

Physics 742:
Statistical Mechanics & Condensed Matter
Notes on statistical mechanics and complex networks

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1 Thermodynamics - a refresher

We'll start with a quick overview of some of the important concepts from thermodynamics. Deriving methods for linking these macroscopic concepts to the microscopic state of a system is the original motivation behind much of statistical mechanics. This section will also recap some content from 315/334.

1.1 Macroscopic laws of thermodynamics

In thermodynamics we study a system — the part of the world that we are interested in — that is separated from its surroundings — the rest of the universe — by some boundary.

We typically classify systems into three types based on the properties of walls that define the system boundary:

- **Adiabatic walls/isolated system:** no energy or matter is transferred in or out of the system.
- **Diathermal walls/closed system:** no matter can be transferred in or out of the system but heat can flow through the walls.
- **(Semi-)Permeable walls/open system:** in addition to heat, one or more chemical species can be transferred through the walls.

The four laws of thermodynamics

Only four laws are required to construct the relationships that control much of classical thermodynamics. These are, in brief:

- **Zeroth law** Defines temperature and thermal equilibrium.
- **First law** Formulates the principle of conservation of energy for thermodynamic systems. (Energy is conserved)
- **Second law** Entropy increases; heat spontaneously flows from high to low temperatures.
- **Third law** The absolute zero of temperature is not attainable.

We'll look the first two laws in a bit more detail and then make use of the second law to derive some familiar properties of heat.

1.2 The zeroth law:

If two systems are each in thermal equilibrium with a third system, then they are also in thermal equilibrium with each other. This implies that they have some property in common. We call this property *temperature*. (In the language of mathematics we would say that thermal equilibrium is a transitive property.)

It is worth noting that thermal equilibrium is not the same as thermodynamic equilibrium. For the latter we also need mechanical equilibrium ($p_1 = p_2$ — there are no unbalanced forces) and chemical equilibrium ($\mu_1 = \mu_2$ — equal rates of reaction).

1.3 The first law:

Heat and work are both forms of energy. The first law of thermodynamics says that energy remains constant for a (collection of) system(s) isolated from their surroundings. We denote work done *on* the system by the surroundings as $W > 0$, similarly, heat supplied to the system is denoted $Q > 0$.)

When considering the change in energy ΔE of a system it is necessary to consider both work and heat. For example, if system A gains energy from system B , $\Delta E_A = -\Delta E_B \implies \Delta E_A + \Delta E_B = 0$. Since, in general, $\Delta E_A \neq W_{B \rightarrow A}$ there will also be a heat flow $Q_{B \rightarrow A}$, due to a temperature difference between systems A and B . So,

$$\Delta E_A = W_{B \rightarrow A} + Q_{B \rightarrow A}$$

$$\Delta E_B = W_{A \rightarrow B} + Q_{A \rightarrow B}.$$

Energy conservation gives

$$\underbrace{(W_{A \rightarrow B} + W_{B \rightarrow A})}_{\text{Work done by the composite system}} + \underbrace{(Q_{A \rightarrow B} + Q_{B \rightarrow A})}_{\text{Heat flow in the composite system}} = 0$$

In an isolated system the first law gives a sort of balance sheet for energy: $W_{A \rightarrow B} + W_{B \rightarrow A} = 0$ and $Q_{A \rightarrow B} + Q_{B \rightarrow A} = 0$.

1.4 Internal energy

The *internal energy* of a system is the energy associated with the internal degrees of freedom of the system. This includes energy that might come from the motion of particles within the system, along with energy due to internal degrees of freedom that might be associated with the molecules that comprise those particles, for example rotational and vibrational degrees of freedom.

The equation of state for internal energy is often written as $E = E(S, V, N)$ where S is entropy, V is the volume of the system and N is the number of particles. (Note - all of these are extensive variables.) Looking at the first of these, we can calculate the total derivative of the E . We get

$$dE = \frac{\partial E}{\partial S} dS + \frac{\partial E}{\partial V} dV + \frac{\partial E}{\partial N} dN.$$

The partial derivatives above must be physically relevant. We identify these as

$$T = \frac{\partial E}{\partial S}, \quad p = -\frac{\partial E}{\partial V}, \quad \text{and} \quad \mu = \frac{\partial E}{\partial N};$$

temperature, pressure and chemical potential, respectively. (Note the sign on the term for pressure.) This also suggests that we could write the equation of state for internal energy using the intensive variables that we found as the dual variables of S and V ; namely $E = E(T, p, N)$.

1.5 Intensive and extensive variables

Intensive variables control the state of a system but are independent of the system size. E.g. T, p .

Extensive, or additive, variables are proportionate to the size of a system (i.e. they depend on N). E.g. V, N, S . One important extensive variable is the entropy (or disorder) of a system. As we just showed above, entropy is related to temperature by the internal energy E .

It is often helpful to think of intensive and extensive variables as coming in dual pairs with the intensive variable being the derivative of the internal energy with respect to the corresponding extensive variable given all other extensive variables are held constant. For example, $T = \frac{\partial E}{\partial S}|_{V, N}$.

Given two systems 1 and 2, an extensive variable for the composite system $1 \cup 2$ can be found by simply adding the individual extensive variables. E.g. $N_{1 \cup 2} = N_1 + N_2$, $V_{1 \cup 2} = V_1 + V_2$, $E_{1 \cup 2} = E_1 + E_2$ (E = internal energy).

Actually, the case of internal energy is not entirely correct since there is often an interaction term between the two systems at the interface. I.e. $E_{1 \cup 2} = E_1 + E_2 + E_{int}$. Since E_{int} depends on the interface between the two systems it scales like an area as a function of system size, while E_1 and E_2 will scale like a volume, so typically $\frac{E_{int}}{E_1 + E_2} \rightarrow 0$ as the system size gets big. E.g. an oil water interface (insert an image?).

But this assumption clearly depends on the structural details of the interface. Imagine the oil-water interface in an emulsion like mayonnaise — in this case the ratio $\frac{E_{int}}{E_1 + E_2}$ is more like $\mathcal{O}(1)$. Similarly, if there are significant long range interactions, it may not be true that we can treat the extensive variables as being truly additive.

1.6 Heat capacity:

Heat flowing into a system causes a change in temperature (except in the case of phase transitions). The amount of change in temperature for a given amount of heat is the *heat capacity* of the system. The heat

capacity of a system depends, in part on the experimental conditions of the system under consideration. Two important cases are constant volume and constant pressure.

For constant volume we have

$$C_V = \left. \frac{\delta Q}{dT} \right|_V$$

and similarly, for constant pressure,

$$C_p = \left. \frac{\delta Q}{dT} \right|_p.$$

The first law of thermodynamics (conservation of energy) implies that $\delta Q = dE - \delta W = dE + pdV$ (since $W = \int F dx$ and $F = p \times \text{Area} \implies W = p dV$). So, since $dV = 0$ for constant volume,

$$C_V = \left. \frac{\delta Q}{dT} \right|_V = \left. \frac{\partial E}{\partial T} \right|_V.$$

In the constant pressure case

$$C_p = \left. \frac{\delta Q}{dT} \right|_p = \left. \frac{\partial E}{\partial T} \right|_p + p \left. \frac{\partial V}{\partial T} \right|_p.$$

If we have constant pressure with a change in heat, it implies that there is a change in volume. We define the *volumetric thermal expansivity* as $\alpha_p = \frac{1}{V} \left. \frac{\partial V}{\partial T} \right|_p$ and hence $V \alpha_p = \left. \frac{\partial V}{\partial T} \right|_p$.

We can now write $C_p = \left. \frac{\partial E}{\partial T} \right|_p + \alpha_p p V$.

We can also find an expression for C_p without resorting to introducing thermal expansivity. Constant pressure implies that $d(pV) = p dV$ so that $\delta Q = dE - dW = dE + p dV = d(E + pV)$. We define the composite quantity $E + pV$ as *enthalpy*, H .

The infinitesimal for enthalpy is $dH = \underbrace{dE + p dV}_{\delta Q} + V dp = \delta Q + V dp|_p$. In the constant pressure case,

the last term is zero, so we get

$$C_p = \left. \frac{\delta Q}{dT} \right|_p = \left. \frac{\partial H}{\partial T} \right|_p.$$

1.7 The central problem of thermodynamics

Given the initial state of equilibrium for several thermodynamic systems that are allowed to interact, we want to be able to determine the final thermodynamic state of equilibrium. The boundaries of the systems — adiabatic, closed, open — determine the types of interactions that are allowed, while the four laws of thermodynamics (and the first and second law in particular) determine how the composite system evolves. We want to be able to describe a final thermodynamic equilibrium state from amongst the space of all possible states for the composite system.

Entropy plays a special role in this problem due to the entropy postulate — the second law of thermodynamics.

The entropy postulate: there exists a function S of the extensive variables X_1, X_2, \dots, X_r , called the entropy, that assumes a maximum value for a state of equilibrium among the space of possible states.

Entropy has the following properties:

1. Extensivity: If 1 and 2 are thermodynamic systems then $S_{1 \cup 2} = S_1 + S_2$.
2. Convexity: If $X^1 = (X_0^1, X_1^1, \dots, X_r^1)$ and $X^2 = (X_0^2, X_1^2, \dots, X_r^2)$ are two thermodynamic states of the same system then for $0 \leq \alpha \leq 1$

$$S((1 - \alpha)X^1 + \alpha X^2) \geq (1 - \alpha)S(X^1) + \alpha S(X^2).$$

That is, the entropy of a linear combination of states for a single system is greater than or equal to the same linear combination of entropies of the individual states. A consequence of this is that if we take derivatives with respect to α and then evaluate at $\alpha = 0$ we get

$$\text{LHS: } \frac{\partial}{\partial \alpha} S((1 - \alpha)X^1 + \alpha X^2) = \sum_{i=0}^r \frac{\partial S}{\partial X_i} (X_i^2 - X_i^1)$$

and

$$\text{RHS: } \frac{\partial}{\partial \alpha} [(1 - \alpha)S(X^1) + \alpha S(X^2)] = S(X^2) - S(X^1).$$

$$\text{Hence, } \sum_{i=0}^r \frac{\partial S}{\partial X_i} (X_i^2 - X_i^1) \geq S(X^2) - S(X^1).$$

Mathematically, this means that the entropy surface (as a function of the other extensive variables) is always below the tangent plane of a point on the surface.

3. Monotonicity: $S(E, X_1, \dots, X_r)$ is a monotonically increasing function of the internal energy E . That is, $\frac{\partial S}{\partial E}|_{X_1, \dots, X_r} = \frac{1}{T} > 0$.

Using these three properties, it is possible to find the final equilibrium thermodynamic state amongst the space of possible states of a system. The equilibrium state is the state with maximum entropy that satisfies the constraints on the system.

An example: Consider two closed systems, 1 and 2, in thermal contact such that they can exchange energy, but nothing else (i.e. no other extensive quantities change). The space of possible states is defined by

$$E^1 + E^2 = X_0^1 + X_0^2 = E = \text{const.}$$

$$X_i^1 = \text{const.} \quad i = 1, 2, \dots, r$$

$$X_i^2 = \text{const.} \quad i = 1, 2, \dots, r.$$

We want to find the maximum of S as a function of E^1 (we could just as well use E^2). Start by taking the derivative of S .

$$\begin{aligned} \frac{\partial S}{\partial E^1} &= \frac{\partial}{\partial E^1} \left(S^1(E^1, X_1^1, X_2^1, \dots, X_r^1) + S^2(\underbrace{E - E^1}_{E=E^1+E^2}, X_1^2, X_2^2, \dots, X_r^2) \right) \\ &= \left. \frac{\partial S^1}{\partial E^1} \right|_{E^1} - \left. \frac{\partial S^2}{\partial E^2} \right|_{E^2=E-E^1}. \end{aligned}$$

For E^1 at equilibrium we will write E_{eq}^1 (sim. for E^2). Then to maximise S we must have

$$\left. \frac{\partial S}{\partial E^1} \right|_{E_{eq}^1} = \left. \frac{\partial S^1}{\partial E^1} \right|_{E_{eq}^1} - \left. \frac{\partial S^2}{\partial E^2} \right|_{E-E_{eq}^1} = 0$$

hence

$$\left. \frac{\partial S^1}{\partial E^1} \right|_{E_{eq}^1} = \left. \frac{\partial S^2}{\partial E^2} \right|_{E_{eq}^2}.$$

We already have our first result — the equilibrium state of the composite system is the one where the change in entropy of the two systems, with respect to, E is equal. Since $\frac{\partial E}{\partial S} = \frac{1}{T}$ (by monotonicity) this means that the equilibrium occurs when the temperatures of the two sub-systems are equal.

Where does the heat flow to in the system in order to reach this equilibrium? The system started in an initial state with $E = E_{in}^1 + E_{in}^2$. Since entropy increases to reach the maximum value at equilibrium we have

$$S^1(E_{eq}^1) + S^2(E_{eq}^2) \geq S^1(E_{in}^1) + S^2(E_{in}^2),$$

so,

$$S^1(E_{in}^1) - S^1(E_{eq}^1) + S^2(E_{in}^2) - S^2(E_{eq}^2) \geq 0.$$

The convexity property of entropy means that both systems 1 and 2 have

$$\left. \frac{\partial S}{\partial E} \right|_{E_{in}} (E_{eq} - E_{in}) \geq S(E_{eq}) - S(E_{in})$$

and from the previous expression, the RHS of this inequality is bounded below by zero so we have

$$\left. \frac{\partial S^1}{\partial E^1} \right|_{E_{in}^1} (E_{eq}^1 - E_{in}^1) + \left. \frac{\partial S^2}{\partial E^2} \right|_{E_{in}^2} \underbrace{(E - E_{eq}^1 - E_{in}^2)}_{=E_{eq}^2} \geq 0$$

But E is conserved so $E = E_{in}^1 + E_{in}^2$ and hence $E - E_{eq}^1 - E_{in}^2 = -(E_{eq}^1 - E_{in}^1)$. Therefore

$$\left[\left. \frac{\partial S^1}{\partial E^1} \right|_{E_{in}^1} - \left. \frac{\partial S^2}{\partial E^2} \right|_{E_{in}^2} \right] (E_{eq}^1 - E_{in}^1) \geq 0.$$

This implies that energy flows to the system with higher $\frac{\partial S}{\partial E}$ and since $\frac{\partial S}{\partial E} = \frac{1}{T}$ this means that energy flows into the system with lowest temperature, until the temperatures are equal.

2 Probability

This section gives an introduction to some tools and ideas from probability theory. One of the main reasons for including it is that it lets us give background to the final sub-section on Brownian motion, but we'll also use some of the methods from here in the final section of the course on complex networks.

Physical variables can be either continuous or discrete. Continuous variables, like the spatial position of a particle $\mathbf{r} = (x, y, z)$; $x \rightarrow x + dx$ allow us to apply tools from differential calculus. Discrete variables (like the spin of an electron, the face of a playing card, or the energy level of a quantum particle) require a different approach. As the number of discrete states gets large, it is possible to approximate a discrete system by a continuous one. However, there are often statistical reasons why this is not always the best approach. (For one example of why it is not always good to approximate a discrete distribution with a continuous one, see the article “Power-Law Distributions in Empirical Data” by Clauset, Shalizi, and Newman, SIAM Review (2009) doi:10.1137/070710111)

Here we are going to look at some simple discrete distributions, including one of the simplest — the binomial distribution which describes the probability of events that can take one of only two possible outcomes. The tool we will employ to study these distributions is *probability generating functions* (PGFs) (or generating functions for short). Although PGFs can be somewhat cumbersome, they provide a powerful general tool that can be applied to a wide range of situations involving discrete probabilities. In this sense they are like the Taylor series of discrete probability.

2.1 Probability generating functions

A probability generating function (PGF) is a polynomial whose coefficients are the probabilities associated with each outcome of some random process. For example, the PGF for an equal probability die is

$$\mathcal{G}(x) = 0x^0 + \frac{1}{6}x^1 + \frac{1}{6}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{6}x^5 + \frac{1}{6}x^6.$$

As an example of how PGFs are a handy general tool, say we now wanted to find the PGF for the sum when we roll two such dice. We can just multiply the PGFs to get $\mathcal{G}'(x) = \mathcal{G}(x)\mathcal{G}(x) = 0x^0 + 0x^1 + \frac{1}{36}x^1x^1 + \frac{2}{36}x^1x^2 + \dots + \frac{1}{36}x^6x^6$.

More generally, the set of probabilities associated with any discrete distribution can be used as coefficients of $\mathcal{G}(x)$:

$$\mathcal{G}(x) = P(X = r_0)x^0 + P(X = r_1)x^1 + \dots = \sum_{n=0}^{\infty} P(X = r_n)x^n.$$

2.2 Properties of generating functions

$$\begin{aligned}\mathcal{G}(0) &= P(X = 0) \\ \mathcal{G}(1) &= \text{sum over all probabilities} = 1\end{aligned}$$

The two properties above are easy to see and are not unexpected. More interesting is the fact that derivatives of PGFs can be used to obtain expressions for the mean, and higher order moments, of the distribution associated with the PGF.

$$\begin{aligned}\frac{d}{dx}\mathcal{G}(x) = \mathcal{G}'(x) &= \sum_r rP_r x^{r-1}, \quad \text{and} \\ \mathcal{G}''(x) &= \sum_r r(r-1)P_r x^{r-2}.\end{aligned}$$

If we evaluate the first derivative of the PGF at 1 we get

$$\mathcal{G}'(1) = \sum_r rP_r.$$

The expected value of an arbitrary function $f(r)$ is given by $\langle f(r) \rangle = \sum_r f(r)P(X=r)$, therefore $\mathcal{G}'(1) = \langle r \rangle$ — the expected value, or mean, of the distribution.

Similarly, Evaluating $\mathcal{G}''(x)$ at 1 gives

$$\mathcal{G}''(1) = \sum_r r(r-1)P(X=r) = \langle X(X-1) \rangle.$$

Why would we care about being able to find $\langle X(X-1) \rangle = \langle X^2 - X \rangle$? Because the variance of X , is given by

$$V(X) = \langle X^2 \rangle - \langle X \rangle^2.$$

We are therefore able to construct $V(X)$ from combinations of \mathcal{G} , \mathcal{G}' , and \mathcal{G}'' . We can use the fact that $\langle X^2 - X \rangle = \langle X^2 \rangle - \langle X \rangle$ I.e. $\sum (r^2 - r)P(X=r) = \sum r^2 P(X=r) - \sum r P(X=r)$ to get

$$V(X) = \langle X^2 \rangle - \langle X \rangle^2 \tag{1}$$

$$= \underbrace{\langle X^2 \rangle - \langle X \rangle}_{\mathcal{G}''(1)} + \underbrace{\langle X \rangle}_{\mathcal{G}'(1)} - \underbrace{\langle X \rangle^2}_{\mathcal{G}'(1)^2} \tag{2}$$

$$= \mathcal{G}''(1) + \mathcal{G}'(1) - \mathcal{G}'(1)^2 \tag{3}$$

2.3 The binomial distribution

The binomial distribution describes the probability of getting r identical outcomes from a sequence of N events where each event has only two possible states, e.g. number of heads from a sequence of coin tosses, number of Au atoms from deposition of Au and Ag atoms in some sputtering process, or number of spin up electrons in an uncorrelated system.

The probabilities for the binomial distribution are given by

$$P(X=r) = \binom{N}{r} p^r q^{N-r}, \text{ where } \binom{N}{r} = \frac{N!}{r!(N-r)!}, \quad r = 0, 1, \dots, N$$

and where (mostly) $p + q = 1$. The PGF for the binomial distribution is therefore given by $\mathcal{G}(x) = \sum_{r=0}^N \binom{N}{r} p^r q^{N-r} x^r$. The coefficients in the binomial distribution are (not surprisingly) the coefficients from the binomial expansion theorem. This means that instead of writing the PGF as a sum, we can easily express it (and its derivatives) as a polynomial:

$$\begin{aligned} \mathcal{G}(x) &= (xp + q)^N \\ \mathcal{G}'(x) &= N(xp + q)^{N-1} p \\ \mathcal{G}''(x) &= N(N-1)(xp + q)^{N-2} p^2. \end{aligned}$$

Evaluating the above at $x = 1$ gives:

$$\begin{aligned} \mathcal{G}(1) &= (p + q)^N = 1 \text{ if } p + q = 1 \\ \mathcal{G}'(1) &= N(p + q)^{N-1} p = Np \text{ if } p + q = 1 \\ \mathcal{G}''(1) &= N(N-1)(p + q)^{N-2} p^2 \\ &= N(N-1)p^2 \text{ if } p + q = 1. \end{aligned}$$

As expected, we have $\langle X \rangle = Np$ for the event associated with the probability p . Similarly, we can calculate the variance of the binomial distribution using equation 3

$$\begin{aligned} V(X) &= \mathcal{G}''(1) - \mathcal{G}'(1) - \mathcal{G}'(1)^2 \\ &= N^2 p^2 - Np^2 + Np - N^2 p^2 \\ &= N(p - p^2) \\ &= Np(1 - p) = Npq. \end{aligned}$$

If you are feeling enthusiastic, compare the calculation above with the usual method for finding the variation of the binomial distribution without using a PGF.

2.4 The expected long-run outcome of a binary process

What is the net outcome we can expect from a sequence of N binary events? What is the variance in this net outcome? How long can we expect the net outcome to remain positive?

Think of a sequence of coin tosses where heads implies $+1$ and tails implies -1 . We will denote the number of heads from N events as n_h and the number of tails as $n_t = N - n_h$, then the net outcome is $d = n_t - n_h = N - 2n_h$.

If $p = q = \frac{1}{2}$ then the expected value of d is $\langle d \rangle = 0$. (It's easy to check this: $\langle d \rangle = \langle N - 2n \rangle = N - 2\langle n \rangle = N - \frac{2}{2}N = 0$.)

We want to find the variation in d , or rather, the root-mean-squared (rms) deviation

$$d_{rms} = (\langle d^2 \rangle - \langle d \rangle^2)^{\frac{1}{2}} = \langle d^2 \rangle^{\frac{1}{2}}.$$

At this point we'll drop the subscript in n since we only have n_h and since the probabilities are equal. From $d = N - 2n$ we have

$$\langle d^2 \rangle = \langle (N - 2n)^2 \rangle = \langle N^2 - 4Nn + 4n^2 \rangle.$$

We know that $\langle n \rangle = \mathcal{G}'(1) = Np = N/2$ and from the expressions for $\mathcal{G}''(1)$ and $\mathcal{G}'(1)$ we get $\langle n^2 \rangle = \mathcal{G}''(1) + \mathcal{G}'(1)$ where $\mathcal{G}''(1) = N^2 p^2 = N^2/4 - N/4$. We can now calculate $\langle d^2 \rangle$.

$$\begin{aligned} \langle d^2 \rangle &= N^2 - 4N \frac{N}{2} + 4 \left(\frac{N^2}{4} - \frac{N}{4} + \frac{N}{2} \right) \\ &= N^2 - 2N^2 + N^2 + N \\ &= N. \end{aligned}$$

So, $d_{rms} = \sqrt{N}$ and the variance in the net expected outcome scales like \sqrt{N} as N increases. I.e. after N steps we expect to be a rms distance of $\pm\sqrt{N}$ away from the mean of zero.

[You may want to add a figure of this, with N as the independent variable.]

We can now look at how this would change if the binary process had some small bias $\alpha = |p - q| > 0$. In this case, at each step, the expected outcome is α and the bounds for the rms distance from the mean are $\alpha N \pm \sqrt{N}$.

[You might want to add a figure for this too.]

2.5 Brownian motion

Brownian motion is a model that describes the behaviour of particles in suspension (e.g. of dust in air, fine silt in dirty water) that have some velocity but which are constantly changing direction as they are buffeted by other particles. Their motion can be understood through a calculation similar to that for the net outcome of the binary process above, but in 3D.

The conceptual model that is often used when discussing Brownian motion is that of a *random walk*. Consider a random walker who after n steps is at the position $\mathbf{r} = (x, y, z)$. Each step is given by $\mathbf{s}_{n+1} = \mathbf{r}_{n+1} - \mathbf{r}_n$ and has step length s_n in a random and uncorrelated direction $\hat{\mathbf{s}}_n$. Again (by symmetry) $\langle \mathbf{s} \rangle = 0$. We want to find how far a particle might have gone after N steps. We will define $\lambda^2 = \langle \mathbf{s}^2 \rangle$ — the average square step-length. We write \mathbf{r}_N for the position of the walker (or the particle) after N steps, then

$$\begin{aligned} \langle \mathbf{r}_N^2 \rangle &= \left\langle \left(\sum_{n=1}^N \mathbf{s}_n \right)^2 \right\rangle \\ &= \left\langle \sum_{n=1}^N \mathbf{s}_n^2 + 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^N \mathbf{s}_n \cdot \mathbf{s}_m \right\rangle \\ &= \underbrace{\sum_{n=1}^N \langle \mathbf{s}_n^2 \rangle}_{=\lambda^2} + 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^N \underbrace{\langle \mathbf{s}_n \cdot \mathbf{s}_m \rangle}_{=\langle \mathbf{s}_n \rangle \cdot \langle \mathbf{s}_m \rangle = 0} \end{aligned}$$

The first term in the expression is the mean squared step-length while the second term is zero since there is no correlation between the directions of each of the steps. Hence $\langle \mathbf{r}_N \rangle = N\lambda^2$ and the rms displacement is $\sqrt{N}\lambda$. If the number of steps is proportionate to the length of time that the particles move for, then we have that the mean-square displacement scales like \sqrt{t} . This is one of the key features of diffusive processes.

3 Student notes from assignment one