(I)

Mathematical defn of probability :-

Let A be an event and S be the sample space. Then the probability of an event A is denoted by P(A).

It is defined as

$$\boxed{P(A) = \frac{|A|}{|S|}} \quad \text{where } A \subseteq S.$$

$$= \frac{\text{No. of favourable cases to the event A}}{\text{Total no. of possible outcomes.}}$$

Axiomatic Theory of probability.

(I) $0 \leq P(A) \leq 1$

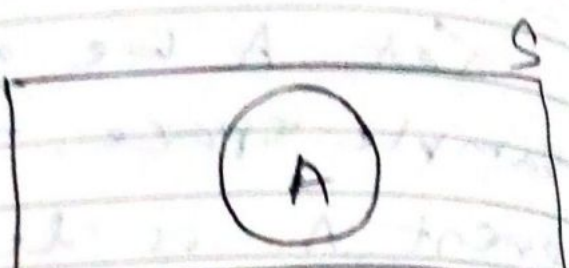
(II) $P(\phi) = 0$

(III) $P(S) = 1$

(IV) $P(A') = 1 - P(A)$. i.e. $P(A) + P(A') = 1$.

Q(1) Show that $0 \leq P(A) \leq 1$.
where 'A' is an event in S.

Pf Let A be an event
'S' is the sample space



from the figure.

$$\boxed{A \subset S} \quad \text{— } (1)$$

$$\Rightarrow |A| \le |S|$$

since 'S' is sample space $|S|>0$.

Dividing $|S|$ both sides.

$$\frac{|A|}{|S|} \le \frac{|S|}{|S|}$$

$$\Rightarrow \boxed{P(A) \le 1} \quad (\text{A/Defn}) \quad \text{— } (1)$$

then A is an empty set

$$A = \phi$$

$$|A| = |\phi| = 0$$

(3)

Dividing |S| both sides we get

$$\frac{|A|}{|S|} = \frac{0}{|S|}$$

$\Rightarrow \boxed{P(A) = 0}$ —— (2)

If A is non empty set i.e. $A \neq \varphi$.

$|A| > 0$.

Dividing |S| both sides we get

$$\frac{|A|}{|S|} > \frac{0}{|S|}$$

$\Rightarrow \boxed{P(A) > 0} \Rightarrow \boxed{0 < P(A)}$ —— (3)

Combining (2) and (3) we get

$\boxed{0 \leq P(A)}$ —— (4)

From (1) and (4) we can write

$\boxed{0 \leq P(A) \leq 1}$ (Proved)

Q-(II) Show that $\boxed{P(\phi)=0}$

Soln we know from set theory, empty set is subset of every set.

i.e. $\phi \subseteq S$

where $\phi = \{\}$

$|\phi| = 0$ Also $|S| > 0$.

Dividing $|S|$ both sides we get

$$\frac{|\phi|}{|S|} = \frac{0}{|S|}$$

$\Rightarrow \boxed{P(\phi)=0}$ (~~A~~ 1st defn)

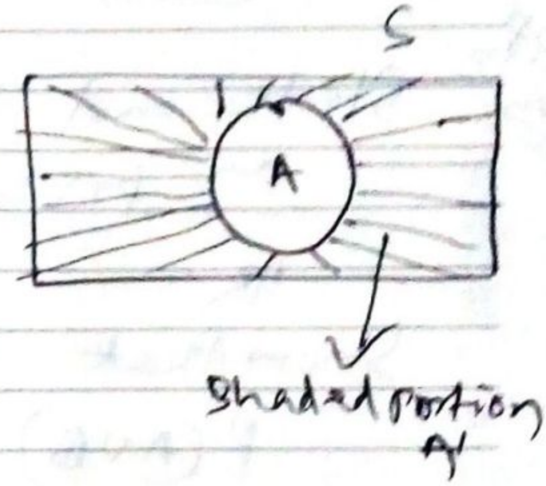
Q-(III) Show that $P(S)=1$

Soln Since 'S' is the sample space it consists of all possible out comes.

$$\therefore \boxed{P(S) = \frac{|S|}{|S|} = 1}$$

Q-(IV) Show that $\boxed{P(A') = 1 - P(A)}$

Soln



from the venn-diagram it is clear that

$$\boxed{A \cup A' = S} \quad \text{— (I)}$$

$$\boxed{A \cap A' = \phi} \quad \text{— (II)}$$

Now $|A \cup A'| = |S|$

$\Rightarrow |A| + |A'| = |S|$

Dividing $|S|$ both sides we get

$$\frac{|A| + |A'|}{|S|} = \frac{|S|}{|S|}$$

$$\Rightarrow \frac{|A|}{|S|} + \frac{|A'|}{|S|} = 1$$

(6)

$\rightarrow P(A) + P(A') = 1$ (A/Defn)

$\rightarrow \boxed{P(A') = 1 - P(A)}$ (proved)

Imp Q. State and prove addition theorem on probability.

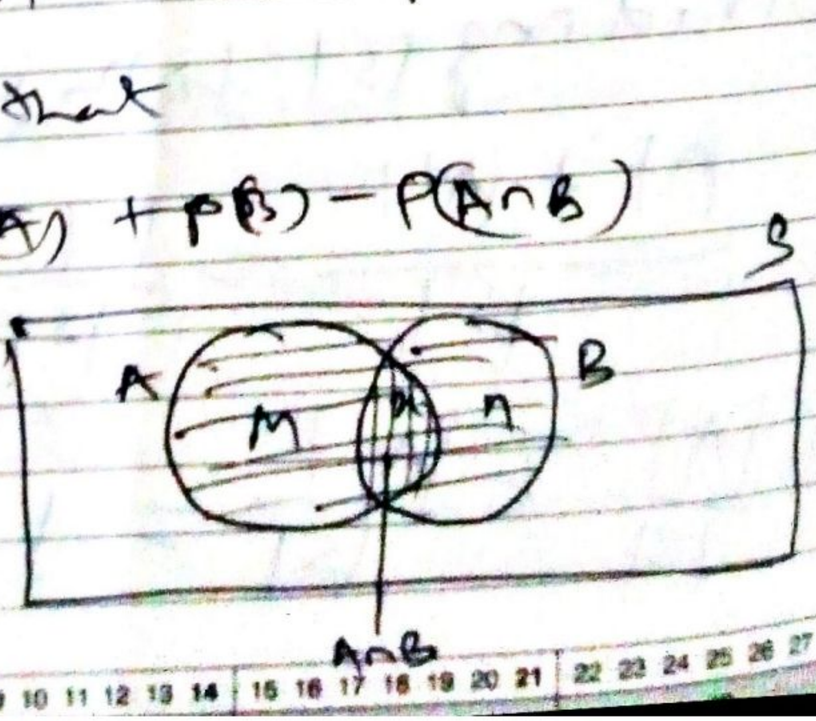
OR

Show that
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
where A and B are two events in S.

Soln

Statement:- "If A and B are two events in the sample space 'S' then prove that

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



from venn-diagram

$$|A| = m+x \qquad |B| = n+x$$

$$|A\cap B| = x \qquad |A\cup B| = m+x+n.$$

Now $|A\cup B| = m+n+x$.

Also $|A| + |B| - |A\cap B| = (m+x) + (n+x) - x$

$$= m+x+n$$

from above we get

$$\therefore |A\cup B| = |A| + |B| - |A\cap B|$$

Since 'S' is the sample space ($|S|>0$)

$\therefore$ Dividing $|S|$ both sides we get

$$\frac{|A\cup B|}{|S|} = \frac{|A| + |B| - |A\cap B|}{|S|}$$

$$\Rightarrow \frac{|A\cup B|}{|S|} = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A\cap B|}{|S|}$$

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$ (By defn)

which is called addition theorem on probability.

Q. If A and B are mutually exclusive events then show that

$$\boxed{P(A\cup B) = P(A) + P(B)}$$

Soln Given that A and B are mutually exclusive events.

A/ Defn $A\cap B = \phi$.

$$\therefore |A\cap B| = |\phi| = 0$$

$$\therefore |A\cap B| = 0$$

Dividing $|S|$ both sides

$$\frac{|A\cap B|}{|S|} = \frac{0}{|S|} = 0$$

⇒ $\boxed{P(A \cap B) = 0}$ (A/Defn)

we know addition theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

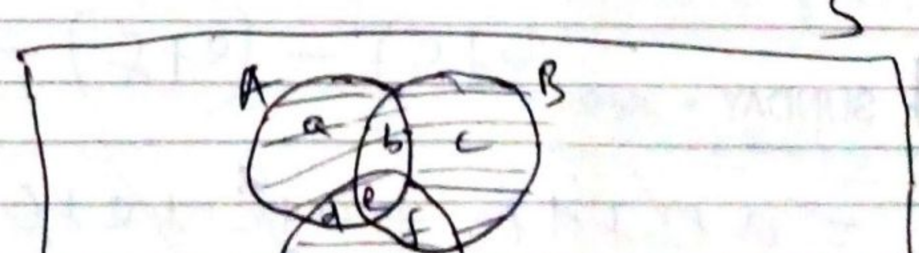
⇒ $P(A \cup B) = P(A) + P(B) - 0$

⇒ $\boxed{P(A \cup B) = P(A) + P(B)}$ (Proved)

Q. If A, B, C are three events then show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Pf Let A, B, C are three events in 'S', where 'S' is the sample space



Shaded portion of $A \cup B \cup C$

from venn diagram it is clear that

$|A \cup B \cup C| = a+b+c+d+e+f+g.$

$|A| = a+b+d+e$

$|B| = b+c+e+f$

$|C| = d+e+f+g$

$|A \cap B| = b+e \quad |B \cap C| = e+f$

$|C \cap A| = d+e \quad |A \cap B \cap C| = e$

Now $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A|$
$+ |A \cap B \cap C|$

$= (a+b+d+e) + (b+c+e+f) + (d+e+f+g)$

$- (b+e) - (e+f) - (d+e) + e$

$= a + \cancel{b} + \cancel{d} + \cancel{e} + b + c + \cancel{e} + \cancel{f} + \cancel{d} + \cancel{e} + f + g$
$- \cancel{b} - \cancel{e} - \cancel{e} - \cancel{f} - \cancel{d} - \cancel{e} + \cancel{e}$

$= a+b+c+d+e+f+g$

$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Dividing $|S|$ both sides we get

$$\frac{|A \cup B \cup C|}{|S|} = \frac{|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|}{|S|}$$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} + \frac{|C|}{|S|} - \frac{|A \cap B|}{|S|} - \frac{|B \cap C|}{|S|} - \frac{|C \cap A|}{|S|} + \frac{|A \cap B \cap C|}{|S|}$$

$\Rightarrow P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(proved)

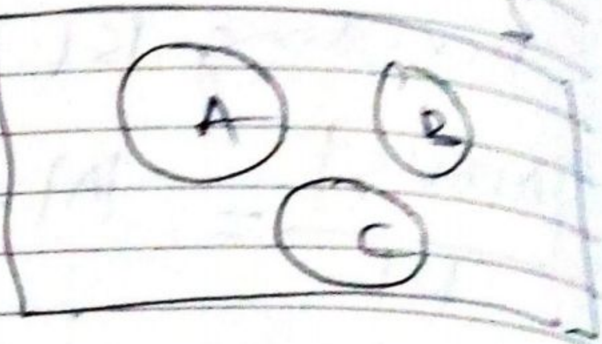
Q. If A, B, C are three mutually exclusive events then show that $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Sol^n Given that A, B, C are three mutually exclusive events and

i.e. $A \cap B = \phi$

$B \cap C = \phi$

$C \cap A = \phi$



$\therefore |A \cap B| = |\phi| = 0$

$|B \cap C| = |\phi| = 0$

$|C \cap A| = |\phi| = 0$

$A \cap B \cap C = \phi$

$|A \cap B \cap C| = |\phi| = 0$

$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{0}{|S|} = 0$

$P(B \cap C) = \frac{|B \cap C|}{|S|} = \frac{0}{|S|} = 0$

$P(C \cap A) = \frac{|C \cap A|}{|S|} = \frac{0}{|S|} = 0$

$P(A \cap B \cap C) = \frac{|A \cap B \cap C|}{|S|} = \frac{0}{|S|} = 0$

(13)

we know

~~P(A∩B∩C)~~ $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

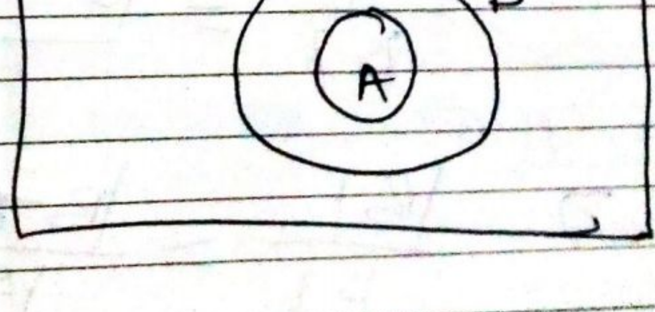
$\Rightarrow P(A \cup B \cup C) = P(A) + P(B) + P(C) - 0 - 0 - 0 + 0$

$\Rightarrow \boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C)}$

Q If $A \subset B$ then (I) $P(A) \leq P(B)$.

(II) $P(B - A) = P(B) - P(A)$

Sol Given that A and B are two events in 'S' such that $\boxed{A \subset B}$



Taking cardinality both sides

$|A| \leq |B|$

(IM)

Dividing $|S|$ both sides

$$\frac{|A|}{|S|} \le \frac{|B|}{|S|}$$

$$\Rightarrow \boxed{P(A) \le P(B)}$$

(II) from venn diagram it is clear that

$$B = (B-A) \cup A$$

$$\therefore |B| = |(B-A) \cup A|$$ taking cardinality both sides

$$\Rightarrow |B| = |B-A| + |A|$$

Dividing $|S|$ both sides.

$$\frac{|B|}{|S|} = \frac{|B-A|+|A|}{|S|}.$$

$$\Rightarrow \frac{|B|}{|S|} = \frac{|B-A|}{|S|} + \frac{|A|}{|S|}$$

$$P(B) = P(B-A) + P(A)$$

$$\Rightarrow P(B) - P(A) = P(B-A)$$

$$\therefore \boxed{P(B-A) = P(B) - P(A)}$$

A[I] T.S.T $P(A) \leq P(B)$

W.K.T probability of an event always greater than equal to zero and less than or equal to '1'

$$\therefore P(B-A) \geq 0$$

$$\Rightarrow P(B) - P(A) \geq 0 \quad \because P(B-A) = P(B) - P(A)$$

$$\Rightarrow P(B) \geq P(A)$$

$$\Rightarrow \boxed{P(A) \leq P(B)} \quad \text{(Proved)}$$

Note If $A_1, A_2, A_3 \ldots A_n$ are n mutually exclusive events in 'S' then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$$