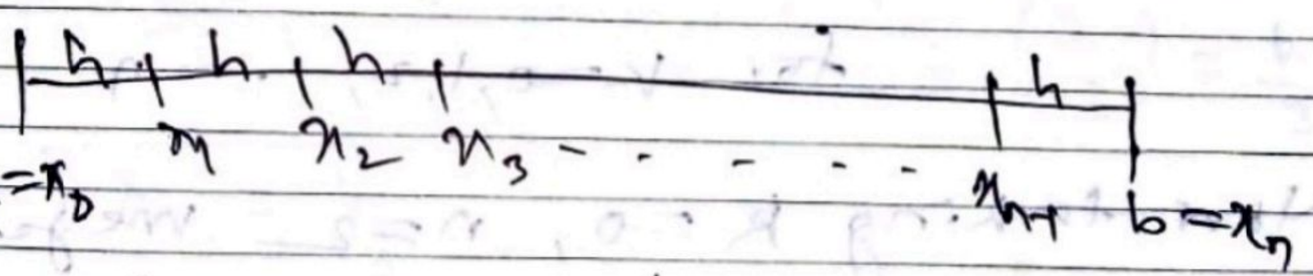
①

Simpson's $\frac{1}{3}$rd Rule :-

When $\boxed{n=2}$ in Newton-Cote's quadrature formula we get Simpson's $\frac{1}{3}$rd Rule.

Let $f(x)$ be a function which is defined at $(n+1)$ points such as $x_0, x_1, x_2, \ldots x_n$, which are equispaced.



$h \to$ step size / step length.

Now $\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx$

Then by Newton-Cote's quadrature formula we get

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$$\int_a^b f(x)\,dx = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 \quad \text{---(1)}$$

where $\lambda_0, \lambda_1, \lambda_2$ can be determined by following formula

$$\lambda_k = \frac{(-1)^{n-k} \cdot h}{k!\,(n-k)!} \int_0^n s(s-1)(s-2)\cdots(s-(k-1))\,(s-(k+1))\cdots(s-n)\,ds$$

for $k=0,1,2,\ldots n$

Now taking $k=0$, $n=2$ we get

$$\lambda_0 = \frac{(-1)^{2-0} \cdot h}{0! \cdot (2-0)!} \int_0^2 (s-1)(s-2)\,ds$$

$$= \frac{(-1)^2 \cdot h}{1 \cdot 2!} \int_0^2 (s^2-3s+2)\,ds$$

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$$= \frac{h}{2}\left[\int_0^2 s^2\,ds - 3\int_0^2 s\,ds + 2\int_0^2 ds\right]$$

$$= \frac{h}{2}\left[\left[\frac{s^3}{3}\right]_0^2 - 3\left[\frac{s^2}{2}\right]_0^2 + 2\left[s\right]_0^2\right]$$

$$= \frac{h}{2}\left[\left(\frac{2^3}{3} - 0\right) - 3\left(\frac{2^2}{2} - 0\right) + 2(2-0)\right]$$

$$= \frac{h}{2}\left[\frac{8}{3} - 6 + 4\right]$$

$$= \frac{h}{2}\left[\frac{8}{3} - 2\right] = \frac{h}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right) = \frac{h}{3}$$

$$\boxed{\lambda_0 = \frac{h}{3}}$$

Next taking $n=2$, $k=1$ we get

$$\lambda_1 = \frac{(-1)^{2-1}\cdot h}{1!\,(2-1)!}\int_0^2 s(s-2)\,ds$$

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$$= \frac{(-1) \cdot h}{1 \cdot 1} \int_0^2 (s^2 - 2s)\, ds$$

$$= -h \left[\int_0^2 s^2\, ds - 2\int_0^2 s\, ds \right]$$

$$= -h \left[\left[\frac{s^3}{3} \right]_0^2 - 2\left[\frac{s^2}{2} \right]_0^2 \right]$$

$$= -h \left[\left(\frac{2^3}{3} - 0 \right) - \left(2^2 - 0 \right) \right]$$

$$= -h \left[\frac{8}{3} - 4 \right]$$

$$= -h \left(-\frac{4}{3} \right) = \frac{4h}{3}$$

$$\therefore \boxed{\lambda_1 = \frac{4h}{3}}$$

Again taking $k = 2$, $n = 2$ integ,

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$$\lambda_2 = \frac{(-1)^{2-2} \cdot h}{2!\,(2-2)!} \int_0^2 s(s-1)\,ds$$

$$= \frac{1 \cdot h}{2 \cdot 0!} \int_0^2 (s^2 - s)\,ds$$

$$= \frac{h}{2}\left[\int_0^2 s^2\,ds - \int_0^2 s\,ds\right]$$

$$= \frac{h}{2}\left[\left[\frac{s^3}{3}\right]_0^2 - \left[\frac{s^2}{2}\right]_0^2\right]$$

$$= \frac{h}{2}\left[\left(\frac{2^3}{3} - 0\right) - \left(\frac{2^2}{2} - 0\right)\right]$$

$$= \frac{h}{2}\left[\frac{8}{3} - 2\right] = \frac{h}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right) = \frac{h}{3}$$

$$\therefore \boxed{\lambda_2 = \frac{h}{3}}$$

Putting the value of $\lambda_0, \lambda_1, \lambda_2$ in eq (1) we get,

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$$\int_a^b f(x)\,dx = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2$$

$$= \frac{h}{3} f_0 + \frac{4h}{3} f_1 + \frac{h}{3} f_2$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\therefore \boxed{\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]}$$

Which is called simpson's $\frac{1}{3}$rd Rule.

OR

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}$$

$$\boxed{\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}$$

Note :- It works on even number of intervals. (n is multiple of 2)
i.e. no. of ordinates must be odd.

Composite Simpson's $\frac{1}{3}$rd Rule :-

The composite Simpson's $\frac{1}{3}$ rd Rule can be written as

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx$$

$$= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx$$

$$+ \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

where n must be even.

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$+ \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$+ \frac{h}{3}\left[f(x_4) + 4f(x_5) + f(x_6)\right]$$

$$+ \ldots\ldots\ldots$$

$$+ \frac{h}{3}\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{3}\Big[f(x_0) + 4f(x_1) + 4f(x_3) + 4f($$

$$+ \ldots + 4f(x_{n-1}) + 2f(x_2)$$

$$+ 2f(x_4) + 2f(x_6) + \ldots$$

$$+ 2f(x_{n-2}) + f(x_n)\Big]$$

$$= \frac{h}{3}\left[[f(x_0) + f(x_n)] + 4[f(x_1) + f(x_3) + \cdots + f(x_{n-1})] + 2[f(x_2) + f(x_4) + \cdots f(x_{n-2})] \right]$$

$$\therefore \boxed{\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) \right]}$$

Which is called composite Simpson's $\frac{1}{3}$rd Rule.

It can be also expressed as

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left[A + 4B + 2C \right]}$$

where

$A \rightarrow$ Sum of 1st and last ordinates.

$B \rightarrow$ Sum of odd ordinates

$C \rightarrow$ Sum of even ordinates.

Q. Using Simpson's 1/3 rd Rule, Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ where $h=1$.

Sol: Given that

$$\int_0^6 \frac{1}{1+x^2} dx = \int_a^b f(x) dx = \int_{x_0}^{x_n} f(x) dx$$

Here $f(x) = \frac{1}{1+x^2}$

$\therefore a = x_0 = 0$, $b = x_n = 6$, $h = 1$.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	1	2	3	4	5	6
$y=f(x)$	1	0.5	0.2	0.1	0.059	0.038	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

As $y_0 = f(0) = \frac{1}{1+0} = \frac{1}{1} = 1$

$y_1 = f(1) = \frac{1}{1+1^2} = \frac{1}{2} = 0.5$

$y_2 = f(2) = \frac{1}{1+2^2} = \frac{1}{5} = 0.2$

$y_3 = f(3) = \frac{1}{1+3^2} = \frac{1}{10} = 0.1$

$y_4 = f(4) = \frac{1}{1+4^2} = \frac{1}{17} = 0.059$

$y_5 = f(5) = \frac{1}{1+5^2} = \frac{1}{26} = 0.038$

$y_6 = f(6) = \frac{1}{1+6^2} = \frac{1}{37} = 0.027$

Here $n=6$ which is even,

no. of ordinates are odd,

W.K.T

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+\dots+y_{n-1})+2(y_2+y_4+\dots+y_{n-2})\Big]$$

Here

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{1}{3}\Big[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\Big]$$

$$= \frac{1}{3}\Big[(1+0.027)+4(0.5+0.1+0.038)+2(0.2+0.059)\Big]$$

$$= \frac{1}{3}\Big[1.027+4(0.638)+2(0.259)\Big]$$

$$= \frac{1}{3}\Big[1.027+2.552+0.518\Big]$$

$= \frac{1}{3}[4.097] = 1.366$

$\therefore \boxed{\int_0^6 \frac{1}{1+x^2}\,dx = 1.366}$

Q. Using Simpson's $\frac{1}{3}$rd rule evaluate $\int_0^\pi \sin x\,dx$ taking 10 equal parts.

Soln Given that

$$\int_0^\pi \sin x\,dx = \int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx$$

Here $f(x) = \sin x$

& $a = 0 = x_0\,,\quad b = \pi = x_n$

$$\therefore h = \frac{x_n - x_0}{n} = \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10}$$

0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}

x	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
y	0	0.309	0.588	0.809	0.951	1	0.951	0.809	0.588	0.309	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Here $f(x) = \sin x$

$f(0) = \sin 0 = 0$

$f\left(\frac{\pi}{10}\right) = \sin\frac{\pi}{10} = 0.309$

$f\left(\frac{2\pi}{10}\right) = \sin\left(\frac{\pi}{5}\right) = 0.588$

$f\left(\frac{3\pi}{10}\right) = \sin\left(\frac{3\pi}{10}\right) = 0.809$

$f\left(\frac{4\pi}{10}\right) = \sin\left(\frac{2\pi}{5}\right) = 0.951$

$f\left(\frac{5\pi}{10}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$f\left(\frac{6\pi}{10}\right) = \sin\left(\frac{3\pi}{5}\right) = 0.951$

$f\left(\frac{7\pi}{10}\right) = \sin\left(\frac{7\pi}{10}\right) = 0.809$

$f\left(\frac{8\pi}{10}\right) = \sin\left(\frac{4\pi}{5}\right) = 0.588$

$f\left(\frac{9\pi}{10}\right) = \sin\left(\frac{9\pi}{10}\right) = 0.309$

$f(\pi) = \sin(\pi) = 0$

Since $n=10$ and no. of ordinates are odd.

we apply composite simpson's $\frac{1}{3}$rd rule.

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

Here

$$\int_0^{\pi} \sin x \, dx = \frac{\pi/10}{3}\Big[(y_0+y_{10}) + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\Big]$$

$$= \frac{\pi}{30}\Big[(0+0) + 4(0.309+0.809+1+0.809+0.309) + 2(0.588+0.951+0.951+0.588)\Big]$$

$$= \frac{\pi}{30}\Big[0 + 4(3.235) + 2(3.078)\Big]$$

$$= \frac{\pi}{30}\big[12.944 + 6.156\big]$$

$$= \frac{\pi}{30}\big[19.1\big]$$

$$= (0.105)\times(19.1) = 2.006$$

observed value

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But actual integration

$$\int_0^{\pi} \sin x \, dx = \left[-\cos x\right]_0^{\pi}$$

$$= -\left[\cos x\right]_0^{\pi}$$

$$= -\left[\cos \pi - \cos 0\right]$$

$$= -\left[-1-1\right]$$

$$= -(-2) = 2 \text{ (Exact value)}$$

Error = Exact value − observed value

$$= 2 - 2.006$$

$$= -0.006$$

Absolute error $= |-0.006| = 0.006$.

(18)

H.W

Q Using Simpson's $\frac{1}{3}$rd rule evaluate $\int_0^1 \frac{x^2}{1+x^3}\,dx$ where $h = 0.25$. Also find the value of $\log(2)^{1/3}$ and compare it.

Soln: $\int_0^1 \frac{x^2}{1+x^3}\,dx = 0.231$

Actual integration

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \left[\frac{1}{3}\log(1+x^3)\right]_0^1$$

$$= \frac{1}{3}\log 2 = \log 2^{1/3}$$

$$= 0.231$$

error is negligible