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Fixed point iteration Method

Let us consider an eqn $f(x)=0$ —— (1)

Let 'α' be the exact root of eqn (1)

$\therefore f(\alpha)=0.$

The eqn eqn (1) can be expressed as $\boxed{x=\varphi(x)}$ ←—— (2)

Let I be the interval containing the root 'α' i.e. $\alpha \in I$.

If $|\varphi'(x)|<1 \;\; \forall x \in I$, then the sequence of approximation $x_0, x_1, x_2, \dots x_n$ will converge to α, when the initial approximation 'x_0' is choosen in the interval I.

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Working Rule

Step-I :- First express the given eqn (1) in the form of

$$\boxed{x = \varphi(x)} \quad \text{--- (2)}$$

where $|\varphi'(x)| < 1$

Step-II :- Choosing x_0 be the initial approximation to the eqn (2).

Step-III :- We get

$$\boxed{x_1 = \varphi(x_0)}$$

which is 1st approximation

Step-IV :- The 2nd approximation

$$x_2 = \varphi(x_1)$$

and 3rd approximation

$$x_3 = \varphi(x_2)$$

(3)

Similarly proceeding in this way we get

$$\boxed{x_n = \varphi(x_{n-1})}$$

Step-(iv): The sequence of approximation $x_1, x_2, x_3 \dots x_n$ which will converge to α.

or $|x_n - x_{n-1}| \le \epsilon$.

or error tolerance is negligible.

Note: (I) Then condition for convergence of fixed point iteration method is $|\varphi'(x)| < 1$. which is the sufficient condition.

(II) The rate of convergence of fixed point iteration method is $\underline{1}$ ie linearly convergent.

(4)

Q Using fixed pt iteration method find a real root of the eqⁿ $\cos x = 3x-1$ which is correct upto 4 decimal places.

Sol:- Given that

$$\cos x = 3x - 1$$

$$\Rightarrow \cos x - 3x + 1 = 0 \quad \text{--- (1)}$$

which is in the form of

$$f(x) = 0$$

Here $f(x) = \cos x - 3x + 1$.

Now eqⁿ (1) can be written as

$$3x = 1 + \cos x$$

$$\boxed{x = \frac{1+\cos x}{3}} \quad \text{--- (2)}$$

which is in the form of

$$x = \phi(x)$$

Here $\phi(x) = \frac{1+\cos x}{3}$

Now $\varphi'(x) = -\frac{1}{3}\sin x$

$\Rightarrow |\varphi'(x)| = \left|-\frac{1}{3}\sin x\right| = \left|-\frac{1}{3}\right| |\sin x|$

$\Rightarrow \boxed{|\varphi'(x)| \le \frac{1}{3}} < 1$

$(\because |\sin x| \le 1)$

$\therefore |\varphi'(x)| < 1$

which is condition of convergent.

Here the eqn

$$\boxed{x = \frac{1}{3}(1+\cos x)}$$

Let us choose $x_0 = \frac{1}{2}$

$\therefore$ 1st approximation

$x_1 = \frac{1}{3}[1+\cos x_0]$

$= \frac{1}{3}[1+\cos(\frac{1}{2})]$

$$x_1 = \frac{1}{3}(1.8776) = 0.6258$$

$$\boxed{x_1 = 0.6258}$$

which is 1st approximation.

The 2nd approximation

$$x_2 = \varphi(x_1) = \frac{1}{3}[1+\cos x_1]$$

$$= \frac{1}{3}[1+\cos(0.6258)]$$

$$= \frac{1}{3}[1+0.8105]$$

$$= 0.6035$$

$$\boxed{x_2 = 0.6035}$$

3rd approximation

$$x_3 = \varphi(x_2) = \frac{1}{3}[1+\cos x_2]$$

$$= \frac{1}{3}[1+\cos(0.6035)]$$

$$= \frac{1}{3}[1+0.8234] = 0.6078$$

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$$\boxed{x_3 = 0.6078}$$

4th approximation

$$x_4 = \varphi(x_3) = \frac{1}{3}\left[1+\cos x_3\right]$$

$$= \frac{1}{3}\left[1+\cos(0.6078)\right]$$

$$= \frac{1}{3}\left[1.8209\right]$$

$$= \cancel{0.6069}\ 0.6070$$

~~5th approximation~~

$$x_5 = \varphi(x_4) = \frac{1}{3}\left[1+\cos x_4\right]$$

$$= \frac{1}{3}\left[1+\cos(0.6070)\right]$$

$$= \frac{1}{3}\left[1+0.8214\right]$$

$$\boxed{x_5 = 0.6071}$$

6th. approximation

$$x_6 = \varphi(x_5)$$

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$$x_6 = \frac{1}{3}[1 + \cos x_5]$$

$$= \frac{1}{3}[1 + \cos(0.6071)]$$

$$= \frac{1}{3}[1 + 0.8213]$$

$$= 0.6071$$

$$\boxed{x_6 = 0.6071}$$

$$\therefore |x_6 - x_5| = 0.0000$$

$\therefore$ The root $= 0.6071$
which is correct upto
4 decimal place.

⇒ Using fixed pt iteration
method find the root of the
eqn $x^3 - 2x - 1 = 0$ which is
correct upto 3 significant figures