HOMEWORK-3
MTH 373/573-SCIENTIFIC COMPUTING
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Ans-1

Ans 1 ① Given :- A is a triangular matrix and $A \in R^{n \times n}$

⇒ To show :- Eigenvalues of A is all its diagonal elements.

⇒ So, we will take A as upper triangular matrix but same steps goes for lower triangular matrix also.

$$A = \begin{bmatrix} a_{11} & - & - & a_{1n} \\ 0 & \ddots & & | \\ | & & \ddots & | \\ 0 & - & - & a_{nn} \end{bmatrix}_{n \times n}$$

⇒ Now for finding eigenvalues λ we know,

⇒ $\det(A - \lambda I) = 0$ — ①

⇒ $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & - & - & a_{1n} \\ | & \ddots & & \\ | & & \ddots & \\ 0 & - & - & a_{nn}-\lambda \end{bmatrix}$

⇒ Now we will prove that determinant of triangular matrix will be multiplication of its diagonal elements,

∴ Let's B be a upper triangular matrix,

$$B = \begin{bmatrix} b_{11} & - & - & b_{1n} \\ 0 & b_{22} & & | \\ | & & \ddots & | \\ 0 & - & - & b_{nn} \end{bmatrix}$$ → M

⇒ Now, $\det(B) = b_{11}\det(M)$ (∵ accounting determinant from 1st column)

$= b_{11} \det(M)$

∵ Again for $\det(M)$,

∵ $\det(M) = b_{22} \times \det(M_1)$

∵ And similarly as go till last diagonal elements we have,

∵ $\det(B) = b_{11} \times b_{22} - - b_{nn}$.

∴ Hence determinant of triangular matrix will be multiplication of its diagonal elements.

∵ Now using above result, $\det(A-\lambda I)$ will be :-

∵ $\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda) - - (a_{nn}-\lambda)$

∵ Put $\det(A-\lambda I)$ in (I) :-

∴ $(a_{11}-\lambda)(a_{22}-\lambda) - - (a_{nn}-\lambda) = 0$.

∵ Solving above eqn we have $\lambda = a_{11}, a_{22}, a_{33} - - a_{nn}$

∴ Eigenvalues will be all diagonal elements of A.

∵ And, similarly goes for lower triangular matrix.

∴ Eigenvalues of A is all its diagonal elements. [Hence Proved]

Ans-2

Ans ② Given :- $A \in \mathbb{R}^{n\times n}$ is a non-defective matrix.

⇒ To Prove :- Rank of A is equal to the numbers of non-zero eigenvalues of this matrix.

⇒ So, by theorem of Diagonalizibility i.e., A matrix $A \in \mathbb{C}^{n\times n}$ is non-defective if and only if it has an eigenvalue decomposition of the form $A = X \Lambda X^{-1}$ where $X \in \mathbb{C}^{n\times n}$ is a non-singular matrix and $\Lambda \in \mathbb{C}^{n\times n}$ is a diagonal matrix. The eigenvalues of A are the diagonal entries of Λ.

⇒ Therefore we have given that A is non-defective matrix,

$$\therefore \quad A = X \Lambda X^{-1}$$

⇒ Now, X is non-singular so X is full rank and product of two non-singular matrix also gives a non-singular because let A and B are two non-singular,

⇒ $C = AB \Rightarrow \det(C) = \det(A)\det(B) > 0$.

⇒ So rank(Λ) will be of rank(A),

⇒ Now from theorem we know eigenvalues of A will be the diagonal entries of Λ and also rank of diagonal matrix is equals to its rank of diagonalization.

∴ We can conclude that rank(Λ) will be equals to number of non-zero eigenvalues and by this rank(A) also equals to number of non-zero eigenvalues. [Hence Proved].

Ans-3

Ans - ③ Given :- $u, v \in R^n$ such that $u^T v = 1$ and $A = uv^T$.

⇒ Now, we have to find eigenvalues of A,

⇒ So, from previous question we know that non-zero eigen-values count will be equal to rank of the matrix. So,

rank (A) = rank (uv^T).

⇒ we know, rank $(XY) \leq \min(\text{rank}(X), \text{rank}(Y))$,

∴ rank (A) = min (rank (u), rank (v^T))

⇒ u, v are column matrix and we know that rank of a non-zero column matrix is 1, as $u^T v = 1$ which shows both are non-zero column matrix.

∴ $\boxed{\text{rank}(A) = 1.}$

⇒ Now we know as rank(A) = 1 then A has only 1 non-zero eigenvalue,

⇒ $An = \lambda n$ (standard expression).

⇒ $uv^T n = \lambda n$

⇒ Put $n = u$,

⇒ $uv^T u = \lambda u$ $[u^T v = v^T u = 1]$

⇒ $u = \lambda u \Rightarrow \boxed{\lambda = 1}$

⇒ So, A has 1 as a non-zero eigenvalue and corresponding eigenvector is u.

⇒ Now, In power iteration method we know,

⇒ If $v_1, v_2 --- v_n$ eigenvectors then corresponding eigenvalues be $\lambda_1, \lambda_2 --- \lambda_n$, and we have x_0,

$x_0 = c_1 v_1 + --- + c_n v_n$ where c_i's are constant

then, $x_1 = Ax_0 = c_1 \lambda_1 v_1 + --- + c_n \lambda_n v_n$.

⇒ Now in our case, $\lambda_1 = 1$ and all others are 0,

∴ $x_1 = c_1 u$ and it's converge at this iteration because,

⇒ $x_m = \lim_{m \to \infty} c_1 \lambda_1^m u$.

$= \lim_{m \to \infty} c_1 (1)^\infty u = c_1 u$.

⇒ So we didn't need more than 1 iteration as $(1)^m$ is independent of m.

∴ Only 1 iteration power iteration takes to converge to the dominant eigenvalue - eigenvector pair for this matrix. Ans

Ans-4

Ans-④ Given :- let A, be real symmetric positive definite matrix,
⇒ So, we have $x^T A x > 0$ where $A \in R^{n \times n}$ and $x \in R^n$
⇒ To Prove :- All eigenvalues are real and strictly greater than 0, i.e., $\lambda_i > 0$, where λ_i is eigenvalues for A

⇒ Now, we have $x^T A x > 0$,
⇒ we know that if λ is some eigenvalue of matrix M then
∴ $\boxed{Mx = \lambda x}$, where $x \notin 0_{n \times 1}$
∴ Similarly, $Ax = \lambda x$ where λ is some eigenvalue of A.
⇒ So, we can write $x^T A x = x^T \lambda x > 0$,
⇒ Now, λ is some constant value, therefore,
⇒ $x^T \lambda x = \lambda x^T x$,
⇒ Suppose, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, [note :- If $x \in \mathbb{C}$ then $x = \begin{bmatrix} a_1 + ib_1 \\ \vdots \\ a_n + ib_n \end{bmatrix}$, So $x^T = x^*$ and again we will get $x^* x > 0$]
⇒ $x^T x = x_1^2 + x_2^2 + \cdots + x_n^2$.
⇒ $x^T x$ will always be greater than 0 because sum of square of real values is positive.
∴ $\lambda x^T x > 0 \Rightarrow \lambda > 0$.
∴ Our λ is general eigenvalue for A so this shows that all eigenvalues will be strictly greater than 0.
⇒ Now we will check whether all eigenvalues are real or not,
∴ $Ax = \lambda x$, ——— ②
⇒ Take the transpose conjugate,
⇒ $x^* A^* = x^* \lambda^*$, ——— ①
⇒ Now $A^* = A$ from given,
⇒ $x^* A = x^* \lambda^*$, now, multiply x^* both side in ②,
⇒ $x^* A x = x^* \lambda x \Rightarrow x^* \lambda^* x = x^* \lambda x$
⇒ $(\lambda^* - \lambda) x^* x = 0 \Rightarrow \lambda^* = \lambda$ which mean λ is real. [Hence Proved]

Ans-5

Ans ⑤ Given :- $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite.

⇒ A is non-defective,

⇒ Also, $e^A = \sum_{k=0} \frac{1}{k!} \times A^k$,

⇒ Now, we have to find an alternative expression for e^A in terms of the eigenvalues of A.

⇒ We will denote eigenvalues of A as $\lambda_i \in \mathbb{R}$ and eigenvectors of A as $u_i \in \mathbb{R}^n$.

⇒ We know some property of symmetric matrix as :-

- All eigenvalues are real.
- eigenvectors corresponding to distinct eigenvalues are orthogonal.
- The n eigenvectors of a symmetric $n \times n$ matrix are all linearly independent. Thus they can be a basis for $\mathbb{R}^n$.

⇒ Now by theorem if A is non-defective then A can be $X \Lambda X^{-1}$ where Λ is a diagonal matrix with eigenvalues as diagonal elements.

∴ $\Lambda = \text{diag}(\lambda_1 \lambda_2 \ldots \lambda_n)$

⇒ Now from the property of symmetric matrix X matrix can be form by n eigenvectors as they basis for $\mathbb{R}^n$. and all are linearly independent

⇒ $X = [u_1 \; u_2 \; \ldots \; u_n]$

⇒ Now this X matrix is orthogonal as all eigenvectors are orthogonal.

∴ $X^{-1} = X^T$, $X^T X = I_n$.

$\Rightarrow$ Putting $A = X\Lambda X^{-1}$ in e^A,

$$\Rightarrow e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{(X\Lambda X^{-1})^k}{k!}$$

$$\Rightarrow e^A = \sum_{k=0}^{\infty} \frac{(X\Lambda X^{-1})^k}{k!},$$

$\Rightarrow$ Now, $(X\Lambda X^{-1})^k = (X\Lambda X^{-1})^{k-1}(X\Lambda X^{-1})$

$$\Rightarrow (X\Lambda X^{-1})^k = (X\Lambda X^{-1})^{k-2}(X\Lambda \underbrace{\boxed{X^{-1}X}}_{I_n}\Lambda X^{-1})$$

$$= (X\Lambda X^{-1})^{k-2}(X\Lambda^2 X^{-1})$$

$\Rightarrow$ Similarly as expand more we have,

$$\Rightarrow (X\Lambda X^{-1})^k = X\Lambda^k X^{-1},$$

$$\therefore e^A = \sum_{k=0}^{\infty} \frac{X\Lambda^k X^{-1}}{k!}$$

$$\therefore e^A = X e^{\Lambda} X^{-1}$$

$\Rightarrow$ Now Λ is a diagonal matrix,

$\Rightarrow$ So, $\Lambda^k = \text{diag}(\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k)$ as when we multiply diagonal matrix many times only diagonal elements of same cell multiply itself.

$\therefore$ Alternative expression for e^A will be :-

$$\Rightarrow e^A = [u_1, u_2 \ldots u_n] \begin{bmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots \\ & & & e^{\lambda_n} \end{bmatrix} [u_1, u_2 \ldots u_n]^T$$

Ans-6

Maximum Magnitude Eigenvalue and corresponding Eigenvector :-

```
Eigenvalue :-
[11.]
Eigenvector :-
[[0.5 ]
 [1.  ]
 [0.75]]
```

Minimum Magnitude Eigenvalue and corresponding Eigenvector :-

```
Eigenvalue :-
[2.]
Eigenvector :-
[[-0.2]
 [-0.4]
 [ 1. ]]
```

Output From inbuilt coroutine :-

```
Result from np.linalg.eig()
[11. -2. -3.]
[[ 3.71390676e-01  1.82574186e-01  2.17732649e-17]
 [ 7.42781353e-01  3.65148372e-01 -5.54700196e-01]
 [ 5.57086015e-01 -9.12870929e-01  8.32050294e-01]]
```

So, we are getting desired result as magnitude wise from the output of np.linalg.eig() we are getting 11 as maximum magnitude eigenvalue and 2 as minimum magnitude eigenvalue and same result we are getting from our normalized power iteration algorithm and Inverse iteration algorithm and also corresponding eigenvector are similar but they have little bit of deviation.

Ans-7

Output from numpy.linalg.eigh() :-

```
Eigenvalues :-
[0.57893339 2.13307448 7.28799214]

Eigenvectors :-
[[-0.0431682  -0.49742503 -0.86643225]
 [-0.35073145  0.8195891  -0.45305757]
 [ 0.9354806   0.28432735 -0.20984279]]
```

Output from our Algorithm :-

```
Eigenvalue :-
[2.13307448]
Eigenvector :-
[[-0.60692002]
 [ 1.        ]
 [ 0.34691451]]
```

So, we are getting 2.13307448 eigenvalue which is nearest to 2 which we can see from the inbuilt result and our task is also to compute the eigenvalue which is nearest to 2.Normalized eigenvector from shifted inversion algorithm and from inbuilt is having deviation to the eigenvector which we get from inbuilt output.

Ans-8

Expected Result from library routine :-

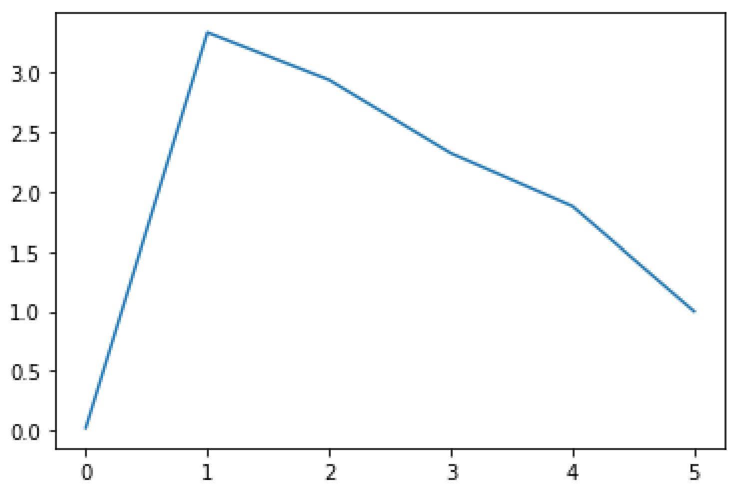
```
Result from np.linalg.eig()
[11. -2. -3.]
[[ 3.71390676e-01  1.82574186e-01  2.17732649e-17]
 [ 7.42781353e-01  3.65148372e-01 -5.54700196e-01]
 [ 5.57086015e-01 -9.12870929e-01  8.32050294e-01]]
```

Result from our Algorithm :-

```
Rate :-  1.0
Eigenvector :-
[[0.5 ]
 [1.  ]
 [0.75]]
Eigenvalue :-  [[11.]]
```

We are getting the max magnitude eigenvalue and we have our corresponding normalized eigenvector and our eigenvalue is exactly equal to our maximum eigenvalue from inbuilt and also our output eigenvector from Rayleigh Quotient also similar to inbuilt but having little bit of deviation.

Plot for rate of convergence :-



Ans - 9

Expected Eigenvalues for Problem 6 :-

```
[11. -2. -3.]
```

Expected Eigenvalues for Problem 7:-

```
[0.57893339 2.13307448 7.28799214]
```

Results from our Algorithm :-

```
Result For Problem 6 :-
[[11.          0.72199487 -6.18642278]
 [ 0.         -3.         -3.8996021 ]
 [ 0.          0.         -2.        ]]


Result For Problem 7 :-
[[ 7.28799214e+00  5.53782393e-16 -1.20642700e-16]
 [ 0.00000000e+00  2.13307448e+00  1.34800587e-16]
 [ 0.00000000e+00  0.00000000e+00  5.78933386e-01]]
```

So, we are getting correct eigenvalues for both the problems as diagonal elements represent all the eigenvalues of the given matrix.