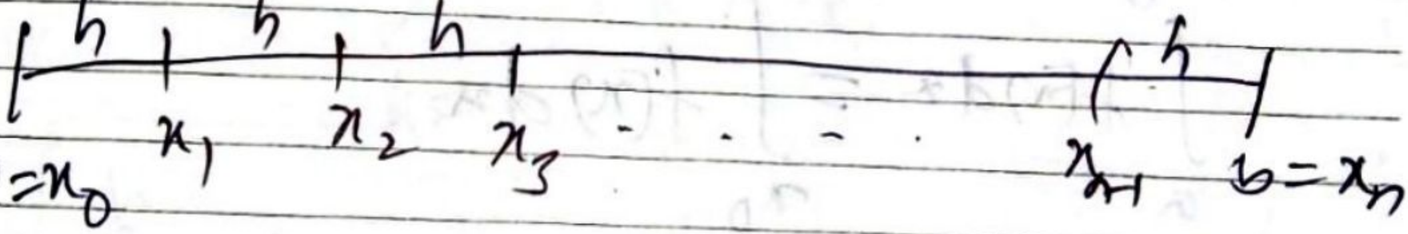
M – II

Numerical Integration :– (1)

Newton's – Cotes Quadrature formula :–

Let us consider a function $f(x)$ which is continuous on $[a,b]$. Also it is defined at $(n+1)$ distinct points such as $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n =$ which are equispaced.



From definite integration we have

$$\int_a^b f(x)\,dx = \Big[\phi(x) + c\Big]_a^b$$

$$= \Big[\phi(x) + c\Big]_{x=b} - \Big[\phi(x) + c\Big]_{x=a}$$

2

$$= [\varphi(b) + c] - [\varphi(a) + c]$$

$$= \varphi(b) + c - \varphi(a) - c$$

$$= \varphi(b) - \varphi(a)$$

$$\therefore \boxed{\int_a^b f(x)\,dx = \varphi(b) - \varphi(a)}$$

Here

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx$$

$$= \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \ldots$$

$$+ \ldots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

It can be expressed as

③

$$\int_a^b f(x)\,dx = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n$$

$$= \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n)$$

Where $\lambda_0, \lambda_1, \lambda_2 \cdots \lambda_n$ are weight functions.

$$= \sum_{k=0}^{n} \lambda_k f(x_k)$$

$$\therefore \boxed{\int_a^b f(x)\,dx = \sum_{k=0}^{n} \lambda_k f(x_k)} \rightarrow (*)$$

where λ_k is obtained by using following formula

$$\boxed{\lambda_k = \frac{(-1)^{n-k} \cdot h}{k!\,(n-k)!} \int_0^n s(s-1)(s-2)\cdots(s-(k-1))(s-(k+1))\cdots(s-n)\,ds}$$

(4)

Which is called Newton-Cote's quadrature formula.

Finding $\lambda_0, \lambda_1, \lambda_2 - - - \lambda_n$ and putting in the above formula ⊛ we get the required result.

Trapezoidal Rule :-

When $\boxed{n=1}$ in Newton-Cote's Quadrature formula, we get Trapezoidal Rule. It is given by

$$\boxed{\int_a^b f(x) \cdot dx = \lambda_0 f_0 + \lambda_1 f_1}$$

5

Here $f(x)$ is defined at two distinct pts x_0 and x_1

$$\therefore \boxed{\int_{x_0}^{x_1} f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)} \quad \text{A}$$

where $a = x_0$, $b = x_1$.

To find the value of λ_0 and λ_1 we use the formula

$$\lambda_k = \frac{(-1)^{n-k} \cdot h}{k!\,(n-k)!} \int_0^n s(s-1)(s-2)\cdots(s-(k-1))\,(s-(k+1))\cdots(s-n)\,ds$$

putting $k=0$, $n=1$

$$\lambda_0 = \frac{(-1)^{1-0} \cdot h}{0!\,(1-0)!} \int_0^1 (s-1)\,ds$$

(6)

$$\lambda_0 = \frac{(-1).h}{1!\,1}\left[\int_0^1 s\,ds - \int_0^1 1.ds\right]$$

$$= (-1)h\left[\left[\frac{s^2}{2}\right]_0^1 - [s]_0^1\right]$$

$$= (-1).h\left[\left(\frac{1^2}{2}-0\right) - (1-0)\right]$$

$$= (-h)\left(\frac{1}{2}-1\right) = (-h).\left(\frac{-1}{2}\right) = \frac{h}{2}$$

$$\boxed{\lambda_0 = \frac{h}{2}}$$

Again taking $k=1$, $n=1$ we get

$$\lambda_1 = \frac{(-1)^{1-1}.h}{1!\,(1-1)!}\int_0^1 s.ds$$

$$= \frac{(-1)^0.h}{1!\,0!}\left[\frac{s^2}{2}\right]_0^1$$

$$= h\left[\frac{1^2}{2}-0\right] = \frac{h}{2}$$

(7)

$$\boxed{\lambda_1 = \frac{h}{2}}$$

putting the value of λ_0, λ_1 in eqn (a) we get.

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\cdot f(x_0) + \frac{h}{2} f(x_1)$$

$$\boxed{\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right]}$$

where $h = x_1 - x_0$

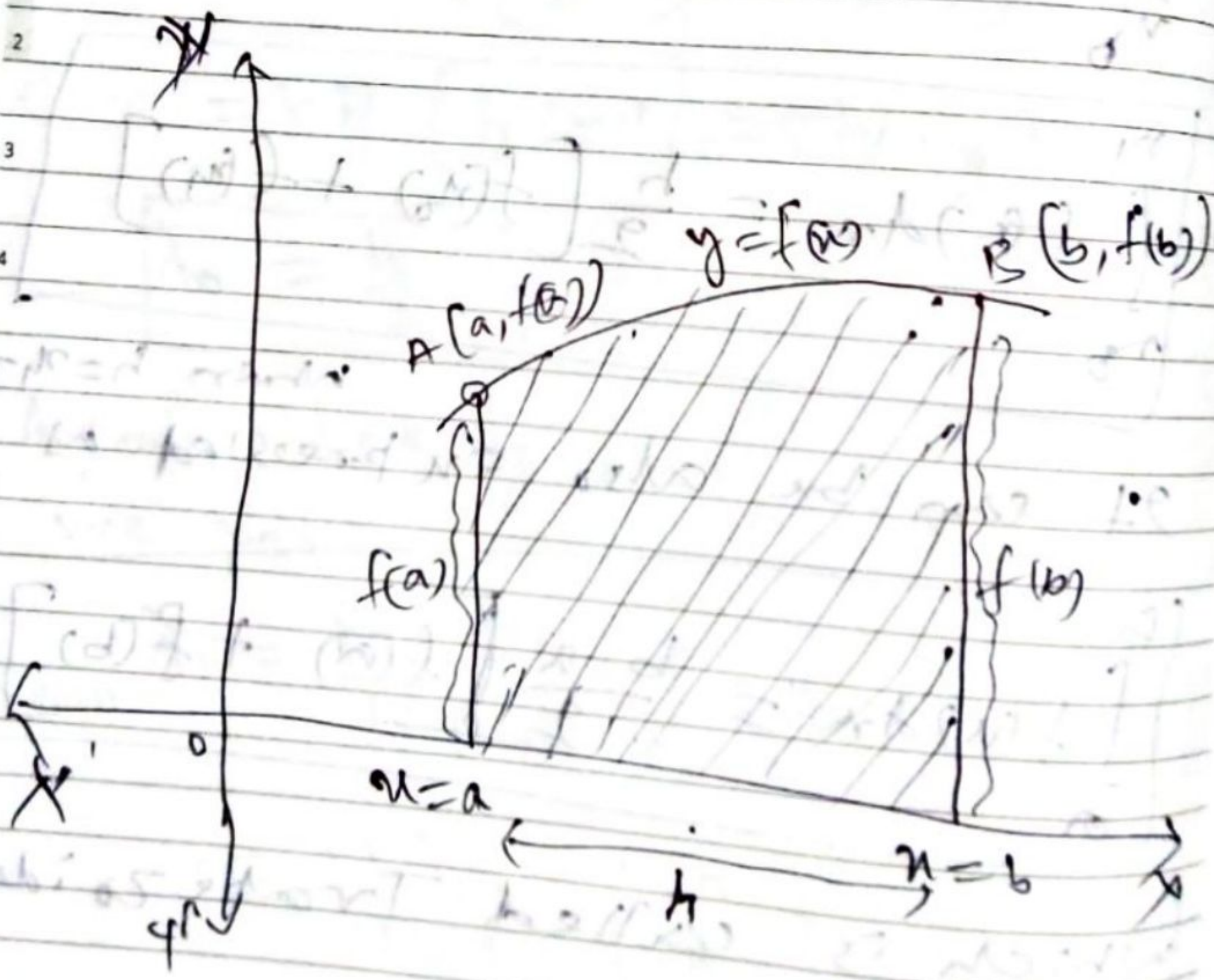
It can be also expressed as

$$\boxed{\int_{a}^{b} f(x)\,dx = \frac{b-a}{2}\left[f(a) + f(b)\right]}$$

Which is called Trapezoidal Rule.

(8)

As we find the area of trapezium so it is called Trapezoidal Rule. which is given in the following figure:



9

Composite Trapezoidal Rule :-

Let $f(x)$ be a continuous function on $[a, b]$. which is defined at $(n+1)$ pts such as $x_0, x_1, x_2 \cdots x_n$ which satisfies

$$\boxed{x_i = x_{i-1} + h} \quad \text{for } i = 1, 2, \cdots n$$

i.e. $x_1 = x_0 + h$

$x_2 = x_1 + h$ and so on.

The Composite Trapezoidal Rule is given by

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx$$

$$+ \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

10

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right]$$

$$+ \frac{h}{2}\left[f(x_2) + f(x_3)\right] + \cdots$$

$$+ \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1) + f(x_1) + f(x_2)\right.$$

$$+ f(x_2) + f(x_3) + \cdots$$

$$\left. + f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

11

$$= \frac{h}{2}\left[f(x_0) + 2\{f(x_1) + f(x_2) + \cdots + f(x_{n-1})\} + f(x_n)\right]$$

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[\left[f(x_0) + f(x_n)\right] + 2\left(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right)\right]}$$

OR

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[A + 2B\right]}$$

where $A = f(x_0) + f(x_n)$

$B = f(x_1) + f(x_2) + \cdots + f(x_{n-1})$

OR

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

(12)

Q-1 Using Trapezoidal Rule, evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ where $h = 0.2$.

Soln Given that

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_a^b f(x)\,dx.$$

Here $\boxed{a=0}$, $\boxed{b=1}$

$$\boxed{f(x) = \frac{1}{1+x^2}}$$

$$\boxed{h = 0.2}$$

x_0	x_1	x_2	x_3	x_4	x_5
0	0.2	0.4	0.6	0.8	1

(13)

Now

	x_0	x_1	x_2	x_3	x_4	x_5
x	0	0.2	0.4	0.6	0.8	1
$y=f(x)$	1	0.96	0.86	0.74	0.61	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

$$y_0 = f(0) = \frac{1}{1+(0)^2} = 1$$

$$y_1 = f(0.2) = \frac{1}{1+(0.2)^2} = 0.96$$

$$y_2 = f(0.4) = \frac{1}{1+(0.4)^2} = 0.86$$

$$y_3 = f(0.6) = \frac{1}{1+(0.6)^2} = 0.74$$

$$y_4 = f(0.8) = \frac{1}{1+(0.8)^2} = 0.61$$

$$y_5 = f(1) = \frac{1}{1+(1)^2} = 0.5$$

We know Traperzoidal Rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots y_{n-1})\right]$$

Here,

$$\int_{x_0}^{x_5} f(x)dx = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{0.2}{2}\left[(1+0.5)+2(0.96+0.86+0.74+0.61)\right]$$

$$= 0.1\left[1.5+2(3.17)\right]$$

$$= 0.1\left[1.5+6.34\right]$$

$$= 0.1\left[7.84\right]$$

$$= 0.784$$

$$\therefore \boxed{\int_0^1 \frac{1}{1+x^2}dx = 0.784}$$

15

But the actual integration

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4} = 0.785$$

(exact value)

i.e. Error = Exact value − observed value

$$= 0.785 - 0.784$$

$$\boxed{\text{Error} = 0.001}$$

Note:- It works on any number of intervals.

16

H.W

Q Using Trapezoidal Rule evaluate $\int_{2}^{10} \frac{1}{1+x} dx$

where $h=1$.

Ans 1.2965 observed value

Exact value
1.299

Error $= 1.299 - 1.296$

$= 0.003$