Sardar Patel University,Balaghat(M.P)
School of Engineering & Technology
Department of CSE
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Module 1

Numerical Methods Solution of polynomial and transcendental equations Bisection method, Newton Raphson method and Regula Falsi method.
Finite differences, Relation between operators, Interpolation using Newton's forward and backward Difference formulae. Interpolation with unequal intervals: Newton's divided difference and Lagrange's formulae.

Introduction

In this unit we will discuss one of the most basic problems in numerical analysis. The problem is called a root-finding problem and consists of finding values of the variable x (real) that satisfy the equation f(x) = 0, for a given function f. Let f be a real-value function of a real variable. Any real number · for which f(·) = 0 is called a root of that equation or a zero of f. We shall confine our discussion to locating only the real roots of f(x), that is, locating non-real complex roots of f(x) = 0 will not be discussed. This is one of the oldest numerical approximation problems. The procedures we will discuss range from the classical Newton-Raphson method developed primarily by Isaac Newton over 300 years ago to methods that were established in the recent past.

Myriads of methods are available for locating zeros of functions and in first section we discuss bisection methods and fixed point method. In the second section, Chord Method for finding roots will be discussed. More specifically, we will take up regula-falsi method (or method of false position), Newton-Raphson method, and secant method. In section 3, we will discuss error analysis for iterative methods or convergence analysis of iterative method.

We shall consider the problem of numerical computation of the real roots of a given equation

$$f(x) = 0$$

which may be algebraic or transcendental. It will be assumed that the function f(x) is continuously differentiable a sufficient number of times. Mostly, we shall confine to simple roots and indicate the iteration function for multiple roots in case of Newton Raphson method.

All the methods for numerical solution of equations discussed here will consist of two steps. First step is about the location of the roots, that is, rough approximate value of the roots are obtained as initial approximation to a root. Second step consists of methods, which improve the rough value of each root.

A method for improvement of the value of a root at a second step usually involves a process of successive approximation of iteration. In such a process of successive approximation a sequence $\{X_n\}$ n = 0, 1, 2, … is generated by the method used starting with the initial approximation x_o of the root · obtained in the first step such that the sequence $\{X_n\}$ converges to · as n · · . This x_n is called the nth approximation of nth iterate and it gives a sufficiently accurate value of the root · .

For the first step we need the following theorem:

Theorem 1: If f(x) is continuous in the closed internal [a, b] and f(a) are of opposite signs, then there is at least one real root ξ of the equation f(x) = 0 such that a < ξ < b.

If further f(x) is differentiable in the open interval (a, b) and either f′(x) < 0 or f′(x) > 0 in (a, b) then f(x) is strictly monotonic in [a, b] and the root ξ is unique.

We shall not discuss the case of complex roots, roots of simultaneous equations nor shall we take up cases when all roots are targeted at the same time, in this unit.

Bisection Method (Bolzano Method)

In this method we find an interval in which the root lies and that there is no other root in that interval. Then we keep on narrowing down the interval to half at each successive iteration. We proceed as follows:

(1) Find interval $I = (x_1, x_2)$ in which the root of f(x) = 0 lies and that there is no other root in I.

(2) Bisect the interval at $x = \dfrac{x_1 + x_2}{2}$ and compute f(x). If | f(x) | is less than the desired accuracy then it is the root of f(x) = 0.

Otherwise check sign of f(x). If sign {f(x)} = sign {f(x_2)} then root lies in the interval [x_1, x] and if they are of opposite signs then the root lies in the interval [x, x_2]. Change x to x_2 or x_1 accordingly. We may test sign of $f(x) \cdot f(x_2)$ for same sign or opposite signs.

(3) Check the length of interval | $x_1 - x_2$ |. If an accuracy of say, two decimal places is required then stop the process when the length of the interval is 0.005 or less. We may take the midvalue $x = \dfrac{x_1 + x_2}{2}$ as the root of f(x) = 0. The convergence of this method is very slow in the beginning.

Example

Find the positive root of the equation $x^3 + 4x^2 - 10 = 0$ by bisection method correct upto two places of decimal.

Solution

$$f(x) = x^3 + 4x^2 - 10 = 0$$

Let us find location of the + ive roots.

x	0	1	2	> 2
f(x)	-10	-5	14	
Sign f(x)	$-$	$-$	+	+

There is only one + ive root and it lies between 1 and 2. Let $x_1 = 1$ and $x_2 = 2$; at x = 1, f(x) is – ive and at x = 2, f(x) is + ive. We examine the sign of f(x) at $x = \frac{x_1 + x_2}{2} = 1.5$ and check whether the root lies in the interval (1, 1.5) or (1.5, 2). Let us show the computations in the table below :

Iteration No.	$x = \frac{x_1 + x_2}{2}$	Sign f(x)	Sign f(x) · f(x_2)	x_1	x_2
1	1.5	+ 2.375	+	1	1.5
2	1.25	− 1.797	−	1.25	1.5
3	1.375	+ 0.162	+	1.25	1.375
4	1.3125	− 0.8484	−	1.3125	1.375
5	1.3438	− 0.3502	−	1.3438	1.375
6	1.3594	− 0.0960	−	1.3594	1.375
7	1.367	− 0.0471	−	1.367	1.375
8	1.371	+ 0.0956	+	1.367	1.371

We see that $| x_1 - x_2 | = 0.004$. We can choose the root as

$$x = \frac{1.367 + 1.371}{2} = 1.369 .$$

Regula-Falsi Method (or Method of False Position)

In this method also we find two values of x say x_1 and x_2 where function f(x) has opposite signs and there is only one root in the interval (x_1, x_2). Let us express the function of y = f(x) and we are interested in finding the value of x where curve y = f(x) intersects x-axis i.e. y = 0. We identify two points (x_1, y_1) and (x_2, y_2) on the curve. Then we approximate the curve by a straight line joining these two points. We find the point on the x-axis where this line cuts the x-axis. The equation of the straight line passing through (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The point on x-axis where y = 0 is given by

$$x = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}$$

Now we check the sign of f(x) and proceed like in the bisection method. That is, if f(x) has same sign as $f(x_2)$ then root lies in the interval (x_1, x) and if they have opposite signs, then it lies in the interval (x, x_2). See Figure 1.

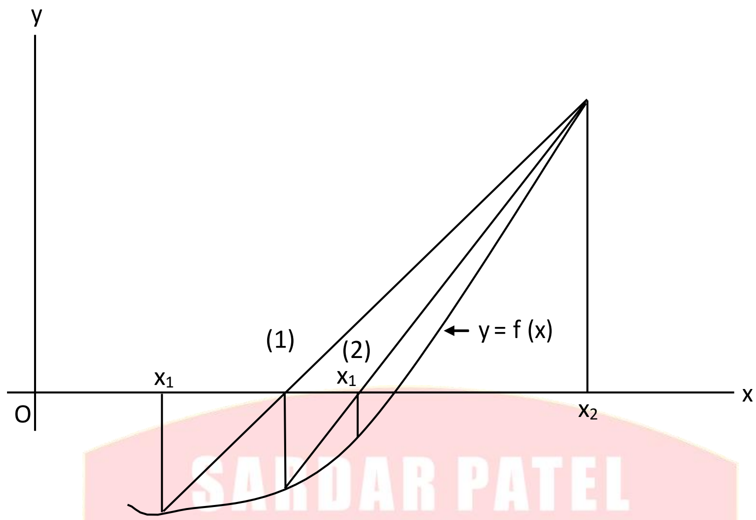

Figure 1 : Regula-Falsi Method, Superscript Shows Iteration Number

Example

Find positive root of $x^3 + 4x^2 - 10 = 0$ by Regula-Falsi method. Compute upto the two decimal places only.

Solution

It is the same problem as given in the previous example. We start by taking $x_1 = 1$ and $x_2 = 2$. We have $y = x^3 + 4x^2 - 10$; $y_1 = -5$ and $y_2 = 14$. The point on the curve are (1, – 5) and (2, 14). The points on the x-axis where the line joining these two pints cuts it, is given by

I-Iteration

$$x = \frac{1 \times 14 - 2 \times (-5)}{14 + 5} = \frac{24}{19} = 1.26$$

$$y = f(x) = -1.65$$

II-Iteration

Take points (1.26, – 1.65) and (2, 14)

$$x = \frac{1.26 \times 14 - 2 \times (-1.65)}{14 + 1.65} = 1.34$$

$$y = f(x) = -0.41$$

III-Iteration

Take two points (1.34, – 0.41) and (2, 14)

$$x = \frac{1.34 \times 14 - 2 \times (-0.41)}{14 + 0.41} = 1.36$$

$$y = f(x) = -0.086$$

IV-Iteration

Take two points (1.36, – 0.086) and (2, 14)

$$x = \frac{1.36 \times 14 - 2 \times (-0.086)}{14 + 0.086} = 1.36$$

Since value of x repeats we take the root as x = 1.36.

Example: Find the real root of the equation $x\log_{10} x - 1.2=0$ by method of False position, correct to four decimal places.

Solution

$f(x)=x\log_{10} x - 1.2=0$

f(2)=-0.5979,f(3)=0.2313

Therefore root lies between 2 & 3

by method of False position

$$x_1 = \frac{a(b) - b(a)}{(b) - (a)}$$

$$x_1 = 2.7210, (x_1) = -0.01709$$

Now the roots lie between $x_1 = 2.7210$ & 3, because the function are of opposite sign.

$$x_2 = 2.7402, (x_2) = 0.00038$$

Root lies between 2.7402 & 3,

$x_3 = 2.7406$hence the root 2.7406

Newton-Raphson (N-R) Method

The Newton-Raphson's method or commonly known as N-R method is most popular for finding the roots of an equation. Its approach is different from all the methods discussed earlier in the sense that it uses only one value of x in the neighbourhood of the root instead of two. We can explain the method geometrically as follows :

Let us suppose we want to find out the root of an equation f(x) = 0 while y = f(x) represents a curve and we are interested to find the point where it cuts the x-axis. Let x = x_0 be an initial approximate value of the root close to the actual root. We evaluate $y(x_0) = f(x_0) = y_0$ (say).

Then point (x_0, y_0) lies on the curve y = f(x). We find $\frac{dy}{dx} = f'(x)$ for x = x_0, say $f'(x_0)$. Then we may draw a tangent at (x_0, y_0) given as,

$$y - y_0 = f'(x_0)(x - x_0)$$

The point where the tangent cuts the x-axis (y = 0) is taken as the next estimate x = x_1 for the root, i.e.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, see Figure 3

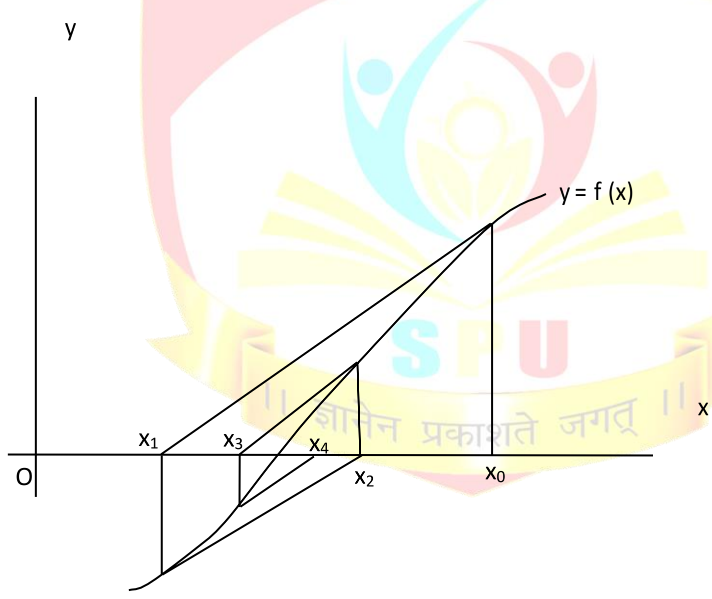

Figure 2 : Newton-Raphson Method

Theoretically, the N-R method may be explained as follows :

Let α be the exact root of f(x) = 0 and let $\alpha = x_0 + h$ where h is a small number to be determined. From Taylor's series as have,

$$f(\alpha) = f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \dots = 0$$

Neglecting h^2 and higher powers we get an approximate value of h, as $h = -\frac{f(x_0)}{f'(x_0)}$. Hence, an approximation for the exact root α may be written as,

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general the N-R formula may be written as,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots$$

It is same as derived above geometrically. It may be stated that the root of convergence of N-R method is faster as compared to other methods. Further, comparing the N-R method with method of successive substitution, it can be seen as iterative scheme for

$$x = x - \frac{f(x)}{f'(x)}$$

where $\phi(x) = x - \frac{f(x)}{f'(x)}$

The condition for convergence $|\phi'(\alpha)| < 1$ in this case would be

$$\phi'(x) = 1 - \frac{\{f'(x)\}^2 - f(x) f''(x)}{\{f'(x)\}^2} = \frac{f(x) f''(x)}{\{f'(x)\}^2}, \text{ at } x = \alpha.$$

This implies that $f'(\alpha) \neq 0$.

Example

Write N-R iterative scheme to find inverse of an integer number N. Hence, find inverse of 17 correct upto 4 places of decimal starting with 0.05.

Solution

Let inverse of N be x, so that we the equation to solve as,

$$x = N^{-1} \qquad \text{or} \qquad x^{-1} - N = 0$$

$$f(x) = x^{-1} - N \; ; \; f'(x) = -\frac{1}{x^2}$$

N-R scheme is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}$$

$$= x_n(2 - Nx_n) = x_n(2 - 17x_n) \qquad \because N = 17.$$

We take $x_0 = 0.05$.

Substituting in the formula, we get

$x_1 = 0.0575$; $x_2 = 0.0588$; $x_3 = 0.0588$

Hence, $\frac{1}{17} = 0.0588$.

Example

Write down N-R iterative scheme for finding q^{th} root of a positive number N. Hence, find cuberoot of 10 correct upto 3 places of decimal taking initial estimate as 2.0.

Solution

We have to solve $x = N^{\frac{1}{q}}$ or $x^q - N = 0$

$$f(x) = x^q - N \;;\; f'(x) = q\,x^{q-1}$$

The N-R iterative scheme may

$$x_{n+1} = x_n - \frac{x_n^q - N}{q\,x_n^{q-1}} = \frac{(q-1)x_n^q + N}{q\,x_n^{q-1}}$$

For cuberoot of 10 we have N = 10, q = 3.

Hence, $x_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$

Taking $x_0 = 2.0$ we get the following iterated values

$$x_1 = \frac{16}{2} = 2.167 \;;\; x_2 = \frac{30.3520}{14.0877} = 2.154 \;;\; x_3 = \frac{29.9879}{13.9191} = 2.154$$

Hence, we get $10^{\frac{1}{3}} = 2.154$.

Example

Using N-R method find the root of the equation x – cos x = 0 correct upto two places of decimal only. Take the starting value as $\frac{\pi}{4}$ (π = 3.1416, π radian = 180°).

Solution

f (x) ' x ' cos x ; f'(x) ' 1 ' sin x

N-R scheme is given by

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n}{1 + \sin x_n}$$

Taking $x_0 = \frac{\pi}{4}$

$$x_1 = \frac{\frac{\pi}{4}\sin\frac{\pi}{4} + \cos\frac{\pi}{4}}{1 + \sin\frac{\pi}{4}} = \frac{\frac{\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}$$

$$= \frac{\frac{\pi}{4} + 1}{\sqrt{2} + 1} = \frac{1.7854}{2.4142} = 0.7395$$

$$x_2 = \frac{0.7395 \sin (0.7395) + \cos (0.7395)}{1 + \sin (0.7395)}$$

$$= \frac{0.7395 \times 0.6724 + 0.7449}{1 + 0.6724} = 0.7427$$

Upto two places of decimal the root is 0.74.

Note : If starting value is not given, we can plot graphs of y = x and y = cos x and locate their point of intersection which will be root of x – cos x = 0. See Figure 4

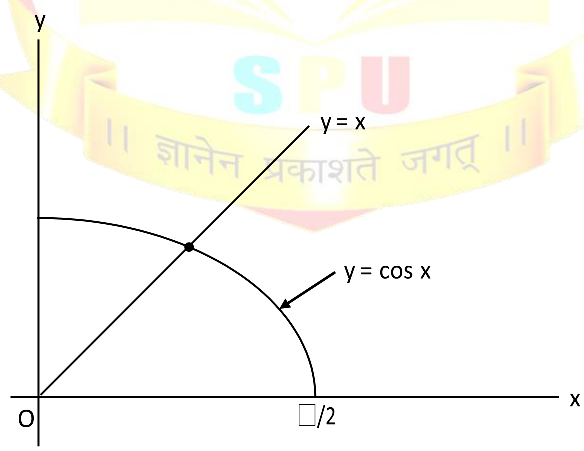

Figure 3 : Intersection of y = x and y = cos x

Example

Find one real root of $3x = \cos x + 1$ by Newton Raphson method.

Solution

Let $(x) = 3x - cosx - 1$

$$'(x) = 3 + \sin x$$

By Newton Raphson method

$$x_{n+1} = x_n \frac{(x_n)}{'(x_n)}$$

$$(0) = -2, (1) = 2.540$$

Since f(0)=-2 is nearer to 0,therefore $x_0 = 0$ is our first approximation.

$x_1 = x_0 - \frac{(x_0)}{'(x_0)}$=0.6

$$x_2 = 0.6071$$

$$x_3 = 0.6071$$

The real root of equation correct to four decimal places 0.6071.

Finite Differences and Interpolation

Suppose we are given the following values of y = f(x) for a set of values of x :

x :	x_0	x_1	x_2	……	x_n
y :	y_0	y_1	y_2	……	y_n

The process of finding the values of y corresponding to any value of $x=x_i$ between x_0 and x_n is called interpolation.

(1) The technique of estimating the value of a function for any intermediate value of the independent variable is called interpolation.

(2) The technique of estimating the value of a function outside the given range is called extrapolation.

(3) The study of interpolation is based on the concept of differences of a function.

(4) Suppose that the function y=f(x) is tabulated for the equally spaced values $x = x_0$, $x_1=x_0+h$, $x_2=x_0+2h$, …, $x_n=x_0+nh$ giving $y = y_0, y_1, y_2, \ldots, y_n$. To determine the values of f(x) and f '(x) for some intermediate values of x, we use the following three types of differences

1.Forward differences

2.Backward differences

Forward differences: The forward differences are defined and denoted by $\Delta f(x)=f(x+h)-f(x)$,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2$$
$$\ldots\ldots\ldots\ldots$$
$$\Delta y_r = y_{r+1} - y_r$$
$$\ldots\ldots\ldots\ldots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

These are called the first forward differences and Δ is the forward difference operator.
Similarly the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r.$$

In general

$$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r ,$$

p^{th} forward differences.

The forward differences systematically set out in a table called forward difference table.

Value of x	Value of y	1^{st} diff. Δ	2^{nd} diff. Δ^2	3^{rd} diff. Δ^3	4^{th} diff. Δ^4	5^{th}diff. Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Backward Differences: The backward differences are defined and denoted by f(x)= f(x)-f(x-h),

$$y_1 = y_1 - y_0$$
$$y_2 = y_2 - y_1$$
$$y_3 = y_3 - y_2$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\nabla y_r = y_r - y_{r-1}$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\nabla y_n = y_n - y_{n-1}.$$

These are called the first backward differences and ∇ is the backward difference operator.
Similarly the second backward differences are defined by

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}.$$

In general

$$\nabla^p y_r = \nabla^{p-1} y_r - \nabla^{p-1} y_{r-1} ,$$

p^{th} backward differences. The backward differences systematically set out in a table called backward

Example#1. Evaluate

(i) $\Delta \tan^{-1}x$

(ii) $\Delta (e^x \log 2x)$

(iii) $\Delta^2 \cos 2x$

Sol. From the definition of forward differences $\Delta f(x) = f(x+h) - f(x)$.

(i) Let $f(x) = \tan^{-1}x$, then

$\Delta \tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x$

$$= \tan^{-1}\left[\frac{x+h-x}{1+(x+h)x}\right] = \tan^{-1}\left[\frac{h}{1+hx+x^2}\right].$$

(ii)

$$\begin{aligned}\Delta(e^x \log 2x) &= e^{x+h}\log 2(x+h) - e^x \log 2x \\ &= e^{x+h}\log 2(x+h) - e^{x+h}\log 2x + e^{x+h}\log 2x - e^x \log 2x \\ &= e^{x+h}\log\left(\frac{x+h}{x}\right) + (e^{x+h} - e^x)\log 2x \\ &= e^x\left[e^h \log\left(1+\frac{h}{x}\right) + (e^h - 1)\log 2x\right].\end{aligned}$$

(iii) $\Delta^2 \cos 2x = \Delta[\Delta \cos 2x]$

$= \Delta[\cos 2(x+h) - \cos 2x]$

$= \Delta \cos 2(x+h) - \Delta \cos 2x$

$= \cos 2(x+2h) - \cos 2(x+h) - [\cos 2(x+h)) - \cos 2x]$

$= -2\cos(2x+3h)\sin h + 2\sin(2x+h)\sin h$

$= -2\sin h[\sin(2x+3h) - \sin(2x+h)]$

$= -2\sin h[2\cos(2x+2h)\sin h]$

$= -2\sin^2 h \cos 2(x+h)$.

Example#2. Evaluate the following, with interval of difference being unity

(i) $\Delta^2 (ab^x)$ (ii) $\Delta^n e^x$

Sol. From the definition of forward differences $\Delta f(x) = f(x+h) - f(x)$.

(i) $\Delta(ab^x) = a\,\Delta b^x = a(b^{x+1} - b^x) = ab^x(b-1)$

$\Delta^2 (ab^x) = \Delta[\Delta(ab^x)]$

$= \Delta\, ab^x(b-1) = a(b-1)\,\Delta(b^x)$

$= a(b-1)(b^{x+1} - b^x)$

$= a(b-1)^2 b^x$.

(ii) $\Delta e^x = e^{x+1} - e^x = e^x(e-1)$

$\Delta^2 e^x = \Delta[\Delta e^x] = \Delta[e^{x+1} - e^x] = (e-1)\Delta e^x$

$= (e-1)e^x(e-1) = (e-1)^2 e^x$.

Similarly $\Delta^2 e^x = (e-1)^2 e^x$, $\Delta^3 e^x = (e-1)^3 e^x$, … and $\Delta^n e^x = (e-1)^n e^x$.

Factorial Notation: A product of the form x(x – 1) (x – 2) …(x – r +1) is denoted by $[x]^r$ and is called a factorial. In particular,

$$[x] = x,\ [x]^2 = x(x-1)\ ,\ [x]^3 = x(x-1)(x-2),\ \ldots [x]^n = x(x-1)(x-2)\ldots(x-n+1).$$

If the interval of difference is h, then

$$[x]^n = x(x-h)(x-2h)\ldots(x-(n-1)h).$$

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation ($[x]^r$ as x^r).

To express a polynomial of nth degree in the factorial notation, we use the following two steps

1. Arrange the coefficients of the powers of x in descending order, replacing missing powers by zeros.
2. Using detached coefficients divide by x, x – 1, x – 2, … x – (n – 1) successively.

Example#4. Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ in a factorial notation and hence find all differences.

Sol. Let $f(x) = A[x]^3 + B[x]^2 + C[x] + D$. Then

	x^3	x^2	x	
1	2	-3	3	-10 = D
	–	2	-1	
2	2	-1	2 = C	
	–	4		
3	2	3 = B		
	–			
	2 = A			

Hence $f(x) = 2[x]^3 + 3[x]^2 + 2[x] - 10$. Therefore,

$\Delta f(x) = 6[x]^2 + 6[x] + 2$

$\Delta^2 f(x) = 12[x] + 6$

$\Delta^3 f(x) = 12.$

Other Difference Operators:

(1) Shift operator : Shift operator E is the operation of increasing the argument x by h so that

$$E\, f(x) = f(x+h),\ E^2 f(x) = f(x+2h), \ldots$$
$$E^n f(x) = f(x+nh).$$

The inverse operator E^{-1} is defined by

$$E^{-1} f(x) = f(x-h).$$

Similarly

$$E^{-n} f(x) = f(x-nh).$$

(2) Averaging operator : Averaging operator μ is defined by the equation

$$\mu f(x) = ½ [f(x + h/2) + f(x - h/2)].$$

In the difference calculus, Δ and E are regarded as the fundamental operators and , δ and μ can be expressed in terms of these.

Relations Between the Operators :

1. $\Delta = E - 1$
2. $\cdot \; = 1 - E^{-1}$
3. $\delta = E^{1/2} - E^{-1/2}$
4. $\mu = ½ [E^{1/2} + E^{-1/2}]$
5. $\Delta = E \cdot = \cdot E = \delta E^{1/2}$
6. $E = e^{hD}$.

Example#1. Determine the missing values in the following table:

x	45	50	55	60	65
y	3	?	2	?	-2.4

Sol. Let p and q be the missing values in the given table, then the difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3			
		$p - 3$		
50	p		$5 - 2p$	
		$2 - p$		$3p + q - 9$
55	2		$p + q - 4$	
		$q - 2$		$3.6 - p - 3q$
60	q		$-0.4 - 2q$	
		$-2.4 - q$		
65	-2.4			

Since three entries are given, the function y can be represented by a second degree polynomial. Therefore, $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$. Thus $3p + q - 9 = 0$ and $3.6 - p - 3q = 0$. Solving these equations, we get $p = 2.925$ and $q = 0.225$.

Example#2. Determine the missing values in the following table without using difference table.

x	45	50	55	60	65
y	3	?	2	?	-2.4

Sol. Given that $y_0 = 3$, $y_2 = 2$ and $y_4 = -2.4$ and missing values be taken as $y_1 = p$ and $y_3 = q$. Since three entries are given, the function y can be represented by a second degree polynomial.

Therefore, $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$.

$$(E - 1)^3 y_0 = 0 \qquad (E - 1)^3 y_1 = 0$$
$$(E^3 - 3E^2 + 3E - 1)y_0 = 0 \qquad (E^3 - 3E^2 + 3E - 1)y_1 = 0$$
$$y_3 - 3y_2 + 3y_1 - y_0 = 0 \qquad y_4 - 3y_3 + 3y_2 - y_1 = 0$$
$$q - 3(2) + 3p - 3 = 0 \qquad -2.4 - 3q + 3(2) - p = 0$$
$$3p + q - 9 = 0 \qquad 3.6 - p - 3q = 0.$$

Solving these equations, we get $p = 2.925$ and $q = 0.225$.

Newton's Forward Interpolation Formulae:

Let the function $y=f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_1, x_2, \ldots$ of x. Suppose it is required to evaluate f(x) for $x=x_0+ph$, p is any real number.

For any real number p, we have defined E such that

$E^p f(x) = f(x_0+ph)$

$y_p = f(x_0+ph) = E^p f(x_0) = (1+\Delta)^p y_0$

$= [1+p\Delta+p(p-1)/2!\ \Delta^2 + p(p-1)(p-2)/3!\ \Delta^3 + \ldots]\ y_0$

$= y_0 + p\ \Delta y_0 + p(p-1)/2!\ \Delta^2 y_0 + p(p-1)(p-2)/3!\ \Delta^3 y_0 + \ldots$

It is called Newton's forward interpolation formulae.

Newton's Backward Interpolation Formulae:

Suppose it is required to evaluate f(x) for $x=x_n+ph$, where p is any real number.

$E^p f(x) = f(x_n+ph)$

$y_p = f(x_n+ph) = E^p f(x_n) = (1-\nabla)^{-p}\ y_n$

$= [1+p\nabla+p(p+1)/2!\ \nabla^2 + p(p+1)(p+2)/3!\ \nabla^3 + \ldots]\ y_n$

$= y_n + p\nabla y_n + p(p+1)/2!\ \nabla^2 y_n + p(p+1)(p+2)/3!\ \nabla^3 y_n + \ldots$

It is called Newton's backward interpolation formulae.

Choice of Newton's Interpolation formulae:

- Newton's forward interpolation formulae is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward of y_0.
- Newton's backward interpolation formulae is used for interpolating the values of y near the end of a set of tabulated values and also extrapolating values of y a little ahead of y_n.

Example#1. The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface :

x=height	100	150	200	250	300	350	400
y=distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) x= 218 ft. (ii) x= 410 ft.

Sol. The difference table is

x	**y**	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.4			
150	13.03		-0.39		
		2.01		0.15	
x_0=200	**15.04**		-0.24		-0.07
		1.77		0.08	
250	16.81		**-0.16**		-0.05
		1.61		**0.03**	

300	18.42		-0.13		**-0.01**
		1.48		**0.02**	
350	19.90		**-0.11**		
		1.37			
x_n=400	**21.27**				

(i) If we take x_0=200, then y_0=15.04, Δy_0=1.77, $\Delta^2 y_0$=-0.16, $\Delta^3 y_0$=0.03, $\Delta^4 y_0$=-0.01.

Since x=218, step length h=50 and $p=(x-x_0)/h$ =18/50 = 0.36.

By Newton's forward interpolation formula, we have

$y(218) = y_0 + p\,\Delta y_0 + p(p-1)/2!\,\Delta^2 y_0 + p(p-1)(p-2)/3!\,\Delta^3 y_0 + p(p-1)(p-2)(p-3)/4!\,\Delta^4 y_0$

$= 15.04 + 0.36\,(1.77) + 0.36(0.36-1)/2\,(-0.16) + 0.36(0.36-1)(0.36-2)/6\,(0.03)$

$+ 0.36(0.36-1)(0.36-2)(0.36-3)/24\,(-0.01)$

$=15.04+0.6372+0.0184+0.0018+ 0.00041 = 15.69741$

$\approx$ 15.7 nautical miles.

(ii) If we take x_n=400, then y_n=21.27, ∇y_n=1.37, $\nabla^2 y_n$=-0.11, $\nabla^3 y_n$=0.02, $\nabla^4 y_n$=-0.01.

Since x=410, step length h=50 and $p=(x-x_n)/h$ =10/50 = 0.2.

By Newton's backward interpolation formula, we have

$y(410) = y_n + p\,\nabla y_n + p(p+1)/2!\,\nabla^2 y_n + p(p+1)(p+2)/3!\,\nabla^3 y_n + p(p+1)(p+2)(p+3)/4!\,\nabla^4 y_n$

$= 21.27 + 0.2\,(1.37) + 0.2(0.2+1)/2\,(-0.11) + 0.2(0.2+1)(0.2+2)/6\,(0.02)$

$+ 0.2(0.2+1)(0.2+2)(0.2+3)/24\,(-0.01)$

$=21.27+0.274-0.0132+0.0017- 0.0007 = 21.5318$

$\approx$ 21.53 nautical miles.

Interpolation with unequal intervals:

The disadvantage for the previous interpolation formulas is that, they are used only for equal intervals. The following are the interpolation with unequal intervals;

1) Lagrange's formula for unequal intervals,
2) Newton's divided difference formula.

Lagrange's interpolation formula: If $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots, x_n$, then

$$f(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)} y_1 + \cdots$$

$$+ \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})} y_n,$$

which is known as Lagrange's formula.

Divided Differences: If $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ are given points, then the first divided differences for the argument x_0, x_1 is defined by

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}.$$

Similarly

$$[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}, [x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2}, \ldots, [x_{n-1}, x_n] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

The second divided differences for x_0, x_1, x_2 is

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}.$$

The third divided differences for x_0, x_1, x_2, x_3 is

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}.$$

And so on, the nth divided differences for $x_0, x_1, x_2, \ldots, x_n$ is

$$[x_0, x_1, x_2, \ldots, x_n] = \frac{[x_1, x_2, \ldots, x_n] - [x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

All the divided differences systematically set out in a table called divided difference table.

Value of x	Value of y	1^{st} divided difference	2^{nd} divided difference	3^{rd} divided difference	4^{th} divided difference	5^{th} divided difference
x_0	y_0					
		$[x_0,x_1]$				

x_1	y_1		$[x_0,x_1,x_2]$			
		$[x_1,x_2]$		$[x_0,x_1,x_2,x_3]$		
x_2	y_2		$[x_1,x_2,x_3]$		$[x_0,x_1,x_2,x_3,x_4]$	
		$[x_2,x_3]$		$[x_1,x_2,x_3,x_4]$		$[x_0,x_1,x_2,x_3,x_4,x_5]$
x_3	y_3		$[x_2,x_3,x_4]$		$[x_1,x_2,x_3,x_4,x_5]$	
		$[x_3,x_4]$		$[x_2,x_3,x_4,x_5]$		
x_4	y_4		$[x_3,x_4,x_5]$			
		$[x_4,x_5]$				
x_5	y_5					

Newton's divided difference formula: If $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots, x_n$, then

$$f(x) = y_0+(x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + \ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})[x_0, x_1, x_2, \ldots, x_n],$$

which is known as Newton's general interpolation formula with divided differences.

Example#1. Given the values

x :	5	7	11	13	17
f(x):	150	392	1452	2366	5202

Evaluate f(9), using

(i) Lagranges formula

(ii) Newton's divided difference formula.

Sol. Let $y = f(x)$, then from the given data, we have

$x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$ and $y_0 = 150$, $y_1 = 392$, $y_2 = 1452$, $y_3 = 2366$, $y_4 = 5202$.

(i) By Lagrnge's interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_n)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4.$$

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}\times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}\times 392$$

$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-8)(11-7)(11-13)(11-17)}\times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}\times 2366$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\times 5202 = -\frac{50}{3}+\frac{3136}{15}+\frac{3872}{3}-\frac{2366}{3}+\frac{578}{5}=810.$$

(ii) The divided difference table is

Value of x	Value of y	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference
5	150				
		121			
7	392		**24**		
		265		**1**	
11	1452		32		**0**
		457		1	
13	2366		42		
		709			
17	5202				

By Newton divided difference formula

$f(x) = y_0+(x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] +(x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2,x_3]$
$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)[x_0, x_1, x_2,x_3,x_4].$

$f(9) = 150 + (9 – 5)\times121 + (9 – 5) (9 – 7)\times24 + (9 – 5)(9 – 7)(9 – 11)\times1$
$+ (9 – 5)(9 – 7)(9 – 11)(9 – 13)\times0$
$= 150 + 484 + 192 – 16 + 0$
$= 810.$

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Module2: Numerical Methods–2

Contents
Numerical Differentiation ,Numerical integration: TrapezoidalruleandSimpson’s1/3rdand3/8rules.SolutionofSimultaneousLinearAlgebraic EquationsbyGauss’sElimination,Gauss’sJordan,Crout’smethods,Jacobi’s,Gauss-Seidal,and Relaxation method.,

Numerical Differentiation

In numerical analysis, numerical differentiation describes algorithms for estimating the derivative of a mathematical function or function subroutine using values of the function and perhaps other knowledge about the function. i.e. Numerical different ion is the process of calculating the derivative of a function at some particular value of the independent variable by means of a set of given values of that function.

[A] **Derivative at any point** :- The general method of numerical differentiation of a function consists in obtaining an explicit analytical relation $y = f\,x$ with the help of an interpolation

[B]

[C] formula ,and then differentiating y with respect to x as many times as required.

Let the function $y = f\,x$ be obtained by the Newton Gregory’s forward interpolation formula

$$y = y_0 + u\,\Delta y_0 + \frac{u\,(u-1)}{2!}\Delta^2 y_0 + \frac{u\,(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

Where $u = \frac{x-x_0}{h}$,

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots \dots$$

and $\frac{du}{dx} = \frac{1}{h}$

Therefore, $\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$

$$\frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots \dots\right] \dots .(2)$$

The second derivative $\frac{d^2y}{dx^2}$ may be determined by differentiating (2) with respect to x

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}[\Delta^2y_0+u-1\,\Delta^3y_0+\frac{6u^2-18u+11}{12}\Delta^4y_0+\frac{2u^3-12u^2+21u-10}{12}\Delta^5y_0+\cdots\ldots]$$

Similarly, derivatives of other higher orders may be obtained.

Example: A rod is rotating in a plane about one of it's ends .If the following table gives the angle radians throughh ,which the rod has turned for different values of time t seconds, find it's angular velocity and angular acceleration ,when $t=0.7$ second.

t seconds	0.0	0.2	0.4	0.6	0.8	1.0
radians	0.0	0.12	0.48	1.10	2.0	3.20

Solution: First , we will construct the difference table :

t seconds	**radians**	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0.0	0.0	0.12				
0.2	0.12	0.36	0.24	0.02	0.00	0.00
0.4	0.48	0.62	0.26	0.02	0.00	
0.6	1.10	0.90	0.28	0.02		
0.8	2.0	1.20	0.30			
1.0	3.20					

Then the angular velocity $\frac{d\theta}{dt}$ is

$$=\frac{1}{h}[\Delta_0+\frac{2u-1}{2!}\Delta^2_0+\frac{3u^2-6u+2}{3!}\Delta^3_0+\frac{4u^3-18u^2+22u-6}{4!}\Delta^4_0+\cdots\ldots]\text{.........(1)}$$

Where $u=\frac{t-t_0}{h,}=\frac{t-0.0}{0.2}=5t$

At $t=0.7$,then$u=3.5$

Therefore, the angular velocity $\frac{d\theta}{dt}=4.496$radian per second.

The angular acceleration $\frac{d^2}{dt^2}$ is obtained with the help of equation (1),

$$\frac{d^2}{dt^2}=\frac{1}{h}\left[0+\frac{2}{2!}\Delta^2_0+\frac{6u-6}{3!}\Delta^3_0\right]\frac{du}{dt}=\frac{1}{h^2}[\Delta^2_0+u-1\,\Delta^3_0]$$

At $t = 0.7$,then$u = 3.5$

Therefore,the angular acceleration

$$\frac{d^2}{dt^2} = \frac{1}{(0.2)^2}\ 0.24 + 3.5 - 1\ 0.02 = 7.25 \text{radianper } second^2.$$

[D] **Derivative at tabulated points**: -In particular case ,When the derivatives are required at one of the tabulated points x_0 ,, y_0 , , x_1 ,, y_1 , ... x_n ,, y_n , and not at the intermediate points, the following formula may be used with advantage**.**

If D denotes the differential operator $\frac{d}{dx}$ and Δ is the difference operator defined by

$\Delta f\ x = f\ x + \square\ - f\ x$,then there holds the operational relation

$\square D = log\ 1 + \Delta$

i.e. $D \equiv \frac{1}{\square}[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \cdots \ldots \ldots .]$

Where, $\square$ is the interval between any two successive values of x ,at which the values of y are prescribed.

Similarly,.$D^2 \equiv \frac{1}{h^2}[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots \ldots \ldots .]$

Example :A slider in a machine moves along a fixed straight rod. It's distance x cm along the road are given in the following table for various values of the time t seconds

t seconds	0.0	0.1	0.2	0.3	0.4	0.5
x **cm**	30.1	31.6	32.9	33.6	40.0	33.8

Find the velocity and acceleration,when$t\ = 0.3$ second

Solution : First we will construct the difference table :

t seconds	x **cm**	Δx	Δ^2x	Δ^3x	Δ^4x	Δ^5x
0.0	30.1	1.5				
0.1	31.6	1.3	-0.2	-0.4	6.7	-31.3
0.2	32.9	0.7	-0.6	6.3	-24.6	
0.3	33.6	6.4	5.7	-18.3		
0.4	40.0	-6.2	-12.6			
0.5	33.8					

Then the velocity $\frac{dx}{dt}$ is

$$. D \equiv \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \cdots \ldots \ldots .\right]$$

Where $h = 0.1$

Therefore, the velocity $\frac{dx}{dt} = Dx$ at $t = 0.3$ is given by

$$Dx_3 = \frac{1}{0.1}\left[\Delta x_3 - \frac{1}{2}\Delta^2 x_3 + \frac{1}{3}\Delta^3 x_3 - \cdots \ldots \ldots .\right] = 10\left[\Delta x_3 - \frac{1}{2}\Delta^2 x_3\right] = 127 \text{ cm per second}$$

Since other differences are not available in the table.

Similarly, acceleration,when $t = 0.3$ second

$$D^2 x_3 = \frac{1}{h^2}\left[\Delta^2 x_3 - \Delta^3 x_3 + \frac{11}{12}\Delta^4 x_3 - \frac{5}{6}\Delta^5 x_3 + \cdots \ldots \ldots .\right] = 100[\Delta^2 x_3 - 0] = -1260$$

cm per $second^2$.

Numerical integration

The process of computing the value of a definite integral $\int_a^b f(x)\, dx$ from a set of numerical values of the integrand is called Numerical integration. When applied to the integration of a function of one variable, the process is known as quadrature.

A General Quadrature Formula for Equidistant Ordinates:-

.We shall, now establish a general formula for the numerical integration, from which other special formulas will be deduced. For this, we assume that the values of x are at equal intervals .i.e $a = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ .... $x_n = x_0 + nh = b$

.Let the function $y = f(x)$ be given by Newton-Gregory's forward interpolation formula's as

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots \quad (1)$$

Where $u = \frac{x - x_0}{h}$, $\Rightarrow x = x_0 + uh$

Integrate (1) on the both sides w.r.t. x over the limits e $a = x_0$ to $x_n = x_0 + nh = b$

$$\int_a^b f(x)\, dx = \int_{x_0}^{x_0+nh} f(x)\, dx$$

But $= x_0 + uh$, $\therefore$ dx = h du , When $x = x_0$, then $u = 0$ and $x = x_0 + n\text{h}$, then u = n

$$\int_{x_0}^{x_0+nh} f(x)\,dx = h\int_0^n \left[y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots \dots \dots \dots \dots \right] du$$

$$\Rightarrow \int_{x_0}^{x_0+nh} f(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \dots\dots\dots\dots\dots\right]$$

..............(2) Where $h = \frac{b-a}{n}$

This is required general quadrature formula.

Trapezoidal rule :- If ,we put $n = 1$ in (2) ,Then there are only two ordinates y_0, y_1, and only one finite difference $\Delta y_0 = y_1 - y_0$ exists, all other higher order finite differences become zero.

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}[y_0 + y_1]$$

The geometrical meaning of this result is that the area between the curve $y = f(x)$,the x-axis and the ordinates y_0 and y_1 is approximated by the area of the trapezium who parallel sides are y_0 and y_1 , and whose breadth is $h = x_1 - x_0$,

Similarly $\int_{x_1}^{x_2} f(x)\,dx = \frac{h}{2}[y_1 + y_2]$,

$$\int_{x_2}^{x_3} f(x)\,dx = \frac{h}{2}[y_2 + y_3]$$

And so on

$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \frac{h}{2}[y_{n-1} + y_n]$$

Therefore, adding all the above results, We get

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots \dots\dots\dots + y_{n-1})] \text{ Where } h = \frac{b-a}{n}$$

This is required **Trapezoidal rule.**

[I] Simpson's 1/3rd rule : If ,we put $n = 2$ in (2) ,Then there are only three ordinates y_0, y_1, y_2, and therefore only two finite differences Δy_0 *and* $\Delta^2 y_0$ are exist and all other differences become zero.

$$\int_{x_0}^{x_2} f(x)\, dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

The geometrical meaning of this result is that the curve $y = f(x)$ between the lines $x = x_0$ *and* $x = x_2$ is approximated by the area of the parabola whose equation is

$y = y_0 + \frac{x-x_0}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 y_0$ and which passes through the points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$.

Proceeding in a similar way, we will get

$$\int_{x_2}^{x_4} f(x)\, dx = \frac{h}{3}[y_2 + 4y_3 + y_4],$$

$$\int_{x_4}^{x_6} f(x)\, dx = \frac{h}{3}[y_4 + 4y_5 + y_6],$$

And so on ……….

$$\int_{x_{n-2}}^{x_n} f(x)\, dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

Therefore, adding all the above results, We get

$$\int_a^b f(x)\, dx = \frac{h}{3}[\, (y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\,]$$

This is required **Simpson's 1/3rd rule.** Where $h = \frac{b-a}{n}$

[II] Simpson's 3/8rule :- **If ,we put** $n = 3$ in (2) ,Then there are only four ordinates y_0, y_1, y_2, y_3, and therefore only three finite differences Δy_0 , $\Delta^2 y_0$, *and* $\Delta^3 y_0$ are exist and all other differences become zero

$$\int_{x_0}^{x_3} f(x)\, dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2) + y_3]$$

The geometrical meaning of this result is that the curve $y = f(x)$ between the lines $x = x_0$ and $x = x_3$ is approximated by the area of the cubical parabola whose equation is

$y = y_0 + \frac{x-x_0}{h} \Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\, h^2} \Delta^2 y_0 + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!\, h^2} \Delta^3 y_0$ and which passes through the points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and (x_3, y_3).

Proceeding in a similar way, we will get

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3}{8}[y_3 + 3(y_4 + y_5 + y_6)]$$

And so on ……….

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3}{8}[y_{n-3} + 3(y_{n-2} + y_{n-1} + y_n)]$$

Therefore, adding all the above results, We get

$$\int_a^b f(x)\,dx = \frac{3}{8}[y_0 + y_n + 2(y_3 + y_6 + \cdots .. + y_{n-3}) + 3(y_1 + y_2 + y_4 + \cdots + y_{n-1})]$$

This is required **Simpson's 3/8th rule.** Where $h = \frac{b-a}{n}$

Example : Evaluate the integral $\int_0^1 \frac{1}{1+x^2} dx$ by taking no. of subinterval 4 through, Trapezoidal rule, Simpson's 1/3rd rule and Simpson's 3/8th rule.

Solution: We know that
Where $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

x	$y = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = 1$
$x_1 = x_0 + h = \frac{1}{4}$	$y_1 = \frac{16}{17}$
$x_2 = x_0 + 2h = \frac{1}{2}$	$y_2 = \frac{4}{5}$
$x_3 = x_0 + 3h = \frac{3}{4}$	$y_3 = \frac{16}{25}$
$x_4 = x_0 + 4h = 1$	$y_4 = \frac{1}{2}$

We know that by Trapezoidal rule

$$\int_a^b f\ x\ dx = \frac{\ }{2}[\,y_0 + y_n\,, + 2(y_1 + y_2\,, + y_3\,, + \cdots \ldots\ldots\ldots + y_{n-1}]$$

Hence from above table the formula becomes

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{\ }{2}[\,y_0 + y_4\,, + 2(y_1 + y_2\,, + y_3\,,)]$$ **=0.782775**

Now by Simpson's 1/3rd rule

$$\int_a^b f\ x\ dx = \frac{\ }{3}[\,y_0 + y_n\,, + 4\ y_1\,, + y_3\,, + \cdots . + y_{n-1}\,, + 2\ y_2\,, + y_4\,, + \cdots\ y_{n-2}\,,\ ]$$

Hence from the above table the formula becomes

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{\ }{3}y_0 + y_4\,, + 4\ y_1\,, + y_3\,, + 2\ y_2\,, = 0.7854.$$

Now by Simpson's 3/8th rule

$$\int_a^b f\,x\,dx = \frac{3}{8}[\,y_0 + y_n\,, + 2\ y_3\,, + y_6\,, + \cdots .. + y_{n-3}\,, + 3\ (y_1\,, + y_2\,, + y_4\,, + \cdots + y_{n-1,}\,]$$

Hence from above table the formula becomes

$$\int_a^b f\,x\,dx = \frac{3}{8}y_0 + y_4\,, + 2\ y_3\,, + 3\ (y_1\,, + y_2\,, = 0.7503$$

Numerical Solution of Simultaneous Linear Algebraic Equations

Simultaneous Linear Algebraic Equations are very common in various fields of Engineering and Science. We used matrix inversion method or Cramer's rule to solve these equations in general. But these methods prove to be tedious, when the system of equations contain a large number of unknowns. To solve such equations there are other numerical methods, which are particularly suited for computer operations. These are of two types:-

[I] **Direct Method**:

(1) Gauss's Elimination Method.

(2) Gauss's Jordan Method.

(3) Crout's methods .

[II] Iterative Method

(1) Gauss's Jacobi's Method
(2) Gauss-Seidal Method
(3) Relaxation method

[I] Direct Method

(1) Gauss's Elimination Method: In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which, the unknowns arefound by back substitutions. The method is quite general and is well adapted for computer operations.

Let us consider a system of m **equations and in** n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n =$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \text{--[1]}$$

In this method for solving the above equations, we proceed stepwise asfollows :

Step-1:- Elimination of x_1 from the second, third,..... nth equation. We assume here that the order of the equation and the order of unknowns in each equations are such that $a_{11} \neq 0$. The variable x_1 can then be eliminated from the second equation by subtracting $(\frac{a_{21}}{a_{11}})$ times the first equation from the second equation $(\frac{a_{31}}{a_{11}})$ times the first equation from the third equation, e.t.c. This gives new system say as follows:

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + \dots\dots\dots. + a'_{2n}x_n = b'_2$$
$$\dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \qquad \text{-------------------------------- [2]}$$
$$a'_{n2}x_2 + \dots\dots\dots. + a'_{nn}x_n = b'_n$$

Here the first equation is called the pivotal equation and a_{11} is called the first pivot.

Step-2. Now, Elimination of x_2 from the third, ….. $n\,th$ equation in (2).

If the coefficient a'_{22} , … …. . , a'_{nn} in (2) are not all zero,we may assume that the order of equation and the unknowns is such that $a'_{22} \neq 0$ .Then ,we may eliminate x_2 from the third, ….. $n\,th$ equation of (2) by subtracting

$(\frac{a'_{33}}{a'_{22}})$times the second equation from the third equation,

$(\frac{a'_{42}}{a'_{22}})$times the second equation from the fourth equation e.t.c.

The further steps are now obvious. In the third step, we eliminate x_3 and in the fourth step, we eliminate x_4 e.t.c.

By successive elimination, we arrive at a single equation in the unknown x_n ,which can be solved and substituting this in the preceding equation, we obtain the value of x_{n-1}. In this,manner ,we find x_n when the elimination is completed. Also when the elimination is complete the system takes the form.

$$c_{11}x_1 + c_{12}x_2 + \dots \dots \dots + c_{1n}x_n = d_1$$

$$c_{22}x_2 + \dots \dots \dots . + c_{2n}x_n = d_2$$

…………… …………… ………………

$$c_{nn}\ x_n = d_n$$

In this case there exists a unique solution.The new coefficient matrix is an upper triangular matrix ; the diagonal element c are usually equal to 1.

Example: - Apply Gauss's Elimination Method to solve the equations

$$\mathrm{x} + 4y - \mathrm{z} = -5$$

$$\mathrm{x} + y - 6\mathrm{z} = -12$$

$$3\mathrm{x} - y - \mathrm{z} = 4$$

Solution: Given system of equations can be written in matrix form

$$\approx \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & 6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

$\to R_2 - R_1, \to R_3 - 3R_1$

$$\approx \begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$$

$$\to R_3 - \frac{13}{3} R_2$$

$$\approx \begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$$

This is an upper triangular matrix of coefficient matrix A

Therefore, the algebraic form of above Matrix form is

$$\mathrm{x} + 4y - \mathrm{z} = -5 \text{ .... } (1)$$

$$\frac{71}{3}\mathrm{z} = 148/3 \text{ ....}(2)$$

Hence from above by back substitution, we get the desiredapproximate solution

$$z = 2.0845\ ,\ y = -1.1408\ ,\ x = 1.6479$$

(2) Gauss's Jordan Method: - This is a modification ofGauss's Elimination Method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also. Ultimatelyreducing thesystem to a diagonal matrix. i.e. each equation involving only one unknown. Thus in this method, thelabor of back substitution for finding the unknowns is saved at the cost of additional calculations. This method is well explained by the following example.

Example: - Apply Gauss's Jordan Method to solve the equations

$$\mathrm{x} + y + \mathrm{z} = 9$$

$$2\mathrm{x} - 3y + 4\mathrm{z} = 13$$

$$3\mathrm{x} + 4y + 5\mathrm{z} = 40$$

Solution: Given system of equations is

$$x + y + z = 9 \ldots\ldots\ldots\ldots (1)$$

$$2x - 3y + 4z = 13 \ldots\ldots\ldots (2)$$

$$3x + 4y + 5z = 40 \ldots\ldots\ldots\ldots (3)$$

Step-1:- Operate (2)-2(1) and (3)-3(1) to eliminate x from (2) and (3**).**

$x + y + z = 9$ ....(4)

$$-5y + 2z = -5 \ldots \ldots \ldots (5)$$

$y + 2z = 13$.........(6)

Step-2:- operate (4) +1/5 (5) and (6) +1/5 (5) to eliminate y from (4) and (6**).**

$x + 7/5z = 8$......... (7)

$$-5y + 2z = -5 \ldots \ldots \ldots (8)$$

$\frac{12}{5}z = 12$............ (9)

Step-3:- operate (7) -7/12 (9) and (8) -5/6 (9) to eliminate z from (7) and (8**).**

$$x = 1$$
$$y = 3$$
$$z = 5$$

(4) Crout's method:- Computation scheme : Let usconsider

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The Augmented Matrix of given system of equations is

$$A : B = \begin{array}{cccc} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}$$

Now from the above matrix, the Matrix of 12 unknowns,so called derived matrix or Auxiliary Matrix is $D : M = \begin{array}{cccc} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{array}$ and is to be calculated as follows:

Step-1:- The first column of the derived matrix is identical with the first column of $A : B$.

Step-2: The first row to the right of the first column of the $D : M$ is obtained by dividing the corresponding element in $A : B$ by the leading diagonal element of that row.

Step-3: Remaining second column of $D : M$ is calculated as follows:

$$l_{22} = a_{22} - l_{21} u_{12} ; \quad l_{32} = a_{32} - l_{31} u_{12}$$

Step-4: Remaining elements of second row of $D : M$ is calculated as follows ;

$$u_{23} = \frac{a_{23} - l_{21} u_{13}}{l_{22}} \quad ; y_2 = \frac{b_2 - l_{21} y_1}{l_{22}}$$

Step-5: Remaining element of the third column of $D : M$ is calculated as follows:

$$l_{33} = a_{33} - (l_{31} u_{13} + l_{32} u_{23})$$

Step-6: Remaining element of third row of $D : M$ is calculated as follows ;

Hence, from above $D : M$ $Matrx , t$□e $requred\ soluton\ s$ obtained as follows:

$x_3 = y_3;$

$x_2 = y_2 - u_{23} x_3 ;$

$$x_1 = y_1 - [x_2 u_{12} + x_3 u_{13}]$$

Example: Apply **Crout's** Method to solve the equations

$$2 x_1 - 6 x_2 + 8 x_3 = 24$$

$$5 x_1 + 4 x_2 - 3 x_3 = 2$$

$$3 x_1 + x_2 + 2 x_3 = 16$$

Solution: The Augmented Matrix of given system of equations is

$$A : B = \begin{matrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{matrix}$$

Now from the above matrix, the Matrix of 12 unknowns,so called derived matrix or Auxiliary Matrix is $D : M = \begin{matrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{matrix}$ and is to be calculated as follows:

Step-1:- The first column of the derived matrix $D : M$ is identical with the first column of $A : B$.

i.e $l_{11} = a_{11} = 2; l_{21} = a_{21} = 5; l_{31} = a_{31} = 3$

Step-2: The first row to the right of the first column of the $D : M$ is obtained by dividing the corresponding element in $A : B$ by the leading diagonal element of that row.i.e.

$$u_{12} = \frac{a_{12}}{a_{11}} = \frac{-6}{2} = -3 \; ; u_{13} = \frac{a_{13}}{a_{11}} = \frac{8}{2} = 4 \; ; y = \frac{b_1}{a_{11}} = \frac{24}{2} = 12$$

Step-3: Remaining second column of $D : M$ is calculated as follows:

$$l_{22} = a_{22} - l_{21}\, u_{12} = 4 - 5 -3 = 19 \; ; l_{32} = a_{32} - l_{31}\, u_{12} = 1 - 3 -3 = 10$$

Step-4: Remaining elements of second row of $D : M$ is calculated as follows ;

$$u_{23} = \frac{a_{23} - l_{21}\, u_{13}}{l_{22}} = \frac{-3 - 5(4)}{19} = -\frac{23}{19}; y_2 = \frac{b_2 - l_{21}\, y_1}{l_{22}} = \frac{2 - 5(12)}{19} = -\frac{58}{19}$$

Step-5: Remaining elements of the third column of $D : M$ is calculated as follows:

$$l_{33} = a_{33} - l_{31}\, u_{13} + l_{32}\, u_{23} = 2 - [3\;4 + 10 - \frac{23}{19} = \frac{40}{19}$$

Step-6: Remaining element of third row of $D : M$ is calculated as follows ;

$$y_3 = \frac{b_3 - (l_{31}\, y_1 + l_{32}\, y_2)}{l_{33}} = \frac{16 - [3\;12 + 10(-\frac{58}{19})}{\frac{40}{19}} = 5$$

Hence, from above $D : M$ $Matrx$, t□e $requred$ $soluton$ s obtained as follows:

$x_3 = y_{3=5};$

$$x_2 = y_2 - u_{23}\, x_3 = -\frac{58}{19} - 5 - \frac{23}{19} = 2;$$

$x_1 = y_1 - [\, x_2\, u_{12} + x_3\, u_{13}]$=12- [3(-3)+5(4)] =1

[II] Iterative Methods: In these methods, we start from an initial approximation to the true solution an and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy. For large systems, iterative methods may be faster than the direct methods. Even the round -off errors in iterative methods are smaller. In fact, iteration is a self-correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

(1) **Gauss's Jacobi's Method :** Let us consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2 \ldots\ldots\ldots\ldots\ldots (1)$$
$$a_3x + b_3y + c_3z = d_3$$

If a_1, b_2, c_3 are large as compared to other coefficients, solve the above system for x, y, z respectively. Then system of equations can be written as

$y = \frac{1}{b_2}(d_2 - a_2\ x - c_2\ z)$ .... (2)

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

Let us start with the initial approximation x_0, y_0, z_0 for the values of x, y, z respectively.Substituting these on the right sides of above, the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Now for second approximations, Substituting thex_1,y_1, z_1 on the right hand sides of (2).

$$x_2 = \frac{1}{a_1}(\ d_1 - \ b_1y_1 - \ c_1z_1)$$

$$y_2 = \frac{1}{b_2}(\ d_2 - \ a_2x_1 - \ c_2z_1)$$

$$z_2 = \frac{1}{c_3}(\ d_3 - a_3x_1 - b_3y_1)$$

This process is repeated till the diffence between two consecutive approximations is negligible.

Note :In the absence of any better estimates for x_0, y_0, z_0,these may each be taken as zero.

Example : Solve by **Gauss's Jacobi's Method**

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18 \ldots\ldots\ldots\ldots(1)$$

$$2x - 3y + 20z = 25$$

Solution : Then system of equations

$$x = \frac{}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z) \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

For first approximations, substituting $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ for the values of x, y, z respectively in above,

$$x_1 = \frac{1}{20}\ 17 - 0 + 2.0\ \ = 0.85;$$

$$y_1 = \frac{1}{20}\ -18 - 3.0 + 0\ \ = -0.9;$$

Now for second approximations, Substituting the x_1,y_1, z_1 on the right hand sides of (2).

$$z_1 = \frac{1}{20}\left(25 - 2.0 + 3.0\right) = 1.25.$$

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Now for second approximations, Substituting the x_1, y_1, z_1 on the right hand sides of (2).

$$x_2 = \frac{1}{20}\ 17 - y_1 + 2\,z_1 = 1.02;$$

$$y_2 = \frac{1}{20}\ -18 - 3\,x_1 + z_1 = -0.965;$$

$$z_2 = \frac{1}{20}\ 25 - 2x_1 + 3y_1 = 1.03.$$

Now for third approximations, Substituting the x_2, y_2, z_2 on the right hand sides of (2).

$$x_3 = \frac{1}{20}\ 17 - y_2 + 2\,z_2 = 1.00125;$$

$$y_3 = \frac{1}{20}\ -18 - 3\,x_2 + z_2 = -1.0015;$$

$$z_3 = \frac{1}{20}\ 25 - 2x_2 + 3y_2 = 1.00325.$$

Now for fourth approximations, substituting these values on the right hand sides of (2).

$$x_4 = \frac{1}{20}\ 17 - y_3 + 2\,z_3 = 1.0004;$$

$$y_4 = \frac{1}{20}\ -18 - 3\,x_3 + z_3 = -1.000025;$$

$$z_4 = \frac{1}{20}\ 25 - 2x_3 + 3y_3 = 0.9965.$$

Now for fifth approximations, substituting these values on the right hand sides of (2).

$$x_5 = \frac{1}{20}\ 17 - y_4 + 2\,z_4 = 0.999966;$$

$$y_5 = \frac{1}{20}\ -18 - 3\,x_4 + z_4 = -1.000078;$$

$$z_5 = \frac{1}{20}\ 25 - 2x_4 + 3y_4 = 0.999956.$$

Now for sixth approximations, substituting these values on the right hand sides of (2).

$$x_6 = \frac{1}{20}\, 17 - y_5 + 2\, z_5 = 1.0000;$$

$$y_6 = \frac{1}{20}\, -18 - 3\, x_5 + z_5 = -0.999997;$$

$$z_6 = \frac{1}{20}\, 25 - 2x_5 + 3y_5 = 0.999992.$$

As the values in the 5th and 6th approximations, i.e. Iterations being practically the same.We can stop now, Hence the solution is $x = 1;\quad y = -1;\quad z = 1$

(2) **Gauss-Seidal Method** : This is a modification of Jacobi's method .

Let us consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2 \text{.................(1)}$$

$$a_3x + b_3y + c_3z = d_3$$

If a_1, b_2, c_3 are large as compared to other coefficients, solve the above system for x, y, z respectively. Then system of equations can be written as

$$x = \frac{1}{a_1}(\, d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(\, d_2 - a_2\, x - c_2\, z) \text{ .... (2)}$$

$$z = \frac{1}{c_3}(\, d_3 - a_3x - b_3y)$$

Let us start with the initial approximation x_0, y_0, z_0 for the values of x, y, z respectively. Which may each be taken as zero .Now,
Substituting $y = y_0, z = z_0$on the right sides of above first equation, the first approximations are given by

$$x_1 = \frac{1}{a_1}(\, d_1 - b_1y_0 - c_1z_0)$$

Then putting $x = x_1, z = z_0$ in the second of the equation of (2),we obtain

$$y_1 = \frac{1}{b_2}(\, d_2 - a_2x_1 - c_2z_0)$$

Next putting $x = x_1, y = y_1$ in the third of the equation of (2),we obtain

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

And so on i.e. as soon as a new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is repeated till the values of x, y, z are obtained to desired degree of accuracy.

Note :- (1) Since the most recent approximations of the unknowns are used,while proceeding to the next step,the convergence in the **Gauss-Seidal Method is twice as fast as in Gauss's Jacobi's Method.**

(3) Gauss's Jacobi's Method and Gauss-Seidal Methods converge for any choice of the initial approximations if in each equation of the system, the absolute value of the largest coefficient is almost equal to or in at least one equation greater than the sum of the absolute values of all the remaining coefficients**.**

Example: Apply Gauss-Seidal Method

$$3x + 20y - z = -18 \text{................}(1)$$

$$2x - 3y + 20z = 25$$

Solution: Then system of equations can be written as

$x = \frac{1}{20}(17 - y + 2z)$..............(i)

$y = \frac{1}{20}(-18 - 3x + z)$...........(ii)

$$z = \frac{1}{20}(25 - 2x + 3y) \text{ ....(iii)}$$

For First approximations / Iteration,

Substituting $y = y_0 = 0, z = z_0 = 0$ for the values of y, z respectively in (i),

$$x_1 = \frac{1}{20} 17 - 0 + 2.0 = 0.8500;$$

Substituting $x = x_1, z = z_0$ for the values of x , z respectively in (ii)

$$y_1 = \frac{1}{20} -18 - 3x_1 + z_0 = -1.0275;$$

Substituting $x = x_1, y = y_1$ for the values of x , y respectively in (iii)

$$z_1 = \frac{1}{20} 25 - 2x_1 + 3y_1 = 1.0109$$

For Second approximations / Iteration

Substituting $y = y_1, z = z_1$ for the values of y, z respectively in (i),

$$x_2 = \frac{1}{20} 17 - y_1 + 2z_1 = 1.0025;$$

Substituting $x = x_2, z = z_1$ for the values of x , z respectively in (ii)

$$y_2 = \frac{1}{20} -18 - 3x_2 + z_1 = -0.9998;$$

Substituting $x = x_2, y = y_2$ for the values of x , y respectively in (iii)

$$z_2 = \frac{1}{20} 25 - 2x_2 + 3y_2 = 0.9998$$

For Third approximations / Iteration

Substituting $y = y_2, z = z_2$ for the values of y, z respectively in (i),

$$x_3 = \frac{1}{20} 17 - y_2 + 2z_2 = 1.0000;$$

Substituting $x = x_3, z = z_2$ for the values of x , z respectively in (ii)

$$y_3 = \frac{1}{20} -18 - 3x_3 + z_2 = -1.0000;$$

Substituting $x = x_3, y = y_3$ for the values of x , y respectively in (iii)

$$z_3 = \frac{1}{20} 25 - 2x_3 + 3y_3 = 1.0000$$

Hence,the values in 2nd and 3rd Approximations being practically the same, Now ,we can stop,

The solution is $x = 1$; $y = -1$; $z = 1$.

Therefore,we seen that the convergence is quite fast in **Gauss-Seidal Method as compared to Gauss's Jacobi's Method.**

(3) Relaxation method: This method was original developed by R.V. South well in 1935,for application to Structural engg. Problems.

Let us consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2 \ldots\ldots\ldots\ldots\ldots(1)$$

$$a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x , R_y R_z , by the relations

$$R_x = d_1 - a_1x - b_1y - c_1z$$

$$R_y = d_2 - a_2x - b_2y - c_2z \ldots\ldots\ldots\ldots\ldots (2)$$

$$R_z = d_3 - a_3x - b_3y - c_3z$$

To start with we assume $x = 0, y = 0, z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step, by giving increments to the variables. For this purpose, we construct the following operation table:

	R_x	R_y	R_z
$x = 1$	$-a_1$	$-a_2$	$-a_3$
$y = 1$	$-b_1$	$-b_2$	$-b_3$
$z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (2) that if x is increased by 1 (keepingy, z constant), R_x , R_y and R_z decrease by a_1 , a_2, a_3 respectively.This is shown in the above table along with the effects on the residuals when $y, and\ z$ are given unit increments (The table is the transpose of the coefficient matrix)

At each step, the numerically largest residual is reduced to almost zero,To reduce a particular residual, the value of the corresponding variable is changed.

When all the residuals have been reduced to almost zero, the increments in $x, y, and\ z$ are added separately to give the desired solution.

Example: Apply Relaxation method

$$10x - 2y - 3z = 205$$

$$-2x + 10y - 2z = 154 \ldots\ldots\ldots\ldots\ldots(1)$$

$$-2x - y + 10z = 120$$

Solution: We define the residuals R_x , R_y R_z , by the relations

$$R_x = 205 - 10x + 2y + 3z$$

$$R_y = 154 + 2x - 10y + 2z \ldots\ldots\ldots\ldots\ldots (2)$$

$$R_z = 120 + 2x + y - 10z$$

The operation table is

	R_x	R_y	R_z
$x = 1$	$-a_1$=-10	$-a_2 = 2$	$-a_3 = 2$
$y = 1$	$-b_1 = 2$	$-b_2 = 10$	$-b_3$=1
$z = 1$	$-c_1 = 3$	$-c_2 = 2$	$-c_3$=-10

The **Relaxation table is**

	R_x	R_y	R_z
$x = 0, y = 0, z = 0$	205	154	120
$x = 20$	5	194	160
$y = 19$	43	4	179
$z = 18$	97	40	-1
$x = 10$	-3	60	19
$y = 6$	9	0	25
$z = 2$	15	4	5
$x = 2$	-5	8	9
$y = 1$	-2	10	-1
$z = 1$	0	0	0

$$R_x = 32, \qquad R_y = 26, \qquad R_z = 21$$

Hence The solution is $x = 32$; $y = 26$; $z = z$

Module3:Numerical Methods–3

Contents

Ordinary differential equations : Taylor's series, Euler and modified Euler's methods. Runge Kutta method of fourth order for solving first and second order equations.Milne'sandAdam'spredicator-correctormethods.Partialdifferential equations:FinitedifferencesolutiontwodimensionalLaplaceequationandPoissionequation, Implicitandexplicitmethodsforonedimensionalheatequation(Bender-SchmidtandCrank-Nicholsonmethods), Finite difference explicit method for wave equation

Ordinary differential equations: Many problems in Engineering and Science can be formulated into ordinary differentional equations satisfying certain given conditions. If these conditions are prescribed for one point only, then the differential equation together with the condition is known as an initial value problem. If the conditions are prescribed for two or more points, then the problem is termed as boundary value problem. A limited number of these differentional equations can be solved by analytical methods. Hence numerical methods play very important role in the solution of differential equations.

We shall discuss some of the following methods for obtaining numerical solution of first order and first degree Ordinary Differential Equation.

[I] Picard'smethod of successive approximation**.**

[II] Taylor"s Seriesmethod.

[III] Euler"s method.

[IV] Modified Euler"s method.

[V] Milne"smethod / Milne"sPredictor-Corrector method.

[VI Runge-Kutta method.

[VII] Adam"s Bash forth Method.

Method-I :Picard's Method of successive approximations for first order and first degree differential equation :--

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \text{..}(1)$$

Subject to $y(x_0) = y_0$. This equation can be written as

$$dy = f(x, y)\, dx$$

Integrating between the limits,

We get $\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$

$$y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \quad \text{----------------------------} \quad (2)$$

Which is an integral equation because the unknown function "y" is present under the integral sign.Such an equation can be solved by successive approximation as follows;

For first approximation y_1,we replace y by y_0 in $f(x, y)$ in the R.H.S. of equation (2),

$$\text{i.e. } y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx \text{--------------------------------} (3)$$

Now, for second approximation y_2,we replace y by y_1 in $f(x, y)$ in the R.H.S. of equation (2),

$$\text{i. e } y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx \text{-------------------------------} (4)$$

Proceeding in this way, we get y_3, y_4, y_5,-------

The n^{th} approximation is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx, \text{n=1,2,3----------}$$

The process is to be stopped when the two values of y, viz y_n, y_{n-1} ,are same to the desired degree of accuracy.

Example-: Find the Picard approximations y_1, y_2, y_3 to the solution of the initial value problem y'=y, y(0) =2 .Use y_3 to estimate the value of y (0.8) and compare it with the exact solution.

Solution: The n^{th} approximation is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx, n=1,2,3 \text{ -------(1)}$$

For First approximation y_1, we replace y by y_0 in $f(x, y)$ in the R.H.S. of equation (1)

Therefore, $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

Here, $y_0 = 2$, the value of y_1 is

$y_1 = 2 + \int_0^x 2\, dx = 2 + 2x$ where $f(x, y_0) = y_0 = 2$

Similarly,

$y_2 = 2 + \int_0^x (2 + 2x)\, dx = 2 + 2x + x^2$

also

$y_3 = 2 + \int_0^x (2 + 2x + x^2)dx = 2 + 2x + x^2 + \frac{x^3}{3}$

At x= 0.8

$y_3 = 2 + 2(0.8) + (0.8)^2 + \frac{1}{3}(0.8)^3$

$= 4.41$

By Direct Method: The solution of the initial-value problem, found by separation of variables, is $y = 2e^x$. At x=0.8

$y = 2e^{0.8} = 4.45$

Method-II :Numerical Solution to Ordinary Differential Equation By Taylor's Series method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{--------------------------} \qquad (1)$$

Subject to $y(x_0) = y_0.$

Then ,we by Taylor"s

$y = y(x) = y(x_0 + x - x_0)$

$$= y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \ldots \ldots \ldots \ldots$$

If $x_0 = 0$ then

$$y = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \cdots \ldots \ldots \ldots \ldots$$

Example- Using Taylor's Series method, find the solution of

$\frac{dy}{dx} = xy - 1$ With $y(1) = 2$ correct to five decimal places at $x = 1.02$

Solution: Given that $\frac{dy}{dx} = xy - 1$ with $y(1) = 2$ i.e. $x_0 = 1$ and $y_0 = 2$

i.e. $y' = xy - 1 \qquad \therefore y'(0) = x_0 y_0 - 1 = 1.2 - 1 = 1 = y'(x_0)$

Differentiate with respect to x

$y'' = y + x y' \qquad \therefore y''(0) = y_0 + x_0 y'(0) = 2 + 1.1 = 3 = y''(x_0)$

Again Differentiate with respect to x

$y''' = 2y' + x y'' \qquad \therefore y'''(0) = 2y'(0) + x_0 y''(0) = 2.1 + 1.3 = 5 = y'''(x_0)$

Again Differentiate with respect to x

$y'''' = 3y'' + x y''' \qquad \therefore y''''(0) = 3y''(0) + x_0 y'''(0) = 3.3 + 1.5 = 14 = y''''(x_0)$

andso on

Now by Taylor"s Series method,

$$y = y\,x_0 + \frac{x - x_0}{1\,!}\,y'\,x_0 + \frac{x - x_0{}^{2}}{2\,!}\,y''\,x_0 + \frac{x - x_0{}^{3}}{3\,!}\,y'''\,x_0 + \cdots \ldots \ldots \ldots \ldots$$

$$=> y = 2 + \frac{x-1}{1\,!}.1 + \frac{x-1^{2}}{2\,!}.3 + \frac{x-1^{3}}{3\,!}.5 + \frac{x-1^{4}}{4\,!}.14 + \cdots \ldots \ldots \ldots \ldots$$

At $x = 1.02$

$$=> y = 2 + \frac{1.02-1}{1\,!}.1 + \frac{1.02-1^{2}}{2\,!}.3 + \frac{1.02-1^{3}}{3\,!}.5 + \frac{1.02-1^{4}}{4\,!}.14 + \cdots \ldots \ldots \ldots \ldots$$

$=> y = 2.020606.$

Method-III :Numerical Solution to Ordinary Differential Equation By Euler'smethod

Euler'smethod,as such, is of very little practical importance, but illustrates in simple form ,the basic idea of those numerical methods,which seek to determine the changeΔy in y corresponding to small increase in the argument x.

Let us consider the equation

$$\frac{dy}{dx} = f\;x, y \qquad \text{--------------------------} \qquad (1)$$

Subject to $y\;x_0 = y_0.$

Let $y = g\;x$ be the solution of (1) and let x_0, $x_1 = x_0 + \text{h}$, $x_2 = x_0 + 2\text{h}$,……… be equidistant values of x.

In an small interval, a curve is nearly a straight line. This is the property used in **Euler'smethod.**

The n^{th} approximation is given by

$y_{n+1} = y_n + \text{h f x}_n$, y_n where $n = 0.1,2,3$ … … …

Example- UsingEuler'smethod, find an approximate value of$y$ corresponding to $x = 1$, $given\ that$

$\frac{dy}{dx} = x + y$With $y\;0 = 1$

Solution: Given that$\frac{dy}{dx} = x + y$With $y\;0 = 1$

We have, the n^{th} approximation is given by

$y_{n+1} = y_n + h\, f(x_n, y_n)$ where $n = 0,1,2,3 \ldots \ldots \ldots$ (1)

We taken $n = 10$ and $h = 0.1$, which is sufficiently small. The various calculations are arranged as follows by (1)

x	y	$\frac{dy}{dx} = x + y = f(x, y)$	Old $y + 0.1\left(\frac{dy}{dx}\right) = new\ y$
0.0	1.0	1.0	1.0 +0.1 (1.0) = 1.10
0.1	1.10	1.20	1.10 +0.1 (1.20) = 1.22
0.2	1.22	1.42	1.22+0.1 (1.42) = 1.36
0.3	1.36	1.66	1.36+0.1 (1.66) = 1.53
0.4	1.53	1.93	1.53+0.1 (1.93) = 1.72
0.5	1.72	2.22	1.72+0.1 (2.22) = 1.94
0.6	1.94	2.54	1.94+0.1 (2.54) = 2.19
0.7	2.19	2.89	2.19+0.1 (2.89) = 2.48
0.8	2.48	3.29	2.48+0.1 (3.29) = 2.81
0.9	2.81	3.71	2.81+0.1 (3.71) = 3.18
1.0	3.18		

Thus the required approximate value is $y = 3.18$.

Method-IV :Numerical Solution to Ordinary Differential Equation By Euler's Modified method

Let us consider the equation

$$\frac{dy}{dx} = f(x, y) \quad \text{--------------------------} \quad (1)$$

Subject to $y(x_0) = y_0$.

Let $y = g(x)$ be the solution of (1) and let $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots\ldots$ be equidistant values of x.

In each step of this method, we first compute the auxiliary value y_{n+1} by

$y_{n+1} = y_n + h\, f(x_n, y_n)$ where $n = 0,1,2,3 \ldots \ldots \ldots$ (2)

and then the new value

$$y_{n+1}^{r} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{r-1})\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

where r *denotes number of iteration* $1,2,3 \dots \dots ..$

This Modified method is a Predictor-Corrector method; because in each step,we first predict a value by (2) and then correct it by (3).

Example: Using Euler's modifiedmethod,solvenumerically the equation

$$\frac{dy}{dx} = x + y$$

Subject to $y(0) = 1$ for $0 \leq x \leq 0.6$ in the steps of 0.2

Solution: Given that

$$\frac{dy}{dx} = x + y$$

Subject to $y(0) = 1$ i.e.$x_0 = 0, y_{0.} = 1$ and $h = 0.2$

∴ we ave ,

$x_0 = 0, x_{1.} = 0.2, x_{2.} = 0.4, x_{3.} = 0.6$

$y_{0.} = 1, \quad y_{1.} = ?, \quad y_{2.} = ?, \quad y_{3.} = ?$

Now for this , we have by Euler's modifiedmethod ,Predictor formula

$y_{n+1} = y_n + h f(x_n, y_n)$ where $n = 0.1,2,3$............ (1)

and then the new value by Corrector formula

$y_{n+1}^{r} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{r-1})]$........................(2)

Where r *denotes number of iteration* $1,2,3 \dots \dots ..$

Now by (1), for finding the value of $y_{1.}$ put $n = 0$
∴ we ave ,

$y_1 = y_0 + h f(x_0, y_0) = 1 + 0.2[x_0 + y_0] = 1.\overline{2}$

Now by (2)this value of y_1, *thus obtained is improved or modified as followes*

For First iteration put $r = 1$ and $n = 0$ in (2)

$$y_1^{1} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{0})]$$

$y_1^1 = 1 + \frac{0.2}{2}[x_0 + y_0 + x_1 + y_1^0]$

$=> y_1^1 = 1 + \frac{0.2}{2}[0 + 1 + 0.2 + 1.2$ where $y_1^0 = y_1 = 1.2$

Now for second iteration put $r = 2$ and $n = 0$ in 2, we get

$y_1^2 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^1)] = 1 + \frac{0.2}{2}[x_0 + y_0 + x_1 + y_1^1] = 1.2309$

Similarly third iteration put $r = 3$ and $n = 0$ in 2, we get

$y_1^3 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^2)] = 1 + \frac{0.2}{2}[x_0 + y_0 + x_1 + y_1^2] = 1.2309$

Hence ,from above ,Since $y_1^2 \approx y_1^3 = 1.2309$

∴ w v , At $x_1 = 0.2$,$y_1 = 1.2309$

Again apply **Euler's**Modified **method for more accurate approximations ,**

Hence , by (1), for finding the value of y_2. put $n = 1$
∴ we ave ,

$y_2 = y_1 + h f(x_1, y_1) = 1.2309 + 0.2[x_1 + y_1] = 1.49279$

Now by (2) this value of y_2, thus obtained is improved or modified as followes

For First iteration put $r = 1$ and $n = 1$ in (2)

$y_2^1 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^0)]$

$y_2^1 = 1.2309 + \frac{0.2}{2}[x_1 + y_1 + x_2 + y_2^0]$

$=> y_2^1 = 1.2309 + \frac{0.2}{2}[0.2 + 1.2309 + 0.4 + 1.49279] = 1.52402$

where $y_2^0 = y_2 = 1.49279$

Now for second iteration put $r = 2$ and $n = 1$ in 2, we get

$y_2^{2} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{1})] = 1.2309 + \frac{0.2}{2}[(x_1 + y_1) + (x_2 + y_2^{1})]$

$= 1.525297$

Similarly third iteration put $r = 3$ and $n = 1$ in (2), we get

$y_2^{3} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{2})] = 1.2309 + \frac{0.2}{2}[(x_1 + y_1) + (x_2 + y_2^{2})]$

$= 1.52535$

Also, similarly fourth iteration put $r = 4$ and $n = 1$ in (2), we get

$y_2^3 = 1.52535$

$\therefore$ w v, At $x_2 = 0.4$, $y_2 = 1.52535$

Hence, by again (1), for finding the value of y_3 put $n = 2$

$\therefore$ we ave,

$y_3 = y_2 + h f(x_2, y_2) = 1.52535 + 0.2[x_2 + y_2] = 1.85236$

Now by (2) this value of y_3, thus obtained is improved or modified as followes

For First iteration put $r = 1$ and $n = 2$ in (2)

$y_3^{1} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{0})]$

$y_3^{1} = 1.52535 + \frac{0.2}{2}[(x_2 + y_2) + (x_3 + y_3^{0})]$ where $y_3^{0} = y_3 = 1.85236$

$=> y_3^{1} = 1.52535 + \frac{0.2}{2}[(0.4 + 1.52535) + (0.6 + 1.85236)] = 1.88496$

Similarly by putting $r = 2, 3, 4$ and $n = 2$ in (2), we get

$=> y_3^{2} = 1.88615,\ y_3^{3} = 1.88619,\ y_3^{4} = 1.88619$

$\therefore$ w v, At $x_3 = 0.6$, $y_3 = 1.88619$

Method-V :Numerical Solution to Ordinary Differential Equation By Milne's method / Milne's Predictor-Corrector method

As the name suggests,predictor-corrector method is the method in which,we first predict a value of y_{n+1} by using a certain formula and then correct this value by using a more accurate formula. i.e.

Let us consider the equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{--------------------------} \qquad (1)$$

Subject to $y(x_0) = y_0$.

Then the predictor formula for finding y_{n+1} is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \ldots\ldots\ldots\ldots\ldots (2)$$

And the Correctorformula for finding y_{n+1}

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[2y'_{n-1} + 4y'_n + y'_{n+1}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now from above ,it is clear that only $n = 3,4,5, \ldots \ldots \ldots$ possible i.e. we shall find y_4, y_5, y_6……………………**,we must required four prior values** y_0, y_1, y_2 *and* y_3 **of** y .

Example: UsingMilne'smethod, solve numerically the equation

$$\frac{dy}{dx} = x^2(1 + y)$$

Subject to $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$ *and evalute* $y(1.4)$

Solution: Given that

$$\frac{dy}{dx} = x^2(1 + y)$$

∴ we ave ,

$x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4$

$y_0 = 1, \quad y_1 = 1.233, \quad y_2 = 1.548, y_3 = 1.979, y_4 = ?$

We have by Milne'smethod,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \ldots\ldots\ldots\ldots\ldots \qquad (1)$$

And the Correctorformula for finding y_{n+1}

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[\,2\,y'_{n-1} + 4\,y'_{n} + y'_{n+1}\,] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now to find $y_{4.}$, put $n = 3, in\ 1$

$$y_{4,p} = y_0 + \frac{4\,h}{3}[\,2\,y'_1 - y'_2 + 2\,y'_3\,] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

For this ,we have $y_{0.} = 1$,

$y'_1 = x^2{}_1\ 1 + y_{1.} = 2.7019$

$y'_2 = x^2{}_2\ 1 + y_{2.} = 3.6691$

$y'_3 = x^2{}_3\ 1 + y_{3.} = 5.0345$

Hence ,on putting all the above values in (3)

We get $y_{4p} = 1 + \frac{4\,.0.1}{3}[\,2\,.\,2.7019 - 3.6691 + 2.\,5.0345] = 2.5738$

$$y'_4 = x^2{}_4\ 1 + y_{4.} = 7.0046$$

Now ,we shall correct this value of $y_{4.}$ **of** y by the corrector formula (2) as

put$n = 3, in\ 2$

$$y_{4,c} = y_2 + \frac{h}{3}[\,2\,y'_2 + 4\,y'_3 + y'_4\,] = 2.5750$$

i.e. $y\ 1.4 = 2.5750$

Method-VI:Numerical Solution to Ordinary Differential Equation By Runge-Kutta method

Runge-Kutta method is more accurate method of great practical importance**.** The working procedure is as follows,

$$\text{To solve } \frac{dy}{dx} = f\ x, y \qquad \text{--------------------------} \qquad (1)$$

Subject to $y\ x_0 = y_{0.}$ by Runge-Kutta method, compute $y_{1.}$ as followes

$k_{1.} = h\,f\,x_0\,,\,y_{0.}$

$k_{2.} = h\,f\,x_0 + \frac{h}{2}\,,\,y_0 + \frac{k_{1.}}{2}$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\,f\left(x_0 + □, y_0 + k_3\right)$$

∴ w v

$$K = \frac{k_1 + 2\,(k_2 + k_3) + k_4}{6}$$

Hence, $y_1 = y_0 + K = y(x_1)$

Similarly, we can compute $y(x_2) = y_2$

$$k_1 = h\,f(x_1, y_1)$$

$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = h\,f\left(x_1 + □, y_1 + k_3\right)$$

∴ w v

$$K = \frac{k_1 + 2\,(k_2 + k_3) + k_4}{6}$$

Hence, $y_2 = y_1 + K = y(x_2)$ and so on for succeeding intervals.

Therefore , we can notice that the only change in the formulae for succeeding intervals is in the values of x *and* y.This refinement was carried out by two German mathematicians C.D.T. Runge (1856-1927) and M.W.Kutta (1867-1944).

Example 1. UsingRunge-Kutta method to find y w□en $x = 1.2$ i. e. $y(1.2)$ *in steps* of 0.1

$$\frac{dy}{dx} = x^2 + y^2 \text{ Subject to } y(1) = 1.5$$

Solution: Given that $\frac{dy}{dx} = x^2 + y^2$ Subject to $y(1) = 1.5$

$$\therefore \text{ we ave , } □ = 0.1$$

$x_0 = 1, x_1 = 1.1, x_2 = 1.2,$

$y_0 = 1.5, \quad y_1 = ?, \quad y_2 = ?,$

First ,we will compute $y(1.1)$ i. e. y_1

For this ,we have

$$k_1 = h\,f(x_0, y_0) = 0.1[1^2 + 1.5^2] = 0.325$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1[1.05^2 + 1.6625^2] = 0.3866$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1[1.05^2 + 1.6933^2] = 0.3969$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.1[1.1^2 + 1.8969^2] = 0.4808$$

∴ w v

$$K = \frac{k_1 + 2(k_2 + k_3) + k_4}{6} = 0.3954$$

Hence, $y_1 = y_0 + K = y(x_1) = 1.5 + 0.3954 = 1.8954$

Similarly, we can compute $y(x_2) = y_2$

For this, we have

$$k_1 = h\,f(x_1, y_1) = 0.1[1.1^2 + 1.8954^2] = 0.4802$$

$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1[(1.1 + 0.05)^2 + (1.8954 + 0.2401)^2] = 0.5882$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.6116$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = 0.7725$$

∴ w v

$$K = \frac{k_1 + 2(k_2 + k_3) + k_4}{6} = 0.60815$$

Hence, $y_2 = y_1 + K = y(x_2) = 2.5035$

Method-VII:Numerical Solution to Ordinary Differential Equation By Adam's Bash forth method

It is thepredictor-corrector method ,in which, we first predict a value of y_1 by using a certain formula and then correct this value by using a more accurate formula. i.e.

Let us consider the equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{--------------------------} \qquad (1)$$

Subject to $y(x_0) = y_0$.

Then the predictor formula for finding y_1 is

$$y_{1,p} = y_0 + \frac{h}{24}[55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}] \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

It is called **Adam's Bashforth**predictor formula.

And the Correctorformula for finding y_1

$$y_{1,c} = y_0 + \frac{h}{24}[9f_1 + 19f_0 - 5f_{-1} + f_{-2}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now from above, it is clear that,for applying this method , we require four starting values of y.

Note :-In practice ,the Adam's Bashforthmethod together with fourth order Runge-Kutta method have been found to be most useful.

Example: UsingAdam'sBashforthmethod, solve the equation

$$\frac{dy}{dx} = x^2(1 + y)$$

Subject to $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$ *and evalute* $y(1.4)$

Solution: Given that

$$\frac{dy}{dx} = x^2(1 + y)$$

∴ we ave ,

$x_0 = 1.0, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4$

$y_{-3} = 1.000$, $y_{-2} = 1.233$, $y_{-1} = 1.548$, $y_0 = 1.979$, $y_4 = .$ e. $y_1 = ?$

Hence , $f_{-3} = (1.0)^2(1 + 1.000) = 2.000$

Similarly $f_{-2} = (1.1)^2(1 + 1.233) = 2.702$, $f_{-1} = (1.2)^2(1 + 1.548) = 3.669$, $f_0 = (1.3)^2(1 + 1.979) = 5.035$

Now,the predictor formula for finding y_1 is

$$y_{1,p} = y_0 + \frac{h}{24}[55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}] = 2.573$$

and $f_1 = (1.4)^2(1 + 2.573) = 7.004$

And the Correctorformula for finding y_1

$$y_{1,c} = y_0 + \frac{h}{24}[9f_1 + 19f_0 - 5f_{-1} + f_{-2}]$$

$$= 1.979 + \frac{0.1}{24}[9\times 7.004 + 19 \times 5.035 - 5 \times 3.669 + 2.702] = 2.575$$

i.e. $y(1.4) = 2.575$.

Numerical Solution of Partialdifferential equations:**[**Finite difference solution two dimensional Laplace equation and Poission equation, Implicit and explicit methods for one dimensional heat equation (Bender-Schmidt and Crank- Nicholson methods), Finite difference explicit method for wave equation]:Many physical problems are mathematically modeled as boundary value problems ,associated with second order Partial differential equations. These second order Partial differential equations are classified into three distinct types. They are elliptic, parabolic and hyperbolic. A general second order linear Partial differential equation is of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \text{ ....(1)}$$

Where A , B, C, D, E, F, G are all functions of x, y.

[I]If $B^2 - 4AC < 0$ at a point in the x, y plane, then the equation (1) is called Elliptic

The standard examples are

(1) TwodimensionalLaplaceequation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(2) TwodimensionalPoissionequation

$$\frac{^2u}{x^2} + \frac{^2u}{y^2} = f(x, y)$$

[II]If $B^2 - 4\,AC = 0$ at a point in the x, y plane, then the equation (1) is called parabolic.

The standard example is one dimensional heat equation

$$\frac{^2u}{x^2} = \frac{1}{c^2}\frac{u}{t}$$

[III]If $B^2 - 4\,AC > 0$ at a point in the x, y plane, then the equation (1) is called hyperbolic.

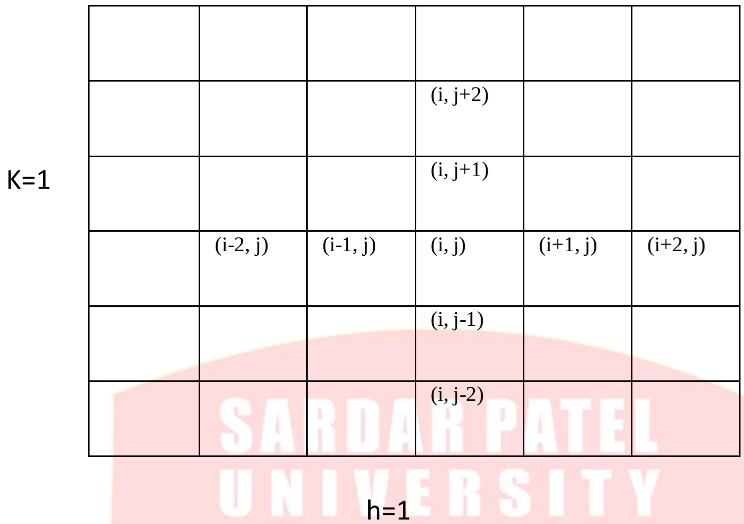
The standard example is one dimensional wave equation

$$\frac{^2u}{x^2} = \frac{1}{c^2}\frac{^2u}{t^2}$$

Numerical methods for solving boundary value problems for each three types are slightly different from one another.Most commonly used numerical method to solve such problems is termed as finite difference method or net method. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equation is changed into a system of linear equations, which are solved by iterative procedures.This process is slow but produces good resultsin many boundary value problems.

Geometrical representation of Partial Difference Quotients:

The x, y plane is divided into a series of rectangles of sides $\Delta x = \square$ and $\Delta y = k$ by equidistant lines drawn parallel to the axis of coordinates. As shown in above figure (1).



Now ,we can interpret the above idea in a different notation by drawing two sets of parallel lines $x = \mathrm{i}\square$ and $y = \mathrm{j}k$, $\mathrm{i} = \mathrm{j} = 0,1,2, \dots \dots \dots$

The point of intersection of these family of lines are called mesh points or lattice points. The point i, j is called the grid point and is surrounded by the neighboring points as shown in below figure (2). If u is a function of two variables x, y then the values of $u\ \ x, y$ at the point i, j is denoted by $u_{\mathrm{i,j}}$

We can write

$$\frac{u}{x} = u_x = \frac{u_{\mathrm{i+1,j}} - u_{\mathrm{i,j}}}{\square}$$

$$= \frac{u_{\mathrm{i,j}} - u_{\mathrm{i-1,j}}}{\square}$$

$$= \frac{u_{\mathrm{i+1,j}} - u_{\mathrm{i-1,j}}}{2\square},$$

$$\frac{u}{y} = u_y = \frac{u_{\mathrm{i,j+1}} - u_{\mathrm{i,j}}}{k}$$

$$= \frac{u_{\mathrm{i,j}} - u_{\mathrm{i,j-1}}}{k}$$

$$= \frac{u_{\mathrm{i,j+1}} - u_{\mathrm{i,j-1}}}{2k}$$

$$\frac{^2u}{x^2} = \frac{u_{\mathrm{i-1,j}} - 2u_{\mathrm{i,j}} + u_{\mathrm{i+1,j}}}{\square^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Now replacing the derivatives in any partial differential equation by their corresponding difference approximations, we obtain the finite difference analogues of the given equations.

[I] **Numerical Solution ofElliptic equations by Finite difference method**:-

An important equation of the **Elliptic type is** Two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This type of equation arises in potential and steady state flow problems. The solution u x, y of (1) is satisfied at every point of the region subject to given boundary conditions on the closed curve. Consider a rectangular region R for which u x, y is known at the boundary. Divide this region R for which u x, y is Known at the boundary.Divide this region into a network of square mesh of side h.

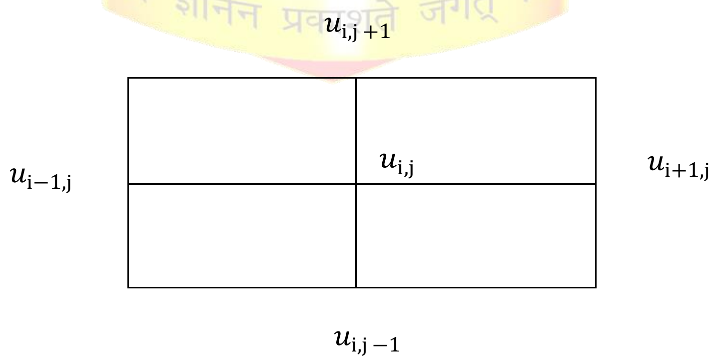
Now, replacing the derivatives in (1) by their difference approximations,

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$$

Since $k = h$,

$$= 0$$

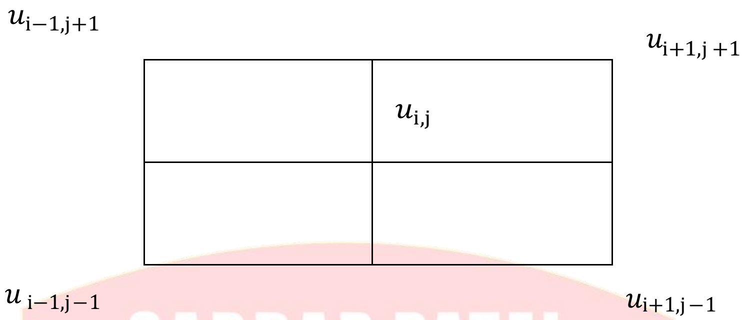
$$\Rightarrow u_{i,j} = \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$



This shows that the value of $u_{i,j}$ = u x, y at any interior mesh point is the average of it‟s values at four neighboring points to left, right, above and below.This is called the Standard five point formula (SFPF) or as Lineman‟s averaging procedure.

Sometimes a formula similar to (2) is used which is given by

$$=> u_{i,j} = \frac{u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$



This shows that the value of $u_{i,j} = u(x, y)$ is the average (or Arithmetic mean) of it‟s values at the four neighboring diagonal mesh points.This is called the Diagonal five point formula (DFPF).
Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points. In the iteration procedure, but wheneverpossible formula (2) is preferred in comparison to formula (3).

Now to solve a Laplace equation by finite difference method,we adapt the following set procedure

Suppose that the given boundary values are a_1, a_2, a_3…………., a_{16} .Now ,we are to determine initial values of $u_{i,j} = u(x, y)$ at the interior mesh points region R. Since the value u_5 at the center an therefore the values u_1, u_3, u_7, u_9, are computed by using Diagonal five point formula (DFPF), therefore ,we have

$$u_5 = \frac{1}{4}[a_1 + a_9 + a_{13} + a_5] \; ; \; u_1 = \frac{1}{4}[a_{15} + a_{11} + a_{13} + u_5] \; ; \; u_3 = \frac{1}{4}[u_5 + a_9 + a_{11} + a_7] \; ;$$

$$u_7 = \frac{1}{4}[a_1 + u_5 + a_{15} + a_3] \; ; \; u_9 = \frac{1}{4}[a_3 + u_5 + a_7 + a_5]$$

The values at the remaining interior mesh points i.e. the values of u_2, u_4, u_6, u_8, are computed by using, Standard five point formula (SFPF) therefore, we have

$$u_2 = \frac{1}{4}[u_1 + u_3 + a_{11} + u_5]; \; u_4 = \frac{1}{4}[a_{15} + u_5 + u_1 + u_7] \; ; \; u_6 = \frac{1}{4}[u_5 + u_7 + u_3 + u_9];$$

$$u_8 = \frac{1}{4}[u_7 + u_9 + u_5 + a_3]$$

Thus ,we have computed all values $u_1, u_2, u_3, \ldots, u_9$ once. The accuracy of $u_1, u_2, u_3, \ldots, u_9$ are improved by the repeated application of the any one of the following iterative formulae.

(i) Gauss sJacobi‟s iterative method

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n}\right]$$

Where $n = 0,1,2, \dots \dots$ be the no. of iterations.

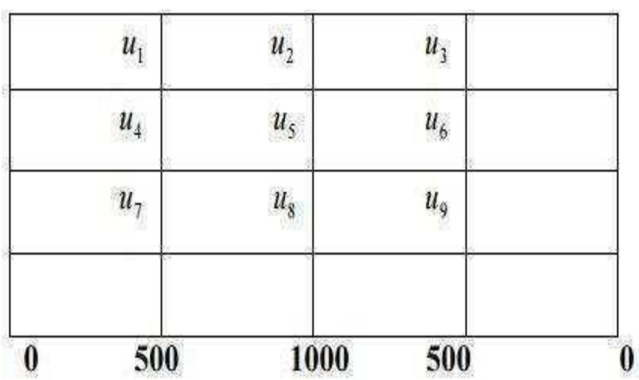
(ii) Gauss s Seidal „s iterative method

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{n+1} + u_{i+1,j}^{n} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n}\right]$$

Where $n = 0,1,2, \dots \dots$ be the no. of iterations.

This method of finite difference is well explained by Example.

1.Solve the equation $\nabla^2 u = 0$ for the following mesh,with boundary vaues as shown using Leibmann's iteration process.



Sol:

Sol:

Let u_1 , u_2 ……u_9 be the values of u at the interior mesh points of the given region.By symmetry about the lines AB and the line CD,we observe

$u_1 = u_3$ $\quad$ $u_1 = u_7$

$u_2 = u_8$ $\quad$ $u_4 = u_6$

$u_3 = u_9$ $\quad$ $u_7 = u_9$

$u_1 = u_3 = u_7 = u_9 , u_2 = u_8 , u_4 = u_6$

Hence it is enough to find u_1 , u_2 , u_4 , u_5

Calculation of rough values

u_5 =1500

$u_1 = 1125$
$u_2 = 1187.5$
$u_4 = 1437.5$
Gauss-seidel

$$u_1 = \frac{1}{4}[1500 + u_2 + u_4]$$

$$u_2 = \frac{1}{4}[2u_1 + u_5 + 1000]$$

$$u_4 = \frac{1}{4}[2000 + u_5 + u_4]$$

$$u_5 = \frac{1}{4}[2u_2 + 2u_4]$$

.

The iteration values are tabulated as follows

Iteration No k	u_1	u_2	u_4	u_5
0	**1500**	**1125**	**1187.5**	**1437.5**
1	**1031.25**	**1125**	**1375**	**1250**
2	**1000**	**1062.5**	**1312.5**	**1187.5**
3	**968.75**	**1031.25**	**1281.25**	**1156.25**
4	**953.1**	**1015.3**	**1265.6**	**1140.6**
5	**945.3**	**1007.8**	**1257.8**	**1132.8**
6	**941.4**	**1003.9**	**1253.9**	**1128.9**
7	**939.4**	**1001.9**	**1251.9**	**1126.9**
8	**938.4**	**1000.9**	**1250.9**	**1125.9**
9	**937.9**	**1000.4**	**1250.4**	**1125.4**
10	**937.7**	**1000.2**	**1250.2**	**1125.2**
11	**937.6**	**1000.1**	**1250.1**	**1125.1**
12	**937.6**	**1000.1**	**1250.1**	**1125.1**

$u_1 = u_3 = u_7 = u_9 = 937.6, u_2 = u_8 = 1000.1, u_4 = u_6 = 1250.1,\ u_5 = 1125.1$

scheme

When steady state condition prevail, the temperature distribution of the plate is represented by Laplace equation $u_{xx}+u_{yy}=0$.The temperature along the edges of the square plate of side 4 are given by along x=y=0,u=x^3 along y=4 and u=16y along x=4, divide the square plate into 16 square meshes of side h=1 ,compute the temperature a iteration process.

Solution of Poisson equation:-

An equation of the type $\nabla^2 u = f(x,y)$ i.e., is called $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ poisson's equation

where f(x,y) is a function of x and y.

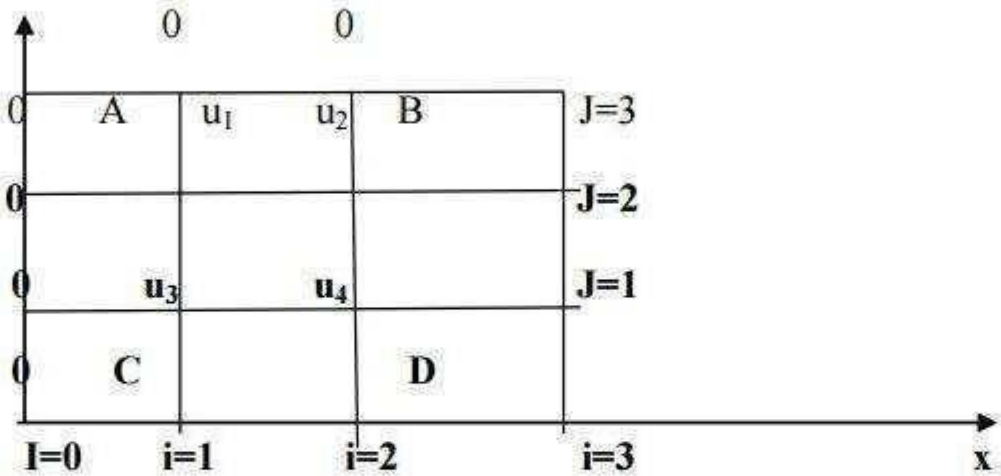
$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

This expression is called the replacement formula.Applying this equation at each internal mesh point ,we get a system of linear equations in u_i,where u_i are the values of u at the internal mesh points.Solving the equations, the values u_i are known.

Problems

1.Solve the poisson equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides x=0,y=0,x=3,y=3 and u=0 on the boundary .assume mesh length h=1 unit.

Sol:



Here the mesh length $\Delta x = h = 1$

Replacement formula at the mesh point (i,j)

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \quad (1)$$

$$u_2 + u_3 - 4u_1 = -150$$
$$u_1 + u_4 - 4u_3 = -120$$
$$u_2 + u_3 - 4u_4 - 150$$
$$u_1 = u_4 = 75, u_2 = 82.5, u_3 = 67.5$$

2.Solve the poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -81xy, 0 < x < 1; 0 < y < 1$ and u(0,y)=u(x,0)=0,u(x,1)=u(1,y)=100 with the square meshes ,each of length h=1/3.

u(0,y)=u(x,0)=0,u(x,1)=u(1,y)=100 with the square meshes ,each of length h=1/3.

[II] Solution of One dimensional heat equation:-

In this session, we will discuss the finite difference solution of one dimensional heat flow equation by Explicit and implicit method

Explicit Method (Bender-Schmidt method)

Consider the one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. This equation is an example of parabolic equation.

$$u_{i,j+1} = \lambda u_{i+1,j} + 1 - 2\lambda u_{i,j} + \lambda u_{i-1,j} \qquad (1)$$

Where $\lambda = \frac{k}{ah^2}$

Expression (1) is called the explicit formula and it valid for $0 < \lambda \leq \frac{1}{2}$

If λ=1/2 then (1) is reduced into

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + \lambda u_{i-1,j}] \qquad (2)$$

This formula is called Bender-Schmidt formula.

This formula is called Bender-Schmidt formula.

Implicit method (Crank-Nicholson method)

$$-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j}$$

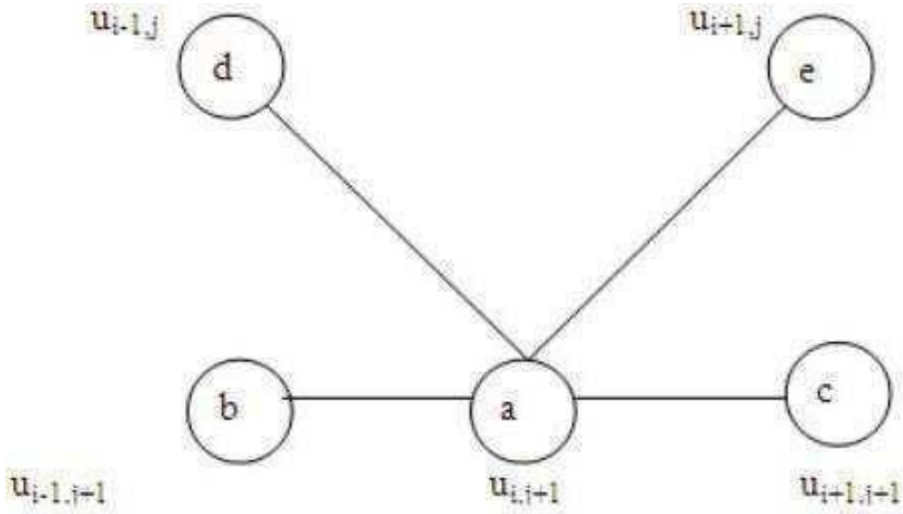
This expression is called Crank-Nicholson’s implicit scheme.

This expression is called Crank-Nicholson’s implicit scheme. We note that Crank Nicholson’s scheme converges for all values of λ

When λ=1, i.e., k=ah^2 the simplest form of the formula is given by

$$\Rightarrow u_{i,j+1} = \frac{1}{4}[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}]$$

The use of the above simplest scheme is given below.



The value of u at A=Average of the values of u at B, C, D, E

Note

In this scheme, the values of u at a time step are obtained by solving a system of linear equations in the unknown’su_i.

Solved Examples when u(0,t)=0,u(4,t)=0 and with initial condition u(x,0)=x(4-x) upto t=sec
By initial conditions, u(x,0)=x(4-x) ,we have

1. Solve $u_{xx} = 2u_t$ **assuming Δx=h=1**

Sol:
By Bender-Schmidt recurrence relation,

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + \lambda u_{i-1,j}] \qquad (1)$$

$$k = \frac{ah^2}{2}$$

For applying eqn(1) ,we choose
Here a=2,h=1.Then k=1
By initial conditions, u(x,0)=x(4-x) ,we have

$$u_{i,0} = i(4-i)i = 1,2,3$$
$$,u_{1,0} = 3, u_{2,0} = 4, u_{3,0} = 3$$

By boundary conditions, u(0,t)=0,u_0=0,u(4,0)=0 $\Rightarrow u_{4,j} = 0 \forall j$

Values of u at t=1

$$u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$$
$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = 2$$
$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = 3$$
$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = 2$$

The values of u up to t=5 are tabulated below.

j\i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0	0.75	0.5	0

2.Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ **subject to the conditions u(0,t)=u(5,t)=0 and** $u(x,0) = x^2(25 - x^2)$ **taking h=1 and k=1/2,tabulate the values of u upto t=4 sec.**

Sol:

Here a=1,h=1

For λ=1/2,we must choose $k=ah^2/2$

K=1/2

By boundary conditions

$$u(0,t)=0 \Rightarrow u_{0,j}=0 \forall j$$
$$u(5,t)=0 \Rightarrow u_{5,j}=0 \forall j$$
$$u(x,0)=x^2(25-x^2)$$
$$\Rightarrow u_{i,0}=i^2(25-i^2), i=0,1,2,3,4,5$$
$$u_{1,0}=24, u_{2,0}=84, u_{3,0}=144, u_{4,0}=144, u_{5,0}=0$$

By Bender-schmidt realtion,

$$u_{i,j+1}=\frac{1}{2}[u_{i+1,j}+u_{i-1,j}]$$

The values of u upto 4 sec are tabulated as follows

[III] Solution of One dimensional wave equation:-

Introduction

j\i	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	144	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0

The one dimensional wave equation is of hyperbolic type. In this session, we discuss the finite difference solution of the one dimensional wave equation

$$u_{tt} = a^2 u_{xx}.$$

Explicit method to solve $u_{tt} = a^2 u_{xx}$

$$u_{i,j+1} = 2(1-\lambda^2 a^2)u_{i,j} + \lambda^2 a^2 u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad \textbf{(1)}$$

Where λ=k/h

Formula (1) is the explicit scheme for solving the wave equation.

Problems

1.Solve numerically , $4u_{xx} = u_{tt}$ **with the boundary conditions u(0,t)=0,u(4,t)=0 and the initial conditions** $u_t(x,0) = 0 \,\&\, u(x,0) = x(4-x),$ **taking h=1.Compute u upto t=3sec.**

Sol:

Here a^2=4

A=2 and h=1

We choose k=h/a$\Rightarrow$k=1/2

The finite difference scheme is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$
$$u(0,t) = 0 \Rightarrow u_{0,j} \,\&\, u(4,t) = 0 \Rightarrow u_{4,j} = 0 \forall j$$
$$u(x,0) = x(4-x) \Rightarrow u_{i,0} = i(4-i), i = 0,1,2,3,4$$
$$u_{0,0} = 0, u_{1,0} = 3, u_{2,0} = 3, u_{4,0} = 0$$
$$u_{1,1} = 4 + 0/2 = 2$$
$$u_{2,1} = 3, u_{3,1} = 2$$

The values of u for steps t=1,1.5,2,2.5,3 are calculated using (1) and tabulated below.

j\i	**0**	**1**	**2**	**3**	**4**
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	3	-2	0
4	0	-3	-4	3	0
5	0	-2	-3	-2	0
6	0	0	0	0	0

2.Solve $u_{xx} = u_{tt}$ given u(0,t)=0,u(4,t)=0,u(x,0)= $u(x,0) = \dfrac{x(4-x)}{2}$ & $u_t(x,0) = 0$. Take h=1.Find the solution upto 5 steps in t-direction.

Sol:

Here a^2=4

A=2 and h=1

We choose k=h/a$\Rightarrow$k=1/2

The finite difference scheme is

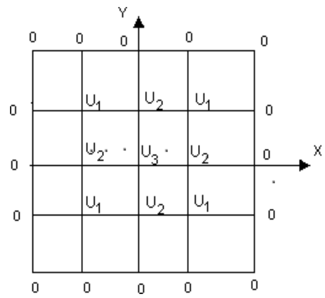
$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$
$$u(0,t) = 0 \Rightarrow u_{0,j} \ \& \ u(4,t) = 0 \Rightarrow u_{4,j} = 0 \forall j$$
$$u(x,0) = x(4-x)/2 \Rightarrow u_{i,0} = i(4-i)/2, i = 0,1,2,3,4$$
$$u_{0,0} = 0, u_{1,0} = 1.5, u_{2,0} = 2, u_{3,0} = 1.5, u_{4,0} = 0$$
$$u_{1,1} = 1$$
$$u_{2,1} = 1.5, u_{3,1} = 1$$

The values of u upto t=5 are tabulated below.

j\i	**0**	**1**	**2**	**3**	**4**
0	0	1.5	2	1.5	0
1	0	1	1.5	1	0
2	0	0	0	0	0
3	0	-1	-1.5	-1	0
4	0	-1.5	-2	-1.5	0
5	0	-1	-1.5	-1	0

Practice Problems:

Example: Solve the equation u_{xx} ' u_{yy} ' $8x^2 y^2$ over the square mesh of following figure with u x, y =0 on the boundary and mesh length=1.



Example: Solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10)$ over the square with sides

$x = y = 0$, to $x = y = 3$ with $u(x, y) = 0$ on the boundary and mesh length is one.

Module -4

Transform Calculus

Contents

Laplace Transform, Laplace Transform of elementary function, Properties of Laplace Transform,Change of Scale property , First and Second Shifting Properties , Laplace Transform of periodic functions , Laplace Transform of Derivatives and Integrals, Inverse Laplace Transform and Its Properties, Convolution Theorem, Application of Laplace Transform in Solving the Ordinary Differential Equations. Fourier transforms.

1. Motivation: Laplace transform a very powerful technique is that it replaces operations of calculus by operations of algebra. Laplace transform is an integral transform method which is particularly useful in solving linear ordinary differential equations. It finds very wide applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing .Laplace transforms help in solving complex problems with a very simple approach.

2. Prerequisite:

Function, the concept of limit, continuity, ordinary derivative of function, rules and formulae of differentiation and integration of function of one independent variable.

3. Objective: The Laplace transform method solves differential equations and corresponding initial

4. and boundary value problems. The Laplace transforms reduce the problem of solving a differential equation to an algebraic problem. It is also useful in problems where the mechanical or electrical driving force has discontinuities, is impulsive or is a complicated periodic function.

The Laplace transform also has the advantage that it solves problems directly,initial and boundary value problems without determining a general solution.

5. Key Notations:

(1) $L\{f(t)\}$:Laplace transform of a function

(2) $L^{-1}\{f(t)\}$:Inverse Laplace transform of a function

6. Key Definitions:

(1) LAPLACE TRANSFORM: Let $f(t)$ be a function defined for all positive values of t , then

$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$ provided the integral exists, is called the Laplace Transform of $f(t)$.

It is denoted as

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

(2) INVERSE LAPLACE TRANSFORM: If $L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$ then $f(t)$ is called the Inverse Laplace transform of $\bar{f}(s)$.

It is denoted as $L^{-1}\{F(s)\} = f(t)$.

7. Important Formulae/ Theorems / Properties:

LAPLACE TRANSFORM:

STANDARD FORMULAE:

1) $L(e^{at}) = \dfrac{1}{s-a} \quad (s > a)$

2) $L(1) = \dfrac{1}{s} \quad (s > 0)$

3) $L(\sin at) = \dfrac{a}{s^2 + a^2}$

4) $L(\cos at) = \dfrac{s}{s^2 + a^2}$

5) $L(\sinh at) = \dfrac{a}{s^2 - a^2} \quad (s > |a|)$

6) $L(\cosh at) = \dfrac{s}{s^2 - a^2} \quad (s > |a|)$

7) $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$

8. SAMPLE PROBLEMS:

I.Exercise can be solved based on following sample problem.

LAPLACE TRANSFORM BY DEFINITION:

Ex. Find the Laplace transform of

$$f(t) = \begin{cases} \cos t & \text{for } 0 < t < \pi \\ \sin t & \text{for } t > \pi \end{cases}$$

Solution: By the definition of Laplace transform we have,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$$

$$\text{But } \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{(a^2 + b^2)}\left(a \cos bx + b \sin bx\right)$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{(a^2 + b^2)}\left(a \sin bx - b \cos bx\right)$$

$$L\{f(t)\} = \left[\frac{1}{s^2+1} e^{-st}\left(-s\cos t + \sin t\right)\right]_0^\pi + \left[\frac{1}{s^2+1} e^{-st}\left(-s\sin t - \cos t\right)\right]_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(s) - (-s)\right] + \frac{1}{s^2+1}\left[-e^{-\pi s}\right]$$

$$= \frac{1}{s^2+1}\left[s + (s-1)e^{-\pi s}\right]$$

Unsolved Problem

Find the Laplace transform of following functions.

1) $f(t) = (t-2)^2 \quad t > 2, \quad f(t) = 0 \quad 0 < t < 2$ $\quad Ans: \frac{2}{s^3} e^{-2s}$

2) $f(t) = t \quad 0 < t < a$
$\quad = b \quad t > a$ $\quad Ans: \frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right] e^{-as}$

3) $f(t) = t, 0 < t < 3$
$\quad = 6, t > 3$ $\quad Ans: \frac{1}{s^2} + \left[\frac{3}{s} - \frac{1}{s^2}\right] e^{-3s}$

II Exercise can be solved based on following sample problem.

LAPLACE TRANSFORM BY LINEARITY PROPERTY:

$$L\{k_1 f(t) + k_2 g(t)\} = k_1 L\{f(t)\} + k_2 L\{g(t)\}$$

Ex. Find the Laplace transform of $\sin(\omega t + \alpha)$

Solution: By linearity property, we have

$$L\{\sin(\omega t + \alpha)\} = L\{\sin \omega t \cos \alpha + \cos \omega t \sin \alpha\}$$

$$= \cos \alpha\, L\{\sin \omega t\} + \sin \alpha\, L\{\cos \omega t\}$$

$$= \cos \alpha \frac{\omega}{s^2 + \omega^2} + \sin \alpha \frac{s}{s^2 + \omega^2}$$

Unsolved Problem

Find the Laplace transform of following functions.

1) $t^2 + e^{-2t} + \cosh^2 3t$ $\quad Ans: \frac{2}{s^3} + \frac{1}{(s+2)} + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-6^2}\right]$

2) $(\sin 2t - \cos 2t)^2$ $\quad Ans: \frac{1}{s} - \frac{4}{s^2+4^2}$

3) $\cos(wt + b)$ $\quad Ans: \frac{s}{s^2+w^2}\cos b - \frac{w}{s^2+w^2}\sin b$

4) $\sin(5t + 3)$ $\quad Ans: \frac{5}{s^2+5^2}\cos 3 + \frac{s}{s^2+5^2}\sin 3$

5) $\cos t \cos 2t \cos 3t$ $\quad Ans: \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+2^2} + \frac{s}{s^2+4^2} + \frac{s}{s^2+6^2}\right]$

6) $\sin^5 t$ $\quad Ans: \frac{5!}{(s^2+1)(s^2+9)(s^2+25)}$

CHANGE OF SCALE PROPERTY:

If $L[f(t)] = \phi(s)$ then $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Ex. If $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$, find $L\{f(2t)\}$.

Solution: By change of scale property, we have

$$L\{f(2t)\} = \frac{1}{2}\phi\left(\frac{s}{2}\right)$$

$$L\{f(2t)\} = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Unsolved Problem

If $L[f(t)] = \phi(s)$ **then** $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

1) Find $L[f(2t)]$ if $L[f(t)] = \log\left(\frac{s+3}{s+1}\right)$ $\quad Ans: \frac{1}{2}\log\frac{(s+6)}{(s+2)}$

2) Find $L[(erf\, 2\sqrt{t})]$, If $L\left[erf\sqrt{t}\right] = \frac{1}{s\sqrt{s+1}}$ $\quad Ans: \frac{2}{s\sqrt{s+4}}$

3) Find $L\{\cos 4t\}$, If $L\{\cos t\} = \frac{s}{s^2+1}$

FIRST SHIFTING THEOREM:

If $L[f(t)] = \phi(s)$ then $L[e^{-at} f(t)] = \phi(s+a)$ & $L[e^{at} f(t)] = \phi(s-a)$

Ex. Find the Laplace transform of $\sin 2t \cos t \cosh 2t$.

Solution: We know that

$$\sin 2t \cos t = \frac{1}{2} 2\sin 2t \cos t = \frac{1}{2}(\sin 3t + \sin t)$$

$$\cosh 2t = \frac{e^{2t} + e^{-2t}}{2}$$

$$\therefore \sin 2t \cos t \cosh 2t = \frac{1}{2}(e^{2t} + e^{-2t})(\sin 3t + \sin t)$$

$$\therefore \sin 3t = \frac{3}{s^2 + 9}$$

$$\therefore L(e^{2t} \sin 3t) = \frac{3}{(s-2)^2 + 9}, \quad L(e^{-2t} \sin 3t) = \frac{3}{(s+2)^2 + 9}$$

$$\therefore L(e^{2t} \sin 3t) + L(e^{-2t} \sin 3t) = 3\left[\frac{1}{} + \frac{1}{}\right]$$

$$= \frac{3.2(s^2 + 13)}{s^4 + 10s^2 + 13^2}$$

$$Now \ \sin t = \frac{1}{s^2 + 1}$$

$$\therefore L(e^{2t} \sin t) = \frac{1}{(s-2)^2 + 1}, \quad L(e^{-2t} \sin t) = \frac{1}{(s+2)^2 + 1}$$

$$L(e^{2t} \sin t) + L(e^{-2t} \sin t) = \frac{2(s^2 + 5)}{s^4 - 6s^2 + 5^2}$$

Form (1), (2) and (3), we get

$$L(\sin 2t \cos t \cosh 2t) = \frac{3(s^2 + 13)}{s^4 + 10s^2 + 13^2} + \frac{(s^2 + 5)}{s^4 - 6s^2 + 5^2}$$

Unsolved Problem

If $L[f(t)] = \phi(s)$ then $L[e^{-at} f(t)] = \phi(s+a)$ & $L[e^{at} f(t)] = \phi(s-a)$

Find the Laplace transform of following functions.

1) $e^{-3t} t^4$ $Ans: \dfrac{4!}{(s+3)^5}$

2) $\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)$ $Ans: \frac{\sqrt{3}\,s}{2(s^4+s^2+1)}$

3) $\frac{\cos 2t \sin t}{e^t}$ $Ans: \frac{(s^2+2s-2)}{(s^2+2s+10)(s^2+2s+2)}$

4) $e^{-4t}\sinh t \sin t$ $Ans: \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}$

5) $e^{t}\sin 2t \sin 3t$ $Ans: \frac{12s}{(s^2-2s+2)(s^2-2s+26)}$

6) $e^{3t}\cosh 5t \sin 4t$ $Ans: \frac{4(s^2-6s+50)}{(s^2+4s+20)(s^2-16s+80)}$

7) $\sin 2t \cos t \cosh 2t$ $Ans: \frac{3(s^2+13)}{(s^4+10s^2+13^2)}+\frac{(s^2+5)}{(s^4-6s^2+5^2)}$

8) $e^{-4t}\cosh t \sin t$ $Ans: \frac{(s^2+8s+18)}{(s^2+6s+10)(s^2+10s+26)}$

EFFECT OF MULTIPLICATION BY t^n:

$If\ L[f(t)] = \bar{f}(s)\ then\ L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

Ex. Find the Laplace transform of $\sqrt{1+\sin t}$.

Solution:

$$\sqrt{1+\sin t} = \sqrt{\sin^2\frac{t}{2}+\cos^2\frac{t}{2}+2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \sqrt{\left(\sin\frac{t}{2}+\cos\frac{t}{2}\right)^2} = \sin\frac{t}{2}+\cos\frac{t}{2}$$

$$\therefore L\left[\sqrt{1+\sin t}\right] = L\left[\sin\frac{t}{2}+\cos\frac{t}{2}\right]$$

$$= \frac{\frac{1}{2}}{s^2+\left(\frac{1}{2}\right)^2}+\frac{s}{s^2+\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{4}{4s^2+1}+\frac{4s}{4s^2+1}$$

$$\frac{4s+2}{(4s^2+1)} = \frac{2(2s+1)}{(4s^2+1)}$$

$$L\left[t\sqrt{1+\sin t}\right] = -\frac{d}{ds}\left[\frac{2(2s+1)}{4s^2+1}\right]$$

$$= -2\left[\frac{(4s^2+1)2-(2s+1)8s}{(4s^2+1)^2}\right]$$

$$= -2\left[\frac{-8s^2-8s+2}{(4s^2+1)^2}\right]$$

$$= 4\left[\frac{4s^2+4s-1}{(4s^2+1)^2}\right]$$

Unsolved Problem

If $L[f(t)] = \bar{f}(s)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

Find the Laplace transform of following functions.

1) $t\sin^3 t$ Ans: $\frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$

2) $t\sin 2t\cosh t$ Ans: $2\left[\frac{(s-1)}{(s^2-2s+5)^2}+\frac{(s+1)}{(s^2+2s+5)^2}\right]$

3) $t\cos^2 t$ Ans: $\frac{1}{2s^2}+\frac{s^2-2^2}{2(s^2+2^2)^2}$

4) $te^{3t}\sin 4t$ Ans: $\frac{8(s-3)}{(s^2-6s+25)^2}$

5) $te^{3t}\sin t$ Ans: $\frac{(2s-6)}{(s^2-6s+10)^2}$

6) $t\sqrt{1+\sin t}$ Ans: $\frac{4(4s^2+4s-1)}{(4s^2+1)^2}$

7) $te^{3t}\sin 2t$ Ans: $\frac{4(s-3)}{(s^2-6s+13)^2}$

8) $t^2\sin 3t$ Ans: $18\frac{(s^2-3)}{(s^2+9)^3}$

EFFECT OF DIVISION BY t

If $L[f(t)] = \bar{f}(s)$ then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \bar{f}(s)\,ds$

Ex. Find $L\left\{\dfrac{\sin^2 t}{t^2}\right\}$

Solution: We know that

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

By effect of division, we have

$$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\int_s^\infty\left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \left[\frac{1}{2}\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= -\frac{1}{4}\log\frac{s^2}{s^2+4}$$

$$= \frac{1}{4}\log\frac{s^2+4}{s^2}$$

$$L\left\{\frac{\sin^2 t}{t^2}\right\} = \int_s^\infty \frac{1}{4}\log\frac{s^2+4}{s^2}\,ds$$

Integrating by parts

$$L\left\{\frac{\sin^2 t}{t^2}\right\} = \frac{1}{4}\left[\log\frac{s^2+4}{s^2}\cdot s - \int s\cdot\frac{s^2}{s^2+4}\cdot\frac{s^2\,2s - (s^2+4)\,2s}{s^4}\,ds\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\frac{s^2+4}{s^2} + 8\int\frac{ds}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\frac{s^2+4}{s^2} + 2\tan^{-1}\frac{s}{2}\right]_s^\infty$$

Unsolved Problem :

If $L[f(t)] = \bar{f}(s)$ then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \bar{f}(s)\,ds$

Find the Laplace transform of following functions.

1) $\frac{1}{t}(1-\cos t)$ $Ans: \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$

2) $\frac{1}{t}(e^{-at}-e^{-bt})$ $Ans: \log\left(\frac{s+b}{s+a}\right)$

3) $\frac{\sin^2 2t}{t}$ $Ans: \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

4) $\frac{e^{-2t}\sin 2t \cosh t}{t}$ $Ans: \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)$

5) $\frac{\sin^2 t}{t^2}$ $Ans: \frac{1}{2}\left[2\cot^{-1}\left(\frac{s}{2}\right) - s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$

6) $\frac{1-\cos t}{t^2}$ $Ans: \frac{\pi}{2} - \frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right) - \tan^{-1}s$

LAPLACE TRANSFORM OF DERIVATIVE

If $L[f(t)] = \bar{f}(s)$ *then*

$$L[f'(t)] = s\bar{f}(s) - f(0)$$

$$L[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0)$$

$$L[F'''(t)] = s^3\bar{f}(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$L[f^n(t)] = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - f^{(n-1)}(0)$$

Ex. Find $L[f(t)]$ and $L[f'(t)]$, where $f(t) = \frac{\sin t}{t}$

Solution: $\because L[f(t)] = \bar{f}(s)$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L[\sin t]\,ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1}s\right]_s^\infty$$

$$= \cot^{-1}s$$

$$\therefore \bar{f}(s) = \cot^{-1}s$$

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

$$= s\cot^{-1}s - \lim_{t\to 0}\frac{\sin t}{t}$$

But $f(0) = \lim_{t\to 0}\frac{\sin t}{t}$ is an indeterminate form, which can be solved by L'Hospital's Rule

$$\therefore \lim_{t\to 0}\frac{\sin t}{t} = \lim_{t\to 0}\frac{\cos t}{1} \quad \textit{(By differentiating numerator and deonimnatorseparately)}$$

$$= 1$$

$$\therefore \quad L\{f'(t)\} = L\{f(t)\} - f(0)$$

$$= s\cot^{-1}s - 1$$

Unsolved Problem

1) Find $L\{f'(t)\}$ and $L\{f''(t)\}$

i) If $f(t) = \frac{\sin t}{t}$ $\quad$ $Ans: s\cot^{-1}s - 1$

ii) $f(t) = 3,\ 0 < t < 5$ $\quad$ $Ans: \frac{3}{s}\left(1 - e^{-5s}\right)$

$\quad = 0,\ t > 5$

iii) $f(t) = t,\ 0 < t < 3$ $\quad$ $Ans: \frac{1}{s^2} - e^{-3s}\left(\frac{3}{s} + \frac{1}{s^2}\right),\ \left(\frac{1}{s} - e^{-3s}\left(3 - \frac{1}{s}\right)\right)$

$\quad = 6,\ t > 3$

2) If $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$, Show that $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

3) If $L\{t\sin\omega t\} = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$, evaluate i) $L\{\omega t\cos\omega t + \sin\omega t\}$ ii) $L\{2\cos\omega t - \omega t\sin\omega t\}$

Ans: i) $\frac{2\omega s^2}{\left(s^2+\omega^2\right)^2}$, ii) $\frac{2s^3}{\left(s^2+\omega^2\right)^2}$

LAPLACE TRANSFORM OF INTEGRAL

If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{\bar{f}(s)}{s}$

Ex. Find the Laplace transform of $\int_0^t e^{-4t}\sin 3t\,dt$

Solution:

$$L\{t\sin 3t\} = -\frac{d}{ds}L\{\sin 3t\}$$

$$= -\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\{te^{-4t}\sin 3t\} = \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t te^{-4t}\sin 3t\,dt\right\} = \frac{1}{s}L\{te^{-4t}\sin 3t\}$$

$$= \frac{6(s+4)}{s(s^2+8s+25)^2}$$

Unsolved Problem

Find the Laplace transform of following functions.

1) $\int_0^t t\cosh t\,dt$ $\quad Ans: \frac{(s^2+a^2)}{s(s^2-a^2)^2}$

2) $\int_0^t t\cos^2 t\,dt$ $\quad Ans: \frac{1}{2s^3}+\frac{1}{2}\frac{s^2-2^2}{s(s^2+2^2)^2}$

3) $\int_0^t \frac{1-e^{-t}}{t}\,dt$ $\quad Ans: \frac{1}{s}\log\left(\frac{s+1}{s}\right)$

4) $\int_0^t \frac{\sin t}{t}\,dt$ $\quad Ans: \frac{1}{s}\cot^{-1} s$

5) $\int_0^t\int_0^t\int_0^t t\sin t\,dt\,dt\,dt$ $\quad Ans: \frac{2}{s^2(s^2+1)^2}$

EVALUATION OF INTEGRAL USING LAPLACE TRANSFORMS

Ex. Evaluate $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt$

Solution: By comparing the given integral $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt$ with the Definition of Laplace transform

L [f (t)] = $\int_0^\infty e^{-st} f(t)\,dt$ we get,

s=1 and $f(t)=\int_0^t \frac{\sin u}{u}\,du$

$$L\{\sin u\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin u}{u}\right\} = \int_s^\infty L\{\sin u\}\,ds$$

Now,
$$= \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s} L\left\{\frac{\sin u}{u}\right\}$$

$$= \frac{1}{s}\cot^{-1} s$$

Now, $\displaystyle\int_0^\infty e^{-st}\left(\int_0^t \frac{\sin u}{u}\,du\right) dt = \frac{1}{s}\cot^{-1} s$

Putting s=1, we get

$$\int_0^\infty e^{-t}\left(\int_0^t \frac{\sin u}{u}\,du\right) dt = \cot^{-1} 1 = \frac{\pi}{4}$$

Unsolved Problem

1) Show that $\displaystyle\int_0^\infty e^{-2t}\sin^3 t\,dt = \frac{6}{65}$

2) Show that $\displaystyle\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t\sinh t}{t}\,dt = \frac{\pi}{8}$

4) Show that $\displaystyle\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{te^t}\right) dt = \frac{3\pi}{4}$

5) Show that $\displaystyle\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}\,dt = \frac{\pi}{3}$

6) Evaluate $\displaystyle\int_0^\infty t^3 e^{-t}\sin t\,dt$

7) Evaluate $\displaystyle\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt$

8) If $\displaystyle\int_0^\infty e^{-2t}\cos(t+\alpha)\sin(t-\alpha)\,dt = \frac{3}{8}$ find α. $Ans: \alpha = \frac{\pi}{4}$

INVERSE LAPLACE TRANSFORM:

STANDARD FORMULAE:

1) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

' ' S' a' '

2) $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

3) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

4) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

5) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

6) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

7) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

8) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$

III Exercise can be solved based on following sample problem.

INVERSE BY DIRECT FORMULAE

Ex. Find the inverse Laplace transform of $\frac{3s+4}{s^2+16}$

Solution:

$$L^{-1}\left\{\frac{3s+4}{s^2+16}\right\} = 3L^{-1}\left\{\frac{s}{s^2+16}\right\} + L^{-1}\left\{\frac{4}{s^2+16}\right\}$$

$$= 3L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + L^{-1}\left\{\frac{4}{s^2+4^2}\right\}$$

$$= 3\cos 4t + \sin 4t$$

Unsolved Problem

Find the inverse Laplace transform of following function.

1) $\frac{1}{s^2+9}$ $\qquad$ $Ans: \frac{\sin 3t}{3}$

2) $\frac{s^2+3s+4}{s^3}$ $\qquad$ $Ans: 2t^2+3t+1$

3) $\frac{3s-4\sqrt{3}}{s^2+7}$ $\qquad$ $Ans: \cos\sqrt{7}t - \frac{4\sqrt{3}}{\sqrt{7}}\sin\sqrt{7}t$

INVERSE BY FIRST SHIFTING THEOREM

$$L^{-1}\{\phi(s-a)\} = e^{at}L^{-1}\{\phi(s)\}$$

Ex. Find the inverse Laplace transform of $\frac{4s+12}{s^2+8s+12}$

Solution:

$$L^{-1}\left[\frac{4s+12}{s^2+8s+12}\right]=L^{-1}\left[\frac{4(s+4)-2^2}{(s+4)^2-2^2}\right]$$

$$=4L^{-1}\left[\frac{s+4}{(s+4)^2-2^2}\right]-L^{-1}\left[\frac{2^2}{(s+4)^2-2^2}\right]$$

By First shifting theorem, we have

$$4e^{-4t}L^{-1}\left[\frac{s}{s^2-2^2}\right]-4e^{-4t}L^{-1}\left[\frac{2^2}{s^2-2^2}\right]$$

$$=4e^{-4t}\cosh 2t-4e^{-4t}\frac{1}{4}\sinh 2t$$

$$=e^{-4t}\left(4\cosh 2t-\sinh 2t\right)$$

Unsolved Problem

Find the inverse Laplace transform of following function.

1) $\dfrac{2s+2}{s^2+2s+10}$ $\quad$ $Ans: 2e^{-t}\cos 3t$

2) $\dfrac{s+2}{s^2+4s+7}$ $\quad$ $Ans: e^{-2t}\cos\sqrt{3}t$

3) $\dfrac{2s+3}{s^2+2s+2}$ $\quad$ $Ans: 2e^{-t}\cos t+e^{-t}\sin t$

INVERSE BY PARTIAL FRACTION

Ex. Find the inverse Laplace transform of $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

Solution: $L^{-1}\left[\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right]=L^{-1}\left[\dfrac{(s+1)^2+2}{\left((s+1)^2+2^2\right)\left((s+1)^2+1^2\right)}\right]$

$$=e^{-t}L^{-1}\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]$$

Let

$s^2=x$

And hence

$$\frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{x+2}{(x+4)(x+1)}$$

$$\frac{x+2}{(x+4)(x+1)} = \frac{a}{x+4} + \frac{b}{x+1}$$

$$x+2 = a(x+1) + b(x+4)$$

When x=-1, 1=3b; when x=-4, -2=-3a

$$\frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{2}{3}\frac{1}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1}$$

$$L^{-1}\left\{\frac{s^2+2}{(s^2+4)(s^2+1)}\right\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s^2+4}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{2}{3}\cdot\frac{1}{2}\sin 2t + \frac{1}{3}\sin t$$

$$\therefore L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right\} = \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

Unsolved Problem

Find the inverse Laplace transform of following function.

1) $\dfrac{3s+1}{(s+1)(s^2+2)}$ $\quad Ans: -\dfrac{2}{3}e^{-t} + \dfrac{2}{3}\cos\sqrt{2}t + \dfrac{7}{3\sqrt{2}}\sin\sqrt{2}t$

2) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ $\quad Ans: \dfrac{1}{a^2-b^2}(a\sin at - b\sin bt)$

4) $\dfrac{s+2}{s^2(s+3)}$ $\quad Ans: \dfrac{1}{9}\left(1+6t-e^{-3t}\right)$

5) $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$ $\quad Ans: \dfrac{e^{-t}}{3}(\sin 2t + \sin t)$

6) $\dfrac{s}{(s^2+1)(s^2+4)}$ $\quad Ans: \dfrac{1}{3}(\cos t - \cos 2t)$.

7) $\dfrac{s}{1+s^2+s^4}$ $\quad Ans: \dfrac{2}{\sqrt{3}}\sin\dfrac{\sqrt{3}t}{2}\sinh\dfrac{t}{2}$

8) $\dfrac{21s-33}{(s+1)(s-2)^3}$ $\quad Ans: 2e^{-t} - 2e^{2t} + 6te^{2t} + \dfrac{3}{2}t^2e^{2t}$

11) $\dfrac{s^2}{(s-a)^3}$ $\quad e^{at}\left(1+2at+\dfrac{a^2t^2}{2}\right)$

12) $\dfrac{1}{s^2(s+1)}$ $Ans:$ $-1+t+e^{-t}$

13) $\dfrac{s}{s^4+4a^4}$ $Ans:$ $\dfrac{1}{2a^2}\sin at \sinh at.$

INVERSE BY CONVOLUTION THEOREM

Let $L[f_1(t)] = \phi_1(s)$ *and* $L[f_2(t)] = \phi_2(s)$ *then*

$$L^{-1}[\phi_1(s).\phi_2(s)] = \int_0^t f_1(u).f_2(t-u)\,du$$

where $f_1(t) = L^{-1}[\phi_1(s)]$ & $f_2(t) = L^{-1}[\phi_2(s)]$

Ex. Find the inverse Laplace transform of $\dfrac{(s+2)^2}{\left[(s+2)^2+2^2\right]^2}$

Solution: By convolution theorem, we have

$$L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = L^{-1}\left[\frac{(s+2)^2}{\left[(s+2)^2+2^2\right]^2}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s^2}{(s^2+2^2)^2}\right]$$

$$L^{-1}\frac{s}{s^2+2^2} = \cos 2t$$

$$\therefore L^{-1}\left[\frac{s}{s^2+2^2}\cdot\frac{s}{s^2+2^2}\right] = \int_0^t \cos 2u \cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left[\cos 2t + \cos(4u-2t)\right]du$$

$$= \frac{1}{2}\left[u\cos 2t + \frac{1}{4}\sin(4u-2t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right]$$

$$\therefore L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = \frac{e^{-2t}}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right]$$

$$= \frac{e^{-2t}}{4}\left[2t\cos 2t + \sin 2t\right]$$

Unsolved Problem

Find the inverse Laplace transform of following function.

1) $\dfrac{s^2}{(s^2-a^2)^2}$ $\quad Ans\ \dfrac{1}{2a}\left(\sinh at + at\cosh at\right)$

2) $\dfrac{(s+1)}{(s-2)^4(s+3)}$ $\quad Ans: \dfrac{e^{-3t}}{625} - e^{2t}\left(\dfrac{1}{625} - \dfrac{t}{125} + \dfrac{t^2}{50} - \dfrac{t^3}{30}\right)$

3) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ $\quad Ans: \dfrac{e^{-2t}}{4}[\sin 2t + 2t\cos 2t]$

4) $\dfrac{1}{(s-2)(s+2)^2}$ $\quad Ans: \dfrac{1}{16}[e^{2t} - e^{-2t} - 4te^{-2t}]$

5) $\dfrac{s^2}{(s^2+1)(s^2+4)}$ $\quad Ans: \dfrac{1}{3}(2\sin 2t - \sin t)$

6) $\dfrac{s}{(s^2+a^2)^2}$ $\quad Ans: \dfrac{t\sin at}{2a}$

7) $\dfrac{1}{(s^2+1)^3}$ $\quad Ans: \dfrac{1}{8}\{(3-t^2)\sin t - 3t\cos t\}$

8) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$ $\quad Ans: \dfrac{1}{5}(3\cos t + \sin t - 3e^{-t}\cos t - e^{-t}\sin t)$

HEAVISIDE UNIT STEP FUNCTION

LAPLACE TRANSFORM OF HEAVISIDE UNIT STEP FUNCTION:

$$L\{H(t-a)\} = \frac{1}{s}e^{-as}$$

$$L\{H(t)\} = \frac{1}{s}$$

$$L\{f(t)H(t-a)\} = e^{-as}L\{f(t+a)\}$$

$$L\{f(t)H(t)\} = L\{f(t)\}$$

Ex 1. Express the function $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi < t < 2\pi \end{cases}$ as Heaviside's unit step functions and find their Laplace transform. $\cos 3t \quad t \; 2$

Solution: By the formulae of Heaviside's unit step function, we have

$$f(t) = \cos t\,[H(t) - H(t-\pi)] + \cos 2t\,[H(t-\pi) - H(t-2\pi)] + \cos 3t\,H(t-2\pi)$$

$$= \cos t\,H(t) + [\cos 2t - \cos t]\,H(t-\pi) + [\cos 3t - \cos 2t]\,H(t-2\pi)$$

$$L\{f(t)\} = L\{\cos t\} + e^{-\pi s}L\{\cos 2(t+\pi) + \cos(t+\pi)\} + e^{-2\pi s}L\{\cos 3(t+2\pi) - \cos 2(t+2\pi)\}$$

$$= L\{\cos t\} + e^{-\pi s}L\{\cos 2t - \cos t\} + e^{-2\pi s}L\{\cos 3t - \cos 2t\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+4} - \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$$

Ex 2. Find the Laplace transform of $(1+2t-3t^2+4t^3)H(t-2)$.

Solution: Here

$$f(t) = 1+2t-3t^2+4t^3 \text{ and } a = 2$$

$$\therefore f(t+2) = 1+2(t+2)-3(t+2)^2+4(t+2)^3$$

$$= 4t^3+21t^2+38t+25$$

$$L\{f(t+2)\} = L\{4t^3+21t^2+38t+25\}$$

$$= 4\frac{3!}{s^4}+21\frac{2!}{s^3}+38\frac{1}{s^2}+25\frac{1}{s}$$

$$L\{f(t)H(t-2)\} = e^{-2s}\left[\frac{24}{s^4}+\frac{42}{s^3}+\frac{38}{s^2}+\frac{25}{s}\right]$$

Unsolved Problem

1) Prove that $L\{H(t-a)\} = \dfrac{e^{-as}}{s}$

2) Prove that $L\{H(t-a)f(t-a)\} = e^{-as}\bar{f}(s)$

3) Evaluate $L\left\{\sin t\, H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right)\right\}$ $\quad Ans: \dfrac{se^{-\frac{\pi s}{2}}}{s^2+1} - \dfrac{e^{-\frac{3\pi s}{2}}}{s}$

4) Evaluate

$L\{(1+2t-3t^2+4t^3)H(t-2)\}$ and hence evaluate $\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)H(t-2)\,dt$

$Ans: e^{-2s}\left[\dfrac{25}{s}+\dfrac{38}{s^2}+\dfrac{42}{s^3}+\dfrac{24}{s^4}\right], \dfrac{129}{e^2}$

6) $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$ $\quad Ans: \dfrac{e^{-s}}{s^2} - \dfrac{2e^{-2s}}{s^2} + \dfrac{e^{-3s}}{s^2}$

7) $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \sin t, & \pi<t<2\pi \end{cases}$ $\quad Ans: \dfrac{1}{s^2+1}\left[s - e^{-\pi s}(s+1) + e^{-2\pi s}\right]$

LAPLACE TRANSFORM OF DIRAC-DELTA (UNIT IMPULSE) FUNCTIONS

$L\{\delta(t-a)\} = e^{-as}$

$L\{\delta(t)\} = 1$

Ex. Find the Laplace transform of $\sin 2t\,\delta(t-2)$.

Solution: By taking $f(t) = \sin at$ and $a = 2$, we have

$$L\{\sin 2t\,\delta(t-2)\} = L\{f(t)\,\delta(t-2)\}$$
$$= e^{-as} f(a)$$
$$= e^{-2s}\sin 4$$

Unsolved Problem

1) Prove that $L\{\delta(t-a)\} = e^{-as}$

2) Prove that $L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$

3) Find $L\{t^4 H(t-2)\}$ Ans: $e^{-2s}\left(\frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{24}{s^5}\right)$

4) Prove that $L\{t^2 H(t-2) - \cosh t\,\delta(t-4)\} = \frac{2e^{-2s}}{s^3}\left(1 + 2s + 2s^2\right) - e^{-4s}\cosh 4$

5) Prove that $\int_0^\infty t^2 e^{-t}\sin t\,\delta(t-2)\,dt = 4e^{-2}\sin 2$

LAPLACE TRANSFORM OF PERIODIC FUNCTION

If f (t) is a periodic function of period T then $L[f(t)] = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\,dt$

Ex. Find the Laplace transform of $f(t) = \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}$ and f (t) is periodic with period 2a.

Solution: Since f (t) is periodic with period 2a.

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(1)\,dt + \int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right\}$$

$$= \frac{1}{s\left(1-e^{-2as}\right)}\left(1-e^{-as}\right)^2 = \frac{1}{s}\,\frac{1-e^{-as}}{1+e^{-as}}$$

$$= \frac{1}{s}\,\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}$$

$$= \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

Unsolved Problem

If $f(t)$ is a periodic function of period T then $L[f(t)] = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\,dt$

1) Find Laplace Transform of $f(t) = kt,\ 0<t<1$ $Ans: \frac{k}{s^2} - \frac{ke^{-s}}{s(1-e^{-s})}$

2) Find Laplace Transform of $f(t) = t,\ 0<t<1$

$= 0,\ 1<t<2$

3) Find Laplace Transform of

$f(t) = a\sin pt \text{ for } 0<t<\frac{\pi}{p}$

$= 0 \quad \text{for } \frac{\pi}{p}<t<\frac{2\pi}{p}$

$Ans: \frac{ap}{1-e^{-\frac{\pi s}{p}}}\cdot\frac{1}{s^2+p^2}$

and $f(t)$ *is periodic with the period* $\frac{2\pi}{p}$

INVERSE LAPLACE TRANSFORM BY HEAVISIDE UNIT STEP FUNCTION:

$L^{-1}\left[\frac{1}{s}\right] = H(t)$

$L^{-1}\left[\frac{1}{s}e^{-as}\right] = H(t-a)$

$L^{-1}\left[e^{-as}\phi(s)\right] = f(t-a)H(t-a)$

Ex. Find the inverse Laplace transform of $\frac{e^{4-3s}}{(s+4)^{5/2}}$

Solution: Here $\phi(s) = \frac{1}{(s+4)^{5/2}}$, We know that

$$f(t) = L^{-1}[\phi(s)] = L^{-1}\left[\frac{1}{(s+4)^{5/2}}\right]$$

$$= e^{-4t}L^{-1}\left[\frac{1}{s^{5/2}}\right] \quad (\textit{By first shifting theorem})$$

$$= e^{-4t}\frac{t^{3/2}}{\Gamma(5/2)},$$

$$= \frac{e^{-4t}t^{3/2}}{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma(1/2)}$$

$$= \frac{4e^{-4t}t^{3/2}}{3\sqrt{\pi}}$$

$$\therefore L^{-1}\left[\frac{e^{4-3s}}{(s-4)^{\frac{5}{2}}}\right] = \frac{4e^4}{3\sqrt{\pi}} e^{4t}(t-3)^{\frac{3}{2}} H(t-3)$$

Unsolved Problem

Evaluate

1) $L^{-1}\left[\frac{e^{4-3s}}{(s-4)^{\frac{5}{2}}}\right]$ $\quad$ $Ans: \frac{4}{3\sqrt{\pi}} e^{4(t-4)}(t-3)^{\frac{3}{2}} H(t-3)$

2) $L^{-1}\left[\frac{(s+1)e^{-s}}{s^2+s+1}\right]$ $\quad$ $Ans: e^{-\frac{(t-1)}{2}}\left[\cos\frac{\sqrt{3}(t-1)}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}(t-1)}{2}\right] H(t-1)$

3) $L^{-1}\left[\frac{e^{-\pi s}}{s^2(s^2+1)}\right]$ $\quad$ $Ans: (t-\pi) - \sin(t-\pi).H(t-\pi)$

4) $L^{-1}\left[\frac{se^{-\frac{s}{2}} + \pi e^{-s}}{s^2+\pi^2}\right]$ $\quad$ $Ans: \sin \pi t\left[H\left(t-\frac{1}{2}\right) - H(t-1)\right]$

IV Exercise can be solved based on following sample problem.

APPLICATIONS OF LAPLACE TRANSFORM : Application of Laplace transforms to solve ordinary differential equations :-

The following are steps to solve ordinary differential equations using the Laplace transform method

(A) Take the Laplace transform of both sides of ordinary differential equations.

(B) Express $Y(s)$ as a function of s.

(C) Take the inverse Laplace transform on both sides to get the solution.

Ex.1. Solve the following equation by using Laplace transform

$$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t, \quad given\ that\ \ y(0) = 1.$$

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform on both the sides, we get

$$L(y') + 2L(y) + L\left[\int_0^t y\,dt\right] = L[\sin t]$$

But

$$L(y') = sL(y) - y(0) = s\bar{y} - 1$$

$$L\left[\int_0^t y\,dt\right] = \frac{1}{s}L(y) = \frac{1}{s}\bar{y}$$

$$L[\sin t] = \frac{1}{s^2+1}$$

$\therefore$ The equation becomes

$$\Rightarrow \left(\frac{s^2+2s+1}{s}\right)\bar{y} = \frac{s^2+2}{s^2+1}$$

$$\Rightarrow \bar{y} = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

Let $$\frac{s(s^2+2)}{(s+1)^2(s^2+1)} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{cs+d}{s^2+1}$$

$$\Rightarrow s(s^2+2) = a(s+1)(s^2+1) + b(s^2+1) + (cs+d)(s+1)^2$$

Putting $s = -1$, $-3 = 2b$ $\Rightarrow$ $b = -\frac{3}{2}$

Putting $s = 0$, $0 = a + b + d$

Equating the coefficients of s^2 and s^3, we get

$0 = a + b + 2c + d$ and $1 = a + c$

$\Rightarrow b = -\frac{3}{2}$, $a + d = \frac{3}{2}$

and $a + 2c + d = \frac{3}{2}$

But $a + d = \frac{3}{2}$ $\Rightarrow 2c = 0$ $\Rightarrow c = 0$

$\therefore 1 = a + c$ and $c = 0$ $\Rightarrow a = 1$

$a + d = \frac{3}{2}$ and $a = 1$ $\Rightarrow d = \frac{1}{2}$

$\therefore a = 1,\ b = -\frac{3}{2},\ c = 0,\ d = \frac{1}{2}$

$$\therefore \bar{y} = \frac{1}{s+1} - \frac{3}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{(s^2+1)}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{s+1}\right] - \frac{3}{2}e^{-t}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s^2+1)}\right]$$

$$\Rightarrow y = e^{-t} - \frac{3}{2}te^{-t} + \frac{1}{2}\sin t$$

Ex. 2. Solve by using Laplace transform

$(D^2 + 2D + 5)Y = e^{-t}\sin t,$ when $y(0) = 0, y'(0) = 1$

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform on both the sides, we get

$$L(y'') + 2L(y') + 5L(y) = L\{e^{-t}\sin t\}$$

But

$$L(y') = sL(y) - y(0) = s\bar{y}$$
$$L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$$
$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

∴ The equation becomes

$$\Rightarrow \quad (s^2\bar{y} - 1) + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

$$(s^2+2s+5)\,\bar{y} = 1 + \frac{1}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}$$

$$\Rightarrow \quad \bar{y} = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $\bar{y} = \dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \dfrac{as+b}{(s^2+2s+5)} + \dfrac{cs+d}{(s^2+2s+2)}$

After simplication, we get

$$\Rightarrow \quad \bar{y} = \frac{2}{3}\frac{1}{(s^2+2s+5)} + \frac{1}{3}\frac{1}{(s^2+2s+2)} = \frac{2}{3}\frac{1}{(s+1)^2+2^2} + \frac{1}{3}\frac{1}{(s+1)^2+1^2}$$

Taking inverse Laplacetransform

$$\Rightarrow \quad y = \frac{2}{3}e^{-t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\} + \frac{1}{3}e^{-t}L^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$y = \frac{2}{3}e^{-t}\cdot\frac{1}{2}\sin 2t + \frac{1}{3}e^{-t}\sin t = \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

Ex.3. Solve

$$3\frac{dy}{dx} + 2y = e^{-x}, \quad y(0) = 5$$

Solution : Taking the Laplace transform of both sides, we get

$$L\left\{3\frac{dy}{dx} + 2y\right\} = L\{e^{-x}\}$$

$$3[sY(s) \cdot \quad y(0)] \cdot \quad 2Y(s) \cdot \quad \frac{1}{s \cdot \quad 1}$$

Using the initial condition, $y(0) = 5$ we get

$$3[sY(s) - 5] - 2Y(s) = \frac{1}{s-1}$$

$$(3s-2)Y(s) = \frac{1}{s-1} + 15$$

$$(3s-2)Y(s) = \frac{15s-16}{s-1}$$

$$Y(s) = \frac{15s-16}{(s-1)(3s-2)}$$

Writing the expression for $Y(s)$ in terms of partial fractions

$$\frac{15s-16}{(s-1)(3s-2)} = \frac{A}{s-1} + \frac{B}{3s-2}$$

$$\frac{15s-16}{(s-1)(3s-2)} = \frac{3As-2A+Bs-B}{(s-1)(3s-2)}$$

$$15s-16 = 3As-2A+Bs-B$$

Equating coefficients of s^1 and s^0 gives

$$3A + B = 15$$
$$2A + B = 16$$

The solution to the above two simultaneous linear equations is

$$A = -1$$
$$B = 18$$

$$Y(s) = \frac{-1}{s-1} + \frac{18}{3s-2}$$
$$= \frac{-1}{s-1} + \frac{6}{s-0.666667}$$

Taking the inverse Laplace transform on both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{-1}{s-1}\right\} + L^{-1}\left\{\frac{6}{s-0.666667}\right\}$$

Since

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

The solution is given by

$$y(x) = -e^{x} + 6e^{0.666667x}$$

Unsolved Problem :

1) Solve $3\frac{dy}{dt} - 2y = e^{3t}, \quad y = 1 \ at\ t = 0$ $\quad Ans: \frac{10}{11}e^{\frac{2}{3}t} + \frac{1}{11}e^{3t}$

2) Solve $\frac{dy}{dt} + 3y = 2 + e^{-t}, \quad y = 1\ at\ t = 0$ $\quad Ans: \frac{2}{3} + \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$

3) Solve $\frac{dx}{dt} + x = \sin wt, x(0) = 2$ $\quad Ans: \frac{1}{1+w^2}[(2w^2+w+2)e^{-t} - w\cos wt + \sin wt]$

4) Solve $(D^2 - 3D + 2)y = 4e^{2t}$, with $y(0) = -3, y'(0) = 5$ $\quad Ans: \; y = -7e^t + 4e^{2t} + 4te^{2t}$

5) Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 1$, where $y(0) = 0, y'(0) = 1$ $\quad Ans: \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t$

6) Solve $\frac{d^2y}{dt^2} + y = t$, where $y(0) = 1, y'(0) = 0$ $Ans: t + \cos t - \sin t$

$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$, given that $y(0) = 1$. $Ans:$ $y = e^{-t} \ldots 3 e^{-t} \ldots t \ldots \frac{1}{2}\sin t$

7) Solve $\frac{d^2y}{dt^2}$ ...

8) Solve $\frac{d^2y}{dt^2} + 9y = \cos 2t$ with $y(0) = 1$ & $y\left(\frac{\pi}{2}\right) = -1$ $y = \frac{1}{5}\cos 2t + \frac{1}{5}\cos 3t + \frac{4}{5}\sin 3t$

9) Solve $\frac{d^2y}{dt^2} - 4y = 3e^t$ where $y(0) = 0$ & $y'(0) = 3$ $Ans: e^t$ $-\frac{e^{2t}}{2}$ $-\frac{e^{-2t}}{2}$

Fourier Transform:

It is denoted by $F\,f(x) = F(s)$ and defined as

$$F\,f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inverse Fourier Transform:

It is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds$$

Fourier Sine Transform & Inverse Fourier Sine Transform:

$$F_s\, f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$$

IFST $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx\, ds$

Fourier Cosine Transform & Inverse Fourier Cosine Transform:

$$F_c\, f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$$

IFST $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx\, ds$

Q.1: Find Fourier Cosine Transform of the function $f(x) = e^{-x}, x \geq 0$

Solution: By definition of Fourier Cosine Transform

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos sx\, dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2}(-\cos sx - s\sin sx)\right]$$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+s^2}$$

Which is the required transform.

Q.2 : Find Inverse Fourier Sine Transform of $F_s(s) = \frac{s}{1+s^2}$

Solution: By definition of Inverse Fourier Sine Transform

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(s)\, Sn\, sx\, ds$$

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{1+s^2}\, Sn\, sx\, ds$$

Multiply and divide by s

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s^2}{s(1+s^2)}\, Sn\, sx\, ds$$

Add and subtract 1

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s^2+1-1}{s(1+s^2)}\, Sn\, sx\, ds$$

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{s}\, Sn\, sx\, ds - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s^2+1-1}{s(1+s^2)}\, Sn\, sx\, ds$$

$$f_s(x) = \frac{\pi}{2}\sqrt{\frac{2}{\pi}} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{s(1+s^2)}\, Sn\, sx\, ds$$

Differentiate bw.r.to x

$$\frac{df_s(x)}{dx} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{s(1+s^2)} \cos sx\, ds$$

Again Differentiate bw.r.to x

$$\frac{d^2 f_s(x)}{dx^2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{(1+s^2)} \sin sx\, ds$$

$$\frac{d^2 f_s(x)}{dx^2} = f_s(x)$$

$$(D^2 - 1) f_s(x) = 0$$

$$f_s(x) = a e^{x} + b e^{-x}$$

After solving , we get

$$f_s(x) = \sqrt{\frac{\pi}{2}}\, e^{-x}$$

Q.3: Find the Fourier sine transform of $\frac{e^{-ax}}{x}$. Hence find Fourier sine transform of $\frac{1}{x}$.

Solution: By definition of Fourier sine transform

$$F_s\, f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\, \sin sx\, dx$$

$$F_s f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\,\text{Sin}sx\,dx$$

Differentiating w.r.to s we get

$$\frac{dF_s x}{ds} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\,x\cos sx\,dx$$

$$\frac{dF_s x}{ds} = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}$$

Now integrating w.r.to s,

We have

$$F_s x = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right) + c$$

But for s=0

F(0)=0

And therefore c=0

Hence,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\,\text{Sin}sx\,dx = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

Taking a=0, both the sides

∞

Application of Fourier Transform in solving the ordinary differential equation:

The method of Fourier transform can be applied in solving some ordinary differential equation.

Procedure

Firstly take the Fourier transform of both sides of the given differential equation. Thus we shell get an algebraic equation. Simplify and get Fourier transform.

Now take the inverse Fourier transform. Apply initial condition and then required solution is obtained.

Remark: If we take the Fourier transform of e^x or e^{-x} then we shell find that the integral diverges and hence Fourier transform does not exist.

Example: Solve $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$

With Boundary conditions:1) V(0,t)=0

2) V(x,0)=e^{-x}, $x > 0$

3) V(x,t) is bounded , x>0,t>0

Solution: Taking Fourier Sine Transform of both the sides of given equation, we have

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial V}{\partial t}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 V}{\partial x^2}\sin sx\,dx$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^\infty V\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left\{\sin sx\frac{\partial V}{\partial x} - s\int_0^\infty \frac{\partial V}{\partial x}\cos sx\,dx\right\}$$

$$\Rightarrow \frac{dV_s}{dt} = \frac{2}{\pi}\{s[V \cos sx] - s^2 \int_0^\infty V \sin sx \, dx\}$$

$$\Rightarrow \frac{dV_s}{dt} = \frac{2}{\pi}\{s[V \cos sx] - s^2 \int_0^\infty V \sin sx \, dx\}$$

By using condition, we have

$$\Rightarrow \frac{dV_s}{dt} + 2s^2 V_s = 0$$

$$\Rightarrow V_s = A e^{-2s^2 t}$$

Now at t=0,

$$\Rightarrow V_s = A$$

$$\Rightarrow V_s(s,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty V(x,0) \sin sx \, dx$$

$$\Rightarrow V_s(s,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$$\Rightarrow V_s(s,0) = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2}$$

Comparing both values of V_s, we get

$$\Rightarrow A = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2}$$

Therefore,

$$\Rightarrow V_s = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2} e^{-2s^2 t}$$

Now taking inverse fourier sine transform,

$V(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2} e^{-2s^2 t} \sin sx \, ds$

Module 5

Concept of Probability: Probability Mass Function, Probability density function.

Discrete Distribution: Binomial, Poisson's.

Continuous Distribution: Normal distribution, Exponential distribution.

Concept of Probability

Random Variable:

A real valued function defined on a sample space is called a Random Variable or a Discrete Random Variable.

A Random Variable assumes only a set of real values & the values which variable takes depends on the chance.

For Example:

a) X takes only a set of discrete values 1,2,3,4,5,6.
b) The values which x takes depends on the chance.

The set values 1,2,3,4,5,6 with their probabilities 1/6 is called the **Probability Distribution** of the variate x.

Continuous Random Variable:

When we deal with variates like weights and temperature then we know that these variates can take an infinite number of values in a given interval. Such type of variates are known as **Continuous Random Variable.**

OR

A Variable which is not discrete i.e. which can take infinite number of values in a given interval $a \le x \le b$, is called **Continuous Random Variable**

Example: Sinx between $(0,\pi)$, x is a **Continuous Random Variable.**

Probability Mass Function:

Suppose that X: $S \to A$ is a discrete random variable defined on a sample space S. Then the probability mass function $p(x)$: $A \to [0, 1]$ for X is defined as:

a) $P(x_i) \ge 0$,for every i=1,2,3..
b) $\sum_{i=1}^{\infty} p(x_i) = 1$

The sum of probabilities over all possible values of a discrete random variables must be

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equal to 1.

Thinking of probability as mass helps to avoid mistakes since the physical mass is conserved as is the total probability for all hypothetical outcomes *x*.

- The following exponentially declining distribution is an example of a distribution with an infinite number of possible outcomes—all the positive integers:

$$p(x_i) = \frac{1}{2^i}, i = 1,2,3, \ldots$$

Despite the infinite number of possible outcomes, the total probability mass is 1/2 + 1/4 + 1/8 + ... = 1, satisfying the unit total probability requirement for a probability distribution.

Probability Density Function:

Let X be a continuous random variable and let the probability of X falling in the infinite interval $(x - \frac{1}{2}dx, x + \frac{1}{2}dx)$ be expressed by f(x)dx, i.e.

P$(x - \frac{1}{2}dx, x + \frac{1}{2}dx)$= f(x)dx

Where f(x) is a continuous function of X & satisfies the following condition:

a) $f(x) \geq 0$
b) $\int_a^b f(x)dx = 1$ f $a \leq x \leq b$

Then the function is called probability density function of the continuous random variable X.

Continuous Probability Distribution:

The Probability distribution of continuous random variate is called the continuous probability distribution and it is expressed in terms of probability density function.

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Cumulative Distribution Function:

The probability that the value of a random variate X is 'x or less than x' is called the Cumulative distribution function of X and is usually denoted by F(x). and it is given by

$F(x)=P(X\leq x)=\sum_{x\leq x} p(x)$

The cumulative distribution functionof a continuous random variable is given by

$$F(x) = P(\leq x) = \int_{-\infty}^{x} f(x)dx$$

Some properties of Cumulative Distribution Function:

a) $F(-\infty) = 0$
b) F(X) is non-decreasing function
c) For a distribution variate

$$P(a < x < b) = F(b) - F(a)$$

d) $F(+\infty) = 1$
e) F(x) is a discontinuous function for a discontinuous variate and F(x) is continuous function foe a continuous variate.

Examples:

1) Let X be a random variable with PDF given by

$$f(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & otherwse \end{cases}$$

a. Find the constant c

. b. Find *EX* and Var(*X*)

. c. Find *P*(*X*≥12).

Solution: **To find *c*, we can use** $\int_{-\infty}^{\infty} f(x)dx$=1:

$$1=\int_{-1}^{1} cx^2dx$$

$$1=\frac{2}{3}c$$

Therfore C=$\frac{3}{2}$

To find EX, we can write $\int_{-1}^{1} xf(x)dx$=o

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In fact, we could have guessed $EX=0$ because the PDF is symmetric around $x=0$. To find Var(X), we have

Var(X)

$=EX2-(EX)2=EX2$

$=\int_{-1}^{1} xf(x)dx$

$=3/5$

To Find $P(X\geq 12)$:

$P(X\geq 12)=\frac{3}{2}\int_{\frac{1}{2}}^{1} x^2 dx$

$=7/16.$

Example: If $f(x)=cx^2, 0<x<1$. Find the value of c and determine the probability that $\frac{1}{3}<x<\frac{1}{2}$

Solution: By property of p.d.f. we have, = 1

So $\int_0^1 cx^2\, dx = 1$, or $c[\frac{x^3}{3}]_0^1 = 1$, so $c = 3$

Consequently $f(x) = 3x^2 : 0< x < 1$

Again $P(\frac{1}{3}< X <\frac{1}{2}) = \int_{\frac{1}{3}}^{\frac{1}{2}} 3x^2\, dx = \frac{1}{8}-\frac{1}{27}=\frac{19}{216}$

Example: For the distribution $dF = sinx\, dx, 0 \leq x \leq {}^{\pi}/_{2.}$ Find Mode and Median.

Solution: Here $f(x) = sinx$, $0\leq x \leq \frac{\pi}{2}$

(a) For Mode: $f'(x) = 0$ & $f''(x) < 0$, $f'(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$ &

$[f''(x)]_{x=\frac{\pi}{2}} = -1< 0$, Hence mode = $\frac{\pi}{2}$

Let M_d be median, then $\int_0^{M_d} sinx\, dx = \frac{1}{2} \Rightarrow M_d=\pi/3$

(b) Mean = $\mu_1' = \int_0^{\frac{\pi}{2}}(x-0)f(x)dx = \int_0^{\frac{\pi}{2}} xsinx\, dx = 1$ &

Variance = $\mu_2 = \int_0^{\frac{\pi}{2}}(x-1)^2 sinx\, dx = \pi - 3$

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Continuous random variable – infinite number of values with no gaps between the values. [You might consider drawing a line, the sweeping hand on a clock, or the analog speedometer on a car.]

In this section, we restrict our discussion to discrete probability distributions. Each probability distribution must satisfy the following two conditions.

1. $\sum P(x) = 1$ where x assumes all possible values of the random variable

2. $0 \le P(x) \le 1$ for every value of x

As we found the mean and standard deviation with data in descriptive statistics, we can find the mean and standard deviation for probability distributions by using the following formulas.

1. $\mu = \sum [x \cdot P(x)]$ **mean** of probability distribution

2. $\sigma^2 = \sum [(x - \mu)^2 \cdot P(x)]$ **variance** of probability distribution

3. $\sigma^2 = \sum [x^2 \cdot P(x)] - \mu^2$ **variance** of probability distribution

4. $\sigma = \sqrt{\sum [x^2 \cdot P(x)] - \mu^2}$ **standard deviation** of probability distribution

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Theoretical Distributions

Definition : When frequency distribution of some universe are not based on actual observation or experiments , but can be derived mathematically from certain predetermined hypothesis , then such distribution are said to be theoretical distributions.

Types of Theoretical Distributions: Following two types of Theoretical Distributions are usually used in statistics:

1) Discrete Probability Distribution
    a) Binomial Distribution
    b) Poisson Distribution
2) Continuous Probability Distribution
    Normal Distribution

Binomial Distribution:

1. The procedure has a **fixed number of trials**. [n trials]

2. The trials must be **independent**.

3. Each trial is in **one of two mutually exclusive categories**.

4. The **probabilities remain**

Notations:

P(success) = P(S) = p probability of success in one of the n trials

P(failure) = P(F) = 1 – p = q probability of failure in one of the n trials

n = fixed number of trials; x = number of successes, where $0 \le x \le n$

P(x) = probability of getting exactly x successes among the n trials

$P(x \le a)$ = probability of getting x-values less than or equal to the value of a.

$P(x \ge a)$ = probability of getting x-values greater than or equal to the value of a.

NOTE: Success (failure) does not necessarily mean good (bad).

Formula for Binomial Probabilities: $P(x) = \dfrac{n!}{(n-x)!x!} p^x q^{n-x} \quad for\ x = 0,1,2,\ldots,n$

Factorial definition: $n! = n(n-1)(n-2)\cdots 2\cdot 1$; $0! = 1$; $1! = 1$

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Example (Formula): Find the probability of 2 successes of 5 trials when the probability of success is 0.3.

$$P(x = 2) = \frac{5!}{(5-2)!2!} 0.3^2 0.7^{5-2} = \frac{5 \cdot 4 \cdot 3!}{3! \, 2!}(0.09)(0.343) = 10(0.03087) = 0.3087$$

Moment about the origin:

1) **First moment about the origin:**

$$\mu_1' = \sum_{r=0}^{n} r \cdot (nCr)\, p^r q^{n-r}$$
$$= np$$

2) Second moment about the origin:

$$\mu_2' = \sum_{r=0}^{n} r^2 \cdot (nCr)\, p^r q^{n-r}$$
$$= npq + n^2p^2$$

Moment about the Mean:

1) First moment about the mean is 0.
2) Second moment about the mean or variance is given by=npq

   Standard deviation=$\sqrt{npq}$

Examples:

(1) Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?

Solution:) We know that when a die is thrown , the probability to show a 5 or 6 = 2/6 = 2/3 = p (say)

q= 1-p = 1- (1/3) = 2/3

The probability to show a 5 or 6 in at least 3 dice

= $\sum_{x=3}^{6} p(x)$ = p(3) + p(4) + p(5) +p(6), where p(x) is the probability to show 5 or 6

$= {}^6C_3 q^3p^3 + {}^6C_4\, p^4q^2 + {}^6C_5\, p^5q + {}^6C_6\, p^6 = \frac{233}{729}$ = p (say)

SO the required no. = np = 233

(2) The mean and variance of a binomial variate are 16 & 8. Find i) P (X= 0)

ii) P(X ≥ 2)

Mean = np = 16

Variance = npq = 8

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$npq / np = 8/16 = 1/2$

ie, $q = \frac{1}{2}$

$p = 1 - q = \frac{1}{2}$

$np = 16$ ie, $n = 32$

i) $P (X = 0) = nC_0\, p^0 q^{n-0}$

$= (\frac{1}{2})^0 (1/2)^{32}$

$= (1/2)^{32}$

ii) $P(X \geq 2) = 1 - P (X < 2)$

$= 1 - P(X = 0 ,1)$

$= 1 - P(X = 0) - P(X = 1)$

$= 1- 33\ (1/2)^{32}$

3) Six dice are thrown 729 times.Howmany times do you expect atleast 3 dice to show a 5 or 6 ?

Solution : Here $n = 6$,$N = 729$

$P(x \geq 3) = 6C_x\, p^x q^{n-x}$

Let p be the probability of getting 5 or 6 with 1 dice

ie, $p = 2/6 = 1/3$

$q = 1 - 1/3 = 2/3$

$P(x \geq 3) = P (x = 2,3,4,5,6)$

$= p(x=3)+p(x=4)+p(x=5)+p(x=6)$

$=0 .3196$

number of times = 729*0.3196 = 233

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4) A basket contains 20 good oranges and 80 bad oranges . 3 oranges are drawn at random from this basket . Find the probability that out of 3 i) exactly 2 ii)atleast 2 iii)atmost 2 are good oranges.

Solution: Let p be the probability of getting a good orange

$$ie,\ p = \frac{80C_1}{100C_1} = 0.8$$

$$q = 1 - 0.8 = 0.2$$

i) $p(x=2) = 3C2\,(0.8)^2(0.2)^1 = 0.384$

ii) $p(x\geq 2) = P(2) + p(3) = 0.896$

iii) $p(x\leq 2) = p(0) + p(1) + p(2) = 0.488$

5) In a sampling a large number of parts manufactured by a machine , the

mean number of defective in a sample of 20 is 2. Out of 1000 such

6) samples howmany would expected to contain atleast 3 defective parts.

$n=20 \quad np = 2$

ie , $p = 1/10 \qquad q = 1-p = 9/10$

$$p(x \geq 3) = 1 - p(x < 3)$$

$$= 1 - p(x = 0,1,2) = 0.323$$

Number of samples having at least 3 defective parts = 0.323 * 1000

= 323

The process of determining the most appropriate values of the parameters from the given observations and writing down the probability distribution function is known as fitting of the binomial distribution.

Problems

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1) Fit an appropriate binomial distribution and calculate the theoretical distribution

x :	0	1	2	3	4	5
f :	2	14	20	34	22	8

Here n = 5 , N = 100

Mean = $\frac{\Sigma x f}{\Sigma f}$ = 2.84

np = 2.84

p = 2.84/5 = 0.568

q = 0.432

$p(r) = 5C_r\,(0.568)^r\,(0.432)^{5-r}$, r = 0,1,2,3,4,5

Theoretical distributions are

R	p(r)	N* p(r)
0	0.0147	1.47 = 1
1	0.097	9.7 =10
2	0.258	25.8 =26
3	0.342	34.2 =34
4	0.226	22.6 =23
5	0.060	6 =6
		Total = 100

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Poisson Distribution :.

The **Poisson distribution** is a discrete distribution. It is often used as a model for the number of events (such as the number of telephone calls at a business, number of customers in waiting lines, number of defects in a given surface area, airplane arrivals, or the number of accidents at an intersection) in a specific time period.

The mean is λ. The variance is λ.

Therefore the P.D. is given by

$P(r) = \dfrac{e^{-m}\, m^r}{r!}$ where r=0,1,2,3...

m is the parameter which indicates the average number of events in the given time interval.

Poisson distribution examples

1.

The number of road construction projects that take place at any one time in a certain city follows a Poisson distribution with a mean of 3. Find the probability that exactly five road construction projects are currently taking place in this city. (0.100819)

2. The number of road construction projects that take place at any one time in a certain city follows a Poisson distribution with a mean of 7. Find the probability that more than four road construction projects are currently taking place in the city. (0.827008)

3. The number of traffic accidents that occur on a particular stretch of road during a month follows a Poisson distribution with a mean of 7.6. Find the probability that less than three accidents will occur next month on this stretch of road. (0.018757)

4. The number of traffic accidents that occur on a particular stretch of road during a month follows a Poisson distribution with a mean of 7. Find the probability of observing exactly three accidents on this stretch of road next month. (0.052129)

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5. The number of traffic accidents that occur on a particular stretch of road during a month follows a Poisson distribution with a mean of 6.8. Find the probability that the next two months will both result in four accidents each occurring on this stretch of road. (0.009846)

Examples:In a certain factory turning razor blades, there is a small chance (1/500) for any blade to be defective. The blades are in packets of 10. Use Poisson's distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets.
Solution: Here p = 1/500, n = 10 , N = 10,000 so m = np = 0.02

Now $e^{-m} = e^{-0.02} = 0.9802$

The respective frequencies containing no defective, 1 defective & 2 defective blades are given

As follows

Ne^{-m}, $Ne^{-m}.m$, $Ne^{-m}.\frac{1}{2}m^2$

i.e. 9802 ; 196; 2

Normal Distribution:

The normal (or Gaussian) distribution is a continuous probability distribution that frequently occurs in nature and has many practical applications in statistics.

Characteristics of a normal distribution

- Bell-shaped appearance
- Symmetrical
- Unimodal
- Mean = Median = Mode
- Described by two parameters: mean (μ_x) and standard deviation (σ_x)
- Theoretically infinite range of x: ($-\infty < x < +\infty$)
- The normal distribution is described by the following formula:

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

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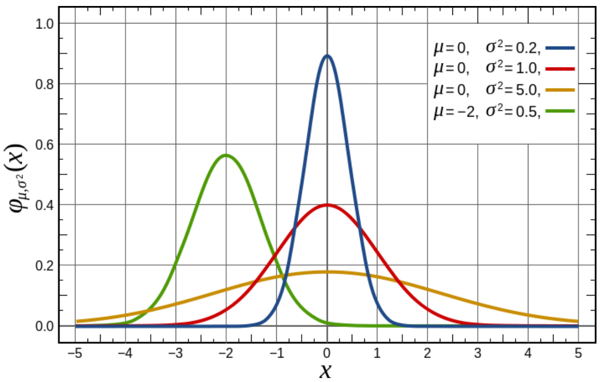
where the function f(x) defines the probability density associated with X = x. That is, the above formula is a probability density function

Because μ_x and σ_x can have infinitely many values, it follows there are infinitely many normal distributions:



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A standard normal distribution is a normal distribution rescaled to have $\mu_x = 0$ and $\sigma_x = 1$. The *pdf* is:

$$f(z;0,1) = \frac{1}{\sqrt{2\Box}} e^{\frac{z^2}{2}} \quad ' \; < x < \; '$$

The ordinate of the standard normal curve is no longer called *x*, but *z*.

For a normal curve, approximately 68.2%, 95.4%, and -99.7% of the observations fall within 1, 2, and 3 standard deviations of the mean, respectively.

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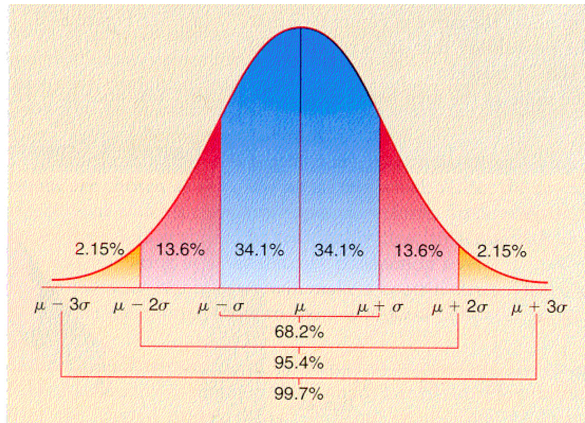

Areas Under the Normal Curve

By standardizing a normal distribution, we eliminate the need to consider μ_x and σ_x; we have a standard frame of reference.

Areas Under the Standard Normal Curve

X (x values) of a normal distribution map into Z (z-values) of a standard normal distribution with a 1-to-1 correspondence.

If *X* is a normal random variable with mean μ_x and σ_x, then the standard normal variable (normal deviate) is obtained by:

$$= \frac{-\mu}{\sigma}$$

Example 1: What is the probability that Z falls z = 1.11 and z = 2.34?

Pr(1.11 < *z* < 2.34)

= area from *z* = 2.34 to *z* = 1.11

= area from (–∞ to *z* = 2.34) minus area from (-∞ to *z* = 1.11)

= .9904 – .8665 = .1239

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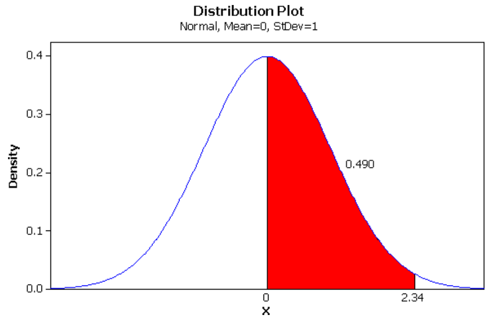

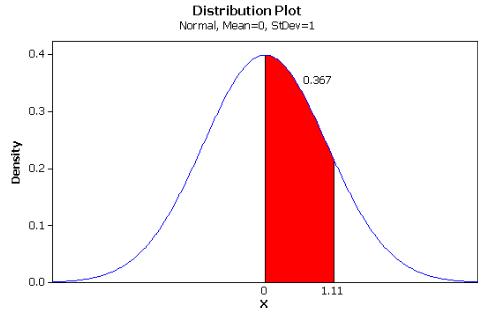

Note: figures above should also shade region from –inf to 0.

Table of the standard normal distribution values (*z* ' 0)

Practice Question:

Q.1 (a): From a pack of 52 cards ,6 cards are drawn at random. Find the probability of the following events:

(a) Three are red and three are black cards (b) three are king and three are queen

Q.1 (b): Out of 800 families with 4 children each, how many families would be expected to

have?

(a) 2 boys & 2 girls (b) at least one boy (c) no girl (d) at most 2 girls?

Assume equal probabilities for boys and girls.

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Q.2(a):One bag contain 4 white,6 red & 15 black balls and a second bag contains 11 white, 5 red& 9 black balls. One ball from each bag is drawn. Find the probability of the following events:(a) both balls are white (b) both balls are red (c) both balls are black (d)both balls are of the same colour.

Q.2(b): Find mean and variance of Binomial distribution .

Q.3(a): In a bolt factory, machines A,B & C manufacture respectively 25%,35% & 40% of the total. Of their output 5,4,2 percent of defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machine A, B or C.

Q.3(b): In a normal distribution 31% of the items are under 45and 8% are over 64. Find the mean and S.D. of the distribution.

Q.4(a): A sample of 100 dry battery cells tested to find the length of life produced the following results:

μ=12hours , σ=3 hours

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life.

(i) more than 15 hours (ii) less than 6 hours (iii) between 10 & 14 hours

Q.4(b): Define the following:

(a) Random Variable (b) Mathematical Expectation (c) Moment generating function

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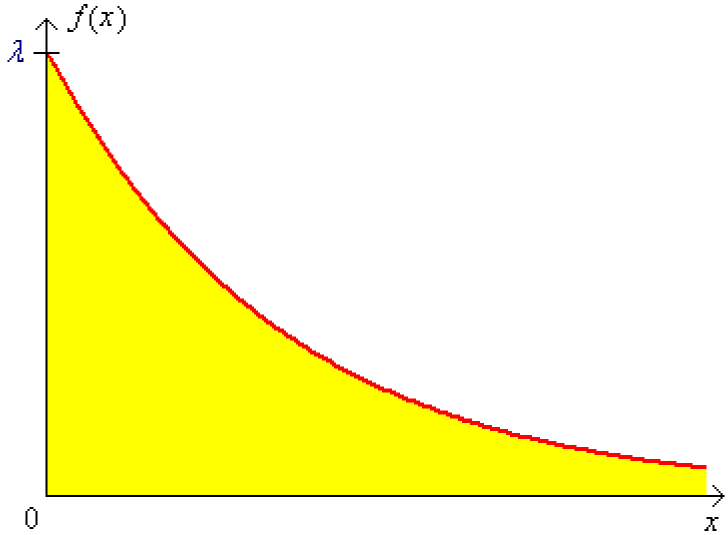
The Exponential Distribution

This continuous probability distribution often arises in the consideration of lifetimes or waiting times and is a close relative of the discrete Poisson probability distribution.

The probability density function is

$$f(x) = \lambda e^{-\lambda x} \quad (x \geq 0)$$

$$= 0 \quad (x < 0)$$



The cumulative distribution function is

$$F(x) = \mathrm{P}[X \leq x] = \int_{-\infty}^{x} f(t)\,dt = 0 + \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x} \quad (x \geq 0)$$

$$\therefore \quad \mathrm{P}[X > x] = e^{-\lambda x} \quad (x \geq 0)$$

Also $\mu = \mathrm{E}[X] = \frac{1}{\lambda}$

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'

and

' = '

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Reason:

E[X^2] – (E[X])^2

V[X] =

OR

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Example

The random quantity X follows an exponential distribution with parameter ' = 0.25 .

Find ' , ' and P[X > 4] .

$$\frac{1}{\,} \quad \frac{1}{.25} \quad \mathbf{4}$$

$$P\; X \; 4 \quad e^{\;x} \quad e^{\;1\;4} \quad e^{\;1} \quad .367879$$

Note: For *any* exponential distribution, P[X > '] ' .368 .

Example

The waiting time T for the next customer follows an exponential distribution with a mean waiting time of five minutes. Find the probability that the next customer waits for at most ten minutes.

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$$\lambda = \frac{1}{\mu} = \frac{1}{5} = .2$$

$$P[T \le 10] = F(10) = 1 - P[T > 10] = 1 - e^{-\frac{1}{5}\cdot 10}$$

$$= 1 - e^{-2} = 1 - .135335$$

- $P[T \le 10] \approx \underline{\underline{.865}}$

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Note:

$$P[X > \mu + 2\sigma] = e^{-\lambda(\mu + 2\sigma)} = e^{-\lambda((1/\lambda)+(2/\lambda))} = e^{-3} = .049787$$

Therefore $P[X > \mu + 2\sigma] \approx 5.0\%$ for <u>all</u> exponential distributions.

Also $\mu - \sigma = \frac{1}{\lambda} - \frac{1}{\lambda} = 0 \Rightarrow P[X < \mu - \sigma] = 0 = P[X < \mu - 2\sigma]$

Therefore $P[\,|X - \mu| > 2\sigma\,] \approx 5.0\%$, a result similar to the normal distribution, except that *all* of the probability is in the upper tail only.
