HOMEWORK-2
MTH373/573 - Scientific Computing
Sourav Goyal
2020341

Ans-1

Scientific Computing - MTH 373/573
Homework - 2

Ans - ① (a) Minimize $x_1^2 + 2x_2^2 + 3x_3^2 + (x_1 - x_2 + x_3 - 1)^2 + (-x_1 - 4x_2 + 2)^2$

$\Rightarrow$ So first take :- minimization of $x_1^2 + 2x_2^2 + 3x_3^2$ which is,

$\Rightarrow$ $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$

$\Rightarrow$ $b_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $A_1 x \cong b_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ Now minimize :- $(x_1 - x_2 + x_3 - 1)^2 + (-x_1 - 4x_2 + 2)^2$

$\Rightarrow$ $A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 4 & 0 \end{bmatrix}$

$\Rightarrow$ $b_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cong \begin{bmatrix} +1 \\ -2 \end{bmatrix}$

$\therefore$ Joining A_1 and A_2 and b_1 and b_2, we have,

$\Rightarrow$ $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\Rightarrow$ $Ax \cong b$

$\Rightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \\ 1 & -1 & 1 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$ Ans

(b) Minimize $(-6x_2+4)^2 + (-4x_1+3x_2-1)^2 + (x_1+8x_2-3)^2$

=> So here we have our x as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

=> an A matrix will be,

=> $$A = \begin{bmatrix} 0 & -6 \\ -4 & 3 \\ 1 & 8 \end{bmatrix}$$

=> and, b will be :-

$$b = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

∴ our $Ax \approx b$ equation will be;

$$\begin{bmatrix} 0 & -6 \\ -4 & 3 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix} \quad \text{Ans}$$

(c) minimize $2(-6x_2+4)^2+3(-4x_1+3x_2-1)^2+4(x_1+8x_2-3)^2$

Modifying the given eqn :-

$\Rightarrow$ minimize $(-6\sqrt{2}x_2+4\sqrt{2})^2+(-4\sqrt{3}x_1+3\sqrt{3}x_2-\sqrt{3})^2+(2x_1+16x_2-6)^2$

So here we have our new eqn :- Now A matrix will be,

$\therefore \quad A = \begin{bmatrix} 0 & -6\sqrt{2} \\ -4\sqrt{3} & 3\sqrt{3} \\ 2 & 16 \end{bmatrix}$

we have our $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

and $b = \begin{bmatrix} -4\sqrt{2} \\ \sqrt{3} \\ 6 \end{bmatrix}$

$\therefore$ our $Ax \simeq b$ equation will be,

$$\begin{bmatrix} 0 & -6\sqrt{2} \\ -4\sqrt{3} & 3\sqrt{3} \\ 2 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4\sqrt{2} \\ \sqrt{3} \\ 6 \end{bmatrix}$$ Ans

(d) minimize $x^T x + \|Bx - d\|_2^2$ where $B \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$ are given.

$\Rightarrow$ So, we will break equation in two subpart :-

First, $\|Bx - d\|_2^2$

Now, $x \in \mathbb{R}^n$ infer from $\|Bx - d\|_2^2$,

Secondly, $x^T x$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$x^T x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 \cdots x_n^2$$

$\therefore$ We have our A matrix for $x^T x$:-

$$A_1 = \begin{bmatrix} 1 & - & - & 0 \\ \vdots & \ddots & & \vdots \\ & & & 1 \end{bmatrix}_{n \times n} = I_n$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_n = 0_n$$

$\therefore$ Joining both Parts we have,

$$A = \begin{bmatrix} B \\ I_n \end{bmatrix}_{(p+n) \times n}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$b = \begin{bmatrix} d \\ 0_n \end{bmatrix}_{(p+n) \times 1}$$

$\therefore$ We have our $Ax \cong b$ equation will be,

$$\begin{bmatrix} B \\ I_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d \\ 0_n \end{bmatrix}$$

(e) Minimize $x^T D x + \|Bx - d\|_2^2$ where $D \in R^{n \times n}$ is a diagonal matrix with positive diagonal elements, $B \in R^{p \times n}$ and $d \in R^p$ are given,

So, we have $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$

Again break given eqn in two parts,

First, $x^T D x$,

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{bmatrix}_{n \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = D_1 x_1^2 + D_2 x_2^2 - - D_n x_n^2$$

So, we have A_1 matrix for $x^T D x$,

$$A_1 = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{bmatrix} = D$$

and $b_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_n = 0_n$

Secondly, $\|Bx - d\|$.

$\therefore$ Our final A matrix will be,

$$A = \begin{bmatrix} D \\ B \end{bmatrix},$$

Our b matrix will be,

$$b = \begin{bmatrix} 0_n \\ d \end{bmatrix}$$

$\therefore$ We have our $Ax \approx b$ equation that is,

$$\begin{bmatrix} D \\ B \end{bmatrix}_{(p+n) \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} d \\ 0_n \end{bmatrix}_{(p+n) \times 1}$$

Ans

Ans-2

Ans-2 Given:-

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \simeq \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

⇒ Solving this problem using necessary condition for existence of a minimum, we obtained Normal equations:

$$A^T A x = A^T b$$

⇒ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

⇒ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

⇒ $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

⇒ $x_1 + x_2 = 2$

⇒ $x_1 + 2x_2 = 3$

⇒ $x_2 = 1$, $x_1 = 1$.

∴ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

⇒ Now, verifying using the sufficiency condition that our solution is minimum by proving $A^T A$ is positive definite,

⇒ Condⁿ for positive definite for a matrix A ⇒ $\boxed{x^T A x > 0.}$

∴ $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

⇒ $x^T A^T A x = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= 5 > 0.$

∴ We have a minimum solution.

⇒ Now for residual vector we have,

⇒ $r = b - Ax$

⇒ $r = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

⇒ $r = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

⇒ $r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

⇒ Euclidean norm of r,

⇒ $\|r\| = \sqrt{0^2 + 0^2 + 1^2}$

⇒ $\|r\| = 1.$ Ans

Ans-3

Ans ③(a) Given:- $A \in R^{m \times n}$, $m \geq n$ with linearly independent column

• $(m+n) \times (m+n)$ matrix say M :-

$$M = \begin{bmatrix} I & A \\ A^T & O \end{bmatrix}.$$

To show :- M is non-singular.

⇒ We will show that $|\det(M)| > 0$ for proving M is non-singular,

⇒ $|\det(M)| = |\det(IO) - \det(A^TA)|$

$= |\det(A^TA)|$

⇒ Now we will prove that A^TA is non-singular i.e., $|\det(A^TA)| > 0$.

⇒ Let's take, $A^TAx = 0$

"

⇒ From given statement we know A matrix has linearly independent columns so if $Ax = 0$ implies $x = 0$.

∴ Null space of $A^TA = \{0\}$,

" Dimension of $A^TA = n \times n$

" So, A^TA is a square matrix and null space of $A^TA = \{0\}$ then A^TA is invertible.

" Now, we know invertible matrix are non-singular.

∴ $|\det(A^TA)| > 0$.

" And, by that we have $|\det(M)| > 0$.

∴ M is a non-singular. (Proved) P/

Ans- 3(b) Given, $\hat{x} = b - Ax$ and $\hat{y} = x$ and x is the soln of $Ax \simeq b$,

$\Rightarrow$ <u>To show</u> :-
$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$\Rightarrow$ So,
$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b - Ax \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b - Ax \\ x \end{bmatrix},$$

$\Rightarrow$ $Ax \simeq b$ (Given),

$\Rightarrow$ So, by necessary condition for existence of a minimum, we have normal equation,

$\therefore$ $A^T A x = A^T b$.

$\Rightarrow$ $A^T(Ax - b) = 0$ — (1)

$\Rightarrow$
$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} b - Ax \\ x \end{bmatrix} = \begin{bmatrix} b - \cancel{Ax} + \cancel{Ax} \\ A^T b - A^T A x \end{bmatrix}$$

$\Rightarrow$
$$= \begin{bmatrix} b \\ A^T b - A^T A x \end{bmatrix}$$

$\Rightarrow$ Now from eq (1) we have, $A^T b - A^T A x = 0$,

$\therefore$
$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$ (Proved). Ans

Ans-5

Ans-2 5(a) Given, $\frac{e^{\alpha t_i+\beta}}{1+e^{\alpha t_i+\beta}} \approx y_i$, $i=1, \dots m$. where $m=50$,

$0<y_i<1$ for all $i=1, \dots, m$.

$\Rightarrow$ Taking logarithm both side,

$$\Rightarrow \log \frac{e^{\alpha t_i+\beta}}{1+e^{\alpha t_i+\beta}} \approx \log y_i$$

$$\Rightarrow \log \frac{1}{e^{-(\alpha t_i+\beta)}+1} \approx \log y_i$$

$$\Rightarrow \log 1 - \log(e^{-(\alpha t_i+\beta)}+1) \approx \log y_i$$

$$\Rightarrow -\log(e^{-(\alpha t_i+\beta)}+1) \approx \log y_i$$

$$\Rightarrow (e^{-(\alpha t_i+\beta)}+1) \approx \frac{1}{y_i}$$

$$\Rightarrow e^{-(\alpha t_i+\beta)} \approx \frac{1}{y_i}-1$$

$$\Rightarrow \frac{1}{e^{\alpha t_i+\beta}} \approx \frac{1-y_i}{y_i}$$

$$\Rightarrow \alpha t_i+\beta \approx \log \frac{y_i}{1-y_i}$$

$\Rightarrow$ Take $\log \frac{y_i}{1-y_i} = y_i'$,

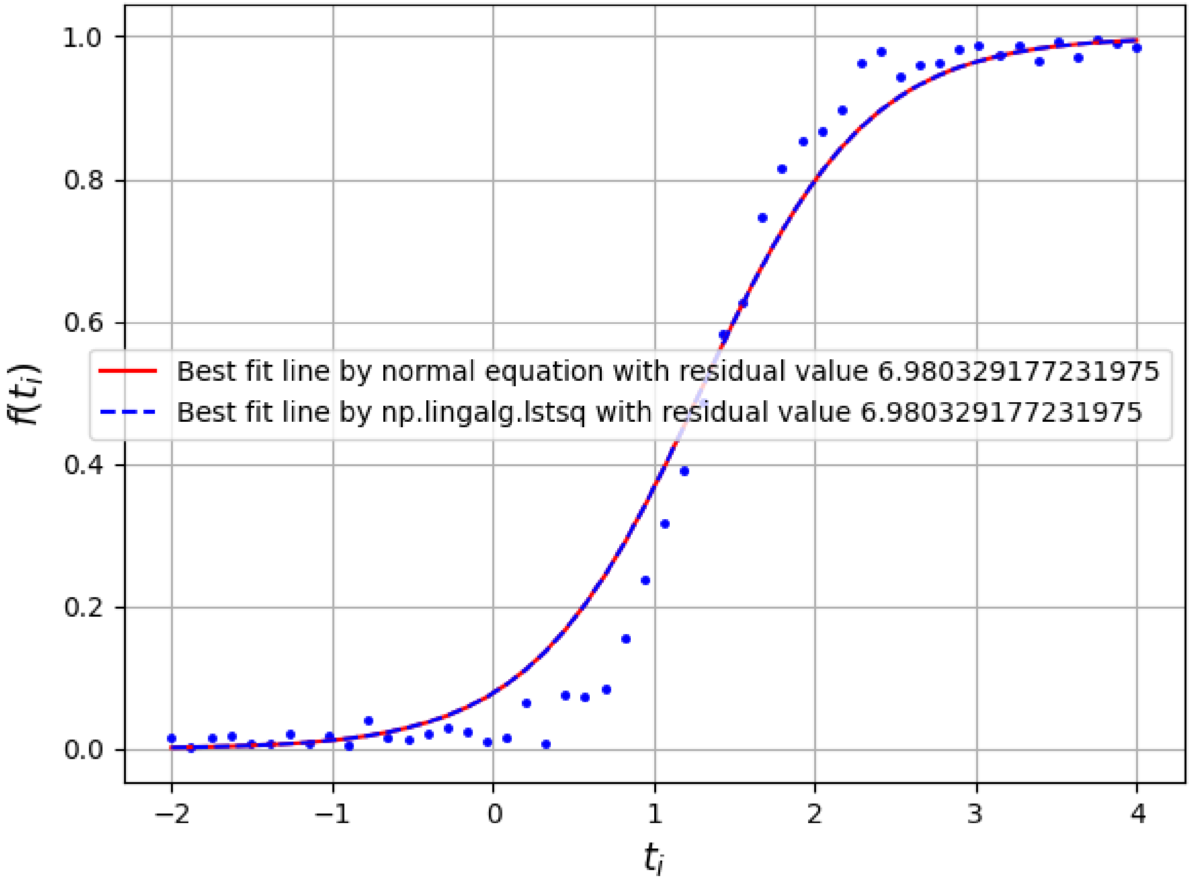
$\therefore \boxed{\alpha t_i+\beta = y_i'}$ Ans

(b)

Error by normal equation : 6.980329177231975

Error by np.linalg.lstsq : 6.980329177231975

Plot :-



Best fit line by normal equation with residual value 6.980329177231975
Best fit line by np.lingalg.lstsq with residual value 6.980329177231975
$f(t_i)$
t_i
1.0
0.8
0.6
0.4
0.2
0.0
−2
−1
0
1
2
3
4

Ans-6

Ans = 6 (a) $A = \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix}$, $b = \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix}$

$\Rightarrow$ Normal equation :- $A^T A x = A^T b$,

$$\therefore \begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+10^{-2k} & 1 \\ 1 & 1+10^{-2k} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10^{-2k} \\ -10^{-2k} \end{bmatrix}$$

$\Rightarrow x_1 + 10^{-2k} x_1 + x_2 = 10^{-2k}$

$\Rightarrow x_1 + x_2 + 10^{-2k} x_2 = -10^{-2k}$

$\Rightarrow x_2 = \dfrac{-10^{-2k} - x_1}{1+10^{-2k}}$

$\Rightarrow x_1(1+10^{-2k}) = 10^{-2k} + \dfrac{10^{-2k} + x_1}{1+10^{-2k}}$ $\Rightarrow x_1 = 1$ and,

$\Rightarrow x_2 = -\cancel{\left(\dfrac{1+10^{-2k}}{1+10^{-2k}}\right)}$

$\Rightarrow x_2 = -1$

$\Rightarrow x_1 = 1$ and $x_2 = -1$

$\therefore x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) Result :-

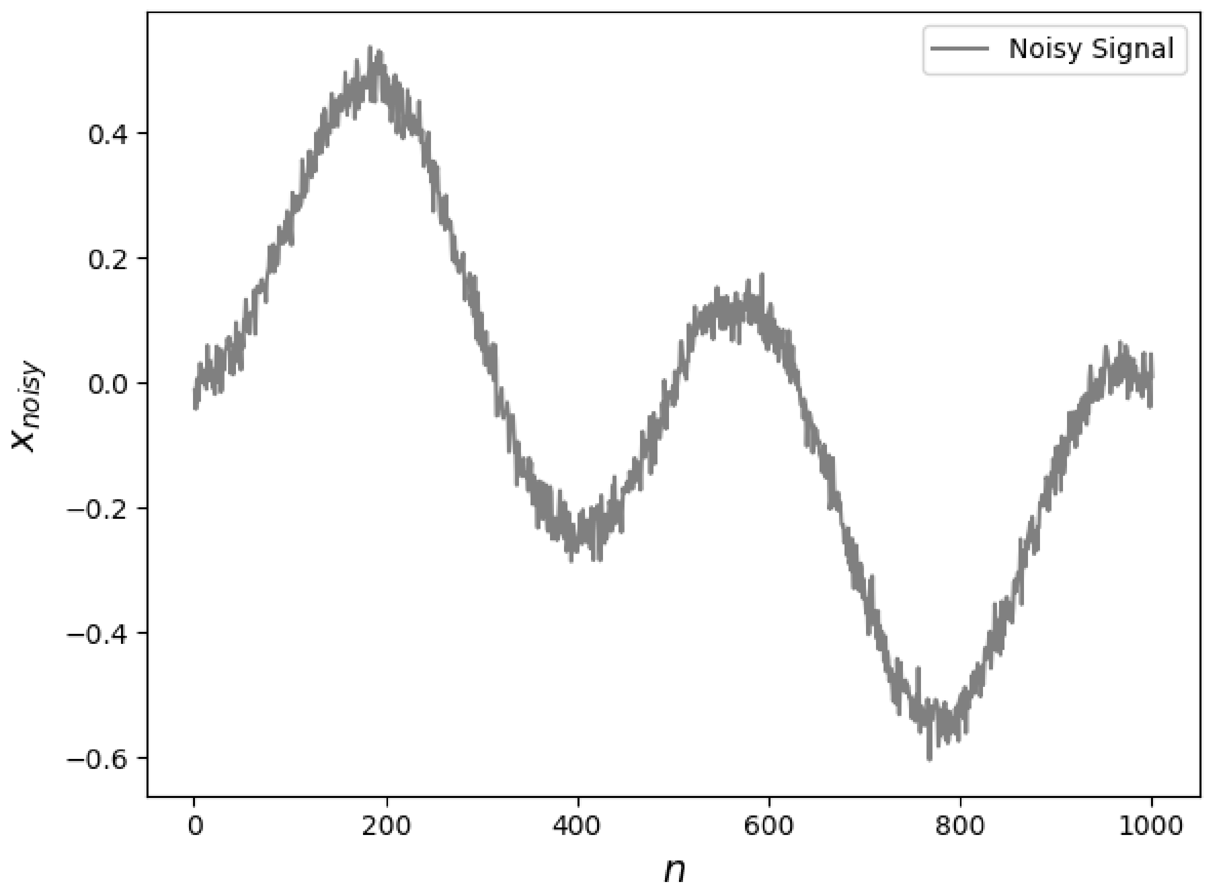
```
X for k = 6 using QR factorization : [[ 1.]
 [-1.]]

X for k = 7 using QR factorization : [[ 1.]
 [-1.]]

X for k = 8 using QR factorization : [[ 1.00000001]
 [-1.00000001]]

X for k = 9 using QR factorization : [[ 1.00000005]
 [-1.00000005]]

X for k = 10 using QR factorization : [[ 1.00000052]
 [-1.00000052]]

X for k = 11 using QR factorization : [[ 1.00002329]
 [-1.00002329]]

X for k = 12 using QR factorization : [[ 0.99991338]
 [-0.99991338]]

X for k = 13 using QR factorization : [[ 0.99936385]
 [-0.99936385]]

X for k = 14 using QR factorization : [[ 1.01270964]
 [-1.01270964]]

X for k = 15 using QR factorization : [[ 0.86355085]
 [-0.86355085]]
```

(c) Result :-

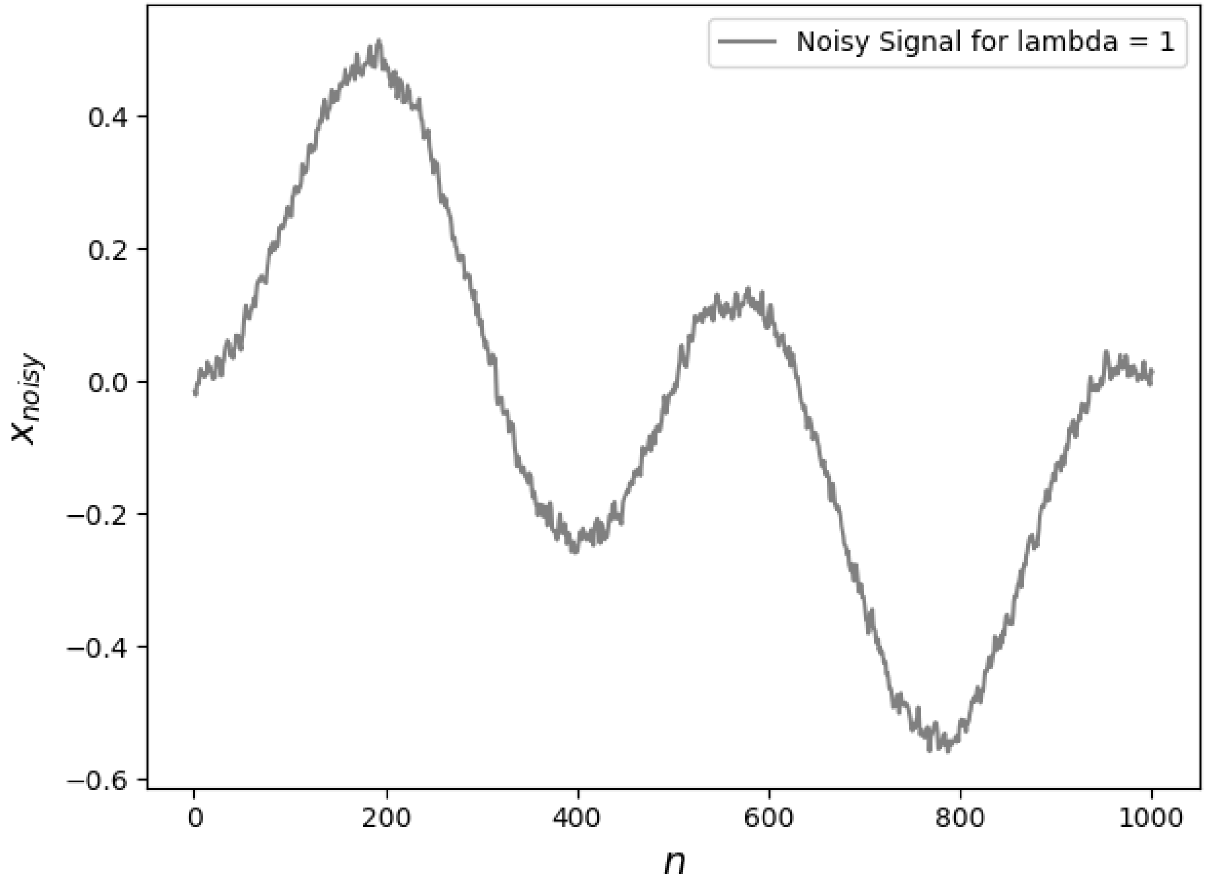
```
X for k = 6 using normal equation : [[ 0.99991111]
 [-0.99991111]]

X for k = 7 using normal equation : [[ 1.00079992]
 [-1.00079992]]

XTX becomes singular
XTX becomes singular
XTX becomes singular
XTX becomes singular
XTX becomes singular
XTX becomes singular
XTX becomes singular
```

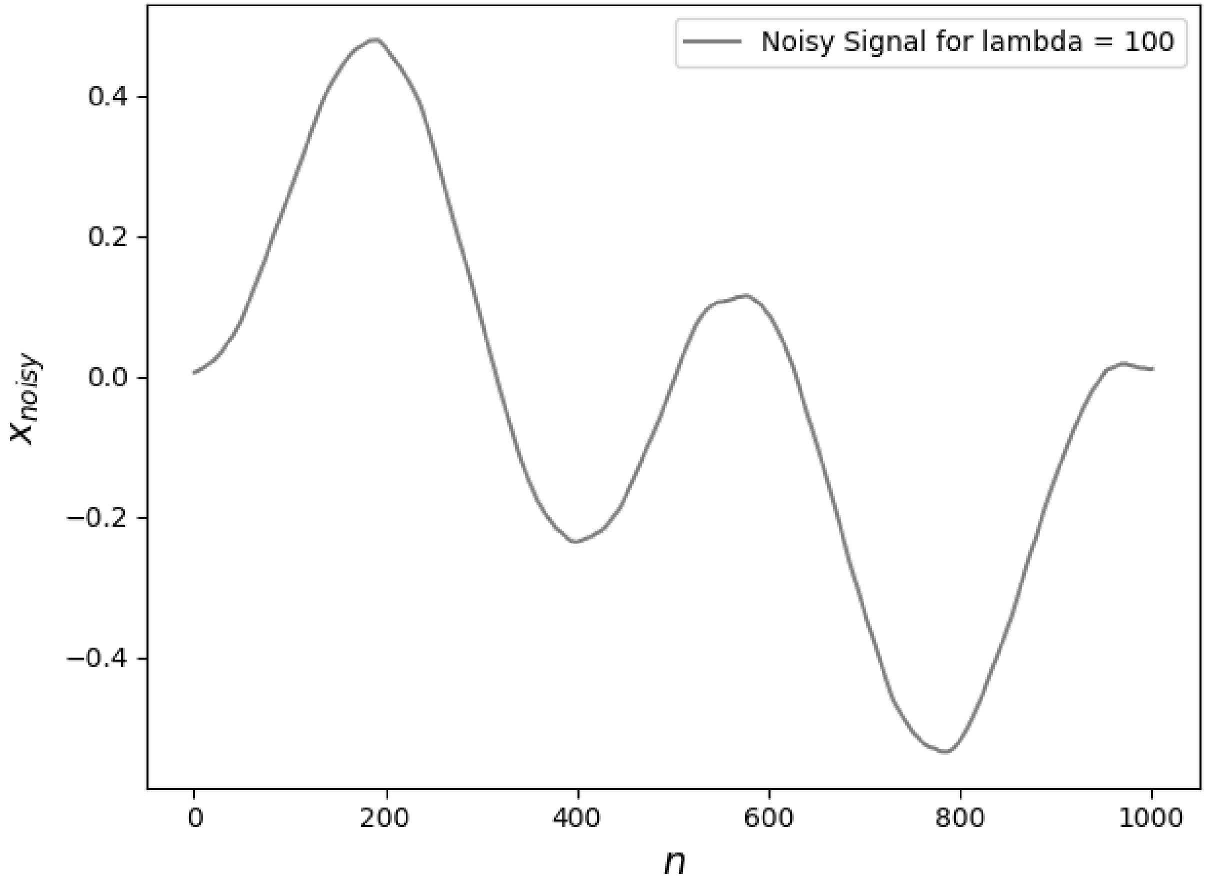
Compare between all three result :-
In normal equation we have x value only for k = 6 and k = 7 because of floating point at k >= 8 10^{-2k} consider as 0 which makes the ATA matrix singular and by that we haven't solution for that but according to our analytical solution we have x independent of k which equals to [[1,-1]]
For k = 6 we have x = [[0.99991111,-0.99991111]] by normal equation but our desired solution is [[1,-1]] and this also happens due to floating point system and same as for k = 7 we get value closer to 1 and -1 but we haven't exact value for k = 7 also.
In QR decomposition we have correct result for k = 6 and 7 which is more accurate than solution we getting from normal equation computation and after k >= 8 our ATA becomes singular so no solution we are getting from normal equation computation but in QR we have solution because R is an upper triangular matrix and Q is orthogonal so inverse exist for both but due to floating point system we have value closer to our solution but not accurate that much.

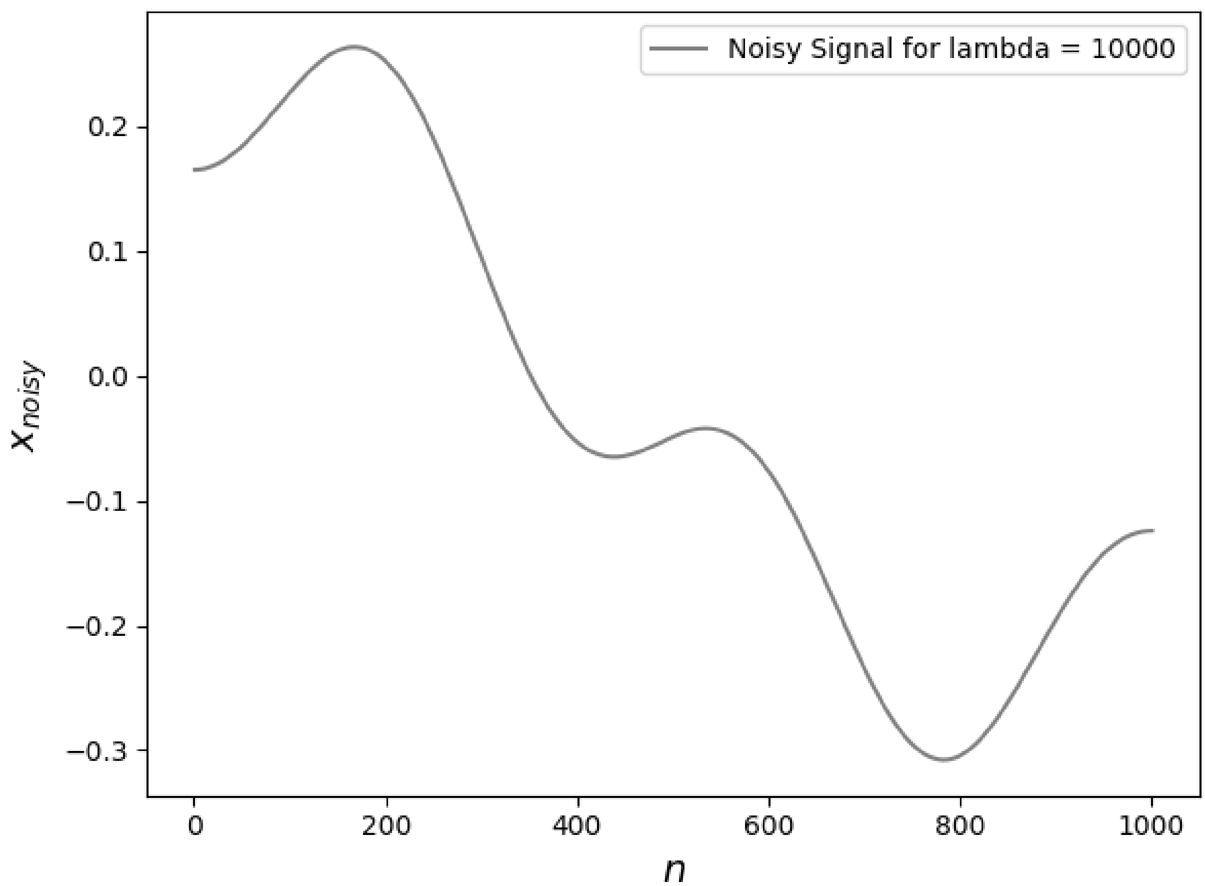
Ans-7
Real Data :-



Effect of λ on the quality of the estimate x :-


Noisy Signal for lambda = 1
0.4
0.2
0.0
−0.2
−0.4
−0.6
x_{noisy}
0
200
400
600
800
1000
n





Observations :-

- As lambda increases :-

    - The noisiness in the curve decreases.
    - The minimum value increases and maximum value decreases.
    - The number of local minimum and local maximum decreases
    - The curve takes the shape of a straight line or shrinking of the graph.

- So from all of the above observation we can conclude that lambda = 100 can be a optimal value because at lambda = 100 graph will be smoothened and not much deviated from original path but at lambda > 100 it’s start more deviating (which we observe from our above observation) which increases the error.