Continuous Probability Distribution - Definition:

Let 'x' be a Continuous random variable which takes the values between '$-\infty$ to $+\infty$'. A function $f(x=x)$ is said to be Continuous probability distribution function if it satisfies the two conditions,

$$\left.\begin{array}{l}(i)\ f(x) \geq 0, \text{ for all } x \\ (ii)\ \int_{-\infty}^{\infty} f(x)\,dx = 1\end{array}\right\} \longrightarrow (1)$$

It is also called as probability density function (p.d.f).

* The probability of a Continuous random variable 'x' that lies in (a, b) is defined as

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx \longrightarrow (2)$$

Cumulative distribution function (c.d.f)

If X is a Continuous random variable with probability density function $f(x)$, then the cumulative distribution function is denoted by '$F(x)$' and it is defined as

$$\left.\begin{array}{l}F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \\ F(x) = P(-\infty \leq X \leq x)\end{array}\right\} \longrightarrow (3)$$

* $\frac{d}{dx}[F(x)] = f(x)$
* $P(x \geq \gamma) = \int_\gamma^\infty f(x)\,dx$, γ is any real number.
* $P(x < \gamma) = 1 - P(x \geq \gamma)$

$$P(x < \gamma) = 1 - \int_\gamma^\infty f(x)\,dx$$

* Mean of a Continuous random variable $(X)(\mu) = \int_{-\infty}^{\infty} x\, f(x)\,dx$
* Variance of a Continuous random variable$(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

(or)

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

Problems:

① Find which of the following functions is a Probability density function.

(a) $f_1(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise.} \end{cases}$ (b) $f_2(x) = \begin{cases} 2x, & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f_3(x) = \begin{cases} |x|, & \text{[redacted]} \ |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$ (d) $f_4(x) = \begin{cases} 2x, & 0<x\le 1 \\ 4-4x, & 1<x<2 \\ 0, & \text{otherwise.} \end{cases}$

Sol: To say the function $f(x)$ is probability density function if it satisfies (i) $f(x) \ge 0, \forall x$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(a) Given $f_1(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise.} \end{cases}$

(i) $\because f_1(x) \ge 0$, for all x

(ii) Consider, $\int_{-\infty}^{\infty} f_1(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 2x\,dx + \int_{1}^{\infty} 0\,dx$

$= 0 + 2\int_0^1 x\,dx + 0$

$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_0^1$

$= 1 - 0$

$\int_{-\infty}^{\infty} f_1(x)\,dx = 1$

$\therefore f_1(x)$ satisfies two conditions of p.d.f.

Hence $f_1(x)$ is a probability density function.

(b) Homework $\longrightarrow$ not a p.d.f ($\because f_2(x) \ngeq 0$ for $-1<x<1$)

(d) Homework $\longrightarrow$ not a p.d.f ($\because f_4(x) \ngeq 0$ for $1<x<2$)

(c) Given $f_3(x) = \begin{cases} |x|, & \text{[redacted]} \ |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$

(i) $f_3(x) \ge 0$ for all x ($\because f_3(x) = |x|$, [redacted] $|x| \le 1$ and $|x|$ is positive)

(ii) Consider,

$$\int_{-\infty}^{\infty} f_3(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx + \int_{1}^{\infty} 0\,dx$$

$$= 0 - \left(\frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2}\right)_{0}^{1} + 0$$

$$= -\left[0 - \frac{(-1)^2}{2}\right] + \left(\frac{1}{2} - 0\right) + 0$$

$$= -\left[-\tfrac{1}{2}\right] + \frac{1}{2}$$

$$= \tfrac{1}{2} + \tfrac{1}{2}$$

$$\int_{-\infty}^{\infty} f_3(x)\,dx = 1$$

Definition of modulus function

$$\because |x| = \begin{cases} x, & 0 < x < \infty \\ -x, & -\infty < x < 0 \end{cases}$$

and $\because f_3(x) = |x|,\ |x| \le 1$

$$= |x|,\ -1 \le x \le 1$$

$$= \begin{cases} -x, & -1 \le x < 0 \\ x, & 0 < x \le 1 \end{cases}$$

Hence $f_3(x)$ satisfied two conditions of p.d.f. So $f_3(x)$ is a p.d.f.

② Find the constant 'k' such that $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f. Also compute (i) $P(1 < x < 2)$ (ii) $P(x \le 1)$ (iii) $P(x > 1)$ (iv) Mean (v) Variance.

Sol Given $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f.

$\Rightarrow$ k must be positive

and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{0}^{3} kx^2\,dx = 1$ $\quad\because f(x) = 0$ in $-\infty < x < 0$ and $3 < x < \infty$

$\Rightarrow k\left[\frac{x^3}{3}\right]_0^3 = 1$

$\Rightarrow \frac{k}{3}\left[3^3 - 0\right] = 1 \Rightarrow \frac{k(\cancel{27}^{9})}{\cancel{3}} = 1 \Rightarrow \boxed{k = 1/9}$

$\therefore f(x) = \begin{cases} \frac{x^2}{9}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$

(i) $P(1 < x < 2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{x^2}{9} dx = \frac{1}{9}\left(\frac{x^3}{3}\right)_1^2$

$= \frac{1}{27}(2^3 - 1^3) = \frac{7}{27}$

$\therefore \boxed{P(1 < x < 2) = \frac{7}{27}}$

(ii) $P(x \leq 1) = \int_{-\infty}^{1} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^1 \frac{x^2}{9} dx = 0 + \frac{1}{9}\left(\frac{x^3}{3}\right)_0^1$

$= \frac{1}{27}(1 - 0)$

$\boxed{P(x \leq 1) = \frac{1}{27}}$

(iii) $P(x > 1) = \int_1^{\infty} f(x)\,dx = \int_1^3 \frac{x^2}{9} dx + \int_3^{\infty} 0\,dx = \frac{1}{9}\left(\frac{x^3}{3}\right)_1^3 + 0$

$= \frac{1}{27}(3^3 - 1^3) = \frac{26}{27}$

$\therefore \boxed{P(x > 1) = \frac{26}{27}}$

(iv) Mean $(\mu) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^3 x \frac{x^2}{9} dx + \int_3^{\infty} 0\,dx$

$= 0 + \frac{1}{9}\int_0^3 x^3 dx + 0$

$\mu = \frac{1}{9}\left(\frac{x^4}{4}\right)_0^3 = \frac{1}{36}(3^4 - 0) = \frac{81}{36} = \frac{9}{4}$

$\boxed{\mu = \frac{9}{4}}$

(v) Variance $(V) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$= \int_0^3 x^2 \frac{x^2}{9} dx - \left(\frac{9}{4}\right)^2$ $\quad \because f(x) = 0, \; -\infty < x < 0, \; 3 < x < \infty$

$= \frac{1}{9}\int_0^3 x^4 dx - \frac{81}{16}$

$V = \sigma^2 = \frac{1}{9}\left(\frac{x^5}{5}\right)_0^3 - \frac{81}{16} = \frac{27}{80}$

(3) A random variable x has the following density function

$$P(x) = \begin{cases} Kx^2 & , -3 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$$, Evaluate 'K' and find

(i) $P(1 \le x \le 2)$ (ii) $P(x \le 2)$ (iii) $P(x > 1)$

Sol Home work. Here p.d.f $[f(x)] = P(x)$, take $f(x)$ as $P(x)$

(4) Find 'k' so that the following function can serve as a probability density function of a random variable.

$$f(x) = \begin{cases} 0 & , \text{for } x \le 0 \\ kx e^{-4x^2} & , \text{for } x > 0 \end{cases}$$

sol: Given $f(x)$ is p.d.f, so

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \quad (\because \text{by the definition})$$

$$\Rightarrow \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} kx e^{-4x^2} dx = 1$$

put $4x^2 = t$

$4(2x)\, dx = dt$

$x\, dx = \frac{dt}{8}$

limits

as $x \to 0$, $t \to 0$

$x \to \infty$, $t \to \infty$

$$\Rightarrow 0 + \int_{0}^{\infty} k e^{-t} \frac{dt}{8} = 1$$

$$\frac{k}{8}\left(\frac{e^{-t}}{-1}\right)_0^{\infty} = 1$$

$$\frac{-k}{8}\left(e^{-\infty} - e^{-0}\right) = 1$$

$$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} \to 0$$

$$\frac{-k}{8}(0-1) = 1$$

$$\boxed{K = 8}$$

⑤ Find the value of 'c' such that $f(x) = \begin{cases} \frac{x}{6} + c, & 0 \le x \le 3 \\ 0 & , \text{elsewhere} \end{cases}$

is a p.d.f. Also find $P(1 \le x \le 2)$

Sol Homework. $c = \frac{1}{12}$, $P(1 \le x \le 2) = 1/3$

⑥ Find K such that $f(x) = \begin{cases} Kxe^{-x}, & 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$ is a p.d.f!

Find mean.

Sol: $K = \frac{e}{e-2}$, $\mu = \frac{2e-5}{e-2}$

⑦ A Continuous Random variable has the distribution function

$F(x) = \begin{cases} 0 & , x \le 1 \\ c(x-1)^4, & 1 \le x \le 3 \\ 1 & , x > 3 \end{cases}$ Find 'c' and also the p.d.f

Sol We know that p.d.f, $f(x) = \frac{d}{dx}[F(x)]$

$$= \frac{d}{dx}\begin{cases} 0 & , x \le 1 \\ c(x-1)^4, & 1 \le x \le 3 \\ 1 & , x > 3 \end{cases}$$

$$f(x) = \begin{cases} 0 & , x \le 1 \\ 4c(x-1)^3, & 1 \le x \le 3 \\ 0 & , x > 3 \end{cases}$$

Since $f(x)$ is p.d.f, so by the definition,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_1^3 4c(x-1)^3\, dx = 1$$

$$\Rightarrow \cancel{4}c\left[\frac{(x-1)^4}{\cancel{4}}\right]_1^3 = 1 \Rightarrow c\left[(3-1)^4 - 0\right] = 1$$

$$\Rightarrow c(2^4) = 1$$

$$\Rightarrow \boxed{c = 1/16}$$

8) Find the c.d.f for the following p.d.f of a random variable x.

(i) $f(x) = \begin{cases} 6x - 6x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(ii) $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & 0 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$

Sol (i) By the definition of c.d.f,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_0^1 (6x - 6x^2)\, dx = \left\{ \cancel{6}^{3}\left(\frac{x^2}{\cancel{2}}\right) - \cancel{6}^{2}\left(\frac{x^3}{\cancel{3}}\right) \right\}_0^x$$

$$= \left[3(x)^2 - 2(x)^3\right] - \left[3(0) - 2(0)\right]$$

$$= 3x^2 - 2x^3 \quad , \; 0 \leq x \leq 1$$

$$\Rightarrow F(x) = 3x^2 - 2x^3, \; 0 \leq x \leq 1$$

(ii) $F(x) = 1 - e^{-x/2} - \frac{x}{2} e^{-x/2}$, if $0 < x < \infty$