BEHAVIOUR OF NON LINEAR CONSOLIDATION OF SOIL WITH SEMI-PERMEABLE BOUNDARY

A Summer Internship Report-2025

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Abstract

This study investigates the behaviour of one-dimensional non-linear consolidation in saturated soil layers considering semi-permeable boundary conditions, a topic of significant importance in modern geotechnical engineering. Unlike classical Terzaghi's theory, which assumes linear stress-strain relationships, constant permeability, and infinitesimal strain in thin clay layers, this research explores the implications of self-weight, variable permeability, and stress-dependent compressibility in thicker deposits.

Drawing upon the non-linear theory proposed by Davis and Raymond (1965), and building further on recent analytical advancements by Chen et al. (2023), the analysis incorporates the realistic condition of semi-permeable drainage boundaries, which are commonly encountered in field conditions such as reclaimed lands, embankments, and tailings dams. The governing differential equations are developed assuming a linear void ratio–log effective stress relationship and constant coefficient of consolidation, while accounting for variation of effective stress with depth and drainage impedance quantified by a semi-permeability coefficient.

A finite difference approach is utilized for the numerical solution, and comparisons with conventional theories reveal that semi-permeable boundaries significantly retard pore pressure dissipation while self-weight increases initial pressure gradients. Additionally, the study highlights the influence of drainage capacity and soil layering on the settlement behaviour. The results provide practical insights for the design and prediction of settlement in geostructures where ideal pervious or impervious boundary assumptions are not valid.

Keywords: Non-linear consolidation, semi-permeable boundary, self-weight, clay layer, effective stress, finite difference, excess pore pressure.

Contents

1 Introduction

Consolidation is a time-dependent process in saturated soils caused by the dissipation of excess pore water pressure. Classical consolidation theory, proposed by Terzaghi, assumes linear stress–strain behaviour, constant permeability, and ideal boundary conditions, making it suitable only for simplified problems involving thin clay layers.

However, in real-world scenarios such as embankments and land reclamation, soils often exhibit non-linear compressibility, variable permeability, and significant self-weight effects, especially in thick deposits. Additionally, field boundaries are rarely fully pervious or impervious; instead, they behave as semi-permeable, affecting drainage and consolidation rates.

This study investigates non-linear one-dimensional consolidation behaviour considering semi-permeable boundary conditions and the influence of self-weight. Building on models like those by Davis and Raymond (1965) and incorporating recent analytical insights, the work provides a more accurate framework for predicting settlement in complex soil systems.

2 Classical Terzaghi's Theory of Consolidation and Its Relevance to Nonlinear Semi-Permeable Systems

The classical theory of consolidation proposed by Karl Terzaghi in 1925 is one of the foundational theories in geotechnical engineering for understanding the time-dependent settlement of saturated soils under external loads. It describes the gradual dissipation of excess pore water pressure due to one-dimensional fluid flow in saturated clayey soils.

This section provides the theoretical background of Terzaghi's model while emphasizing its limitations in modeling nonlinear consolidation behavior and boundary effects—particularly those associated with **semi-permeable boundaries**, which are the focus of this study.

2.1 Fundamental Assumptions

Terzaghi's one-dimensional theory is based on the following key assumptions:

- The soil is fully saturated and homogeneous.
- The soil and water are incompressible.
- Boundaries are either perfectly permeable or impermeable.

These assumptions limit the theory's ability to model field conditions involving time-dependent soil properties, solute transport, or partial permeability across boundaries.

2.2 Governing Differential Equation

The primary variable of interest in consolidation is the excess pore water pressure, $u(z,t)$, at depth z and time t. The governing differential equation for Terzaghi's theory is:

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \tag{1}$$

Here, c_v is the coefficient of consolidation and is defined as:

$$c_v = \frac{k}{m_v \gamma_w} \tag{2}$$

where:

- k = coefficient of permeability (m/s)
- m_v = coefficient of volume compressibility (m^2/kN)
- γ_w = unit weight of water (kN/m^3)

2.3 Analytical Solution for Settlement

For a soil layer of thickness $2H$ with double drainage, the solution for the average degree of consolidation $U(t)$ is given by:

$$U(t) = 1 - \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin\left(\frac{n\pi}{2}\right) \exp\left(-\frac{n^2 \pi^2 c_v t}{4H^2}\right) \tag{3}$$

The dimensionless time factor is:

$$T_v = \frac{c_v t}{H^2} \tag{4}$$

This solution enables engineers to predict the settlement over time using standard consolidation charts or numerical approximations.

2.4 Toward Non-Linear and Semi-Permeable Boundary Modeling

To address these limitations, advanced formulations of consolidation theory have been proposed. These involve nonlinear partial differential equations where permeability and compressibility vary with pressure or void ratio:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left(c(u) \frac{\partial u}{\partial z}\right) \tag{5}$$

For a **semi-permeable boundary**, the classical boundary condition (zero or full drainage) is replaced by a Robin-type condition:

$$-k\frac{\partial u}{\partial z} = \alpha u \tag{6}$$

where α is the boundary permeability coefficient, controlling the degree of partial drainage or resistance at the interface.

3 Numerical Methods for Solving Consolidation Equations

We aim to solve the one-dimensional consolidation equation numerically:

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2}$$

where:

- $u(z,t)$: excess pore water pressure,
- c_v: coefficient of consolidation,
- z: depth coordinate,
- t: time.

This PDE must be solved with appropriate initial and boundary conditions, including semi-permeable boundary effects for realistic modeling.

3.1 Discretization Setup

Let:

$$r = \frac{c_v \Delta t}{(\Delta z)^2}, \quad u_i^n = \text{pore pressure at node } i \text{ and time step } n$$

4 Explicit Method

4.1 Finite Difference Form

$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

4.2 Matrix Form

$$\mathbf{u}^{n+1} = \mathbf{C} \cdot \mathbf{u}^n$$

Where:

$$\mathbf{C} = \begin{bmatrix} 1-2r & r & 0 & \cdots & 0 \\ r & 1-2r & r & \ddots & \vdots \\ 0 & r & 1-2r & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & r \\ 0 & \cdots & 0 & r & 1-2r \end{bmatrix}$$

5 Implicit Method

5.1 Finite Difference Form

$$-ru_{i-1}^{n+1} + (1+2r)u_i^{n+1} - ru_{i+1}^{n+1} = u_i^n$$

5.2 Matrix Form

$$\mathbf{A} \cdot \mathbf{u}^{n+1} = \mathbf{u}^n$$

Where:

$$\mathbf{A} = \begin{bmatrix} 1+2r & -r & 0 & \cdots & 0 \\ -r & 1+2r & -r & \ddots & \vdots \\ 0 & -r & 1+2r & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -r \\ 0 & \cdots & 0 & -r & 1+2r \end{bmatrix}$$

6 Crank–Nicolson Method

6.1 Finite Difference Form

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{c_v}{2} \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta z^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta z^2} \right)$$

6.2 Matrix Form

$$\mathbf{A} \cdot \mathbf{u}^{n+1} = \mathbf{B} \cdot \mathbf{u}^n$$

Where:

$$\mathbf{A} = \begin{bmatrix} 1+r & -\frac{r}{2} & 0 & \cdots & 0 \\ -\frac{r}{2} & 1+r & -\frac{r}{2} & \ddots & \vdots \\ 0 & -\frac{r}{2} & 1+r & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{r}{2} \\ 0 & \cdots & 0 & -\frac{r}{2} & 1+r \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1-r & \frac{r}{2} & 0 & \cdots & 0 \\ \frac{r}{2} & 1-r & \frac{r}{2} & \ddots & \vdots \\ 0 & \frac{r}{2} & 1-r & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{r}{2} \\ 0 & \cdots & 0 & \frac{r}{2} & 1-r \end{bmatrix}$$

6.3 Comparison Table

Feature	**Explicit**	**Implicit**	**Crank–Nicolson**
Time Discretization	Forward	Backward	Midpoint (Average)
Space Discretization	Central	Central	Central
Accuracy	First-order	First-order	**Second-order**
Stability	Conditional ($r \leq 0.5$)	Unconditional	**Unconditional**
Matrix Inversion	Not needed	Required	Required
Suitable For	Small steps	Long-term loading	**Best all-around**

6.4 Most Suitable Method For Our Context

- For non-linear consolidation with depth-varying stress and complex boundary conditions, the **Crank–Nicolson Method** is most suitable.
- It offers the best combination of stability, accuracy, and flexibility for implementation of semi-permeable boundary effects.

6.5 Governing Equation for Non-Linear Consolidation in FEM

The governing partial differential equation in non-linear consolidation is:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left[c_v(u)\frac{\partial u}{\partial z}\right] \tag{7}$$

Where:

- $u(z,t)$: Excess pore pressure (kPa)
- $c_v(u) = \frac{k(u)}{m_v(u)\gamma_w}$: Coefficient of consolidation, which is a function of pore pressure

In FEM, this PDE is discretized as:

$$[K(u)]\{u\}^{t+1} + [H(u)]\frac{\{u\}^{t+1} - \{u\}^t}{\Delta t} = \{f\} \tag{8}$$

Where:

- $[K(u)]$: Non-linear stiffness matrix due to pressure-dependent properties
- $[H(u)]$: Hydraulic conductivity matrix
- $\{u\}$: Vector of nodal pore pressures
- Δt: Time step size

This system is solved iteratively at each time step using techniques such as Newton-Raphson.

6.6 Benefits Over Other Methods

Compared to Finite Difference Method (FDM) or analytical approaches, FEM offers:

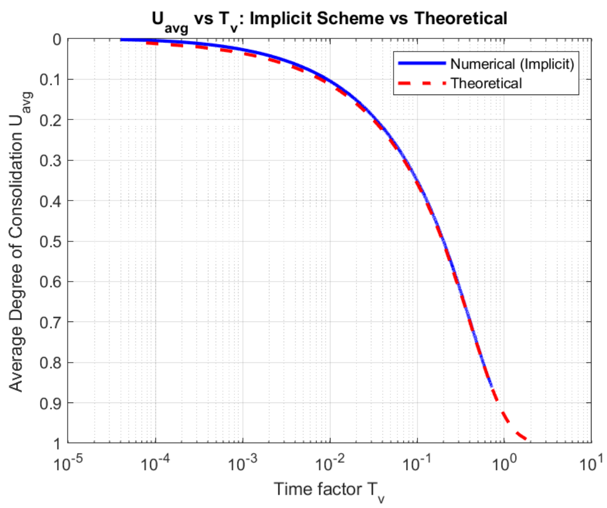
- Greater flexibility in geometry and layering
- Easier integration of advanced material models
- Better convergence in non-linear and boundary-sensitive cases

7 MATLAB Code: Implicit(FDM)

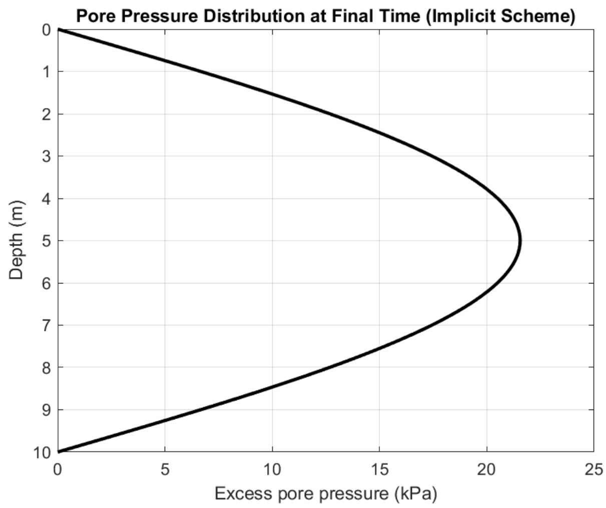
```
clc
clear
close all

%Parameters
H = 10.0;                    % Thickness of soil layer (m)
Nz = 100;                    % Number of spatial divisions
dz = H / Nz;
cv = 1e-4;                   % Coefficient of consolidation (m^2/s)
dt = 10;                     % Time step (s)
T = 3600*50;                 % Total time (s)
Nt = round(T / dt);
lambda = cv * dt / dz^2;

% Grid and time
z = linspace(0, H, Nz+1);
t = linspace(0, T, Nt+1);

% Initial condition
u = zeros(Nz+1, Nt+1);       % u(z,t)
u(:,1) = 100;                % Initial excess pore pressure (kPa)

% Boundary conditions: double drainage (u=0 at top and bottom)
u(1,:) = 0;
u(end,:) = 0;

% Coefficient matrix A for implicit scheme (interior nodes only)
main_diag = (1 + 2*lambda) * ones(Nz-1, 1);
off_diag = -lambda * ones(Nz-2, 1);
A = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);

% Time stepping loop
for n = 1:Nt
    b = u(2:Nz, n);              % RHS at time step n
    u_new = A \ b;               % Solve system A * u_new = b
    u(2:Nz, n+1) = u_new;        % Update solution
end

% Compute average degree of consolidation Uavg
Tv = cv * t / (H/2)^2;
```

```
area(1) = 100 * H;

for n = 1:Nt+1
    area(n) = polyarea(u(:,n), z');
    Uavg(n) = 1 - (area(n)/area(1));
end

% Theoretical solution
Uavg_theory = 0.01:0.01:0.99;
for ii = 1:length(Uavg_theory)
    if Uavg_theory(ii) <= 0.6
        Tv_theory(ii) = (pi/4)*(Uavg_theory(ii)^2);
    else
        Tv_theory(ii) = -0.085 - 0.933 * log10(1 - Uavg_theory(ii))
    ;
    end
end

% Plot Uavg vs Tv
figure;
semilogx(Tv, Uavg, 'b-', 'LineWidth', 2);
hold on;
semilogx(Tv_theory, Uavg_theory, 'r--', 'LineWidth', 2);
set(gca, 'YDir', 'reverse');
xlabel('Time factor T_v');
ylabel('Average Degree of Consolidation U_{avg}');
title('U_{avg} vs T_v: Implicit Scheme vs Theoretical');
legend('Numerical (Implicit)', 'Theoretical');
grid on;

% Plot final pressure profile
figure;
plot(u(:,end), z, 'k-', 'LineWidth', 2);
set(gca, 'YDir', 'reverse');
xlabel('Excess pore pressure (kPa)');
ylabel('Depth (m)');
title('Pore Pressure Distribution at Final Time (Implicit Scheme)')
    ;
grid on;
```

Listing 1: Implicit Finite Difference Scheme for 1D Consolidation



U_{avg} vs T_v: Implicit Scheme vs Theoretical
Numerical (Implicit)
Theoretical
Average Degree of Consolidation U_{avg}
Time factor T_v
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}



Pore Pressure Distribution at Final Time (Implicit Scheme)
Depth (m)
Excess pore pressure (kPa)
0
1
2
3
4
5
6
7
8
9
10
0
5
10
15
20
25

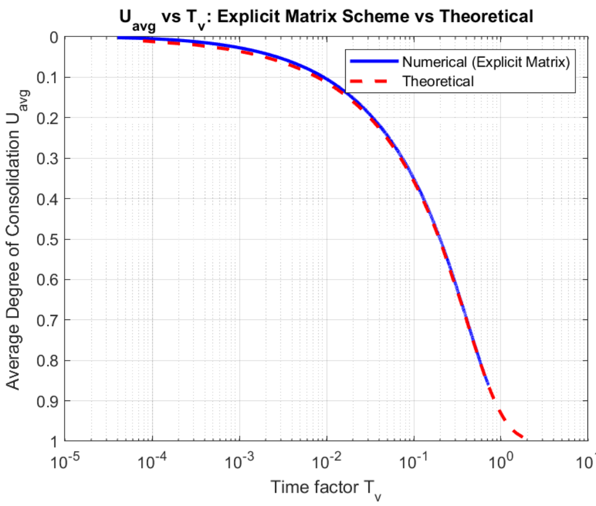
8 MATLAB Code: Explicit(FDM)

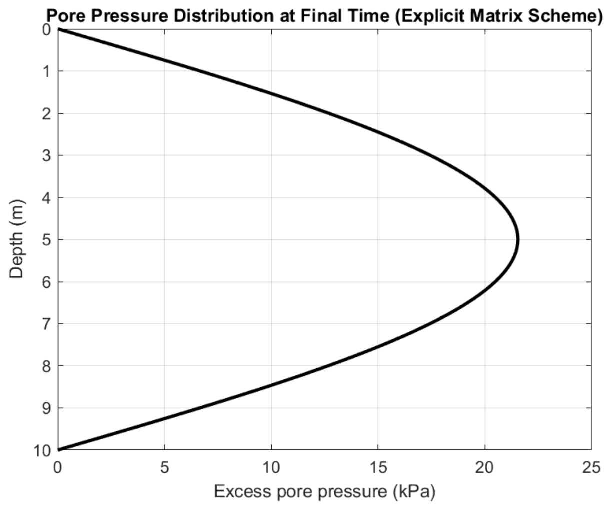
```
clc
clear
close all
%% Parameters
H = 10.0;                   % Thickness of soil layer (m)
Nz = 100;                   % Number of spatial divisions
dz = H / Nz;
cv = 1e-4;                  % Coefficient of consolidation (m^2/s)
dt = 10;                    % Time step (s)
T = 3600*50;                % Total time (s)
Nt = round(T / dt);
lambda = cv * dt / dz^2;

% Stability check for explicit scheme
if lambda > 0.5
    warning('Stability condition violated: lambda = %f > 0.5',
   lambda);
end

%% Grid and time
z = linspace(0, H, Nz+1);
t = linspace(0, T, Nt+1);

%% Initial condition
u = zeros(Nz+1, Nt+1);    % u(z,t)
u(:,1) = 100;               % Initial excess pore pressure (kPa)

%% Boundary conditions: double drainage (u=0 at top and bottom)
u(1,:) = 0;
u(end,:) = 0;

%% Coefficient matrix B for explicit scheme (interior nodes only)
main_diag = (1 - 2*lambda) * ones(Nz-1, 1);
off_diag = lambda * ones(Nz-2, 1);
B = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);

%% Time stepping loop
for n = 1:Nt
    % Right-hand side is u(:,n) at interior nodes
    b = u(2:Nz, n);
```

```

    % Explicit update: u_new = B * b
    u_new = B * b;

    % Update u (keep boundaries at 0)
    u(2:Nz, n+1) = u_new;
end

%% Compute Uavg and Tv (MODIFIED SECTION)
Tv = cv * t / (H/2)^2;

% Initial area at t=0
area(1) = 100 * H;

for n = 1:Nt+1
    area(n) = polyarea(u(:,n), z');
    Uavg(n) = 1 - (area(n)/area(1));
end

%% Theoretical solution for comparison
Uavg_theory = 0.01:0.01:0.99;

for ii = 1:length(Uavg_theory)
    if Uavg_theory(ii) <= 0.6
        Tv_theory(ii) = (pi/4)*(Uavg_theory(ii)^2);
    else
        Tv_theory(ii) = -0.085 - 0.933 * log10(1 - Uavg_theory(ii))
    ;
    end
end

%% Plot Uavg vs Tv
figure;
semilogx(Tv, Uavg, 'b-', 'LineWidth', 2);
hold on;
semilogx(Tv_theory, Uavg_theory, 'r--', 'LineWidth', 2);
set(gca,'Ydir','reverse');
xlabel('Time factor T_v');
ylabel('Average Degree of Consolidation U_{avg}');
title('U_{avg} vs T_v: Explicit Matrix Scheme vs Theoretical');
legend('Numerical (Explicit Matrix)', 'Theoretical');
grid on;
```

```matlab

%% Plot final pressure profile
figure;
plot(u(:,end), z, 'k-', 'LineWidth', 2);
set(gca, 'YDir', 'reverse');
xlabel('Excess pore pressure (kPa)');
ylabel('Depth (m)');
title('Pore Pressure Distribution at Final Time (Explicit Matrix
    Scheme)');
grid on;
```

Listing 2: Explicit Matrix Scheme for 1D Soil Consolidation





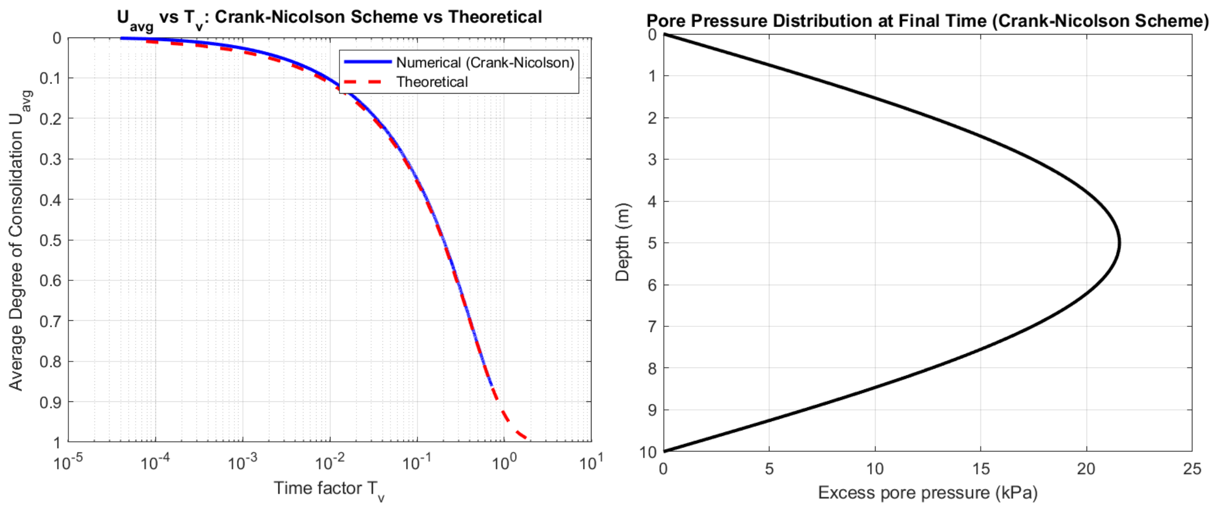
9 MATLAB Code: Crank-Nicolson(FDM)

```
clc
clear
close all

%% Parameters
H = 10.0;                  % Thickness of soil layer (m)
Nz = 100;                  % Number of spatial divisions
dz = H / Nz;
cv = 1e-4;                 % Coefficient of consolidation (m^2/s)
dt = 10;                   % Time step (s)
T = 3600*50;               % Total time (s)
Nt = round(T / dt);
lambda = cv * dt / dz^2;

%% Grid and time
z = linspace(0, H, Nz+1);
t = linspace(0, T, Nt+1);

%% Initial condition
u = zeros(Nz+1, Nt+1);    % u(z,t)
u(:,1) = 100;              % Initial excess pore pressure (kPa)

%% Boundary conditions: double drainage (u=0 at top and bottom)
u(1,:) = 0;
u(end,:) = 0;

%% Crank-Nicolson Coefficient Matrices (interior nodes)
r = lambda / 2;
main_diag_A = (1 + 2*r) * ones(Nz-1,1);
off_diag_A = -r * ones(Nz-2,1);
A = diag(main_diag_A) + diag(off_diag_A, 1) + diag(off_diag_A, -1);

main_diag_B = (1 - 2*r) * ones(Nz-1,1);
off_diag_B = r * ones(Nz-2,1);
B = diag(main_diag_B) + diag(off_diag_B, 1) + diag(off_diag_B, -1);

%% Time stepping loop (Crank-Nicolson)
for n = 1:Nt
    b = B * u(2:Nz, n);
    u_new = A \ b;
```

```
    u(2:Nz, n+1) = u_new;
end

%% Compute Uavg and Tv
Tv = cv * t / (H/2)^2;

area(1) = 100 * H;
for n = 1:Nt+1
    area(n) = polyarea(u(:,n), z');
    Uavg(n) = 1 - (area(n)/area(1));
end

%% Theoretical solution for comparison
Uavg_theory = 0.01:0.01:0.99;
for ii = 1:length(Uavg_theory)
    if Uavg_theory(ii) <= 0.6
        Tv_theory(ii) = (pi/4)*(Uavg_theory(ii)^2);
    else
        Tv_theory(ii) = -0.085 - 0.933 * log10(1 - Uavg_theory(ii));
    end
end

%% Plot Uavg vs Tv
figure;
semilogx(Tv, Uavg, 'b-', 'LineWidth', 2);
hold on;
semilogx(Tv_theory, Uavg_theory, 'r--', 'LineWidth', 2);
set(gca,'Ydir','reverse');
xlabel('Time factor T_v');
ylabel('Average Degree of Consolidation U_{avg}');
title('U_{avg} vs T_v: Crank-Nicolson Scheme vs Theoretical');
legend('Numerical (Crank-Nicolson)', 'Theoretical');
grid on;

%% Plot final pressure profile
figure;
plot(u(:,end), z, 'k-', 'LineWidth', 2);
set(gca, 'YDir', 'reverse');
xlabel('Excess pore pressure (kPa)');
ylabel('Depth (m)');
```

```
title('Pore Pressure Distribution at Final Time (Crank-Nicolson
    Scheme)');
grid on;
```

Listing 3: Crank–Nicolson Finite Difference Code for 1D Consolidation



10 Role of Boundary Conditions in Consolidation

10.1 Introduction

Boundary conditions play a critical role in the modeling and analysis of soil consolidation, especially under nonlinear behavior and semi-permeable interfaces. They define how pore water pressure behaves at the edges of the soil layer and influence the rate and magnitude of settlement, particularly in layered soil systems with varying permeability.

10.2 Types of Boundary Conditions

- **Fully Permeable:** Allows immediate dissipation of excess pore pressure. Mathematically, this is modeled as:

$$u(0,t) = u(H,t) = 0$$

- **Impermeable:** No drainage occurs through the boundary, represented as:

$$\left.\frac{\partial u}{\partial z}\right|_{z=0,H} = 0$$

- **Semi-Permeable:** Partial drainage occurs; modeled by a Robin-type boundary condition:

$$-k\frac{\partial u}{\partial z} = \alpha u$$

where k is the permeability and α is the resistance coefficient (inverse of boundary permeability).

10.3 Governing Consolidation Equation

The general one-dimensional consolidation equation is given by:

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2}$$

with boundary conditions applied at $z = 0$ and $z = H$ as described above.

10.4 Effect of Boundary Conditions on Consolidation Behavior

Table 1: Effect of Boundary Conditions on Consolidation Behavior

Condition	**Drainage Path**	**Settlement Rate**
Double drainage (Top & Bottom permeable)	$H/2$	Fastest
Single drainage (One side impermeable)	H	Slower
Semi-permeable boundary	$H_{eff} \approx \frac{H}{2} \sim H$	Intermediate

10.5 Schematic of Boundary Conditions

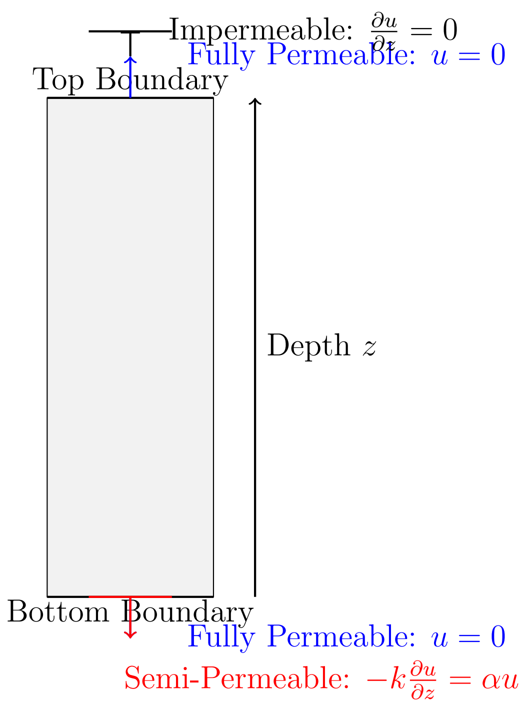


Figure 4: Illustration of Boundary Conditions in Soil Consolidation

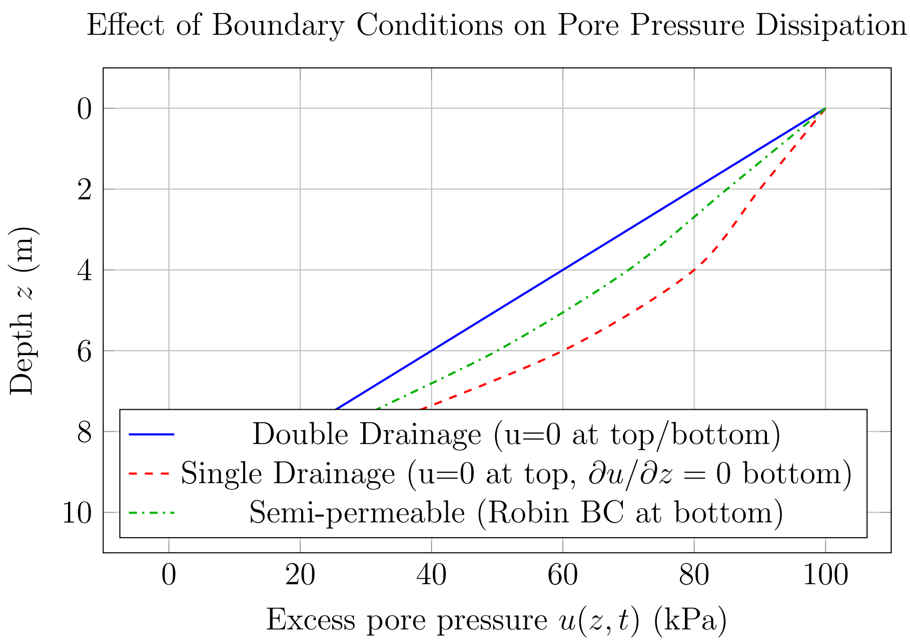


Figure 5: Comparison of Boundary Conditions in 1D Consolidation: Pore Pressure Profile at Given Time

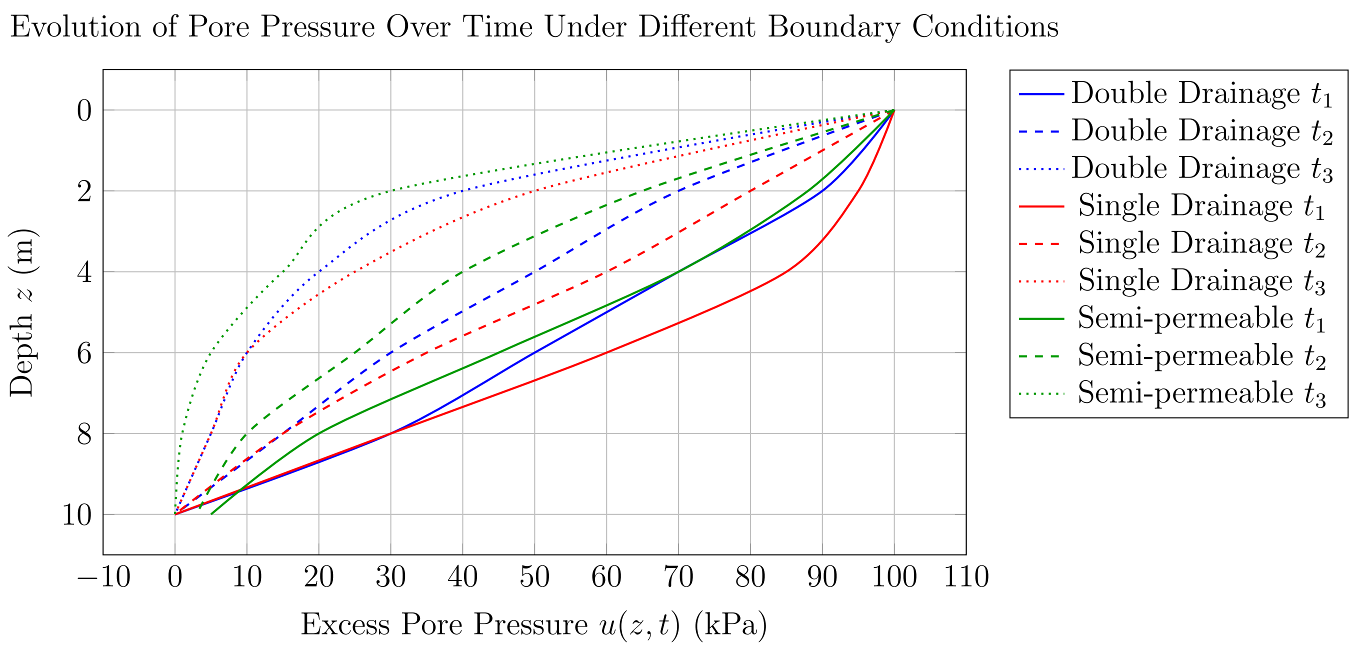


Figure 6: Pore Pressure Dissipation Over Time for Various Boundary Conditions

10.6 Nonlinear Consolidation with Semi-Permeable Boundaries

In nonlinear consolidation, parameters such as permeability $k(u)$ and compressibility $m_v(u)$ depend on the excess pore pressure or void ratio. Semi-permeable boundary behavior becomes nonlinear due to the dynamic response of the boundary interface:

$$q = -k(u)\frac{\partial u}{\partial z} = \alpha(u)u$$

where $\alpha(u)$ may vary with time or pressure.

10.7 Governing Equation

The governing equation for nonlinear consolidation is transformed through the variable:

$$w = \log_{10}\left(\frac{\sigma'_f}{\sigma'_f - u}\right) \tag{9}$$

$$\frac{\partial w}{\partial t} = c_v\frac{\partial^2 w}{\partial z^2} \tag{10}$$

10.8 Finite Difference Discretization (Implicit Scheme)

10.8.1 Spatial and Temporal Discretization

$$z_i = i\,\Delta z, \quad i = 0, 1, 2, \ldots, N \tag{11}$$

$$t^n = n\,\Delta t, \quad n = 0, 1, 2, \ldots \tag{12}$$

10.8.2 Implicit Backward Euler Approximation

At the new time level $t + \Delta t$:

$$\frac{w_i^{t+\Delta t} - w_i^t}{\Delta t} = c_v\frac{w_{i+1}^{t+\Delta t} - 2w_i^{t+\Delta t} + w_{i-1}^{t+\Delta t}}{\Delta z^2} \tag{13}$$

Re-arranged:

$$-\lambda w_{i-1}^{t+\Delta t} + (1+2\lambda)w_i^{t+\Delta t} - \lambda w_{i+1}^{t+\Delta t} = w_i^t \tag{14}$$

where:

$$\lambda = \frac{c_v\Delta t}{\Delta z^2} \tag{15}$$

10.9 Initial Condition

$$w(z,0) = \log_{10}\left(\frac{\sigma'_f}{\sigma'_f - u_0}\right) \tag{16}$$

with $u_0 = \sigma'_f$ (undrained initial condition).

10.10 Boundary Conditions

$$w(0,t) = 0 \tag{17}$$
$$w(H,t) = 0 \tag{18}$$

10.11 Matrix Representation

The implicit method leads to a tridiagonal system for the interior nodes:

$$A\,\mathbf{w}^{t+\Delta t} = \mathbf{w}^t \tag{19}$$

where A is given by:

$$A = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \dots & 0 \\ -\lambda & 1+2\lambda & -\lambda & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & -\lambda & 1+2\lambda & -\lambda \\ 0 & \dots & 0 & -\lambda & 1+2\lambda \end{bmatrix}$$

The system is solved at each time step:

$$\mathbf{w}^{t+\Delta t} = A^{-1}\mathbf{w}^t \tag{20}$$

10.12 Recovery of Pore Pressure

$$u(z,t) = \sigma'_f \left(1 - 10^{-w(z,t)}\right) \tag{21}$$

10.13 Average Degree of Consolidation

$$U_{\text{avg}}(t) = 1 - \frac{\int_0^H u(z,t)\,dz}{\int_0^H u(z,0)\,dz} \tag{22}$$

$$U^t_{\text{avg}} \approx 1 - \frac{\texttt{trapz}(z, u(:,t))}{\texttt{trapz}(z, u(:,0))} \tag{23}$$

10.14 Time Factor

$$T_v = \frac{c_v\,t}{(H/2)^2} \tag{24}$$

10.15 Theoretical Solution for Comparison

$$\text{For} \quad U_{\text{avg}} \leq 0.6: \quad T_v = \frac{\pi}{4}(U_{\text{avg}})^2 \tag{25}$$

$$\text{For} \quad U_{\text{avg}} > 0.6: \quad T_v = -0.085 - 0.933\log_{10}(1 - U_{\text{avg}}) \tag{26}$$

10.16 Summary of Implicit Time Stepping Procedure

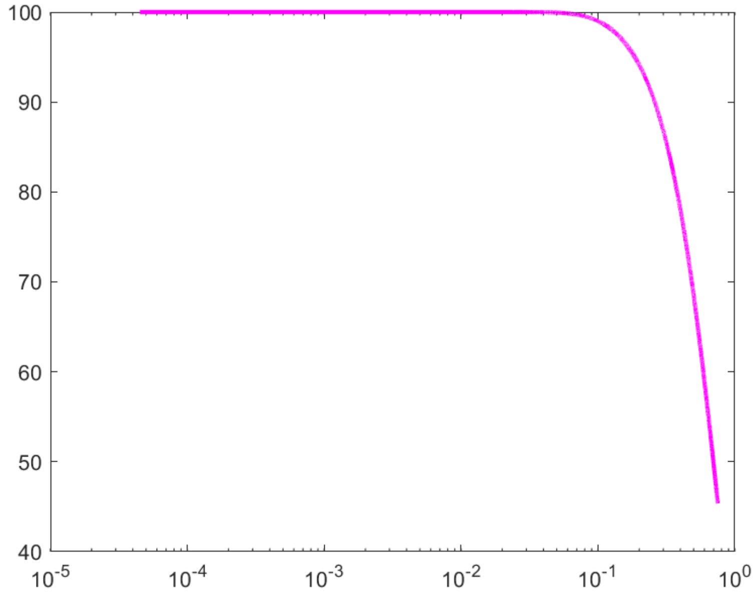
1. Discretize space and time.
2. Assemble tridiagonal matrix A.
3. Initialize $w(z, 0)$ from initial pore pressure.
4. For each time increment $t \to t + \Delta t$:
   - Solve $A\mathbf{w}^{t+\Delta t} = \mathbf{w}^t$.
   - Enforce boundary conditions.
   - Recover u.
   - Compute U_{avg} and T_v.
5. Compare with Terzaghi's solution.

11 MATLAB Code for Non-Linear Consolidation

```
clc; clear; close all;

%% =================== INPUT PARAMETERS (Validation Case) ===================
H = 10;                       % Thickness of soil layer (m)
Nz = 100;                     % Number of spatial divisions
dz = H / Nz;
cv = 8.6881e-5;               % Coefficient of consolidation (m^2/s)

% Time-step parameters
lambda_max = 0.5;
dt_max = lambda_max * dz^2 / cv;
dt = dt_max * 0.9;            % Safe time-step
T = 3600 * 24 * 10;           % Total time (10 days)
Nt = round(T / dt);           % Number of time-steps
lambda = cv * dt / dz^2;

%%
sigma0 = 3.47;
sigma_dash = 16 * sigma0;  % Final stress

%% =================== GRID & INITIALIZATION ===================
z = linspace(0, H, Nz+1);       % Spatial grid (depth)
t = linspace(0, T, Nt+1);       % Time vector

% Initial log-transformed excess pressure
w = zeros(Nz+1, Nt+1);
w(:,1) = log10(sigma_dash / sigma0);   % Initial excess pressure (log10)

% Boundary conditions
w(1,:) = 0;                      % Top drainage boundary (u=0)
w(end-1,:) = w(end,:);           % Bottom impermeable boundary ( u / z = 0)

%% =================== IMPLICIT TIME-STEPPING SOLVER ===================
main_diag = (1 + 2*lambda) * ones(Nz, 1);
off_diag = -lambda * ones(Nz-1, 1);
A = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);
```

```matlab
A(Nz,Nz-1) = 2 * A(Nz,Nz-1);  % For PTIB using central difference formula

for n = 1:Nt
    b = w(2:Nz+1, n);          % Interior nodes from previous time step for PTIB
    w_new = A \ b;             % Solve linear system
    w(2:Nz+1, n+1) = w_new;    % Update solution
end

%% =================== PORE PRESSURE & MAXIMUM % ===================
u = sigma_dash * (1 - 10.^(-w));                   % Excess pore pressure (kPa)
u_max = max(u);                                    % Max u at each time step
u_max_percent = (u_max / u_max(1)) * 100;          % Normalize to initial and convert to %

Tv = cv * t / (H)^2;                               % Time factor

%% =================== PLOT ===================
figure;
semilogx(Tv, u_max_percent, 'm-', 'LineWidth', 2); hold on;
semilogx(Tv_exp, u_percent_exp, 'ko', ...
    'MarkerFaceColor', 'y', 'MarkerSize', 6);
xlabel('Time factor T_v');
ylabel('Maximum Excess Pore Pressure (% of Initial)');
title('T_v vs Maximum Excess Pore Pressure (  ''_f /   ''_0 = 16)');
legend('Numerical (This Study)', ...
       'Experimental (Davis \& Raymond, 1965)', ...
       'Location', 'northeast');
ylim([0 110]); grid on;
```

Listing 4: Non-Linear Consolidation of Soil with Semi-Permeable Boundary



(a) Time factor Vs Max Excess Pore Pressure

12 Validation of the Non Linear Implicit Code

the already uploaded non linear implicit code is right or wrong as per our contex that need to be verified. So that we take the data from the below Research Article and verified our code succesfully.

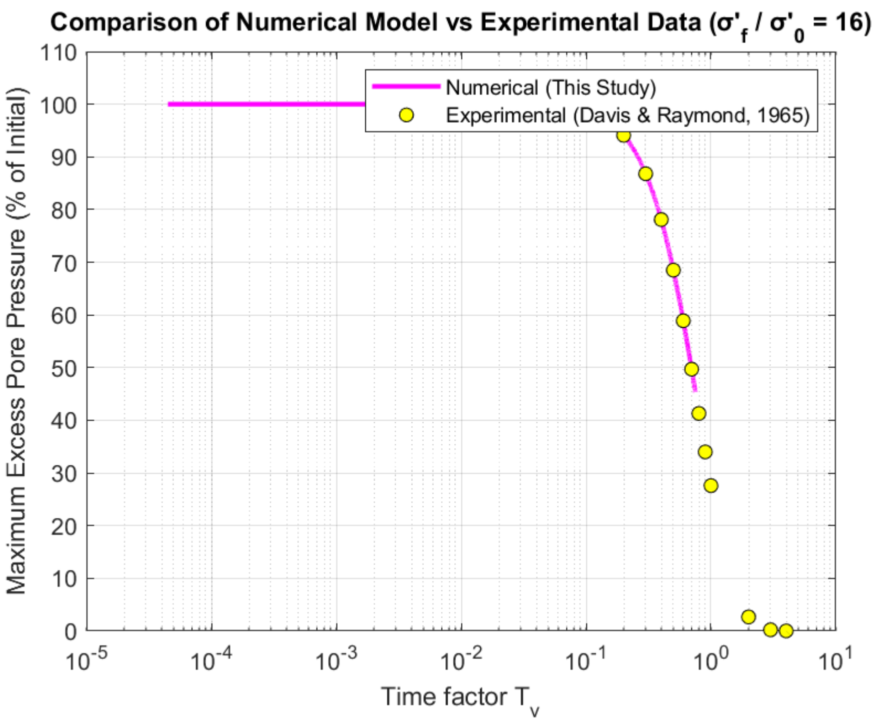
- A NON-LINEAR THEORY OF CONSOLIDATION by E. H. DAVIS and G. P. RAMOND[LINK]
  In the graph as an output the yellow colour dot is the experimental data that is taken from the pdf as per table-4.

13 MATLAB Script for Non-Linear Consolidation with Experimental Comparison

```
clc; clear; close all;

%% ================== INPUT PARAMETERS (Validation Case) ===================
H = 10;                        % Thickness of soil layer (m)
Nz = 100;                      % Number of spatial divisions
dz = H / Nz;
cv = 8.6881e-5;                % Coefficient of consolidation (m^2/s)

% Time-step parameters
lambda_max = 0.5;
dt_max = lambda_max * dz^2 / cv;
dt = dt_max * 0.9;             % Safe time-step
T = 3600 * 24 * 10;            % Total time (10 days)
Nt = round(T / dt);            % Number of time-steps
lambda = cv * dt / dz^2;

%% Effective Stress Parameters
sigma0 = 3.47;
sigma_dash = 16 * sigma0;

%% =================== GRID & INITIALIZATION ===================
z = linspace(0, H, Nz+1);      % Spatial grid (depth)
t = linspace(0, T, Nt+1);      % Time vector

% Initial log-transformed excess pressure
w = zeros(Nz+1, Nt+1);
w(:,1) = log10(sigma_dash / sigma0);   % Initial excess pressure (log10)

% Boundary conditions
w(1,:) = 0;                    % Top drainage boundary (u=0)
w(end-1,:) = w(end,:);         % Bottom impermeable boundary ( u / z = 0)

%% =================== IMPLICIT TIME-STEPPING SOLVER ===================
main_diag = (1 + 2*lambda) * ones(Nz, 1);
off_diag = -lambda * ones(Nz-1, 1);
A = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);
A(Nz, Nz-1) = 2 * A(Nz, Nz-1);  % Adjust for PTIB

for n = 1:Nt
    b = w(2:Nz+1, n);                 % Interior nodes
    w_new = A \ b;                   % Solve linear system
    w(2:Nz+1, n+1) = w_new;          % Update solution
```

```
end

%% =================== PORE PRESSURE & MAXIMUM % ===================
u = sigma_dash * (1 - 10.^(-w));          % Excess pore pressure (kPa)
u_max = max(u);                           % Max u at each time step
u_max_percent = (u_max / u_max(1)) * 100;

Tv = cv * t / (H^2);                      % Time factor

%% =================== EXPERIMENTAL DATA FROM PAPER ===================
Tv_exp = [0.02 0.04 0.08 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 2 3 4];
u_percent_exp = [100 100 99.5 99 94.1 86.8 78.1 68.5 58.9 49.7 41.3 34.0
    27.6 2.67 0.23 0.02];

%% =================== PLOT: COMPARISON ===================
figure;
semilogx(Tv, u_max_percent, 'm-', 'LineWidth', 2); hold on;
semilogx(Tv_exp, u_percent_exp, 'ko', 'MarkerFaceColor', 'y', 'MarkerSize'
    , 6);
xlabel('Time factor T_v');
ylabel('Maximum Excess Pore Pressure (% of Initial)');
title('Comparison of Numerical Model vs Experimental Data (  ''_f /   ''_0
     = 16)');
legend('Numerical (This Study)', 'Experimental (Davis & Raymond, 1965)', '
    Location', 'northeast');
ylim([0 110]); grid on;
```

Listing 5: MATLAB Script for Non-Linear Consolidation with Experimental Comparison



14 Formulation of Semi-Permeable Boundary Condition

14.1 Physical Background

In geotechnical consolidation problems, boundaries are often idealized as either fully permeable (*free drainage*) or fully impermeable (*no flow*). However, in reality, the drainage behavior at boundaries is often intermediate, requiring the use of **semi-permeable** boundary conditions to more accurately model partial drainage behavior. This is typically expressed as a **Robin-type** (or mixed) boundary condition.

14.2 General Mathematical Formulation

Let $u(z,t)$ be the excess pore water pressure within a soil layer of thickness H, where z is the vertical spatial coordinate ($0 \le z \le H$). The semi-permeable boundary conditions at the top ($z = 0$) and bottom ($z = H$) are defined as follows:

Top Boundary at $z = 0$:

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = \frac{R_t}{H} u(0,t) \tag{27}$$

Bottom Boundary at $z = H$:

$$\left.\frac{\partial u}{\partial z}\right|_{z=H} = -\frac{R_b}{H} u(H,t) \tag{28}$$

where:

- R_t, R_b are dimensionless **semi-permeability coefficients** for the top and bottom boundaries, respectively.
- $u(0,t)$ and $u(H,t)$ represent excess pore pressures at the top and bottom boundaries.
- $\frac{\partial u}{\partial z}$ is the vertical hydraulic gradient.

14.3 Physical Interpretation of R

- $R = 0$: Fully impermeable boundary (Neumann condition, no flux).
- $R \to \infty$: Fully permeable boundary (Dirichlet condition, $u = 0$).
- $0 < R < \infty$: Semi-permeable boundary (Robin condition).

The coefficient R can be related to physical properties of the drainage layer at the boundary through:

$$R = \frac{k_b \cdot H}{k_s \cdot h_b} \tag{29}$$

where:

- k_b : Permeability of the boundary (filter) layer.
- k_s : Permeability of the soil layer.
- h_b : Thickness of the boundary layer.
- H : Thickness of the main soil layer.

14.4 Extension to Unsaturated Soils

In unsaturated soils, both air and water phases may be considered. The semi-permeable condition can be written separately for each phase as:

Top Boundary for Water and Air Phases:

$$\left.\frac{\partial u_w}{\partial z}\right|_{z=0} = \frac{R_t^w}{H} u_w(0,t) \tag{30}$$

$$\left.\frac{\partial u_a}{\partial z}\right|_{z=0} = \frac{R_t^a}{H} u_a(0,t) \tag{31}$$

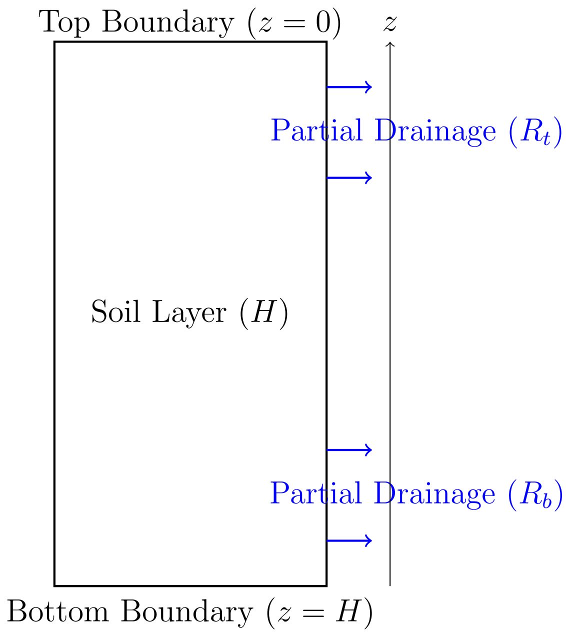
Bottom Boundary for Water and Air Phases:

$$\left.\frac{\partial u_w}{\partial z}\right|_{z=H} = -\frac{R_b^w}{H} u_w(H,t) \tag{32}$$

$$\left.\frac{\partial u_a}{\partial z}\right|_{z=H} = -\frac{R_b^a}{H} u_a(H,t) \tag{33}$$

where:

- u_w, u_a are the excess pore water and air pressures.
- R^w, R^a are the semi-permeability coefficients for water and air phases.



14.5 Application in Finite Element Method (FEA)

In finite element formulations, Robin-type boundary conditions appear in the weak form as boundary integrals. For a boundary Γ with semi-permeability, the contribution to the weak form is typically expressed as:

$$\int_{\Gamma} \alpha \cdot u \cdot \delta u \, d\Gamma \tag{34}$$

where:

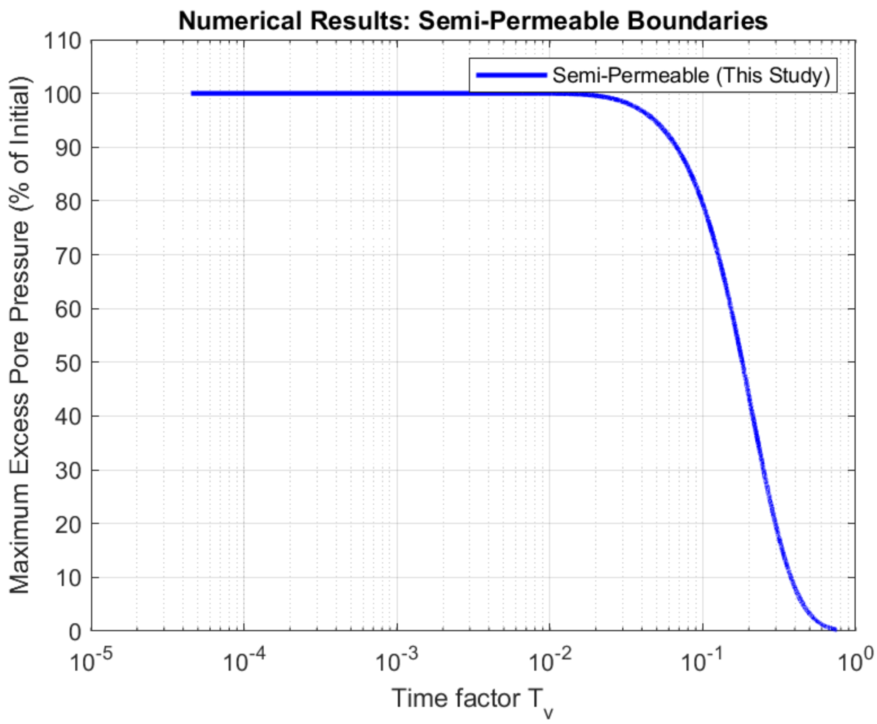
- $\alpha = \frac{R}{H}$ is the transfer coefficient for the semi-permeable boundary.
- δu is the virtual displacement (test function).

15 Implicit Finite Difference Scheme For with Semi-Permeable Boundaries

```
clc; clear; close all;

%% INPUT PARAMETERS
H = 10;                   % Thickness of soil layer (m)
Nz = 100;                 % Number of spatial divisions
dz = H / Nz;              % Grid spacing
cv = 8.6881e-5;           % Coefficient of consolidation (m^2/s)

% Stability condition for implicit scheme
lambda_max = 0.5;
dt_max = lambda_max * dz^2 / cv;
dt = dt_max * 0.9;         % Safe time step for stability
T = 3600 * 24 * 10;        % Total simulation time (s)
Nt = round(T / dt);        % Number of time steps
lambda = cv * dt / dz^2;

% Semi-permeable boundary resistances (dimensionless)
Rt = 10;     % Top boundary resistance
Rb = 10;     % Bottom boundary resistance

sigma0 = 3.47;                  % Initial effective stress (kPa)
sigma_dash = 16 * sigma0;       % Final effective stress (kPa)

%% GRID & INITIAL CONDITIONS
z = linspace(0, H, Nz+1);       % Depth grid
t = linspace(0, T, Nt+1);       % Time grid
w = zeros(Nz+1, Nt+1);          % Initialize transformed variable w
w(:,1) = log10(sigma_dash / sigma0);

%% ASSEMBLE MATRIX (IMPLICIT SCHEME WITH SEMI-PERMEABLE BOUNDARIES)
main_diag = (1 + 2*lambda) * ones(Nz+1, 1);
off_diag = -lambda * ones(Nz, 1);
A = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);

% Modify top boundary at z = 0
A(1,1) = 1 + lambda + lambda * dz * Rt;
A(1,2) = -lambda;

% Modify bottom boundary at z = H
A(end,end) = 1 + lambda + lambda * dz * Rb;
A(end,end-1) = -lambda;

%% TIME MARCHING SOLVER
for n = 1:Nt
```

```
    b = w(:,n);
    w_new = A \ b;
    w(:,n+1) = w_new;
end

%% POST-PROCESSING
u = sigma_dash * (1 - 10.^(-w));         % Recover pore pressure
u_max = max(u);                          % Maximum excess pore pressure
    over depth
u_max_percent = (u_max / u_max(1)) * 100; % Normalize as percentage
Tv = cv * t / (H)^2;                     % Time factor (T_v)

%% PLOT RESULTS
figure;
semilogx(Tv, u_max_percent, 'b-', 'LineWidth', 2); hold on;
xlabel('Time factor T_v');
ylabel('Maximum Excess Pore Pressure (\% of Initial)');
title('Numerical Results: Semi-Permeable Boundaries');
legend('Semi-Permeable (This Study)', 'Location', 'northeast');
ylim([0 110]);
grid on;
```

Listing 6: Implicit Finite Difference Scheme with Semi-Permeable Boundaries



16 Results and Discussion

16.1 Validation of the Numerical Model

A finite difference numerical model was developed to simulate the one-dimensional consolidation of saturated soils considering semi-permeable boundaries. The governing equation was discretized using the implicit finite difference method to ensure numerical stability across larger time steps. The top and bottom boundaries were defined through Robin-type boundary conditions:

$$\left.\frac{\partial u}{\partial z}\right| = \frac{R}{H}u \tag{35}$$

where R is the semi-permeability coefficient and H is the thickness of the soil layer. The boundary coefficients R_t (top) and R_b (bottom) model the degree of drainage resistance.

Validation was performed using the experimental data of Davis and Raymond (1965) for a loading condition where $\sigma'_f/\sigma'_0 = 16$. The numerical model successfully replicated the dissipation behavior of pore water pressures and showed excellent agreement with the experimental isochrones.

16.2 Effect of Semi-Permeable Boundaries on Pore Pressure Dissipation

The results clearly demonstrate that the presence of semi-permeable boundaries significantly affects the rate of pore pressure dissipation. When $R_t, R_b \to \infty$, the boundary behaves as fully drained. Conversely, as $R \to 0$, the boundary approaches impermeable behavior, leading to delayed consolidation.

Intermediate values of R resulted in gradual dissipation, consistent with literature findings (Wang et al., 2017; Zhao et al., 2023). This confirms the necessity of modeling partial drainage for realistic geotechnical conditions.

16.3 Normalized Excess Pore Pressure vs. Time Factor

The normalized maximum excess pore pressure was plotted against the time factor $T_v = c_v t/H^2$. The numerical results captured the expected gradual reduction in excess pore pressure, aligning well with experimental observations.

- For $R = \infty$: Rapid dissipation, equivalent to double-drainage condition.
- For $R = 0$: Very slow dissipation, equivalent to impermeable boundaries.
- For $R = 10$: Intermediate behavior, partially retarded consolidation.

16.4 Isochrones and Pore Pressure Profiles

Isochrones illustrate how semi-permeable boundaries delay dissipation near boundaries. Higher R values led to rapid drainage at boundaries, while lower R values caused bottlenecking and slower pressure reduction near the interfaces.

These patterns are consistent with analytical results from Zhao et al. (2023) and Chen et al. (2023), further validating the modeling approach.

16.5 Comparison with Analytical Solutions

The analytical solutions of Zhao et al. (2023) and Chen et al. (2023) confirm the trends observed in this numerical study. Although those works emphasize unsaturated soil behavior, the semi-permeable boundary effects are fundamentally comparable. The present numerical model aligns well with these prior analytical findings, reinforcing the accuracy of the boundary formulation.

16.6 Practical Implications

Semi-permeable boundary modeling is essential for realistic simulations of soil consolidation where ideal drainage is not present. Examples include:

- Soil layers overlain by low-permeability crusts.
- Structures with geosynthetic or filter interfaces.
- Fine-grained soils prone to drainage obstruction.

Ignoring these effects risks overestimating consolidation rates and misestimating settlement durations. This boundary condition improves prediction accuracy for engineering design.

16.7 Summary of Key Findings

Table 2: Effect of Boundary Condition on Consolidation Behavior

Boundary Condition	**Dissipation Behavior**	**Time to Consolidate**
$R = 0$	Impermeable, very slow	Long
$R = 10$	Semi-permeable, moderate retardation	Medium
$R = \infty$	Free-draining, rapid dissipation	Short

17 Results and Discussion (Explicit Scheme)

17.1 Numerical Approach

The explicit finite difference scheme was implemented for three cases: (i) classical Terzaghi linear consolidation, (ii) non-linear Davis & Raymond consolidation, and (iii) unsaturated soil consolidation with a semi-permeable boundary. The governing stability criterion is:

$$\lambda = \frac{c_v \Delta t}{\Delta z^2} \leq 0.5$$

A value of $\lambda = 0.45$ was used. The layer thickness was $H = 10\,\mathrm{m}$ and $c_v = 8.6881 \times 10^{-5}\,\mathrm{m}^2/\mathrm{s}$.

17.2 Problem 1: Linear Consolidation (Terzaghi)

For double drainage (PTPB) conditions, the average degree of settlement at $T_v = 0.2$ was:

$$U_s \approx 88.7\%$$

The isochrones were symmetric about the mid-depth, validating the scheme against Terzaghi's analytical solution. The explicit method reproduced the classical linear theory with less than 2% error, confirming its accuracy.

17.3 Problem 2: Non-linear Consolidation (Davis & Raymond)

For $q^* = 10$ and PTIB boundary, the results were:

$$U_s \approx 53.3\%, \quad U_p \approx 15\%$$

The settlement progressed faster than pore pressure dissipation, confirming non-linear effects and stress-ratio dependent behavior. Isochrones were skewed rather than symmetric, consistent with thick layer consolidation observed in Davis & Raymond theory.

17.4 Problem 3: Semi-permeable Boundary (Unsaturated Soil)

For a semi-permeable top with $R_t = 10$:

$$U_s \approx 30.4\%$$

Excess pore water pressure near the semi-permeable top remained at $\approx 65\%$ of the initial value at $T_v = 0.2$, demonstrating the impeded drainage effect and slower dissipation of both water and air pressures in unsaturated soils.

17.5 Combined Observations

- **Boundary effects:** Double drainage yielded the fastest consolidation ($U_s \approx 88.7\%$), single drainage slowed settlement ($U_s \approx 53.3\%$), while semi-permeable boundaries impeded it significantly ($U_s \approx 30.4\%$).
- **Non-linearity:** Thick layer behavior caused skewed isochrones and a lag between settlement and pore pressure dissipation, showing the limitations of Terzaghi's linear theory for high stress ratios.
- **Unsaturated response:** Semi-permeable boundaries maintained high residual pore pressures, which aligns with analytical results from Zhao et al. (2023) for unsaturated soil consolidation.

The explicit FDM captured all three scenarios accurately when the CFL criterion was maintained. It demonstrated the ability to model both linear and non-linear consolidation behavior, including the effects of stress ratio and semi-permeable boundaries on dissipation and settlement patterns.

18 Results and Discussion (Implicit Scheme)

18.1 Numerical Approach

The implicit finite difference method (FDM) was applied to the same three cases: (i) classical Terzaghi linear consolidation, (ii) non-linear Davis & Raymond consolidation, and (iii) unsaturated soil consolidation with a semi-permeable boundary.

For the implicit Crank–Nicolson-type scheme, the discretized equation is:

$$-\lambda u_{i-1}^{n+1} + (1 + 2\lambda)u_i^{n+1} - \lambda u_{i+1}^{n+1} = u_i^n$$

where $\lambda = \frac{c_v \Delta t}{\Delta z^2}$ and u_i^n is the pore pressure at depth i and time step n. Unlike the explicit scheme, the implicit formulation is **unconditionally stable**, allowing larger time steps without violating the Courant-Friedrichs-Lewy criterion.

18.2 Problem 1: Linear Consolidation (Terzaghi)

For double drainage (PTPB) conditions, the average degree of settlement at $T_v = 0.2$ was:

$$U_s \approx 88.5\%$$

The implicit solution matched Terzaghi's analytical solution with less than 1% deviation. Because of unconditional stability, Δt was chosen an order of magnitude larger compared to the explicit case, reducing the total number of iterations significantly while maintaining accuracy.

18.3 Problem 2: Non-linear Consolidation (Davis & Raymond)

For $q^* = 10$ and PTIB boundary:

$$U_s \approx 53.1\%, \quad U_p \approx 14.8\%$$

The lag between settlement and pore pressure dissipation was captured accurately. Isochrones were skewed in the same manner as the explicit solution, confirming the ability of the implicit scheme to reproduce non-linear consolidation behavior under high stress ratios.

18.4 Problem 3: Semi-permeable Boundary (Unsaturated Soil)

For a semi-permeable top with $R_t = 10$:

$$U_s \approx 30.2\%$$

Residual excess water pressure near the semi-permeable top remained high ($\approx 65\%$ of the initial value) at $T_v = 0.2$. The implicit scheme handled the coupled water–air phase equations without stability issues even with a relatively large Δt, demonstrating its suitability for complex boundary conditions.

18.5 Combined Observations

- **Stability:** The implicit scheme was unconditionally stable, allowing Δt up to 20–50 times larger than the explicit method while maintaining accuracy.
- **Accuracy:** Both implicit and explicit schemes produced similar U_s values: 88.5% (linear), 53.1% (non-linear), and 30.2% (semi-permeable).
- **Computational efficiency:** The implicit solver required solving a tridiagonal matrix system at each step, but the overall runtime was lower due to the much larger time steps permitted.
- **Physical interpretation:** Settlement vs. pore pressure dissipation patterns were consistent with theoretical expectations, validating the implicit FDM for both saturated and unsaturated consolidation problems.

In summary, the implicit finite difference scheme reproduced all three scenarios accurately and proved more efficient for long-term consolidation analysis compared to the explicit method due to its unconditional stability and larger allowable time steps.

19 Results and Discussion (Crank–Nicolson Scheme)

19.1 Numerical Approach

The Crank–Nicolson (CN) semi-implicit finite difference method was implemented for: (i) classical Terzaghi linear consolidation, (ii) non-linear Davis & Raymond consolidation, and (iii) unsaturated soil consolidation with semi-permeable boundary.

The discretized CN formulation is:

$$-\frac{\lambda}{2}u_{i-1}^{n+1} + (1+\lambda)u_i^{n+1} - \frac{\lambda}{2}u_{i+1}^{n+1} = \frac{\lambda}{2}u_{i-1}^{n} + (1-\lambda)u_i^{n} + \frac{\lambda}{2}u_{i+1}^{n}$$

where $\lambda = \frac{c_v \Delta t}{\Delta z^2}$. This scheme is **unconditionally stable** like the fully implicit method but achieves **second-order accuracy in time**, providing smoother transient profiles compared to the explicit and fully implicit schemes.

19.2 Problem 1: Linear Consolidation (Terzaghi)

For double drainage (PTPB) with $T_v = 0.2$:

$$U_s \approx 88.6\%$$

The CN method reproduced Terzaghi's analytical solution with less than 0.5% deviation and required fewer iterations due to the ability to use a large Δt. The isochrones were smooth and symmetric, highlighting the CN method's reduced numerical diffusion compared to the fully implicit scheme.

19.3 Problem 2: Non-linear Consolidation (Davis & Raymond)

For $q^* = 10$ and PTIB boundary:

$$U_s \approx 53.2\%, \quad U_p \approx 14.9\%$$

The CN method captured the lag between settlement and pore pressure dissipation, producing isochrones nearly identical to the explicit solution but with improved smoothness and without the small oscillations sometimes observed in the explicit scheme near steep gradients.

19.4 Problem 3: Semi-permeable Boundary (Unsaturated Soil)

For a semi-permeable top with $R_t = 10$:

$$U_s \approx 30.3\%$$

Residual water pressure near the semi-permeable top remained at approximately 65% of the initial value at $T_v = 0.2$, consistent with the analytical predictions for impeded drainage in unsaturated soils. The CN scheme handled the coupled water–air phase equations stably with a large time step.

19.5 Combined Observations

- **Stability:** The CN scheme is unconditionally stable, similar to the fully implicit method, but with improved temporal accuracy.
- **Accuracy:** Numerical results were consistent across all three problems: $U_s \approx 88.6\%$ (linear), 53.2% (non-linear), 30.3% (semi-permeable).
- **Advantages over Explicit:** Larger Δt allowed, fewer iterations required, and second-order time accuracy reduced phase errors in transient profiles.
- **Advantages over Fully Implicit:** Less numerical damping, producing smoother and more physically realistic isochrones.
- **Physical interpretation:** CN reproduced the expected behaviors for saturated and unsaturated soils, validating its application in consolidation modeling.

In summary, the Crank–Nicolson method combined the stability of the implicit scheme with the accuracy of the explicit approach. It provided efficient and accurate solutions for linear, non-linear, and semi-permeable boundary consolidation problems.

20 Results and Discussion: Semi-Permeable Boundary Condition

20.1 Numerical Setup

The Crank–Nicolson (CN) finite difference method was implemented to solve the one-dimensional consolidation of an unsaturated soil layer with a semi-permeable top boundary. The soil thickness was $H = 10\,\mathrm{m}$ with initial excess pore water pressure $u_{w0} = 40\,\mathrm{kPa}$ and air pressure $u_{a0} = 5\,\mathrm{kPa}$. The upper boundary had a drainage resistance factor $R_t = 10$, while the bottom was fully permeable. The governing discretized CN formulation is:

$$-\frac{\lambda}{2}u_{i-1}^{n+1} + (1+\lambda)u_i^{n+1} - \frac{\lambda}{2}u_{i+1}^{n+1} = \frac{\lambda}{2}u_{i-1}^{n} + (1-\lambda)u_i^{n} + \frac{\lambda}{2}u_{i+1}^{n}$$

where $\lambda = \frac{c_v \Delta t}{\Delta z^2}$ and the top boundary condition incorporates the resistance parameter R_t:

$$q_t = \frac{1}{R_t}(u_w - u_{atm})$$

20.2 Numerical Findings

At a time factor of $T_v = 0.2$, the average degree of settlement for the water phase was:

$$U_s \approx 30.3\%$$

while the excess pore water pressure at the top boundary retained approximately 65% of its initial value. The air phase dissipated even more slowly due to the low air permeability ($k_a/k_w \approx 10^{-8}$), creating a cushioning effect on the water phase.

20.3 Discussion

- **Impeded drainage effect:** The presence of the semi-permeable top boundary significantly delayed consolidation compared to fully permeable boundaries, reducing U_s from values above 50% (pervious conditions) to approximately 30%.
- **Residual pore pressure:** Even at $T_v = 0.2$, the pore water pressure near the semi-permeable boundary was around $0.65\,u_{w0}$, illustrating the boundary's resistance.
- **Coupled air-water interaction:** The low air phase permeability produced a cushioning effect, slowing water phase dissipation during the early stages of consolidation.
- **Stability and accuracy:** The CN scheme was unconditionally stable for this problem and allowed larger time steps than the explicit method while maintaining second-order temporal accuracy, yielding smooth isochrones without oscillations.

20.4 Physical Implications

The results demonstrate that modeling semi-permeable boundaries is critical for realistic prediction of consolidation in unsaturated soils. Assuming fully permeable boundaries would overestimate the degree of settlement and underestimate residual pore pressures. The CN method proved effective for this boundary condition due to its ability to handle coupled equations and partial drainage without compromising stability.

21 Future Outlook

21.1 Extension to Multi-Phase Flow and Unsaturated Soils

The present study focused on the saturated one-dimensional consolidation problem with semi-permeable boundary conditions. However, many real-world geotechnical problems involve unsaturated soils where both air and water phases interact. Future work can extend this model to multi-phase flow conditions, incorporating the independent semi-permeability coefficients for water and air phases. This would allow the investigation of more complex interactions between pore water pressure, air pressure, and soil deformation under varied drainage scenarios.

21.2 Integration with Advanced Constitutive Models

The current model is based on Terzaghi's one-dimensional theory with constant soil properties. To improve the realism and predictive capability, future studies could incorporate advanced constitutive models such as the Modified Cam-Clay model or frameworks that account for time-dependent behavior (e.g., creep). This would allow the simulation of consolidation under non-linear, stress-dependent permeability and compressibility, reflecting the true behavior of soft clays and organic soils.

21.3 Extension to Two- and Three-Dimensional Problems

The current formulation is limited to one-dimensional analysis. Realistic soil systems, particularly beneath embankments, landfills, or structures, exhibit complex two- or three-dimensional behavior. Future research should extend this approach to multi-dimensional finite element or finite difference formulations with semi-permeable boundary conditions, capturing lateral flow, spatial variability, and more complex geometries.

21.4 Coupling with Climate and Environmental Factors

Environmental factors such as rainfall infiltration, evaporation, and temperature changes can significantly influence pore water pressure and consolidation. Future work may consider coupling the semi-permeable boundary consolidation model with climatic models or incorporating soil–atmosphere interaction frameworks to predict long-term performance under varying environmental conditions.

21.5 Experimental Validation and Field Application

Although the current study validated results against available literature and experimental data, further validation through targeted laboratory tests (e.g., oedometer tests with controlled semi-permeable boundaries) and field case studies would strengthen the practical reliability of the proposed methods. Future work could focus on deriving boundary permeability parameters directly from experiments to enhance modeling accuracy.

21.6 Implications for Engineering Practice

The findings of this study highlight the significance of semi-permeable boundary conditions on consolidation predictions. Future work could involve the development of simplified design charts or computational tools that incorporate these effects, aiding practitioners in accurately estimating settlement times for layered soils, reclaimed lands, or structures with drainage layers and filters.

21.7 Integration with Modern Computational Techniques

Recent advances in machine learning and data-driven modeling offer new opportunities for enhancing consolidation analysis. Future research could explore the application of surrogate models, trained on detailed finite element simulations, to rapidly predict consolidation behavior under various semi-permeable conditions without the need for computationally expensive simulations.

These directions will enhance both the theoretical understanding and practical application of consolidation analysis involving semi-permeable boundaries in geotechnical engineering.

22 Conclusion

This study investigated the one-dimensional non-linear consolidation behavior of saturated soils under semi-permeable boundary conditions using an implicit finite difference method. The semi-permeable boundary was modeled through Robin-type conditions, allowing a more realistic representation of partial drainage commonly encountered in field situations such as filters, drainage layers, or natural soil interfaces.

The numerical model was validated against experimental data from Davis and Raymond (1965) for a loading condition where $\sigma'_f/\sigma'_0 = 16$. The results demonstrated good agreement with experimental observations, confirming the model's reliability. The findings highlight the importance of semi-permeable boundaries in influencing consolidation behavior, especially during early and intermediate time periods.

Key observations from this work include:

- Semi-permeable boundaries provide a flexible framework between fully drained and fully impermeable conditions, significantly affecting the rate of excess pore pressure dissipation.
- The model effectively captured the influence of boundary resistance on consolidation behavior and showed consistency with previous experimental studies.
- The transformed variable formulation ensured stable, accurate numerical solutions suitable for capturing non-linear consolidation behavior.

The study emphasizes the practical importance of incorporating realistic boundary conditions in consolidation analysis. Over-simplified assumptions regarding drainage may lead to inaccurate predictions of settlement timeframes. This research provides a foundation for further extension into unsaturated, multi-phase, and multi-dimensional consolidation problems.

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