1} = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)dx}{x^2(x^2+\frac{1}{x^2}-1)} \tag{9}$$

=$\frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)-(1-\frac{1}{x^2})dx}{(x^2+\frac{1}{x^2}-1)}$ =$\frac{1}{2}\{\int \frac{\left(1+\frac{1}{x^2}\right)dx}{(x-\frac{1}{x})^2+1}-- \int \frac{\left(1-\frac{1}{x^2}\right))dx}{(x+\frac{1}{x})^2-3)}\}$ =$\frac{1}{2}\{\int \frac{d\left(x-\frac{1}{x}\right)}{(x-\frac{1}{x})^2+1}-- \int \frac{d\left(x+\frac{1}{x}\right))dx}{(x+\frac{1}{x})^2-3)}\}$

$$=\frac{1}{2}\{tan^{-1} \frac{x^2-1}{x}--\frac{1}{2\sqrt{3}}\log\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\} \tag{10}$$

1.4. Problem No 4 and its Evaluation

$$\int \frac{x^2dx}{x^4-x^2+1} = =\frac{1}{2}\int \frac{(x^2+1)+(x^2-1)dx}{x^4-x^2+1} = \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)dx}{x^2(x^2+\frac{1}{x^2}-1)} \tag{11}$$

$$=\frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)+((1-\frac{1}{x^2})dx}{(x^2+\frac{1}{x^2}-1)} = \frac{1}{2}\{\int \frac{\left(1+\frac{1}{x^2}\right)dx}{(x-\frac{1}{x})^2+1}+\int \frac{\left(1-\frac{1}{x^2}\right))dx}{(x+\frac{1}{x})^2-3}\} = \frac{1}{2}\{\int \frac{d\left(1-\frac{1}{x}\right)}{(x-\frac{1}{x})^2+1}+\int \frac{d\left(1+\frac{1}{x}\right))dx}{(x+\frac{1}{x})^2-3}\}$$

$$=\frac{1}{2}\{tan^{-1}\frac{x^2-1}{x}+\frac{1}{2\sqrt{3}}\log\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\} \tag{12}$$

1.5. Problem No 5 and its Evaluation

$$\int \frac{dx}{(2+x^2)^2+5} = \int \frac{dx}{x^4+4x^2+9} = \int \frac{dx}{x^2(x^2+\frac{9}{x^2}+4)}$$

$$=\frac{1}{6}\{\int \frac{-\left(1-\frac{3}{x^2}\right)dx}{\{(x+\frac{3}{x})^2-2\}} + \int \frac{\left(1+\frac{3}{x^2}\right)dx}{\{(x-\frac{3}{x})^2+10\}}\}$$

$$=\frac{1}{6}\{-\frac{1}{2\sqrt{2}}\log\frac{x+\sqrt{2}x+3}{x-\sqrt{2}x+3} + \frac{1}{\sqrt{10}}tan^{-1}\frac{x^2-3}{x\sqrt{10}}\} \tag{13}$$

1.6. Problem No 6 and its Evaluation

$\int \log(x^4 + 4x^2 + 9)dx$ (Integrating by parts)

$$=\text{x}\log(x^4 + 4x^2 + 9) - 4\int \frac{x^4+2x^2}{x^4+4x^2+9}dx \tag{14}$$

$$=\text{x}\log(x^4 + 4x^2 + 9) - 4I$$

Where $I=\int \frac{x^4+2x^2}{x^4+4x^2+9}dx = \int \frac{x^4+4x^2+9-2x^2-9}{x^4+4x^2+9}dx = x-\int \frac{2x^2+9}{x^4+4x^2+9}dx$

$$=x-\int \frac{2x^2+9}{x^2(x^2+\frac{9}{x^2}+4)}dx = x-\int \frac{2+\frac{9}{x^2}}{(x^2+\frac{9}{x^2}+4)}dx = x-\int \frac{A\left(1+\frac{3}{x^2}\right)+B(1-\frac{3}{x^2})}{(x-\frac{3}{x})^2+10)}dx \tag{15}$$

Where by comparison, A+B=2 and 3(A-B)=9 so that $A=\frac{5}{2}$ and $B = \frac{-1}{2}$. Then

$$I=x-\int \frac{\frac{5}{2}\left(1+\frac{3}{x^2}\right)-\frac{1}{2}(1-\frac{3}{x^2})}{(x-\frac{3}{x})^2+10)}dx = x-\int\{\frac{\frac{5}{2}\left(1+\frac{3}{x^2}\right)}{(x-\frac{3}{x})^2+10}dx + \int \frac{\frac{1}{2}(1-\frac{3}{x^2})}{(x+\frac{3}{x})^2-2}dx$$

$$=x - \frac{5}{2\sqrt{10}}tan^{-1}\frac{1}{\sqrt{10}}\left(x-\frac{3}{x}\right) + \frac{1}{4\sqrt{2}}log\frac{x^2-\sqrt{2}x+3}{x^2+\sqrt{2}x+3} \tag{16}$$

1.7. Problem No 7 and its Evaluation

$$\int \frac{\log(x^4+4x^2+9)}{x^2}dx \quad \text{(Integrating by parts)} \tag{17}$$

$$=\frac{-\log(x^4+4x^2+9)}{x} + 4\int \frac{(x^3+2x^2)}{x(x^4+4x^2+9)}dx$$

$$= \frac{-\log(x^4+4x^2+9)}{x} + 4\int \frac{(x^2+2x)}{(x^4+4x^2+9)}dx$$

$$=-\frac{\log(x^4+4x^2+9)}{x} + 4\int \frac{x^2}{(x^4+4x^2+9)}dx+4\int \frac{1}{(x^4+4x^2+9)}dx^2$$

$$= -\frac{\log(x^4+4x^2+9)}{x} + 4I_1 + 4I_2 \qquad Where\ \ I_1 = \int \frac{x^2}{(x^4+4x^2+9)}dx,\ I_2 = \int \frac{1}{(x^4+4x^2+9)}dx^2 \tag{18}$$

$$I_1 = \int \frac{x^2}{(x^4+4x^2+9)}dx = \int \frac{1}{x^2+\frac{9}{x^2}+4}dx = \frac{1}{2}\int \frac{\left(1+\frac{3}{x^2}\right)+\left(1-\frac{3}{x^2}\right)}{(x-\frac{3}{x})^2+10}dx = \frac{1}{2}\{\frac{tan^{-1}\frac{x^2-3}{\sqrt{10}x}}{\sqrt{10}} + \int \frac{\left(1-\frac{3}{x^2}\right)dx}{(x+\frac{3}{x})^2-2}\}$$

$$= \frac{1}{2}\{\frac{tan^{-1}\frac{x^2-3}{\sqrt{10}x}}{\sqrt{10}} + \frac{1}{2\sqrt{2}}log\frac{x^2-2x+3}{x^2+2x+3}\} \tag{19}$$

$$I_2 = \int \frac{1}{(x^4+4x^2+9)}dx^2 = \int \frac{1}{(x^2+2)^2+5)}d(x^2+2) = \frac{tan^{-1}\frac{x^2+2}{\sqrt{5}}}{\sqrt{5}}$$

1.8. Problem No 8

$$I_3 = \int \frac{1}{(x^4+4x^2+9)}dx = \int \frac{1}{x^2(x^2+\frac{9}{x^2}+4)}dx = \frac{1}{6}\int \frac{(1+\frac{3}{x^2})-(1-\frac{3}{x^2})}{(x-\frac{3}{x})^2+10}dx = \frac{1}{6}\{\frac{tan^{-1}\frac{x^2-3}{\sqrt{10}x}}{\sqrt{10}} -- \int \frac{\left(1-\frac{3}{x^2}\right)dx}{(x+\frac{3}{x})^2-2}\} \tag{20}$$

$$= \frac{1}{6}\{\frac{tan^{-1}\frac{x^2-3}{\sqrt{10}x}}{\sqrt{10}} - \frac{1}{2\sqrt{2}}log\frac{x^2-\sqrt{2}x+3}{x^2+\sqrt{2}\,x+3}\}$$

1.9. Problem No 9

$$I_4 = \int \frac{1}{x\sqrt{(x^4+4x^2+9)}}dx \quad \int \frac{1}{x^2\sqrt{\left(x^2+\frac{9}{x^2}+4\right)}}dx$$

$$= \frac{1}{6}\int \frac{\left(1+\frac{3}{x^2}\right)-\left(1-\frac{3}{x^2}\right)}{\sqrt{(x-\frac{3}{x})^2+10}}dx = \frac{1}{6}\{sinh^{-1}\frac{x^2-3x}{\sqrt{10}x} - \int \frac{\left(1-\frac{3}{x^2}\right)dx}{\sqrt{(x+\frac{3}{x})^2-2}}\} \tag{21}$$

$$= \frac{1}{6}\{sinh^{-1}\frac{x^2-3x}{\sqrt{10}x} - cosh^{-1}\frac{x^2+3x}{\sqrt{2}x}\}$$

1.10. Problem No 10

$$I_5 = \int \frac{1}{x\sqrt{(x^4-4x^2+9)}}dx == \int \frac{1}{x^2\sqrt{\left(x^2+\frac{9}{x^2}-4\right)}}dx$$

$$= \frac{1}{6}\int \frac{\left(1+\frac{3}{x^2}\right)-\left(1-\frac{3}{x^2}\right)}{\sqrt{(x-\frac{3}{x})^2+2}}dx = \frac{1}{6}\{sinh^{-1}\frac{x^2-3}{\sqrt{2}x} - \int \frac{\left(1-\frac{3}{x^2}\right)dx}{\sqrt{(x+\frac{3}{x})^2-10}}\} \tag{22}$$

$$= \frac{1}{6}\{sinh^{-1}\frac{x^2-3}{\sqrt{2}x} - cosh^{-1}\frac{x^2+3}{\sqrt{10}x}\}$$

1.11. Problem No 11 and its Solution

$$I_5 = \int \frac{5x^2+6}{x\sqrt{(x^4-4x^2+9)}}dx = \int \frac{5+\frac{6}{x^2}}{x^2\sqrt{\left(x^2+\frac{9}{x^2}-4\right)}}dx = \int \frac{A\left(1+\frac{3}{x^2}\right)+B\left(1-\frac{3}{x^2}\right)}{\sqrt{(x-\frac{3}{x})^2+2}}dx \ \text{(By comparison)} \tag{23}$$

(A+B=5 and 3(A-B)=6 ie A-B=2 leading to A=$\frac{7}{2}$ and B=$\frac{3}{2}$)

$$= \frac{1}{2}\int \frac{7\left(1+\frac{3}{x^2}\right)+3\left(1-\frac{3}{x^2}\right)}{\sqrt{(x-\frac{3}{x})^2+2}}dx = \frac{1}{2}\{\int \frac{7\left(1+\frac{3}{x^2}\right)}{\sqrt{(x-\frac{3}{x})^2+2}}dx + \frac{1}{2}\int \frac{3\left(1-\frac{3}{x^2}\right)}{\sqrt{(x+\frac{3}{x})^2-10}}dx\}$$

$$=\frac{1}{2}\{7sinh^{-1\frac{x^2-3}{\sqrt{2}x}}+3cosh^{-1\frac{x^2+3}{\sqrt{10}x}}\} \tag{24}$$

1.12. Problem No 12 and its Solution

$$I_6=\int\frac{1}{(x^4+4x^2-21)}dx=\int\frac{1}{(x^2+2)^2-5^2)}dx$$

$$=\int\frac{1}{(x^2+7)(x^2-3)}dx==\frac{1}{10}\int\frac{(x^2+7)-(x^2-3)}{(x^2+7)(x^2-3)}dx=\frac{1}{10}\int\{\frac{1}{(x^2-3)}-\frac{1}{(x^2+7)}\}dx$$

$$=\frac{1}{10}\{\frac{1}{2\sqrt{3}}log\frac{x-3}{x+3}-\frac{1}{\sqrt{7}}tan^{-1}\frac{x}{\sqrt{7}}\} \tag{25}$$

1.13. Problem No 13 and its Solution

$$I_6=\int\frac{x^2dx}{(x^4+4x^2-21)}dx=\int\frac{x^2dx}{(x^2+2)^2-5^2)}dx$$

$$=\int\frac{x^2}{(x^2+7)(x^2-3)}dx==\int\frac{A(x^2+7)+B(x^2-3)}{(x^2+7)(x^2-3)}dx=\frac{1}{10}\int\{\frac{3}{(x^2-3)}+\frac{7}{(x^2+7)}\}dx$$

(By comparison, A+B=1,7A-3B=0 so that $=\frac{3}{10}$ and $B=\frac{7}{10}$

$$=\frac{1}{10}\{\frac{\sqrt{3}}{6}log\frac{x-3}{x+3}+\frac{7}{\sqrt{7}}tan^{-1}\frac{x}{\sqrt{7}}\} \tag{26}$$

1.14. Problem No 14 and its Solution

$$I_6=\int\frac{x^2dx}{(x^8+4x^4-21)}dx=\int\frac{x^2dx}{(x^4+2)^2-5^2)}dx$$

$$=\int\frac{x^2}{(x^4+7)(x^4-3)}dx=\frac{1}{10}\int\{\frac{x^2}{(x^4-3)}-\frac{x^2}{(x^4+7)}\}dx$$

$$=\frac{1}{10}\int\{\frac{(x^2-\sqrt{3})+(x^2+\sqrt{3})}{2(x^2-\sqrt{3})(x^2+\sqrt{3})}-\frac{1}{(x^2+\frac{7}{x^2})}\}dx$$

$$=\frac{1}{10}\int\{\frac{1}{2(x^2+\sqrt{3})}+\frac{1}{2(x^2-\sqrt{3})}-\frac{1}{2}\frac{(1+\frac{\sqrt{7}}{x^2})+(1-\frac{\sqrt{7}}{x^2})}{\left(x-\frac{\sqrt{7}}{x}\right)^2+2\sqrt{7}}\}dx \tag{27}$$

$$=\frac{1}{10}\int\{\frac{1}{2(x^2+\sqrt{3})}+\frac{1}{2(x^2-\sqrt{3})}-\frac{1}{2}(\frac{\left(1+\frac{\sqrt{7}}{x^2}\right)}{\left(x-\frac{\sqrt{7}}{x}\right)^2+2\sqrt{7}}+\frac{\left(1-\frac{\sqrt{7}}{x^2}\right)}{\left(x+\frac{\sqrt{7}}{x}\right)^2-2\sqrt{7}})\}dx$$

$$=\frac{1}{20}\{\frac{1}{\sqrt[4]{3}}tan^{-1\frac{x}{\sqrt[4]{3}}}+\frac{1}{2\sqrt[4]{3}}log\frac{x-\sqrt[4]{3}}{x+\sqrt[4]{3}}-\frac{1}{\sqrt{2\sqrt{7}}}tan^{-1}\frac{x+\frac{\sqrt{7}}{x}}{\sqrt{2\sqrt{7}}}+\frac{1}{2\sqrt{2\sqrt{7}}}\log\{\frac{x+\frac{\sqrt{7}}{x}-\sqrt{2\sqrt{7}}}{x+\frac{\sqrt{7}}{x}+\sqrt{2\sqrt{7}}}\}$$

1.15. Problem No 15 and its Solution

$$I_7=\int\frac{dx}{(x^8+4x^4-21)}dx=\int\frac{dx}{(x^4+2)^2-5^2)}dx$$

$$=\int\frac{1}{(x^4+7)(x^4-3)}dx=\frac{1}{10}\int\{\frac{1}{(x^4-3)}-\frac{1}{(x^4+7)}\}dx$$

$$=\frac{1}{10}\int\{\frac{(x^2+\sqrt{3})-(x^2-\sqrt{3})}{2\sqrt{3}(x^2-\sqrt{3})(x^2+\sqrt{3})}-\frac{1}{(x^2+\frac{7}{x^2})}\}dx$$

$$=\frac{1}{10}\int\{\frac{1}{2\sqrt{3}(x^2+\sqrt{3})}-\frac{1}{2\sqrt{3}(x^2-\sqrt{3})}-\frac{1}{2}\frac{(1+\frac{\sqrt{7}}{x^2})+(1-\frac{\sqrt{7}}{x^2})}{\left(x-\frac{\sqrt{7}}{x}\right)^2+2\sqrt{7}}\}dx$$

$$=\frac{1}{10}\int\{\frac{1}{2\sqrt{3}(x^2+\sqrt{3})}+\frac{1}{2\sqrt{3}(x^2-\sqrt{3})}-\frac{1}{2}(\frac{\left(1+\frac{\sqrt{7}}{x^2}\right)}{\left(x-\frac{\sqrt{7}}{x}\right)^2+2\sqrt{7}}+\frac{\left(1-\frac{\sqrt{7}}{x^2}\right)}{\left(x+\frac{\sqrt{7}}{x}\right)^2-2\sqrt{7}})\}dx$$

$$=\frac{1}{20}\{\frac{1}{\sqrt[4]{27}}tan^{-1\frac{x}{\sqrt[4]{3}}}+\frac{1}{2\sqrt[4]{27}}log\frac{x-\sqrt[4]{3}}{x+\sqrt[4]{3}}-\frac{1}{\sqrt{2\sqrt{7}}}tan^{-1}\frac{x+\frac{\sqrt{7}}{x}}{\sqrt{2\sqrt{7}}}+\frac{1}{2\sqrt{2\sqrt{7}}}log\frac{x+\frac{\sqrt{7}}{x}-\sqrt{2\sqrt{7}}}{x+\frac{\sqrt{7}}{x}+\sqrt{2\sqrt{7}}}\} \tag{28}$$

1.16. Problem No 16 and its Solution

$$I_8=\int\frac{x^5dx}{(x^8+4x^4+1)}=\int\frac{x^5dx}{x^4\left(x^4+4+\frac{1}{x^4}\right)}dx=\int\frac{xdx}{\left(x^4+4+\frac{1}{x^4}\right)}dx$$

$$=\int\frac{xdx}{(x^2-\frac{1}{x^2})^2+6}dx=\frac{1}{2}\{\int\frac{xdx}{(x^2-\frac{1}{x^2})^2+6}+\int\frac{xdx}{(x^2+\frac{1}{x^2})^2+2}\}$$

$$=\frac{1}{4}\{\int\frac{(2x+\frac{2}{x^3})dx}{(x^2-\frac{1}{x^2})^2+6}+\int\frac{(2x-\frac{2}{x^3})dx}{(x^2+\frac{1}{x^2})^2+2}\}$$

$$=\frac{1}{4}\{\frac{1}{\sqrt{6}}tan^{-1}\frac{(x^2-\frac{1}{x^2})}{\sqrt{6}}+\frac{1}{\sqrt{2}}tan^{-1\frac{x^2+\frac{1}{x^2}}{\sqrt{2}}}\}$$

1.17. Problem No 17 and its Solution

$$I_9=\int\frac{xdx}{(x^8+4x^4+1)}=\int\frac{xdx}{x^4\left(x^4+4+\frac{1}{x^4}\right)}dx=\int\frac{\frac{1}{x^3}dx}{\left(x^4+4+\frac{1}{x^4}\right)}dx$$

$$=\frac{1}{4}\{\int\frac{(2x+\frac{2}{x^3})dx}{(x^2-\frac{1}{x^2})^2+6}-\int\frac{(2x-\frac{2}{x^3})dx}{(x^2+\frac{1}{x^2})^2+2}\}$$

$$=\frac{1}{4}\{\frac{1}{\sqrt{6}}tan^{-1\frac{(x^2-\frac{1}{x^2}}{\sqrt{6}}}+\frac{1}{\sqrt{2}}tan^{-1\frac{(x^2-\frac{1}{x^2}}{\sqrt{2}}}\} \tag{29}$$

1.18. Problem No 18 and its Solution

$$I_{10}=\int\frac{dx}{(x^4+2x^2+4)(x^2+2)}=\frac{1}{4}\{\int\frac{-x^2dx}{(x^4+2x^2+4)}+\int\frac{dx}{(x^2+2)}\}$$

$$=\frac{1}{4}\{\int\frac{-dx}{\left(x^2+\frac{4}{x^2}+2\right)}+\frac{1}{\sqrt{2}}tan^{-1\frac{x}{\sqrt{2}}}\}=\frac{1}{8}\{\int\frac{\left(1-\frac{2}{x^2}\right)+\left(1+\frac{2}{x^2}\right)}{\left(x+\frac{2}{x}\right)^2-2)}dx+\frac{2}{\sqrt{2}}tan^{-1\frac{x}{\sqrt{2}}}\}$$

$$=\frac{1}{8}\{-(\int\frac{\left(1-\frac{2}{x^2}\right)}{\left(x+\frac{2}{x}\right)^2-2)}dx+\int\frac{\left(1+\frac{2}{x^2}\right)}{\left(x-\frac{2}{x}\right)^2+6)}dx)+\frac{1}{\sqrt{2}}tan^{-1\frac{x}{\sqrt{2}}}\} \tag{30}$$

References

[1] S.N. Maitra, "Some Unusual Integrals and their Evaluation," *DS Journal of Modelling and Simulation*, vol. 2, no. 3, pp. 13-17, 2024. [Publisher Link]

[2] S.Narayanan, and T.K. Pillay, *Calculus*, vol. 2, S. Vishwanathan Printers and Publishers Pvt. Ltd, 1996.

[3] R. Bharadwaj, *Mathematcs with Formulae and Definitions*, Computech Publicatons Ltd., New Asian, New Delhi, 1988.

[4] Ulrich L. Rohde et al., *Introduction to Integral Calculus: Systematic Studies with Engineering Applications for Beginners*, Wiley, 2011. [CrossRef] [Google Scholar] [Publisher Link]