This handout includes space for every question that requires a written response. Please feel free to use it to handwrite your solutions (legibly, please). If you choose to typeset your solutions, the `README.md` for this assignment includes instructions to regenerate this handout with your typeset LaTeX solutions.

1.a

Since $g'(z) = g(z)(1 - g(z))$ and $h(x) = g(\theta^T x)$, it follows that $\partial h(x)/\partial \theta_k = h(x)(1 - h(x))x_k$.
Letting $h_\theta(x^{(i)}) = g(\theta^T x^{(i)}) = 1/(1 + \exp(-\theta^T x^{(i)})$, we have

$$\frac{\partial \log h_\theta(x^{(i)})}{\partial \theta_k} =$$
$$\frac{\partial \log(1 - h_\theta(x^{(i)}))}{\partial \theta_k} =$$

Substituting into our equation for $J(\theta)$, we have

$$\frac{\partial J(\theta)}{\partial \theta_k} =$$

Consequently, the (k, l) entry of the Hessian is given by

$$H_{kl} = \frac{\partial^2 J(\theta)}{\partial \theta_k \partial \theta_l} =$$

Using the fact that $X_{ij} = x_i x_j$ if and only if $X = xx^T$, we have

$$H =$$

To prove that H is positive semi-definite, show $z^T H z \geq 0$ for all $z \in \mathbb{R}^d$.

$$z^T H z =$$

1.c

For shorthand, we let $\mathcal{H} = \{\phi, \Sigma, \mu_0, \mu_1\}$ denote the parameters for the problem. Since the given formulae are conditioned on y, use Bayes rule to get:

$$\begin{aligned} p(y=1|x;\mathcal{H}) &= \frac{p(x|y=1;\mathcal{H})p(y=1;\mathcal{H})}{p(x;\mathcal{H})} \\ &= \frac{p(x|y=1;\mathcal{H})p(y=1;\mathcal{H})}{p(x|y=1;\mathcal{H})p(y=1;\mathcal{H}) + p(x|y=0;\mathcal{H})p(y=0;\mathcal{H})} \\ &= \end{aligned}$$

1.d

First, derive the expression for the log-likelihood of the training data:

$$\begin{aligned}\ell(\phi, \mu_0, \mu_1, \Sigma) &= \log \prod_{i=1}^{n} p(x^{(i)}|y^{(i)}; \mu_0, \mu_1, \Sigma) p(y^{(i)}; \phi) \\ &= \sum_{i=1}^{n} \log p(x^{(i)}|y^{(i)}; \mu_0, \mu_1, \Sigma) + \sum_{i=1}^{n} \log p(y^{(i)}; \phi) \\ &=\end{aligned}$$

Now, the likelihood is maximized by setting the derivative (or gradient) with respect to each of the parameters to zero.

For ϕ:

$$\frac{\partial \ell}{\partial \phi} =$$

Setting this equal to zero and solving for ϕ gives the maximum likelihood estimate.

For μ_0:
Hint: Remember that Σ (and thus Σ^{-1}) is symmetric.

$$\nabla_{\mu_0} \ell =$$

Setting this gradient to zero gives the maximum likelihood estimate for μ_0.

For μ_1:
Hint: Remember that Σ (and thus Σ^{-1}) is symmetric.

$$\nabla_{\mu_1} \ell =$$

Setting this gradient to zero gives the maximum likelihood estimate for μ_1.

For Σ, we find the gradient with respect to $S = \Sigma^{-1}$ rather than Σ just to simplify the derivation (note that $|S| = \frac{1}{|\Sigma|}$). You should convince yourself that the maximum likelihood estimate S_n found in this way would correspond to the actual maximum likelihood estimate Σ_n as $S_n^{-1} = \Sigma_n$.

Hint: You may need the following identities:

$$\nabla_S |S| = |S|(S^{-1})^T$$

$$\nabla_S b_i^T S b_i = \nabla_S tr\left(b_i^T S b_i\right) = \nabla_S tr\left(S b_i b_i^T\right) = b_i b_i^T$$

$$\nabla_S \ell =$$

Next, substitute $\Sigma = S^{-1}$. Setting this gradient to zero gives the required maximum likelihood estimate for Σ.

1.f

1.g

1.h

2.a

2.b

2.c

The log-likelihood of an example $(x^{(i)}, y^{(i)})$ is defined as $\ell(\theta) = \log p(y^{(i)}|x^{(i)}; \theta)$. To derive the stochastic gradient ascent rule, use the results in part (a) and the standard GLM assumption that $\eta = \theta^T x$.

$$\begin{aligned}\frac{\partial \ell(\theta)}{\partial \theta_j} &= \frac{\partial \log p(y^{(i)}|x^{(i)}; \theta)}{\partial \theta_j} \\ &= \frac{\partial \log \left(\frac{1}{y^{(i)}!} \exp(\eta^T y^{(i)} - e^{\eta}) \right)}{\partial \theta_j} \\ &=\end{aligned}$$

Thus the stochastic gradient ascent update rule should be:

$$\theta_j := \theta_j + \alpha \frac{\partial \ell(\theta)}{\partial \theta_j},$$

which reduces here to: