Intermediate value Theorem

If $f(x)$ is continuous on $[a,b]$ and $f(a)\cdot f(b)<0$, then the eqn $f(x)=0$ has atleast one real root or odd number of roots in the interval (a,b).

Pf Consider an eqn $f(x)=0$.

$f(x)$ is continuous $[a,b]$.

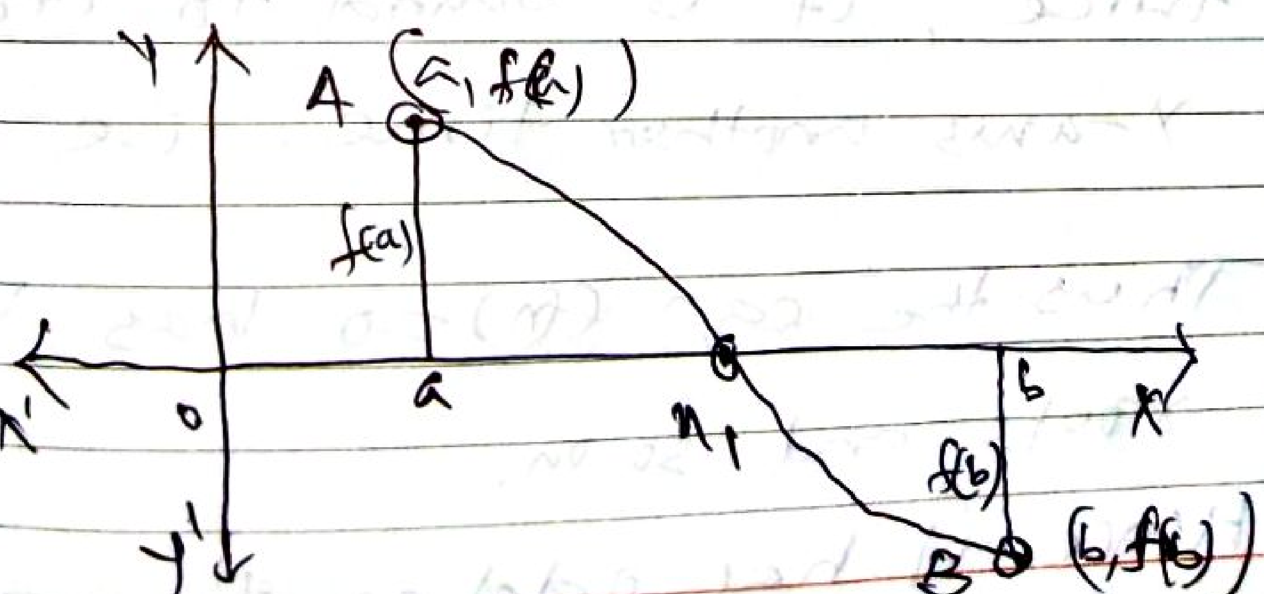
Also $\boxed{f(a)\cdot f(b)<0}$

$\therefore$ $f(a)$ and $f(b)$ are opposite sign.

Hence we have two cases

(I) $f(a)>0$ and $f(b)<0$

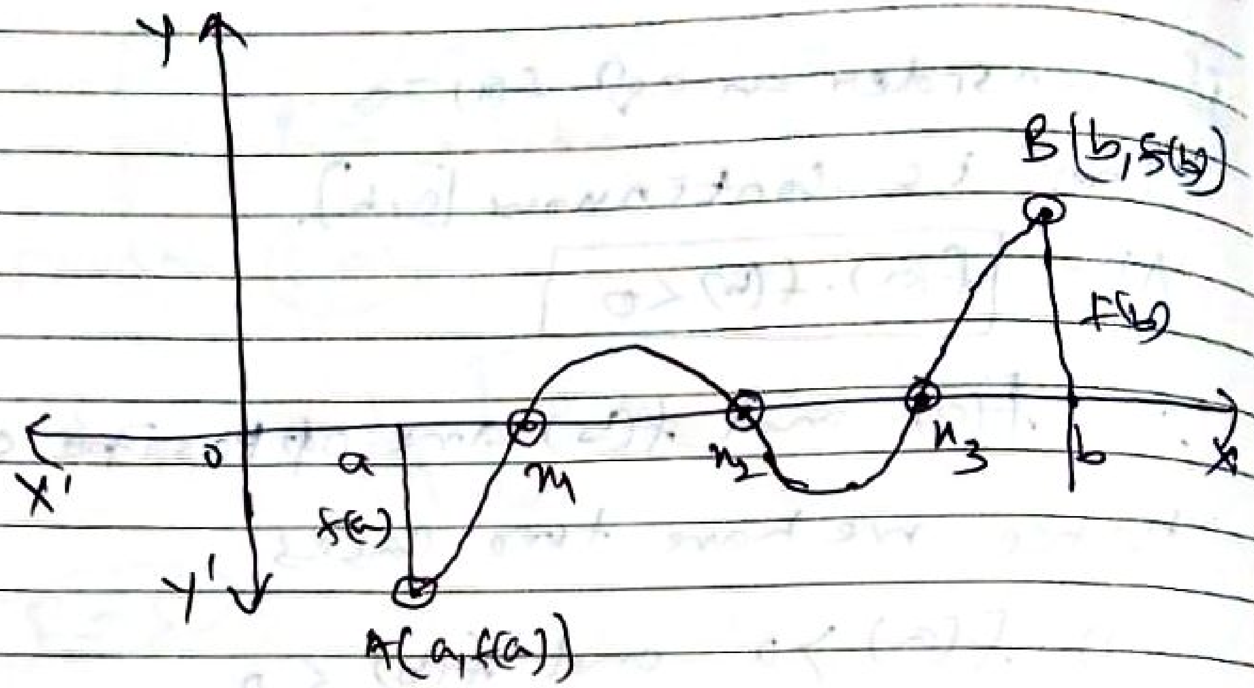
(II) $f(a)<0$ and $f(b)>0$



②

While moving from A to B, the curve will meet x-axis at some where.

That point is called root of the eqn $f(x)=0$.



Suppose the curve crosses the x-axis twice, it is bound to cross the x-axis another time. i.e. thrice.

Thus the eqn $f(x)=0$ has three roots and so on.

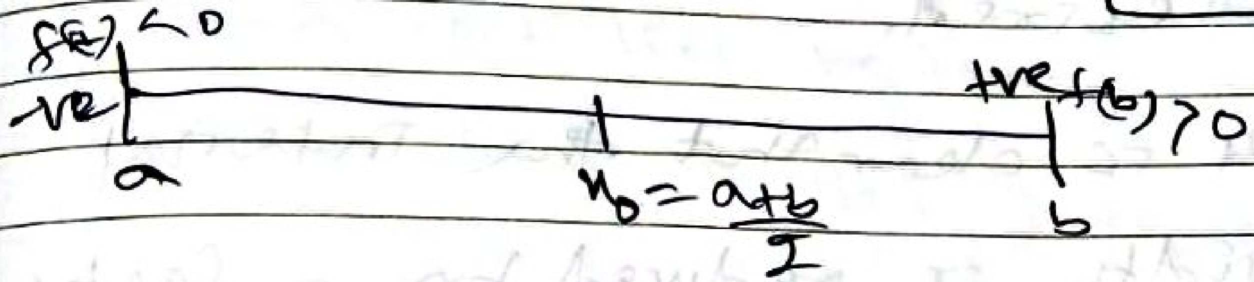
 Hence it has odd no. of roots.

Bisection Method

This method is based on the repeated application of intermediate value theorem.

Let $f(x)$ is continuous on the $[a, b]$ and $f(a) \cdot f(b) < 0$. Then there exists atleast one root between a and b.

Let $f(a) < 0$ and $f(b) > 0$
Then root lies between a and b.
its approximate value is $\boxed{x_0 = \frac{a+b}{2}}$



If $f(x_0) = 0$, we conclude that x_0 is a root of the eqn $f(x) = 0$
otherwise the root lies either between "a" and "x_0" or "x_0" and "b".
depending on $f(x_0)$ ~~negative~~ positive or negative.

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we take the new interval $[a_1, b_1]$ whose length is $\frac{b-a}{2}$.

As before, this is bisected at x_1. and the new interval will be exactly half the length of previous one.

Repeat this process until the latest interval is as small as desired.

It is clear that the interval width is reduced by a factor of one half at each step and at the end of nth step. Then the new interval $[a_n, b_n]$ it's length is $\frac{|b-a|}{2^n}$.

Working Rule ⑤

Step-① :- choose two real numbers a and b such that

$$\boxed{f(a) \cdot f(b) < 0}$$

Step-⑪ :-
put $\boxed{x_r = \frac{a+b}{2}}$

for $r = 0, 1, 2, 3 \ldots$

Step-⑪ :- Find $f(x_r)$.
If $f(a) \cdot f(x_r) < 0$
then root lies between a and x_r
Then set $x_r = b$. go to step-⑪

Step-Ⓘⓥ :- If $f(a) \cdot f(x_r) > 0$
then root lies between
x_r and b then set $x_r = a$
go to step-Ⓘ.

Step-ⓥ :- Repeat this procedure
until getting required decimal
places.

(6)

If the two successive roots are equal.
or error is negligible.

Q Using BSM find a root of the eqn $x^4 - x - 10 = 0$ which is correct upto 2 decimal places.

Soln Given that

$$x^4 - x - 10 = 0 \quad \text{——— (1)}$$

which is on the form of

$$f(x) = 0$$

Here $f(x) = x^4 - x - 10$

Taking $x = 0$

$$f(0) = 0^4 - 0 - 10 = -10 < 0 \ (-ve)$$

Taking $x = 1$

$$f(1) = 1^4 - 1 - 10 = -10 < 0 \ (-ve)$$

∴ root does not lie between 0 and 1

Putting $x=2$

$$f(2) = 2^4 - 2 - 10 = 4 > 0 \text{ +ve}$$

As $\boxed{f(1) \cdot f(2) < 0}$

$\therefore$ root lies between 1 and 2

$$x_0 = \frac{1+2}{2} = 1.5$$

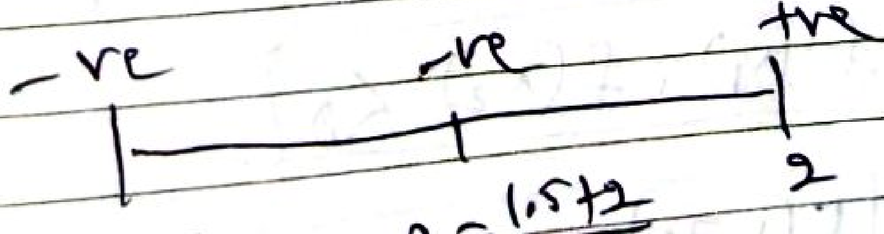
$$\therefore x_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$$

$$f(x_0) = f(1.5) = (1.5)^4 - 1.5 - 10$$
$$= 5.0625 - 11.5$$
$$= -6.4375 < 0$$

Now $\boxed{f(x_0) \cdot f(2) < 0}$

i.e. $f(1.5) \cdot f(2) < 0$

$\therefore$ root lies between 1.5 and 2



$$x_1 = \frac{1.5+2}{2}$$
$$= 1.75$$

(5)

$\therefore x_1 = \frac{1.5+2}{2} = 1.75$

$f(x_1) = f(1.75) = (1.75)^4 - 1.75 - 10$

$= 9.3789 - 11.75$

$= -2.3711 < 0$

-ve   +ve   +ve

1.75 $x_2 = \frac{1.75+2}{2}$ 2

$= 1.875$

$\therefore x_2 = \frac{1.75+2}{2} = 1.875$

Now $f(x_2) = f(1.875)$

$= (1.875)^4 - 1.875 - 10$

$= 12.3596 - 11.875$

$= 0.4846 > 0$

$\therefore \boxed{f(x_1) \cdot f(x_2) < 0}$

i.e. $\boxed{f(1.75) \cdot f(1.875) < 0}$

$\therefore$ root lies between x_1 and x_2

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$-ve$ $\qquad$ $-ve$ $\qquad$ $+ve$

$x_1 = 1.75$ $\quad$ $x_3 = 1.8125$ $\quad$ $x_2 = 1.875$

$$\therefore x_3 = \frac{x_1+x_2}{2} = \frac{1.75 + 1.875}{2}$$

$$= \frac{3.625}{2} = 1.8125$$

$$f(x_3) = (1.8125)^4 - 1.8125 - 10$$

$$= 10.7923 - 11.8125$$

$$= -1.0202 < 0$$

$$\therefore \boxed{f(x_3) \cdot f(x_2) < 0}$$

ie

$$f(1.8125) \cdot f(1.875) < 0$$

root lies between x_3 and x_2

$-ve$ $\qquad$ $+ve$

$x_3 = 1.8125$ $\quad$ $x_4 = 1.8438$ $\quad$ $x_2 = 1.875$

$$\therefore x_4 = \frac{1.8125 + 1.875}{2} = \frac{3.6875}{2} = 1.84375$$

(18)

$\therefore |x_4 - x_3| = |1.8438 - 1.8125|$

$= 0.0$

$\therefore$ root $= 1.8$.

HW

Q-1 Using BSM find a real root of $x^3 - 4x - 9 = 0$ which is correct up to two significant figure

Q-2 Using BSM find a real root of $xe^x - 1 = 0$ which is correct upto 3 decimal places

Q-3 Using BSM find a root of $\cos x - xe^x = 0$ which is correct up to 2 decimal places.