VIP Refresher: Linear Algebra and Calculus

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General notations

❏ **Vector** – We note $x \in \mathbb{R}^n$ a vector with n entries, where $x_i \in \mathbb{R}$ is the ith entry:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

❏ **Matrix** – We note $A \in \mathbb{R}^{m \times n}$ a matrix with m rows and n columns, where $A_{i,j} \in \mathbb{R}$ is the entry located in the ith row and j^{th} column:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Remark: the vector x defined above can be viewed as a n × 1 matrix and is more particularly called a column-vector.

❏ **Identity matrix** – The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix with ones in its diagonal and zero everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Remark: for all matrices $A \in \mathbb{R}^{n \times n}$, we have $A \times I = I \times A = A$.

❏ **Diagonal matrix** – A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is a square matrix with nonzero values in its diagonal and zero everywhere else:

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix}$$

Remark: we also note D as diag($d_1,...,d_n$).

Matrix operations

❏ **Vector-vector multiplication** – There are two types of vector-vector products:

- inner product: for $x,y \in \mathbb{R}^n$, we have:

$$x^T y = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$$

- outer product: for $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, we have:

$$xy^T = \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{pmatrix} \in \mathbb{R}^{m \times n}$$

❏ **Matrix-vector multiplication** – The product of matrix $A \in \mathbb{R}^{m \times n}$ and vector $x \in \mathbb{R}^n$ is a vector of size $\mathbb{R}^m$, such that:

$$Ax = \begin{pmatrix} a_{r,1}^T x \\ \vdots \\ a_{r,m}^T x \end{pmatrix} = \sum_{i=1}^{n} a_{c,i} x_i \in \mathbb{R}^m$$

where $a_{r,i}^T$ are the vector rows and $a_{c,j}$ are the vector columns of A, and x_i are the entries of x.

❏ **Matrix-matrix multiplication** – The product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ is a matrix of size $\mathbb{R}^{n \times p}$, such that:

$$AB = \begin{pmatrix} a_{r,1}^T b_{c,1} & \cdots & a_{r,1}^T b_{c,p} \\ \vdots & & \vdots \\ a_{r,m}^T b_{c,1} & \cdots & a_{r,m}^T b_{c,p} \end{pmatrix} = \sum_{i=1}^{n} a_{c,i} b_{r,i}^T \in \mathbb{R}^{n \times p}$$

where $a_{r,i}^T, b_{r,i}^T$ are the vector rows and $a_{c,j}, b_{c,j}$ are the vector columns of A and B respectively.

❏ **Transpose** – The transpose of a matrix $A \in \mathbb{R}^{m \times n}$, noted A^T, is such that its entries are flipped:

$$\forall i,j, \qquad A_{i,j}^T = A_{j,i}$$

Remark: for matrices A,B, we have $(AB)^T = B^T A^T$.

❏ **Inverse** – The inverse of an invertible square matrix A is noted A^{-1} and is the only matrix such that:

$$AA^{-1} = A^{-1}A = I$$

Remark: not all square matrices are invertible. Also, for matrices A,B, we have $(AB)^{-1} = B^{-1}A^{-1}$

❏ **Trace** – The trace of a square matrix A, noted $\text{tr}(A)$, is the sum of its diagonal entries:

$$\text{tr}(A) = \sum_{i=1}^{n} A_{i,i}$$

Remark: for matrices A,B, we have $tr(A^T) = tr(A)$ and $tr(AB) = tr(BA)$

❏ **Determinant** – The determinant of a square matrix $A \in \mathbb{R}^{n \times n}$, noted $|A|$ or $\det(A)$ is expressed recursively in terms of $A_{\backslash i, \backslash j}$, which is the matrix A without its ith row and j^{th} column, as follows:

$$\det(A) = |A| = \sum_{j=1}^{n} (-1)^{i+j} A_{i,j} |A_{\backslash i, \backslash j}|$$

Remark: A is invertible if and only if $|A| \neq 0$. Also, $|AB| = |A||B|$ and $|A^T| = |A|$.

Matrix properties

❑ **Symmetric decomposition** – A given matrix A can be expressed in terms of its symmetric and antisymmetric parts as follows:

$$A = \underbrace{\frac{A + A^T}{2}}_{\text{Symmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{Antisymmetric}}$$

❑ **Norm** – A norm is a function $N : V \longrightarrow [0, +\infty[$ where V is a vector space, and such that for all $x, y \in V$, we have:

- $N(x + y) \leqslant N(x) + N(y)$

- $N(ax) = |a| N(x)$ for a scalar

- if $N(x) = 0$, then $x = 0$

For $x \in V$, the most commonly used norms are summed up in the table below:

Norm	Notation	Definition	Use case
Manhattan, L^1	$||x||_1$	$\displaystyle\sum_{i=1}^{n} |x_i|$	LASSO regularization
Euclidean, L^2	$||x||_2$	$\displaystyle\sqrt{\sum_{i=1}^{n} x_i^2}$	Ridge regularization
p-norm, L^p	$||x||_p$	$\displaystyle\left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}}$	Hölder inequality
Infinity, L^∞	$||x||_\infty$	$\displaystyle\max_i |x_i|$	Uniform convergence

❑ **Linearly dependence** – A set of vectors is said to be linearly dependent if one of the vectors in the set can be defined as a linear combination of the others.
Remark: if no vector can be written this way, then the vectors are said to be linearly independent.

❑ **Matrix rank** – The rank of a given matrix A is noted $\text{rank}(A)$ and is the dimension of the vector space generated by its columns. This is equivalent to the maximum number of linearly independent columns of A.

❑ **Positive semi-definite matrix** – A matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite (PSD) and is noted $A \succeq 0$ if we have:

$$\boxed{A = A^T} \quad \text{and} \quad \boxed{\forall x \in \mathbb{R}^n, \quad x^T A x \geqslant 0}$$

Remark: similarly, a matrix A is said to be positive definite, and is noted $A \succ 0$, if it is a PSD matrix which satisfies for all non-zero vector x, $x^T A x > 0$.

❑ **Eigenvalue, eigenvector** – Given a matrix $A \in \mathbb{R}^{n \times n}$, λ is said to be an eigenvalue of A if there exists a vector $z \in \mathbb{R}^n \backslash \{0\}$, called eigenvector, such that we have:

$$\boxed{Az = \lambda z}$$

❑ **Spectral theorem** – Let $A \in \mathbb{R}^{n \times n}$. If A is symmetric, then A is diagonalizable by a real orthogonal matrix $U \in \mathbb{R}^{n \times n}$. By noting $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n)$, we have:

$$\boxed{\exists \Lambda \text{ diagonal}, \quad A = U \Lambda U^T}$$

❑ **Singular-value decomposition** – For a given matrix A of dimensions $m \times n$, the singular-value decomposition (SVD) is a factorization technique that guarantees the existence of U $m \times m$ unitary, Σ $m \times n$ diagonal and V $n \times n$ unitary matrices, such that:

$$\boxed{A = U \Sigma V^T}$$

Matrix calculus

❑ **Gradient** – Let $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ be a function and $A \in \mathbb{R}^{m \times n}$ be a matrix. The gradient of f with respect to A is a $m \times n$ matrix, noted $\nabla_A f(A)$, such that:

$$\boxed{\left(\nabla_A f(A) \right)_{i,j} = \frac{\partial f(A)}{\partial A_{i,j}}}$$

Remark: the gradient of f is only defined when f is a function that returns a scalar.

❑ **Hessian** – Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $x \in \mathbb{R}^n$ be a vector. The hessian of f with respect to x is a $n \times n$ symmetric matrix, noted $\nabla_x^2 f(x)$, such that:

$$\boxed{\left(\nabla_x^2 f(x) \right)_{i,j} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}}$$

Remark: the hessian of f is only defined when f is a function that returns a scalar.

❑ **Gradient operations** – For matrices A, B, C, the following gradient properties are worth having in mind:

$$\boxed{\nabla_A \text{tr}(AB) = B^T} \qquad \boxed{\nabla_{A^T} f(A) = (\nabla_A f(A))^T}$$

$$\boxed{\nabla_A \text{tr}(ABA^T C) = CAB + C^T AB^T} \qquad \boxed{\nabla_A |A| = |A|(A^{-1})^T}$$

VIP Refresher: Probabilities and Statistics

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Introduction to Probability and Combinatorics

❏ **Sample space** – The set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by S.

❏ **Event** – Any subset E of the sample space is known as an event. That is, an event is a set consisting of possible outcomes of the experiment. If the outcome of the experiment is contained in E, then we say that E has occurred.

❏ **Axioms of probability** – For each event E, we denote $P(E)$ as the probability of event E occuring. By noting $E_1,...,E_n$ mutually exclusive events, we have the 3 following axioms:

$$(1) \quad \boxed{0 \leqslant P(E) \leqslant 1} \quad (2) \quad \boxed{P(S) = 1} \quad (3) \quad \boxed{P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)}$$

❏ **Permutation** – A permutation is an arrangement of r objects from a pool of n objects, in a given order. The number of such arrangements is given by $P(n, r)$, defined as:

$$\boxed{P(n, r) = \frac{n!}{(n-r)!}}$$

❏ **Combination** – A combination is an arrangement of r objects from a pool of n objects, where the order does not matter. The number of such arrangements is given by $C(n, r)$, defined as:

$$\boxed{C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}}$$

Remark: we note that for $0 \leqslant r \leqslant n$, we have $P(n,r) \geqslant C(n,r)$.

Conditional Probability

❏ **Bayes' rule** – For events A and B such that $P(B) > 0$, we have:

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

Remark: we have $P(A \cap B) = P(A)P(B|A) = P(A|B)P(B)$.

❏ **Partition** – Let $\{A_i, i \in [\![1,n]\!]\}$ be such that for all i, $A_i \neq \varnothing$. We say that $\{A_i\}$ is a partition if we have:

$$\boxed{\forall i \neq j, A_i \cap A_j = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{n} A_i = S}$$

Remark: for any event B in the sample space, we have $P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$.

❏ **Extended form of Bayes' rule** – Let $\{A_i, i \in [\![1,n]\!]\}$ be a partition of the sample space. We have:

$$\boxed{P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}}$$

❏ **Independence** – Two events A and B are independent if and only if we have:

$$\boxed{P(A \cap B) = P(A)P(B)}$$

Random Variables

❏ **Random variable** – A random variable, often noted X, is a function that maps every element in a sample space to a real line.

❏ **Cumulative distribution function (CDF)** – The cumulative distribution function F, which is monotonically non-decreasing and is such that $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$, is defined as:

$$\boxed{F(x) = P(X \leqslant x)}$$

Remark: we have $P(a < X \leqslant B) = F(b) - F(a)$.

❏ **Probability density function (PDF)** – The probability density function f is the probability that X takes on values between two adjacent realizations of the random variable.

❏ **Relationships involving the PDF and CDF** – Here are the important properties to know in the discrete (D) and the continuous (C) cases.

Case	CDF F	PDF f	Properties of PDF
(D)	$F(x) = \sum_{x_i \leqslant x} P(X = x_i)$	$f(x_j) = P(X = x_j)$	$0 \leqslant f(x_j) \leqslant 1$ and $\sum_{j} f(x_j) = 1$
(C)	$F(x) = \int_{-\infty}^{x} f(y)dy$	$f(x) = \dfrac{dF}{dx}$	$f(x) \geqslant 0$ and $\int_{-\infty}^{+\infty} f(x)dx = 1$

❏ **Variance** – The variance of a random variable, often noted $\text{Var}(X)$ or σ^2, is a measure of the spread of its distribution function. It is determined as follows:

$$\boxed{\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2}$$

❏ **Standard deviation** – The standard deviation of a random variable, often noted σ, is a measure of the spread of its distribution function which is compatible with the units of the actual random variable. It is determined as follows:

$$\boxed{\sigma = \sqrt{\text{Var}(X)}}$$

❐ **Expectation and Moments of the Distribution** – Here are the expressions of the expected value $E[X]$, generalized expected value $E[g(X)]$, k^{th} moment $E[X^k]$ and characteristic function $\psi(\omega)$ for the discrete and continuous cases:

Case	$E[X]$	$E[g(X)]$	$E[X^k]$	$\psi(\omega)$
(D)	$\sum_{i=1}^{n} x_i f(x_i)$	$\sum_{i=1}^{n} g(x_i) f(x_i)$	$\sum_{i=1}^{n} x_i^k f(x_i)$	$\sum_{i=1}^{n} f(x_i) e^{i\omega x_i}$
(C)	$\int_{-\infty}^{+\infty} x f(x) dx$	$\int_{-\infty}^{+\infty} g(x) f(x) dx$	$\int_{-\infty}^{+\infty} x^k f(x) dx$	$\int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx$

Remark: we have $e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$.

❐ **Revisiting the kth moment** – The kth moment can also be computed with the characteristic function as follows:

$$E[X^k] = \frac{1}{i^k} \left[\frac{\partial^k \psi}{\partial \omega^k} \right]_{\omega=0}$$

❐ **Transformation of random variables** – Let the variables X and Y be linked by some function. By noting f_X and f_Y the distribution function of X and Y respectively, we have:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

❐ **Leibniz integral rule** – Let g be a function of x and potentially c, and a, b boundaries that may depend on c. We have:

$$\frac{\partial}{\partial c} \left(\int_a^b g(x) dx \right) = \frac{\partial b}{\partial c} \cdot g(b) - \frac{\partial a}{\partial c} \cdot g(a) + \int_a^b \frac{\partial g}{\partial c}(x) dx$$

❐ **Chebyshev's inequality** – Let X be a random variable with expected value μ and standard deviation σ. For $k, \sigma > 0$, we have the following inequality:

$$P(|X - \mu| \geqslant k\sigma) \leqslant \frac{1}{k^2}$$

Jointly Distributed Random Variables

❐ **Conditional density** – The conditional density of X with respect to Y, often noted $f_{X|Y}$, is defined as follows:

$$f_{X|Y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

❐ **Independence** – Two random variables X and Y are said to be independent if we have:

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

❐ **Marginal density and cumulative distribution** – From the joint density probability function f_{XY}, we have:

Case	Marginal density	Cumulative function
(D)	$f_X(x_i) = \sum_j f_{XY}(x_i, y_j)$	$F_{XY}(x,y) = \sum_{x_i \leqslant x} \sum_{y_j \leqslant y} f_{XY}(x_i, y_j)$
(C)	$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y) dy$	$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x',y') dx' dy'$

❐ **Distribution of a sum of independent random variables** – Let $Y = X_1 + ... + X_n$ with $X_1, ..., X_n$ independent. We have:

$$\psi_Y(\omega) = \prod_{k=1}^{n} \psi_{X_k}(\omega)$$

❐ **Covariance** – We define the covariance of two random variables X and Y, that we note σ_{XY}^2 or more commonly $\text{Cov}(X,Y)$, as follows:

$$\text{Cov}(X,Y) \triangleq \sigma_{XY}^2 = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y$$

❐ **Correlation** – By noting σ_X, σ_Y the standard deviations of X and Y, we define the correlation between the random variables X and Y, noted ρ_{XY}, as follows:

$$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$$

Remarks: For any X, Y, we have $\rho_{XY} \in [-1, 1]$. If X and Y are independent, then $\rho_{XY} = 0$.

❐ **Main distributions** – Here are the main distributions to have in mind:

Type	Distribution	PDF	$\psi(\omega)$	$E[X]$	$\text{Var}(X)$
(D)	$X \sim \mathcal{B}(n, p)$ Binomial	$P(X = x) = \binom{n}{x} p^x q^{n-x}$ $x \in [\![0, n]\!]$	$(pe^{i\omega} + q)^n$	np	npq
	$X \sim \text{Po}(\mu)$ Poisson	$P(X = x) = \frac{\mu^x}{x!} e^{-\mu}$ $x \in \mathbb{N}$	$e^{\mu(e^{i\omega} - 1)}$	μ	μ
(C)	$X \sim \mathcal{U}(a, b)$ Uniform	$f(x) = \frac{1}{b - a}$ $x \in [a, b]$	$\frac{e^{i\omega b} - e^{i\omega a}}{(b - a)i\omega}$	$\frac{a + b}{2}$	$\frac{(b - a)^2}{12}$
	$X \sim \mathcal{N}(\mu, \sigma)$ Gaussian	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2}$ $x \in \mathbb{R}$	$e^{i\omega\mu - \frac{1}{2}\omega^2\sigma^2}$	μ	σ^2
	$X \sim \text{Exp}(\lambda)$ Exponential	$f(x) = \lambda e^{-\lambda x}$ $x \in \mathbb{R}_+$	$\frac{1}{1 - \frac{i\omega}{\lambda}}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Parameter estimation

❏ **Random sample** – A random sample is a collection of n random variables $X_1, ..., X_n$ that are independent and identically distributed with X.

❏ **Estimator** – An estimator $\hat{\theta}$ is a function of the data that is used to infer the value of an unknown parameter θ in a statistical model.

❏ **Bias** – The bias of an estimator $\hat{\theta}$ is defined as being the difference between the expected value of the distribution of $\hat{\theta}$ and the true value, i.e.:

$$\boxed{\text{Bias}(\hat{\theta}) = E[\hat{\theta}] - \theta}$$

Remark: an estimator is said to be unbiased when we have $E[\hat{\theta}] = \theta$.

❏ **Sample mean and variance** – The sample mean and the sample variance of a random sample are used to estimate the true mean μ and the true variance σ^2 of a distribution, are noted $\overline{X}$ and s^2 respectively, and are such that:

$$\boxed{\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i} \quad \text{and} \quad \boxed{s^2 = \hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

❏ **Central Limit Theorem** – Let us have a random sample $X_1, ..., X_n$ following a given distribution with mean μ and variance σ^2, then we have:

$$\boxed{\overline{X} \underset{n\to+\infty}{\sim} \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)}$$