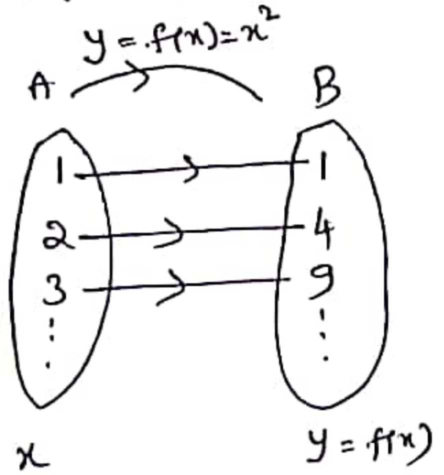
# Conformal mapping

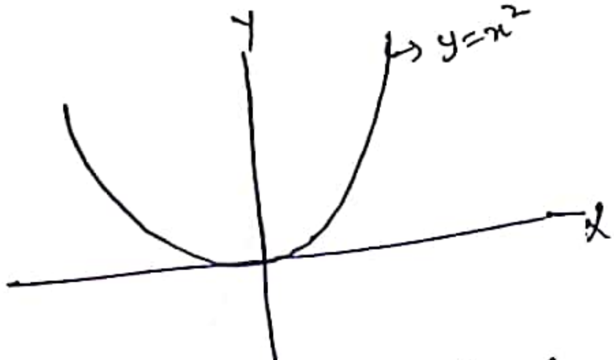
## Introduction

### Mapping :-

Ex: $f(x) = x^2$




If we plot the same function in 2-Dimension ($xy$-plane) we get




Like this we can map complex functions also.

For example :

$w = f(z) = z^2$ , $\boxed{z = x + iy}$

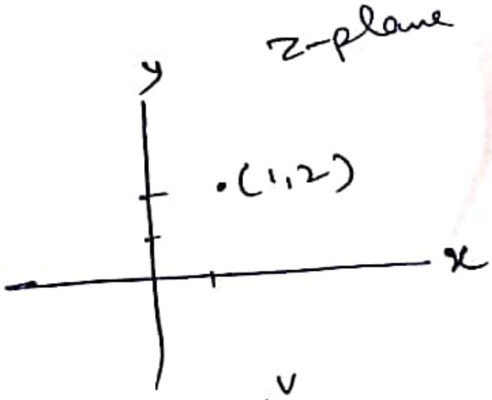
$= (x+iy)^2$

$= x^2 - y^2 + 2ixy$

$w = f(z) = (x^2 - y^2) + i(2xy)$

$= u + iv$

$\boxed{w = f(z) = u(x,y) + iv(x,y)}$



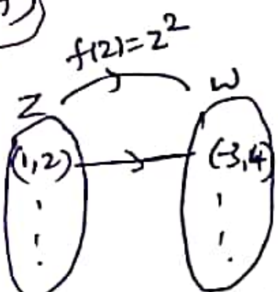

If $z = 1 + i2$

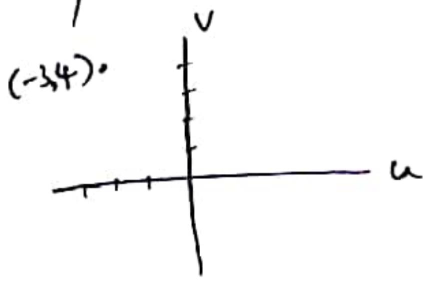
$w = (1+i2)^2$

$= 1 - 4 + i4$

$w = -3 + i4$

ie




Here we can't place 4 points in 2-D plane. So that we should consider two planes (ie z-plane & w-plane)



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# Conformal transformation

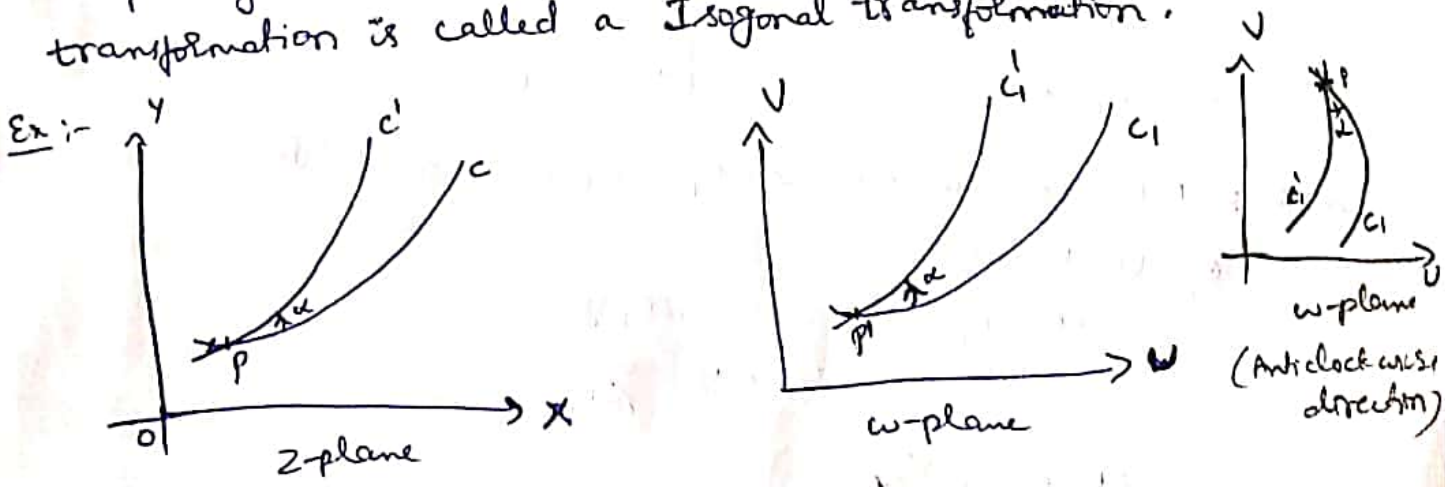
**Definitions :-**

Consider a complex valued function $w = f(z)$.
Putting $z = x+iy$, $w = f(z) = u(x,y) + i v(x,y)$. The complex quantities $z = z(x,y)$, $w = w(u,v)$ are represented in two separate planes namely z-plane & the w-plane respectively. A point $(x,y)$ in the z-plane corresponds to a point $(u,v)$ in the w-plane. If a set of points $(x,y)$ traces a curve $C$ in z-plane and the corresponding points $(u,v)$ traces a curve $C'$ in the w-plane, then we can say that the curve $C$ is transformed / mapped onto the curve $C'$ under the transformation $w = f(z)$. The corresponding set of points in the two planes are called 'images' of each other.

In a transformation the angle b/w any two curves, both in magnitude and sense (direction) are same then it is called a conformal transformation.

If only the magnitude of the angle is same then the transformation is called a Isogonal transformation.

Ex :-




In the above figure, the curves $c, c'$ in the z-plane intersect at the point $P$ and the corresponding curves $c_1$ & $c_1'$ in the w-plane intersect at $P'$. If the angle of intersection of the curves at $P$ is same as the angle of intersection of the curves at $P'$ in magnitude & sense then the transformation is said to be ~~conformal~~ conformal.



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**Note :-**

**Property :-** If $w = f(z)$ is an analytic function of $z$ in a region of the z-plane then $w = f(z)$ is conformal at all points of the region where $f'(z) \neq 0$.

**Discussion of conformal transformation :-**

For a given transformation $w = f(z)$, first we put $z = x+iy$ (or) $z = re^{i\theta}$ to obtain $u$ & $v$ as functions of $x, y$ (or) $r, \theta$. Then by using those functions we can find the image in w-plane corresponding to the given curve in the z-plane. Sometimes we need to make some judicious elimination from $u$ & $v$ for obtaining the image in the w-plane.

1) Discussion of $w = z^2$

**Soln :-** Consider $w = z^2$, where $w = u+iv$, $z = x+iy$

$$u+iv = (x+iy)^2$$

$$u+iv = x^2 - y^2 + i2xy$$

$$\therefore u = x^2 - y^2 \quad \text{and} \quad v = 2xy \longrightarrow (1)$$

**Case (i) :-** Let us consider $x = c_1$, $c_1$ is a constant.

$\therefore$ eqn (1) becomes

$$u = c_1^2 - y^2, \quad v = 2c_1 y$$

$$\Rightarrow y = v/2c_1.$$

$$u = c_1^2 - \left(v^2/4c_1^2\right)$$

$$\Rightarrow \frac{v^2}{4c_1^2} = u - c_1^2$$

$$\Rightarrow v^2 = -4c_1^2(u - c_1^2).$$

This is a parabola in the w-plane symmetric about the real axis with its vertex is at $(c_1^2, 0)$ and focus at the origin. It may be observed that the line $x = -c_1$ is also transformed into the same parabola.



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Case(ii) :- Let us consider $y = c_2$, $c_2$ is constant

$\therefore$ eqn ① becomes

$$u = x^2 - c_2^2 \quad , \quad v = 2xc_2$$

$$\Rightarrow x = \frac{v}{2c_2}$$

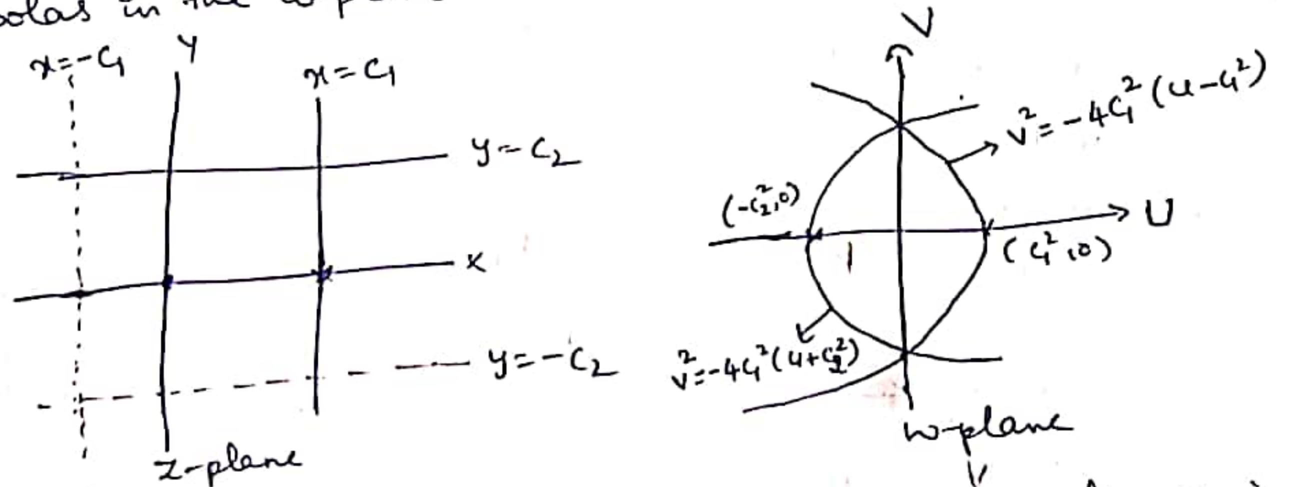
$$\therefore \ u = \left(\frac{v^2}{4c_2^2}\right) - c_2^2$$

$$\Rightarrow \frac{v^2}{4c_2^2} = u + c_2^2$$

$$v^2 = 4c_2^2(u + c_2^2)$$

This is also a parabola in the w-plane symmetrical about the real axis whose vertex is at $(-c_2^2, 0)$ and focus at the origin. Also $y = -c_2$ is transformed into the same parabola.

Hence from these two cases we conclude that the straight lines parallel to the co-ordinate axes in the z-plane map onto parabolas in the w-plane.



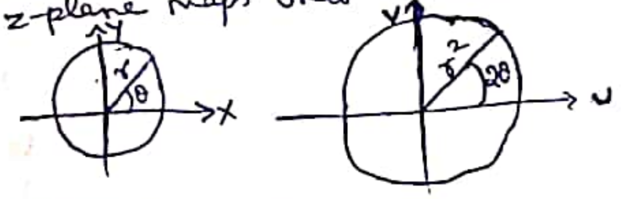

Case(iii) :- Let us consider a circle with centre origin & radius $r$ in the z-plane.

ie $|z| = r$, $\therefore z = re^{i\theta} \Rightarrow w = z^2 = (re^{i\theta})^2$

$$\therefore w = r^2 e^{i2\theta} = Re^{i\phi} \quad \text{where } R = r^2, \ \phi = 2\theta$$

This is also a circle in the w-plane having radius $r^2$ and subtending an angle $2\theta$ at the origin.

Hence we conclude that a circle with centre origin and radius $r$ in the z-plane maps onto a circle with centre origin & radius $r^2$ in the w-plane.



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Example: Find the images in the w-plane corresponding to the straight lines $x = c_1$, $x = c_2$, $y = k_1$, $y = k_2$ under the transformation $w = z^2$. Indicate the region with sketches.

$$w = z^2$$

$$u + iv = (x + iy)^2$$

$$= x^2 - y^2 + i2xy$$

$$u = x^2 - y^2, \quad v = 2xy \quad \text{—} (1)$$

(i) if $x = c_1$

$$(1) \Rightarrow u = c_1^2 - y^2, \quad v = 2c_1 y$$

$$y = v/2c_1$$

$$\therefore u = c_1^2 - \frac{v^2}{4c_1^2}$$

$$\therefore v^2 = -4c_1^2(u - c_1^2)$$ This is a parabola in w-plane with vertex $(c_1^2, 0)$

(ii) Similarly if $x = c_2$

$$v^2 = -4c_2^2(u - c_2^2)$$

(iii) if $y = k_1$

$$(1) \Rightarrow u = x^2 - k_1^2, \quad v = 2xk_1$$

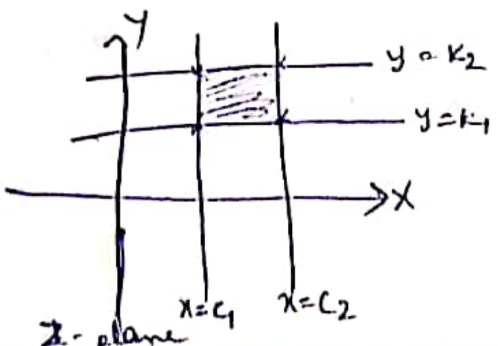
$$x = \frac{v}{2k_1}$$

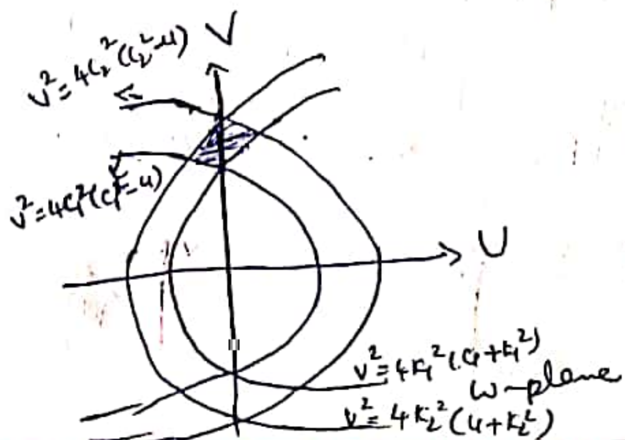
$$u = \frac{v^2}{4k_1^2} - k_1^2$$

$$\Rightarrow v^2 = 4k_1^2(u + k_1^2)$$

(iv) Similarly if $y = k_2$,

$$v^2 = 4k_2^2(u + k_2^2)$$



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## 2) Discussion of $w = e^z$

Consider $w = e^z$

$$u + iv = e^{x+iy}$$
$$= e^x \cdot e^{iy}$$
$$= e^x(\cos y + i \sin y)$$
$$u + iv = e^x \cos y + i e^x \sin y \Rightarrow \left.\begin{array}{l} u = e^x \cos y \\ v = e^x \sin y \end{array}\right\} \rightarrow (1)$$

We shall find the image in the w-plane corresponding to the straight lines parallel to the co-ordinates axes in the z-plane. That is $x$ = constant, $y$ = constant.

Here by using equation (1) we can't find any values. So that first let us eliminate $x$ & $y$ separately from (1)

Squaring & adding equations in (1),

$$u^2 + v^2 = e^{2x} \longrightarrow (2)$$

Also dividing the equations in (1)

$$\frac{v}{u} = \tan y \longrightarrow (3)$$

**Case(i):** Let $x = c_1$, $c_1$ is a constant

$\therefore$ eq$^n$ (2) $\Rightarrow u^2 + v^2 = e^{2c_1} = r^2$ | where $r^2$ = constant $= e^{2c_1}$

$$\Rightarrow u^2 + v^2 = r^2$$

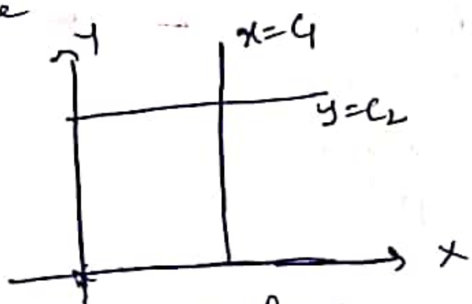
This represents a circle with centre origin & radius $r$ in the w-plane.

**Case(ii):** Let $y = c_2$, $c_2$ is constant

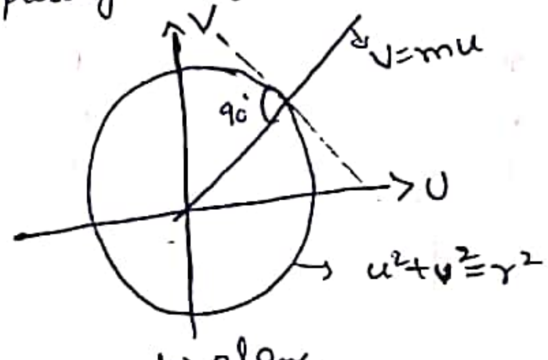
$\therefore$ eq$^n$ (3) $\Rightarrow \frac{v}{u} = \tan c_2 = m$, | where $m = \tan c_2$

$$\Rightarrow v = mu$$

This represents a straight line passing through the origin in the w-plane




Z-plane




w-plane

Conclusion :- The straight line parallel to the x-axis ($y = c_2$) in the z-plane maps onto a straight line passing through the origin in the w-plane. The straight line parallel to the y-axis ($x = c_1$) in the z-plane maps onto a circle with centre origin and radius $r$ when $r = e^{c_1}$ in the w-plane. Suppose we draw a tangent at the point of intersection of these two curves in the w-plane, the angle subtended is equal to 90°. Hence these two curves can be regarded as orthogonal trajectories of each other.

Question :- Show that the transformation $w = e^z$ map straight line parallel to the co-ordinate axes in the z-plane onto orthogonal trajectories in the w-plane and sketch the region.

(Answer in above discussion).

Example :- Discuss the transformation $w = e^z$ with respect to the lines represented as co-ordinate axes in the z-plane

Soln. The co-ordinate axes in the z-plane are represented by $x = 0$, $y = 0$.

Given $w = e^z$

$$u + iv = e^{x+iy}$$

$$u + iv = e^x \cos y + i e^x \sin y$$

$$\Rightarrow u = e^x \cos y, \quad v = e^x \sin y \rightarrow (1)$$

Also we have $u^2 + v^2 = e^{2x} \rightarrow (2)$

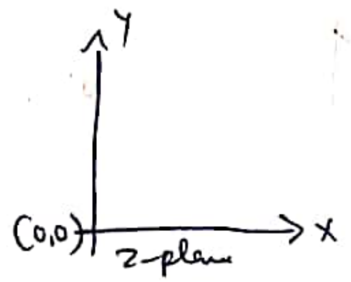
$$\frac{v}{u} = \tan y \rightarrow (3)$$

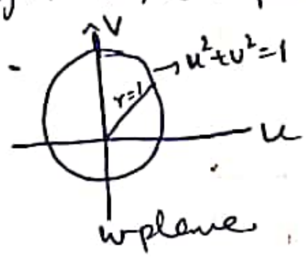
when $y = 0$, (3) $\Rightarrow \frac{v}{u} = 0 \Rightarrow v = 0$

∴ the x-axis in z-plane is mapped onto the u-axis in the w-plane

when $x = 0$, (2) $\Rightarrow u^2 + v^2 = 1$

∴ the y-axis in the z-plane is mapped onto ~~the u-axis~~ a unit circle with centre origin in the w-plane.



3) Discussion of $w = z + \frac{1}{z}$, $z \neq 0$.

consider, $w = z + \frac{1}{z}$

put $z = re^{i\theta}$

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$\Rightarrow u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta \longrightarrow (1)$$

Now we shall eliminate $r$ & $\theta$ respectively from (1).

To eliminate $\theta$, put (1) in the form

$$\frac{u}{\left(r + \frac{1}{r}\right)} = \cos\theta, \quad \frac{v}{\left(r - \frac{1}{r}\right)} = \sin\theta$$

squaring and adding, we get

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1, \quad r \neq 1. \longrightarrow (2)$$

To eliminate $r$, put (1) in the form.

$$\frac{u}{\cos\theta} = r + \frac{1}{r}, \quad \frac{v}{\sin\theta} = r - \frac{1}{r}$$

squaring and subtracting, we get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4$$

$$\Rightarrow \frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \longrightarrow (3)$$

Since $z = re^{i\theta}$, $|z| = r$ & $\operatorname{amp} z = \theta$

$|z| = r \Rightarrow \sqrt{x^2 + y^2} = r$

(or) $r^2 = x^2 + y^2$

This represents a circle with centre origin and radius $r$ in the z-plane when $r$ is a constant.



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$\arg z = \theta$

$\Rightarrow \tan^{-1}(y/x) = \theta$

$\Rightarrow y/x = \tan\theta$

This represents a straight line in the z-plane, when $\theta$ is constant.

Now we shall discuss the image in the w-plane, corresponding to $r =$ constant (circle) & $\theta =$ constant (straight line) in the z-plane.

Case(i) :- Let $r =$ Constant

Equation (2) is of the form

$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$ where $A = r + \frac{1}{r}$, $B = r - \frac{1}{r}$

This represents an ellipse in the w-plane with foci $(\pm\sqrt{A^2-B^2}, 0)$
$= (\pm 2, 0)$

(Since $\sqrt{A^2-B^2} = \sqrt{(r+\frac{1}{r})^2 - (r-\frac{1}{r})^2} = \sqrt{4} = \pm 2$.)

Hence we conclude that the circle $|z| = r =$ constant in the z-plane, maps onto an ellipse in the w-plane with foci $(\pm 2, 0)$.

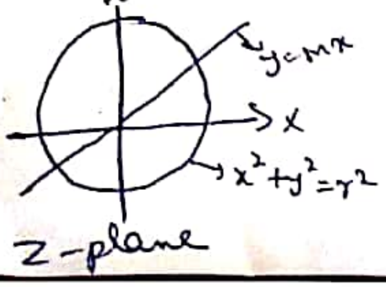
Case(ii) :- Let $\theta =$ constant

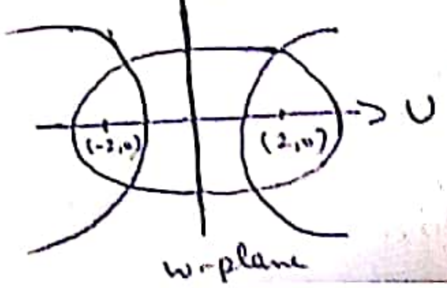
Eq$^{n}$ (3) is of the form

$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$ where $A = 2\cos\theta$, $B = 2\sin\theta$.

This represents a hyperbola in the w-plane with foci
$(\pm\sqrt{A^2+B^2}, 0) = (\pm 2, 0)$.

Hence we conclude that the straight line passing through the origin in the z-plane maps onto a hyperbola in the w-plane with foci $(\pm 2, 0)$.
Since both these circles (ellipse & hyperbola) have the same foci independent of $r, \theta$ they are called confocal conics.



## Bilinear Transformation (BLT) :-

The transformation $w = \frac{az+b}{cz+d}$, where a, b, c, d are real (or) complex constants such that $ad-bc \neq 0$ is called bilinear transformation

**Note :**

1) The condition $ad-bc \neq 0$ is the conformal property of BLT
2) If a point z maps onto itself that is $w=z$ under the bilinear transformation then the point is called an invariant point (or) a fixed point of the BLT.
3) Bilinear transformation is also called mobius transformation.

**Property 1:-** There exists a bilinear transformation that maps three given distinct points $z_1, z_2, z_3$ onto three given distinct points $w_1, w_2, w_3$ respectively.

By this property, if we solve the equation w in terms of z

ie
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$ ~~the equation~~

we obtain the bilinear transformation that transforms $z_1, z_2, z_3$ onto $w_1, w_2, w_3$ respectively.

**Property 2:-** Bilinear transformation preserve (do not alter) the cross-ratio of four points.

ie
$$\frac{(w_4-w_1)(w_2-w_3)}{(w_4-w_3)(w_2-w_1)} = \frac{(z_4-z_1)(z_2-z_3)}{(z_4-z_3)(z_2-z_1)}$$ this shows that

the cross ratio of the points $w_1, w_2, w_3, w_4$ is equal to the cross ratio of the points $z_1, z_2, z_3, z_4$. Thus BLT preserve the cross ratio.



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Problems:

1) Find the bilinear transformation which map the points $z = 1, i, -1$ into $w = i, 0, -i$. Under this transformation find the images of $|z| < 1$.

Sol$^n$: Let $w = \dfrac{az+b}{cz+d}$ be the required transformation.

Let $z_1 = 1$, $z_2 = i$, $z_3 = -1$ & $w_1 = i$, $w_2 = 0$, $w_3 = -i$

$\therefore$ The required BLT is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

ie $$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)\cancel{i}}{(w+i)(-\cancel{i})} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{w-i}{w+i} = -\frac{(i+1)}{(i-1)}\frac{(z-1)}{(z+1)}$$

$$= -\frac{(i+1)^2}{(i-1)(i+1)}\frac{(z-1)}{(z+1)}$$

$$= -\frac{(\cancel{i^2}+\cancel{1}-2i)}{i^2-1^2}\frac{(z-1)}{(z+1)}$$

$$= \frac{\cancel{-2}i}{\cancel{-2}}\frac{(z-1)}{(z+1)}$$

$$\frac{w-i}{w+i} = i\frac{(z-1)}{(z+1)}$$

$$(w-i)(z+1) = i(w+i)(z-1)$$

$$w(z+1) - i(z+1) = iw(z-1) + i^2(z-1).$$

$$w\{(z+1) - i(z-1)\} = i(z+1) - (z-1)$$

$$w = \frac{i(z+1)-(z-1)}{(z+1)-i(z-1)}$$



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$$w = \frac{iz + i - z + 1}{z + 1 - iz + i}$$

$$= \frac{z(i-1) + (1+i)}{z(1-i) + (1+i)}$$

multiply & ÷ by $(1-i)$

$$= \frac{-z(1-i)^2 + (1+i)(1-i)}{z(1-i)^2 + (1+i)(1-i)} = \frac{-z(1+i^2-2i) + (1^2-i^2)}{z(1+i^2-2i) + (1^2-i^2)}$$

$$= \cancel{\frac{-z(1+i^2+2i) + 1 - i^2}{z(1+i^2-2i) + 1 - i^2}}$$

$$= \frac{-z(-2i) + 2}{-z(2i) + 2}$$

$$= \frac{2zi + 2}{-2zi + 2}$$

$$\boxed{w = \frac{1 + iz}{1 - iz}}$$

This is the required transformation.

To find the image of $|z| < 1$ ;

Consider, $w = \dfrac{1+iz}{1-iz}$

$$w(1-iz) = 1 + iz$$

$$w - iwz = 1 + iz$$

$$-iz - iwz = 1 - w$$

$$-zi(1+w) = 1 - w$$

$$z = \frac{(1-w)}{-i(1+w)}$$

$$z = i\frac{(1-w)}{(1+w)} \qquad 1 \because \frac{1}{i} = -i \; \& \; -\frac{1}{i} = i$$

If $|z| < 1$ this expression yields

$$|i| \left| \frac{1-w}{1+w} \right| < 1$$



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(or) $||1-w|^2 < |1+w^2|$

ie $|1-(u+iv)|^2 < |1+(u+iv)|^2$

ie $|(1-u)+iv|^2 < |(1+u)+iv|^2$

ie $(1-u)^2+v^2 < (1+u)^2+v^2$

ie $\cancel{1}+\cancel{u^2}-2u+\cancel{v^2} < \cancel{1}+\cancel{u^2}+2u+\cancel{v^2}$

$\Rightarrow \quad -2u < 2u$

$\Rightarrow \quad 4u > 0$

$\Rightarrow \quad u > 0.$

Thus $u>0$ is the image of $|z|<1$.

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__Alternate method__ :

__Soln__: Let $w = \dfrac{az+b}{cz+d}$ be the required BLT.

Substitute the given values of $z$ & $w$, we get three equations

$z=1,\ w=i \Rightarrow i = \dfrac{a+b}{c+d}$

$\Rightarrow a+b-ci-di=0 \rightarrow$ (1)

$z=i,\ w=0,\quad 0 = \dfrac{ai+b}{ci+d}$

$\Rightarrow ai+b=0 \rightarrow$ (2)

$z=-1,\ w=-i,\quad -i = \dfrac{-a+b}{-c+d}$

$\Rightarrow -a+b-ci+di=0 \rightarrow$ (3)

(1) + (3) $\Rightarrow 2b-2ci=0$

$b-ci=0 \rightarrow$ (4)

Solve (2) & (4) by writing

$ia+1b+0c=0$

$0a+1b-ic=0$

Applying the rule of cross multiplication, we have

$$\frac{a}{\begin{vmatrix}1 & 0\\ 1 & -i\end{vmatrix}} = \frac{-b}{\begin{vmatrix}i & 0\\ 0 & -i\end{vmatrix}} = \frac{c}{\begin{vmatrix}i & 1\\ 0 & 1\end{vmatrix}}$$

ie $\dfrac{a}{-i} = \dfrac{-b}{-i^2} = \dfrac{c}{i}$

(or) $\dfrac{a}{-i} = \dfrac{b}{-1} = \dfrac{c}{i} = k$ (say)

$\Rightarrow a=-ki,\ b=-k,\ c=ik$

Substituting these in (1), we get

$-ik-k-i^2k-di=0$

$-ik-\cancel{k}+\cancel{k}-di=0$

$-(di+ik)=0$

$di=-ik$

$\boxed{d=-k}$

$\therefore$ Sub. $a, b, c, d$ in $w$, we get

$w = \dfrac{-kiz-k}{ikz-k} = \dfrac{-\cancel{k}(1+iz)}{-\cancel{k}(1-iz)}$

$w = \dfrac{1+iz}{1-iz}$

**2)** Find the bilinear transformation which map the points $z = 1, i, -1$ into $w = 2, i, -2$. Also find the invariant points of the transformation.

**Sol:**

(a) Let $w = \frac{az+b}{cz+d}$ be the required BLT.

Now $z=1,\ w=2,\ \Rightarrow\ 2 = \frac{a+b}{c+d}$

$$a+b-2c-2d=0 \longrightarrow (1)$$

$z=i,\ w=i\ \Rightarrow\ i = \frac{ai+b}{ci+d}$

$$ai+b+c-di=0 \longrightarrow (2)$$

$z=-1,\ w=-2,\ \Rightarrow\ -2 = \frac{-a+b}{-c+d}$

$$-a+b-2c+2d=0 \longrightarrow (3)$$

$(1)+(3) \Rightarrow \cancel{a}+b-2c-\cancel{2d}-\cancel{a}+b-2c+\cancel{2d}=0$

$$2b-4c=0$$

$$b-2c=0 \longrightarrow (4)$$

$(2)+i\times(3) \Rightarrow ai+b+c-di+i(-a+b-2c+2d)=0$

$$\cancel{ai}+b+c-di-\cancel{ai}+bi-2ci+2di=0$$

$$(1+i)b+(1-2i)c+id=0 \longrightarrow (5)$$

Solve (4) & (5)

$$b-2c+0d=0$$

$$(1+i)b+(1-2i)c+id=0$$

Applying the rule of cross multiplication:

$$\frac{b}{\begin{vmatrix} -2 & 0 \\ (1-2i) & i \end{vmatrix}} = \frac{-c}{\begin{vmatrix} 1 & 0 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 1 & -2 \\ (1+i) & (1-2i) \end{vmatrix}}$$

$$\Rightarrow \frac{b}{-2i} = \frac{-c}{i} = \frac{d}{(1-2i)+2(1+i)}$$

$$\Rightarrow \frac{b}{-2i} = \frac{-c}{i} = \frac{d}{3}$$



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$\therefore\ b = -2i, \quad c = -i, \quad d = 3.$

Substitute these values in (1),

$$a - 2\cancel{i} + 2\cancel{i} - 6 = 0$$

$$\Rightarrow \quad \boxed{a = 6}$$

$\therefore$ The required BLT is

$$w = \frac{6z - 2i}{-iz + 3}$$

Further, the invariant points of this transformation are obtained by taking $w = z$.

ie $$z = \frac{6z - 2i}{-iz + 3}$$

$$-iz^2 + 3z - 6z + 2i = 0$$

$$-iz^2 - 3z + 2i = 0$$

Applying the quadratic formula,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-i)(2i)}}{-2i}$$

$$= \frac{3 \pm \sqrt{9-8}}{-2i}$$

$$z = \frac{3 \pm 1}{-2i}$$

$$\Rightarrow \quad z = \frac{3+1}{-2i} \quad \& \quad z = \frac{3-1}{-2i}$$

$$\therefore\ z = \frac{4}{-2i} = \frac{2}{-i}, \qquad z = \frac{2}{-2i} = \frac{1}{-i} \quad \Rightarrow \quad z = \frac{2}{-i} = 2i,$$

$$z = \frac{1}{-i} = i$$

Thus $\boxed{z = 2i,\ i}$ are the invariant points.



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** 3) Find the BLT, which maps $z_1=-1$, $z_2=0$, $z_3=1$ into $w_1=0$, $w_2=i$, $w_3=3i$

Sol^n: Let $w=\dfrac{az+b}{cz+d}$ be the required BLT.

Now let

$z_1=-1,\ w_1=0 \Rightarrow 0=\dfrac{-a+b}{-c+d}$

$\Rightarrow -a+b=0 \rightarrow (1)$

$z_2=0,\ w_2=i \Rightarrow i=\dfrac{0+b}{0+d}$

$b-id=0 \rightarrow (2)$

$z_3=1,\ w_3=3i \Rightarrow 3i=\dfrac{a+b}{c+d}$

$a+b-3ci-3di=0 \rightarrow (3)$

$(1)-(2) \Rightarrow -a+\cancel{b}-\cancel{b}+id=0$

$-a+id=0 \rightarrow (4)$

Solve (2) & (4)

$0a+b-id=0$

$-a+0b+id=0$

Applying rule of cross multiplication,

$$\frac{a}{\begin{vmatrix}1 & -i\\ 0 & i\end{vmatrix}}=\frac{-b}{\begin{vmatrix}0 & -i\\ -1 & i\end{vmatrix}}=\frac{d}{\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}}$$

$\Rightarrow \dfrac{a}{i}=\dfrac{-b}{-i}=\dfrac{d}{1}$

$\Rightarrow a=i,\ b=i,\ d=1$

Sub. all the values in (3), we get

$i+i-3ci-3i=0$

$-i-3ci=0$

$-i(1+3c)=0 \Rightarrow 1+3c=0 \Rightarrow c=-1/3$

$\therefore w=\dfrac{iz+i}{-\frac{1}{3}z+1} \Rightarrow w=\dfrac{3i(z+1)}{-z+3}$ (or) Also $z=\dfrac{3i(z+1)}{i(iz-3i)}=\dfrac{3z+3}{iz-3i}$

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4) Find the bilinear transformation which maps $z = \infty, i, 0$ into $w = -1, -i, 1$. Also the fixed points of the transformation.

**Sol$^n$:** Let $w = \dfrac{az+b}{cz+d}$ is the required BLT.

Now, $z = \infty$, $w = -1$, the BLT can be written in the form

$$w = \frac{z(a+b/z)}{z(c+d/z)} = \frac{a+b/z}{c+d/z}$$

$$\Rightarrow -1 = \frac{a+0}{c+0} \qquad (1/z = 0 \text{ when } z=\infty)$$

$$a + c = 0 \longrightarrow (1)$$

$z = i$, $w = -i$, $\Rightarrow$ $-i = \dfrac{ai+b}{ci+d}$

$$ai + b - c + di = 0 \rightarrow (2)$$

$z = 0$, $w = 1$ $\Rightarrow$ $1 = \dfrac{0+b}{0+d}$

$$b - d = 0 \longrightarrow (3)$$

(1) + (2) $\Rightarrow$ $a + \cancel{c} + ai + b - \cancel{c} + di = 0$

$$(1+i)a + b + di = 0 \rightarrow (4)$$

Solve (3) & (4),

$$0a + b - d = 0$$

$$(1+i)a + b + di = 0$$

Applying cross multiplication rule,

$$\frac{a}{\begin{vmatrix} 1 & -1 \\ 1 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -1 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 0 & 1 \\ 1+i & 1 \end{vmatrix}}$$

$$\frac{a}{i+1} = \frac{-b}{1+i} = \frac{d}{-(1+i)} \Rightarrow \frac{a}{1} = \frac{-b}{1} = \frac{d}{-1}$$

$\therefore$ $a = 1$, $b = -1$, $d = -1$, in & from (1) $\Rightarrow$ $a + c = 0$

$1 + c = 0$

$\boxed{c = -1}$

$$\therefore w = \frac{z-1}{-1\cdot z-1} \Rightarrow w = \frac{z-1}{-z-1} \Rightarrow \boxed{w = \frac{1-z}{1+z}}$$



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Further, the the fixed points (or) invariant points are obtained by taking $w=z$,

$$\therefore \quad z = \frac{1-z}{1+z}$$

$$z+z^2 = 1-z$$

$$z+z^2-1+z=0$$

$$z^2+2z-1=0$$

$$\therefore \quad z = \frac{-2\pm\sqrt{4+4}}{2} = \frac{-2\pm2\sqrt{2}}{2} = -1\pm\sqrt{2}$$

$\therefore$ The invariant points are $-1+\sqrt{2}$, $-1-\sqrt{2}$

5) Find the BLT which maps $z=\infty, i, 0$ and $w=0, i, \infty$

Sol$^n$: Let $w = \frac{az+b}{cz+d}$ be the required BLT.

Now $z=\infty$, $w=0$, the BLT can be written as

$$w = \frac{z(a+b/z)}{z(c+d/z)} = \frac{a+b/z}{c+d/z}$$

$$\therefore \quad 0 = \frac{a+0}{c+0}$$

$$\Rightarrow a=0 \rightarrow (1)$$

$z=i$, $w=i$, $\Rightarrow$ $i = \frac{ai+b}{ci+d}$

$$ai+b+c-di=0 \rightarrow (2)$$

$z=0$, $w=\infty$, The BLT is written as

$$\frac{1}{w} = \frac{cz+d}{az+b}$$

$$0 = \frac{0+d}{0+b} \qquad \left|\because \frac{1}{w}=0, \text{ if } w=\infty.\right.$$

$$\Rightarrow d=0 \rightarrow (3)$$

Now by using $a=0,\ d=0$ in ② we get

$$0+b+c+0=0$$
$$b=-c$$

Choose $c=1$, we get $b=-1$

Substitute $a=0,\ b=-1,\ c=1,\ d=0$, the required BLT is

$$w=\frac{0+(-1)}{1\cdot z+0}$$

$$w=-\frac{1}{z}$$

6) Find the bilinear transformation which map the points $z=0,1,\infty$ into the points $w=-5,-1,3$ respectively. What are the invariant points in this transformation?

Sol$^n$: Let $w=\dfrac{az+b}{cz+d}$,

$z=0,\ w=-5 \Rightarrow$ $-5=\dfrac{0+b}{0+d}$

$$b+5d=0 \Rightarrow b=-5d \rightarrow ①$$

$z=1,\ w=-1 \Rightarrow$ $-1=\dfrac{a+b}{c+d}$

$$a+b+c+d=0 \rightarrow ②$$

$z=\infty,\ w=3 \Rightarrow$

$$w=\frac{a+b/z}{c+d/z}$$

$$\Rightarrow 3=\frac{a+0}{c+0}$$

$$a=3c \rightarrow ③$$

Sub. ① & ③ in ② we get

$$4c-4d=0 \Rightarrow c=d$$

Choose $c=1 \Rightarrow d=1$. $\therefore a=3,\ b=-5$

$\therefore$ Sub. $a=3,\ b=-5,\ c=1,\ d=1$ in required BLT, we get

$$w=\frac{3z-5}{z+1}$$



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The invariant points are obtained by taking $w=z$,

i.e $z = \frac{3z-5}{z+1}$

$z^2+z = 3z-5$

$z^2-2z+5=0$

$\therefore z = \frac{-(-2)\pm\sqrt{4-20}}{2} = \frac{2\pm\sqrt{-16}}{2} = \frac{2\pm 4i}{2} = 1\pm 2i$

$\therefore z = 1+2i,\ 1-2i$ are the invariant point.

7) Find the BLT which maps the points $z=1, i, -1$ into $w=0, 1, \infty$.

Ans: $a=-(1+i)$, $b=1+i$, $c=1-i$, $d=1-i$

$w = i\left(\frac{1-z}{1+z}\right)$.

8) Find the B.LT which maps $z=0, -i, 2i$ onto $w=5i, \infty, \frac{-i}{3}$ respectively. what are the invariant points of the transformation?

Ans: $a=3i$, $b=5$, $c=-1$, $d=-i$

$w = \frac{-3z+5i}{-iz+1}$

& $z = i, -5i$ are invariant points.

9) Find the BLT which map the points $z=1, i, -1$, to $w=0, i, \infty$

Ans: $w = \frac{z-1}{z+1}$

10) Find the invariant points of the BLT

(i) $w = \frac{z-1+i}{z+2}$ Ans: $-i$, $(i-1)$

(ii) $w = \frac{3z-4}{z-1}$ Ans: $2, 2$



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11) Find the map of the real axis of the Z-plane in the w-plane under the transformation $w = \frac{1}{z+i}$

Sol$^n$: The equation of the real axis of the z-plane is $y=0$ and we have by data

$$w = \frac{1}{z+i}$$

$$\Rightarrow z+i = \frac{1}{w}$$

$$z = \frac{1}{w} - i$$

Hence we have

$$x+iy = \frac{1}{u+iv} - i = \frac{u-iv}{(u+iv)(u-iv)} - i$$

$$x+iy = \frac{u-iv}{u^2+v^2} - i$$

$$x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2} - 1\right)$$

Equating ~~real~~ imaginary parts, we get

$$y = \frac{-v}{u^2+v^2} - 1$$

but $y=0$

$$\Rightarrow \frac{-v}{u^2+v^2} - 1 = 0$$

$$-v - u^2 - v^2 = 0$$

$$u^2+v^2+v = 0$$

$$(u-0)^2 + \left(v+\frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow \cancel{(u-0)^2 + \left(v+\frac{1}{2}\right)}$$

$$\Rightarrow (u-0)^2 + (v-(-1/2))^2 = (1/2)^2$$

This is a circle in the w-plane with centre $(0, -1/2)$ & radius $1/2$.

Thus we conclude that the map of the real axis of the z-plane is a circle in the w-plane.

## Complex integration :-

Let $f(z)$ be a continuous function of the complex variable $z = x+iy$ and $C$ be a curve in the $x$-$y$ plane. The integral of $f(z)$ along the path $C$ is called the complex line integral, usually denoted by $\int_C f(z)dz$.
If $C$ is any simple closed curve, the notation $\oint_C f(z)dz$ is also used.

## Properties :-

1) If $-C$ denotes the curve traversed in the anticlockwise direction, then $\int_{-C} f(z)dz = -\int_C f(z)dz$.

2) If $C$ is split into a number of parts $C_1, C_2, C_3, \ldots$, then $$\int_C f(z)dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \cdots$$

3) If $\lambda_1$ & $\lambda_2$ are constants then, $$\int_C [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\,dz = \lambda_1 \int_C f_1(z)\,dz \pm \lambda_2 \int_C f_2(z)dz.$$

## Line integral of a complex valued function :-

Let $f(z) = u(x,y) + i\,v(x,y)$ be a complex valued function defined over a region $R$ and $C$ be a curve in the region.
Then $\int_C f(z)dz = \int_C (u+iv)(dx+idy)$

ie $$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C v\,dx + u\,dy$$

This shows that the evaluation of a line integral of a complex valued function is nothing but the evaluation of line integral of real ~~f~~ valued function.



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Problems:

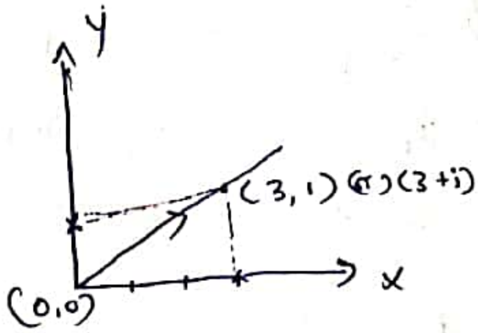
1) Evaluate $\int_C z^2\,dz$

(i) along the straight line from $z=0$ to $z=3+i$

(ii) along the curve made up of two line segments, one from $z=0$ to $z=3$ and another from $z=3$ to $z=3+i$.

Soln: Given

(a) $$\int_C z^2\,dz = \int_{z=0}^{3+i}$$




Here $z$ varies from $0$ to $3+i$.

ie $(x,y)$ varies from $(0,0)$ to $(3,1)$.

For $(0,0)$ & $(3,1)$ the equation of line joining two points is given by

$$\left[\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \quad \text{for } (x_1,y_1) \text{ \& } (x_2,y_2)\right]$$

ie $$\frac{y-0}{x-0} = \frac{1-0}{3-0}$$

$$\Rightarrow \quad y = \frac{x}{3} \quad (\text{or}) \quad x = 3y$$

Now $z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy)$ & $dz = dx + i\,dy$

$$\therefore \int z^2\,dz = \int_{(0,0)}^{(3,1)} \{(x^2-y^2) + i(2xy)\}(dx+i\,dy)$$

$$= \int_{(0,0)}^{(3,1)} \{(x^2-y^2)dx - 2xy\,dy\} + i\int_{(0,0)}^{(3,1)} \{2xy\,dx + (x^2-y^2)dy\}$$

Now convert these integrals into the variable $y$, & then integrate w.r.t $y$ from $0$ to $1$. $\because y = \frac{x}{3}$ & $dx = 3dy$.

$\Rightarrow x = 3y$

$$\therefore \int_C z^2\,dz = \int_{y=0}^{1} \{(9y^2-y^2)3dy - 2(3y)\cdot y\,dy\} + i\int_{y=0}^{1} \{2(3y)\cdot y\cdot 3dy + (9y^2-y^2)dy\}$$

$$= \int_{y=0}^{1} (24y^2-6y^2)dy + i\int_{y=0}^{1} (18y^2+8y^2)\,dy$$

$$= \int_0^1 18y^2\,dy + i\int_0^1 26y^2\,dy = 18\left.\frac{y^3}{3}\right|_0^1 + 26i\left.\frac{y^3}{3}\right|_0^1$$

$$\boxed{\int_C z^2\,dz = 6 + \frac{26}{3}i}$$ along the given path.



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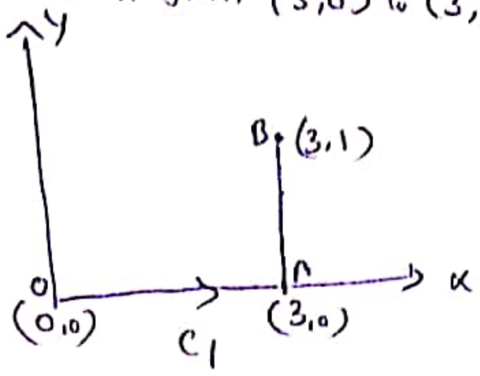
b) Line segments from $z=0$ to $z=3$ & then from $z=3$ to $3+i$

ie $(x,y)$ varies from $(0,0)$ to $(3,0)$ and then from $(3,0)$ to $(3,1)$




$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz \rightarrow ①$$

Now along $C_1$: $y=0 \Rightarrow dy=0$ and $x$ varies from 0 to 3.

Along $C_2$: $x=3 \Rightarrow dx=0$ & $y$ varies from 0 to 1.

$$\int_C z^2 dz = \int_{x=0}^{3} x^2 dx + i\int_{y=0}^{1} (3+iy)^2 dy$$

$$= \frac{x^3}{3}\Big|_0^3 + i\int_{y=0}^{1}(9-y^2+6iy)\,dy$$

$$= \frac{(3)^3}{3} + i\left(9y - \frac{y^3}{3} + 6i\frac{y^2}{2}\right)\Big|_0^1$$

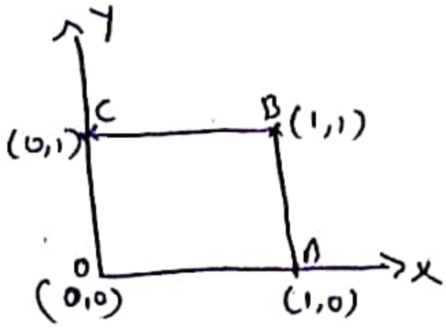
$$= 9 + i\left(9 - \frac{1}{3} + 3i\right)$$

$$= 9 + 9i - \frac{1}{3}i - 3$$

$$\int_C z^2 dz = 6 + \frac{26}{3}i \quad \text{along the given path.}$$

2) Evaluate $\int_C |z|^2 dz$, where $C$ is a square with the following vertices $(0,0)\,(1,0)\,(1,1)\,(0,1)$.

Sol: The curve $C$ is as shown in the figure.




$$\int_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz \rightarrow ①$$

We have $|z|^2 dz = (x^2+y^2)(dx+idy)$

(i) Along $OA\,(C_1)$: $y=0 \Rightarrow dy=0$ & $x$ varies from 0 to 1. $\therefore |z|^2 dz = x^2 dx$ (i.e. $0 \le x \le 1$)

(ii) Along $AB\,(C_2)$: $x=1 \Rightarrow dx=0$ & $y$ varies from 0 to 1. $\therefore |z|^2 dz = (1+y^2)\,i\,dy$ $(0 \le y \le 1)$

(iii) Along BC ($C_3$): $y=1 \Rightarrow dy=0$ & $x$ varies from 1 to 0.

and $|z|^2 dz = (x^2+1)\,dx$ $(1 \le x \le 0)$

(iv) Along CO ($C_4$):- $x=0 \Rightarrow dx=0$ & $y$ varies from 1 to 0.

$\therefore |z|^2 dz = y^2 i\,dy$ $(1 \le y \le 0)$

Use these in eqn ① we get

$$\int_C |z|^2 dz = \int_{x=0}^{1} x^2 dx + \int_{y=0}^{1}(1+y^2)i\,dy + \int_{x=1}^{0}(x^2+1)dx + \int_{y=1}^{0} y^2 i\,dy$$

$$= \frac{x^3}{3}\Big|_0^1 + i\left[y+\frac{y^3}{3}\right]_0^1 + \left[\frac{x^3}{3}+x\right]_1^0 + i\left[\frac{y^3}{3}\right]_1^0$$

$$= \frac{1}{3} + i\left[1+\frac{1}{3}\right] + \left(-\frac{1}{3}-1\right) + \frac{i}{3}[0-1]$$

$$= \cancel{\frac{1}{3}} + i + \cancel{\frac{i}{3}} - \cancel{\frac{1}{3}} - 1 - \cancel{\frac{i}{3}}$$

$\int_C |z|^2 dz = \underline{\underline{-1+i}}$ along the given path.

3) Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along the following paths : ⓐ The parabola $x=2t$, $y=t^2+3$ ⓑ The straight line from $(0,3)$ to $(2,4)$.

Soln:- Let $I = \int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)\,dy$, $x$ varies from 0 to 2 & $y$ varies from 3 to 4.

ⓐ Given $x=2t$, $y=t^2+3$. $\Rightarrow dx = 2dt$, $dy = 2t\,dt$

If $x=0$, $\Rightarrow 0=2t$ i.e. $t=0$ | (or) $y=3$, $3=t^2-$

$x=2 \Rightarrow 2=2t$ i.e. $t=1$

$$\therefore I = \int_{t=0}^{1}\left[2(t^2+3)+(2t)^2\right]2dt + \left[3\cdot 2t-(t^2+3)\right]2t\,dt$$

$$= \int_0^1 (2t^2+6+4t^2)2\,dt + (6t-t^2-3)2t\,dt$$

$$= \int_{t=0}^{1}(12t^2+12+12t^2-2t^3-6t)\,dt$$

$$= \int_{t=0}^{1} (24t^2 - 2t^3 - 6t + 12)\,dt$$

$$= \left[ 24\frac{t^3}{3} - 2\frac{t^4}{4} - 6\frac{t^2}{2} + 12t \right]_0^1$$

$$= 8 - \frac{1}{2} - 3 + 12$$

$$= 17 - \frac{1}{2}$$

$$I = \frac{33}{2}$$

(b) Equation of the straight line joining $(0,3)$ and $(2,4)$ is given by $\frac{y-3}{x-0} = \frac{4-3}{2-0}$

$$\frac{y-3}{x} = \frac{1}{2} \Rightarrow x = 2y - 6$$

$$dx = 2\,dy$$

$$\therefore I = \int_{y=3}^{4} [2y + (2y-6)^2]\,2\,dy + [3(2y-6) - y]\,dy$$

$$= \int_{y=3}^{4} [2y + 4y^2 + 36 - 24y]\,2\,dy + [6y - 18 - y]\,dy$$

$$= \int_{y=3}^{4} (8y^2 + 72 - 44y + 5y - 18)\,dy$$

$$= \int_{y=3}^{4} (8y^2 - 39y + 54)\,dy$$

$$= \left[ 8\frac{y^3}{3} - 39\frac{y^2}{2} + 54y \right]_3^4$$

$$= \frac{8}{3}[64 - 27] - \frac{39}{2}[16 - 9] + 54(4-3)$$

$$= \frac{296}{3} - \frac{273}{2} + 54 = \frac{97}{6}$$

$$I = \frac{97}{6}$$

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4) Evaluate $\int_{1-i}^{2+i}(2x+iy+1)dz$ along the following paths:

(a) $x=t+1$ , $y=2t^2-1$

(b) straight line joining $(1-i)$ and $(2+i)$

Sol: Given $I=\int_{1-i}^{2+i}(2x+iy+1)\,dz$ [ie $x$ varies from $(1,2)$, $y$ varies from $(-1,1)$

(a) $x=t+1$ , $y=2t^2-1 \Rightarrow dx=dt$, $dy=4t\,dt$

If $x=1$, $1=t+1 \Rightarrow t=0$; $x=2$, $2=t+1 \Rightarrow t=1$ $\}$ $t \to 0$ to $1$. & $dz=dx+idy$, $dz=dt+i4t\,dt$

$$I=\int_{t=0}^{1}[2(t+1)+i(2t^2-1)+1][dt+i4t\,dt]$$

$$=\int_{t=0}^{1}(2t+2+2it^2-i+1)(1+i4t)\,dt$$

$$=\int_{t=0}^{1}(2t+2it^2-i+3+8it^2+8i^2t^3-4i^2t+12it)\,dt$$

$$=\int_{t=0}^{1}(-8t^3+10it^2+6t-i+3+12it)\,dt$$

$$=-8\frac{t^4}{4}+10i\frac{t^3}{3}+6\frac{t^2}{2}-it+3t+12i\frac{t^2}{2}\Big|_0^1$$

$$=-2+\frac{10i}{3}+3-i+3+6i$$

$$=4+\frac{10i}{3}+5i$$

$$I=4+\frac{25i}{3}$$



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b) Equation of the straight line joining $(1,-1)$ and $(2,1)$ is given by $\frac{y+1}{x-1} = \frac{1+1}{2-1}$

$$\Rightarrow \frac{y+1}{x-1} = \frac{2}{1}$$

$$\Rightarrow y+1 = 2x-2$$

$$\Rightarrow \boxed{y = 2x-3}$$

$$\therefore dy = 2\,dx \quad \& \quad dz = dx + i\,dy$$

$$\therefore I = \int_{(1,-1)}^{(2,1)} (2x+iy+1)\,dz$$

$$I = \int_{x=1}^{2} [2x + i(2x-3)+1][dx + i2dx]$$

$$= \int_{x=1}^{2} (2x+2ix-3i+1)(1+i2)dx$$

$$= \int_{x=1}^{2} (2x+2ix-3i+1+4ix-4x+6+2i)\,dx$$

$$= \int_{x=1}^{2} (6ix-2x-i+7)\,dx$$

$$= 6i\frac{x^2}{2} - \cancel{2}\frac{x^2}{\cancel{2}} - ix + 7x \Big|_1^2$$

$$= 3i(2^2-1) - (2^2-1) - i(2-1) + 7(2-1)$$

$$= 9i - 3 - i + 7$$

$$= 8i + 4$$

$$= 4(2i+1)$$

$I = 4(1+2i)$ along the given path



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5) If $c$ is a circle with centre $a$ & radius $r$ then S.T (i) $\int_c \frac{dz}{z-a} = 2\pi i$ (ii) $\int_c (z-a)^n dz = 0$ if $n \neq -1$.

Soln: Given curve is

$|z-a| = r$

$\Rightarrow z-a = re^{i\theta}$

$\Rightarrow dz = ire^{i\theta}$ , $\theta \to 0$ to $2\pi$.

wkt $|z| = r$

$|z-a| = |z|$

$z-a = z$

$z-a = re^{i\theta}$

a) $\int_c \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\int_{\theta=0}^{2\pi} d\theta = i[\theta]_0^{2\pi} = 2\pi i$

$\therefore \int_c \frac{dz}{z-a} = 2\pi i$

(b) Also $\int_c (z-a)^n dz = \int_{\theta=0}^{2\pi} (re^{i\theta})^n \, ire^{i\theta} d\theta$

$= ir^{n+1} \int_{\theta=0}^{2\pi} e^{i(n+1)\theta} d\theta$

$= ir^{n+1} \left[ \frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi}$

$= \frac{r^{n+1}}{n+1} \left[ e^{i(n+1)2\pi} - 1 \right]$.

But $e^{i(n+1)2\pi} = \cos(n+1)2\pi + i\sin(n+1)2\pi = 1 + i(0) = 1$

but $\cos 2k\pi = 1$ & $\sin 2k\pi = 0$ for $k = 1, 2, 3, \ldots$

Hence $\int_c (z-a)^n dz = \frac{r^{n+1}}{n+1}[1-1]$

$\therefore \int_c (z-a)^n dz = 0$ , $n \neq -1$



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Cauchy's theorem :-

State :- If $f(z)$ is analytic at all points inside and on a simple closed curve 'C' then $\int_C f(z)\,dz = 0$.

Pf :- Let $f(z) = u + iv$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \quad \rightarrow (1)$$

we have Green's theorem in a plane stating that if $M(x,y)$ & $N(x,y)$ are two real valued functions having continuous first order partial derivatives in a region R bounded by the curve C, then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Apply this theorem in RHS of eqn (1)

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Since $f(z)$ is analytic, we have C-R eqn

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$ then we have

$$\int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$\int_C f(z)\,dz = 0$$

Hence the theorem proved.

Note :

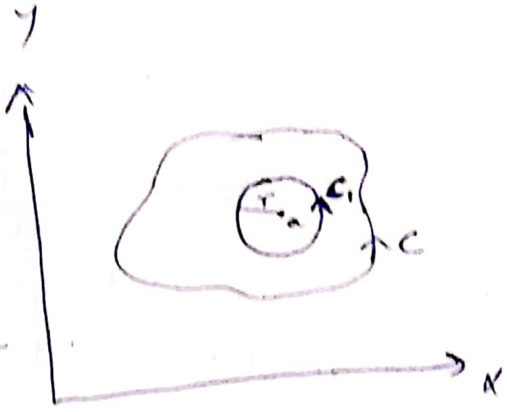
Property :- If $C_1, C_2$ are two simple closed curves such that $C_2$ lies entirely within $C_1$ & if $f(z)$ is analytic on $C_1, C_2$, and in the region bounded by $C_1, C_2$ then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

## *** Cauchy's integral formula :-

**State** :- If $f(z)$ is analytic inside and on a simple closed curve $C$ and if '$a$' is any point within $C$ then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$




**Proof** :- Since '$a$' is a point within $C$, we shall enclose it by a circle $C_1$ with $z=a$ as centre and $r$ as radius such that $C_1$ lies entirely within $C$.

The function $\frac{f(z)}{z-a}$ is a analytic inside and on the boundary of a the annular region b/w $C$ & $C_1$.

By a property, we get

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \longrightarrow ①$$

The equation of $C_1$ (centre with centre $a$ & radius $r$) can be written as

$|z-a| = r$.

$\Rightarrow z-a = re^{i\theta} \Rightarrow z = a + re^{i\theta}, \quad 0 \le \theta < 2\pi$

$$dz = ire^{i\theta}d\theta$$

Using these results in RHS of ①, we obtain

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

This is true for any $r>0$, however small. Hence as $r\to 0$ we get

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta = if(a).\theta\Big|_0^{2\pi} = 2\pi i f(a)$$

$$\Rightarrow \boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$

Hence proved.

## Generalized Cauchy's integral formula:-

**State:-** If $f(z)$ is a analytic inside and on a simple closed curve C and if 'a' is a point on C then

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

**Pf:-** we have Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Diff w.r.t. a

$$\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right) = \frac{\partial}{\partial a}(z-a)^{-1}$$

$$\therefore f'(a) = \frac{1}{2\pi i}\int_C f(z)\frac{\partial}{\partial a}\left[\frac{1}{z-a}\right]dz$$

$$= \frac{1}{2\pi i}\int_C f(z)\left\{(-1)(z-a)^{-2}(-1)\right\}dz$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

Now

$$f''(a) = \frac{1!}{2\pi i}\int_C f(z)\frac{\partial}{\partial a}(z-a)^{-2}\,dz$$

$$= \frac{1!}{2\pi i}\int f(z)(-2)(z-a)^{-3}(-1)\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

Continuing like this, after differentiating n times we obtain

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Here $f^n(a)$ denotes $n^{th}$ derivative of $f(z)$ at $z=a$.

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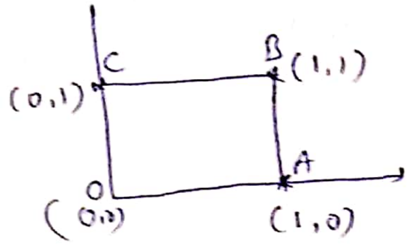
Problems :

1) Verify Cauchy's theorem for the function $f(z) = z^2$ where C is the square having vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Sol$^n$: C is the square OABC and we have Cauchy's theorem

$$\int_C f(z)\,dz = 0 \quad \text{where } f(z) = z^2 = (x+iy)^2$$




$$\Rightarrow \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz = 0$$

$$\int_{OA} z^2dz + \int_{AB} z^2dz + \int_{BC} z^2dz + \int_{CO} z^2dz = 0 \longrightarrow (1)$$

Along OA : $y = 0$, $dy = 0$, $x \to 0$ to $1$

$$\therefore \int_{OA} z^2dz = \int_{x=0}^{1} (x+iy)^2(dx+idy) = \int_{x=0}^{1} x^2\,dx = \frac{x^3}{3}\Big|_0^1 = 1/3$$

$$\int_{OA} z^2dz = \frac{1}{3} \longrightarrow (2)$$

Along AB :- $x = 1 \Rightarrow dx = 0$ & $y$ varies from 0 to 1

$$\int_{AB} z^2dz = \int_{y=0}^{1} (x+iy)^2(dx+idy) = \int_0^1 (1+iy)^2\,i\,dy$$

$$= \int_{y=0}^{1} (1-y^2+2iy)\,i\,dy = i\left[y - \frac{y^3}{3} + 2i\frac{y^2}{2}\right]_0^1 = i\left(1-\frac{1}{3}+i\right)$$

$$\int_{AB} z^2dz = -1 + \frac{2i}{3} \longrightarrow (3)$$

Along BC :- $y = 1 \Rightarrow dy = 0$ & $x$ varies from 1 to 0.

$$\therefore \int_{BC} z^2dz = \int_{x=1}^{0} (x+iy)^2(dx+idy) = \int_{x=1}^{0} (x+i)^2\,dx = \int_1^0 (x^2-1+2xi)\,dx$$

$$= \frac{x^3}{3} - x + 2\frac{x^2}{2}i\Big|_1^0 = 0 - \left[\frac{1}{3} - 1 + i\right] = -\frac{1}{3} + 1 - i$$

$$\int_{BC} z^2dz = \frac{2}{3} - i \longrightarrow (4)$$

Along CO :- $x = 0 \Rightarrow dx = 0$, $y$ varies from 1 to 0.

$$\int_{CO} z^2dz = \int_{y=1}^{0} (iy)^2\,i\,dy = i\cdot i^2\,\frac{y^3}{3}\Big|_1^0 = -i\left[0-\frac{1}{3}\right] = i/3 \longrightarrow (5)$$

Sub. (2), (3), (4) & (5) in eq$^n$ (1)

$$\int_C f(z)dz = \frac{1}{3} - 1 + \frac{2i}{3} + \frac{2}{3} - i + i/3 = \frac{1-3+2}{3} + \frac{2i-3i-i}{3} = 0$$

$\therefore \int f(z)dz = 0$ Hence Cauchy's theorem verified. //

2) Evaluate $\int_C \frac{dz}{z^2-4}$ over the following curve $C$:

(i) $C: |z|=1$ (ii) $C: |z|=3$, (iii) $C: |z+2|=1$.

Sol: Consider $\frac{1}{z^2-4} = \frac{1}{(z-2)(z+2)}$

By partial fraction method,

$$\frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$$

$$1 = (z+2)A + (z-2)B$$

put $z=2$, $1 = A(4) \Rightarrow A = 1/4$

$z=-2$, $1 = B(-4) \Rightarrow B = -1/4$

$$\therefore \frac{1}{(z-2)(z+2)} = \frac{1}{4}\cdot\frac{1}{z-2} - \frac{1}{4}\,\frac{1}{z+4}$$

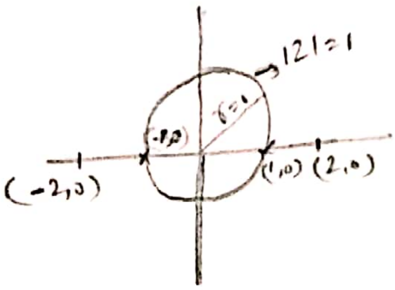
$$\therefore \int_C \frac{dz}{z^2-4} = \int \frac{1}{(z-2)(z+2)}\,dz = \frac{1}{4}\int_C \frac{dz}{z-2} - \frac{1}{4}\int_C \frac{dz}{z+(2)} \longrightarrow (1)$$

$$\int_C \frac{f(z)}{z-a} \qquad \int_C \frac{f(z)}{z-a}$$

(a) $C: |z|=1$, $z=a=2$ & $z=a=-2$

Here $|z|=1$ is a circle with centre origin & radius 1.

The points $z=a=2$, ie $(2,0)$ & $z=a=-2$ $(-2,0)$ lies out side the circle $|z|=1$.




Thus by Cauchy's theorem

$$\int_C \frac{dz}{z^2-4} = 0, \quad |z|=1.$$

(b) $C: |z|=3$, $z=a=2$, $z=a=-2$

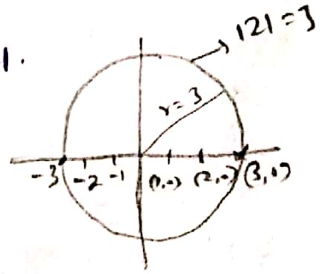
Here $|z|=3$ is a circle with centre origin & radius 3

The points $z=a=2$, $z=a=-2$ lies inside the circle.

Also in each of the integrals as in RHS of (1), $f(z)=1$.

Apply Cauchy's integral formula,

$$\int \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$



$\therefore \int \frac{dz}{z-2} = 2\pi i f(2)$

$= 2\pi i (1)$

$= 2\pi i$

| $z=a$
$\Rightarrow f(z) = f(a) \Rightarrow f(2) = f(2)$
$\Rightarrow f(2) = 1$

also $\int \frac{dz}{z+2} = 2\pi i f(-2)$

$= 2\pi i (1)$

$= 2\pi i$

Substitute these in RHS of (1)

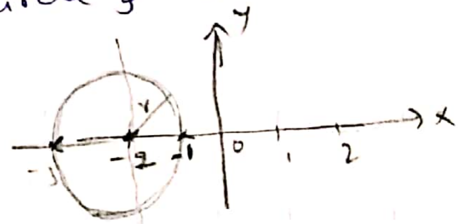
$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i) = 0$$

$\therefore \int_C \frac{dz}{z^2-4} = 0$ , where $C : |z| = 3$.

(c) $C : |z+2| = 1$ This is a circle with centre $(-2, 0)$ & radius 1.

& the points are $z = a = 2$, $z = a = -2$.

Hence the point $z = a = 2$ lies outside the circle & $z = a = -2$ lies inside the circle.




$\therefore$ By Cauchy's theorem, $\int_C \frac{dz}{z-2} = 0$.

Now apply Cauchy's integral formula for 2nd integral in RHS of (1)

ie $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$

$\Rightarrow \int_C \frac{dz}{z-(-2)} = 2\pi i f(-2)$

| $f(z) = 1$
$\Rightarrow f(a) = 1$
$\Rightarrow f(-2) = 1$

$= 2\pi i (1)$

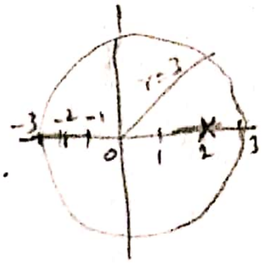
$\therefore \int_C \frac{dz}{z-(-2)} = 2\pi i$

$\therefore$ eqn (1) becomes

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}(0) - \frac{1}{4}(2\pi i)$$

$$\int_C \frac{dz}{z^2-4} = -\frac{\pi i}{2} \quad ; \; C : |z+2| = 1$$

3) Evaluate $\int_C \frac{z^2+z+1}{(z-2)^3}\,dz$ over $|z|=3$.



Sol$^n$: Here $z=a=2$ lies inside the circle $|z|=3$.

we have $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$

$$\Rightarrow \int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

Take $f(z)=z^2+z+1$, $a=2$, $n+1=3$ $\Rightarrow n=2$

$f'(z)=2z+1$

$f''(z)=2 \Rightarrow f''(2)=2$

$$\therefore \int_C \frac{z^2+z+1}{(z-2)^3}\,dz = \frac{\cancel{2}\pi i}{\cancel{2!}} f''(2)$$

$$= \pi i\cdot(2)$$

$$\int_C \frac{z^2+z+1}{(z-2)^3}\,dz = 2\pi i$$

*** 4) Evaluate $\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz$, where $C: |z|=3$.

Sol$^n$: First we shall resolve $\frac{1}{(z+1)^2(z-2)}$ into partial fractions.

ie $\frac{1}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$

$1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$

if $z=-1 \Rightarrow 1=-3B \Rightarrow B=-1/3$

$z=2 \Rightarrow 1=9C \Rightarrow C=1/9$

$z=0 \Rightarrow 1=-2A-2B+C \Rightarrow A=-1/9$

$\therefore \dfrac{1}{(z+1)^2(z-2)} = \dfrac{-1}{9}\cdot\dfrac{1}{z+1} - \dfrac{1}{3}\,\dfrac{1}{(z+1)^2} + \dfrac{1}{9}\cdot\dfrac{1}{z-2}$

multiply $e^{2z}$ and integrate w.r.t $z$ over $c$.

$$\int_c \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = \frac{-1}{9}\int_c \frac{e^{2z}}{z+1}\,dz - \frac{1}{3}\int_c \frac{e^{2z}}{(z+1)^2}\,dz + \frac{1}{9}\int_c \frac{e^{2z}}{z-2}\,dz \longrightarrow (1)$$

Here the points $z=a=-1$ & $z=a=2$ both lies inside the circle $|z|=3$.

Hence use Cauchy's integral formulas in RHS of (1). in the form

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \quad \& \quad \int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a).$$

where $f(z) = e^{2z}$, & $f'(z) = 2e^{2z}$, $n+1=2$, $n=1$

$$\therefore \int_c \frac{e^{2z}}{z+1}\,dz = 2\pi i\, f(-1) = 2\pi i\,(e^{-2}) = \frac{2\pi i}{e^2}$$

$$\& \int_c \frac{e^{2z}}{(z+1)^2}\,dz = \frac{2\pi i}{1!}\cdot f'(-1) = \frac{2\pi i\,(2e^{-2})}{1} = \frac{4\pi i}{e^2}$$

$$\text{Also} \int \frac{e^{2z}}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i\,(e^4) = 2\pi i\, e^4$$

Substitute these values in RHS of (1) we get

$$\int_c \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = \frac{-1}{9}\,\frac{2\pi i}{e^2} - \frac{1}{3}\,\frac{4\pi i}{e^2} + \frac{1}{9}\,2\pi i\, e^4$$

$$= \frac{2\pi i}{e^2}\left[\frac{-1}{9} - \frac{2}{3}\right] + \frac{2\pi i e^4}{9} = \frac{2\pi i}{e^2}\left[\frac{-1-6}{9}\right] + \frac{2\pi i e^4}{9}$$

$$= -\frac{7}{9}\,\frac{2\pi i}{e^2} + \frac{2}{9}\pi i e^4$$

$$\int_c \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = \frac{2\pi i}{9}\left[e^4 - \frac{7}{e^2}\right] \;//$$

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5) Evaluate $\int_C \frac{e^{3z}}{z^2} dz$ over $|z|=1$.

$z^2=(z-0)^2$
$=(z-a)^2$
$a=0$.

Soln: Here the point $z=a=0$ lies with in the circle $|z|=1$.

and we have Cauchy's Generalized integral form,

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{n}(a).$$

where $f(z)=e^{3z}$, $f'(z)=3e^{z}$, $a=0$, $n+1=2$
$n=1$

$$\therefore \int \frac{e^{3z}}{z^2} dz = \frac{2\pi i}{1!} f'(a)$$

$$= \frac{2\pi i}{1!}(3e^{0})$$

$$\int \frac{e^{3z}}{z^2} = 6\pi i$$

6) Evaluate $\int \frac{e^{2z}}{(z+1)(z-2)} dz$ where $C$ is the circle $|z|=3$.

Soln:- Consider $\frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$

$\Rightarrow 1 = A(z-2) + B(z+1)$

If $z=-1$, $1=-3A \Rightarrow A=-1/3$

$z=2$, $1=B(3) \Rightarrow B=1/3$

$$\therefore \frac{1}{(z+1)(z-2)} = -\frac{1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{1}{z-2}$$

Multiply $e^{2z}$ & integrate w.r.t $z$ over $C$

$$\int_C \frac{e^{2z}}{(z+1)(z-2)} dz = -\frac{1}{3}\int_C \frac{1}{z+1} dz + \frac{1}{3}\int_C \frac{1}{z-2} dz \longrightarrow (1)$$

Here the points $z=a=-1$, & $z=a=2$ both lies inside the circle $|z|=3$. Now use Cauchy's integral formula

$$\int \frac{f(z)}{z-a} dz = 2\pi i f(a),$$ where $f(z)=e^{2z}$

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$$\Rightarrow \int_C \frac{e^{2z}}{z+1}\,dz = 2\pi i\, f(a)$$
$$= 2\pi i\, f(-1)$$
$$= 2\pi i\,(e^{-2})$$
$$\int \frac{e^{2z}}{z+1}\,dz = \frac{2\pi i}{e^2}$$

Also,

$$\int_C \frac{e^{2z}}{z-2}\,dz = 2\pi i\, f(2)$$
$$= 2\pi i\, e^{2(2)}$$
$$= 2\pi i e^4$$

Substitute these in eq$^n$ 1,

$$\int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz = -\frac{1}{3}\left[\frac{2\pi i}{e^2}\right] + \frac{1}{3}\left[2\pi i e^4\right]$$
$$= \frac{2\pi i}{3}\left[e^4 - \frac{1}{e^2}\right]$$

***

7) Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\,dz$ where $C$ is the circle with $|z|=3$, $|z|=1/2$, $|z|=3/2$

Sol$^n$:- Consider $\frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$.

$$\Rightarrow 1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

If $z=1 \Rightarrow 1 = -B \Rightarrow B = -1$

$z = 2 \Rightarrow 1 = C \Rightarrow C = 1$

comp. coefficient of $z^2$ : $A + C = 0 \Rightarrow A = -C$
$A = -1$

$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{-1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{z-2}$$

$$\therefore \int_C \frac{f(z)}{(z-1)^2(z-2)}\,dz = -\int_C \frac{f(z)}{z-1}\,dz - \int_C \frac{f(z)}{(z-1)^2}\,dz + \int_C \frac{f(z)}{z-2}\,dz \longrightarrow 1$$

where $f(z) = \sin \pi z^2 + \cos \pi z^2$
$f'(z) = \cos \pi z^2 (2\pi z) - \sin \pi z^2 (2\pi z)$

(i) $C: |z|=3$, The points $z=a=1$, $z=a=2$, lies inside the circle $|z|=3$.

$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a), \quad \int \frac{f(z)}{(z-a)^{n+1}}\,dz = 2\pi i f^n(a)$$

Now, $$\int_C \frac{f(z)}{z-1}\,dz = 2\pi i f(1)$$
$$= 2\pi i(\sin\pi + \cos\pi)$$
$$= 2\pi i(-1)$$
$$= -2\pi i$$

& $$\int \frac{f(z)}{(z-1)^2}\,dz = 2\pi i f'(1)$$
$$= 2\pi i\left(\cos\pi(2\pi) - \cancel{\sin\pi(2\pi)}^{\,0}\right)$$
$$= 2\pi i(-2\pi)$$
$$= -4\pi^2 i$$

Also $$\int \frac{f(z)}{z-2}\,dz = 2\pi i f(2)$$
$$= 2\pi i\left(\cancel{\sin\pi(2^2)}^{\,0} + \cos\pi(2)^2\right) \quad | \cos 4\pi = 1$$
$$= 2\pi i(+1)$$
$$= 2\pi i$$

$\therefore$ (1) $\Rightarrow$ $$\int_C \frac{f(z)}{(z-1)^2(z-2)}\,dz = -(-2\pi i) - (-4\pi^2 i) + 2\pi i$$
$$= 2\pi i + 4\pi^2 i + 2\pi i$$
$$= 4\pi i + 4\pi^2 i$$

$$\int_C \frac{f(z)}{(z-1)^2(z-2)}\,dz = \underline{\underline{4\pi i(1+\pi)}}. \quad , \quad C: |z|=3.$$

(ii) $C: |z|=1/2$, but $z=a=1$, $z=a=2$ lies out side the circle $|z|=1/2$.

Hence by Cauchy's theorem, we obtain

$$\int_C \frac{f(z)}{(z-1)^2(z-2)}\,dz = 0.$$

(iii) $C: |z| = 3/2$, ~~Here~~ the point $z = a = 1$ lies inside & $z = a = 2$ lies outside the circle $|z| = 3/2$.

$$\therefore \; ① \Rightarrow \int \frac{f(z)}{(z-1)^2(z-2)}\,dz = -(2\pi i(-1)) - \frac{2\pi i}{1!}(-2\pi) + 0$$

$$= 2\pi i + 4\pi^2 i$$

$$\therefore \int \frac{f(z)}{(z-1)^2(z-2)}\,dz = 2\pi i(1+2\pi)$$