- Mean number of errors in one micro second.

$$\mu = np = m = 12 \times 0.001 = 0.012$$

The Poisson distribution is $P(x) = \dfrac{m^x e^{-m}}{x!}$

ie., $P(x) = e^{-0.012} \dfrac{(0.012)^x}{x!}$

(i) $P(0) = \boxed{0.988072}$

(ii) $P(1) = e^{-0.012} \times 0.012 = \boxed{0.01186}$

(iii) Probability of atleast one error

$$= 1 - P(0) = 1 - e^{-0.012} = \boxed{0.01193}$$

(iv) $P(2) = e^{-0.012} \dfrac{(0.012)^2}{2!} = \boxed{0.000071}$

(v) Probability of atmost two errors

$$= P(0) + P(1) + P(2)$$

$$= e^{-0.012}\left[1 + 0.012 + \frac{(0.012)^2}{2}\right] = \boxed{0.999999714 \approx 1}$$

$\boxed{3.6}$ Continuous Probability Distrbution- Definition

Binomial and Poisson distributions discussed eariler are discrete probability distributions where in the variate can only take integral values.

We have already defined that a random variable which takes noncountable infinite number of values is called a continuous random variable.

If a variate can take any value in an interval, it will give rise to continuous distribution. When a random variable is identified as continuous, we need to consider various questions connected with the probability of the random variable assuming different values. In this context we need a continuous probability function which is defined as follows.

Definition If for every x belonging to the range of a continuous random variable X, we assign a real number $f(x)$ satisfying the conditions,

(1) $f(x) \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

then $f(x)$ is called a *continuous probability function or probability density function (p.d.f)*

If (a, b) is a subinterval of the range space of X then the probability that x lies in (a, b) is defined to be the integral of $f(x)$ between a and b.

That is, $$P(a \leq x \leq b) = \int_a^b f(x)\,dx \quad \ldots (1)$$

Cumulative distribution function

If X is a continuous random variable with probability density function $f(x)$ then the function $F(x)$ defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \quad \ldots (2)$$

is called the *cumulative distribution function* (c.d.f) of X.

It is evident from (2) that,

$$F(x) = P(X \leq x) = P(-\infty < X \leq x) \text{ and } \frac{d}{dx}[F(x)] = f(x)$$

It should be noted that the probability of a continuous random variable taking a particular value is zero, whereas the probability that it take values in an interval is a positive quantity.

If r is any real number then

$$P(x \geq r) = \int_r^{\infty} f(x)\,dx \quad \ldots (3)$$

$$P(x < r) = 1 - P(x \geq r)$$

ie., $$P(x < r) = 1 - \int_r^{\infty} f(x)\,dx \quad \ldots (4)$$

Remark: $\int_{-\infty}^{\infty} f(x)\,dx = 1$ *geometrically means that the area bounded by the curve $f(x)$ and the x - axis is equal to unity.*

Also, $P(a \le x \le b)$ is equal to the area of the region bounded by the curve $f(x)$, the x - axis and the ordinates $x = a$ and $x = b$.

Mean and Variance

If X is a continuous random variable with probability density function $f(x)$ where $-\infty < x < \infty$, the *mean* (μ) or *expectation* $E(X)$ and the *variance* (σ^2) of X is defined as follows.

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx \qquad \ldots (5)$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx \qquad \ldots (6)$$

We now introduce two continuous probability distributions, namely the *exponential distribution* and the *normal distribution*.

3.7 Exponential Distribution

The continuous probability distribution having the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x > 0 \\ 0 \text{ otherwise,} & \text{where } \alpha > 0 \end{cases}$$

is known as the exponential distribution.

Evidently, $f(x) > 0$ and we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \alpha e^{-\alpha x}\,dx = \alpha\left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^{\infty} = -(0-1) = 1$$

Thus, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$f(x)$ satisfy both the conditions required for a continuous probability function/ probability density function.

3.71 Mean and Standard Deviation of the Exponential Distribution

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} x \cdot \alpha e^{-\alpha x}\,dx$$

$$\mu = \alpha \int_0^{\infty} x e^{-\alpha x}\,dx$$

Applying Bernoulli's rule of integration by parts we have,

$$\mu = \alpha\left[x \cdot \left(\frac{e^{-\alpha x}}{-\alpha}\right) - 1 \cdot \left(\frac{e^{-\alpha x}}{\alpha^2}\right)\right]_0^{\infty}$$

(Here $x / e^{\alpha x} \to 0$ as $x \to \infty$ by L' Hospital rule)

$$\mu = \alpha\left[0 - \frac{1}{\alpha^2}(0-1)\right] = \frac{1}{\alpha}\ ; \quad \mu = \frac{1}{\alpha}$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$$

$$(\sigma^2) = \alpha \int_0^{\infty} (x-\mu)^2 e^{-\alpha x}\,dx$$

Applying Bernoulli's rule we have,

$$\sigma^2 = \alpha\left[(x-\mu)^2 \cdot \left(\frac{e^{-\alpha x}}{-\alpha}\right) - 2(x-\mu) \cdot \left(\frac{e^{-\alpha x}}{\alpha^2}\right) + 2 \cdot \frac{e^{-\alpha x}}{-\alpha^3}\right]_0^{\infty}$$

$$= \alpha\left[\frac{-1}{\alpha}\{0-\mu^2\} - \frac{2}{\alpha^2}\{0-(-\mu)\} - \frac{2}{\alpha^3}\{0-1\}\right]$$

$$= \alpha\left(\frac{\mu^2}{\alpha} - \frac{2\mu}{\alpha^2} + \frac{2}{\alpha^3}\right) \text{ But } \mu = \frac{1}{\alpha}$$

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$\therefore \quad \sigma^2 = \alpha\left(\frac{1}{\alpha^3} - \frac{2}{\alpha^3} + \frac{2}{\alpha^3}\right) = \frac{1}{\alpha^2}$ Hence, $\sigma = \frac{1}{\alpha}$

Thus for the exponential distribution

$$\text{Mean } (\mu) = \frac{1}{\alpha}\ ;\ \text{S.D. } (\sigma) = \frac{1}{\alpha}$$

Remark : *Mean = S.D. for the exponential distribution.*

3.8 Normal Distrubtion

The continuous probability distribution having the probability density function $f(x)$ given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as the normal distribution.

Evidently, $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Putting, $t = \frac{x-\mu}{\sqrt{2}\,\sigma}$ or $x = \mu + \sqrt{2}\,\sigma t$, we have $dx = \sqrt{2}\,\sigma\,dt$

t also varies from $-\infty$ to ∞.

Hence, $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\,\sigma\,dt$

ie, $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{\pi}} \cdot 2\int_{0}^{\infty} e^{-t^2}\,dt$

But, $\int_{0}^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$ by gamma functions.

$\therefore \quad \int_{-\infty}^{\infty} f(x)\,dx = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$

Thus $f(x)$ represents a probability density function.

Note : *Normal distribution can be derived as a limiting case of the binomial distribution when n is large, neither p nor q is very small. We can derive $f(x)$ as $n \to \infty$ and $p = 1/2 = q$.*

3.81 Mean and Standard Deviation of the Normal Distribution

Mean $(\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$\mu = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Putting, $t = \frac{x-\mu}{\sqrt{2}\,\sigma}$ or $x = \mu + \sqrt{2}\,\sigma t$, we have $dx = \sqrt{2}\,\sigma\,dt$

t also varies from $-\infty$ to ∞.

$$\text{Mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\,\sigma t) e^{-t^2} \sqrt{2}\,\sigma\,dt$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt + \sigma\frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2}\,dt$$

$$\text{Mean} = \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2}\,dt + \sigma\frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2}\,dt$$

Here the second integral is zero by a standard property since te^{-t^2} is an odd function.

Hence, mean $= \frac{2\mu}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} + 0 = \mu$

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Thus, mean $= \mu$

Hence we can say that the mean of the normal distribution is equal to the mean of the given distribution.

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Substituting $t = \frac{x-\mu}{\sqrt{2}\,\sigma}$, we have as in the earlier case,

$$\text{Variance} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2} \sqrt{2}\,\sigma\,dt$$

$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\, 2 \int_{0}^{\infty} t^2 e^{-t^2}\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t\,(2t e^{-t^2})\,dt$$

We know that, $\int u v\,dt = u \int v\,dt - \int \left[\int v\,dt \cdot u'\right] dt$

Taking $u = t$, $v = 2te^{-t^2}$ and noting that $\int v\,dt = -e^{-t^2}$ we now have,

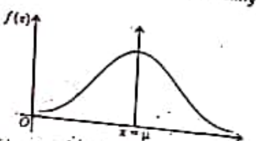
$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \left\{ \left[t(-e^{-t^2}) \right]_0^{\infty} - \int_0^{\infty} -(-e^{-t^2}) \cdot 1\,dt \right\}$$

$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \left\{ 0 + \int_0^{\infty} e^{-t^2}\,dt \right\} = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \sigma^2$$

Thus, Variance $= \sigma^2$

Hence we can say that the Variance / S.D of the normal distribution is equal to the Variance / S.D of the given distribution.

Note : *The graph of the probability function $f(x)$ is a bell shaped curve symmetrical about the line $x = \mu$ and is called the normal probability curve. The shape of the curve is as follows.*



The line $x = \mu$ divides the total area under the curve which is equal to 1 into two equal parts. The area to the right as well as to the left of the line $x = \mu$ is 0.5

3.82 Standard Normal Distribution

We have, $P(a \le x \le b) = \int_a^b f(x)\,dx$

In the case of normal distribution we have,

$$P(a \le x \le b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/2\sigma^2}\,dx$$

The integral in the RHS of this equation cannot be evaluated by known methods of integration and we have to employ the technique of numerical integration which becomes tedious. Hence we think of standardization and the same is as follows.

Putting, $z = \frac{x-\mu}{\sigma}$ or $x = \mu + \sigma z$, we have, $dx = \sigma\,dz$

Let $z_1 = \frac{a-\mu}{\sigma}$ and $z_2 = \frac{b-\mu}{\sigma}$ be the values of z corresponding to $x = a$ and $x = b$. The integral in (1) assumes the following form.

$$P(a \le x \le b) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2}\,dz$$

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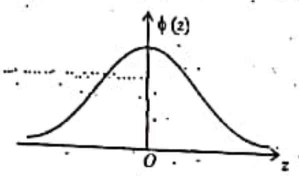
Hence, $P(a \le x \le b) = P(z_1 \le z \le z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2}\, dz$ . . . (2)

If $F(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ (*Standard normal probability density function*), it may be observed that this is same as the *p.d.f* of the normal distribution with $\mu = 0$ and $\sigma = 1$. *Thus we can say that the normal probability density function with $\mu = 0$ and $\sigma = 1$ is the standard normal probability density function.*

$z = \frac{x-\mu}{\sigma}$ *is called the Standard Normal Variate (S.N.V) and F (z) is called the standard normal curve which is symmetrical about the line z = 0. The curve is as follows.*

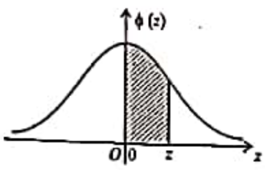


The integral in the RHS of (2) geometrically represents the area bounded by the standard normal curve $F(z)$ between $z = z_1$ and $z = z_2$.

Further in particular if $z_1 = 0$ we have

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2}\, dz$$

This represents the area under the standard normal curve from z = 0 to z.



$\phi(z)$ also denoted by $A(z)$ represents the area (*shaded portion*) as shown in the figure. Since the total area is 1, the area on either side of $z = 0$ is 0.5. Tabulated values which gives the area for different positive values of z are available and this helps us in practical problems. The procedure for using the table will be discussed later.

Table is given at the end of the book

WORKED PROBLEMS

[31] *Find which of the following functions is a probability density function.*

(i) $f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(ii) $f_2(x) = \begin{cases} 2x, & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(iii) $f_3(x) = \begin{cases} |x|, & 0 \,|x|\le 1 \\ 0 & \text{otherwise.} \end{cases}$

(iv) $f_4(x) = \begin{cases} 2x, & 0 < x \le 1 \\ 4 - 4x, & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$

☞ Conditions for a p.d.f are $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

(i) Clearly $f(x) \ge 0$.

$$\int_{-\infty}^{\infty} f_1(x) = \int_0^1 f_1(x)\, dx = \int_0^1 2x\, dx = [x^2]_0^1 = 1$$

$\therefore$ $f_1(x)$ is a p.d.f

(ii) The given fucntion can be written in the form.

$$f_2(x) = \begin{cases} 2x, & -1<x<0 \\ 2x, & 0<x<1 \\ 0 & \textit{otherwise} \end{cases}$$

In $-1<x<0$, $f_2(x) = 2x$ is less than zero.

Further, $\int_{-\infty}^{\infty} f_2(x)\,dx = \int_{-1}^{1} 2x\,dx = [x^2]_{-1}^{1} = 0$

Both the conditions are not satisfied.

$\therefore$ $f_2(x)$ is not a p.d.f

(iii) Evidently $f_3(x) = |x| \geq 0$

$$\int_{-\infty}^{\infty} f_3(x)\,dx = \int_{-1}^{1} f_3(x)\,dx = \int_{-1}^{1} |x|\,dx$$

But, $|x| = \begin{cases} -x, & -1<x<0 \\ +x, & 0<x<1 \end{cases}$

$$\therefore \quad \int_{-\infty}^{\infty} f_3(x)\,dx = \int_{-1}^{0} x\,dx + \int_{0}^{1} x\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

$\therefore$ $f_3(x)$ is a p.d.f

(iv) The given function

$f_4(x) = 2x > 0$ in $0<x\leq 1$

But, $f_4(x) = 4-4x$ is negative in $1<x<2$

The first condition is not satisfied.

$\therefore$ $f_4(x)$ is not a p.d.f

[32] *Find the value of c such that* $f(x) = \begin{cases} \frac{x}{6}+c, & 0 \leq x \leq 3 \\ 0 & \textit{elsewhere} \end{cases}$ *is a p.d.f*

Also find $P(1 \leq x \leq 2)$.

☞ $f(x) \geq 0$ if $c \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie., $\int_0^3 \left(\frac{x}{6}+c\right) dx = 1$

ie., $\left[\frac{x^2}{12}+cx\right]_0^3 = 1$

ie., $\frac{3}{4}+3c = 1$ $\therefore$ $\boxed{c = \frac{1}{12}}$

Now, $P(1 \leq x \leq 2) = \int_1^2 f(x)\,dx$

$$= \int_1^2 \left(\frac{x}{6}+\frac{1}{12}\right)dx$$

$$= \left[\frac{x^2}{12}+\frac{x}{12}\right]_1^2$$

$$= \frac{1}{12}[(4+2)-(1+1)] = \frac{1}{3}$$

Thus, $\boxed{P(1 \leq x \leq 2) = 1/3}$

[33] *Find the constant k such that* $f(x) = \begin{cases} kx^2, & 0<x<3 \\ 0 & \textit{otherwise} \end{cases}$ *is a p.d.f*

Also compute,

(i) $P(1<x<2)$ (ii) $P(x \le 1)$ (iii) $P(x>1)$ (iv) Mean (v) Variance [Dec 2017]

☞ $f(x) \ge 0$ if $k \ge 0$. Also we must have, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie., $\int_0^3 k x^2\,dx = 1$

ie., $\left[\frac{k x^3}{3}\right]_0^3 = 1$ or $9k = 1$ $\therefore \boxed{k = \frac{1}{9}}$

(i) $P(1<x<2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_1^2 = \boxed{\frac{7}{27}}$

(ii) $P(x \le 1) = \int_0^1 f(x)\,dx = \int_0^1 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_0^1 = \boxed{\frac{1}{27}}$

(iii) $P(x>1) = \int_1^3 f(x)\,dx = \int_1^3 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_1^3 = \boxed{\frac{26}{27}}$

(iv) Mean $= \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^3 x \cdot \frac{x^2}{9}\,dx = \left[\frac{x^4}{36}\right]_0^3 = \frac{81}{36} = \boxed{\frac{9}{4}}$

(v) Variance $= V = \int_{-\infty}^{\infty} x^2 f(x)\,dx - (\mu)^2$

$$= \int_0^3 x^2 \cdot \frac{x^2}{9}\,dx - \left(\frac{9}{4}\right)^2$$

$$V = \left[\frac{x^5}{45}\right]_0^3 - \frac{81}{16} = \frac{81}{15} - \frac{81}{16} = \frac{81}{240} = \boxed{\frac{27}{80}}$$

[34] Find k such that $f(x) = \begin{cases} k x e^{-x}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ is a p.d.f. Find the mean.

☞ $f(x) \ge 0$ if $k \ge 0$. Also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie., $\int_0^1 k x e^{-x}\,dx = 1$

Applying Bernoulli's rule we have,

$$k\left[x(-e^{-x}) - (1)(e^{-x})\right]_0^1 = 1$$

ie., $k\left[-\frac{1}{e} - \left(\frac{1}{e} - 1\right)\right] = 1$,

ie., $k\left(1 - \frac{2}{e}\right) = 1$

$\therefore$ $\boxed{k = \frac{e}{e-2}}$

Mean $= \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 x \cdot \frac{e}{e-2} x e^{-x}\,dx$

$$= \frac{e}{e-2}\int_0^1 x^2 e^{-x}\,dx$$

$$= \frac{e}{e-2}\left[x^2(-e^{-x}) - (2x)(e^{-x}) + 2(-e^{-x})\right]_0^1$$

$$\mu = \frac{e}{e-2}\left[-\frac{1}{e} - \frac{2}{e} - 2\left(\frac{1}{e} - 1\right)\right]$$

$$\mu = \frac{e}{e-2}\left[2 - \frac{5}{e}\right] = \boxed{\frac{2e-5}{e-2}}$$

[35] *Is the following function a density function ?*

$$f(x)=\begin{cases} e^{-x}, & x\ge 0\\ 0, & x<0\end{cases}$$

If so, determine the probability that the variate having this density will fall in the interval (1, 2).

☞ We observe $f(x)\ge 0$. Also we must have $\int_{-\infty}^{\infty} f(x)dx=1$.

$$\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{0} f(x)dx+\int_{0}^{\infty} f(x)dx$$

$$=0+\int_0^{\infty} e^{-x}dx=[-e^{-x}]_0^{\infty}=-(0-1)=1$$

Hence $f(x)$ is a probability density function.

The probability that the variate having this density will fall in the interval (1,2) is to compute $P(1<x<2)$.

$$\therefore\quad P(1<x<2)=\int_1^2 f(x)dx=-[e^{-x}]_1^2=-(e^{-2}-e^{-1})$$

Thus, $\boxed{P(1<x<2)=(1/e-1/e^2)=0.2325}$

[36] *A random variable x has the following density function*

$$P(x)=\begin{cases} kx^2, & -3\le x\le 3\\ 0 & elsewhere\end{cases}$$

Evaluate k and find (i) $P(1\le x\le 2)$ *(ii)* $P(x\le 2)$ *(iii)* $P(x>1)$

☞ $P(x)\ge 0$ if $k\ge 0$ and also we must have $\int_{-\infty}^{\infty} P(x)dx=1$

That is, $\int_{-3}^{3} kx^2\,dx=1$

or $\left[\frac{kx^3}{3}\right]_{-3}^{3}=1\quad\therefore\ \boxed{k=\frac{1}{18}}$

(i) $P(1\le x\le 2)=\int_1^2 \frac{x^2}{18}dx=\left[\frac{x^3}{54}\right]_1^2=\frac{1}{54}(8-1)=\boxed{\frac{7}{54}}$

(ii) $P(x\le 2)=\int_{-3}^{2}\frac{1}{18}x^2\,dx=\frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{2}=\frac{1}{54}(8+27)=\boxed{\frac{35}{54}}$

(iii) $P(x>1)=\int_1^3\frac{1}{18}x^2\,dx=\frac{1}{18}\left[\frac{x^3}{3}\right]_1^3=\frac{1}{54}(27-1)=\frac{26}{54}=\boxed{\frac{13}{27}}$

[37] *Find the c.d.f for the following p.d.f of a random variable x.*

(i) $f(x)=\begin{cases} 6x-6x^2, & 0\le x\le 1\\ 0 & otherwise\end{cases}$

(ii) $f(x)=\begin{cases} \frac{x}{4}e^{-x/2}, & 0<x<\infty\\ 0 & otherwise\end{cases}$

(iii) *Exponential distribution.*

☞ If $f(x)$ is the p.d.f. then the c.d.f $=F(x)=\int_{-\infty}^{x} f(x)dx$

(i) $F(x)=\int_{-\infty}^{x} f(x)dx=\int_{-\infty}^{0} f(x)dx+\int_0^x f(x)dx$

$$=0+\int_0^x f(x)dx$$

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$$F(x) = \int_0^x (6x - 6x^2)\,dx = [3x^2 - 2x^3]_0^x = 3x^2 - 2x^3$$

$\therefore$ $\boxed{\text{c.d.f} = 3x^2 - 2x^3 \text{ if } 0 \le x \le 1}$

(ii) $F(x) = \int_0^x \frac{x}{4} e^{-x/2}\,dx$

$$\therefore \quad F(x) = \frac{1}{4}\left[x\frac{e^{-x/2}}{-1/2} - 1\cdot\frac{e^{-x/2}}{1/4}\right]_0^x$$

$$= \frac{1}{4}\left[-2(xe^{-x/2} - 0) - 4(e^{-x/2} - 1)\right]$$

$\therefore$ $\boxed{\text{c.d.f} = 1 - e^{-x/2} - (x/2)e^{-x/2}, \text{ if } 0 < x < \infty}$

(iii) $f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$

is the p.d.f of the exponential distribution.

$$F(x) = \int_{-\infty}^x f(x)\,dx = \int_0^x \alpha e^{-\alpha x}\,dx = -[e^{-\alpha x}]_0^x = 1 - e^{-\alpha x}$$

$\therefore$ $\boxed{\text{c.d.f} = 1 - e^{-\alpha x} \text{ if } 0 < x < \infty}$

[38] *A continuous random variable has the distribution function*

$$F(x) = \begin{cases} 0, & x \le 1 \\ c(x-1)^4, & 1 \le x \le 3 \\ 1, & x > 3 \end{cases}$$ *Find c and also the p.d.f.*

☞ We know that the p.d.f, $f(x) = \frac{d}{dx}[F(x)]$

$$\therefore \quad f(x) = \begin{cases} 0, & x \le 1 \\ 4c(x-1)^3, & 1 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

$f(x) \ge 0$ for $c \ge 0$ and we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Hence we must have, $\int_1^3 4c(x-1)^3\,dx = 1$.

That is, $\left[c(x-1)^4\right]_1^3 = 1$,

ie., $16c = 1$, $\therefore$ $\boxed{c = 1/16}$

Thus the p.d.f $\boxed{f(x) = (x-1)^3/4 \text{ where } 1 \le x \le 3.}$

[39] *Find k so that the following function can serve as a probability density function of a random variable.*

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ kxe^{-4x^2} & \text{for } x > 0 \end{cases}$$

☞ We must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie., $\int_0^{\infty} kxe^{-4x^2}\,dx = 1$

Putting $4x^2 = t$, we have, $8x\,dx = dt$; t also varies from 0 to ∞.

$$\therefore \quad \int_{t=0}^{\infty} ke^{-t}\frac{dt}{8} = 1$$

ie., $\frac{k}{8}\left[-e^{-t}\right]_0^{\infty} = 1$ or $\frac{k}{8} = 1$ $\therefore$ $\boxed{k = 8}$

[40] *If x is an exponential variate with mean 3 find (i)* $P(x>1)$ *(ii)* $P(x<3)$

☞ The p.d.f of the exponential distribution is given by

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$$f(x)=\begin{cases}\alpha e^{-\alpha x}, & 0<x<\infty\\ 0 & \text{otherwise}\end{cases}$$

The mean of this distribution is given by $1/\alpha$.

By data, mean $= 1/\alpha = 3 \quad \therefore\ \alpha = 1/3$

Hence, $$f(x)=\begin{cases}\frac{1}{3}e^{-x/3}, & 0<x<\infty\\ 0 & \text{otherwise}\end{cases}$$

(i) $P(x>1)=1-P(x\le 1)$

$$=1-\int_0^1 f(x)\,dx$$

$$=1-\int_0^1 \frac{1}{3}e^{-x/3}\,dx = 1+\left[e^{-x/3}\right]_0^1 = e^{-1/3}$$

$\therefore\ \boxed{P(x>1)=e^{-1/3}=0.7165}$

(ii) $P(x<3)=\int_0^3 f(x)\,dx$

$$=\int_0^3 \frac{1}{3}e^{-x/3}\,dx$$

$$=-\left[e^{-x/3}\right]_0^3 = 1-\frac{1}{e}=0.6321$$

$\therefore\ \boxed{P(x<3)=0.6321}$

[41] *If x is an exponential variate with mean 5, evaluate the following.*

(i) $P(0<x<1)$ *(ii)* $P(-\infty<x<10)$ *(iii)* $P(x\le 0 \text{ or } x\ge 1)$

☞ $f(x)=\alpha e^{-\alpha x}, 0<x<\infty$; Mean $=\frac{1}{\alpha}=5 \ \therefore\ \alpha=\frac{1}{5}$

Hence, $f(x)=\frac{1}{5}e^{-x/5}, 0<x<\infty$

(i) $P(0<x<1)=\int_0^1 f(x)\,dx$

$$=\int_0^1 \frac{1}{5}e^{-x/5}\,dx = -\left[e^{-x/5}\right]_0^1$$

$P(0<x<1)=1-e^{-1/5}=1-e^{-0.2}=0.1813$

$\therefore\ \boxed{P(0<x<1)=0.1813}$

(ii) $P(-\infty<x<10)=\int_{-\infty}^0 f(x)\,dx+\int_0^{10} f(x)\,dx$

$$=\int_0^{10}\frac{1}{5}e^{-x/5}\,dx=-\left[e^{-x/5}\right]_0^{10}=1-(1/e^2)=0.8647$$

$\therefore\ \boxed{P(-\infty<x<10)=0.8647}$

(iii) $P(x\le 0 \text{ or } x\ge 1)=P(x\le 0)+P(x\ge 1)$

$$P(x\le 0 \text{ or } x\ge 1)=0+\int_1^\infty f(x)\,dx=\int_1^\infty \frac{1}{5}e^{-x/5}\,dx$$

$$P(x\le 0 \text{ or } x\ge 1)=-\left[e^{-x/5}\right]_1^\infty=-(0-e^{-0.2})=e^{-0.2}$$

$\therefore\ \boxed{P(x\le 0 \text{ or } x\ge 1)=0.8187}$

[42] *The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth (i) ends less than 5 minutes (ii) between 5 and 10 minutes.* **[Dec 2017]**

☞ We have, $f(x)=\alpha e^{-\alpha x}, x>0$; Mean $=1/\alpha$

By data, $1/\alpha=5 \quad \therefore\ \alpha=1/5$

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Hence, $f(x) = \frac{1}{5}e^{-x/5}$

(i) $P(x<5) = \int_0^5 f(x)\,dx$

$= \int_0^5 \frac{1}{5}e^{-x/5}\,dx = -\left[e^{-x/5}\right]_0^5$

$= 1-(1/e) = 0.6321$

$\therefore\ \boxed{P(x<5) = 0.6321}$

(ii) $P(5<x<10) = \int_5^{10} f(x)\,dx = \int_5^{10} \frac{1}{5}e^{-x/5}\,dx$

$\therefore\ P(5<x<10) = -\left[e^{-x/5}\right]_5^{10}$

$= (1/e)-(1/e^2) = 0.2325$

$\therefore\ \boxed{P(5<x<10) = 0.2325}$

[43] *In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for :*
(i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 and 12 minutes [June 2019]

☞ The p.d.f of the exponential distribution is given by

$f(x) = \alpha e^{-\alpha x},\ x>0$ and the mean $= 1/\alpha$

By data, $1/\alpha = 5$ $\therefore\ \alpha = 1/5$ and hence $f(x) = \frac{1}{5}e^{-x/5}$

(i) $P(x \geq 10) = \int_{10}^{\infty} \frac{1}{5}\cdot e^{-x/5}\,dx = -\left[e^{-x/5}\right]_{10}^{\infty}$

$\therefore\ P(x \geq 10) = -(0-e^{-2}) = e^{-2} = \boxed{0.1353}$

(ii) $P(x<10) = \int_0^{10} \frac{1}{5}\cdot e^{-x/5}\,dx = -\left[e^{-x/5}\right]_0^{10}$

$\therefore\ P(x<10) = -(e^{-2}-1) = 1-e^{-2} = \boxed{0.8647}$

(iii) $P(10<x<12) = \int_{10}^{12} \frac{1}{5}\cdot e^{-x/5}\,dx = -\left[e^{-x/5}\right]_{10}^{12}$

$\therefore\ P(10<x<12) = -(e^{-12/5}-e^{-2}) = \boxed{0.0446}$

Illustrations on Normal Distribution

We have said that, $\phi(z) = \frac{1}{\sqrt{2\pi}}\int_0^z e^{-z^2/2}\,dz$

represents the area under the standard normal curve from 0 to z. Tabulated values which gives the area for different positive values of z is readily available. Such a table analogous to the format of a logarithmic table is called *normal probability table*. We present a few illustrations geometrically, theoretically and write the value by making use the table (*given at the end of the book*). We also bear in mind the following established results.

(1) $\int_{-\infty}^{\infty} \phi(z)\,dz = 1$ (2) $\int_{-\infty}^{0} \phi(z)\,dz = \int_0^{\infty} \phi(z)\,dz = \frac{1}{2}$

These results in the equivalent form are as follows.

(1) $P(-\infty \leq z \leq \infty) = 1$ (2) $P(-\infty \leq z \leq 0) = 1/2$

(3) $P(0 \leq z \leq \infty)$ or $P(z \geq 0) = 1/2$

Also, $P(-\infty < z < z_1) = P(-\infty < z \leq 0) + P(0 \leq z < z_1)$

ie., $P(z<z_1) = 0.5+\phi(z_1)$ ... (4)

Also, $P(z>z_1) = P(z \geq 0) - P(0 \leq z < z_1)$

ie., $P(z>z_1) = 0.5-\phi(z_1)$ ... (5)

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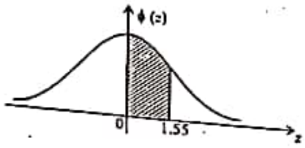
Illustration - 1

To find the area under the standard normal curve between z = 0 and 1.55



Theoretically the area $= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{1.55} e^{-z^2/2}\, dz = \phi(1.55)$

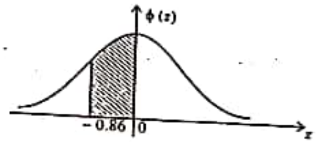
Referring to the table we move vertically down along the column of z to reach 1.5 and then move horitzontally along this line to the value 5 (*regarded* as .05) to intersect with the numerical figure 0.4394

Hence, $\phi(1.55) = 0.4394$

Equivalently we have, $P(0 \le z \le 1.55) = \phi(1.55) = 0.4394$

Illustration - 2

Area of the standard normal curve between $z = -0.86$ and $z = 0$



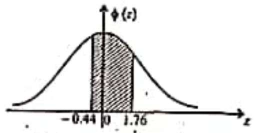
Area $= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-0.86}^{0} e^{-z^2/2}\, dz = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{0.86} e^{-z^2/2}\, dz$ by symmetry.

$\therefore$ the required area $= \phi(0.86) = 0.3051$

Equivalently, $P(-0.86 \le z \le 0) = 0.3051$

Illustration - 3

Area of the standard normal curve between $z = -0.44$ and $z = 1.76$



$$\text{Area} = \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{1.76} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{0} e^{-z^2/2}\, dz + \frac{1}{\sqrt{2\pi}} \int_0^{1.76} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{0.44} e^{-z^2/2}\, dz + \frac{1}{\sqrt{2\pi}} \int_0^{1.76} e^{-z^2/2}\, dz$$

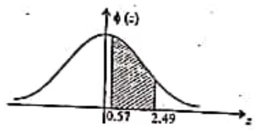
$$= \phi(0.44) + \phi(1.76)$$

$$= 0.1700 + 0.4608 = 0.6308$$

Equivalently, $P(-0.44 \le z \le 1.76) = 0.6308$

Illustration - 4

Area of the standard normal curve between $z = 0.57$ to $z = 2.49$



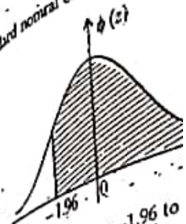
Required area = (Area between $z = 0$ to 2.49) − (Area between $z = 0$ to 0.57)

Required area $= \phi(2.49) - \phi(0.57)$

$= 0.4936 - 0.2157 = 0.2779$

Equivalently, $P(0.57 \le z \le 2.49) = 0.2779$

... standard normal curve to the rig...



... = (Area between $z = -1.96$ to ...

= (Area between $z = 0$ and 1...

$= \phi(1.96) + 0.5 = 0.4750 + 0$...

... $P(z \ge -1.96) = 0.975$

WORKED PRO...

... the following probabilities with ...

(i) $P(z \ge 0.85)$ (ii) $P(-1$...

(iii) $P(z \le -2.43)$ (iv) $P(|z$...

$P(z \ge 0.85) = P(z \ge 0) - P(0 \le z \le$...

$= 0.5 - \phi(0.85)$

$= 0.5 - 0.3023 = 0.1977$

$\therefore \boxed{P(z \ge 0.85) = 0.1977}$

(ii) $P(-1.64 \le z \le -0.88)$

By symmetry, $P(-1.64 \le z \le -0.88) = $...

$\therefore P(-1.64 \le z \le -0.88) = P(0 \le$...

$= \phi(1.6$...

$= 0.449$...

$\therefore \boxed{P(-1.64 \le z \le -0.88) = 0.1389}$

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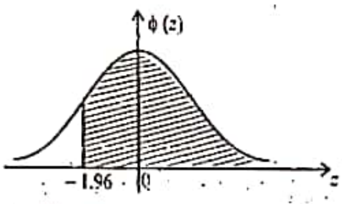
Illustration - 5

Area of the standard nomral curve to the right of $z = -1.96$



Required area = (Area between $z = -1.96$ to 0) + (Area to the right of $z = 0$)

$\therefore$ = (Area between $z = 0$ and 1.96) + 0.5, by symmetry.

$\therefore = \phi(1.96) + 0.5 = 0.4750 + 0.5 = 0.9750$

Equivalently, $P(z \geq -1.96) = 0.975$

WORKED PROBLEMS

[44] *Evaluate the following probabilities with the help of normal probability tables.*

(i) $P(z \geq 0.85)$ (ii) $P(-1.64 \leq z \leq -0.88)$

(iii) $P(z \leq -2.43)$ (iv) $P(|z| \leq 1.94)$

(i) $P(z \geq 0.85) = P(z \geq 0) - P(0 \leq z \leq 0.85)$

$= 0.5 - \phi(0.85)$

$= 0.5 - 0.3023 = 0.1977$

$\therefore$ $\boxed{P(z \geq 0.85) = 0.1977}$

(ii) $P(-1.64 \leq z \leq -0.88)$

By symmetry, $P(-1.64 \leq z \leq -0.88) = P(0.88 \leq z \leq 1.64)$

$\therefore$ $P(-1.64 \leq z \leq -0.88) = P(0 \leq z \leq 1.64) - P(0 \leq z \leq 0.88)$

$= \phi(1.64) - \phi(0.88)$

$= 0.4495 - 0.3106 = 0.1389$

$\therefore$ $\boxed{P(-1.64 \leq z \leq -0.88) = 0.1389}$

Remark : *In this case, using the concept of symmetry we can directly write* $\phi(1.64) - \phi(0.88)$

(iii) $P(z \leq -2.43)$

$P(z \leq -2.43) = P(z \geq 2.43)$

$= P(z \geq 0) - P(0 \leq z \leq 2.43)$

$= 0.5 - \phi(2.43)$

$= 0.5 - 0.4925 = 0.0075$

$\therefore$ $\boxed{P(z \leq -2.43) = 0.0075}$

(iv) $P(|z| \leq 1.94) = P(-1.94 \leq z \leq 1.94)$

$= 2P(0 \leq z \leq 1.94)$

$= 2\phi(1.94) = 2(0.4738) = 0.9476$

$\therefore$ $\boxed{P(|z| \leq 1.94) = 0.9476}$

[45] *If x is a normal variate with mean 30 and standard deviation 5 find the probability that*

(i) $26 \leq x \leq 40$ (ii) $x \geq 45$ **[June 2019]**

We have standard normal variate (s.n.v), $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-30}{5}$

(i) To find $P(26 \leq x \leq 40)$

If $x = 26$, $z = -0.8$; If $x = 40$, $z = 2$

Hence we need to find, $P(-0.8 \leq z \leq 2)$

$P(-0.8 \leq z \leq 2) = P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 2)$

$= P(0 \leq z \leq 0.8) + P(0 \leq z \leq 2)$

$= \phi(0.8) + \phi(2)$

$= 0.2881 + 0.4772 = 0.7653$

$\therefore$ $\boxed{P(26 \leq x \leq 40) = 0.7653}$

(ii) To find $P(x \geq 45)$.

If $x = 45$, $z = 3$ and hence we have to find $P(z \geq 3)$

$$P(z \geq 3) = P(z \geq 0) - P(0 \leq z \leq 3)$$
$$= 0.5 - \phi(3)$$
$$= 0.5 - 0.4987 = 0.0013$$

$\therefore$ $\boxed{P(x \geq 45) = 0.0013}$

[46] *If x is normally distributed with mean 12 and S.D 4, find the following.*
(i) $P(x \geq 20)$ (ii) $P(x \leq 20)$

☞ We have s.n.v, $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-12}{4}$

If $x = 20$, $z = 2$

We have to find $P(z \geq 2)$ and $P(0 \leq z \leq 2)$.

Now, $P(z \geq 2) = P(z > 0) - P(0 \leq z \leq 2)$
$$= 0.5 - \phi(2)$$
$$= 0.5 - 0.4772 = 0.0228$$

Also, $P(z \leq 2) = P(-\infty \leq z \leq 0) + P(0 \leq z \leq 2)$
$$= 0.5 + \phi(2)$$
$$= 0.5 + 0.4772 = 0.9772$$

Thus, $\boxed{P(x \geq 20) = 0.0228 \text{ and } P(x \leq 20) = 0.9772}$

[47] *The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be (i) less than 65 (ii) more than 75 (iii) 65 to 75* **[June 2019]**

☞ Let x represent the marks of students.

By data, $\mu = 70$, $\sigma = 5$. Hence, s.n.v $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-70}{5}$

(i) If $x = 65$, $z = -1$ and we have to find $P(z < -1)$

$$P(z < -1) = P(z > 1)$$
$$= P(z \geq 0) - P(0 \leq z \leq 1)$$
$$= 0.5 - \phi(1) = 0.5 - 0.3413 = 0.1587$$

$\therefore$ number of students scoring less than 65 marks
$= 1000 \times 0.1587 = 158.7 \approx \boxed{159}$

(ii) If $x = 75$, $z = 1$ and we have to find $P(z > 1)$

$$P(z > 1) = P(z \geq 0) - P(0 \leq z \leq 1)$$
$$= 0.5 - \phi(1) = 0.5 - 0.3413 = 0.1587$$

$\therefore$ number of students scoring more than 75 marks
$= 1000 \times 0.1587 = 158.7 \approx \boxed{159}$

(iii) We have to find $P(-1 < z < 1)$

$$P(-1 < z < 1) = 2P(0 < z < 1)$$
$$= 2\phi(1) = 2(0.3413) = 0.6826$$

$\therefore$ number of student scoring marks between 65 and 75
$= 1000 \times 0.6826 = 682.6 \approx \boxed{683}$

[48] *In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D of 60 hours. If a firm purchases 2500 bulbs, find the number of bulbs that are likely to last for*
(i) more than 2100 hours (ii) less than 1950 hours
(iii) between 1900 to 2100 hours **[June 2017]**

☞ By data $\mu = 2000$, $\sigma = 60$

We have, s.n.v $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-2000}{60}$.

(i) To find $P(x > 2100)$

If $x = 2100$, $z = 100/60 \approx 1.67$

$P(x>2100) = P(z>1.67)$

$= P(z \ge 0) - P(0<z<1.67)$

$= 0.5 - \phi(1.67)$

$= 0.5 - 0.4525 = 0.0475$

$\therefore$ number of bulbs that are likely to last for more than 2100 hours is

$2500 \times 0.0475 = 118.75 \approx \boxed{119}$

(ii) To find $P(x<1950)$

If $x = 1950$, $z = -5/6 = -0.83$

$P(x<1950) = P(z<-0.83)$

$= P(z>0.83)$

$= P(z \ge 0) - P(0<z<0.83)$

$= 0.5 - \phi(0.83)$

$= 0.5 - 0.2967 = 0.2033$

$\therefore$ number of bulbs that are likely to last for more than 1950 hours is

$2500 \times 0.2033 = 508.25 \approx \boxed{508}$

(iii) To find $P(1900<x<2100)$

If $x = 1900$, $z = -1.67$ and if $x = 2100$, $z = 1.67$

$P(1900<x<2100) = P(-1.67<z<1.67)$

$= 2P(0<z<1.67)$

$= 2\phi(1.67) = 2 \times 0.4525 = 0.905$

$\therefore$ number of bulbs that are likely to last between 1900 and 2100 hours is

$= 2500 \times 0.905 = 2262.5 \approx \boxed{2263}$

[49] *In a normal distribution 31% of the items are under 45 and 8% of the items are over 64. Find the mean and S.D of the distribution.*

☞ Let μ and σ be the mean and S.D of the normal distribution.

By data, $P(x<45) = 0.31$ and $P(x>64) = 0.08$

We have, s.n.v $z = \dfrac{x-\mu}{\sigma}$

When $x = 45$, $z = \dfrac{45-\mu}{\sigma} = z_1$ *(say)*

$x = 64$, $z = \dfrac{64-\mu}{\sigma} = z_2$ *(say)*

So we have,

$P(z<z_1) = 0.31$ and $P(z>z_2) = 0.08$

ie., $0.5 + \phi(z_1) = 0.31$ and $0.5 - \phi(z_2) = 0.08$

$\Rightarrow$ $\phi(z_1) = -0.19$ and $\phi(z_2) = 0.42$

Refering to the normal probability tables we have

$0.1915 (\approx 0.19) = \phi(0.5)$ and $0.4192 (\approx 0.42) = \phi(1.4)$

$\therefore$ $\phi(z_1) = -\phi(0.5)$ and $\phi(z_2) = \phi(1.4)$

$\Rightarrow$ $z_1 = -0.5$ and $z_2 = 1.4$

ie., $\dfrac{45-\mu}{\sigma} = -0.5$ and $\dfrac{64-\mu}{\sigma} = 1.4$

or $\mu - 0.5\sigma = 45$ and $\mu + 1.4\sigma = 64$

By solving we get $\mu = 50$ and $\sigma = 10$

Thus, $\boxed{\text{Mean} = 50 \text{ and S.D} = 10}$

[50] *In an examination 7% of students score less than 35% marks and 89% of students score less than 60% marks. Find the mean and standard deviation if the marks are normally distributed. It is given that if*

$$P(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz \text{ then } P(1.2263) = 0.39 \text{ and } P(1.4757) = 0.43$$

[June 2018]

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☞ Let μ and σ be the mean and S.D of the normal distribution.

By data we have, $P(x<35) = 0.07$ and $P(x<60) = 0.89$

We have, s.n.v $z = \dfrac{x-\mu}{\sigma}$

When $x = 35$, $z = \dfrac{35-\mu}{\sigma} = z_1$ (say)

$x = 60$, $z = \dfrac{60-\mu}{\sigma} = z_2$ (say)

Hence we have

$P(z<z_1) = 0.07$ and $P(z<z_2) = 0.89$

i.e., $0.5+\phi(z_1) = 0.07$ and $0.5+\phi(z_2) = 0.89$

$\therefore$ $\phi(z_1) = -0.43$ and $\phi(z_2) = 0.39$

Using the given data in the RHS of these we have,

$\phi(z_1) = -\phi(1.4757)$ and $\phi(z_2) = \phi(1.2263)$

$\Rightarrow$ $z_1 = -1.4757$ and $z_2 = 1.2263$

i.e., $\dfrac{35-\mu}{\sigma} = -1.4757$ and $\dfrac{60-\mu}{\sigma} = 1.2263$

or $\mu - 1.4757\sigma = 35$ and $\mu + 1.2263\sigma = 60$

By solving we get $\mu = 48.65$ and $\sigma = 9.25$

Thus $\boxed{\text{Mean} = 48.65 \text{ and S.D} = 9.25}$

ASSIGNMENT

1. Find the mean and the variance for the following probability distribution choosing k suitably.

x	0	1	2	3	4	5
$p(x)$	k	$5k$	$10k$	$10k$	$5k$	k

2. A random variable X has the following probability mass function.

$X = x_i$	0	1	2	3	4	5	6	7	8
$p(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Determine the value of a and hence find

(i) $P(X<3)$ (ii) $P(X\geq 3)$ (iii) $P(0<X<5)$

Also find the distribution of X.

3. Find the value of k such that $p(x) = k/2^x$; $x = 1, 2, 3 \cdots$ represents a probability distribution.

4. Six coins are tossed. Find the probability of getting (i) exactly 3 heads (ii) atleast 3 heads (iii) atleast one head.

5. The probability of germination of a seed in a packet of seeds is found to be 0.7. If 10 seeds are taken for experimenting on germination in a laboratory, find the probability that

   (i) 8 seeds germinate (ii) atleast 8 seeds germinate

6. X is a binomially distributed random variable. If the mean and variance of X are 2 and 3/2 respectively, find the distribution function.

7. 10 coins are tossed 1024 times and the following frequencies are observed. Fit a binomial distribution for the data and calculate the expected frequencies.

No. of heads	0	1	2	3	4	5	6	7	8	9	10
Frequency	2	10	38	106	188	257	226	128	59	7	3

8. A switch board can handle only 4 telephone calls per minute. If the incoming calls per minute follow a Poisson distribution with parameter 3, find the probability that the switch board is over taxed in any one minute.

9. A travel agency has 2 cars which it hires daily. The number of demands for a car on each day is distributed as a Poisson variate with mean 1.5. Find the probability that on a particular day (i) there was no demand (ii) a demand is refused.

10. Fit a Poisson distribution for the following data and calculate the theoretical frequencies.

x	0	1	2	3	4
f	111	63	22	3	1

11. Find the value of k such that the function

$$f(x) = \begin{cases} kx^2, & 1<x<3 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function of a continuous random variable. Also find $P(1.5<x<2.5)$

12. The p.d.f of a continuous random variable is given by

$$P(x) = \begin{cases} kx(1-x)e^{x}, & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find k and hence find the mean and the standard deviation.

13. The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is
(i) less than 200 months (ii) between 100 months and 25 years

14. If z is a standard normal variate, find the following probabilities by using normal probability table.
(i) $P(0.87<z<1.28)$ (ii) $P(-0.34<z<0.62)$
(iii) $P(z>0.85)$ (iv) $P(z>-0.65)$

15. The mean weight of 500 students during a medical examination was found to be 50 kgs and S.D weight 6 kgs. Assuming that the weights are normally distributed, find the number of student having weight
(i) between 40 and 50 kgs. (ii) more than 60 kgs. given that

$\phi(1.67) = 0.4525$ where $\phi(z) = \frac{1}{\sqrt{2\pi}}\int_0^z e^{-z^2/2}\,dz$

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ANSWERS

1. $k = 1/32$, mean $= 5/2$, variance $= 5/4$
2. $a = 1/81, 1/9, 8/9, 8/27$

0	1	2	3	4	5	6	7	8
1/81	4/81	9/81	16/81	25/81	36/81	49/81	64/81	1

3. $k = 1$
4. 5/16, 21/32, 63/64
5. 0.2335, 0.3828
6. $P(x) = {}^{8}C_x (1/4)^x (3/4)^{8-x}$
7. ${}^{10}C_x (1/2)^x (1/2)^{10-x} = \frac{1}{2^{10}} {}^{10}C_x$

Expected frequencies : 1, 10, 45, 120, 252, 210, 120, 45, 10, 1

8. 0.1847
9. 0.2231, 0.1912
10. $P(x) = \dfrac{e^{-0.6}(0.6)^x}{x!}$; 110, 66, 20, 4, 1
11. $k = 3/26$; 49/104
12. $k = 1/3 - e$; mean ≈ 0.55, S.D. ≈ 0.23
13. 0.6321, 0.3834
14. 0.0919, 0.3655, 0.1977, 0.7422
15. 226, 24

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