(1)

Cholesky's Method

Let us consider the system of linear eq^{n}s.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

It can be express into matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

$$\Rightarrow \boxed{A \cdot X = B} \longrightarrow (1)$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$

M	T	W	T	F	S	S	M	T	W	T	F	
1	2	3	4	5	6	7	8	9	10	11	12	
15	16	17	18	19	20	21	22	23	24	25	26	27
29	30	31										

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$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

The necessary condition for Cholesky's method is symmetricity.

i.e. $\boxed{A = A^T}$

Also the sufficient condition for Cholesky's method is positive definiteness,

i.e. $\boxed{X^T A X > 0}$

Now eqn 1 can be written as $\boxed{L L^T X = B}$ — 2

(3)

Let $\boxed{L^T X = Y}$ — (3)

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times1}$

taking $\boxed{LY = B}$ — (4) [from eqn (2)]

we get the value of Y.

i.e. y_1, y_2, y_3.

Finally taking $\boxed{L^T X = Y}$

we get the value of

x_1, x_2, x_3 i.e. X.

which is the required result of Cholesky's method.

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Working Rule :-

To solve the system of linear eqn by Cholesky's method we use following steps.

Step-1 :- Express the given eqn's in to matrix form

if $\boxed{AX = B}$ —— 1

find A, X, B

Step-2 verify

$\boxed{A = A^T}$

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Step-III :- eqn (i) can be written as

$$\boxed{LL^T X = B}$$ where $\boxed{A = LL^T}$

$L \rightarrow$ lower triangular matrix

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

Step-IV Taking $\boxed{A = LL^T}$

find all elements of Lower triangular matrix 'L'.

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Step-V Taking $\boxed{LY = B}$

find the value of Y.

i.e y_1, y_2, y_3 where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Step-VI: Finally taking

$$\boxed{L^T X = Y}$$

find the value of x_1, x_2, x_3

i.e $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

which is required result.

⑦

Q. Solve the system of linear eqn by Cholesky's method.

$$4x_1 + 2x_2 + 14x_3 = 14$$
$$2x_1 + 17x_2 - 5x_3 = -101$$
$$14x_1 - 5x_2 + 83x_3 = 155$$

Soln Given that

$$4x_1 + 2x_2 + 14x_3 = 14 \quad \text{—— (1)}$$
$$2x_1 + 17x_2 - 5x_3 = -101 \quad \text{—— (2)}$$
$$14x_1 - 5x_2 + 83x_3 = 155 \quad \text{—— (3)}$$

The above eqns are expressed into matrix form

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$$\Rightarrow \boxed{A \cdot X = B} \quad \text{—— (4)}$$

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where $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 13 \end{bmatrix}_{3\times3}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1}$

$B = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}_{3\times1}$

Now $A^T = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 13 \end{bmatrix}$ $R_1 \to C_1$ $R_2 \to C_2$ $R_3 \to C_3$ or vice versa.

$\therefore \boxed{A = A^T}$

It satisfies symmetric condition.

Now eqn (4) can be written as

$$\boxed{L.L^T X = B} \quad \text{---(5)}$$

where $A = LL^T$

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$\therefore$ Here $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

$\Rightarrow L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$

Now $\boxed{A = LL^T}$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \cdot \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}.l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}.l_{11} & l_{31}.l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Comparing both sides we get

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$l_{11}^2 = 4$

$\boxed{l_{11} = 2}$

$l_{21} l_{11} = 2$

$l_{21} \cdot 2 = 2$

$\boxed{l_{21} = 1}$

$l_{31} \cdot l_{11} = 14$

$l_{31} \cdot 2 = 14$

$\boxed{l_{31} = 7}$

$l_{11} \cdot l_{21} = 2$

$\boxed{l_{21} = 1}$

$l_{21}^2 + l_{22}^2 = 17$

$(1)^2 + l_{22}^2 = 17$

$l_{22}^2 = 16$

$\boxed{l_{22} = 4}$

$l_{31} \cdot l_{21} + l_{32} l_{22} = -5$

$7 \cdot 1 + l_{32} \cdot 4 = -5$

$l_{32} = -3$

$l_{11} \cdot l_{31} = 14$

$\boxed{l_{31} = 7}$

$l_{21} \cdot l_{31} + l_{22} l_{32} = -5$

$1 \cdot 7 + 4 \cdot l_{32} = -5$

$4 l_{32} = -5 - 7$

$\Rightarrow l_{32} = \frac{-12}{4}$

$\boxed{l_{32} = -3}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 83$

$(7)^2 + (-3)^2 + l_{33}^2 = 83$

$l_{33}^2 = 83 - 58$

$l_{33}^2 = 25$

$\boxed{l_{33} = 5}$

$$\therefore L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$$

$$L^T = \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

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Now taking $\boxed{Ly = B}$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2y_1 \\ y_1 + 4y_2 \\ 7y_1 - 3y_2 + 5y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$$\Rightarrow 2y_1 = 14 \Rightarrow \boxed{y_1 = 7}$$

$$y_1 + 4y_2 = -101$$

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$$\Rightarrow 7 + 4y_2 = -101$$

$$4y_2 = -108$$

$$y_2 = \frac{-108}{4} = -27$$

$$\boxed{y_2 = -27}$$

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Next $7y_1 - 3y_2 + 5y_3 = 155$

$7(7) - 3.(-27) + 5y_3 = 155$

$49 + 81 + 5y_3 = 155$

$5y_3 = 155 - 130 = 25$

$y_3 = 25/5 = 5 \rightarrow \boxed{y_3 = 5}$

Now $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$

Now Taking $\boxed{L^T x = y}$

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 + 7x_3 \\ 4x_2 - 3x_3 \\ 5x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$$

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9 Comparing both sides we get

10 $5x_3 = 5 \Rightarrow \boxed{x_3 = 1}$

11 $4x_2 - 3x_3 = -27$

12 $4x_2 - 3(1) = -27$

1 $4x_2 = -27 + 3 = -24$

2 $x_2 = -24/4 = -6$

3 $\boxed{x_2 = -6}$

And $2x_1 + x_2 + 7x_3 = 7$

5 $2x_1 + (-6) + 7(1) = 7$

6 $2x_1 - 6 + \cancel{7} = \cancel{7}$

7 $\Rightarrow 2x_1 - 6 = 0 \Rightarrow 2x_1 = 6$

$\boxed{x_1 = 3}$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}$$

H.W

Q Using cholesky's method solve

$9x_1 + 6x_2 + 12x_3 = 87$, $6x_1 + 13x_2 + 11x_3 = 118$

$12x_1 + 11x_2 + 26x_3 = 154$

Ans $[3, 6, 2]^T$