MTH 372 : STATISTICAL INFERENCE
ASSIGNMENT-1

Q.1

a) Generation of 1000 samples using rexp() function in R for Exponential Distribution for λ = 1,2,3,4 :-

Values	
data1	num [1:1000] 0.8435 0.5766 1.3291 0.0316 0.0562 ...
data2	num [1:1000] 0.1117 0.0588 1.5258 0.2453 1.0191 ...
data3	num [1:1000] 0.68054 0.00159 0.83791 0.22473 0.03614...
data4	num [1:1000] 0.22 0.021 0.14 0.195 0.24 ...

b) Calculation for Method of Moment and Maximum Likelihood Estimation :-

DELTA Pg No.

* Exponential distribution (λ) :-

$\Rightarrow f(x;\lambda) = \lambda e^{-\lambda x}$

$\Rightarrow$ Maximum Likelihood Estimation :-

$\Rightarrow$ let Likelihood function, $L(\theta)$, and suppose we have n-iid variables or samples of Exponential distribution.

$\therefore L(\theta) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$

$\Rightarrow \quad = \lambda^n e^{-\lambda \Sigma x_i}$

$\Rightarrow$ Now for simplicity we will take log,

$\Rightarrow \log(L(\theta)) = l(\theta) \Rightarrow$ log likelihood function,

$\Rightarrow l(\theta) = -\lambda \Sigma x_i + n \log \lambda$.

$\Rightarrow$ Now we will differentiate for minima or maxima,

$\therefore \frac{d(l(\theta))}{d\theta} = 0$

$\Rightarrow -\Sigma x_i + \frac{n}{\lambda} = 0 \Rightarrow \lambda = \frac{n}{\Sigma x_i} = \frac{1}{\text{mean}} = \frac{1}{\mu}$

$\Rightarrow$ Now for checking whether the point is maxima or minima we again differentiate it,

$\therefore \frac{d^2 l(\theta)}{d\theta^2} = \frac{-n}{\lambda^2}$ which is < 0.

$\therefore \lambda = \frac{1}{\mu}$ is maxima.

$\therefore$ MLE value :- $\left(\frac{n}{\Sigma x_i}\right)^n e^{-n}$

$= \left(\frac{n}{e \Sigma x_i}\right)^n$ Ans.

Method of moment :-

In exponential distribution we have one parameter (λ), so we need only one eqn to solve i.e.,

$$E(x) = \frac{\sum x_i}{n}$$

Expected value of exponential distribution is $\frac{1}{\lambda}$.

$$\therefore \frac{1}{\lambda} = \frac{\sum x_i}{n}.$$

So from here we get $\lambda = \frac{n}{\sum x_i} = \frac{1}{\mu}$. Ans

Results from R :-

For $\lambda = 1$:-

Initial value = Method of moments = 1/mean :-

```
$par
[1] 0.9708933

$value
[1] -1029.539
```

Initial value = 1/mean - 0.4 :-

```
$par
[1] 0.9708933

$value
[1] -1029.539
```

Initial value = 1/mean + 0.4 :-

```
$par
[1] 0.9708933

$value
[1] -1029.539
```

So from all three initial values we get the same output which is our maximum likelihood and as expected our par is close to 1 at which our likelihood function yields maximum value.

For $\lambda = 2$:-
Initial value = Method of moments = 1/mean :-

```
$par
[1] 2.012931

$value
[1] -300.4081
```

Initial value = 1/mean - 0.4 :-

```
$par
[1] 2.012931

$value
[1] -300.4081
```

Initial value = 1/mean + 0.4 :-

```
$par
[1] 2.012931

$value
[1] -300.4081
```

So from all three initial values we get the same output which is our maximum likelihood and as expected our par is close to 2 at which our likelihood function yields maximum value.

For $\lambda = 3$:-
Initial value = Method of moments = 1/mean :-

```
$par
[1] 2.929678

$value
[1] 74.89265
```

Initial value = 1/mean - 0.4 :-

```
$par
[1] 2.929678

$value
[1] 74.89265
```

Initial value = 1/mean + 0.4 :-

```
$par
[1] 2.929678

$value
[1] 74.89265
```

So from all three initial values we get the same output which is our maximum likelihood and as expected our par is close to 3 at which our likelihood function yields maximum value.

For $\lambda = 4$:-

Initial value = Method of moments = 1/mean:-

```
$par
[1] 4.200171

$value
[1] 435.1253
```

Initial value = 1/mean - 0.4 :-

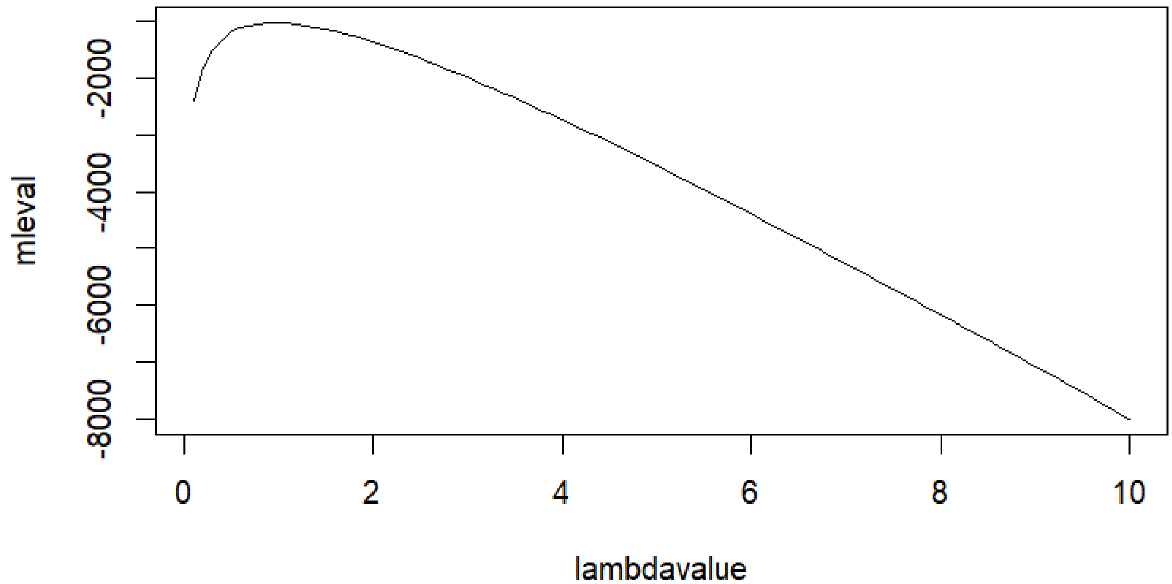
```
$par
[1] 4.200171

$value
[1] 435.1253
```

Initial value = 1/mean + 0.4 :-

```
$par
[1] 4.200171

$value
[1] 435.1253
```

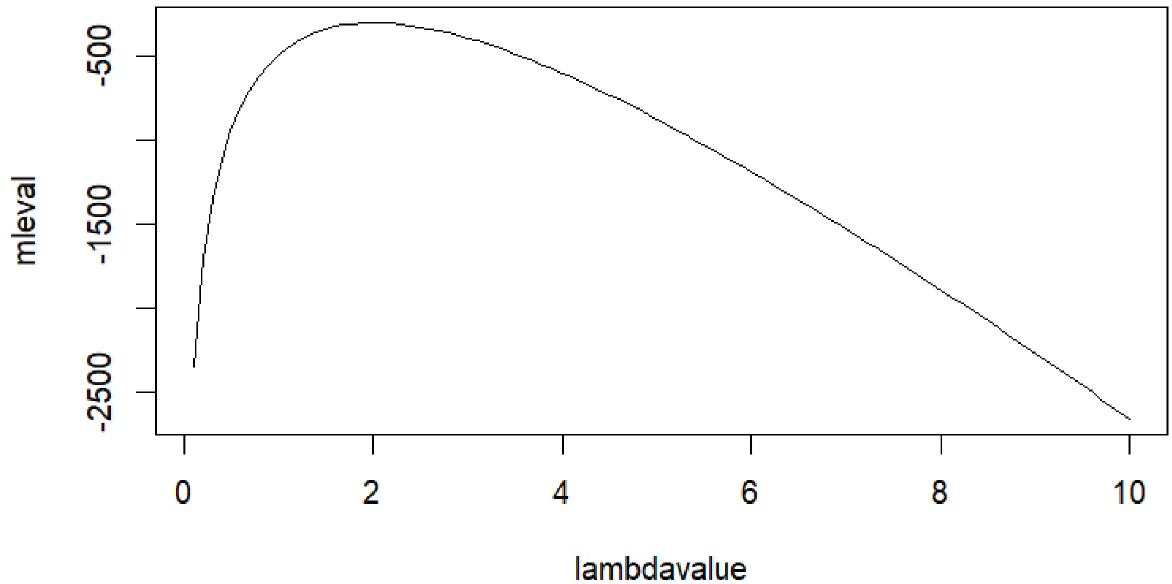
So from all three initial values we get the same output which is our maximum likelihood and as expected our par is close to 4 at which our likelihood function yields maximum value.

c) Plot for $\lambda = 1$:-



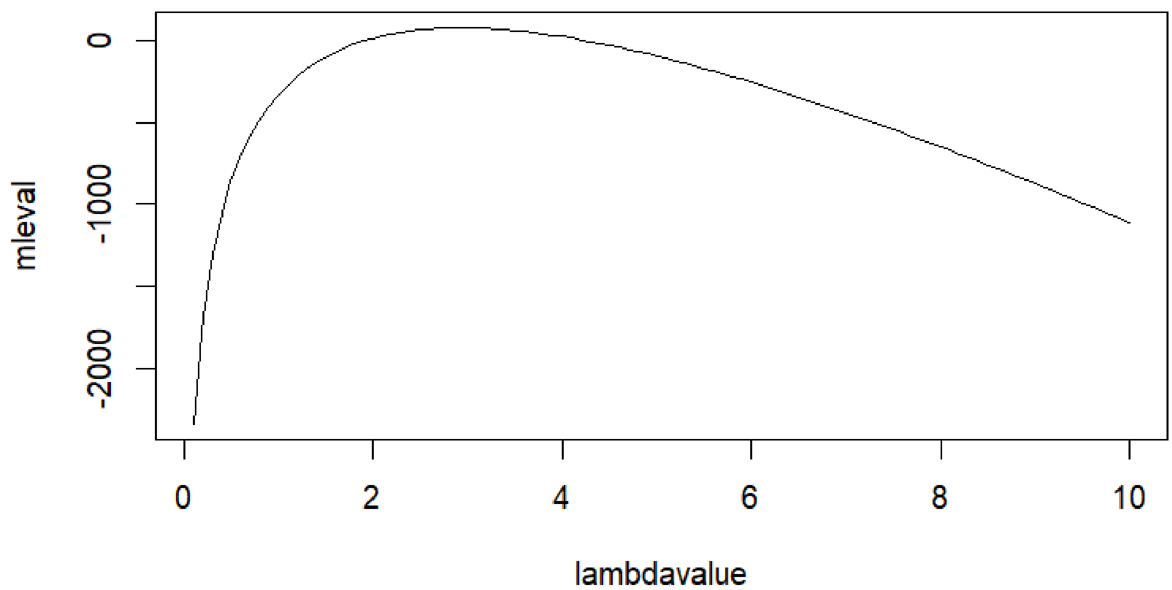
We can clearly see that at λ closes to 1, plot point at maximum mle value. So graphically also our b result for λ = 1 data is verified.

Plot for λ = 2 :-



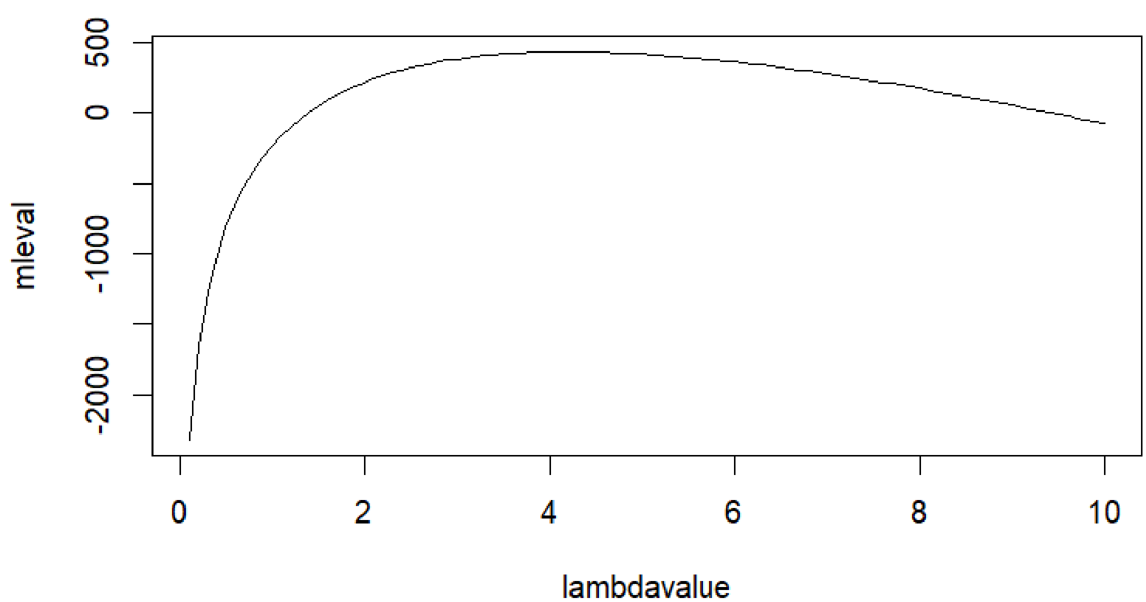
We can clearly see that at λ closes to 2, plot point at maximum mle value. So graphically also our b result for λ = 2 data is verified.

Plot for λ = 3 :-



We can clearly see that at λ closes to 3, plot point at maximum mle value. So graphically also our b result for λ = 3 data is verified.

Plot for λ = 4 :-



We can clearly see that at λ closes to 4, plot point at maximum mle value. So graphically also our b result for λ = 4 data is verified.

Q.2

a) Calculation for Maximum Likelihood Estimation :-

⇒ Normal distribution :- $N(\mu, \sigma^2)$

⇒ Take $\theta_1 = \mu$ and $\sigma^2 = \theta_2$,

$$\therefore \quad f(x) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2\theta_2}(x-\theta_1)^2}$$

⇒ let's take n-iid variables of $N(\mu, \sigma^2)$, so likelihood will be $L(\theta)$,

$$\Rightarrow \quad L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2\theta_2}(x_i-\theta_1)^2}$$

$$\Rightarrow \quad L(\theta) = \frac{1}{(2\pi\theta_2)^{\frac{n}{2}}} e^{-\frac{1}{2\theta_2}\left(\Sigma x_i^2 - 2\theta_1 \Sigma x_i + n\theta_1^2\right)}$$

⇒ Now for simplicity we take log,

⇒ $\log(L(\theta)) = l(\theta)$,

$$\therefore \quad l(\theta) = -\frac{n}{2} \log 2\pi\theta_2 - \frac{1}{2\theta_2}\left(\Sigma x_i^2 - 2\theta_1 \Sigma x_i + n\theta_1^2\right)$$

or

$$l(\theta) = -\frac{n}{2} \log 2\pi\theta_2 - \frac{1}{2\theta_2} \Sigma (x_i - \theta_1)^2 \quad \text{Ans.}$$

⇒ Now for finding maxima or minima, we will differentiate

$$\frac{d(l(\theta))}{d\theta_1} = 0 \quad \text{and} \quad \frac{d\,l(\theta)}{d\theta_2} = 0.$$

$$\therefore \quad \frac{d(l(\theta))}{d\theta_1} = +\frac{1}{\theta_2} \Sigma (x_i - \theta_1) = 0$$

$$= n\theta_1 = \Sigma x_i \Rightarrow \theta_1 = \frac{\Sigma x_i}{n} = \text{sample mean } (\bar{x}).$$

$$\Rightarrow \quad \frac{d(l(\theta))}{d\theta_2} = -\frac{n}{2} \times \frac{1}{\cancel{2\pi}\theta_2} \times \cancel{2\pi} + \frac{1}{2\theta_2^2} \Sigma (x_i - \theta)^2 = 0$$

$$= -\frac{n}{\cancel{2}} + \frac{1}{\cancel{2}\theta_2} \Sigma (x_i - \theta_1)^2 = 0$$

$$= \theta_2 = \frac{\Sigma (x_i - \bar{x})^2}{n} \quad \text{Ans}$$

$\therefore$ We have optimum $\hat{\theta}_1 = \bar{x}$ and $\hat{\theta}_2 = \frac{\sum(x_i - \bar{x})^2}{n}$. Ans

$\Rightarrow \frac{\partial^2 l(\theta)}{\partial(\theta_1)^2} = -\frac{n}{\theta_2}$

$\Rightarrow \frac{\partial^2 l(\theta)}{\partial(\theta_2)^2} = +\frac{n}{2\theta_2^2} - \frac{1}{\theta_2^3}\sum(x_i - \theta_1)^2 = T$

$\Rightarrow \frac{\partial^2 (l(\theta))}{\partial(\theta_1, \theta_2)} = -\frac{1}{\theta_2^2}\sum(x_i - \theta_1)$

$\Rightarrow \frac{\partial^2 (l(\theta))}{\partial(\theta_2 \theta_1)} = -\frac{1}{\theta_2^2}\sum(x_i - \theta_1)$

$\therefore$ Hessian matrix :-

$$H = \begin{bmatrix} -\frac{n}{\theta_2} & -\frac{1}{\theta_2^2}\sum(x_i - \theta_1) \\ -\frac{1}{\theta_2^2}\sum(x_i - \theta_1) & T \end{bmatrix}$$

$\therefore$ for maximum $\frac{\partial^2 l(\theta)}{\partial(\theta_1)^2} < 0$ and here $-\frac{n}{\theta_2} < 0$ because

$\theta_2 = \frac{\sum(x_i - \bar{x})^2}{n}$ which is positive and $T \times -\frac{n}{\theta_2} > 0$

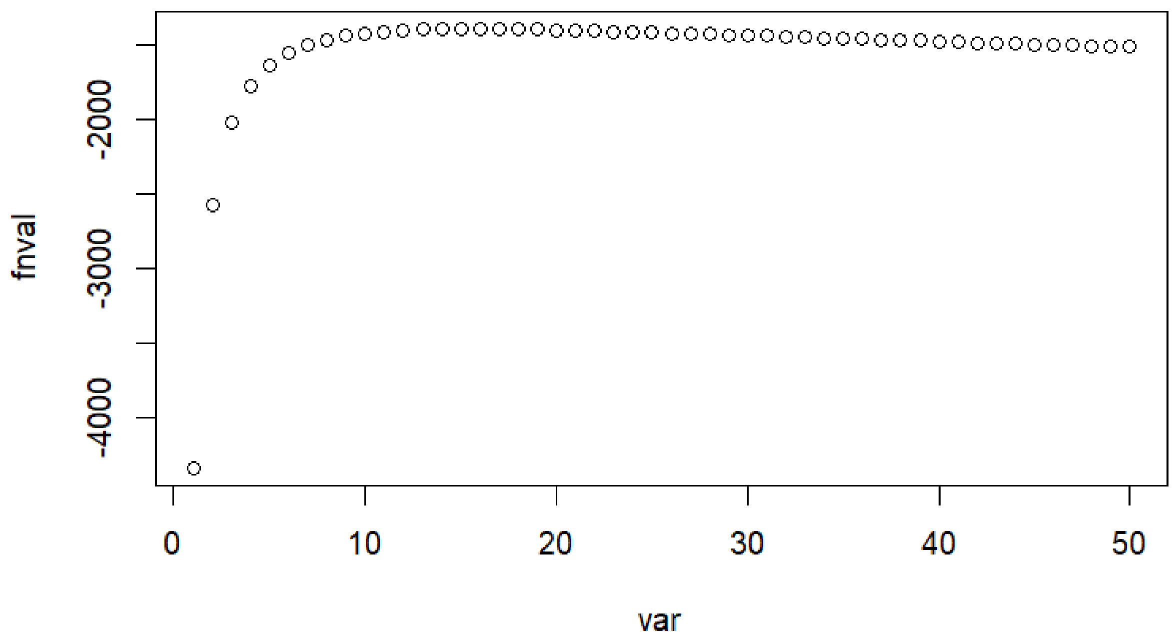
$\Rightarrow \left(\frac{n}{2\theta_2^2} - \frac{n^3}{\sum(x_i - \theta_1)^2}\right)\frac{-n}{\theta_2} = \frac{-1 n^3}{2\sum(x_i - \theta_1)^2} \times \frac{-n}{\theta_2} = \frac{n^4}{2\sum(\theta_1 + x_i)\theta_2} > 0$

$\therefore$ $\hat{\theta}_1$ and $\hat{\theta}_2$ is maximum point or maxima Ans

Output from R code after simulation MLE function :-

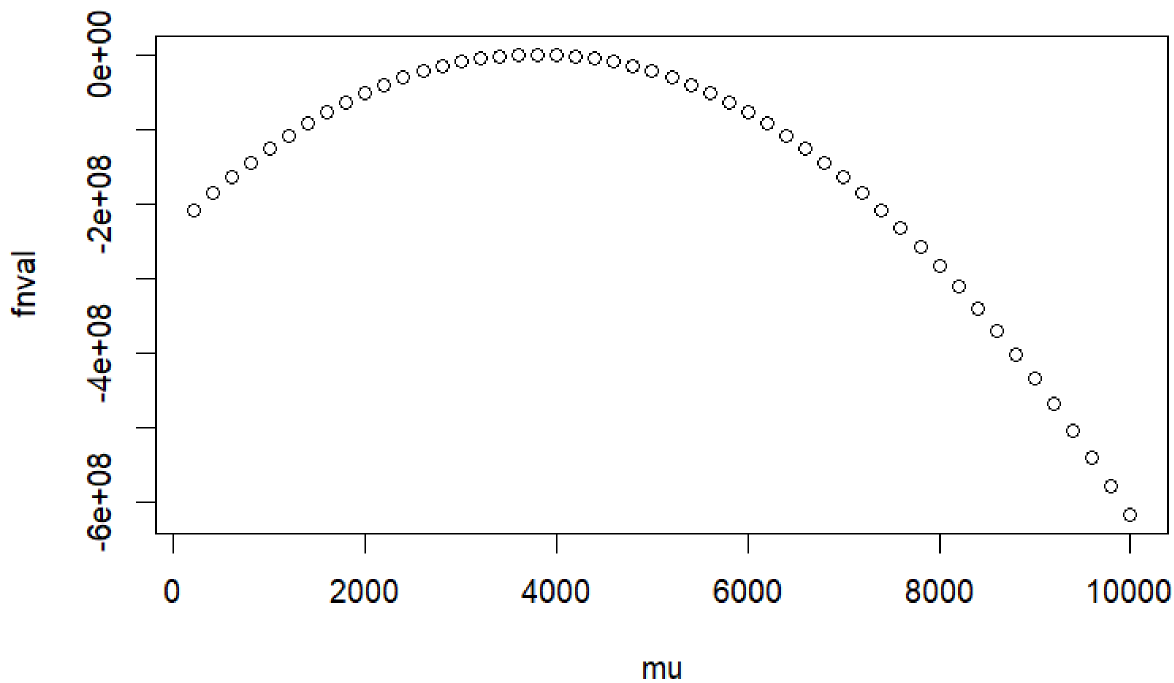
```
$par
[1] 4000.04397   15.53298

$value
[1] -1395.109
```

b) Assuming we know the optimum mean value and varying the variance to different values and we get a plot shown below :-



From the above plot we can clearly see that variance close to 15 gives the maximum likelihood value.

Now,Assuming we know the optimum variance value and varying the mean to different values and we get a plot shown below :-



From the above plot we can clearly see that the mean of the data close to 4000 gives the maximum likelihood value.

c) ML estimate of $e^{-\mu}$:
We know the maximum likelihood value of μ from part a) i.e., μ = 4000.04397, so by invariance property of mle, we have mle of $e^{-\mu}$ is $e^{-4000.04397}$.