Bilinear Transformation (BLT) :-

The transformation $w = \frac{az+b}{cz+d}$, where a, b, c, d are real (or) complex constants such that $ad-bc \neq 0$ is called bilinear transformation

Note :

1) The condition $ad-bc \neq 0$ is the conformal property of BLT
2) If a point z maps onto itself that is $w = z$ under the bilinear transformation then the point is called an invariant point (or) a fixed point of the BLT.
3) Bilinear transformation is also called mobius transformation.

Property 1:- There exists a bilinear transformation that maps three given distinct points z_1, z_2, z_3 onto three given distinct points w_1, w_2, w_3 respectively.

By this property, if we solve the equation w in term of z ~~this equation~~

ie
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

we obtain the bilinear transformation that transforms z_1, z_2, z_3 onto w_1, w_2, w_3 respectively.

Property 2:- Bilinear transformation preserve (do not alter) the cross-ratio of four points.

ie
$$\frac{(w_4-w_1)(w_2-w_3)}{(w_4-w_3)(w_2-w_1)} = \frac{(z_4-z_1)(z_2-z_3)}{(z_4-z_3)(z_2-z_1)}$$
this shows that

the cross ratio of the points w_1, w_2, w_3, w_4 is equal to the cross ratio of the points z_1, z_2, z_3, z_4. Thus BLT preserve the cross ratio.

Problems:

1) Find the bilinear transformation which map the points $z = 1, i, -1$ into $w = i, 0, -i$. Under this transformation find the images of $|z| < 1$.

Sol^n: Let $w = \frac{az+b}{cz+d}$ be the required transformation.

Let $z_1 = 1$, $z_2 = i$, $z_3 = -1$ & $w_1 = i$, $w_2 = 0$, $w_3 = -i$

$\therefore$ The required BLT is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

ie $$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)\cancel{i}}{(w+i)(-\cancel{i})} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{w-i}{w+i} = -\frac{(i+1)}{(i-1)}\frac{(z-1)}{(z+1)}$$

$$= -\frac{(i+1)^2}{(i-1)(i+1)}\frac{(z-1)}{(z+1)}$$

$$= -\frac{(\cancel{i^2}+\cancel{1}-2i)}{i^2-1^2}\frac{(z-1)}{(z+1)}$$

$$= \frac{\cancel{-2}i}{\cancel{-2}}\frac{(z-1)}{(z+1)}$$

$$\frac{w-i}{w+i} = \frac{i(z-1)}{(z+1)}$$

$$(w-i)(z+1) = i(w+i)(z-1)$$

$$w(z+1) - i(z+1) = iw(z-1) + i^2(z-1).$$

$$w\{(z+1) - i(z-1)\} = i(z+1) - (z-1)$$

$$w = \frac{i(z+1)-(z-1)}{(z+1)-i(z-1)}$$

$$w = \frac{iz + i - z + 1}{z + 1 - iz + i}$$

$$= \frac{z(i-1) + (1+i)}{z(1-i) + (1+i)}$$

multiply & $\div$ by $(1-i)$

$$= \frac{-z(1-i)^2 + (1+i)(1-i)}{z(1-i)^2 + (1+i)(1-i)} = \frac{-z(1+i^2-2i) + (1^2-i^2)}{z(1+i^2-2i) + (1^2-i^2)}$$

$$= \cancel{\frac{-z(1+i^2-2i)+1-i^2}{z(1+i^2-2i)+1-i^2}}$$

$$= \frac{-z(-2i) + 2}{-z(2i) + 2}$$

$$= \frac{2zi + 2}{-2zi + 2}$$

$$\boxed{w = \frac{1+iz}{1-iz}}$$

this is the required transformation.

To find the image of $|z| < 1$;

Consider, $w = \dfrac{1+iz}{1-iz}$

$$w(1-iz) = 1 + iz$$

$$w - iwz = 1 + iz$$

$$-iz - iwz = 1 - w$$

$$-zi(1+w) = 1 - w$$

$$z = \frac{(1-w)}{-i(1+w)}$$

$$z = i\frac{(1-w)}{(1+w)} \qquad 1 \because \frac{1}{i} = -i \quad \& \quad -\frac{1}{i} = i$$

If $|z| < 1$ this expression yields

$$|i|\left|\frac{1-w}{1+w}\right| < 1$$

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(or) $|1-w|^2 < |1+w^2|$

ie $|1-(u+iv)|^2 < |1+(u+iv)|^2$

ie $|(1-u)-iv|^2 < |(1+u)+iv|^2$

ie $(1-u)^2+v^2 < (1+u)^2+v^2$

ie $\cancel{1}+\cancel{u^2}-2u+\cancel{v^2} < \cancel{1}+\cancel{u^2}+2u+\cancel{v^2}$

$\Rightarrow \quad -2u < 2u$

$\Rightarrow \quad 4u > 0$

$\Rightarrow \quad u > 0.$

Thus $u > 0$ is the image of $|z| < 1$.

Alternate method :

Sol^n: Let $w = \dfrac{az+b}{cz+d}$ be the required BLT.

Substitute the given values of z & w, we get three equations.

$z=1, w=i \Rightarrow i = \dfrac{a+b}{c+d}$

$\Rightarrow a+b-ci-di = 0 \rightarrow (1)$

$z=i, \quad w=0, \quad 0 = \dfrac{ai+b}{ci+d}$

$\Rightarrow ai+b = 0 \rightarrow (2)$

$z=-1, w=-i, \quad -i = \dfrac{-a+b}{-c+d}$

$\Rightarrow -a+b-ci+di = 0 \rightarrow (3)$

$(1)+(3) \Rightarrow 2b-2ci = 0$

$b-ci = 0 \rightarrow (4)$

Solve (2) & (4) by writing

$ia+1b+0c = 0$

$0a+1b-ic = 0$

Applying the rule of cross multiplication, we have

$$\frac{a}{\begin{vmatrix}1 & 0\\ 1 & -i\end{vmatrix}} = \frac{-b}{\begin{vmatrix}i & 0\\ 0 & -i\end{vmatrix}} = \frac{c}{\begin{vmatrix}i & 1\\ 0 & 1\end{vmatrix}}$$

ie $\dfrac{a}{-i} = \dfrac{-b}{-i^2} = \dfrac{c}{i}$

(or) $\dfrac{a}{-i} = \dfrac{b}{-1} = \dfrac{c}{i} = k$ (say)

$\Rightarrow a = -ki, \quad b = -k, \quad c = ik$

Substituting these in (1), we get

$-ik-k-i^2k-di = 0$

$-ik-\cancel{k}+\cancel{k}-di = 0$

$-(di+ik) = 0$

$di = -ik$

$\boxed{d=-k}$

$\therefore$ Sub. a, b, c, d in w, we get

$w = \dfrac{-kiz-k}{ikz-k} = \dfrac{-\cancel{k}(1+iz)}{-\cancel{k}(1-iz)}$

$w = \dfrac{1+iz}{1-iz}$

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2) Find the bilinear transformation which map the points $z = 1, i, -1$ into $w = 2, i, -2$. Also find the invariant points of the transformation.

Sol:

(a) Let $w = \dfrac{az+b}{cz+d}$ be the required BLT.

Now $z=1, \; w=2, \Rightarrow 2 = \dfrac{a+b}{c+d}$

$$a+b-2c-2d = 0 \longrightarrow (1)$$

$z=i, \; w=i \Rightarrow i = \dfrac{ai+b}{ci+d}$

$$ai+b+c-di = 0 \longrightarrow (2)$$

$z=-1, \; w=-2, \Rightarrow -2 = \dfrac{-a+b}{-c+d}$

$$-a+b-2c+2d = 0 \longrightarrow (3)$$

$(1)+(3) \Rightarrow \cancel{a}+b-2c-\cancel{2d}-\cancel{a}+b-2c+\cancel{2d} = 0$

$$2b-4c = 0$$

$$b-2c = 0 \longrightarrow (4)$$

$(2)+i\times(3) \Rightarrow ai+b+c-di+i(-a+b-2c+2d) = 0$

$$\cancel{ai}+b+c-di-\cancel{ai}+bi-2ci+2di = 0$$

$$(1+i)b+(1-2i)c+id = 0 \longrightarrow (5)$$

Solve (4) & (5)

$$b-2c+0d = 0$$

$$(1+i)b+(1-2i)c+id = 0$$

Applying the rule of cross multiplication:

$$\frac{b}{\begin{vmatrix} -2 & 0 \\ (1-2i) & i \end{vmatrix}} = \frac{-c}{\begin{vmatrix} 1 & 0 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 1 & -2 \\ (1+i) & (1-2i) \end{vmatrix}}$$

$$\Rightarrow \frac{b}{-2i} = \frac{-c}{i} = \frac{d}{(1-2i)+2(1+i)}$$

$$\Rightarrow \frac{b}{-2i} = \frac{-c}{i} = \frac{d}{3}$$

$\therefore b = -2i, \quad c = -i, \quad d = 3.$

Substitute these values in ①,

$$a - 2\cancel{i} + 2\cancel{i}i - 6 = 0$$

$$\Rightarrow \boxed{a = 6}$$

$\therefore$ The required BLT is

$$w = \frac{6z - 2i}{-iz + 3}$$

Further, the invariant points of this transformation are obtained by taking $w = z$.

ie $$z = \frac{6z - 2i}{-iz + 3}$$

$$-iz^2 + 3z - 6z + 2i = 0$$

$$-iz^2 - 3z + 2i = 0$$

Applying the quadratic formula,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-i)(2i)}}{-2i}$$

$$= \frac{3 \pm \sqrt{9 - 8}}{-2i}$$

$$z = \frac{3 \pm 1}{-2i}$$

$$\Rightarrow z = \frac{3+1}{-2i} \quad \& \quad z = \frac{3-1}{-2i}$$

$$\therefore z = \frac{4}{-2i} = \frac{2}{-i}, \quad z = \frac{2}{-2i} = \frac{1}{-i} \Rightarrow z = \frac{2}{-i} = 2i,$$

$$z = \frac{1}{-i} = i$$

Thus $\boxed{z = 2i, \ i}$ are the invariant points.

** 3) Find the BLT, which maps $z_1=-1,\ z_2=0,\ z_3=1$ into $w_1=0,\ w_2=i,\ w_3=3i$

<u>sol</u>: Let $w=\dfrac{az+b}{cz+d}$ be the required BLT.

Now let

$z_1=-1,\ w_1=0 \Rightarrow 0=\dfrac{-a+b}{-c+d}$

$\Rightarrow -a+b=0 \rightarrow (1)$

$z_2=0,\ w_2=i \Rightarrow i=\dfrac{0+b}{0+d}$

$b-id=0 \rightarrow (2)$

$z_3=1,\ w_3=3i \Rightarrow 3i=\dfrac{a+b}{c+d}$

$a+b-3ci-3di=0 \rightarrow (3)$

$(1)-(2) \Rightarrow -a+\cancel{b}-\cancel{b}+id=0$

$-a+id=0 \rightarrow (4)$

Solve (2) & (4)

$0a+b-id=0$

$-a+0b+id=0$

Applying rule of cross multiplication,

$$\frac{a}{\begin{vmatrix}1 & -i\\ 0 & i\end{vmatrix}}=\frac{-b}{\begin{vmatrix}0 & -i\\ -1 & i\end{vmatrix}}=\frac{d}{\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}}$$

$\Rightarrow \dfrac{a}{i}=\dfrac{-b}{-i}=\dfrac{d}{1}$

$\Rightarrow a=i,\quad b=i,\quad d=1$

<u>Sub.</u> all the values in (3), we get

$i+i-3ci-3i=0$

$-i-3ci=0$

$-i(1+3c)=0 \Rightarrow 1+3c=0 \Rightarrow c=-1/3$

$\therefore w=\dfrac{iz+i}{-\frac{1}{3}z+1} \Rightarrow w=\dfrac{3i(z+1)}{-z+3}$ (or) Also $z=\dfrac{3i(z+1)}{i(iz-3i)}=\dfrac{3z+3}{iz-3i}$

4) Find the bilinear transformation which maps $z = \infty, i, 0$ into $w = -1, -i, 1$. Also the fixed points of the transformation.

Sol^n: Let $w = \dfrac{az+b}{cz+d}$ is the required BLT.

Now, $z = \infty$, $w = -1$, the BLT can be written in the form

$$w = \frac{z(a+b/z)}{z(c+d/z)} = \frac{a+b/z}{c+d/z}$$

$$\Rightarrow -1 = \frac{a+0}{c+0} \qquad (1/z = 0 \text{ when } z = \infty)$$

$$a + c = 0 \rightarrow (1)$$

$z = i$, $w = -i$, $\Rightarrow$ $-i = \dfrac{ai+b}{ci+d}$

$$ai + b - c + di = 0 \rightarrow (2)$$

$z = 0$, $w = 1$ $\Rightarrow$ $1 = \dfrac{0+b}{0+d}$

$$b - d = 0 \rightarrow (3)$$

(1) + (2) $\Rightarrow$ $a + \cancel{c} + ai + b - \cancel{c} + di = 0$

$$(1+i)a + b + di = 0 \rightarrow (4)$$

Solve (3) & (4),

$$0a + b - d = 0$$
$$(1+i)a + b + di = 0$$

Applying cross multiplication rule,

$$\frac{a}{\begin{vmatrix} 1 & -1 \\ 1 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -1 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 0 & 1 \\ 1+i & 1 \end{vmatrix}}$$

$$\frac{a}{i+1} = \frac{-b}{1+i} = \frac{d}{-(1+i)} \Rightarrow \frac{a}{1} = \frac{-b}{1} = \frac{d}{-1}$$

$\therefore$ $a = 1$, $b = -1$, $d = -1$, & from (1) $\Rightarrow$ $a + c = 0$

$1 + c = 0$

$\boxed{c = -1}$

$$\therefore w = \frac{z-1}{-1 \cdot z - 1} \Rightarrow w = \frac{z-1}{-z-1} \Rightarrow \boxed{w = \frac{1-z}{1+z}}$$

Further, the the fixed points (or) invariant points are obtained by taking $w=z$.

$\therefore$ $$z = \frac{1-z}{1+z}$$

$$z+z^2 = 1-z$$

$$z+z^2-1+z=0$$

$$z^2+2z-1=0$$

$$\therefore \quad z = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

$\therefore$ The invariant points are $-1+\sqrt{2}$, $-1-\sqrt{2}$

5) Find the BLT which maps $z=\infty, i, 0$ and $w=0, i, \infty$

Sol^n: Let $w = \frac{az+b}{cz+d}$ be the required BLT.

Now $z=\infty$, $w=0$, the BLT can be written as

$$w = \frac{z(a+b/z)}{z(c+d/z)} = \frac{a+b/z}{c+d/z}$$

$$\therefore \quad 0 = \frac{a+0}{c+0}$$

$$\Rightarrow \quad a=0 \longrightarrow (1)$$

$z=i$, $w=i$, $\Rightarrow$ $$i = \frac{ai+b}{ci+d}$$

$$ai+b+c-di=0 \longrightarrow (2)$$

$z=0$, $w=\infty$, The BLT is written as

$$\frac{1}{w} = \frac{cz+d}{az+b}$$

$$0 = \frac{0+d}{0+b} \qquad \left|\because \frac{1}{w}=0, \text{ if } w=\infty.\right.$$

$$\Rightarrow \quad d=0 \longrightarrow (3)$$

Now by using $a=0,\ d=0$ in (2) we get

$$0+b+c+0=0$$
$$b=-c$$

Choose $c=1$, we get $b=-1$

Substitute $a=0,\ b=-1,\ c=1,\ d=0$, the required BLT is

$$w=\frac{0+(-1)}{1\cdot z+0}$$

$$w=-\frac{1}{z}$$

6) Find the bilinear transformation which map the points $z=0,1,\infty$ into the points $w=-5,-1,3$ respectively. What are the invariant points in this transformation?

Solⁿ: Let $w=\dfrac{az+b}{cz+d}$,

$z=0,\ w=-5, \Rightarrow\quad -5=\dfrac{0+b}{0+d}$

$$b+5d=0 \Rightarrow b=-5d \rightarrow (1)$$

$z=1,\ w=-1 \quad\Rightarrow\quad -1=\dfrac{a+b}{c+d}$

$$a+b+c+d=0 \rightarrow (2)$$

$z=\infty,\ w=3 \Rightarrow$

$$w=\frac{a+b/2}{c+d/2}$$

$$\Rightarrow 3=\frac{a+0}{c+0}$$

$$a=3c \rightarrow (3)$$

Sub. (1) & (3) in (2) we get

$$4c-4d=0 \Rightarrow c=d$$

Choose $c=1 \Rightarrow d=1$. $\therefore a=3,\ b=-5$

$\therefore$ Sub. $a=3,\ b=-5,\ c=1,\ d=1$ in required BLT, we get

$$w=\frac{3z-5}{z+1}$$

The invariant points are obtained by taking $w = z$,

ie $$z = \frac{3z-5}{z+1}$$

$$z^2 + z = 3z - 5$$

$$z^2 - 2z + 5 = 0$$

$$\therefore z = \frac{-(-2) \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\therefore z = 1+2i,\ 1-2i$ are the invariant points.

7) Find the BLT which maps the points $z = 1, i, -1$ into $w = 0, 1, \infty$.

Ans: $a = -(1+i)$, $b = 1+i$, $c = 1-i$, $d = 1-i$

$$w = i\left(\frac{1-z}{1+z}\right).$$

8) Find the B.LT which maps $z = 0, -i, 2i$ onto $w = 5i, \infty, \frac{-i}{3}$ respectively. What are the invariant points of the transformation?

Ans: $a = 3i$, $b = 5$, $c = -1$, $d = -i$

$$w = \frac{-3z + 5i}{-iz + 1}$$

& $z = i, -5i$ are invariant points.

9) Find the BLT which map the points $z = 1, i, -1$, to $w = 0, i, \infty$

Ans: $w = \frac{z-1}{z+1}$

10) Find the invariant points of the BLT

(i) $w = \frac{z-1-i}{z+2}$ Ans: $-i$, $(i-1)$

(ii) $w = \frac{3z-4}{z-1}$ Ans: 2, 2

11) Find the map of the real axis of the z-plane in the w-plane under the transformation $w = \frac{1}{z+i}$

Soln. The equation of the real axis of the z-plane is $y=0$ and we have by data

$$w = \frac{1}{z+i}$$

$$\Rightarrow z+i = \frac{1}{w}$$

$$z = \frac{1}{w} - i$$

Hence we have

$$x+iy = \frac{1}{u+iv} - i = \frac{u-iv}{(u+iv)(u-iv)} - i$$

$$x+iy = \frac{u-iv}{u^2+v^2} - i$$

$$x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2} - 1\right)$$

Equating ~~real &~~ imaginary parts, we get

$$y = \frac{-v}{u^2+v^2} - 1$$

but $y=0$

$$\Rightarrow \frac{-v}{u^2+v^2} - 1 = 0$$

$$-v - u^2 - v^2 = 0$$

$$u^2+v^2+v = 0$$

$$(u-0)^2 + \left(v+\frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow \cancel{(u-0)^2 + v + \frac{1}{2}}$$

$$\Rightarrow (u-0)^2 + (v-(-1/2))^2 = (1/2)^2$$

This is a circle in the w-plane with centre $(0, -1/2)$ & radius $1/2$.

Thus we conclude that the map of the real axis of the z-plane is a circle in the w-plane.