Complex integration :-

Let $f(z)$ be a continuous function of the complex variable $z = x+iy$ and C be a curve in the x-y plane. The integral of $f(z)$ along the path C is called the complex line integral, usually denoted by $\int_C f(z)dz$. If C is any simple closed curve, the notation $\oint_C f(z)dz$ is also used.

Properties :-

1) If $-C$ denotes the curve traversed in the anticlockwise direction, then $\int_{-C} f(z)dz = -\int_C f(z)dz$.

2) If C is split into a number of parts $C_1, C_2, C_3, \ldots$, then

$$\int_C f(z)dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \cdots$$

3) If λ_1 & λ_2 are constants then,

$$\int_C [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\,dz = \lambda_1 \int_C f_1(z)\,dz \pm \lambda_2 \int_C f_2(z)dz.$$

Line integral of a complex valued function :-

Let $f(z) = u(x,y) + iv(x,y)$ be a complex valued function defined over a region R and C be a curve in the region. Then

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

ie $$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C v\,dx + u\,dy$$

This shows that the evaluation of a line integral of a complex valued function is nothing but the evaluation of line integral of real valued function.

Problems:

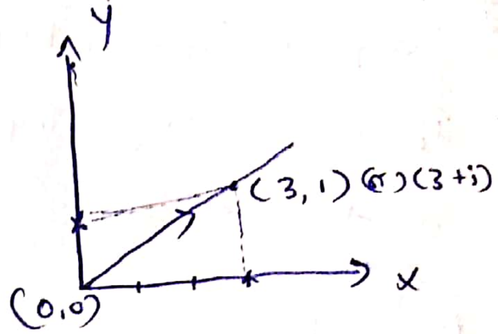
1) Evaluate $\int_C z^2 dz$

(i) along the straight line from $z=0$ to $z=3+i$

(ii) along the curve made up of two line segments, one from $z=0$ to $z=3$ and another from $z=3$ to $z=3+i$.

Soln: Given

(a)
$$\int_C z^2\,dz = \int_{z=0}^{3+i}$$



Here z varies from 0 to $3+i$.

ie (x,y) varies from $(0,0)$ to $(3,1)$.

For $(0,0)$ & $(3,1)$ the equation of line joining two points is given by

$$\left[\ \frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \quad \text{for } (x_1,y_1) \text{ \& } (x_2,y_2)\right]$$

ie
$$\frac{y-0}{x-0} = \frac{1-0}{3-0}$$

$$\Rightarrow y = \frac{x}{3} \quad (\text{or}) \quad x = 3y$$

Now $z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy)$ & $dz = dx + i\,dy$

$$\therefore \int z^2\,dz = \int_{(0,0)}^{(3,1)} \{(x^2-y^2) + i(2xy)\}(dx+i\,dy)$$

$$= \int_{(0,0)}^{(3,1)} \{(x^2-y^2)dx - 2xy\,dy\} + i\int_{(0,0)}^{(3,1)} \{2xy\,dx + (x^2-y^2)dy\}$$

Now convert these integrals into the variable y, & then integrate w.r.t y from 0 to 1. $\because y = \frac{x}{3}$ & $dx = 3dy$.
$\Rightarrow x = 3y$

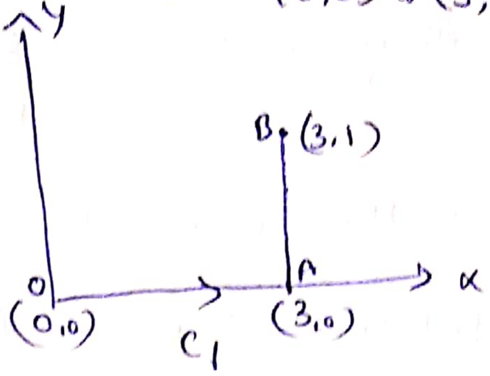
$$\therefore \int_C z^2 dz = \int_{y=0}^{1} \{(9y^2-y^2)3dy - 2(3y)\cdot y\,dy\} + i\int_{y=0}^{1} \{2(3y)\cdot y\cdot 3dy + (9y^2-y^2)dy\}$$

$$= \int_{y=0}^{1} (24y^2-6y^2)dy + i\int_{y=0}^{1}(18y^2+8y^2)\,dy$$

$$= \int_0^1 18y^2\,dy + i\int_0^1 26y^2\,dy = 18\left.\frac{y^3}{3}\right|_0^1 + 26i\left.\frac{y^3}{3}\right|_0^1$$

$$\boxed{\int_C z^2 dz = 6 + \frac{26}{3}i}$$ along the given path.

b) Line segments from $z=0$ to $z=3$ & then from $z=3$ to $3+i$
ie (x,y) varies from $(0,0)$ to $(3,0)$ and then from $(3,0)$ to $(3,1)$



$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz \rightarrow ①$$

Now along C_1: $y=0 \Rightarrow dy=0$ and x varies from 0 to 3.

Along C_2: $x=3 \Rightarrow dx=0$ & y varies from 0 to 1.

$$\int_C z^2 dz = \int_{x=0}^{3} x^2 dx + i\int_{y=0}^{1} (3+iy)^2 dy$$

$$= \frac{x^3}{3}\Big|_0^3 + i\int_{y=0}^{1} (9 - y^2 + 6iy)\, dy$$

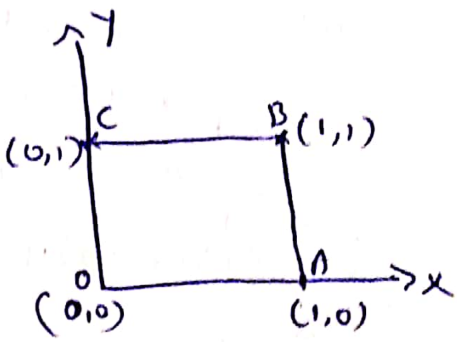
$$= \frac{(3)^3}{3} + i\left(9y - \frac{y^3}{3} + \frac{6iy^2}{2}\right)\Big|_0^1$$

$$= 9 + i\left(9 - \frac{1}{3} + 3i\right)$$

$$= 9 + i9 - \frac{1}{3}i - 3$$

$$\int_C z^2 dz = 6 + \frac{26}{3}i \quad \text{along the given path.}$$

2) Evaluate $\int_C |z|^2 dz$, where C is a square with the following vertices $(0,0)\,(1,0)\,(1,1)\,(0,1)$.



Sol^n: The curve C is as shown in the figure.

$$\int_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz \rightarrow ①$$

We have $|z|^2 dz = (x^2+y^2)(dx + i\,dy)$

(i) Along OA (C_1): $y=0 \Rightarrow dy=0$ & x varies from 0 to 1. $\therefore |z|^2 dz = x^2 dx$ (ie $0 \le x \le 1$)

(ii) Along AB (C_2): $x=1 \Rightarrow dx=0$ & y varies from 0 to 1. $\therefore |z|^2 dz = (1+y^2)\,i\,dy$ $(0 \le y \le 1)$

(iii) Along BC (C_3): $y=1 \Rightarrow dy=0$ & x varies from 1 to 0.

and $|z|^2 dz = (x^2+1)\,dx$ $\quad (1 \le x \le 0)$

(iv) Along CO (C_4):- $x=0 \Rightarrow dx=0$ & y varies from 1 to 0.

$\therefore |z|^2 dz = y^2 i\,dy$ $\quad (1 \le y \le 0)$

Use these in eqn ① we get

$$\int_C |z|^2 dz = \int_{x=0}^{1} x^2\,dx + \int_{y=0}^{1} (1+y^2)\,i\,dy + \int_{x=1}^{0} (x^2+1)\,dx + \int_{y=1}^{0} y^2\,i\,dy$$

$$= \frac{x^3}{3}\Big|_0^1 + i\left[y+\frac{y^3}{3}\right]_0^1 + \left[\frac{x^3}{3}+x\right]_1^0 + i\left[\frac{y^3}{3}\right]_1^0$$

$$= \frac{1}{3} + i\left[1+\frac{1}{3}\right] + \left(-\frac{1}{3} - 1\right) + \frac{i}{3}[0-1]$$

$$= \cancel{\frac{1}{3}} + i + \cancel{\frac{i}{3}} - \cancel{\frac{1}{3}} - 1 - \cancel{\frac{i}{3}}$$

$\int_C |z|^2 dz = \underline{-1+i}$ along the given path.

3) Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along the following paths : ⓐ The parabola $x=2t$, $y=t^2+3$ from $(0,3)$ to $(2,4)$.

ⓑ The straight line from $(0,3)$ to $(2,4)$.

Soln:- Let $I = \int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$, x varies from 0 to 2 & y varies from 3 to 4.

ⓐ Given $x=2t$, $y=t^2+3$. $\Rightarrow dx = 2dt$, $dy = 2t\,dt$

If $x=0 \Rightarrow 0=2t$ i.e $t=0$ $\quad$ | (or) $y=3$, $3=t^2-$

$x=2 \Rightarrow 2=2t$ i.e $t=1$

$$\therefore I = \int_{t=0}^{1} [2(t^2+3)+(2t)^2]\,2\,dt + [3\cdot 2t-(t^2+3)]\,2t\,dt$$

$$= \int_0^1 (2t^2+6+4t^2)\,2\,dt + (6t-t^2-3)\,2t\,dt$$

$$= \int_{t=0}^{1} (12t^2+12+12t^2-2t^3-6t)\,dt$$

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$$= \int_{t=0}^{1} (24t^2 - 2t^3 - 6t + 12)\, dt$$

$$= \left[24\frac{t^3}{3} - 2\frac{t^4}{4} - 6\frac{t^2}{2} + 12t \right]_0^1$$

$$= 8 - \frac{1}{2} - 3 + 12$$

$$= 17 - \frac{1}{2}$$

$$I = \frac{33}{2}$$

(b) Equation of the straight line joining (0,3) and (2,4) is given by $\frac{y-3}{x-0} = \frac{4-3}{2-0}$

$$\frac{y-3}{x} = \frac{1}{2} \Rightarrow x = 2y - 6$$

$$dx = 2\,dy$$

$$\therefore I = \int_{y=3}^{4} [2y + (2y-6)^2]\, 2\,dy + [3(2y-6) - y]\, dy$$

$$= \int_{y=3}^{4} [2y + 4y^2 + 36 - 24y]\, 2\,dy + [6y - 18 - y]\, dy$$

$$= \int_{y=3}^{4} (8y^2 + 72 - 44y + 5y - 18)\, dy$$

$$= \int_{y=3}^{4} (8y^2 - 39y + 54)\, dy$$

$$= \left[8\frac{y^3}{3} - 39\frac{y^2}{2} + 54y \right]_3^4$$

$$= \frac{8}{3}[64 - 27] - \frac{39}{2}[16 - 9] + 54(4-3)$$

$$= \frac{296}{3} - \frac{273}{2} + 54 = \frac{97}{6}$$

$$I = \frac{97}{6}$$

4) Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$ along the following paths:

ⓐ $x = t+1$, $y = 2t^2 - 1$

ⓑ straight line joining $(1-i)$ and $(2+i)$

Sol: Given $I = \int_{1-i}^{2+i} (2x+iy+1)\,dz$ [ie x varies from $(1,2)$ y varies from $(-1,1)$

ⓐ $x = t+1$, $y = 2t^2-1 \Rightarrow dx = dt$, $dy = 4t\,dt$

If $x = 1, 1 = t+1 \Rightarrow t = 0$
$x = 2$, $2 = t+1 \Rightarrow t = 1$ } $t \to 0$ to 1. & $dz = dx + i\,dy$
$dz = dt + i4t\,dt$

$$I = \int_{t=0}^{1} [2(t+1) + i(2t^2-1) + 1][dt + i4t\,dt]$$

$$= \int_{t=0}^{1} (2t+2+2it^2 - i + 1)(1+i4t)\,dt$$

$$= \int_{t=0}^{1} (2t + 2it^2 - i + 3 + 8it^2 + 8i^2t^3 - 4i^2t + 12it)\,dt$$

$$= \int_{t=0}^{1} (-8t^3 + 10it^2 + 6t - i + 3 + 12it)\,dt$$

$$= -\frac{8t^4}{4} + 10i\frac{t^3}{3} + \frac{6t^2}{2} - it + 3t + 12i\frac{t^2}{2} \Big|_0^1$$

$$= -2 + \frac{10i}{3} + 3 - i + 3 + 6i$$

$$= 4 + \frac{10i}{3} + 5i$$

$$I = 4 + \frac{25i}{3}$$

b) Equation of the straight line joining $(1,-1)$ and $(2,1)$ is given by $\frac{y+1}{x-1} = \frac{1+1}{2-1}$

$\Rightarrow \frac{y+1}{x-1} = \frac{2}{1}$

$\Rightarrow y+1 = 2x-2$

$\Rightarrow \boxed{y = 2x-3}$

$\therefore dy = 2\,dx$ & $dz = dx + i\,dy$

$\therefore I = \int_{(1,-1)}^{(2,1)} (2x+iy+1)\,dz$

$I = \int_{x=1}^{2} [2x + i(2x-3)+1][dx + i2dx]$

$= \int_{x=1}^{2} (2x+2ix-3i+1)(1+i2)dx$

$= \int_{x=1}^{2} (2x+2ix-3i+1+4ix-4x+6+2i)dx$

$= \int_{x=1}^{2} (6ix-2x-i+7)\,dx$

$= 6i\frac{x^2}{2} - \cancel{2}\frac{x^2}{\cancel{2}} - ix + 7x \Big|_1^2$

$= 3i(2^2-1) - (2^2-1) - i(2-1) + 7(2-1)$

$= 9i - 3 - i + 7$

$= 8i + 4$

$= 4(2i+1)$

$I = 4(1+2i)$ along the given path

5) If c is a circle with centre a & radius r then S.T (i) $\int_C \frac{dz}{z-a} = 2\pi i$ (ii) $\int_C (z-a)^n dz = 0$ if $n \neq -1$.

Soln: Given curve is

$$|z-a| = r$$

$$\Rightarrow z-a = re^{i\theta}$$

$$\Rightarrow dz = ire^{i\theta}, \quad \theta \to 0 \text{ to } 2\pi.$$

w.k.t $|z| = r$

$|z-a| = |z|$

$z-a = z$

$z-a = re^{i\theta}$

a) $$\int_C \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\int_{\theta=0}^{2\pi} d\theta = i[\theta]_0^{2\pi} = 2\pi i$$

$$\therefore \int_C \frac{dz}{z-a} = 2\pi i$$

(b) Also $$\int_C (z-a)^n dz = \int_{\theta=0}^{2\pi} (re^{i\theta})^n ire^{i\theta} d\theta$$

$$= ir^{n+1}\int_{\theta=0}^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - 1\right].$$

But $e^{i(n+1)2\pi} = \cos(n+1)2\pi + i\sin(n+1)2\pi = 1 + i(0) = 1$

but $\cos 2k\pi = 1$ & $\sin 2k\pi = 0$ for $k = 1, 2, 3, \ldots$

Hence $$\int_C (z-a)^n dz = \frac{r^{n+1}}{n+1}[1-1]$$

$$\therefore \int_C (z-a)^n dz = 0, \quad n \neq -1$$