Exponential Distribution :-

The continuous probability distribution having the Probability density function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x>0 \\ 0, & \text{otherwise} \end{cases} \quad \text{where } \alpha > 0.$$

is known as exponential distribution.

Also $f(x)$ satisfies both the conditions of continuous probability function (or) probability density function.

That is, $f(x) > 0$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} \alpha e^{-\alpha x}\,dx$$

$$= \alpha \left. \frac{e^{-\alpha x}}{-\alpha} \right|_{0}^{\infty}$$

$$= -[0-1]$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

* Mean of the exponential distribution $= \mu = \frac{1}{\alpha}$
* Variance of the exponential distribution $= \sigma^2 = V = \frac{1}{\alpha^2}$
* Standard deviation of the E.D $= \sigma = \frac{1}{\alpha}$

Problems :

1) If x is an exponential variate with mean 3 find (i) $P(x>1)$, (ii) $P(x<3)$.

Sol :- The p.d.f of the exponential distribution is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$$

The mean of this distribution is $1/\alpha$.

$\therefore$ By data, mean $= 1/\alpha = 3 \Rightarrow \alpha = 3$

Hence $$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(x>1) = 1 - P(x \le 1)$

$= 1 - \int_0^1 f(x)\,dx$

$= 1 - \int_0^1 \frac{1}{3} e^{-x/3}\,dx = \cancel{1+}\frac{1}{\cancel{3}}\left[\frac{e^{-x/3}}{-1/\cancel{3}}\right]_0^1$

$= 1 + \left(e^{-x/3}\right]_0^1$

$= 1 + [e^{-1/3} - e^{0}]$

$= \cancel{1} + e^{-1/3} - \cancel{1}$

$P(x>1) = e^{-1/3} = 0.7165$

(ii) $P(x<3) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^3 f(x)\,dx + \int_3^{\infty} f(x)\,dx$

$= 0 + \int_0^3 \frac{1}{3} e^{-x/3}\,dx + 0$

$= \frac{1}{3} \frac{e^{-\frac{x}{3}}}{-1/3} \Big|_0^1$

$= -1[e^{-1/3} - e^{0}]$

$= -[e^{-1/3} - 1]$

$= 1 - e^{-1/3}$

$P(x<3) = 0.6321$

2) If x is an exponential variate with mean 5, evaluate the following: (i) $P(0<x<1)$ (ii) $P(-\infty<x<10)$ (iii) $P(x\le 0$ or $x\ge 1)$

Sol^n:- The pdf of the exponential distribution is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$$

by given data, $\text{mean} = \frac{1}{\alpha} = 5$

$$\Rightarrow \alpha = \frac{1}{5}$$

$$\therefore f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \quad 0<x<\infty$$

(i) $P(0<x<1) = \int_0^1 f(x)\,dx$

$$= \int_0^1 \frac{1}{5} e^{-\frac{x}{5}}\,dx = \frac{1}{5}\left.\frac{e^{-x/5}}{-1/5}\right|_0^1$$

$$= -\left[e^{-x/5}\right]_0^1$$

$$= -\left[e^{-1/5} - 1\right]$$

$$= 1 - e^{-1/5}$$

$$P(0<x<1) = \underline{\underline{0.1813}}$$

(ii) $P(-\infty<x<10) = \int_{-\infty}^{0} f(x)\,dx + \int_0^{10} f(x)\,dx$

$$= 0 + \int_0^{10} \frac{1}{5} e^{-x/5}\,dx = -\left[e^{-x/5}\right]_0^{10} = 1 - e^{-10/5} = 1 - e^{-2}$$

$$= 1 - \frac{1}{e^2}$$

$$= \underline{\underline{0.8647}}$$

(iii) $P(x\le 0 \text{ or } x\ge 1) = P(x\le 0) + P(x\ge 1)$

| $P(A \text{ or } B) = P(A)+P(B)$
| $P(A \text{ and } B) = P(A)\cdot P(B)$

$$= 0 + \int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{5} e^{-x/5}\,dx$$

$$= -(0 - e^{-0.2}) = e^{-0.2}$$

$$P(x\le 0 \text{ or } x\ge 1) = \underline{\underline{0.8187}}$$

** 3) The length of telephone conversation in a both has been exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this ~~both~~ booth (i) ends less than 5 minutes

(ii) between 5 and 10 minutes.

Sol^n:- $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$,

mean $= \frac{1}{\alpha}$

By data, $\frac{1}{\alpha} = 5$, $\therefore \alpha = \frac{1}{5}$

$\therefore f(x) = \frac{1}{5} e^{-\frac{x}{5}}$

(i) $P(x<5) = \int_0^5 \frac{1}{5} e^{-\frac{x}{5}} dx = -\left[e^{-\frac{x}{5}}\right]_0^5 = -\left[e^{-1} - 1\right]$

$= 1 - \frac{1}{e}$

$= 0.6321$

(ii) $P(5<x<10) = \int_5^{10} \frac{1}{5} e^{-\frac{x}{5}} dx$

$= \frac{1}{\cancel{5}} \left[\frac{e^{-x/5}}{-1/\cancel{5}}\right]_5^{10}$

$= -\left[e^{-x/5}\right]_5^{10}$

$= -\left(e^{-2} - e^{-1}\right)$

$= 1/e - \frac{1}{e^2}$

$P(5<x<10) = 0.2325$

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4) In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for :
(i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 & 12 minutes.

Soln :

$$f(x) = \alpha e^{-\alpha x}, \quad x > 0.$$

$$\text{mean} = \frac{1}{\alpha},$$

By data, $\frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$

$$\therefore f(x) = \frac{1}{5} e^{-x/5}$$

$$\left| \; e^{-\infty} = 0 \right.$$

(i) $P(x \geq 10) = \int_{10}^{\infty} \frac{1}{5} e^{-x/5} \, dx = -\left[e^{-x/5}\right]_{10}^{\infty}$

$$= -\left[0 - e^{-2}\right] = \frac{1}{e^2}$$

$$= \underline{\underline{0.1353}}$$

(ii) $P(x < 10) = \int_{0}^{10} \frac{1}{5} e^{-x/5} \, dx = -\left[e^{-x/5}\right]_{0}^{10}$

$$= -\left[e^{-2} - 1\right]$$

$$= 1 - e^{-2}$$

$$= \underline{\underline{0.8647}}$$

(iii) $P(10 < x < 12) = \int_{10}^{12} \frac{1}{5} e^{-x/5} \, dx = -\left(e^{-x/5}\right)_{10}^{12}$

$$= -\left[e^{-12/5} - e^{-2}\right]$$

$$= \underline{\underline{0.0446}}$$