HOMEWORK-4
MTH 373/573-SCIENTIFIC COMPUTING
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2020341

Ans-1

(a)

Ans-① Data points :- $\{(-2,15), (0,-1), (1,0), (3,-2)\}$,

(i) Using the monomial basis :-

⇒ We know, $Vb = y$ where V is Vandermonde matrix and b is the coefficient vector and y is output matrix,

$$\therefore V = \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix}$$

$$\text{and, } y = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

⇒ Clearly, $b_1 = -1$

⇒ Solving above eq^n :- i.e, $\boxed{Vb = y}$

⇒ $b_1 - 2b_2 + 4b_3 - 8b_4 = 15$ — ③

⇒ $b_1 + b_2 + b_3 + b_4 = 0$ — ①

⇒ $b_1 + 3b_2 + 9b_3 + 27b_4 = -2$ — ②

⇒ Subtract ① from ② :-

⇒ $0 + 2b_2 + 8b_3 + 26b_4 = -2$

⇒ Add ③ in above eq^n :-

⇒ $12b_3 + 18b_4 = 13$

⇒ On solving; we have :-

⇒ $b_1 = -1$, $b_2 = -\frac{8}{15}$, $b_3 = \frac{34}{15}$, $b_4 = -\frac{11}{15}$

$$\therefore b = \begin{bmatrix} -1 \\ -\frac{8}{15} \\ \frac{34}{15} \\ -\frac{11}{15} \end{bmatrix} \quad \text{Ans}$$

(b)

DELTA Pg No.

(ii) using the Lagrange for :-

$\Rightarrow$ General term for lagrange $\Rightarrow f(x) = y_0 L_0(x) + y_1 L_1(x) \cdots y_n L_n(x)$

$\Rightarrow$ So now we have given degree of polynomial 3 so our $n = 3$.

$$\therefore L_0(x) = \frac{(x-0)(x-1)(x-3)}{(-2-0)(-2-1)(-2-3)}$$

$$= \frac{x(x-1)(x-3)}{-2 \times -3 \times -5} = \frac{x^3 - 4x^2 + 3x}{-30}$$

$\Rightarrow y_0 = 15,$

$$\Rightarrow \text{Now, } L_1(x) = \frac{(x+2)(x-1)(x-3)}{2 \times (-1) \times (-3)} = \frac{x^3 - 2x^2 - 5x + 6}{6}$$

$\Rightarrow y_1 = -1,$

$$\Rightarrow \text{Now, } L_3(x) = \frac{(x+2)(x)(x-1)}{5 \times 3 \times 2} = \frac{x^3 + x^2 - 2x}{30}$$

$\Rightarrow y_3 = -2,$

$$\therefore f(x) = \frac{15}{-30}(x^3 - 4x^2 + 3x) + \left(\frac{-1}{6}\right)(x^3 - 2x^2 - 5x + 6)$$

$$+ 0(L_2(x)) + \frac{(-2)}{30}(x^3 + x^2 - 2x)$$

$$= \frac{-x^3}{2} + 2x^2 - \frac{3x}{2} - \frac{x^3}{6} + \frac{x^2}{3} + \frac{5x}{6} - 1 + \left(\frac{-x^3}{15}\right) - \frac{x^2}{15}$$

$$+ \frac{2x}{15}$$

$$= -\frac{x^3}{2} - \frac{x^3}{6} - \frac{x^3}{15} + 2x^2 + \frac{x^2}{3} - \frac{x^2}{15} - \frac{3x}{2} + \frac{5x}{6} + \frac{2x}{15} - 1$$

$$\Rightarrow \boxed{f(x) = \frac{-11}{15}x^3 + \frac{34}{15}x^2 - \frac{8}{15}x - 1} \text{ Ans}$$

(c)

(iii) Using the Newton basis :-

⇒ First we calculate x_is from $Ax = b$ where,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 5 & 15 & 30 \end{bmatrix}$$

and $b = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 5 & 15 & 30 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

⇒ $x_1 = 15$

⇒ $x_2 = \frac{-16}{2} = -8$

⇒ $x_3 = 3$ and $x_4 = \frac{-11}{15}$.

⇒ Now we have $f(x) = x_1 + x_2(t-t_1) \ldots - x_n (t-t_1)(t-t_2) \ldots (t-t_{n-1})$

$\therefore y = 15 + (-8)(t+2) + 3(t+2)(t) - \frac{11}{15}(t+2)(t)(t-1)$

⇒ $y = 15 - 8t - 16 + 3t^2 + 6t - \frac{11}{15}(t^3 + t^2 + 2t^2 - 2t)$

⇒ $y = -\frac{11}{15}t^3 + \frac{34}{15}t^2 - \frac{8}{15}t - 1$

⇒ Replacing $t \to x$, $\boxed{y = \frac{-11}{15}x^3 + \frac{34}{15}x^2 - \frac{8}{15}x - 1}$

(d)

A-(d) From monomial basis :-

we have, $b = \begin{bmatrix} -1 \\ -\frac{8}{15} \\ \frac{34}{15} \\ -\frac{11}{15} \end{bmatrix}$

$\therefore$ Corresponding eq^n :- $y = \frac{-11}{15} x^3 + \frac{34}{15} x^2 - \frac{8}{15} x - 1$

$\Rightarrow$ From newton basis :-

we have, $y = \frac{-11}{15} x^3 + \frac{34}{15} x^2 - \frac{8}{15} x - 1$

$\Rightarrow$ From langrange basis :-

we have, $y = \frac{-11}{15} x^3 + \frac{34}{15} x^2 - \frac{8}{15} x - 1$

$\Rightarrow$ So we can clearly observe that the polynomial eq^n from all three methods are similar and coeffi also same i.e, x^3 coeff :- $\frac{-11}{15}$, x^2 coeff :- $\frac{34}{15}$,

x coeff :- $\frac{-8}{15}$, constant = -1.

$\therefore$ All methods give similar polynomial eq^n. Ans

Ans-(2)

Ans-2 ② Integrals using mid point rule :-

⇒ Mid point rule is :-

$$\int_a^b f(t)\,dt \approx (b-a)\, f\left(\frac{a+b}{2}\right)$$

∴ (i) $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$

⇒ $f(x) = \frac{1}{1+x^2}$ and $a=0,\ b=1$.

∴ $\int_0^1 \frac{1}{1+x^2}\,dx \approx 1 \times \frac{1}{1+(0.5)^2} = \frac{4}{5}$ or 0.8.

⇒ Now $\frac{\pi}{4} \approx \frac{3.14}{4} = 0.785$.

∴ Results are very closer for $\int_0^1 \frac{1}{1+x^2}\,dx$.

(ii) $\int_0^1 \sqrt{x}\log x\,dx = -\frac{4}{9}$.

⇒ $f(x) = \sqrt{x}\log x$ and $a=0,\ b=1$

∴ $\int_0^1 \sqrt{x}\log x\,dx \approx 1 \times \sqrt{\frac{1}{2}} \times \log\frac{1}{2} = 0.7 \times (-0.3)$
$= -0.21$

⇒ $\frac{4}{9} = 0.45$.

∴ Results are not very much clear as there a high difference between -0.21 and -0.45.

⇒ Integrals using Simpson's Rule :-

⇒ $\int_a^b f(t)\,dt \approx \left(\frac{b-a}{6}\right) \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$

(i) $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$

⇒ $f(x) = \frac{1}{1+x^2}$ and $a = 0,\ b = 1.$

∴ $\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1}{6}\left(1 + 4 \times \frac{4}{5} + \frac{1}{2}\right)$

$\approx \frac{1}{6}\left(1 + \frac{16}{5} + \frac{1}{2}\right) = \frac{4.7}{6} = 0.783.$

⇒ $\frac{\pi}{4} \approx 0.785,$

∴ Our result is more closer from true value and it is better than from above 2 rules we discuss.

(ii) $\int_0^1 \sqrt{x}\log x\,dx = -\frac{4}{9}.$

⇒ $f(x) = \sqrt{x}\log x$ and $a = 0, b = 1.$

⇒ $\int_0^1 f(x)\,dx \approx \frac{1}{6}\left(0 + 4 \times \frac{1}{\sqrt{2}} \times \log\frac{1}{2} + 0\right)$

$\approx -0.14.$

∴ In this case also we have a high deviated value from true value.

$\Rightarrow$ Integrals using trapezoid rule,

$\because$ Trapezoid rule is :-

$$\int_a^b f(t)\,dt \approx \frac{(b-a)}{2}\,(f(a)+f(b))$$

(i) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$

$\because$ $f(x) = \dfrac{1}{1+x^2}$ and $a=0, b=1$

$\therefore$ $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx \approx \left(\frac{1}{2}\right)(f(0)+f(1)) = \left(1+\frac{1}{2}\right)\times\frac{1}{2} = \frac{3}{4}$

$= 0.75$

$\because$ We saw in previous part $\frac{\pi}{4} \approx 0.785$.

$\therefore$ Our result is very close to actual result.

(ii) $\displaystyle\int_0^1 \sqrt{x}\log x\,dx = -\frac{4}{9}$.

$\because$ $f(x) = \sqrt{x}\log x$ and $a=0, b=1$.

$\therefore$ $\displaystyle\int_0^1 \sqrt{x}\log x\,dx \approx \frac{1}{2}(0+0) = 0$.

$\therefore$ This rule doesn't work for above eqn as this shows a high deviation from actual result, i.e, -0.45.

=> Integrals using 2-point Gaussian quadrature :-

=> We will convert (0, 1) to (-1, 1) because in 2-point quadrature we have $x_1 = \frac{-1}{\sqrt{3}}$ and $x_2 = \frac{1}{\sqrt{3}}$ and $\omega_1 = 1$, $\omega_2 = 1$.

$$\therefore \quad I(f) = \frac{b-a}{\beta-\alpha} \sum_{i=1}^{n} \omega_i f\left(\frac{(b-a)x_i + a\beta - b\alpha}{\beta-\alpha}\right)$$

$\therefore \quad b = 1, a = 0, \beta = 1, \alpha = -1$

$$\Rightarrow I(f) = \frac{1}{2} \sum_{i=1}^{2} \omega_i f\left(\frac{x_i+1}{2}\right)$$

$$\therefore \quad I(f) = \frac{1}{2}\left(f\left(\frac{x_1+1}{2}\right) + f\left(\frac{x_2+1}{2}\right)\right)$$

=> Now for,

(i) $\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$

=> $f(x) = \frac{1}{1+x^2}$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2} dx \approx \frac{1}{2}\left(f\left(\frac{\frac{-1}{\sqrt{3}}+1}{2}\right) + f\left(\frac{\frac{+1}{\sqrt{3}}+1}{2}\right)\right)$$

$$\approx \frac{1}{2}\left(f\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right) + f\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\right)$$

$$\approx \frac{1}{2}\left(\frac{1}{1+\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)^2} + \frac{1}{1+\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)^2}\right)$$

$$\approx 6\left(\frac{32}{256-12}\right) = \frac{192}{244}$$

$$\approx 0.786$$

$\therefore$ We are getting 0.786 which is very close to actual value

DELTA Pg No.

for (ii) $\int_0^1 \sqrt{x} \log x \, dx = -\frac{4}{9}$.

$f(x) = \sqrt{x} \log x$,

$$\therefore \int_0^1 \sqrt{x} \log x \approx \frac{1}{2}\left(f\left(\frac{1-\frac{1}{\sqrt{3}}}{2}\right) + f\left(\frac{1+\frac{1}{\sqrt{3}}}{2}\right)\right)$$

$$\approx \frac{1}{2}\left(f\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right) + f\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\right)$$

$$\approx \frac{1}{2}\left(\frac{\sqrt{\sqrt{3}-1}}{\sqrt{2\sqrt{3}}} \times \log\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right) + \frac{\sqrt{\sqrt{3}+1}}{\sqrt{2\sqrt{3}}} \times \log\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\right)$$

$$\approx \frac{1}{2}\left(0.45 \times (-1.56) + (-0.24) \times 0.88\right)$$

$$\approx \frac{1}{2}\left(-0.702 - 0.2112\right)$$

$$\approx -0.4566.$$

$\therefore$ we get -0.456 which is close to true value $-\frac{4}{9}$ and yes this works best in both the example from our previous three rule.

Results :-

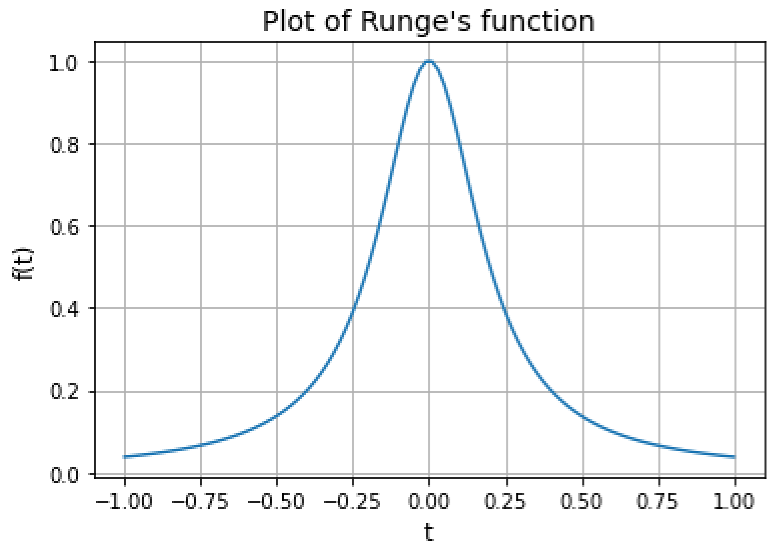
Rule	1st integral Actual value = 0.785	2nd integral Actual value = -0.45
Mid Point	**0.8**	**-0.21**
Simpson	**0.783**	**-0.14**
Trapezoid	**0.785**	**0**
2 Point Gaussian	**0.786**	**-0.4566**

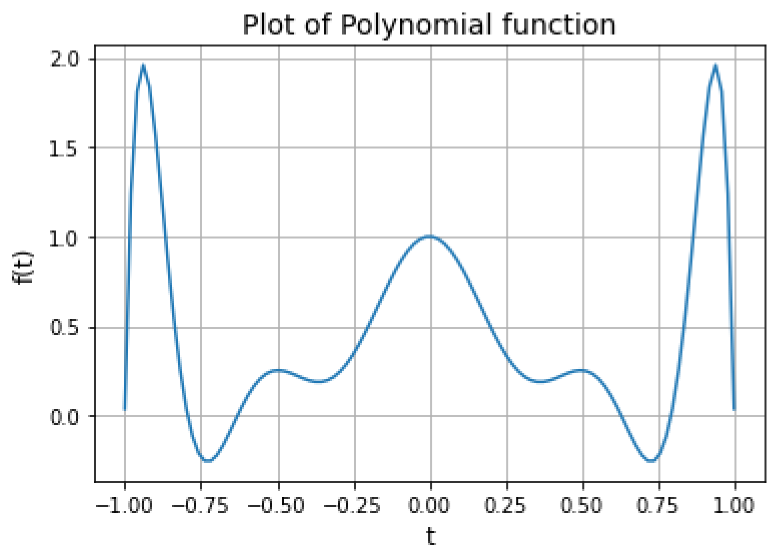
So, 2 Point Gaussian works best.

Ans-3

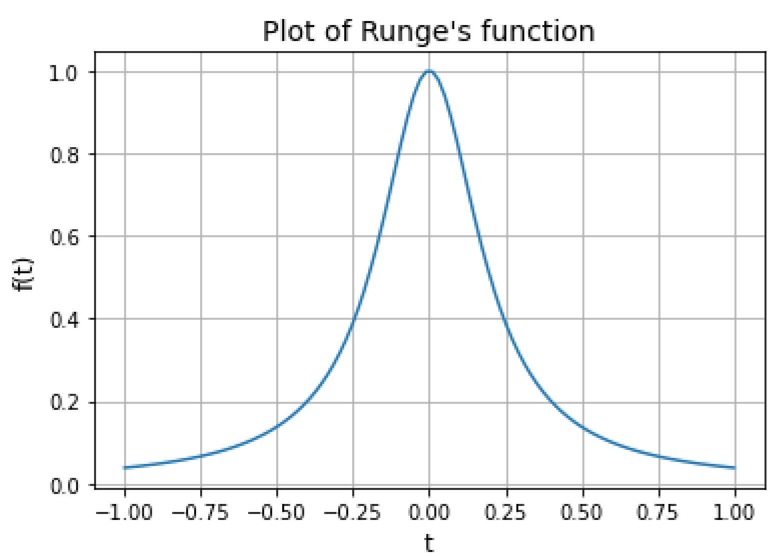
(A) Polynomial interpolation

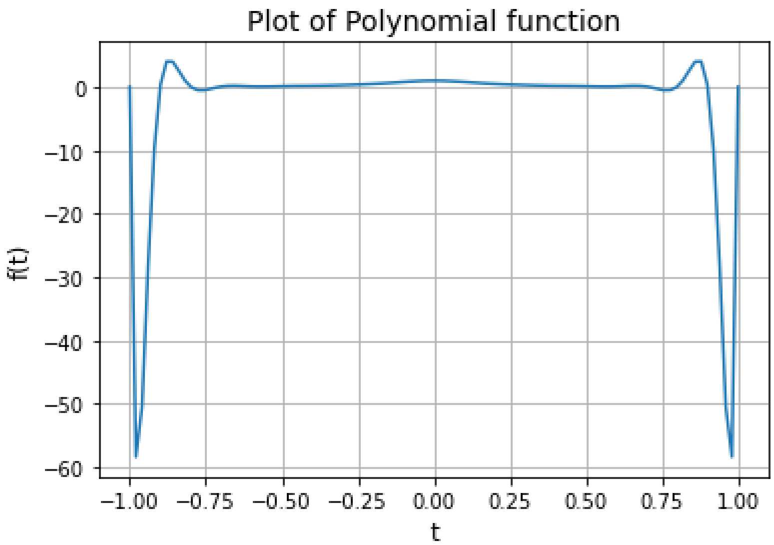
For n = 11 :-


Plot of Runge's function
1.0
0.8
0.6
0.4
0.2
0.0
f(t)
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00
t


Plot of Polynomial function
2.0
1.5
1.0
0.5
0.0
f(t)
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00
t

For n = 21 :-


Plot of Runge's function
1.0
0.8
0.6
0.4
0.2
0.0
f(t)
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00
t



(B) Cubic Spline Interpolation :-

Ans-4

```
For n =  2  Computed Value :  21.589657875842885  Relative Error :  2.1484901032945363
For n =  4  Computed Value :  -9.76123404248495  Relative Error :  0.4807383813951242
For n =  6  Computed Value :  -18.478977278578718  Relative Error :  0.016986621765830726
For n =  8  Computed Value :  -19.340167311023777  Relative Error :  0.028825508977831458
For n =  10  Computed Value :  -19.212121607590415  Relative Error :  0.02201395022093469
For n =  12  Computed Value :  -19.054556800438228  Relative Error :  0.013632084112519826
For n =  14  Computed Value :  -18.95441593250772  Relative Error :  0.0083049595969341
For n =  16  Computed Value :  -18.895837167333262  Relative Error :  0.005188785526322299
For n =  18  Computed Value :  -18.861389454893224  Relative Error :  0.003356294384219156
For n =  20  Computed Value :  -18.840550524795436  Relative Error :  0.00224774022749321
For n =  22  Computed Value :  -18.8275125673081  Relative Error :  0.0015541690172642026
For n =  24  Computed Value :  -18.819080296099532  Relative Error :  0.0011056033157125615
For n =  26  Computed Value :  -18.813456740139127  Relative Error :  0.0008064508973190062
For n =  28  Computed Value :  -18.80960091017638  Relative Error :  0.0006013349766466979
For n =  30  Computed Value :  -18.806890752169185  Relative Error :  0.0004571645748957297
For n =  32  Computed Value :  -18.804943245212424  Relative Error :  0.00035356439379052716
For n =  34  Computed Value :  -18.803515860409295  Relative Error :  0.0002776327912883191
For n =  36  Computed Value :  -18.802451029313186  Relative Error :  0.00022098770767494774
For n =  38  Computed Value :  -18.80164395055663  Relative Error :  0.00017805409706317905
For n =  40  Computed Value :  -18.801023413379088  Relative Error :  0.00014504380985840386
For n =  42  Computed Value :  -18.80054008050214  Relative Error :  0.00011933228497171242
For n =  44  Computed Value :  -18.800159157505913  Relative Error :  9.906858770509814e-05
For n =  46  Computed Value :  -18.799855705170728  Relative Error :  8.292604363210574e-05
For n =  48  Computed Value :  -18.799611580789517  Relative Error :  6.993952770829459e-05
For n =  50  Computed Value :  -18.79941340486647  Relative Error :  5.939730014551699e-05
For n =  52  Computed Value :  -18.79925118600978  Relative Error :  5.0767855781541846e-05
For n =  54  Computed Value :  -18.799117376050088  Relative Error :  4.3649659869756555e-05
For n =  56  Computed Value :  -18.799006211148814  Relative Error :  3.773609747425513e-05
For n =  58  Computed Value :  -18.7989132460125  Relative Error :  3.2790695392270876e-05
For n =  60  Computed Value :  -18.798835020407786  Relative Error :  2.8629381982178344e-05
For n =  62  Computed Value :  -18.798768817555786  Relative Error :  2.5107634635850923e-05
For n =  64  Computed Value :  -18.7987124871472  Relative Error :  2.2111064836170353e-05
```

Ans-5

```
Absolute Error for h = 10^-1 is :  0.06940588094341621
Absolute Error for h = 10^-2 is :  0.005227002682469728
Absolute Error for h = 10^-3 is :  0.0005069717636996818
Absolute Error for h = 10^-4 is :  5.0541282400395904e-05
Absolute Error for h = 10^-5 is :  5.052570714092486e-06
Absolute Error for h = 10^-6 is :  5.052526364512921e-07
Absolute Error for h = 10^-7 is :  4.939506254020287e-08
Absolute Error for h = 10^-8 is :  6.096364579821767e-09
Absolute Error for h = 10^-9 is :  4.941478665143606e-08
Absolute Error for h = 10^-10 is :  6.160751581107960e-08
Absolute Error for h = 10^-11 is :  1.1718305404362361e-06
Absolute Error for h = 10^-12 is :  5.433932069082159e-05
Absolute Error for h = 10^-13 is :  0.0002763839256158529
Absolute Error for h = 10^-14 is :  0.004717276024116479
Absolute Error for h = 10^-15 is :  0.11573957848663208
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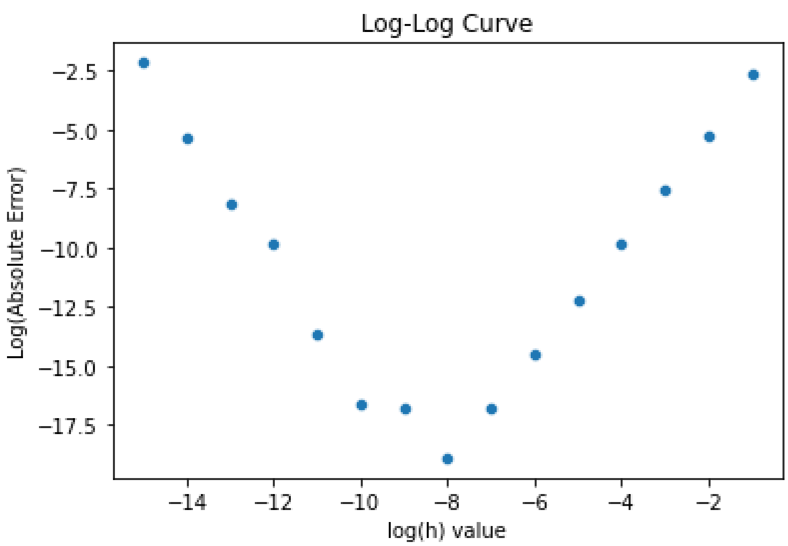

Explanation :-

It doesn't converge to 0 as h smaller because after a particular h value in floating system it will neglect the more smaller values so till h goes to 10^{-8} the absolute error keeps on decreasing but after that floating point system start neglecting h value or start generating error in computation by which derivative in f(x+h) - f(x) start becomes 0 as x+h -> x because of floating point system and compute derivative will be 0 but the actual is non-zero which increase error and doesn't converge to 0.