Using Numerical Methods on Differential Equations Regarding Fish Population

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Introduction

This report will look into using numerical methods in systems of first-order ordinary differential equations to represent the fish population in a lake. The exact solution will be compared to approximations made using the forward Euler method [Bro22] and those made with the modified Euler method [Con23] on a single fish population. Its limiting behaviour will be analysed. Then, a system of ordinary differential equations representing two different fish coexisting will be solved using the modified Euler method. We will also look into how fishing may affect the fish populations in this ecosystem. Visualizations will be used to look further into these various situations.

(a) Small fish [sma]

(b) Big fish [big]

Figure 1: Examples of fish types

Small Fish Population Analysis

The growth rate of the small fish population without considering external factors is given by

$$\frac{du}{dt} = 9u - 9u^2, \quad u(0) = \frac{1}{10} \tag{1}$$

where u is population (one unit is ten thousand) and t is time (one unit is six months). $9u$ represents the growth rate while $-9u^2$ represents the rate of decline. To solve this first-order separable differential equation, we can rearrange it and apply partial fraction decomposition to the LHS to get

$$\int \frac{1}{9u - 9u^2} du \Longrightarrow \int \frac{1}{9u} du + \int \frac{1}{9(1-u)} du = \int dt$$

which gives us

$$\frac{1}{9} \ln|9u| - \frac{1}{9} \ln|1-u| = t + C.$$

This gives the following function of the small fish population in terms of time where

$$u(t) = \frac{81e^{9t+C}}{1 + 81e^{9t+C}}$$

where $C = \ln(\frac{1}{729})$. Using this function, one can determine the exact population of small fish at a given time. This allows us to find the limiting behaviour of the small fish population as $t \longrightarrow \infty$.

$$\lim_{t\to\infty} \frac{81e^{9t+C}}{1+81e^{9t+C}} \implies \lim_{t\to\infty} \frac{729e^{9t+C}}{729e^{9t+C}} = \lim_{t\to\infty} 1 = 1.$$

This tells us that the small fish population will converge to one population unit as time goes on. The limit had an indefinite form, "∞/∞", which allowed the usage of l'hopital's rule. While this is the exact solution, could it be possible to find a sufficient approximation? How accurate would this approximation be and would it be possible to use those numerical methods to solve more complicated systems of ODEs?

Solving Numerically

We will use the forward Euler and higher-order methods to numerically solve (1). The forward Euler method is given by the following iterative scheme

$$u_{n+1} = u_n + \tau \times \left.\frac{du}{dt}\right|_{u_n,t_n}. \tag{2}$$

This is akin to the left-hand rectangle rule to estimate areas underneath curves. The second-order method will be the modified Euler method based on the trapezoidal rule where

$$u_{n+1} = u_n + \frac{\tau}{2} \times \left(\left.\frac{du}{dt}\right|_{u_n,t_n} + \left.\frac{du}{dt}\right|_{u_{n+1},t_n+\tau} \right). \tag{3}$$

However, there is an issue. The formula uses u_{n+1} which is what needs to be calculated. This is where we will use a "predictor" in the form of the forward Euler method to approximate u_{n+1} and use it in (3) which will be our "corrector". Once coded in MATLAB, these are the results.

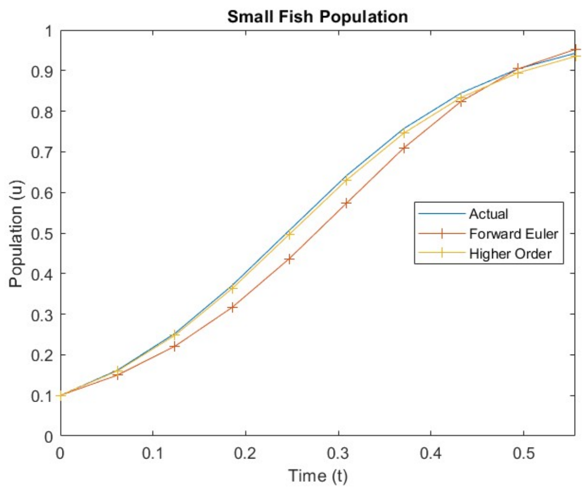

Figure 2: The higher-order approximation is more accurate for the same τ (0.0617 in this case).

These are the error values at $t = 5/9$ at different values of τ.

The forward Euler error seems to be better perhaps due to how the method takes time to catch up with the new derivative. This means the smaller error is merely a coincidence.

The rate of convergence of the error will tell us the order of our numerical method. The error is given by

$$\text{error}(N) = CN^{-p}$$

where error(N) is the error after N timesteps at a certain point, C is a constant and p is the order. Since we have two unknowns, C and p, two equations are required to solve for p. We will consider using N and

Error at $t = 5/9$			
τ	No. of Timesteps	Forward Euler	Modified Euler
0.1389	5	0.3205	0.5523
0.0617	10	0.0096	0.0080
0.0292	20	0.0040	0.0016
0.0142	40	0.0018	0.0004

Table 1: The modified Euler method error is better for most values of τ.

$2N$ timesteps which gives us $\text{error}(N) = CN^{-p}$ and $\text{error}(2N) = C(2N)^{-p}$. We can divide the errors to cancel C as we are not interested in that value which now allows us to focus on p.

$$\frac{\text{error}(2N)}{\text{error}(N)} = \frac{C(2N)^{-p}}{CN^{-p}} = 2^{-p} \Longrightarrow p = \log_2\left(\frac{\text{error}(N)}{\text{error}(2N)}\right) \tag{4}$$

If we calculate p for $N = 20$, we get $p \approx 1.15$ for forward Euler and $p = 2$, which is what we expected knowing ahead of time that forward Euler is first-order and the modified Euler method.

Larger Lake Analysis

Looking at a bigger lake, we see that there are two species of fish one of which predates the other. The behaviour of the total population can be described by

$$\frac{du}{dt} = 9u - 9u^2 - 18uv, \frac{dv}{dt} = 18\left(uv - \frac{v}{30}\right), \quad u(0) = 1, v(0) = \frac{1}{100}$$

where u is small fish population and v is big fish population and t is time. Looking at $\frac{du}{dt}$, the term '$-18uv$' represents the rate of predation which increases when both u and v increase. Looking at $\frac{dv}{dt}$, the term '$18uv$' is the growth rate which matches the predation rate in $\frac{du}{dt}$. The term '$-\frac{v}{30}$' represents the population decline due to natural causes such as old age, limited shelter, etc. of the larger fish which is lower than that of the small fish. This could be due to how the larger fish are more resilient to their surroundings.

To numerically solve this system of first-order nonlinear ODEs, the modified Euler method (3) will be used. It will be done similarly to the small fish population analysis from earlier, however, predictors will be made for both u_{n+1} and v_{n+1} and used in two separate correctors for each of them. This gives us the following plot.

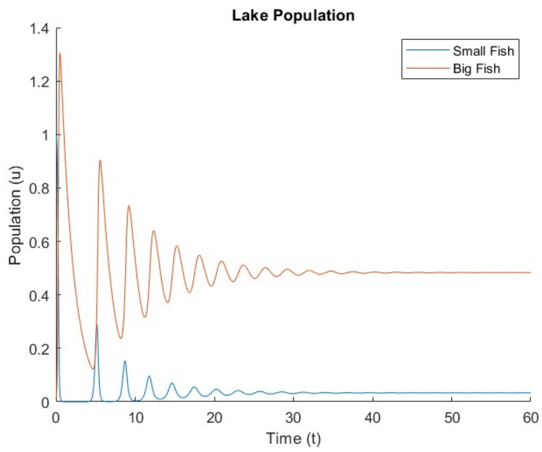

Figure 3: Both populations settle over time.

After looking further into the values, we find that the big fish population converges to 29/60 while the small fish population settles on 1/30.

Effect of Fishing

Say fishermen discovered the lake we were observing and decided to set up their business there. How can we include this in our system of ODEs and what fishing rate should be used so that the fish do not become extinct?

The new equation for the small fish population will be

$$\frac{du}{dt} = 9u - 9u^2 - 18uv - fu$$

where f is the fishing rate while $\frac{dv}{dt}$ stays the same. To find the maximum fishing rate at which our fish populations do not become extinct we employ trial and error to find the approximate value of f. This was found to be at $f = 8.65$ where v converges to around 0.0027 and u tends to 0.0333.

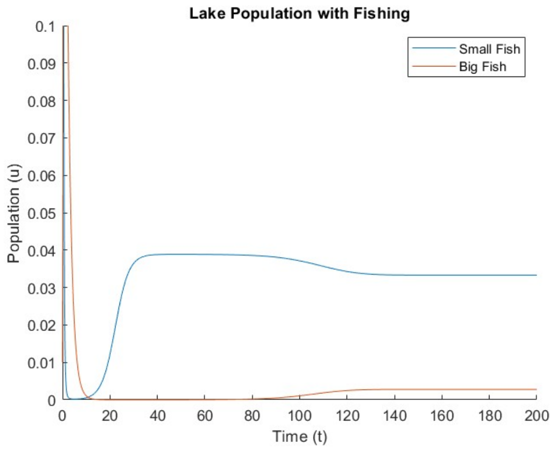

Figure 4: The small fish population is actually larger than that of the big fish.

The values of the populations go dangerously close to zero for a portion of the timeline, however, even with no fishing the populations cannot avoid going very close to zero. As a result, long-term behaviour is looked at instead. This now depends on how the population unit was defined. If one population unit meant one fish, nothing from the very beginning would make sense as a lot of the values are decimals and below one, so if our lake follows the system of ODEs that have been defined then our unit of population would have to be equal to one ten thousand or more.

Conclusion

Fish population can be analysed by looking at their ODE and its analytical solution, however, this is not always possible or could be overly convoluted. In these cases, numerical solutions can be used. Depending on the order of our numerical method we can get answers that are almost identical to the exact analytical solution with an error that is negligible. This was seen through the use of the modified Euler method which was only second order but provided a very accurate result nonetheless. We then leveraged this to numerically solve a system of nonlinear ODEs which would be very long-winded to solve analytically. We then used this to optimize the amount of fishing in the lake and observed the long-term behaviour of the fish populations.

References

[big] URL: `https://wallpapercave.com/w/wp6144875`.

[Bro22] Stuart Brorson, 2022. URL: `https://math.libretexts.org/Bookshelves/Differential_Equations/Numerically_Solving_Ordinary_Differential_Equations_(Brorson)/01%3A_Chapters/1.02%3A_Forward_Euler_method`.

[Con23] Wikipedia Contributors, 2023. URL: https://en.wikipedia.org/wiki/Heun%27s_method#References.

[sma] URL: https://www.publicdomainpictures.net/en/view-image.php?image=51519&picture=ryby-w-jeziorze-3&jazyk=PL.

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