SOLUTIONS OF HARTSHORNE'S ALGEBRAIC GEOMETRY

Contents

1. Varieties

1.1. Affine Varieties.

Exercise 1.1.1.

(a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.
(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.
(c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Proof.

(a) Since $y-x^2$ is an irreducible polynomial, $(y-x^2)$ is a prime ideal of $k[x, y]$, so

$$\begin{aligned} A(Y) &= k[x, y]/I(Y) = k[x, y]/I(Z(y - x^2)) \\ &= k[x, y]/\sqrt{(y - x^2)} = k[x, y]/(y - x^2) \end{aligned}$$

Consider the map

$$\begin{aligned} \varphi : k[x, y]/(y - x^2) &\to k[t] \\ f(x, y) + (y - x^2) &\mapsto f(t, t^2) \end{aligned}$$

If $y-x^2 \mid f(x, y)$, then $f(t, t^2) = 0$ so it is well defined ring homomorphism. And define

$$\begin{aligned} \psi : k[t] &\to k[x, y]/(y - x^2) \\ g(t) &\mapsto g(x) + (y - x^2) \end{aligned}$$

We want to show that $\varphi \circ \psi = 1_{k[t]}$ and $\psi \circ \varphi = 1_{k[x,y]/(y-x^2)}$ so $k[t] \cong k[x, y]/(y - x^2)$. The first one is easy because

$$\varphi(\psi(g(t))) = \varphi(g(x) + (y - x^2)) = g(t)$$

Take any $f(x,y) = \sum_{i,j} a_{ij} x^i y^j \in k[x,y]$. Then,

$$\psi(\varphi(f(x,y) + (y - x^2))) = \psi(f(t,t^2)) = f(x,x^2) + (y - x^2)$$

so we need to show that $f(x,y) - f(x,x^2) \in (y - x^2)$.

$$\begin{aligned} f(x,y) - f(x,x^2) &= \sum_{i,j} a_{ij} x^i y^j - \sum_{i,j} a_{ij} x^i x^{2j} \\ &= \sum_{i,j} a_{ij} x^i (y^j - (x^2)^j) \\ &= \sum_{i,j} a_{ij} x^i (y - x^2) \left(y^{j-1} + y^{j-2}(x^2)^1 + \cdots + y^1 (x^2)^{j-2} + (x^2)^{j-1} \right) \end{aligned}$$

Hence, $A(Y) \cong k[t]$

(b) Note that

$$A(Z) = k[x,y]/(xy - 1)$$

and suppose $\theta : A(Z) \to k[t]$ is a ring isomorphism. Then, $x + (xy-1), y + (xy-1) \in A(Z)$ are unit elements, so $\theta(x + (xy-1)), \theta(y + (xy-1)) \in k[t]$ are unit elements, which implies they are in k. Then there is no element in $A(Z)$ that goes to t through the map θ, which is a contradiction with the fact that θ is an isomorphism.

(c) Let $f = ax^2 + bxy + cy^2 + dx + ey + g$. Since k is algebraically closed, we can decompose $ax^2 + bxy + cy^2$ into the product of two linear terms $u = a'x + b'y$ and $v = c'x + d'y$.

□

Exercise 1.1.2. *The Twisted Cubic Curve.* Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

Proof. □

Exercise 1.1.3. Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Proof. □

Exercise 1.1.4. If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Proof. □

Exercise 1.1.5. Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Proof. □

Exercise 1.1.6. Any nonempty open subset of an irreducible topological space is dense and irreducible If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Proof. □

Exercise 1.1.7.

(a) Show that the following conditions are equivalent for a topological space X :(i) X is noetherian; (ii) every nonempty family of closed subsets has a minimal elements; (iii) X satisfies the ascending chain condition for open subsetes; (iv) every nonempty family of open subsets has a maximal elements.
(b) A noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.
(c) Any subset of a noetherian topological space is noetherian in its induced topology.
(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Proof. □

Exercise 1.1.8. Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \nsubseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$. (See (7.1) for a generalization.)

Proof. □

Exercise 1.1.9. Let $\mathfrak{a} \subseteq A = k[x_1, \cdots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geqslant n-r$.

Proof. □

Exercise 1.1.10.

(a) If Y is any subset of a topological space X, then $\dim Y \leqslant \dim X$.
(b) If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.
(c) Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.
(d) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.
(e) Give an example of a noetherian topological space of infinite dimension.

Proof. □

Exercise 1.1.11. Let $Y \subseteq \mathbb{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x,y,z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection* cf.(Ex. 2.17).

Proof. □

Exercise 1.1.12. Give an example of an irreducible polynomial $f \in \mathbb{R}[x,y]$, whose zero set $Z(f)$ in $\mathbb{A}^2_{\mathbb{R}}$ is not irreducible (cf. 1.4.2).

Proof. □

1.2. **Projective Varieties.**

Exercise 1.2.1.

Proof. □

1.3. **Morphisms.**

Exercise 1.3.1.

Proof. □

1.4. **Rational Maps.**

Exercise 1.4.1.

Proof. □

1.5. **Nonsingular Varieties.**

Exercise 1.5.1.

Proof. □

1.6. **Nonsingular Curves.**

Exercise 1.6.1.

Proof. □

1.7. **Intersections in Projective Space.**

Exercise 1.7.1.

Proof. □