CSCE 625-600 Homework #2 (2016 Fall)

Logic and Theorem Proving

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1 Propositional Logic

In this section, assume P, Q, R, S, T are atoms (propositions).

1.1 Inference rule

Question 1 (10 pts): Using a truth table, show that the resolution inference rule is valid (if the premises are true, the conclusion is also true, or, ((P V Q) ^ (¬ P V R)) -> (Q V R) is valid).

P	Q	R	$(P \vee Q)$	$(\neg P \vee R)$	$(P \vee Q) \wedge (\neg P \vee R)$	$(Q \vee R)$	$((P \vee Q) \wedge (\neg P \vee R)) \rightarrow (Q \vee R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T

1.2 Normal forms

In all of the problems in this section, show each step of the derivation and indicate which axioms (or other rules) you used: For example, distributive law, by definition, etc.

Question 2 (5 pts): Convert (P ^ ¬ (Q -> R)) V ¬ (S -> ¬P) into conjunctive normal form.

$(P \wedge \neg(Q \to R)) \vee \neg(S \to \neg P)$ to CNF

$(P \wedge \neg(\neg Q \vee R)) \vee \neg(\neg S \vee \neg P)$ [∵ $F \to G = \neg F \vee G$]

$(P \wedge (Q \wedge \neg R)) \vee (S \wedge P)$ [By DeMorgan's Laws]

$(P \vee S) \wedge (Q \vee S) \wedge (\neg R \vee S) \wedge (P \vee P) \wedge (Q \vee P) \wedge (\neg R \vee P)$ [By distributive laws]

$(P \vee S) \wedge (Q \vee S) \wedge (\neg R \vee S) \wedge P \wedge (P \vee Q) \wedge (P \vee \neg R)$ (∵ $P \vee P = P$)

is in Conjunctive Normal Form.

Question 3 (5 pts): Convert ¬ ((S -> (P ^ Q)) ^ R) -> T into disjunctive normal form.

$\neg((S \rightarrow (P \wedge Q)) \wedge R) \rightarrow T$ to DNF.

$((S \rightarrow (P \wedge Q)) \wedge R) \vee T$ ($\because A \rightarrow B = \neg A \vee B$)

$((\neg S \vee (P \wedge Q)) \wedge R) \vee T$ ($\because A \rightarrow B = \neg A \vee B$)

$(\neg S \wedge R) \vee (P \wedge Q \wedge R) \vee T$ [By distributive law

is in Disjunctive normal form.

1.3 Proof by Resolution

Given:

1. P -> S
2. ¬ S V R
3. R -> T
4. (R -> ¬S) V (T -> Q)

Show that P -> Q is a logical consequence of the above using **resolution**.

Precisely follow the steps below.

Question 4 (5 pts): Convert the above problem into a form that is suitable for resolution. This may involve converting some expressions into CNF, and other steps such as including the conclusion part P -> Q (don't forget to negate it!).

$P \to S \equiv \neg P \vee S \to$ 1

$\neg S \vee R \to$ 2

$R \to T \equiv \neg R \vee T \to$ 3

$(R \to \neg S) \vee (T \to Q) \equiv \neg R \vee \neg S \vee \neg T \vee Q \to$ 4

$\neg(P \to Q) \equiv \neg(\neg P \vee Q) \equiv P \wedge \neg Q$

$\therefore \quad P \to$ 5

$\neg Q \to$ 6

Question 5 (10 pts): With the resulting resolution problem from the above, prove the theorem using resolution. Show every step.

$\neg P \vee R \to$ 7 From 1, 2

$R \to$ 8 From 5, 7

$S \to$ 9 From 1, 5

$\neg S \vee \neg T \vee Q \to$ 10 From 4, 8

$T \to$ 11 From 3, 8

$\neg T \vee Q \to$ 12 From 9, 10

$Q \to$ 13 From 11, 12.

False From 6, 13

$P \to Q$ is a logical consequence of the given relations

2 First-Order Logic

Important: In this section, assume that w, x, y, z are variables; A, B, C, D are constants; and f(·), g(·), h(·) are functions; and P (·), Q(·), R(·) are predicates.

2.1 Normal forms

Question 6 (9 pts): Convert to prenex normal form (3 points each):

1. $\forall x \neg (\forall y \exists z P(x,y,z))$

2. $\neg \forall x(P(x) \wedge \neg (\exists y Q(x,y)))$

3. $\neg \forall x((\exists y Q(x,y)) \rightarrow (\forall z P(x,z)))$

① $\forall x \neg(\forall y \exists z \; P(x,y,z))$

$\forall x \exists y \neg(\exists z \; P(x,y,z))$

$\forall x \exists y \forall z (\neg P(x,y,z))$

② $\neg \forall x (P(x) \wedge \neg(\exists y \; Q(x,y)))$

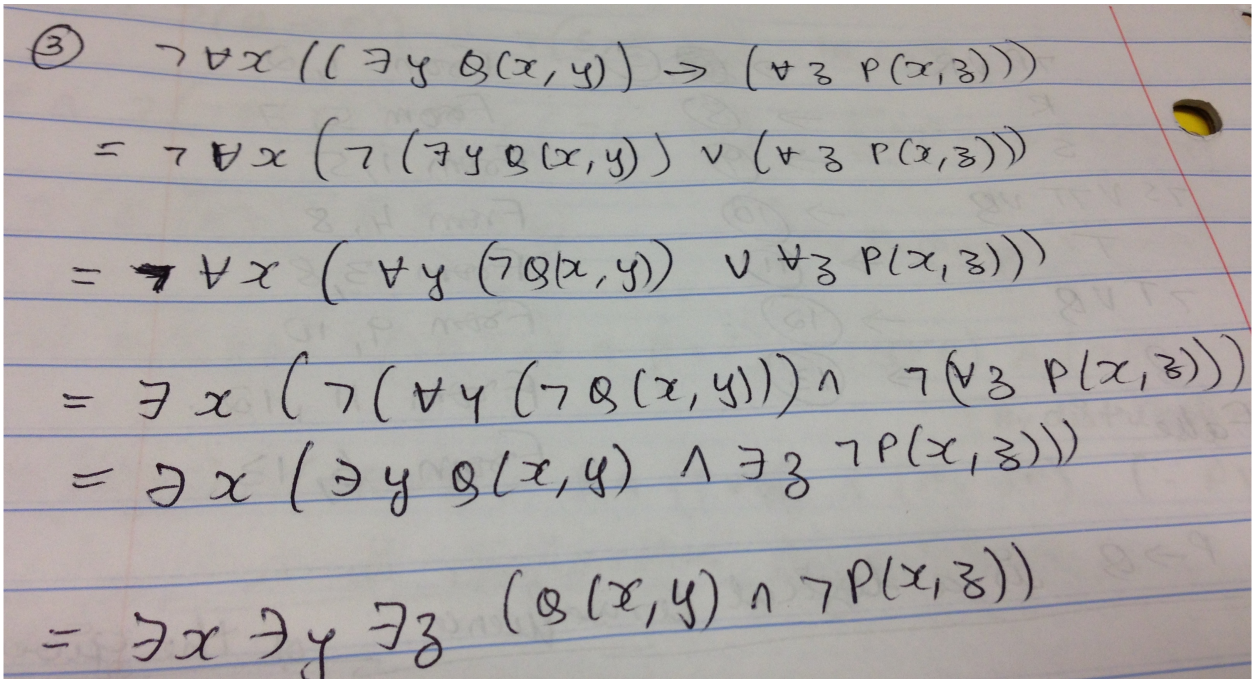
$\neg \forall x (P(x) \wedge \forall y (\neg Q(x,y))$

$\neg \forall x (\cancel{P(x)} \; \forall y (P(x) \wedge \neg Q(x,y))$

$\exists x \neg(\forall y (P(x) \wedge \neg Q(x,y))$

$\exists x \exists y \neg(P(x) \wedge \neg Q(x,y))$

$\exists x \exists y (\neg P(x) \vee Q(x,y))$



③ $\neg \forall x ((\exists y\, Q(x,y)) \rightarrow (\forall z\, P(x,z)))$

$= \neg \forall x (\neg(\exists y\, Q(x,y)) \vee (\forall z\, P(x,z)))$

$= \neg \forall x (\forall y (\neg Q(x,y)) \vee \forall z\, P(x,z)))$

$= \exists x (\neg(\forall y (\neg Q(x,y))) \wedge \neg(\forall z\, P(x,z)))$

$= \exists x (\exists y\, Q(x,y) \wedge \exists z\, \neg P(x,z)))$

$= \exists x\, \exists y\, \exists z\, (Q(x,y) \wedge \neg P(x,z))$

Question 7 (10 pts): Skolemize the expressions (2 points each):

1. ∃ x P(x)

2. ∀x ∃y P(x,y)

3. ∃x ∃y ∀z P(x,y,z) ^ Q(x,z)

4. ∀x ∃y ∀z P(x,z) ^ Q(x,y,z)

5. ∀x ∀y ∃z P(z,y) ^ Q(y,x)

1) $\exists x \; P(x)$

$= P(a)$

2) $\forall x \, \exists y \; P(x,y)$ || Sub y with $f(x)$

$= \forall x \; P(x, f(x))$

3) $\exists x \exists y \, \forall z \; P(x,y,z) \wedge Q(x,z)$ || Sub x with a, y with b

$= \forall z \; P(a,b,z) \wedge Q(a,z)$

4) $\forall x \, \exists y \, \forall z \; P(x,z) \wedge Q(x,y,z)$ || Sub y with $f(x)$

$= \forall x \, \forall z \; P(x,z) \wedge Q(x, f(x), z)$

5) $\forall x \, \forall y \, \exists z \; P(z,y) \wedge Q(y,x)$ || z with $f(x,y)$

$= \forall x \, \forall y \; P(f(x,y), y) \wedge Q(y,x)$

Question 8 (9 pts): Convert the following into a standard form: ∀x [(∃y P(x,y)) -> Q(x)]

$\forall x \, [(\exists y \; P(x,y)) \rightarrow Q(x)]$ to standard form:

$\forall x \, [\neg(\exists y \, P(x,y)) \vee Q(x)]$ $(A \rightarrow B \equiv \neg A \vee B)$

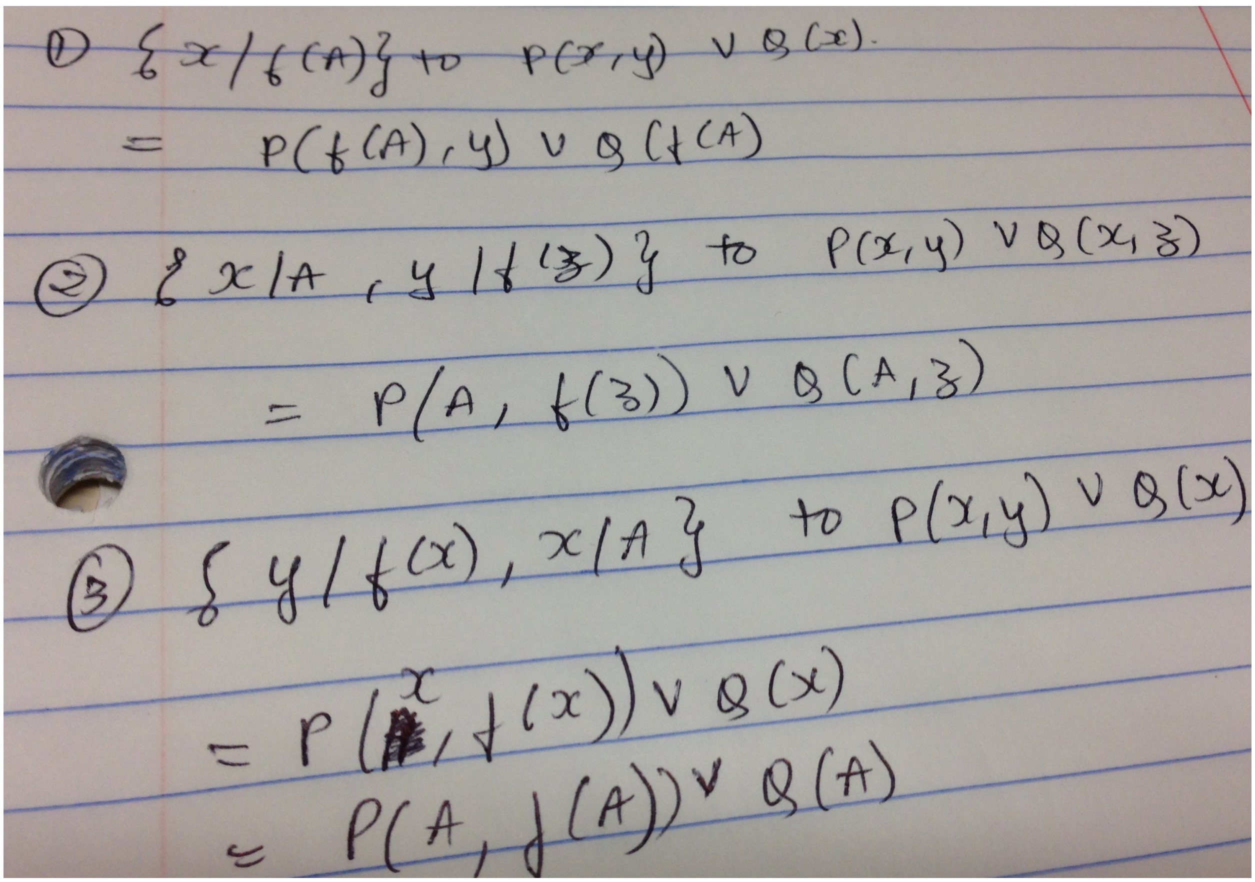
$\forall x \, [\forall y \; \neg P(x,y) \; \vee Q(x)]$

$\forall x \, \forall y \; (\neg P(x,y) \vee Q(x)$

2.2 Substitution and Unification

Question 9 (9 pts): Apply the following substitutions to the expressions (3 point each);

1. Apply {x/f(A)} to P(x,y) V Q(x).
2. Apply {x/A,y/f(z)} to P(x,y) V Q(x,z).
3. Apply {y/f(x),x/A} to P(x,y) V Q(x).


① {x/f(A)} to P(x,y) v Q(x).
= P(f(A),y) v Q(f(A)
② {x/A, y/f(z)} to P(x,y) v Q(x,z)
= P/A, f(z)) v Q(A,z)
③ {y/f(x), x/A} to P(x,y) v Q(x)
= P(~~A~~ x, f(x)) v Q(x)
= P(A, f(A)) v Q(A)

Question 10 (12 pts): For each of the following, (1) find the unifier, and (2) show the unified expression. For example, given P (A) and P (x), the unifier would be {x/A}, and the unified expression P (A). If the pair of expressions is not unifiable, indicate so. (3 points each):

1. P(x,f(B)) and P(A,f(y))

2. P(x,f(A)) and P(y,y)

3. P(x,f(y),y)andP(A,f(g(B)),g(A))

4. P(A,f(y),y,A) and P(x,f(g(x)),g(A),w)

1) P(x, f(B)) and P(A, f(y))

{x/A} : P(A, f(B)) and P(A, f(y))

{y/B} : P(A, f(B)) and P(A, f(B))

unifier: {x/A, y/B} , Expression = P(A, f(B))

2) P(x, f(A)) and P(y, y)

{x/y} : P(y, f(A)) and P(y, y)

{y/f(A)} : P(f(A), f(A) and P(f(A), f(A))

∴ unifier = {x/y, y/f(A)} Expression: P(f(A), f(A))

3) P(x, f(y), y) and P(A, f(g(B)), g(A))

{x/A} : P(A, f(y), y) and P(A, f(g(B)), g(A))

{y/g(B)} : P(A, f(g(B)), g(B) and P(A, f(g(B)), g(A))

We cannot unify, as A & B conflict with each other.

4) P(A, f(y), y, A) and P(x, f(g(x)), g(A), w)

{x/A} : P(A, f(y), y, A) and P(A, f(g(A)), g(A), w)

{y/g(A)} : P(A, f(g(A)), g(A), A) and P(A, f(g(A)), g(A), w)

{w/A} : P(A, f(g(A)), g(A), A) and P(A, f(g(A)), g(A), A)

∴ unifier: {x/A, y/g(A), w/A} Expression: P(A, f(g(A)), g(A), A)

2.3 Proof by resolution

Question 11 (16 pts): (1) Translate the following into first-order logic, (2) convert the resulting formulas into a normal form, and (3) prove the theorem using resolution.

Given:

1. Every child who finds some [thing that is an] egg or chocolate bunny is happy.

2. Every child who is helped finds some egg.

3. Every child who is not young or who tries hard finds some chocolate bunny.

Show that the following is a logical consequence of the above:

4. (Conclusion) If every young child tries hard or is helped, then every child is happy.

Solution:

Notations used:

C(x) : True if x is a child

E(x): True if x is an egg

F(x,y): True if x finds y

CB(x): True if x is a chocolate Bunny

H(x): True if x is helped

Y(x): True if x is young

T(x): True if x tries hard

Hp(x): True if x is helped

After translating into first-order logic:

1) $\forall x \ [C(x) \wedge \exists y [F(x,y) \wedge (E(y) \vee CB(y))] \rightarrow H(x)]$

2) $\forall x \ [(C(x) \wedge Hp(x)) \rightarrow \exists y (F(x,y) \wedge E(y))]$

3) $\forall x \ [C(x) \wedge (\neg Y(x) \vee T(x)) \rightarrow \exists y \ (F(x,y) \wedge CB(y))]$

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$\forall x [((C(x) \wedge Y(x)) \wedge (T(x) \vee H(x))) \rightarrow H(x)]$

After converting the resulting formulas into a normal form and finally proving the result with resolution:

1) $\forall x \exists y [\neg C(x) \vee \neg F(x,y) \vee (\neg E(y) \wedge \neg CB(y))] \vee H(x)$

$\equiv \forall x \exists y (\neg C(x) \vee \neg F(x,y) \vee \neg E(y) \vee H(x)) \wedge (\neg C(x) \vee \neg F(x,y) \vee \neg CB(y) \vee H(x))$ [Distributive law]

Split & Normalize:

$\forall x\ \neg C(x) \vee \neg F(x, f(x)) \vee \neg E(f(x)) \vee H(x) \rightarrow$ 1a

$\forall x\ \neg C(x) \vee \neg F(x, f(x)) \vee \neg CB((f(x)) \vee H(x) \rightarrow$ 1b

~~2) $\forall x [C(x) \wedge Hp(x) \vee (F(x,y) \wedge E(y))]$~~

2) $\forall x [C(x) \wedge Hp(x)] \rightarrow [\exists y\ F(x,y) \wedge E(y)]$

$\forall x \exists y [\neg C(x) \vee \neg Hp(x)] \vee [F(x,y) \wedge E(y)]$

$\therefore\ \forall x\ \neg C(x) \vee \neg Hp(x) \vee F(x, f(x)) \rightarrow$ 2a

$\forall x\ \neg C(x) \vee \neg Hp(x) \vee E(f(x)) \rightarrow$ 2b.

3) $\forall x \exists y\ \neg C(x) \vee (Y(x) \wedge \neg T(x)) \vee (F(x,y) \wedge CB(y))$

$\equiv \forall x \exists y [(\neg C(x) \vee Y(x)) \wedge (\neg C(x) \vee \neg T(x)] \vee [F(x,y)$ $CB(y)]$

P.T.O

Split:

$\forall x \ \neg C(x) \vee Y(x) \vee F(x, f(x)) \rightarrow$ 3a

$\forall x \ \neg C(x) \vee Y(x) \ \vee CB(f(x)) \rightarrow$ 3b

$\forall x \ \neg C(x) \ \vee \neg T(x) \vee CB(f(x)) \rightarrow$ 3c

$\forall x \ \neg C(x) \vee \neg T(x) \vee F(x, f(x)) \rightarrow$ 3d

4) $\neg$ of the conclusion:

$\exists x \ [C(x) \wedge Y(x) \wedge T(x) \vee Hp(x))] \wedge \neg H(x)$.

$\equiv C(a) \wedge Y(a) \wedge (T(a) \vee Hp(a)) \wedge \neg H(a)$

$C(a) \rightarrow$ 4a

$Y(a) \rightarrow$ 4b

$T(a) \vee Hp(a) \rightarrow$ 4c

$\neg H(a) \rightarrow$ 4d

Resolution:

$\neg T(a) \vee CB(f(a)) \rightarrow$ 5 From 3C, 4a

$\neg T(a) \vee F(a, f(a)) \rightarrow$ 6 From 3d, 4a.

$Hp(a) \vee CB(f(a)) \rightarrow$ 7 From 4C, 5.

$Hp(a) \vee F(a, f(a)) \rightarrow$ 8 From 4C, 6.

$\neg F(a, f(a)) \vee \neg CB(f(a)) \rightarrow$ 9 from 1b, 4a, 4d

$Hp(a) \vee \neg CB(f(a)) \rightarrow$ 10 from 8, 9.

$Hp(a) \rightarrow$ 11 from 7, 10

$\neg Hp(a) \vee F(a, f(a)) \rightarrow$ 12 from 2a, 4a

$F(a, f(a)) \rightarrow$ 13 from 11, 12

$\neg Hp(a) \vee E(f(a)) \rightarrow$ 14 from 2b, 4a.

$E(f(a)) \rightarrow$ 15 from 11, 14.

$\neg F(a, f(a)) \vee \neg E(f(a)) \vee H(a) \rightarrow$ 16 from 1a, 4a.

$\neg E(f(a)) \vee H(a) \rightarrow$ 17 from 13, 16

$H(a) \rightarrow$ 18. from 15, 17

false. from 18, 4d