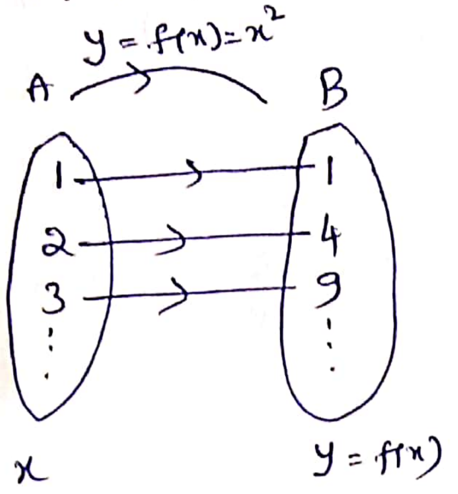
Conformal mapping

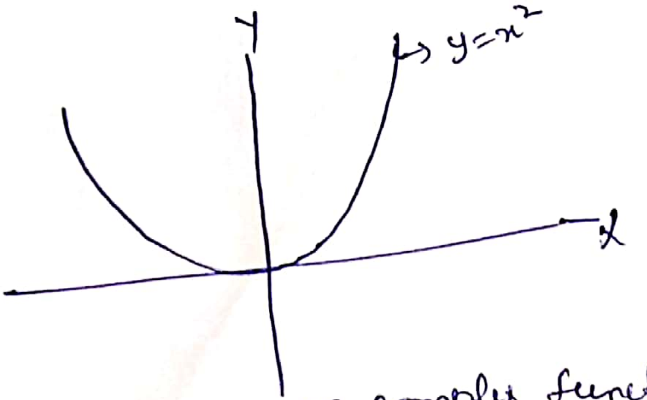
Introduction

Mapping :-

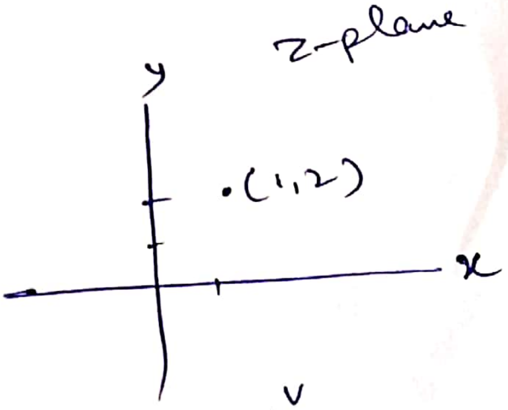
Ex: $f(x) = x^2$



If we plot the same function in 2-Dimension (X Y-plane) we get



Like this we can map complex functions also.



For example:

$w = f(z) = z^2$, $\boxed{z = x + iy}$

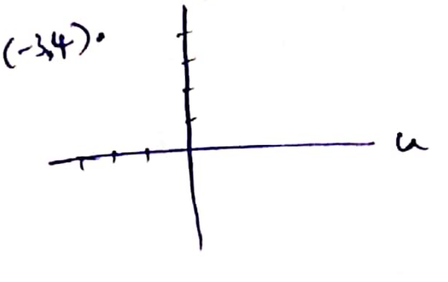
$= (x+iy)^2$

$= x^2 - y^2 + 2ixy$

$w = f(z) = (x^2 - y^2) + i(2xy)$

$= u + iv$

$\boxed{w = f(z) = u(x,y) + iv(x,y)}$

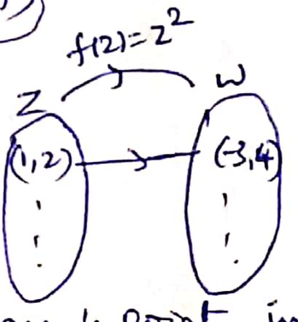


If $z = 1 + i2$ ie

$w = (1+i2)^2$

$= 1 - 4 + i4$

$w = -3 + i4$



Here we can't place 4 points in 2-D plane. So that we should consider two planes (ie z-plane & w-plane)

Conformal transformation

Definitions :-

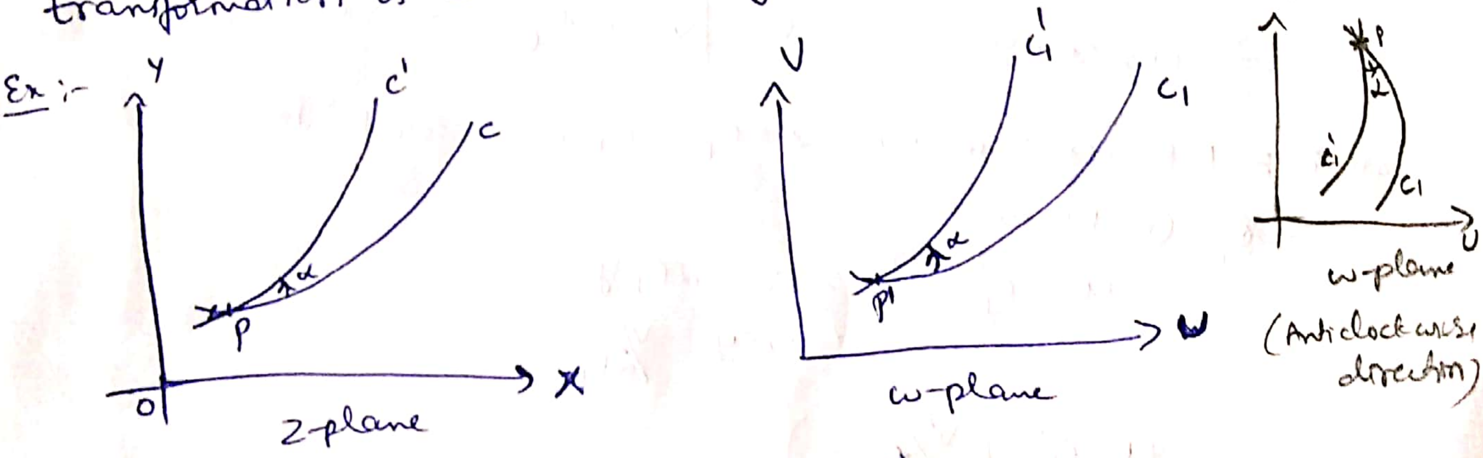
Consider a complex valued function $w = f(z)$.

Putting $z = x+iy$, $w = f(z) = u(x,y) + i v(x,y)$. The complex quantities $z = z(x,y)$, $w = w(u,v)$ are represented in two separate planes namely z-plane & the w-plane respectively. A point (x,y) in the z-plane corresponds to a point (u,v) in the w-plane. If a set of points (x,y) traces a curve C in z-plane and the corresponding points (u,v) traces a curve C' in the w-plane, then we can say that the curve C is transformed / mapped onto the curve C' under the transformation $w = f(z)$. The corresponding set of points in the two planes are called 'images' of each other.

In a transformation the angle b/w any two curves, both in magnitude and sense (direction) are same then it is called a conformal transformation.

If only the magnitude of the angle is same then the transformation is called a Isogonal transformation.

Ex :-



In the above figure, the curves c, c' in the z-plane intersect at the point P and the corresponding curves c_1 & c_1' in the w-plane intersect at P'. If the angle of intersection of the curves at P is same as the angle of intersection of the curves at P' in magnitude & sense then the transformation is said to be ~~conformal~~ conformal.

Note :-

Property :- If $w=f(z)$ is an analytic function of z in a region of the z-plane then $w=f(z)$ is conformal at all points of the region where $f'(z) \neq 0$.

Discussion of conformal transformation :-

For a given transformation $w=f(z)$, first we put $z=x+iy$ (or) $z=re^{i\theta}$ to obtain u & v as functions of x, y (or) r, θ. Then by using those functions we can find the image in w-plane corresponding to the given curve in the z-plane. Sometimes we need to make some judicious elimination from u & v for obtaining the image in the w-plane.

1) Discussion of $w=z^2$

Soln :- Consider $w=z^2$, where $w=u+iv$, $z=x+iy$

$$u+iv = (x+iy)^2$$

$$u+iv = x^2-y^2+i2xy$$

$$\therefore \quad u = x^2-y^2 \quad \text{and} \quad v = 2xy \longrightarrow (1)$$

Case (i) :- Let us consider $x=c_1$, c_1 is a constant.

$\therefore$ eqn (1) becomes

$$u = c_1^2 - y^2, \quad v = 2c_1 y$$

$$\Rightarrow y = v/2c_1$$

$$u = c_1^2 - (v^2/4c_1^2)$$

$$\Rightarrow \quad v^2/4c_1^2 = u - c_1^2$$

$$\Rightarrow v^2 = -4c_1^2(u-c_1^2).$$

This is a parabola in the w-plane symmetric about the real axis with its vertex is at $(c_1^2, 0)$ and focus at the origin. It may be observed that the line $x=-c_1$ is also transformed into the same parabola.

Case (ii) :- Let us consider $y = c_2$, c_2 is constant

$\therefore$ eqn ① becomes

$$u = x^2 - c_2^2 \quad , \quad v = 2xc_2$$

$$\Rightarrow x = \frac{v}{2c_2}$$

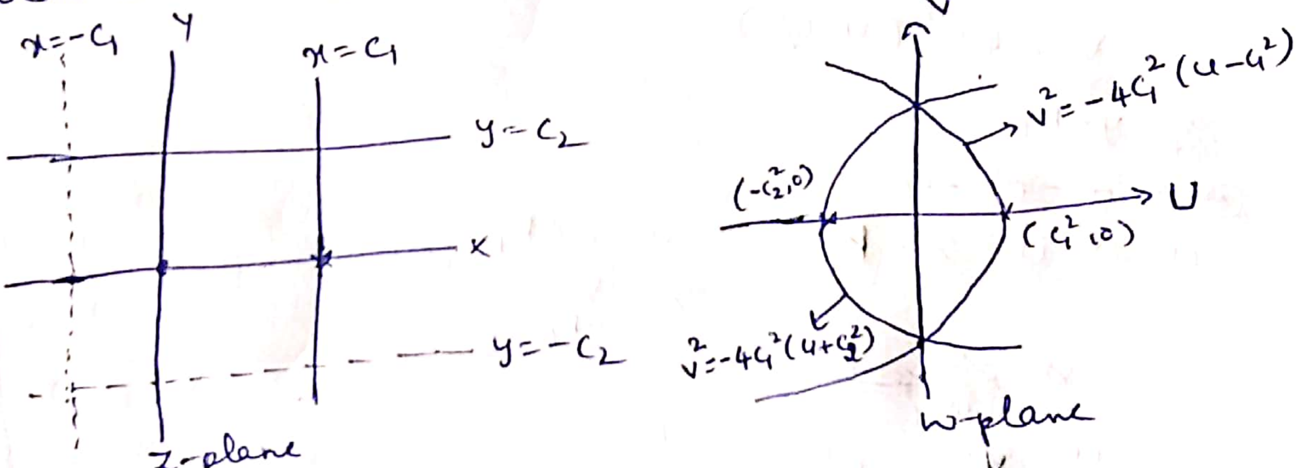
$$\therefore u = \left(\frac{v^2}{4c_2^2}\right) - c_2^2$$

$$\Rightarrow \frac{v^2}{4c_2^2} = u + c_2^2$$

$$v^2 = 4c_2^2(u + c_2^2)$$

This is also a parabola in the w-plane symmetrical about the real axis whose vertex is at $(-c_2^2, 0)$ and focus at the origin. Also $y = -c_2$ is transformed into the same parabola.

Hence from these two cases we conclude that the straight lines parallel to the co-ordinate axes in the z-plane map onto parabolas in the w-plane.



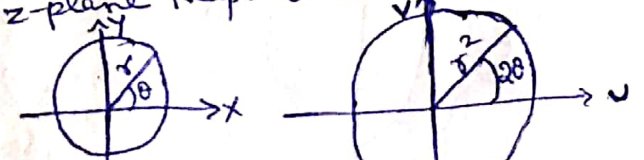
Case (iii) :- Let us consider a circle with centre origin & radius r in the z-plane.

ie $|z| = r$, $\therefore z = re^{i\theta} \Rightarrow w = z^2 = (re^{i\theta})^2$

$$\therefore w = r^2 e^{i2\theta} = Re^{i\phi} \quad \text{where } R = r^2, \ \phi = 2\theta$$

This is also a circle in the w-plane having radius r^2 and subtending an angle 2θ at the origin.

Hence we conclude that a circle with centre origin and radius r in the z-plane maps onto a circle with centre origin & radius r^2 in the w-plane.



Example: Find the images in the w-plane corresponding to the straight lines $x = C_1$, $x = C_2$, $y = k_1$, $y = k_2$ under the transformation $w = z^2$. Indicate the region with sketches.

$$w = z^2$$

$$u + iv = (x+iy)^2$$

$$= x^2 - y^2 + i2xy$$

$$u = x^2 - y^2, \quad v = 2xy \longrightarrow (1)$$

(i) if $x = C_1$

$$(1) \Rightarrow u = c_1^2 - y^2, \quad v = 2c_1 y$$

$$y = v/2c_1$$

$$\therefore u = c_1^2 - \frac{v^2}{4c_1^2}$$

$$\therefore v^2 = -4c_1^2(u - c_1^2)$$ This is a parabola in w-plane with vertex $(c_1^2, 0)$

(ii) Similarly if $x = C_2$

$$v^2 = -4c_2^2(u - c_2^2)$$

(iii) if $y = k_1$

$$(1) \Rightarrow u = x^2 - k_1^2, \quad v = 2xk_1$$

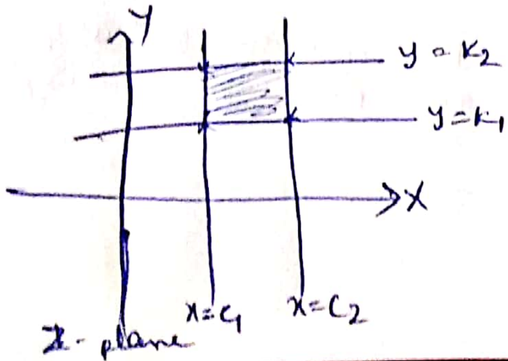
$$x = \frac{v}{2k_1}$$

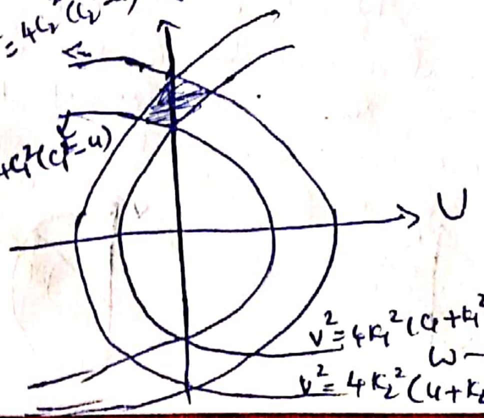
$$u = \frac{v^2}{4k_1^2} - k_1^2$$

$$\Rightarrow v^2 = 4k_1^2(u + k_1^2)$$

(iv) Similarly if $y = k_2$,

$$v^2 = 4k_2^2(u + k_2^2)$$





$v^2 = 4c_2^2(c_2^2 - u)$

$v^2 = 4c_1^2(c_1^2 - u)$

$v^2 = 4k_1^2(u + k_1^2)$

$v^2 = 4k_2^2(u + k_2^2)$

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2) Discussion of $w = e^z$

Consider $w = e^z$

$u + iv = e^{x+iy}$

$= e^x . e^{iy}$

$= e^x(\cos y + i \sin y)$

$u + iv = e^x \cos y + i e^x \sin y \Rightarrow \left.\begin{array}{l} u = e^x \cos y \\ v = e^x \sin y \end{array}\right\} \rightarrow (1)$

we shall find the image in the w-plane corresponding to the straight lines parallel to the co-ordinates axes in the z-plane. That is x = constant, y = constant.

Here by using equation (1) we can't find any values. so that first let us eliminate x & y separately from (1)

Squaring & adding equations in (1),

$u^2 + v^2 = e^{2x} \longrightarrow (2)$

Also dividing the equations in (1)

$\frac{v}{u} = \tan y \longrightarrow (3)$

Case(i): Let $x = c_1$, c_1 is a constant

$\therefore$ eqn (2) $\Rightarrow u^2 + v^2 = e^{2c_1} = r^2$ | where r^2 = constant $= e^{2c_1}$

$\Rightarrow u^2 + v^2 = r^2$

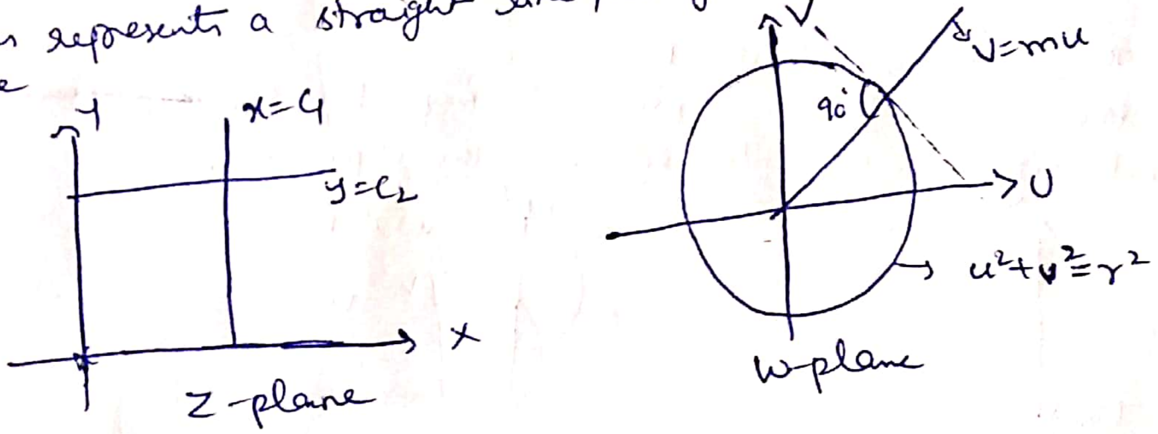
This represents a circle with centre origin & radius r in the w-plane

Case(ii): Let $y = c_2$, c_2 is constant

$\therefore$ eqn (3) $\Rightarrow \frac{v}{u} = \tan c_2 = m$, | where $m = \tan c_2$

$\Rightarrow v = mu$

This represents a straight line passing through the origin in the w-plane



Conclusion :- The straight line parallel to the x-axis ($y = c_2$) in the z-plane maps onto a straight line passing through the origin in the w-plane. The straight line parallel to the y-axis ($x = c_1$) in the z-plane maps onto a circle with centre origin and radius r when $r = e^{c_1}$ in the w-plane. Suppose we draw a tangent at the point of intersection of these two curves in the w-plane, the angle subtended is equal to $90°$. Hence these two curves can be regarded as orthogonal trajectories of each other.

Question :- Show that the transformation $w = e^z$ map straight line parallel to the co-ordinate axes in the z-plane onto orthogonal trajectories in the w-plane and sketch the region.
(Answer is above dissussion).

Example :- Discuss the trasformation $w = e^z$ with respect to the lines represented as co-ordinate axes in the z-plane

Soln: The co-ordinate axes in the z-plane are represented by $x = 0$, $y = 0$.

Given $w = e^z$

$u + iv = e^{x+iy}$

$u + iv = e^x \cos y + i e^x \sin y$

$\Rightarrow u = e^x \cos y \quad , \quad v = e^x \sin y \rightarrow (1)$

Also we have $u^2 + v^2 = e^{2x} \rightarrow (2)$

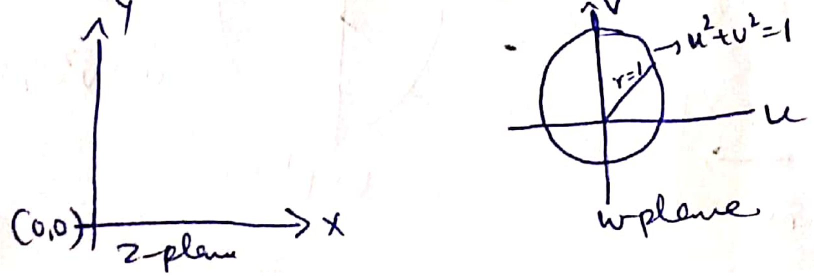
$\frac{v}{u} = \tan y \rightarrow (3)$

when $y = 0$, (3) $\Rightarrow \frac{v}{u} = 0 \Rightarrow v = 0$

$\therefore$ the x-axis in z-plane is mapped onto the u-axis in the w-plane

when $x = 0$, (2) $\Rightarrow u^2 + v^2 = 1$

$\therefore$ the y-axis in the z-plane is mapped onto ~~the~~ a unit circle with centre origin in the w-plane.



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3) Discussion of $w = z + \frac{1}{z}$, $z \neq 0$.

consider, $w = z + \frac{1}{z}$

put $z = re^{i\theta}$

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$\Rightarrow u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta \longrightarrow (1)$$

Now we shall eliminate r & θ respectively from (1).

To eliminate θ, put (1) in the form

$$\frac{u}{\left(r + \frac{1}{r}\right)} = \cos\theta, \quad \frac{v}{\left(r - \frac{1}{r}\right)} = \sin\theta$$

squaring and adding, we get

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1, \quad r \neq 1. \longrightarrow (2)$$

To eliminate r, put (1) in the form.

$$\frac{u}{\cos\theta} = r + \frac{1}{r}, \quad \frac{v}{\sin\theta} = r - \frac{1}{r}$$

squaring and subtracting, we get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4$$

$$\Rightarrow \frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \longrightarrow (3)$$

Since $z = re^{i\theta}$, $|z| = r$ & amp $z = \theta$

$|z| = r \Rightarrow \sqrt{x^2 + y^2} = r$

(or) $r^2 = x^2 + y^2$

This represents a circle with centre origin and radius r in the z-plane when r is a constant.

$\arg z = \theta$

$\Rightarrow \tan^{-1}(y/x) = \theta$

$\Rightarrow y/x = \tan\theta$

This represent a straight line in the z-plane, when θ is constant.

Now we shall discuss the image in the w-plane, corresponding to r = constant (circle) & θ = constant (straight line) in the z-plane.

<u>Case(i)</u> :- Let r = constant

Equation ② is of the form

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \qquad \text{where } A = r+\frac{1}{r}, \quad B = r - \frac{1}{r}$$

This represents an ellipse in the w-plane with foci $(\pm\sqrt{A^2-B^2}, 0)$
$= (\pm 2, 0)$

$\left(\text{Since } \sqrt{A^2-B^2} = \sqrt{(r+\frac{1}{r})^2-(r-\frac{1}{r})^2} = \sqrt{4} = \pm 2.\right)$

Hence we conclude that the circle $|z| = r$ = constant in the z-plane, maps onto an ellipse in the w-plane with foci $(\pm 2, 0)$.

<u>Case(ii)</u> :- Let θ = constant

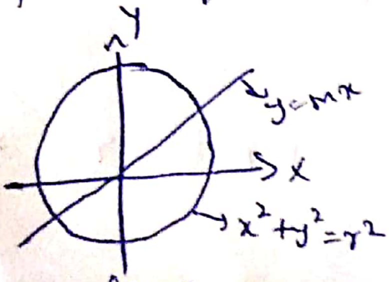
Eqn ③ is of the form

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \qquad \text{where } A = 2\cos\theta, \quad B = 2\sin\theta.$$

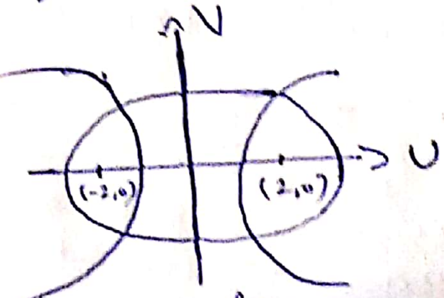
This represents a hyperbola in the w-plane with foci $(\pm\sqrt{A^2+B^2}, 0) = (\pm 2, 0)$.

Hence we conclude that the straight line passing through the origin in the z-plane maps onto a hyperbola in the w-plane with foci $(\pm 2, 0)$.

Since both these circles (ellipse & hyperbola) have the same foci independent of r, θ they are called confocal conics.



z-plane



w-plane